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The Common Link: An Exploration of the Social Cognitive Dimensions of Meaning-Making in Algebra and the Visual Arts Using a Case Study Approach

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The Common Link: An Exploration of the Social Cognitive Dimensions of Meaning-Making in Algebra and the Visual Arts Using a Case Study Approach

by

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Abstract

It is commonplace to hold that algebra and the visual arts are mutually exclusive activities. In this thesis, an attempt was made to connect how we learn in algebra and the visual arts from the social cognitive perspective proposed by Bandura (1986, 1997). That is, the personal, social, and behavioural dimensions of learning in algebra and the visual arts were considered. Also, the issue of a connection between algebra and the visual arts was tackled by taking into account the most recent advances in cognitive science, like the "situated movement," the notion, in a nutshell, that cognition is extended throughout our social relations and practices.

Making the connection between, what Snow (1959) called generally the "two cultures" (cited in Stent, 2001, p. 31) of art and science, has precedence. There have been attempts, as interpreted in this thesis, to consider what learning in the arts and sciences have in common from various quarters, be they philosophical, psychological, or historical.

Identifying the link between algebra and the visual arts involved several things. First, the historical context for the schism between our understanding of learning in algebra and the visual arts was considered. Second, a detailed review-cum-analysis of the literature was undertaken, and this yielded the themes upon which the connections between algebra and the visual arts were made.

Turning to the fieldwork, four probing case studies were utilized to explore how those in algebra or the visual arts learn in their fields. By analyzing the data from the case studies, pattern regularities between learning in algebra and the visual arts were extracted.

Finally, the theoretical and pedagogical consequences of having made the common link between algebra and the visual arts were addressed. Theoretically, by considering the role of, for instance, aesthetics and identity as reasons to pursue algebra or the visual arts, Bandura's (1986, 1997) social cognitive theory was corroborated and enlarged. Practically, recommendations were offered for the pedagogy of algebra and the visual arts.

Dedication

To Savita

For everything we lost,

"*Toota dil kubi murka nehi jurdai*" ([Punjabi], Malik, 2003).

(A broken heart [even] by movement never comes back together.)

"*Hum dekna ha*" ([Punjabi], Anonymous).

(We'll see.)

Acknowledgements

I enjoyed writing this thesis and working with the people to be discussed here; I indicate a link or path to their Web sites in parentheses documented in the "References." My greatest debt is to my supervisor Dr. Ezeife (2010). It is noteworthy that he not only had the presence of mind to ferret out the weak spots in the arguments, but also to take the next step: to hint at what was needed to patch them up. In addition, he also brilliantly had the foresight to allow me time to overcome some challenges by myself! He calibrated his comments to me, not crushing me when the thesis was only a skeleton of an idea. He was always available and spent long hours with me going over the dissertation line-by-line, when needed. He put me at ease during the sometimes unnerving process of fieldwork. I have learned from Dr. Ezeife not only the fundamentals of research in the social sciences, but about the human side of it and teaching.

When I requested Dr. Ezeife's supervision, it was a difficult period in my life with the passing of my father in 2008. Dr. Ezeife had the humanity to console me but still kept my research on-track. To find the qualities on both sides of putative opposing poles, of rigor and humanity, of scholarship and teaching excellence, of idealism and activism, in one person is uncommon. I found the optimization of all of these virtues in Dr. Ezeife. My time with him was a blessing, and its reverberations shall be long and deep. I hope we never lose touch.

The next series of acknowledgements go to my committee, and I list them in alphabetical order. Dr. Bayley has been a lighthouse, carefully reading the thesis and making sagacious suggestions. He has also helped guide my career. When I first enrolled in the Joint PhD program, I was interested in organizational behavior and mathematical truth, enrolling in the administration stream of the program. He shifted me into Cognition and Learning, which best captures what I wanted to do, namely, explore the influence of social groups upon individual behaviour.

Down the road from the educational building at the University of Windsor, Dr. Goodwin (http://www.mindcrafters.net/) was kind enough to participate in this doctoral committee, even though his research is squarely in the field of computer science. However, he is also an intellectual with broad interests including, for example, the cognitive science program. He raised questions from angles that social scientists may often not consider, and for that input the dissertation is much richer.

Some of the very hard questions, in fact, are not technical, but rather deal with the very nature of my project. Dr. Goodwin raised the tough questions time and again. He has also maintained a friendly and supportive role during the grueling, and sometimes solitary, process of producing a dissertation. On this score, producing this thesis has not been too isolating since social science research often involves other people—but Dr. Goodwin made the process all the less nerve-racking.

From Brock University, Dr. Hutchison (http://www.ed.brocku.ca/~dhutchison/) encouraged my attempts to build upon my philosophical education during Core I, resulting in a paper on Aristotle's *Poetics*, which was subsequently accepted for publication (Gupta, in press-a). Dr. Hutchison has the singular virtue of being able to interrogate one's idea without ever making the exchange detrimental; that is, he accomplishes perhaps the most difficult aspect of teaching: he can deliver both positive and negative searching insights into a piece of writing, while at the same time, increasing motivation, enthusiasm, and the quality of the product. His comments have always improved the quality of this thesis, making it a better fit with an educational audience that it would have otherwise been.

I thank my external examiner Dr. Sugarman (http://www.educ.sfu.ca/profiles/?page_id=272), for his willingness to participate from the other side of the coast that touches the Pacific Ocean. With his background and expertise in counseling psychology and educational psychology, he is the good complement to our interdisciplinary committee. Specifically, his cutting-edge research on the socio-cultural dimensions of human agency provides, I believe, unique opportunities for my growth and our, possibly, future collaborations.

In fact, the choice of Dr. Sugarman was advocated for by our own resident educational psychologist, Dr. Maldonado, who inferred from his Web site hosted by Simon Fraser University that he had an interest in humanistic, qualitative research. For always being available and approachable, for her encyclopedic knowledge of qualitative research, and for her original, interpersonal approach to educational psychology, I gratefully thank Dr. Maldonado.

She is a veritable goldmine for qualitative researchers and a first-rate mentor to her colleagues and students. She offered me invaluable tips on conducting qualitative research and advised me on managing the massive data a study of this nature generates. I have learned from Dr. Maldonado the importance of interpersonal relations to

understanding the mind. She modeled for me, in fact, an entirely different way of being—a more interpersonal, interactive way—putting theory into practice before my eyes.

I owe a lot of debt of Dr. Stanley, a specialist in mathematics education in the Faculty of Education, University of Windsor, for helping me vet the initial idea of this thesis, to ensure it would be of interest to educationists, and subsequently assisting me with refining some of my mathematical definitions, specifically, *algebra*.

Many thanks go also to Dean Rogers of the Faculty of Education, University of Windsor. She had the visionary, courageous leadership to build a unique department that is highly competent, collegial, and diverse—the perfect environment for active, far-reaching research. I could not think of a better place to have done this research in Canada (other than, perhaps, in British Columbia, partly because of the weather, mountains, and ocean there).

Turning to my extended circle of teachers and friends (and this distinction has blurred over the years), thanks to Dr. Brook, in the department of philosophy at Carleton University, for continuing to correspond with me since my only course with him on the science of the mind in 1989; to Dr. Hunt, in the department of psychology at Brock University for his encouragement on this arduous path I have trod; and Dr. Allen in the department of philosophy at McMaster University for being available whenever I needed his help; he made me believe in my own instincts. Quite recently, Dr. Chakravartty (2007) of the Institute for the History and Philosophy of Science and Technology at the University of Toronto, was kind enough to correspond with me about his first book on scientific realism. Imagine my surprised delight when I read about his discussion of the relation between art and science at the end of his text!

Ms. Gayle Tait, the graduate secretary in the Faculty of Education, University of Windsor, went above and beyond the call of duty by making me feel welcome; she made sure the paper work was in order for several awards and applications. University of Windsor's Leddy Library, specifically the staff at the Inter-Library Loan office, did commendable work by getting me resources quickly. I would also like to thank Ms. Karina Schneider of the School of Graduate Studies of the University of Windsor. She did excellent work, making sure my ducks were in a row when it came to some matters of copyright. She also helped clarify some of the minutiae of the formatting requirements of Graduate Studies, specifically how far, if it all, I can depart from them.

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My final acknowledgement goes to my wife, Manjit, for having me. Even though so much of my mind rolls around with ideas for the next big writing project; she brings me back to earth, and makes me think about matters that would have otherwise escaped me. She is there when I am successful, making it worthwhile to be so; and she renders my fledgling, and sometimes flopping, attempts more bearable than they would otherwise be. We are both caught up in the common link.

Even though all work has a social context, and a thesis is especially collaborative: all errors are solely mine.

Note on the Use of Artifacts and Ben's Work

I have used samples of participants' work in this dissertation. Many thanks go to Mr. Gene Lewis, General Secretary of the Elementary Teachers' Federation of Ontario (ETFO), who granted me permission (see Appendix F) to reprint one of his organization's images. As required, when I use samples of participants' work to illustrate their cases, I have their approval to do so (see Appendices E-F); and in all cases they are aware of my use of their works in this dissertation.

In fact, all the materials in Appendices G-J relating to Ben, a mathematician and the first participant interviewed are already open-access, in the public domain, and are available on the Internet. A further word is in order about the appendices related to Ben's work.

This is not a dissertation in mathematics. Little attempt is therefore made to explain the technical aspect of Ben's work, going beyond my purposes in this thesis. In fact, Ben's research into algebra goes beyond the elementary type I am using for the purpose of illustration in this thesis. However, it is important to note that his core interest does span algebra and number theory. My aim in reproducing some of his work in the appendices, and discussing them more generally, is to illuminate some of the psycho-social behaviours that I commented upon in this thesis.

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1. Introduction: A Philosophically Minded Prelude

"In the general case, I dislike introductions; what they convey, when not superfluous, is usually premature" (Fodor, 1982, p. 1).

I begin with a unusual challenge. At the most general level—by comparing algebra and the visual arts what can we learn about the mind? Algebra and the visual arts are as far apart as two fields could be in the academy. The fissure between them has manifestations that are socio-cognitive, biological, and philosophical. Socio-cognitively, many students leave mathematics after high school to pursue a humanities education, for example, having had negative experiences in the former. This situation often motivates social scientists to unpack the dynamics of why, and how to change these occurrences (e.g., Bishop, 1988; Ezeife, 1995, 2000, 2003; Robitaille, Beaton, & Plomp, 1999). Biologically, specifically for neurobiological reasons, perhaps, some people are just inclined to mathematics or the arts, but not both. Also, mathematics is often considered abstract, while the arts concrete; a view that is reflected in our philosophical tradition, which I turn to consider now.

As far back as Plato's (trans. 1992) *Republic,* in Book 6, we encounter an ontological, epistemological, and ethical divided line between mathematics and the visual arts. The following definitions may be helpful to navigate the philosophical discussion:

- Ontology is the theory of what *is*. It concerns the issue of the reality of something (e.g., colors, numbers, thoughts, tables and chairs, subatomic particles, and so on).

- Epistemology is a theory of knowledge. That is, how do we come to know, for example, about tomatoes (e.g., do we use perception, mental processes, social interactions, and so on?)
- Ethics concerns human actions related to what we ought to do.

Table 1.

Analysis of Plato's (trans. 1992, 6, 509d) Divided Line in the Republic Adapted from Strawser ([emphasis mine]. 2008)

1. Ontology	**2. Epistemology**	**3. Output**	**4. Source of output**
The Good	(*noesis*) rational intuition/ understanding	↓ Knowledge (the highest type of knowledge is *sophia,* i.e., wisdom) ↑	↓ Intelligible world lit by the form of the good
Higher forms (beauty, justice, truth)	(*dianoia*) reasoning (dialectic)		
Form of mathematics and science			

Particular things, living and artificial	(*pistis*) perception, belief	↓ Opinion (*doxa*)	↑
[*Art*], shadows, images, reflections	(*eikasia*) conjecture, imaging	↑	↓ Sensible world lit by the sun ↑

On the divided line (see Table 1, Column 1, "Ontology"), the Good is at the pinnacle, mathematics in the top half, and art likely at the bottom. Plato in fact does not place *art* on the divided line, but it has been commonplace for commentators (e.g., Strawser, 2008) to infer that it can be placed on it for reasons I will touch on shortly. In any case, the ways we come to know these things also differs: the Good is known through rational intuition, mathematics is known through reasoned understanding, particular things are known through perception, and shadows are known through imaging (see Table 1, Column 2, "Epistemology").

Mathematics, according to Plato, is closer to truth, involving the intellect, than the arts, which requires the senses. Famously, Plato banned poetry from his notional republic. "Imitative art…," Plato explained, "is far removed from truth" (trans. 1992, 10, 598c). According to Plato, we should study mathematics because it removes us from the world of particular things, assisting in our ascent towards universal truth.

Broadly, we can distinguish between the realm of the mind, roughly the upper half of the table; and the realm of the body, roughly the lower half of the table. Further, implicit in Plato's thinking is the idea that by accessing the truth we separate ourselves from the subjective or inter-subjective perspectives generated from dealings with particular things and particular practices—to see things as they really are. On Plato's account, because art involves images, particular things, the body, and social practices, it is standard practice, and indeed is reasonable, to think that art falls to the bottom of the divided line.

