

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/231144957>

# Estimation of domains of attraction in epidemiological models with constant removal rates of infected individuals

Article in *Journal of Physics Conference Series* · December 2007

DOI: 10.1088/1742-6596/90/1/012052

CITATIONS

4

READS

19

3 authors, including:



**Anibal Blanco**

Planta Piloto de Ingeniería Química

45 PUBLICATIONS 154 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:



Development of soft computing techniques for field weed emergence prediction [View project](#)



Development of decision making tools for Integrated weed management: its application to agricultural systems of south-central region of Buenos Aires province, Argentina [View project](#)

All content following this page was uploaded by [Anibal Blanco](#) on 22 July 2015.

The user has requested enhancement of the downloaded file. All in-text references [underlined in blue](#) are added to the original document

## Estimation of domains of attraction in epidemiological models with constant removal rates of infected individuals

Luis G. Matallana, Aníbal M. Blanco, J. Alberto Bandoni

PLAPIQUI (UNS-CONICET), Camino “La Carrindanga” km. 7, (8000) Bahía Blanca, Argentina

E-mail: {lmatallana, ablanco, abandoni} @plapiqui.edu.ar

**Abstract.** The spread of infections is commonly represented through the so-called Susceptible – Infectious – Recovered models (SIR). Treatment based on isolation of infected individuals is often applied to decrease the spread of certain diseases. Such situation is considered in the SIR model through a constant removal rate term. It has been shown that in such cases, the outcome of the disease spread may depend on the position of the initial states for certain range of the model parameters. The estimation of the domains of attraction of the equilibrium points may therefore be useful to understand the dynamic behavior of the infection spread as a function of the initial population distribution. In this contribution, a Lyapunov based approach to estimate the domain of attraction of the endemic steady state point in SIR models is proposed.

### 1. Introduction

Many models to study the dynamics of the spread of infectious diseases have been proposed so far to address particular aspects of epidemic and endemic dynamics. For a comprehensive survey on the subject, see [1]. Such models have been shown useful in other fields, for example in the study of the dynamics of virus [2], ideas [3] and rumors [4].

In particular, the SIR model has been intensively addressed, since it captures the essentials of epidemical dynamics. The SIR model is conformed by the balance equations of Susceptible, Infected and Recovered individuals of a population. The model assumes “horizontal incidence”, meaning that susceptible individuals get infected through contact with infected individuals. The simplest model for incidence rate is homogeneous mixing, represented by a bilinear relation between susceptible and infected members.

Recovery of infected individuals may be due to a natural mechanism or through some treatment strategy. In order to model the isolation treatment strategy (quarantine) to combat disease spread in SIR models, an infected individual removal rate term is included in the equations.

For a bilinear incidence rate, the reduced SIR model is usually described by the following system of nonlinear differential equations [5]:

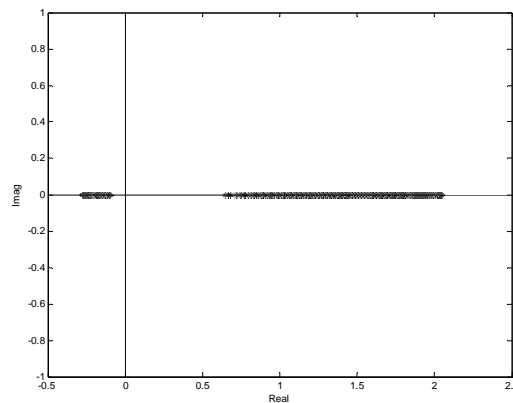
$$\frac{dS}{dt} = A - ds - \lambda SI \quad (1.a)$$

$$\frac{dI}{dt} = \lambda SI - (d + \gamma)I - r \quad (2.a)$$

being  $S$  and  $I$  the numbers of susceptible and infected individuals of a certain population respectively,  $A$  the recruitment rate of the population,  $d$  the natural rate of death of the population,  $\gamma$  the natural recovery rate of the infected individuals and  $\lambda$  the per capita contact rate. Parameter  $r$  represents the capacity of treatment for infected individuals, which plays the role of a constant removal rate term.