At the outset, I need to say a few words about the plausibility of a challenge to Plato's division between algebra and the visual arts. It is sometimes assumed that the epistemological divide between algebra and the arts, has a neurological basis: the two can be distinguished based on hemispheric specialization: algebra is done in the left side of the brain, and art in the right side of the brain. Yet it is important to note that this left-right brain hypothesis in relation to algebra and the arts is unsubstantiated. Contributing a monograph to a series that deals with the revolution in thinking about the relationship between the brain-behaviour-cognition nexus, Zaidel (2005) wrote,

> It is highly likely that both hemispheres modulate many human perceptions and expressions through functional specialization and complementarily, and this includes the multiple components of art, creativity, and the emotions. (p. xiv)

In short, what we do and cognize, in algebra and the arts likely draws upon both hemispheres; as Zaidel explained, for instance: there is no "art center," only dedicated neural networks, which we can infer are distributed, more or less, globally. According to him, neuro-anatomical and neuropsychological underpinnings of creativity need not be different based upon the domain in question.

Going forward, however, I accept that there are also likely differences between learning in algebra and the visual arts. For the cognitive scientist, *learning* is considered a second-order effect: transforming a system capable of certain performances into new ones. Yet, cognitive and neurological accounts of learning may diverge in important ways for these two domains. However, I focus on what is common to learning in these two domains, like the role of interpersonal relationships that has often been overlooked. I do not devote time to discussing the differences between learning in algebra and the visual arts, unless it sharpens the connections I want to make; because opposing the two domains in a bi-polar fashion, and overlooking the role of social context, I believe, are characteristics of the putative view, which I turn to discuss next.

The Platonic epistemological split between mathematics and the arts became solidified in later thinkers in different ways. I undertake a quick tour of the history of some milestones in modern philosophy in order to provide some examples. For Hume (1739-1740/2000), there was a distinction between knowledge derived from sense impressions and knowledge derived from the relation of ideas; and we are apt to think that art is the realm of the senses and algebra the realm of abstract ideas.

Building on Hume's thinking, Kant (1781/1989) distinguished between knowledge that was *a posterior,* from experience; and *a priori,* where knowledge is contained in its concept and necessarily true (e.g., the concept *triangle* must contain the idea of having three sides). Also, Kant claimed that mathematics was *synthetic a priori,* where various combinations of ideas are synthesized as the sources of knowledge. For example, according to him, the number "12" does not contain 7 and 5 in its concept, so it cannot be *a priori.* Because "12" can also be composed by 4 sets of 3, and this must be so, it is *synthetic a* *priori*: that is, we can put together "12" in various ways; there is no one definition based on how it is composed.

Returning to Platonism, the German logician Frege (1892/1970) distinguished a statement's sense, its meaning, from its reference, the object to which it refers; for example, both sides of the expression "2 x 3 = 11 – 5" have a different sense but the same reference, namely both sides are equivalent to "6." More specifically, Frege (1918-19/1984) claimed that the references of mathematical statements are their truth-values that exist in a mind-independent third realm. That is, according to Frege (1884/1953), mathematical truths are abstract in that we discover them, and they are not dependent on the empirical or mental realms. Like Plato's forms, mathematical truths exist independently of all human activity.

Yet there has been some movement to rethink at least the empirical basis of the formal sciences (Jenkins, 2008; Kitcher, 1983; Lakoff & Núñez, 1997, 2001; Mill, 1843/1876). It is commonplace for philosophers of mathematics interested in empiricism to distinguish between two tiers within the development of mathematical structures, the first where we acquire basic concepts through the interaction with the physical world; and the second that builds upon that interaction through the relation of ideas. The renewed interest in empiricism among philosophers of mathematics at least gives us cause to reconsider the bodily basis of our most abstract ideas, like number, and how this process of abstraction may be similar to learning in the arts.

Also, connections between mathematics and the arts via the social context have been explored by historians of different stripes. For example Mumford (1986), a social critic, has linked the development of abstraction in the arts with changes in mathematics. He pointed out that architectural development is dependent on the materials we use (e.g., stone, concrete, steel). According to Mumford (1963), cultural forms of Western civilization have been profoundly influenced by the machine. In the nineteenth century, Mumford (1963) said, science became a "mode of life" (p. 219), where precision and calculation became preeminent values. For instance, said Mumford (1986), the cubist artists exemplified the mechanical approach to man. He stated:

> Calculation, invention, mathematical organization play a special role in the new visual effects produced by the machine.... By a process of abstraction the new paintings finally, in some painters like Mondrain, approached purely geometrical formula, with a mere residue of visual content. (Mumford, 1986, p.85)

For Mumford, the development of the machine relies, at least for its refinement, upon modern mathematics, which impacted the art of that age.

In *Plato's Ghosts: The Modernist Transformation in Mathematics,* Gray (2008) offered a philosophically-minded, historical characterization of the period in terms of its members' preoccupation with formal aspects, and anxiety with this; and the development of non-Euclidian geometries. According to Gray:

> Here, modernism is defined as an autonomous body of ideas, having little or no outward reference, placing considerable emphasis on formal aspects of the work and maintaining a complicated—indeed, anxious—rather than a naive relationship with the day-to-day world. (2008, p. 1)

Gray also, like Mumford (1963, 1986) before him, cited cubists as exemplifying the modernist preoccupation with form in the arts. So, for historically-minded thinkers like

Mumford and Gray, our culture, that is, our shared technologies, patterns of behaviour, and values, do affect both our arts and mathematics.

It would be natural in this thesis to consider in some depth historically how we have taught algebra and the visual arts, and a word is in order at the outset why I do not. I wish to make explicit in advance that the literature review I conduct is theoretical in style, as opposed to documenting the history of learning in algebra and the visual arts, and here is why. It is reasonable to think that most theories of learning that I do discuss and that held sway at different times during the twentieth century were also reflected in pedagogical practices; and documenting them would involve me in a historical study that would warrant a book in its own right.

In lieu of a historical account of the formal educational practices within institutions (henceforth *education* or *educational* practices) in algebra and the visual arts in the North American context in the nineteenth and twentieth centuries, I can at least share what I believe happened. When behaviourism, along the lines of Thorndike's (1925) behaviourist account of learning in arithmetic, was the accepted view, we emphasized modelling, drill-and practice; when the cognitive revolution took root with Piaget's ground-breaking work, we emphasized children's construction of mathematical concepts; and when social cognitive theory came on the scene, there was, for example, a stress on increasing student's self-efficacy beliefs. Because mathematics, and not the visual arts, has been the focus of formal education, these various learning theories had more impact upon mathematics; but to the extent we have taught art, it is reasonable to think the implications for pedagogical practices were similar.

An overview of recent mathematics education in North America bears out my beliefs about the relationship between learning theories and pedagogical practices and often goes like this: mathematics education has moved from rote learning; the October 4, 1957, Russian launch of Sputnik prompting reforms of the 1950s and 1960s, which were widely perceived as a failure; a back to basics movement arrived on the scene; and with revelations that North American students do not fare well on international tests, a more progressive (constructivist) regime began in the 1980s. "That reality [of poor showing on international tests]," wrote Nastu (2007) in *eSchool Newsonline*, "coupled with the confusion about how math is being—and should be—taught in this country, has tossed the topic into the center of a controversy often called the Math Wars" (p. 1). My approach, to contributing to resolving these now iconic conflicts, stems from a concern about learning rooted in contemporary cognitive science—and that, I believe, requires a synthetic view that allows us to pay heed to the role of the social context in learning.

Some contemporary theorists of the mind, influential in education, however, have generally moved along the traditional, philosophical views about the nature of mathematical knowledge whereby it is considered abstract and set against what is concrete. For instance, Piaget defined four stages of intellectual development: the sensory motor stage; preoperational stage; concrete operational stage; and formal operational stage. At the formal operational stage, mental transformations take place without the need for concrete materials (e.g., mathematics). The stages are cumulative.

For example, in explaining how children elaborate the concept of Euclidian space, Piaget (1955) said that added to perception is a system of operations (i.e., a system of internalized actions) that are coordinated into reversible structures (e.g., we can count backwards). Even logic, Piaget (1975) argued, comes from the coordination of actions that allow us to build up schemas.

Piaget (1975) distinguished between to two types of abstraction, simple (sometimes called *empirical*) and reflective two types of experiences, which corresponds to physical and logico-mathematical experiences. First, *simple reflection* moves from an action to operation. *Physical experience* is acting on objects to discover their properties. Second, *reflective abstraction* reorganizes thought for Piaget. *Logico-mathematical experience* is where we gather information, not from the physical properties of objects, but from the coordination of actions.

Piaget retells a story about a friend who was a mathematician, which illustrates these processes. According to Piaget, this friend, when he was four or five years old, amused himself by placing ten pebbles in a straight line, counting them backward and forward, realizing he still found ten. He also found this to be true even when organizing the pebbles in a circle. He continued rearranging the pebbles in all sorts of ways till convinced that the sum, ten, was independent of order; this is simple reflection acting on a physical experience of being able to add 10 pebbles in different ways.

As Piaget (1975) noted, the story of working with pebbles is an example of conservation: "[B]ut conservation itself also gives rise to logico-mathematical experience" (p. 6), like the laws of algebra, which arise from reflective abstraction and in turn provide for logico-mathematical experiences.

Mathematics is governed by laws and is self-regulatory. Intellectual operations, Piaget (1975) claimed, are "interiorized actions" (p. 6). With reflective abstraction we reorganize mental structures, reconstructing them at higher levels drawn from the coordination of actions. So he claimed there was "no contradiction" (p. 8) between concrete and formal modes of thought—because they sit on a continuum, the former being the basis for the latter.

In fact, it has been shown that various types of reasoning discussed by Piaget (e.g., classification, conservation, proportional reasoning, interactional reasoning, correlation reasoning, and propositional reasoning) can be analyzed in terms of his critical distinction between being concrete or formal (Karplus, 1981).

Clark et al. (1997) relied upon Piaget's theory to explain how students know and use a mathematical rule in calculus. Their definitions help us bring Piaget's thought into focus, generally. In the context of their paper, they defined an *action* as a mental or physical transformation of mathematical objects according to a specific algorithm. They said something is *internalized* when the whole action takes place in a person's mind. A *schema*, they said, is a coherent collection of actions, processes, objects, and other schemas to deal with new mathematical situations.

Also, Clark et al. (1997) identified three types of actions on objects in Piaget's thought: intra, inter, and trans. The *intra* concerns one object isolated from other actions. The *inter* deals with many objects and the relation of actions. Finally, the *trans* concerns a coherent schema, encapsulating relations discovered at the inter stage. They concluded that understanding a mathematical rule depends on schema developed through the intra, inter, and trans stages. So even though Piaget sets the tone for a serious consideration of the concrete basis for the exact sciences, he still inclines us to consider them as different from the arts, which are inevitably concrete, requiring our senses to create and appreciate them; and more importantly, there is little explanation of the role of the social context in the process of learning more generally.

However, Vygotsky (1956) with his emphasis that we learn in the zone of proximal development (ZPD), the place where we can solve problems with guidance, emphasized the way cultural practices involving tools and symbols mediate development (Sophian, 2007). Fernyhough (2008), in a paper extending Vygotsky's thought in an attempt to account for social understanding, noted that he must account for (a) observable development, (b) why there are differences in development (e.g., access to adequate interpersonal exchanges), and (c) abnormal development (e.g., autism). He remarked that Vygotsky differs from Piaget in that higher modes of thought are mediated by social artefacts; for example, Piaget rejected Vygotsky's idea of private speech as the vehicle to higher mental functioning.

Fernyhough claimed that there were five salient points to Vygotsky's thought that allow him to fulfill the desideratum, including the ZPD. The other four, then, are as follows: First, *internalization* is the idea that we internalize interpersonal exchanges. Second, *naive participation* is the notion that we participate in activities that we may not understand the reason for. Third, we use *semiotic systems,* like natural languages, as mediators of understanding. Fourth, there is a *dialogical* nature of higher understanding, that is, we encounter multiple perspectives of reality that we attempt to synthesize. Fernyhough also adds *enculturation,* the way we become socialized into group practices. He focused on expanding the semiotic and dialogical aspects to account for social understanding. The important point from this discussion of both Piaget and Vygotsky for our purposes is that we acknowledge that there are indeed empirical and social dimensions to mathematics (discussed further in "Aesthetics and Concept-Rich Mathematics").

Also, Gardner (1985) is well known for proposing seven types of intelligences: linguistic, musical, spatial, bodily-kinaesthetic, interpersonal, intrapersonal, and logico-mathematical; later he added others (naturalist, existential, and moral intelligence) (1993). His theory of multiple intelligences provided a theoretical framework for Gardner's (1993) life-long study of creativity, especially in prodigies, such as Picasso.

However, Hatch and Gardner (1993) have identified three broad forces at work in learning across domains: the personal, local, and social. According to them, all three factors bear on what types of skills are exhibited, how they are developed, and the purpose to which they are put. Further, they distinguished between endogenous factors, like genetics, and exogenous factors, like personal experiences. As they pointed out, we have tended to regard art works as products of individuals, without paying much attention to the role of the social context.

Supporting the Platonic dichotomy, as well, cognitive scientists interested in mathematics and the arts have contributed to two mutually exclusive literatures (see Chapter 2, "Literature Review: Identifying the Common Link"). The idea has been to figure out what is going on in our heads, differently, when we do mathematics (e.g., algebra) and when we engage in the visual arts. Yet both algebra and the visual arts rely on a social context, like conventional symbols usage, to make meaning (Fodor, 1982, 1983, 1989, 2001; Gardner, 1985; Silver, 1987) in ways that, at least, merit exploration together. Little has been done, however, to consider the similarity between meaning-making in mathematics and the visual arts, within the context of the cognitive science program.

Motivating this investigation is the question: *What role does social context play in how we learn in the domains of algebra and the visual arts from a social cognitive point of view?* The terms, *algebra, visual arts,* cognitive *science, social cognitive theory, social context,* and *meaning-making,* are discussed further on in this section; as well as an explanation of why I focus on social cognitive theory and not cognitive science more generally. In a nutshell, social cognitive theory allows me to focus on the social factors involved in learning that preoccupy me in this study, and recent changes within the cognitive science program, hailed as the "situated movement," prompts us to look into the role of the social context in cognition. More about this as we proceed.

A Brief Discussion of Key Terms

Algebra. "Patterning and Algebra" is one of the five strands of mathematics, *The Ontario Curriculum Grades 1-8: Mathematics* (2005a; "Early Math Strategy," 2005). Building on the aforementioned curriculum, in Grades 9 and 10, we have the strand "number sense and algebra," ("The Ontario Curriculum," 2005b, p. 9). Further, in Grades 11 and 12, we encounter "polynomial and rational functions" ("The Ontario Math Curriculum," 2007, p. 91). "In mathematics, a polynomial is an expression of finite length constructed from variables (also known as indeterminates) and constants, using only the operations of addition, subtraction, multiplication, and non-negative, whole-number exponents" ("Polynomial," 2010). A function, in this case, is a way of naming a polynomial, for example, of the "$f(x) =$" form. A *rational function* involves, in addition, a fraction where both the numerator and denominator are polynomials ("Polynomial," 2010).

As one of the five main branches of mathematics, algebra concerns structure, relations, and quantity. As I have indicated, elementary algebra is part of the secondary education curriculum, concerning basic notions of the discipline like the effects of adding and multiplying numbers, the concept of variables, definition of polynomials, as well as factorization and determining the root of polynomials (Algebra, 2007).

Derbyshire (2006) in an account of the history of the discipline, entitled, *Unknown Quantity: A Real and Imaginary History of Algebra*, pointed out that algebra moves us from declarative statements, common to arithmetic ("2 + 3 = 5"), to interrogative ones ("2 + x = 5"), leading to generalization about their relations. Derbyshire (2006) noted that the laws of algebra can be written in five lines. There are nine but they fall into four pairs for addition and multiplication, which, incidentally, were not known until the 1830s in Great Britain and Germany.

$(a + b) = (b + a)$	$a*b = b*a$	(communicative law)
$a + (b + c) = b + (a + c)$	$a * (b * c) = b * (a * c)$	(associative law)
$a + 0 = a$	$a * 1 = a$	(identity law)
$a + (-a) = 0$	for $a \neq 0$, $a * a^{-1} = 1$	(inverse law)
$a *(b + c) = a*b + a*c$		(distributive law)

I consider, then, basic algebra ranging over integers, since it has been expanded to include everything from sets, logic (Boolean algebra), and geometry. The rules cited above do not always hold for other types of numbers either (like rational numbers, real numbers, and complex numbers), a discussion of which would take us beyond the scope of this thesis. Though my focus is algebra, however, I also draw on examples from other areas of mathematics and my reasons for doing so often relate to the nature of the literature (exemplified in Chapter 2, "Literature Review: Identifying the Common Link," in "Algebra") and the case studies (e.g., Chapter 4, "Findings: Four Case Studies").

Visual arts. It is useful to begin with the definition of *visual arts* in *The Ontario Arts Curriculum Grades 1-8* (1998):

> The visual arts include traditional fine arts of drawing, painting, sculpting, printmaking, architecture, and photography, as well as crafts, industrial design, commercial art, performance art, and electronic arts. (p. 28)

Accordingly, all of these are considered visual arts in this thesis, even though most of my examples primarily concern paintings or drawings. By *visual arts*, I indicate artistic works that must be processed by the visual system to be appreciated, and in this thesis, I specifically have in mind paintings, drawings, and so on.

Cognitive science. The term *cognitive science* refers to an interdisciplinary study of the mind; the reigning paradigm is an information processing theory of the mind modeled on the computer (Fodor, 2001; Gardner, 1985, Newell & Simon, 1972, Posner, 1989; Silver, 1987). I need to say a bit about the background of cognitive science because it reflects a plethora of views and programs. Pylyshyn's (1989), paper in *Foundations of Cognitive Science,* provides a good summary for our purposes. He noted that what distinguishes cognitive science from psychology is the influence of computing, though there is no unanimity on what that is. In any case, for cognitive scientists, cognition is a type of computing carried out by a biological mechanism. Due to the complexity of mental processes, abstract mechanisms (e.g., storing, retrieving, and altering tokens of symbolic code) have been thought to provide a model for cognition, whereby a processor writes things into memory then reads them. On this view, we operate on representations that take the form of symbolic codes.

Further, for the *classical symbol system* or *classical computational/cognitive architecture* theorist, we operate on three levels, semantic, symbolic, and biological, according to Pylyshyn. At the knowledge level, we deal with meanings and intentions, and hence semantics. At the symbolic level, we are concerned with how the semantic level is encoded by symbolic expressions. Finally, at the biological level, we are concerned with how a physical system can realize its overall function—cognition. "It says," according to Pylyshyn (1989) summing up the classical view, "that knowledge is encoded by a system of symbolic codes, which themselves are physically realized, and that it is the physical properties of the codes that cause the behaviour in question" (p. 61).

I utilize the term *cognitive science* because a large amount of the literature I draw upon falls squarely under this rubric—or its authors are heavily influenced by it. However, at the level of discourse at which this study occurs, that of observable behaviour, its consequences, and introspective reports—the semantic or knowledge level—I do not endeavour to resolve debates on how the brain processes information. For example, the classical cognitive architecture approach has been challenged by proponents of connectionist models whereby knowledge is represented as patterns or the activation of weights that are distributed through the network—and such debates can perhaps be best resolved by neurobiologists in conjunction with computer scientists, thus falling outside the scope of this study. However, what I have to say about the role of biology in this study is taken up in Chapter 5, "Discussion: Solidifying the Common Link," for reasons that, I believe, are self-evident. (Specifically, to anticipate this point about self-evidence in relation to the findings, some aspects of the analysis of the case studies can perhaps only be accounted for by biology.)

In any case, the reason to retain the discussion of *cognitive science* is because changes in that research program, the so-called "situated movement," has led its proponents to emphasize the semantic level, and I say more about that at the very end of this section when I discuss *representations*. In fact, the social cognitive perspective I adopt coincides with the semantic level of description that is my focus, which I also discuss further on when I deal with that theory. Focusing on the social context of learning provides an opportunity to expand social cognitive theorists' preoccupation with self-efficacy beliefs, discussed further on when I consider that theory. So, by discussing them conjointly in this study, cognitive scientists can learn something from social cognitive theorists and vice versa: we can better understand the role of the invariable common link, socio-cultural context, in learning in algebra and the visual arts. Suffice it to say, then, in this thesis I speak broadly, referring to cognitive science as any attempt to account for human mental processes and behaviour, which can be considered complimentary to even the *classical cognitive architectural* approach because it does occur at the semantic level of description.

Learning theories. We may also refer to theories of learning, hence, psychological, learning theorists: behaviourists, like B. F. Skinner; constructivists, like Piaget; social constructivists, like Vygotsky. Yet the distinction between these three schools of thought has been blurred; and I begin by considering the relationship between Piagetian and Vygotskian schools before returning to make connections to Skinner's thought.

Some of the legacy of Piaget and Vygotsky. Though it has been noted that uniting constructivist views with Vygotsky's is not easy (Bauersfeld, 1995), his work has been thought to be compatible with earlier Piagetian accounts (Cobbs, 1994; Smolucha & Smolucha, 1986; Vidakovic & Martin, 2004). In a collection of papers *New Directions in Piagetian Theory and Practice* (1981), there have been various attempts to develop the social dimensions of Piaget's (1932) thought. This collection is, of course, no longer recent, but does bear upon current trends in cognitive science; so I explore it here. Writing for this collection, Youniss (1981) has re-read Piaget's (1932) *A Moral Judgment of the Child* emphasizing relational theory, the idea that a self is known and dependent on other persons. As Youniss pointed out, even though order is constructed through actions for Piaget, we must take into account interpersonal actions. He argued that the relational position Piaget articulated has been "neglected" (Youniss, 1981, p. 200), and calls this new approach to social development "clear and interesting" (p. 201).

Contributing to the same collection, Sigel (1981) characterized his paper as "an extension and elaboration of Piagetian theory particularly with respect to the role of the social factors in cognitive development" (p. 202), deeming his view "revisionist" (p. 205). Sigel noted that we construct knowledge about the world through interactions with the physical and social world, and are "capable of integrating the past and future" (p. 203). That is to say, not only does cognition require experiences as actions and internalizations of actions with objects, but "what is an additional necessity is a particular set of social experiences—interactions with others" (p. 203).

Smolucha and Smolucha (1986) noted that societies that view logical-scientific thinking as the paragon of cognition tend to devalue the arts. They asked what role imagination plays in both the visual arts and mathematics. They integrated Piaget's talk of stages into a Vygotskian framework, the notion that self-regulative functions of inner speech are internalized through social dialogue. They reversed the Piagetian learning process, beginning with the social realm and then moving to the individual level. We move, according to them, from sensori-motor thought to verbal thought. Symbolic play, however, is first social and then used to organize mental imagery. Higher mental functions, they said, are an internalization of social interactions.

In the visual arts, according to Smolucha and Smolucha (1986), several different types of isomorphism are at play: resemblance of form, like a homology (e.g., a bat's wing and a human arm); resemblance of function (e.g., a dolphin's tail and a person's legs); and resemblances that are coincidental (e.g., the branches of a tree and a hand). They regard isomorphism as social constructions that are selected by artists. They concluded, "We are arguing that artists can learn to consciously attend to visual analogies and condense them into images conveying multiple meanings" (1986, p. 481).

Behaviourism and social cognitive theory. In addition, Piagetian and behaviourist accounts have also been synthesized, a view called *social cognitive theory* (Bandura, 1986, 1997), that is, the notion that our acts are shaped by a history of contingencies that have a meaning for us. Bandura (1997) attempted to fuse the theories of behaviourism and cognitive theory with a reciprocal determinative model represented as a triadic structure.

That is, Bandura takes into account the personal (e.g., cognitive, affective, and biological realms), environmental (e.g., the social realm), and behavioural dimensions of the mind. Consider these examples. Our person—our thoughts, feelings, and biological dispositions—it is commonplace to think, do influence behaviour. Also, the environment can impact behaviour: we often make choices that have value in the community of our peers. Finally, our behaviour can also affect the personal dimension (e.g., doing a task can increase its likeability), and socio-cultural context (e.g., others may wish to imitate our behaviour).

It is helpful to understand Bandura's theory with some background about B. F. Skinner's behaviourism. Skinner (1989) held that the "mind is what the body does" (p. 67). As he explained, behaviourists "solve [epistemological] problems by [reference to] assembling, classifying, arranging, and rearranging things" (Skinner, 1974, p. 223). For him, the meaning of an event is analyzed in terms of the "history of exposure to contingencies in which similar settings have played a part" (Skinner, 1974, p. 90).

For example, Skinner (1989) wrote, "Logic and mathematics presumably arose from simple contingencies of reinforcement" (p. 42). He said, however, that "an organism cannot acquire a large repertoire of behaviours through operant conditioning [i.e., using the consequences to modify behaviour] alone in a non-social environment. Other organisms are important" (Skinner, 1989, p. 51). That is to say, Skinner's (1953) core idea is that behaviour is a learned response to stimuli in the environment.

Bandura sought a middle road between either internal or external determinism. Social cognitive theory, as Bandura's views are also dubbed, is intended to be attentive to both sub-processes of the individual and the collective level. Self-efficacy beliefs, for example, are used to explain why some, in the face of continual failure, persist, as is common among many luminaries throughout history (Bandura, 1997). He wrote, "Hence, the impact of socio-structural factors on organizational performance is mediated through motivational and learning mechanisms operating at the individual levels" (Bandura, 1997, p. 472). In other words, both biology and environmentally drawn experiences have influenced the adoption of certain behaviours.

As Woolfolk, Winne, and Perry (2006) explained in an introductory textbook on educational psychology, Bandura has focused on cognitive factors like beliefs, perceptions, and expectations. They noted that he distinguished enactive and vicarious learning. *Enactive learning* is gaining knowledge through personally experiencing the consequence of actions. Woolfolk, Winne, and Perry (2006) emphasized, however, that this goes beyond operant conditioning: the consequences do not merely strengthen or weaken behaviour without further qualification; rather, they provide information to the learner that creates expectations, influences motivation, and shapes beliefs. *Vicarious learning* is knowledge gained by observing others. If we can learn by watching others, Woolfolk, Winne, and Perry (2006) remarked, this implies that we focus our attention, construct images, remember, analyze and make decisions that affect learning.