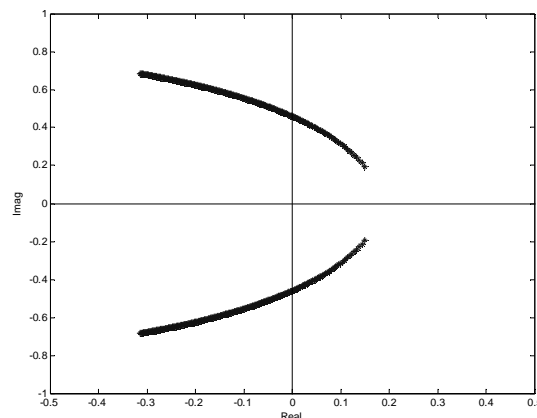
It has been shown [5] that such a term has a significant effect on the resulting dynamics. While for  $r=0$ , system (1) presents globally stable equilibrium points (infection-free or endemic) [6], for  $r > 0$ , system (1) may present two coexisting equilibrium states (E1 and E2).

Consider the following parameterization  $A = 4$ ,  $d = 0.3$ ,  $\gamma = 0.8$  and  $\lambda = 0.3$ , (see [5] for details). Figures 1 and 2 show the bifurcation diagrams of both equilibriums regarding parameter  $r$  varying from 0.5 to 0.91. On figure 1, it can be seen that E1 is always a saddle node, since an eigenvalue remains real positive and the other real negative for increasing  $r$ . Equilibrium E2 presents a stable node to unstable node bifurcation (*Hopf bifurcation*), because the real part of both eigenvalues crosses the imaginary axes as  $r$  increases, (see figure 2).

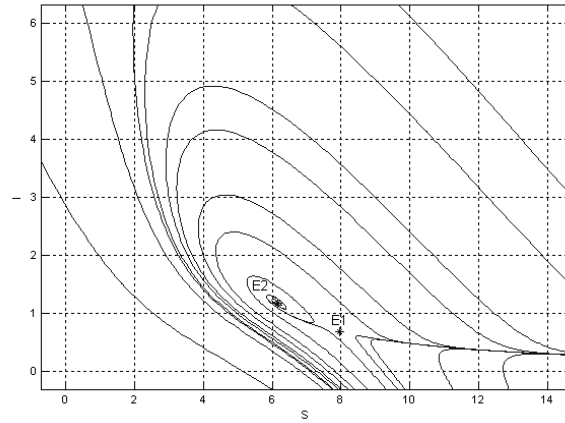


**Figure 1.** Bifurcation diagram for E1

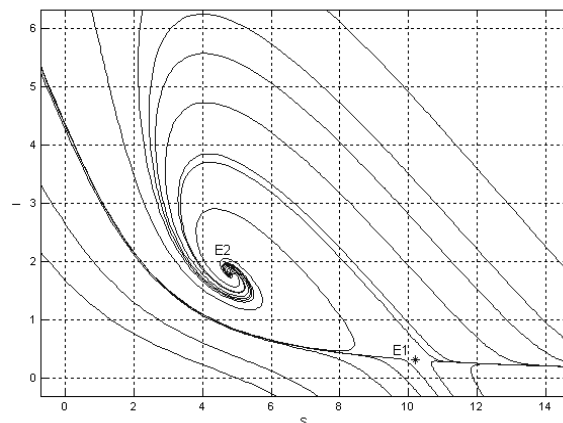
For a large enough removal rate ( $r = 0.87$ ), E2 will remain unstable and the disease will become extinct in finite time (figure 3). For a lower removal rate ( $r = 0.6$ ), equilibrium E2 will become locally asymptotically stable and the final situation of the population will depend on the initial state, meaning the initial distribution of infected and susceptible individuals (figure 4).



**Figure 2.** Bifurcation diagram for E2



**Figure 3.** Phase portrait for  $r=0.87$  E1: (7.95, 0.67); E2: (6.14, 1.16)



**Figure 4.** Phase portrait for  $r=0.60$  E1: (10.21, 0.30); E2: (4.78, 1.78)

The set of initial states whose corresponding trajectories converge to an asymptotically stable equilibrium point as time increases is known as the “stability region” or “Domain of Attraction” (DA) of the equilibrium under study [7]. If the initial state lies within the DA(E2), the disease will evolve towards an endemic state. If the initial state is outside the DA(E2), the system will converge to a disease-free situation.

Some knowledge of the shape and size of such a region is considered valuable to understand the dynamic behavior of the infection spread. In this contribution, a *Lyapunov*-based approach to estimate the DA of the endemic steady state point in SIR models of type (1) is proposed.

The remaining part of the article is structured as follows. In section 2, the basics of the adopted *Lyapunov* based technique is described. In section 3, the proposed technique is applied to the estimation of the DA of a particular instance of model (1) (figure 4). A section for conclusions and future work closes the article.

## 2. Determination of the DA

The problem of determination of DA for dynamical nonlinear systems has been largely addressed in literature due to the important theoretical and practical implications [7]. Among the many existing

approaches [8], a particularly important family of techniques is based on the *Lyapunov* stability theory. In the following, the fundamentals of the *Lyapunov* based method for DA determination will be outlined. For a rigorous and comprehensive treatment, see -for example- [7].