The cognitive science revolution of the 1950s was, in broad outline, the addition of the cognitive, inter-subjective dimension that mediates between stimulus and response. Zimmerman (1981), commenting upon the relation of cognitive behaviourism to Piaget's thought, characterized *cognitive* as the idea that learning is not a peripheral process, that is, treatable in terms of stimulus and responses.

Yet he argued that cognitive activity does not preclude being explained lawfully. For the constructivist, he said, “Humans act on the environment rather than react to it. Information does not happen, but requires self-generated mental activity by the person to make sense of their environment” (p. 40). He noted, however, that Bandura’s social learning theory is cognitive, constructivist, and deterministic. “Social learning theory,” Zimmerman said, “assumes that both behavioural and cognitive theories are partially correct” (p. 42). According to him, the way we construct concepts and stimulus is assumed to be jointly influenced by previous or acquired rules and experiences with the stimulus, that is, reciprocal determinism. For example, as Zimmerman pointed out, a child that has an experience with a red ball that bounces may form a *rule* that “red balls bounce;” this rule has to be modified when confronted by a red billiard ball that does not bounce, or extended, in the case of a blue ball that bounces.

Commenting on the fusion of constructivism with behaviourism, Zimmerman noted that evidence suggests that rules underlying constructive activity are derived from social models. We can, then, be both lawful, taking into account the role of the stimulus; and constructivists, taking into account the cognitive response. In fact, he said, wide arrays of conceptual rules are social (e.g., Piagetian concepts and syntax).

What Bandura (1997) included, by relying on Skinner’s thought, is the link to the social context where behaviour is shaped, which, as we have seen, can also be elaborated with Vygotsky’s thought. Suffice it to note that I do not merely adopt a Vygotskian perspective because his view renders cognitive factors perhaps too subservient to the social context; and the role he attributes to language in learning is at least questionable. In any case, given the generality of the cognitive behavioural framework that Bandura provides, it is reasonable to think that a multitude of theoretical perspectives can be accounted for in it—because he takes into account the personal, environmental, and behavioural dimensions of the mind.

Of course, looking to behaviourists seems peculiar for those interested in mental processes because of a perceived lack of a human dimension in that program that helped usher in the cognitive revolution to begin with. But behaviourists did in fact emphasize the role of the social environment in shaping behaviour (Harman, 1989; Ross & Nisbett, 1991). In bringing the two sides together, as Doris (2002) noted, writing about the importance of empirical studies to making headway in philosophical ethics, "The dynamic theorist can acknowledge the importance of the environment, and any but the most doctrinaire behaviourist can acknowledge the importance of psychological dynamics" (p. 12).

In this study, we may wish to recall, I focus on the personal, environmental, and behavioural dimensions. However, the biological aspects of the personal dimension take us outside of the social sciences and hence beyond the scope of this thesis. In the literature review, however, I consider some biological research, partly because it has a current influence on our ideas about cognition, and as a prelude to discussing the social aspects of learning in algebra and the visual arts that are my focus. I also return to touch upon the biological aspects in the analysis of the data in Chapter 5, "Discussion: Solidifying the Common Link."

In the context of this thesis, I use *social* broadly, to indicate various relations with others, both personal, like family and friends, and cultural, or "supra-social" (Bishop, 1988, p. 14), which includes the ideological (e.g., ideas, beliefs, values), sociological (e.g., customs, institutions, patterns of behaviour), sentimental (attitudes and feelings), and technological (the manufacture of tools and implements) (cited in Bishop, 1988); and some social factors, like ideology and sentiments, for instance. These social factors can readily be considered along the lines of Bandura's tri-deterministic model as part of the personal dimension at the environmental level, too. The reason for the overlap between levels is clear enough: social ideas, social values, and social actions have to be personalized and enacted by individuals, which is what we mean when we speak of enculturation.

Maslow's hierarchy of needs and social cognitive theory. Also influential in educational psychology discussions, and yet to be discussed, has been the work of Maslow (1970), specifically his hierarchy of needs. Although Maslow's theory, it is pointed out (Woolfolk, Winne, & Perry, 2006), has been criticized, it allows us to further focus our understanding of social cognitive theory.

Maslow identified four higher-order needs, namely, knowing, understanding, aesthetic appreciation, and self-actualization; and five lower-order needs, namely, survival, safety, belongingness, love and esteem, which are also called deficiency needs because when they are satisfied, the motivation to fulfill them decreases. As we might expect, lower-order needs are foundational for higher-order needs. Maslow encapsulated the basic idea of the hexarchy of needs in the slogan, without bread we cannot have ideas.

Table 2.

Maslow's Hierarchy of Needs

Type of Need	**Specific Need**
↓ **Higher-Order Needs**	Self- actualization
	Aesthetic appreciation
	Understanding
↑	Knowing
↓ **Lower-Order Needs**	Esteem
(Deficiency Needs)	Love
	Belongingness
	Safety
↑	Survival

Leaving aside disputes about the order or make up of Maslow's hierarchy of needs, we can draw out the following salient points: At the very basis of behaviour is the aim to meet lower-order needs, such as belongingness, which some (Baumeister & Leary,1995) have even considered fundamental. When these lower-order needs are met, we strive to meet higher-order ones, such as self-actualization: Maslow's theory forces us to look at a person holistically. That is to say, the gist of Maslow's approach is not to break down human behaviour into a collection of competing needs; but rather to see it as an assemblage or a totality of needs that function together, sitting on a continuum, all working together to form a self. The fulfillment of higher-order needs, for example, it is reasonable to think, are contingent upon basic needs being met at least at some point.

For social cognitive theorists, the cognitive realm and social realm crosscut both enactive and vicarious learning. In both types of learning, mental processes are engaged that rely upon previous social experiences, to different degrees. Both social cognitive theory and Maslow's hierarchy of needs prompt us to look at a person holistically and, all the while, allow us to emphasize the role of the social context in learning. Taking my point of departure from Bandura's (1997) identification of three various forces, then, I wish to consider how we transport ourselves from our kinaesthetic, hands-on, social experiences, to abstract thought, and back again to the concrete world as we attempt to apply our ideas in specific situations.

The relationship between social cognitive theory and cognitive science. It is worthwhile at this point to consider the relationship between social cognitive theory and cognitive science, which allows me to explain how and why I retain language related to both theoretical approaches throughout this thesis.

Table 3.

An Outline of the Relationship between Bandura's (1986, 1997) Social Cognitive Theory and the Cognitive Science Model of Mind

The Dimensions of Social Cognitive Theory	**The Levels of Description of the Cognitive Science Model of Mind**	**Comment**
↓ *Personal*	Semantic	Personal factors can be realized at a semantic level

		(e.g., our experience of a red tomato).
	Symbolic	Personal influences are encoded: for example, the concept of a *tomato* must be stored in a register, as well as individual association with it (e.g., "It is an edible fruit").
↑	Physical	The cognitive and affective domains must be realized by a physical entity (e.g., a brain or hardware).
↓ *Environment*	Semantic	Environmental influences can be realized at a semantic level (e.g., our experience of a red tomato is shaped by the social context in various ways).

	Symbolic	Socio-cultural influences are encoded (e.g., the conventional concept of a *tomato* must be stored in a register and recalled in order to recognize it as such).
↑	Physical	Environmental influences must be recorded by a physical entity (e.g., a brain or hardware).
↓ *Behaviour*	Semantic	Our behaviour can have a meaning for us (e.g., alternating our self assessment of a task).
	Symbolic	Our behaviour has, and can shape, a pragmatics, that is, specific social meaning, either overtly (e.g., talking) or covertly (e.g., smiling) in

		a given context.
↑	Physical	*Behaviour* will constrain (and be constrained by) the nature of the *physical entity* (e.g., the brain or hardware), impacting what actions (and cognitive functions) are possible and likely.

Depicted in Table 3 (above) are the dimensions of Bandura's tri-deterministic theory mapped onto the traditional levels of the cognitive science model of mind: there is, at the top (see Table 3, Column 1, "The Dimensions"), the personal, followed by the environment and behavioural dimensions; all of which can be described (see Table 3, Column 2, "The Levels") at a semantic, symbolic, or physical level. That is, as I pointed out (see Table 3, Column 3, "Comment"), the personal, environmental, and behavioural dimensions of Bandura's theory are crosscut by both the semantic, symbolic, and physical levels discussed by Pylyshyn (1989).

The connection between the personal and environmental dimensions with the semantic and social levels becomes all the more striking once we emphasize the socio-cultural context along the lines of proponents of the situated movement: our meanings are through and through personal and environmental. Also, we have the biological aspect of the personal dimension, in Bandura's terms, which correlates to the nature (e.g., structure) of the physical level for the cognitive scientist. To reiterate, I am primarily concerned with the personal and environmental factors of learning at the semantic level.

The following recapitulation usefully summarizes some of the milestones, and the trend, in psychology in the twentieth century as I have elaborated it, and which has led me to social cognitive theory in this thesis:

1. Beginning in the 1920s, we had behaviourism and a focus on the role of the *environment* in shaping behaviour.

2. In the 1950s, there was the cognitive science revolution and a focus on the *internal mental mechanisms* that direct behaviour.

3. And towards the end on the twentieth century, social cognitive theorists and the situated movement in cognitive science beckon us to pay heed to both internal and external, *socio-cultural dimensions* of behaviour.

The situated movement (to be discussed in more detail in "The Situated Movement," further on), I have been suggesting, then, tallies with social cognitive theorists' tri-deterministic model, both historically in its motivation and conceptually in its content. However, suffice it to note that traditionally, cognitive scientists, of most stripes, have not paid enough heed to the socio-cultural context in general (Rogoff, 1984, 1993). Since 1956, they have tended to emphasize the individual and internal, symbolic aspects of cognition (Newell & Simon, 1972) to the near exclusion of the social context. And none has considered the socio-cultural context of learning in any depth across domains as different as algebra and the visual arts.

Symbolic architectures and representations. Further, it is commonplace, in the educational context, and in the cognitive science of mathematics and visual arts literature, to speak of *representations* (e.g., how we represent an algebraic problem), which requires saying something about what this means for cognitive scientists. Adding a bit more detail to the picture sketched by Pylyshyn (1989), Newell, Rosenbloom, and Laird (1989), have discussed, in the same anthology *Foundations of Cognitive Science*, how symbolic architectures for cognition give rise to representations; and it is helpful to review that now.

The symbolic architecture functions, following the cognitive scientists' model of the mind, by utilizing a *memory* that contains structures with symbols that persist over time. The system uses *symbols* that provide access to distal structures (i.e., other parts of memory not involved in the initial computation), thus bringing to bear the entire system's knowledge to achieve a goal. The system employs operations that involve the input and output of symbols to compose new structures. Finally, the system must engage in *interpretation* that link symbols to behaviour, which is a process of applying operations. That is to say, there must be an interface with the world because symbols are part of our "internal milieu" (Newell, Rosenbloom, & Laird, 1989, p. 107). Finally, representations are considered a function of the symbolic system as a whole.

In an introductory textbook on cognitive science, Freidendberg and Sliverman (2006) noted that the idea of *representations* are fundamental to this discipline. On the classical cognitive architectural approach, we perform computation on representations, which can be described along the three levels we are already familiar with: a computational (or semantic) level, what they mean to us; an algorithmic (or symbolic) level, what goes on in our skulls that allows their generation; and an implementation (or biological) level, what goes on in our brains when we have a representation of say, our grandmother.

They considered different types of representations, like *concepts* that stand for a group of entities (e.g., *apple*); *propositions* that are statements about the world and can be illustrated by a sentence, often thought to be true or false (at least by classical logicians); *rules* that specify the relations between propositions and are one way of representing procedural knowledge (e.g., how to drive a car); and *analogies* that allow us to make comparisons.

According to Freidendberg and Sliverman (2006), there are four crucial aspects for representations. First, there must be a *representation bearer*; a human or computer must realize the representation. Second, there must be *content* or meaning that stands for one or more objects, which are called *referents*. As they pointed out, since representations stand for something, they are symbols (e.g., "$" stands for money, but is not money). The semantic dimension is sometimes referred to as *intentionality*, which encompasses the notion of *isomorphism,* or similarity between the representation and referent; and a relation between inputs and outputs (e.g., thinking about *food* should cause behaviours or actions related to the referent). Third, representations must be *grounded*: there must be some way of relating the representation and referent. Finally, representations must be *interpretable* by some interpreter.

Meaning-making. I employ the term *meaning-making* instead of *representations*. In a nutshell, since what constitutes even a *computation* does not find unanimity in the field, there is little reason to think there would be widespread agreement on what constitutes a *symbol* or *representation*; and there is not. In fact, even if we standardized the definition (e.g., along the lines of Friedenberg and Silverman, 2006), representations are only a theoretical postulate whose usefulness has been contested; for example, there are attempts to avoid the use of the term *representation* within some quarters of the cognitive sciences (Brooks, 1997)—a debate which would carry us too far afield and into philosophy.

Suffice it to say, within the context of this thesis, I believe that the use of *meaning-making* is at least as equally justified as *representation* because it allows us to emphasize the semantic level of description that is my focus; and this choice tallies with a newfound interest in the role of the social context in cognition among cognitive scientists. As anthropologists remind us: whatever internal mechanisms allow us to cognize, they would be retarded without the social context in which thinking needs to develop; and the social context often contains the reasons we do so—to influence the behaviour of our fellows (D' Andrade, 1989).

Meaning-making, in the context of this thesis, is considered in terms of two levels:

(a) The macro-level, comprising *external factors* such as socio-cultural modalities

(b) The micro-level, constituted by the *internal factors* that include the cognitive dimensions.

Both levels are related. As Cobb, Gravemeijer, Yackel, McClain, and Whitenack (1997) commented, in a paper about how we make meaning in a first-grade mathematics classroom, cognitive scientists interested in the social context have an "emergent perspective" (p. 152). That is, they want to do more than articulate the cognitive process of socialization. For them, the goal is to view the interaction between the individual and social context reflexively: how the social influences the individual and vice-versa.

In the context of this thesis, I focus on how the interaction of individuals with socio-cultural processes allows them to make meaning in relation to algebra and the visual arts, by considering other related literature on situated learning (e.g., Lave, 1988; Pea, 1987a). Learning styles are often distinguished as being high-context, where the setting is prominent, and low-context, which is more abstract or context-free. The goal of traditional schools has been to teach general knowledge that is thought to be applicable to a vast range of situations. We have tended towards valuing knowledge of higher levels of abstraction, even sometimes (e.g., in the thought of Nicholas Bourbaki (Stewart, 1995)) jettisoning concrete learning.

Meaning-making is also discussed cognitively: first, as bottom-up learning, the way we form generalizations to create abstractions (e.g., algebra allows us to generalize over arithmetical operations); and second, as top-down learning, the way we apply the said abstractions to specific cases. That is to say, I consider, for example, meaning-making in algebra both as it relates to concrete objects and as a formal system of symbols governed by rules. These rules are often viewed as generalizations of the relations of quantities (Baroody, 1999; English, 1999; Russell, 1999). Summing up matters from a Baundurian perspective, in this study, I consider meaning-making as focusing on the environmental and cognitive factors that influence learning.

Within the history of modern philosophy, it would be natural to infer that art concerned sense impressions, and mathematics the relation of ideas. The moral that has often been drawn is familiar: Mathematics was something we did in our heads and art something we did with our hands. And we have valued the head over the hand, splitting apart the two (Scribner, 1984). Some social critics (Mumford, 1934) and philosophical historians of mathematics (Gray, 2008), however, have pointed out that mathematics and the arts can be considered to be an expression of the prevailing social milieu—they do, therefore, share something in common.

However, turning our attention once again to theories of learning, in the case of Piaget, we can still distinguish between the arts that concern the senses, and mathematics that concerns the relations of ideas (Karplus, 1981). Subsequent readings and elaborations of Piaget's work (Glick, 1981; Sigel, 1981; Youniss, 1981; Zimmerman, 1981) prompt us to consider the social dimension in learning in these domains. Yet even Vygotsky's work, alone or combined with other theories (Bauersfeld, 1995), has not led to an in-depth comparison of the role of the social context in learning in algebra and the visual arts. Also, in Gardner's and associates' (Hatch & Gardner, 1993) writings after the development of the theory of multiple intelligences, we get a hint of the role of the socio-cultural context in which abilities may be developed. But this has yet to be explored in detail across domains as different as algebra and the visual arts.

Admittedly, Vygotsky's (1956) or Gardner's and associates' work (Hatch & Gardner, 1993), for example, could be developed to help us explore the role of the socio-cultural context in learning in algebra and the visual arts. I have chosen to use social cognitive theory because of the synthesis it brings to the major psychology theories of the twentieth century; its pre-eminence in the field of educational psychology; and the fact that it is either consistent with or can subsume other theories. For example, I have attempted to show that social cognitive theory tallies with the situated turn within cognitive science. Nevertheless, I do not contend that there are not other theoretical approaches we could use to make the socio-cultural link between learning in algebra and the visual arts, but that would be another thesis, not this one.

According to my social cognitive view, then, I aim to pay heed to the idea that all three factors—the personal, social, and biological—shape behaviour and thought process, emphasizing, in this study, the role of the social context in learning. In fact, the focus of Bandura's research program was mathematics education. So even though social cognitive theorists did explore the role of the social context in learning, they have not done that extensively across domains as different as algebra and the visual arts. And even when focusing on mathematics, researchers inspired by Bandura have interpreted the social and cognitive factors rather narrowly, often in terms of the environmental impact on self-efficacy beliefs, not delving into specifics. For example, they have not considered how prior knowledge, family experiences, school experiences, or the role of social models, shape behaviour. This is the task I set out to explore in this study.

In our Western culture, we carry with us the Platonic schism. Mathematics we have been led to believe is generally an elitist activity, anti-social, and abstract;. By contrast, the arts are often considered egalitarian, social, and concrete. However, I believe that there are aspects to both mathematics and arts that can be considered egalitarian and elitist; social and anti-social; and concrete and abstract. In this investigation, I call into question the putative bifurcation in how we have attempted to understand learning in algebra and the visual arts from a generically cognitive science point of view. Let me explain.

In an introduction to an anthology, *Situated Cognition: Social, Semiotic, and Psychological Perspectives*, Kirshner and Whitson (1997) noted that the central philosophical assumptions of traditional cognitive science were (a) mind-body dualism (i.e., the mind and body are ontologically distinct entities), and (b) devaluating lived experience in favour of higher abstract thought.

Also, Agre (1997), summarizing Lave's (1988) work, a key exponent of situated cognition, noted that we have used a problem-solving approach traditionally, whereby: (a) life situations are transformed into formal cognitive structures, "problems;" (b) the structures are manipulated through mental processes, "solutions;" and (c) outcomes are interpreted within larger life situations as actions. In this thesis, I focus on revising the Platonic epistemology for learning in algebra and the visual arts, not his metaphysics (though it is assumed that a revised epistemology would likely entail emendations to his metaphysics as well).

The situated movement. There has also been a transformation in cognitive science often referred to as the "situated movement," which gives us cause to reflect upon its relationship to this project. Writing about situated rationality, Brighton and Todd (2009) suggested a "mixed toolbar" (p. 323), that is, the idea that basic (innate) heuristics are shaped by the situation, and they cited the work of Smith (1999) to make their point, which helps us understand what is meant by the *social context,* and I turn to that next.

Writing the entry "Situated/Embeddedness," for the *MIT Encyclopedia of Cognitive Science*, Smith (1999, pp. 769-770) noted that the interest in situated cognition represents a challenge to GOFAI (Good Old Fashioned Artificial Intelligence) of the 1980s by viewing intelligent behaviour as engaged, socially and materially embodied, arising from specific concrete details of natural settings, rather than detached, general purpose processes of formal rationality. In Smith's account of GOFAI we have the following characteristics: cognition is viewed as *individual*, in the sense the locus is in the person; *rational*, because conceptual thought was considered primary; *abstract*, since the implementation and environment are secondary; *detached*, in that thinking is separated from perception and action; and *general,* for the reason that we sought the universal principles of intellection for all people in all circumstances.

Conversely, Smith (1999) wrote that the *situated movement* can be characterized as having the following approach to cognition: *social*, being located in humanly constructed settings and among human communities; *embodied*, our bodies are taken to be pragmatically and theoretically significant; *concrete*, in the sense that physical constraints of realization are taken to be important; *located,* because context-dependence is enabling of all human endeavours; *engaged*, since on-going interaction with environment is primary; and *specific*, in that what we do is viewed as varying, depending upon contingent facts about our circumstances.

As Smith observed, views within the *situated movement* vary with the degree to which they incorporate context-dependence within their larger, classical frameworks, with more radical suggestions having substantial methodological and metaphysical commitments. Suffice it to note that my social cognitive view falls broadly within the turn towards situated cognition by cognitive scientists (discussed further in "Connecting Algebra and Art with the Common Link"). I call into question, then, the putative view that learning algebra is an abstract, intellectual pursuit and that learning in the arts is a social, bodily one.

Problem

It is commonplace to hold that algebra and the visual arts are mutually exclusive activities. There have been attempts, as interpreted in this thesis, to consider what learning in the arts and sciences have in common (Arnheim, 1965, 1969, 1986; Csikszentmihalyi & Robinson, 1990; Dewey 1934/1958, 1938/1997; Fry, 1947, 1956; Gray, 2008; Holt, 1971; Mumford, 1986; Perkins & Leondard, 1977; Pfenninger & Shubik, 2001; Smolucha & Smolucha, 1986; Snow, 1959; Stent, 2001; Tolstoy, 1899). Yet, there has been no in-depth, recent attempt to make the link between what is common to learning in algebra and the visual arts that relies on widely held, current psychological accounts like Bandura's (1986, 1997) social cognitive theory.

Further, social cognitive theorists (Bandura 1986, 1997) have explored the role of the social context in a rather narrow way, focusing on the role of self-efficacy beliefs in learning. But they do not consider in depth the role of culture and identity, for instance. That is to say, Bandura-inspired theorists have not explored robustly, other than, for instance, the idea of self-efficacy beliefs, the interaction of the social, personal, and behavioural dimensions, which, it is reasonable to think, can obscure more broadly what algebra and the visual arts share. Nor have we paid enough attention to the role that identity plays in the learning of algebra or the visual arts.

In addition, cognitive scientists (Newell & Simon, 1972) have focused heavily upon procedures of various tasks (e.g., algebraic or artistic tasks); not paying as much attention to what may be the common role the social context serves in the motivation of behaviour, related to algebraic and visual arts abilities. Also, cognitive scientists, under the influence of the computer model, where roughly, the brain is the hardware and the mind the software, have focused upon the mind's symbol-processing ability (Newell & Simon, 1972), at the expense of the social context, which, it is reasonable to think, can obscure what algebra and the visual arts share.

With the foregoing in view, this study explores how our interaction with the social context facilitates meaning-making in algebra and the visual arts: how our prior knowledge, family and educational experiences, shape the learning process; and, how social models, that is, aspiring to be a certain type of person—our identity—affects the learning process.

Research Questions

The following research questions, then, guided this study.

1) What role does the social context play in motivating meaning-making in the acquisition of algebraic and visual arts skills?

2) How does prior knowledge, family and educational experiences, shape the learning process?

3) How does becoming who we are, our identity, shape the learning process?

4) Are there any implications for pedagogical practice for considering the role of the socio-cultural context, prior knowledge, family and educational experiences, and identity in learning in algebra and the visual arts? For example, with respect to:

(a) Increasing the motivation to engage in algebra or the visual arts?

(b) The effectiveness of the acquisition of algebraic or visual arts skills?

(c) Designing and differentiating curricula for algebra and the visual arts that would take into account the role of identity formation in the learning process?

Purpose

The purpose of this investigation is threefold.

(a) To explore the role of various modalities of the social context in meaning-making in algebra and visual arts from a social cognitive point of view.

(b) To offer suggestions on how we can develop pedagogical regimes for algebra and the visual arts that build upon the way we make meaning in various social contexts.

(c) For the purpose of the design and differentiation of curricula for students in algebra and the visual arts, to consider some factors that contribute to identity formation (e.g., prior knowledge, family and educational experiences).

Significance

Taking together what I have said so far, I believe the originality of this thesis to be as follows:

- The divided epistemological line, held since Plato, between algebra and the visual arts is called into question
- Along with historical scholars (e.g., Mumford, Gray), the idea that reason could ever be pure (i.e., independent of the social context) is put into question
- Psychological theories are discussed in a synthetic fashion (e.g., Skinner, Piaget, Vygotsky, Maslow)

- Four participants are studied to explore social cognitive theory comparatively across the disparate domains of algebra and the visual arts
- The focus of psychological theorists and social cognitive theorists interested in learning in mathematics or only the visual arts are extended to the other domain
- A detailed, comparative literature review is provided with regard to the hitherto largely distinct areas of research into algebra and the visual arts
- Thinkers (e.g., Plato, Bandura) are brought into dialogue with contemporary debates about learning in cognitive science, across disparate domains
- The most recent trend in cognitive science, "the situated movement," is utilized to tie together the otherwise disparate mentalist studies into algebra and the visual arts
- Participants are studied holistically using the case study approach
- Social cognitive theory is corroborated and developed for the integration of those dimensions considerations of learning into our pedagogical practices.

This thesis is significant because social cognitive theorists have considered self-efficacy beliefs to be instrumental to the learning process across multiple domains. Also, cognitive scientists interested in mathematics and the arts have considered the way we make meaning in algebra and the visual arts through generalization, the cognitive processes we use, and the role the social context plays in the learning process with regard to the two fields of study (see Chapter 2, "Literature Review: Identifying the Common Link"). However, little has been done to consider what broad similarities there are, if any, between algebra and the visual arts, within the context of the social cognitive paradigm or cognitive science program.