Consider the following autonomous nonlinear dynamic system:

$$\frac{d\mathbf{x}}{dt} = [\mathbf{f}(\mathbf{x}(t))] \quad (2)$$

being the origin of the state space,  $\mathbf{x} = \mathbf{0}$ , an asymptotically stable equilibrium point. The DA of the origin is defined by the set of initial conditions  $\mathbf{x}_0$ , such that the trajectories initiated at  $\mathbf{x}_0$  (denoted by  $\mathbf{x}(\mathbf{x}_0, t)$ ) converge to the origin as time increases:

$$DA(\mathbf{0}) = \{\mathbf{x}_0 : \mathbf{x}(\mathbf{x}_0, t) \rightarrow 0 \text{ as } t \rightarrow \infty\} \quad (3)$$

Consider that a function  $V(\mathbf{x})$  exists such that:

$$V(\mathbf{x}) > 0 \text{ in } R(\mathbf{0}) \quad (4.a)$$

$$\frac{dV(\mathbf{x})}{dt} < 0 \text{ in } R(\mathbf{0}) \quad (4.b)$$

Equation (4a) means that  $V(\mathbf{x})$  is positive definite in some region  $R(\mathbf{0})$  around the origin and (4b) that the time derivative of  $V(\mathbf{x})$  is negative definite in that region. Positive (negative) definition of a function in some region of the state space implies that the function is strictly positive (negative) in all  $\mathbf{x} \neq \mathbf{0}$  in  $R(\mathbf{0})$  and 0 in  $\mathbf{x} = \mathbf{0}$ . If (4a) and (4b) hold for system (2) then  $V(\mathbf{x})$  is known as a *Lyapunov* function of (2) at the origin.

Let  $V(\mathbf{x}) = c$  be the level set of  $V(\mathbf{x})$  at value  $c$ , this is, the projection on the state space of  $V(\mathbf{x})$  at value  $c$ . Consider the region  $S(\mathbf{0}) = \{\mathbf{x} : V(\mathbf{x}) - c \leq 0\}$  is contained in  $R(\mathbf{0})$ :

$$S(\mathbf{0}) \subset R(\mathbf{0}) \quad (5)$$

then  $S(\mathbf{0})$  belongs to the DA of the origin and can be considered an estimation of  $DA(\mathbf{0})$ . Clearly, the larger the  $c$ , the better the estimation of the DA.

The main result of the *Lyapunov* stability theory establishes that a quadratic-type *Lyapunov* function,  $V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$ , being  $\mathbf{P}$  a symmetric positive definite matrix, always exists for an asymptotically stable equilibrium point of a dynamic system. Therefore, an estimation of the corresponding DA can be always calculated through (5) (see [7] for details). However, the estimations based on quadratic *Lyapunov* functions are usually very conservative for meaningful analysis and better approaches are usually desired.

A particularly successful methodology is based on the so-called maximal *Lyapunov* functions. A powerful technique for the estimation of domains of attraction based on maximal *Lyapunov* functions has been proposed in [9]. In the remaining part of this section, the basics of such a technique will be outlined. The reader is referred to [9] for the details.

Consider the following maximal rational *Lyapunov* "candidate" for  $n \geq 2$ :

$$V(\mathbf{x}) = \frac{N(\mathbf{x})}{D(\mathbf{x})} = \frac{\sum_{i=2}^n R_i(\mathbf{x})}{1 + \sum_{i=1}^{n-2} Q_i(\mathbf{x})} \quad (6)$$

conformed by a numerator  $N(\mathbf{x})$  and a denominator  $D(\mathbf{x})$  and being  $R_i(\mathbf{x})$  and  $Q_i(\mathbf{x})$  homogeneous functions of degree  $i$ . Consider also the system of nonlinear equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}(t)] = \sum_{i=1}^{\infty} F_i(\mathbf{x}) \quad (7)$$

being  $F_i(\mathbf{x})$  homogeneous functions of degree  $i$ .