That is, little has been done to see how a general theory of mind can be used to account for learning in algebra and the visual arts. For example, Piaget's classification of knowledge is one way to explain why learning differs by domain (e.g., Kamii & Housman, 2006); but his views have also been the subject of much criticism for not paying adequate attention to the social context in the learning process (e.g., as noted in Gardner, 1985). We are inclined to think that a general theory of mind and learning would have some implications across the board for learning, regardless of the domain in question.

This study fills the lacuna, by building a bridge between the way we understand learning in algebra and the visual arts. We need a theoretical rationale for promoting a pedagogical regime that includes identifiable modalities, like prior knowledge, family and educational experiences, and identity, of meaning-making. I propose that we utilize our social understandings to form abstractions, in both algebra and the visual arts, and to generate new meanings in both fields. One abstraction is identity. Wanting to be a certain type of person, I also propose, then, plays a role in learning in these two fields.

The issue of identity has become important to teachers as they attempt to instruct students that are male and female; and increasingly, come from different socio-cultural backgrounds. We need to consider how we can design curricula to meet the needs of students from different parts of the world. For instance, it is important to understand the socio-cultural background of students and make reference to it in designing curricula in algebra and the visual arts.

Plan of this Work

Each part of this thesis is aimed to accomplish specific tasks that are stated at the outset of each chapter. Chapter 2, “Literature Review: Identifying the Common Link,” is about making the theoretical case that the cognitive science of algebra and the visual arts can be understood in common, social cognitive terms. Chapter 3, “Methodology: The Case Study Approach,” provides an explanation of, and rationale for, using the case study approach.

Chapter 4, “Findings: The Four Case Studies,” are constituted as narratives. Finally, Chapter 5, “Discussion: Solidifying the Common Link,” takes us deeper into the common links between algebra and the visual arts, the limitations of doing so, and the theoretical and practical significance of having done so. The references, the appendices, and the vita auctoris are at the end.

2. Literature Review: Identifying the Common Link

> "However, my uncle [a mathematician] saw art and mathematics as totally separate fields whereas, for me, they were always indistinguishable" (Mandelbrot, 2001, p. 192) [founder of fractal geometry].

This literature review is intended not only to provide the rationale for my study, but it is also aimed to fuse together discussions that have been going on among, for example, cognitive scientists interested in algebra, and independently, cognitive scientists interested in the visual arts. I aim to demonstrate that there is a basis to study the common social cognitive dimensions of learning in these two disparate domains through my analysis of the literature. In fact, on occasion, some discussion of the visual arts will occur in the section dedicated to algebra and vice versa; the reason for this overlap, I believe, will become obvious as we proceed.

I argue that what are common to the cognitive science literature on the two domains in question are the social cognitive dimensions of how we use our bodies, minds, and interpersonal interactions within a larger socio-cultural context to develop and learn—to produce algebra and works of visual arts.

First to be considered is the literature dealing with algebra, then the visual arts, followed by a recapitulation and an analysis; emergent questions appear at the end.

Algebra

> "Algebra represents one of humankind's greatest intellectual achievements — the use of symbols to capture abstractions and generalizations, and to provide analytic power over a wide range of situations, both pure and applied" (Shoenfeld, 2008, p. 506).

Aesthetics and concept-rich mathematics. A concept-rich approach to learning arises from the cognitive tradition of Piaget, where we emphasize the way in which we construct meanings. Ben-Hur (2006), discussing his program for instruction in mathematics, said that concepts are to be developed through meta-cognition, that is, "knowledge about our own thinking processes" (Woolfolk, Winne, & Perry, 2006, p. 540). Ben-Hur explained that concepts are not to be memorized. He also remarked that in concept-rich instruction we emphasize the critical role of mediation in conceptual development and underscore the importance of verbal and reflective features of classroom activity.

Many ideas about aesthetics also impact the way we think about mathematics concepts, and I turn to touch on some of them next. In Tolstoy's (1899) famous book, *What is Art*, he claimed that the founder of aesthetics, Baumgarten (1714 – 1762), identified reason that concerns truth, beauty that concerns the sensuous, and the good that concerns moral will. However, Tolstoy criticised the idea that art is about what pleases us. Rather, for him, universal art arises when someone has an appropriate experience that she tries to communicate. According to him, art should unite us in common feelings.

Tolstoy retains the idea that words deal with thoughts and the arts with feelings. So, for Tolstoy, whereas science transmits truth, art conveys perceptions through the emotions. But the key point is that art and science, for Tolstoy, are concerned with goodness—increasing our cohesion with our fellows through the communication of appropriate thoughts and emotions.

Fry (1947), a former curator at the Metropolitan Museum in New York, positioned himself as diametrically opposed to Tolsoty's views about art, and in so doing anticipated many of the debates about art that would be taken up by contemporary authors. Fry was sceptical of the idea that art often provides a valid reflection of the ethos of the time. He noted that our reactions to works often rely, to some extent, on how we would interact with the objects they depict. Yet he conceded that art is an organ of our spiritual life.

More than anything, Fry emphasized that each part of the process of artistic production is accompanied by pleasure. Fry (1956) wrote, "It is true in nearly all works of art agreeable sensations form the texture of the work" (p. 5). In fact, according to Fry (1956), beauty is that which evokes a positive response, what he characterizes as "the aesthetic state of mind" (p. 2).

Further, the emotion of unity in science, he claimed, is similar to that of the arts. He conjectured that the emotional state that arises from the apprehension of the relation of ideas, in mathematics, for example, is similar to aesthetic apprehension. However, he qualified his view by pointing out that whereas in art the emotional response may be the end, in mathematics it may be the validity of the relationships that is the end.

However, Gestalt theorists emphasizing that perception is holistic, self-organizing, and form-forming, provide a counterweight to the Frian split between the cognitive and affective domains. Influenced by the Gestalt theorists, Arnheim (1969), who was trained at the University of Berlin in 1928 as a psychologist, contended that artistic activity is a form of reasoning, hence the title of his famous book *Visual Thinking*. Reasoning and perception are inextricably intertwined, he thought. Art provides a case, he claimed, where we can clearly see that perception and thinking are fused together.

So Arnheim (1969) ignored, in his words, the "property lines" (p. vii) between art and science. Art, he said, is aimed at a "pattern of directed forces that are being balanced, ordered, and unified. A work of art is a statement about the nature of reality" (1965, p. 5). Art, he went on, shows us the play between an "inner and outer world" (1965, p. 360).

As he noted, we may think, in theory, that the mind must gather information (perception) and processes it (thinking); however, in practice, he explained, the two are intertwined. As Arnehim (1969) put it,

> My contention is that the cognitive operations called thinking are not the privilege of mental processes above and beyond perception but essential ingredients of it. I am referring to such operations as active exploration, selection, grasping of the essentials, simplification, abstraction, analysis and synthesis, completion, correction, comparison, problem solving, as well as combining, separating, and putting in context. (p. 13)

For example, he pointed out that perception already involves problem solving. According to him, meaningful images help us identify, interpret and supplement perception. As he observed, we identify an elephant by our visual concept of it. In contemporary psychological jargon, we use both top-down and bottom-up processing. "Perceiving and thinking require each other," as Arnheim (p. 1986, p. 135) asserted.

He admitted that the essential trait of our unitary cognitive process is abstraction. But, as he argued, even concepts are particular; abstraction would be useless, unless we selected relevant traits. The upshot, for him, is that the senses and abstraction are inextricably intertwined. So, according to Arnheim (1989), then, artists use their knowledge of the meanings of various perceptions to influence the audience emotionally (Goleman, 1995) and intellectually.

Csikszentmihalyi and Robinson (1990) have explored the idea of aesthetics using interviews and quantitative methods aimed at art curators. They wanted to see how art experts describe their aesthetic experiences, by first letting them talk about them.

Relying on Beardsley's (1982) account of aesthetics, Csikszentmihalyi and Robinson characterized experience as having the following features: (a) an investment of attention (focus); (b) felt freedom (harmony); (c) a detached effect (as if we were separate from what is depicted); (e) active discovery (exhilaration); and (f) wholeness (integration and a sense of self-acceptance). As they went on to explain, these five characteristics mirror Csikszentmihalyi's (1990) idea of "flow" experiences, where we are lost in the moment, as it were. Also, in flow experiences, they pointed out, there is an autotelic experience where people do the said activity for its own reward. According to Csikszentmihalyi and Robinson (1990), like other types of flow experiences, the aesthetics experience is personally meaningful and becomes more complex with time. But how is the aesthetic experience discussed in art and mathematics similar, if at all?

Writing about the connection between mathematics and beauty, Sinclair (2006) noted that we often think that aesthetics does not involve concepts. As she pointed out, however, mathematics is based on the recognition of patterns and cycles of repetition.

Drawing upon Dewey's (1934/1958) work, she claimed that aesthetics serves a motivational role: we choose problems and solutions we find aesthetically pleasing.

Because Dewey's work has been so influential in educational circles, and cross-cuts and foreshadows my discussion of the arts (in "Visual Arts"), I shall digress to discuss it here in order to illuminate Sinclair's allusion to him. Dewey calls art a language, and as such is a mode of communication. Art gives us the "power to experience the common world," said Dewey (1934/1958, p. 131). Art is the extension of the rites and ceremonies of human beings. Through art, he tells us, we are aware of our "union with one another in origin and destiny" (Dewey, 1934/1958, p. 271). In fact, he calls art a mode of prediction of possible human relations.

His social view of knowledge is reflected in his views on education. Dewey (1938/1997), writing about education, said, "[W]e live from birth to death in a world of persons and things which in large measure is what it is because of what has been done and transmitted from previous human activities" (p. 34). We live "in" (1938/1997, p. 41) a situation—in an inter-subjective world. His views on education, namely, that educators need to take account of the past as well as our present lived experiences, reflect the idea that we are caught more generally between the precarious and stable.

Writing on the pragmatist's theory of mind Godfrey-Smith (1996), pointed out that Dewey was an interactional empiricist; he thought knowledge arose from the collision of the mind and world. As Godfrey-Smith put it: (a) there is a complex environment, (b) we need to render it reliable for prediction, and (c) there is to-and-fro between the two. That is to say, we have consciousness to deal with anomalies in the flow of events, and attempt to control them, which allows for the automation of behaviour and thinking.

So what Dewey brings to Sinclair's discussion of aesthetics is an understanding of the social dramatics underlying the consensus that aesthetic experiences can give rise to. Part of what we do in learning mathematics, according to Sinclair (2006), is developing an aesthetic sense of "fit" (p. 135).

Let me say a bit more about what a mathematician could mean by an aesthetic sense of fit in order to elaborate on this idea of consensus. In his now-famous *A Mathematician's Apology,* Hardy (1967) took the aesthetic point of view in mathematics to the extreme. He argued that beauty is the first test of the value of a mathematical idea: there is no place for ugly mathematics. As he explained, a mathematical idea is significant because it can connect in an original way to other ones. He cited as examples of significant mathematical theorems the fundamental theorem of arithmetic (i.e., any integer greater than 1 can be written as a *unique* product, up to ordering of the terms, of prime numbers, or is a prime number, e.g., $666 = 2 \cdot 3 \cdot 3 \cdot 37$); and Fermat's last theorem (i.e., $a^n + b^n = c^n$ is not true for integers when n is greater than 2). These theorems allow us insight into the nature of numerical relations: for instance, the fundamental theorem of arithmetic has led us to consider primes as building blocks of the natural numbers because all numbers greater than 1 can be reduced to their products.

According to Hardy, a serious theorem is more than a rule, however. He noted, for example, that there are several numbers whose sum of the cubes of the numbers of which it is composed equals the number itself:

$$153 = 1^3 + 5^3 + 3^3$$

$$371 = 3^3 + 7^3 + 1^3$$

$$370 = 3^3 + 7^3 + 0^3$$

$$407 = 4^3 + 0^3 + 7^3$$

These are trivial mathematical truths—they do not connect ideas in the field and can be likened to coincidences in everyday parlance.

With the distinction between the beautiful and trivial in view, he speaks of the depth and beauty of a serious theorem apart from its utility, calling mathematics “wholly useless” (Hardy, 1967, p. 119). He does not deny that mathematics may find an application (e.g., in physics), but that real mathematics, as opposed to trivial mathematics, cannot be justified other than being “an art” (Hardy, 1967, p. 139).

At the Eighth International Conference on Mathematical Education (ICME-8) in Seville, Spain, in July 1996, several papers were presented that related to how we learn in mathematics. They have been collected together in a publication entitled, *Forms of Mathematical Knowledge: Learning and Teaching with Understanding.* Fischbein (1999) focused, in his paper in this collection, on intuitions and schema in mathematical reasoning, noting that mental compression allows us to generalize about a vast amount of information that enables us to see the big picture. In Piaget’s thought, according to Gray, Pinto, Pitta, and Tall (1999), we can distinguish among three types of abstraction, not just two that are often remarked upon. The three types of abstraction are as follows: empirical abstraction, which allows us to derive knowledge from the properties of objects; pseudo-abstraction, which lets us tease out properties we have introduced into objects; and reflective abstraction that allows us to use existing structures to build new ones (Kamii & Housman, 2000).

Also participating at this conference, Ernest (1999) had pointed out that Ryle's (1949) distinction between knowing *how* and knowing *that* has come under attack; basically, philosophers of science often emphasize, contra Ryle—knowledge is not just a product we can separate from the process by which we construct it. Knowing something often requires knowing how to do something—or at least that must be the case somewhere in the genesis of knowledge.