In order to ensure condition (4b), the time derivative of  $V(\mathbf{x})$  is equated to a negative definite function. Consider  $\mathbf{Q}$  a positive definite matrix, then:

$$\frac{dV(\mathbf{x})}{dt} = \nabla^T V \frac{d\mathbf{x}}{dt} = -\mathbf{x}^T \mathbf{Q} \mathbf{x} \quad (8)$$

From (6) and (8) it follows that:

$$\left[ \left( I + \sum_{i=1}^n Q_i \right) \sum_{i=2}^n \nabla^T R_i - \left( \sum_{i=2}^n \nabla^T Q_i \right) \sum_{i=2}^n R_i \right] \sum_{i=1}^n F_i = -\mathbf{x}^T \mathbf{Q} \mathbf{x} \left( I + \sum_{i=1}^n Q_i \right)^2 \quad (9)$$

By considering the homogeneous function representation of  $N(\mathbf{x})$ ,  $D(\mathbf{x})$  and  $\mathbf{f}(\mathbf{x})$  (eqs (6) and (7)), equating the coefficients of the degrees until  $n$  at both sides of (9) and minimizing the norm of the terms larger than  $n$  by manipulating the coefficients of  $R_i$  and  $Q_i$ , it can be proved [9] that the resulting  $V(\mathbf{x})$  (6) is a *Lyapunov* function of (7).

Once a *Lyapunov* function is obtained by the above described procedure, an estimation  $S(\mathbf{0})$  of  $DA(\mathbf{0})$  can be calculated by finding the largest positive value of  $c$  that verifies (5). Several strategies can be adopted for such a purpose ([9] and [10]). For the present contribution it is proposed the solution of the following optimization problem:

$$\begin{aligned} \max_{c, \mathbf{X}} \quad & c \\ \text{s.t.} \quad & \mathbf{X} \subset \{V(\mathbf{x}) - c \leq 0\} \quad a) \\ & \frac{dV(\mathbf{x})}{dt} \leq 0 \quad b) \end{aligned} \quad (10)$$

$\mathbf{X}$  represents a set of discrete points contained in  $S(\mathbf{0})$ . For an initial  $c$  ( $c^0$ ) a grid of points ( $\mathbf{X}^0$ ), which verifies (10a) is generated. Then, constraint (10b) is evaluated for each point of  $\mathbf{X}^0$ . If  $dV(\mathbf{x})/dt \leq 0$  for all points of  $\mathbf{X}^0$ ,  $c$  is updated ( $c^1$ ) and a new set of points ( $\mathbf{X}^1$ ) is calculated. The algorithm proceeds until no feasible improvements on  $c$  are possible regarding constraint (10b).

The described approach usually produces very good estimates of  $DA(\mathbf{0})$  with respect to classic quadratic *Lyapunov* function, if  $n$  is large enough due to the “blowing up effect” of  $D(\mathbf{x})$  in (6).

### 3. Computational results

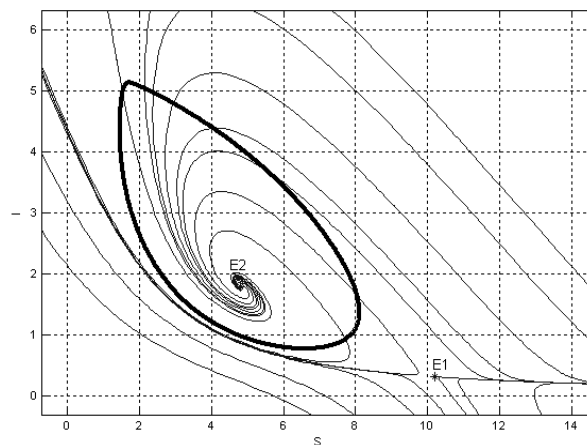
As can be appreciated from figure 4, the  $DA(E2)$  is an infinite region in the state space bounded by the stable manifold of equilibrium  $E1$ . By applying the procedure described in the previous section for  $r = 0.6$ ,  $n = 4$  and  $\mathbf{x} = [x_1, x_2] = [S, I]$ , the following maximal *Lyapunov* function was attained:

$$V(\mathbf{x}) = \frac{R_2(\mathbf{x}) + R_3(\mathbf{x}) + R_4(\mathbf{x})}{1 + Q_1(\mathbf{x}) + Q_2(\mathbf{x})} \quad (11)$$

where:

$$\begin{aligned}
 R_2(\mathbf{x}) &= 1.8205x_1^2 + 3.8124x_1x_2 + 6.6580x_2^2 \\
 R_3(\mathbf{x}) &= -0.0296x_1^3 + 0.3893x_1^2x_2 + 0.5026x_1x_2^2 - 0.0956x_2^3 \\
 R_4(\mathbf{x}) &= -0.0191x_1^4 - 0.0516x_1^3x_2 - 0.1948x_1^2x_2^2 - 0.1248x_1x_2^3 - 0.1341x_2^4 \\
 Q_1(\mathbf{x}) &= 0.1850x_1 + 0.7683x_2 \\
 Q_2(\mathbf{x}) &= -0.0390x_1^2 - 0.0281x_1x_2 - 0.0888x_2^2
 \end{aligned}$$

By solving problem (10), a value of  $c = 13$  is obtained for *Lyapunov* function (11). The corresponding level surface is shown in solid line in figure 5.