As the result of 15 years of research of the Algebra Working Group (AWG) at the University of Wisconsin Center for Educational Research, we have a collection of papers, *Algebra in the Early Years,* that summarizes some key findings. Contributing to this collection of papers, Kaput (2008), also one of the editors, claims that mathematics evolves as a cultural artefact in terms of the symbols that embody it. He claimed that we need to describe the structure and function of mathematical coding in mathematically mature individuals.

Kaput, Blanton, and Moreno (2008) considered the character of algebra, noting the commonplace observation that generalization and symbolization are at the heart of algebraic thinking (Krshner & Awtry, 2004; Mason, 2008). As they noted, the only way to make a single statement that ranges over many instances is to form a unifying expression (e.g., "$x > y \equiv y < x$," i.e., x is greater than y if and only if y is less than x). "Generalizing," Kaput and Blanton (2008) said, "is the act of creating that symbolized object" (Kaput, Blanton, & Moreno, 2008, p. 20). In fact, they define mathematics as being concerned with (a) generalizing and expressing (those generalizations); and (b) a specialized system of symbols to reason about generalizations.

However, symbolization, Kaput, Blanton, and Moreno (2008) said, cannot be separated from conceptualization. Symbolic generalization, they noted, is not the only kind of abstraction. For example, we could generalize in natural languages. But symbolic generalization allows treating expressions as objects in their own right, in an "upward spiral of abstraction and mathematical power" (p. 23).

However, algebra is not a one-way ticket to ever more abstraction; it is a two-way street. As Smith and Thompson (2008) pointed out, in a paper on algebraic reasoning also in this collection, we need to reason about the relation between quantities, not just calculations. They observed three aspects to mathematics: (a) *generalization*; (b) the *socio-cultural*, for example, rules for using a system of notation; and (c) an *integrated aspect*, where we interweave representational thinking with children's constructions of mathematical generalizations.

In considering the ways in which mathematics learning is a situated activity, Ozmantar and Munaghan (2008) noted that the idea of abstraction is drawn from empiricist philosophers. It is often assumed that an abstraction can be applied in any context. They pointed out, however, that abstractions are dependent on the context, hence, situated; that is, they are shaped by the cultures and technologies through which they arise.

According to them, the development of algebra is dialectic: we move from the concrete to the abstract, and then back again to the concrete as we apply our generalizations to specific problems. "Mathematical abstractions," they said, "develop in personal-cultural-historical space" (p. 117). Mathematics, then, is *situated*—in the sense that its abstractions arise and are applied in contexts. Rogoff and Gardner (1984) claimed that even transference—applying knowledge to new situations—occurs by relying on more familiar contexts.

Rogoff (1995), for instance, discussed how we can consider the sociocultural context in three planes: participatory appropriation, which concerns how individuals change with their environments; guided participation, which concerns systems of involvements as people participate; and apprenticeship, which concerns community activity. She said that the goal is to seek patterns in the organization of socio-cultural activities. The goal of isolating the three planes is to explore how individual groups and communities transform as they constitute and are constituted by socio-cultural activity.

In fact, this new way of thinking about mathematics—in relation to concrete objects and contexts—ushers in a new philosophy of mathematics, or at least resurgence and reworking of some older empiricist ideas. Lakoff and Núñez (1997) claimed that traditionally we have considered the mind as disembodied, asocial, abstract, and de-contextualized; but now we know better, which they said gives rise to a new philosophy—a cognitive science philosophy of mathematics.

They analyzed mathematics in terms of grounding metaphors drawn from everyday experiences, and linking metaphors, that connect different branches of mathematics. They have articulated their ideas at length in *Where Mathematics Comes From* (2001), and their work can be read in conjunction with Lakoff's other major collaborator, Johnson (1986), who independently worked on the bodily basis of mental processes.

Johnson (1986) challenged traditional model-theoretic semantics whereby meaning is the result of a correspondence between words and things, popular among analytic philosophers working in the tradition of Frege (1884/1953). Rather, Johnson (1986) emphasized the role of understanding as underpinning a theory of meaning (i.e., an account of when a statement is meaningful), relying upon the work of Continental philosophers like Heidegger.

The notion of anchoring meaning in the socio-cultural context has been taken up by others in a variety of contexts, like cognitive science and clinical psychiatry, too (Brooks, 1993; Frie, 1997, 2007, 2008a, 2008b; Frie & Coburn, in press; Martin & Sugarman, 1997a, 1997b, 1999b, 2001a, 2001b, 2001c, 2003; Martin, Prupas, & Sugarman, 1998; Sugarman, 2008; Sugarman & Martin, in press; Sugarman, Martin, & Hickenbottom, 2009; Sugarman, Martin, & Thompson, 2003; Winograd & Flores, 1987). According to Johnson, for instance, breaking down images into propositions cannot fully capture transformations (e.g., scanning an image).

He argued that there must be order to having a meaningful experience, for actions, perceptions, and conceptions. His alternative, cognitive semantics, takes shared understandings as the basic way in which we order the world (discussed further in "The Cultural Context").

In sum, an embodied approach to mathematics has practical consequences, providing a further reason to embrace the notion of concept-rich learning. Specifically, the embodied framework allows us to build upon a Piagetian approach whereby we emphasize how we construct meanings from our initial interactions with physical objects, unto reflecting upon our thinking processes themselves. Also consistent with the embodied approach, is the importance of aesthetics, which it is reasonable to think, is grounded in our embodied state, both biologically and socially.

Based on the writings of Tolstoy (1899), Fry (1956), Hardy (1967), Holt (1971), Arnheim (1965, 1969, 1986, 1989), Csikszentmihalyi and Robinson (1990), and Sinclair (2006), we have a reason to think that there is an aesthetic sense at play in both algebra and the visual arts: aesthetics contributes to what we find valuable in each field. Further, Dewey (1934/1958) set the stage to also consider the role of the social context in the construction of meanings; but before we turn to that idea, I delve further into the interiors of the mind, and discuss how cognitive scientists have thought we make meaning in algebra.

Cognitive science and algebra. I shall discuss several prominent, traditional cognitive science accounts of learning mathematics (Fuson, 1998; Resnick, 1987; Baroody, 1987; Clements & Samara, 2004; Shoenfeld, 1987; Thorndike, 1923) as a prelude to considering some of the more recent situated approaches. In fact, even Thorndike's (1923) famous behaviourist account of arithmetic contains oft-overlooked cognitive features, which had been well known by Piaget (1956, 1959, 1964, 1970, 1985).

More recently than Piaget's writings, however, in a collection of papers, *Cognitive Science and Mathematics Education*, Shoenfeld (1987) pointed out that there is no single cognitive science account of meaning-making. He had argued, however, that we can discuss mathematics in these terms: "Mathematics is a coherent, sense-making activity. Actions are not arbitrary; one does what one does for good reason. If one understands these reasons, then everything fits together" (Shoenfled, 2008, p. 506).

Davis' (1984) *Learning Mathematics: The Cognitive Science Approach to Mathematics Education* provided some guideposts to the cognitive science of meaning-making, which have been reiterated, in various ways (e.g., Silver, 1987). According to Davis, memory in a computer can be said to be kept in a *register,* of which he distinguished several types. Davis argued that human memory can be seen to have two knowledge structures, passive memory (remembering a list of names) and an active component (a recipe from a cookbook), which are often intertwined in practice (Kieran, 1992; Polya, 1953).

At the very basis of the cognitive science approach to mathematics and meaning-making is symbol manipulation. The first type of meaning-making we deal with, in algebra, is linguistic (e.g., "if you buy 12 pencils for 16 cents, how much do 15 pencils cost?"). Fuson (1988) has considered the linguistic aspects of numbers. According to Fuson, we move from linking works with objects, to them being representational tools that allow more flexibility in application.

She distinguished between the following usages of numbers: cardinality, which concerns how many are in a set; ordinals, which concerns discrete and ordered (e.g., relative position to each other, "5 > 3"); sequence (e.g., "2, 4, 6…"); measure, which is continuous (e.g., length, volume, time); as well as symbolic (numerical), and non-numerical (e.g., a telephone number). Over time, said Fuson (1988), the sequence, counting, and ordinal meaning of number words become "closely related and eventually integrated" (p. ix). We must consider a counting situation, she said, which leads to understanding a cardinal, ordinal, sequence, or measured usages.

Resnick and Ford (1981), also taking a cognitive science perspective, were interested in how we think mathematically. They pointed out that from an information processing point of view all such activity occurs through a series of different types of memories (each capable of different types of storage and processing, constituting the system). These memories are: (a) the sensory intake register, (b) working memory, and (c) long-term memory. They investigated the way "the body of mathematical knowledge is internally organized and interrelated" (1981, p. 98). They claimed that the info-processing perspective is significant because it addressed both the skills of performance and the nature of the comprehension underlying them.

At the very origin of algebra, according to Resnick, Cauzinille-Marmeche, and Mathieu (1987) who wrote on it and cognition, is meaning. Meanings in algebra, they said, arise not from memorizing rules but reasoning about quantities and their relations. They identified two broad types of meanings—formal and representational. In a nutshell, formal meanings concern manipulation of expressions that are already algebraic; whereas representational meanings concern the generation of algebraic expressions. Between them, these two broad types incorporate three sub-types of meanings, namely transformational rules, generalizations, and quantitative relationships; I shall describe each in turn. (a) *Transformational rules*, are a sub-type of formal meaning, and can be applied to an initial expression, derivable from a set of axioms.

$$a - (b + c) = a - b - c$$

We can apply transformational rules deductively to all cases that take a given form (though we still must abide by other rules, e.g., if exponents were involved).

The next two sub-types of meanings are, according to Resnick and associates, representational. (b) Generalizations "can be viewed as statements of the relationship that hold between numbers and operations on them in *general*" (Resnick et al., 1987, p. 171), like the distributive law. Finally, (c) situations that can be represented mathematically by way of their quantitative relationships, requiring us to write appropriate equations. As Resnick and associates (1987) put it:

> The challenge of learning algebra, then, is both to relate the formalism to the situations and mathematical principles that give referential meaning, and to construct an understanding of algebra as a powerful formal system that contains its own internal meaning. (p. 201)

Baroody (1987) who has concentrated on children's mathematical thinking is important to consider because of his pre-eminence in the field. According to Baroody, for an absorption theorist, knowledge is impressed from without, internalized, and copied. For the cognitive theorist, however, he noted, learning is a collection of facts and habits worked on from within in order to make meaning. He advocated cultivating mathematical thinking and understanding. He also counselled that we match external and internal factors (like needs).

Baroody (1987) also pointed out that for the absorption theorist an error, is a deficiency. For the cognitive theorist, however, he remarked, an error is an attempt to make meaning or solve a problem where we want to know. He noted that even understanding the cardinality rule requires knowing that the last number counted represents the entire set.

Finally, focusing on young children, Samara and Clements (2002, 2004) have provided another basis for a cognitive science approach to mathematics. Samara and Clements (2002) pointed out that for Piaget, mathematical skills should be based on logical skills required to classify, order, and conserve. They noted, however, that children acquire mathematical competency with numbers directly. To resolve the dispute, they involved two groups of students, one instructed with logical operations and the other group with number concepts. From their findings, Samara and Clements concluded, "It can be concluded that children benefit by engaging in meaningful number activities, many of which involve classifying and ordering" (2002, p. 165).

For adherents of GOFAI (Good Old Fashioned Artificial Intelligence), the attempt to explain how we make meaning in algebra becomes a problem of explaining how we process information, which is to be modeled along the lines of a computer, using language common there, like storage, retrieval, coding, decoding, and so on. Yet, as I have already intimated, the big change in cognitive science has been the situated moment, and I turn to consider the implications for understanding meaning-making in algebra for its adherents.

The cultural context. The *situated movement* holds important consequences for mathematical pedagogical practice. For example, writing for educators, Hatfield, Edwards, and Bitter (1997) provided the following useful definition: "Culturally relevant mathematics is a term that shows awareness on the part of educators that all subjects, including mathematics, occur in an environment that must not be ignored" (p. 69). Specifically, they point out that mathematics is part of every culture and that it is the right of all to have access to it for success in the modern world. They distinguish between field-dependent learning, where problems have a role within a whole; and field-independent learning, where problems are abstracted from contexts.

The importance of field-dependent learning has been emphasized by educators working in an aboriginal context. Aboriginal education, especially in Canada, has been a major focus of the State (in the Canadian context, specifically, provinces) and social science researchers. Since I am considering the role of the socio-cultural context in learning, it is important to briefly consider the work of Cajete, who has written mostly about science education in an aboriginal context; his ideas could well apply to both the visual arts and mathematics, as we shall see.

The title of his book *Look to the Mountain* (1994) suggests to Native Americans to be concerned about the future impact of our actions on future generations. Cajete (1994) observed that natives focus on learning by doing, which has traditionally been denigrated. As he pointed out, modern and traditional education remain separate both historically and contextually. He sought to integrate modern learning with the cultural basis of knowledge that supports aboriginal ways of life. Otherwise, as he said, Western education can erode aboriginal culture.