**Figure 5.** Level set  $V(x) = 13$ ,  $r = 0.60$ , E1: (10.21, 0.30); E2: (4.78, 1.78)

It can be observed from figure 5 that a finite estimation  $S(E2)$  of the infinite  $DA(E2)$  was produced with the considered maximal *Lyapunov* function. Although the level set underestimates the real domain, it closely approximates the region near the stability boundary (the stable manifold of E1) in an important portion of the state space. Such information may be useful to decide if an endemic situation will arise depending on the initial distribution of the population for a certain infected individuals removal capacity: any initial condition of infected and susceptible individuals lying within  $S(E2)$  will reach the endemic state in finite time.

Nothing can be concluded regarding the final state of the system if the initial condition lies outside  $S(E2)$

#### 4. Conclusions and future work

In this contribution, a maximal *Lyapunov* function approach has been applied to the estimation of the  $DA$  of a stable endemic equilibrium point of an epidemiological model with a constant removal rate of infected individuals. The estimated region  $S(E2)$  approximates the actual  $DA(E2)$  near its stability boundary, and it can be ensured that any initial state within  $S(E2)$  will end up in an endemic situation.

Larger estimations of  $DA(E2)$  can be produced by increasing the number of terms in the *Lyapunov* candidate (6).

The proposed technique can also be applied to other type of epidemiological models with complex dynamics, for example, those arising from incidence rate models ( $f(I,S)$ ) different than the simple classic bilinear relationship to account for homogeneous mixing of the population assumed in (1):  $f(I,S) = \lambda I S$ . When homogeneous mixing is no longer a valid assumption due to the population structure, other nonlinear incidence rates should be considered. Such situations are usually associated

to cooperation or community effects which might be better represented by incidence rates of the form  $f(I,S) = \lambda I^p S^q$  for  $p > 1$  and  $q > 0$ .

In such cases, two or more asymptotically stable equilibrium points may coexist [6]. In such situations, the dominating steady state (infection-free or endemic) will depend on the initial distribution of individuals (susceptible, exposed and infected) in the population and estimations of the corresponding DA would be valuable to analyze the spread outcome of the disease.

### References

- [1] H. Hethcote, "The Mathematics of Infectious Diseases", *Siam Review*, vol. 42, No 4, pp. 599-653, 2000.
- [2] S. Bonhoeffer, R. M. May, G. M. Shaw and M. A. Nowak, "Virus Dynamics and Drug Therapy", *Proc. Natl. Acad. Sci. USA*, vol. 94, pp. 6971-6976, 1997.
- [3] L. M. Bettencourt, A. Cintrón Arias, D. I. Kaiser and C. Castillo Chávez, "The Power of a Good Idea: Quantitative Modeling of the Spread of Ideas from Epidemiological Models", *Physica A*, vol. 364 pp. 513-536, 2006.
- [4] K. Thompson, R. Castro Estrada, D. Daugherty, and A. Cintrón Arias, "A Deterministic Approach to the Spread of Rumors", *Cornell University, Department of Biological Statistics and Computational Biology, Technical Report BU-1642-M*, 2003.
- [5] W. Wang and S. Ruan, "Bifurcations in an Epidemic Model with Constant Removal Rate of Infectives", *Journal of Mathematical Analysis and Applications*, vol. 291, pp. 775-793.
- [6] A. Korobeinikov and P. Maini, "Non-linear incidence and Stability of Infectious Disease Models", *Mathematical Medicine and Biology*, vol. 22, pp. 113-128, 2005.
- [7] H. K. Kahlil, *Nonlinear Systems*, Prentice Hall, 1996.
- [8] R. Genesio, M. Tartaglia and A. Vicino, "On the Estimation of Asymptotic Stability Regions: State of the Art and New Proposals", *IEEE Transactions on Automatic Control*, vol. AC30, No 8, pp. 747-755, 1985.
- [9] A. Vannelli and M. Vidyasagar, "Maximal Lyapunov Functions and Domains of Attraction for Autonomous Nonlinear Systems", *Automatica*, vol. 21, No1, pp. 69-80, 1985.
- [10] O. Hachicho, "A Novel LMI-based Optimization Algorithm for the Guaranteed Estimation of the Domain of Attraction Using Rational Lyapunov Functions", *Journal of the Franklin Institute*, 344, pp. 535-552.