The Western approach, according to Cajete is objectivist, deemphasizing the role of the subject. However, Native American's knowledge is personal and relational. Learning is a creative act of making meaning. More than this, for the Native American, learning requires participation with others in collaborative activities where everybody plays specified roles.

According to Cajete, we need to teach with an awareness that we live in an increasingly multi-cultural setting. Education must be, he said, for "life's sake" (1994, p. 26). That is, what we learn must connect to the world we live in. Learning, he contended, is a subjective experience tied to a place environmentally, socially, and spiritually.

Taking the example of art, he noted that for the Native American there is no word for it; the closest is "making" or "completing." As he observed, the way we talk about place reflects how we feel, see, understand, and think. As Cajete (1994) put it, "We are the natural world" (p. 80). And again, "We are all related ['*Mitakuye oyasin*,' as the Lakota people said (Cajete, 2000, p. 86)]" (1994, p. 89).

Another important aspect of Native American knowledge is story. Their stories add context and meaning to content. They also use visions to learn. He characterized the Native American use of visions as holistic, evolving through relationships with others, having endemic patterns, teaching us about learning, about our strengths, weaknesses, and potential.

In his book *Mathematical Enculturation: A Cultural Perspective on Mathematics Education,* Bishop (1988) of the Department of Education, Cambridge University, emphasized that mathematics is "a way of knowing" (p. 3). As he put it, he advocated educating about mathematics, through mathematics, and with mathematics. Traditionally, he pointed out, math educators have focused on technique, how to do so-and-so. However, as he noted, a doing-approach (e.g., learning by rote) is impersonal and ignores personal meanings. For example, Bishop (1988) wrote,

> What the teacher really needs is not a text, but activities and resources that help learners develop. What the learner needs is not a text, but an involving, warm, sympathetic, and intellectually challenging environment. (p. 11)

The traditional approach, he commented, takes on a top-down form, characterized by technique, an impersonal approach, and the text.

Based on Lancy's (1983) cross-cultural studies of mathematics education, Bishop pointed out that though the first of the Piagetian stages, the motor-sensory stage, of development has a genetic basis—it is universal—the others (e.g., the formal operations stage) are culturally bound and determined by such factors as technology. So, according to Bishop, there is no escaping the role of culture in mathematics education.

As a counterpoint to the traditional approach, then, mathematical enculturation, he characterized as: (a) interpersonal and interactional; (b) a significant account of the social context is provided; (c) a formal, institutionalized, intentional, and accountable setting. The point of mathematical enculturation, according to him, is to move away from pedagogy of technique, to meaning. And, according to him, mathematical enculturation involves the teacher and student, along with peers, in a joint enterprise of mathematical meaning-making. In other words, it is for all—teachers, learners, and their peers alike.

The educational moral of the situated movement, epitomized by Cajete's (1994, 2000) account of Native American knowledge, Hatfield's, Edwards', and Bitter's (1997) emphasis on culturally relevant pedagogy, and Bishop's (1988) call for mathematical enculturation, is that we need to challenge traditional in-the-head learning, looking in the direction of real life contexts (Sawyer & Greeno, 2009). In Chapter 1, "Introduction: A Philosophically Minded Prelude," I have already considered Smith's (1999) characterization of the situated movement. Here I want to explore it in more depth and at length. In what follows, I consider the historical background of the situated cognition movement and some of its key exponents.

The historical background of the situated movement. Lerman (2000) spoke of the "social turn" in our understanding of mathematics. Resnick (1993) spoke of a "sea change" (p. 3) in cognitive science in the direction of the social context, and away from the individual (Cole, 1993; Donald, 2001, 2006; Martin & Sugarman, 1998, 1999a, 2002; Sugarman & Martin, in press; Wertsch, 1993). As Resnick (1993) explained, we look to the social context to understand cognition because constructivists emphasized that knowledge is assembled with the help of schemas — that are social.

Participating in the 1992 symposium of the annual meeting of the American Research Association, Krishner and Whitson (1997) claimed that the shift in cognitive science towards situated cognition is as profound as that which occurred 35 years earlier from behaviourism to cognitive approaches; and that the changes are most obvious in mathematics.

However, they cite several reasons that the shift to situated cognition has met with such resistance: (a) individualist approaches accord with our common sense assumptions about thinking and being in the Western tradition; (b) individualist approaches are recognized by the scientific community; and (c) postmodern influences have provided a critique of modernism, that is, basically a shadow of doubt has been cast upon the idea that we can use reason to hook on to an objective reality.

Also, Lubin and Forbes (1984), reflecting on reasoning in children, argued that changing the motivational background in which we apply social skills is important to understanding structural development of the skills themselves. For example, reflecting upon the work of Piaget, White and Siegel (1984) noted that research in developmental psychology is tied to "distal bureaucracies," preparing people for specialized occupational roles. According to them, we assume that development moves from the concrete to abstract; are trained to think at a distance; and to be able to speak about our goals and motives. For example, as they noted, Piaget (1952) articulated three types of symbol manipulation: symbolic play, language, and drawing and painting. Symbols require the translation of external relations into internal relations, taking us beyond the here and now. Yet, they concluded, it is the contexts in which we learn that help us understand how we develop.

Yet Glick (1981), in considering Piaget's developmental model, has suggested that there is mounting evidence of a greater role of intra and inter cultural variability. Glick claimed that perhaps there are "multiple rationalities" and that Piaget offered a precise description of only "one type of rationality" (1981, p. 226). In fact, Walkerdine (1988, 1997), in a critique of rationality, pointed out that modern scientific accounts of cognition like Piaget's can be seen as implicated in the production of modern forms of government. As she claimed, minimal changes in the context result in huge changes in performance.

For example, Mayer (1986), summarizing research on mathematics education for *Cognition and Instruction,* pointed out that, when solving an algebraic problem, the solution strategy that we adopt depends on the format of the problem (e.g., is it presented as an equation). He noted that with algebraic problems, students sometimes adopt an *isolate strategy,* getting all the x's on one side; or a *reduce strategy,* clearing parentheses as soon as possible. He concluded that algebraic problems require the development of a level of automaticity. Thus, he suggested we allow opportunities for practice, planning, and monitoring our plans. Students, he said, need to learn how to identify problem types and represent them in different ways (e.g., drawing a picture corresponding to the problem, delete irrelevant information, underline essential information, and sort problems into types). So in addition to behaviourism, he also has emphasized cognitive approaches: techniques for representing and planning.

In fact, according to Walkerdine, relational dynamics create intersubjectivity. For example, even an interest in calculation also occurs within the regulation of practices, with a positioned subject, and further, the activity has emotional significance (Goleman, 1995).

Writing for a collection of papers on socially shared cognition, Wertsch (1993) has pointed out that this area of study could be traced back to Marx: the idea that understanding the individual requires understanding the social relations in which the individual exists. According to Wertsch, what Vygotsky adds that is novel to Marx's thought is the notion of "mediation:" the way artefacts, for instance, help mediate the relationship between the individual and the social world.

As Wertsch (1995) explained, the goal of sociocultural research is to understand the relationship between human mental functioning and the cultural, historical, and intersubjective setting. He proposed, however, that mental functioning and the socio-cultural context are dialectically related.

As a result of the 1990 American Education Research Conference in Boston, a collection of papers has been complied in *Distributed Cognitions: Psychological and Educational Considerations,* which was motivated by the realization that memory studied in the laboratory is different from what happens in real life contexts.

Pea (1993) remarked that the conceptions of reasoning and intelligence have largely been thought to be the property of minds and individuals. As she puts it, we have been concerned with a solitary intelligence. But, as she observed, the mind rarely works alone. "The intelligence revealed through these practices [of cognition]," Pea (1993) said, "are distributed—across minds, persons, and the symbolic and physical environments, both natural and artificial" (p. 47). As she pointed out, intelligence is distributed in designed artefacts (e.g., physical tools, representations like diagrams, and computer interfaces for complex tasks). According to her, intelligence is distributed by "off-loading" (p. 48), that is, as action constrains in the physical or symbolic environments; in other words, the environment participates in our intelligence.

And, as Pea (1993) pointed out, our environments are "thick" (p. 48) with artefacts used to structure activities. The artefacts we engage with, as Pea (1993) noted, already are embedded with patterns of previous reasoning. Constructivists, she said, miss the fact that intelligence is already built into artefacts and situations (e.g., LEGO blocks can only be assembled in so many ways). She traced the interest in situated cognition to Simon's (1969/1996) *Sciences of the Artificial,* which requires a brief explanation.

In that book, Simon (1969/1996) observed that if we sketch a path made by an ant, we cannot attribute the complexity to the ant. Rather, the complexity of the path is the result of the interaction between the ant's goals and the environment. He noted that we are connected to the environment in two ways, which he dubs: *Afferent*, we intake sensory input; and *efferent*, we act on the environment, both of which require having built up associations about the world. He provided the following example of the interaction:

> The painting process is a process of cyclical interaction between painter and canvas in which current goals lead to new applications of the paint, while the gradually changing pattern suggests new goals. (Simon, 1969/1996, p. 163)

Yet educationalists that rely on a solitary paradigm of cognition tend to decontextualize intelligence from the context in which it functions. *Distributed* means for Pea (1993) "resources that shape and enable activity are distributed in configuration across people, environment, and situations" (p. 49). She stressed, then, that intelligence is foremost an activity, something accomplished not possessed. Relying on the work of Gibson (1979), she said that the artefacts in the environment provide "affordances" (p. 51) for its support and actualization. Pea suggested striving towards a reflectively and intentionally distributed intelligence, where we are inventors of distributed-intelligence-as-tools, rather than receivers of intelligence-as-substance, which she also thinks affords us the ability to adapt to and contribute to change.

Moll, Tapia, and Witmore (1993), in the same collection, attempted to present an ethnographic view of cultural dynamics of activity settings in households and classrooms. They are not interested in the immediate environment where learning takes place, but the broader social system that helps define the nature of the environment and what tools and resources are available to participants. They speak, for example, of the "social distribution" (Moll, Tapia, & Witmore, 1993, p. 140) of cultural resources. For instance, Hutchins (1993a, 1993b 1995) has documented situated cognition on navel vessels. Echoing a common refrain, Hutchins (1995) criticized the tendency to understand cognitive process in-the-head as motivated by analytic purposes.

One important source for the situated movement is ecological psychology. Writing the entry for "ecological psychology," also in the *MIT Encyclopedia of the Cognitive Sciences*, Neisser (1999) noted that these theorists reject the traditional notion of *stimulus* where we are bombarded by sense data. Rather, relying on the work of J. J. Gibson (1979), they hold that the environment presents us with *affordances,* or information that we can use in a specific situation. "The ecological self, for example," Neisser (1993) explained, "is the individual considered as an active agent in the immediate environment" (p. 4). According to him, we perceive ourselves as both here and now; and conceptually that relies on social forms, like "professor," "husband," "father," and so on. He wrote, "Each of us has been an interpersonal self just as long as — perhaps longer as—an ecological self" (Neisser, 1993, p. 18).

Neisser (1999) cited Reed (1996a, 1996b) as a contemporary source for ecological psychology and I consider his views briefly for the purpose of exemplification. As Reed (1996a) explained our dilemma, psychologists have been caught between the neurosciences and the historical cognitive sciences; and this harkens back to an older tension between physiologists and interpretive psychologists. Basically, on the one hand we have a reductive tendency; and on the other, we attempt to look at our ordinary, everyday experiences.

Against a mechanical, reductive worldview, and also taking his point of departure from Gibson (1979), Reed (1996a) emphasized that we actively encounter an environment: "Cognition is not either copying or constructing the world. Cognition is, instead, the process that keeps us active, changing creatures in touch with an eventful, changing world" (Reed, 1996a, p. 13). He considered his ecological view "completely consistent" (1996a, p. 181) with Lave's (1990) work (discussed in "Core Contributions to the Situated Cognition Literature"). The key point of contact is that the ecological approach, like situated theorists, take a holistic approach to human actions; we do not attempt to decompose human actions into the associations between the stimulus and response.

We may wish to recall that in Chomsky's (1956/1969) famous critique of behaviourism, he pointed out that the notion of what can count as a stimulus in B. F.

Skinner's behaviourism is too vague. Even, "the Vietnam war," he noted, could be considered a stimulus, for example. But looking at the stimulus holistically we are returned to understanding along the lines of ecological psychologists. That is to say, a stimulus can be conceived as an everyday object or event, opposed to sense data, which prompts a response. Suffice it to say that looking at things this way, contra Chomsky, may stand to make behaviourism more attractive, not less because it includes the personal dimension of meaning.

Since a broader philosophical discussion of situated cognition will further help lay the groundwork for making links to the visual arts, I shall consider the views of Robbins and Aydede (2009), authors of the introductory chapter for *The Cambridge Handbook of Situated Cognition.* They identified three theses often held by proponents of situated cognition: the embodied, embedded, and extended theses. First, *the embodied thesis* is that cognition depends on the body. For example, we can explain how symbols become meaningful by understanding the role of sensory motor basis of cognition. Further, they distinguished between on-line cognition when we are engaged in the world; and off-line cognition when we are not engaged (e.g., when we are reflecting).

Second, *the embedded thesis* is that cognitive activity exploits structure in the natural and physical worlds. As Warneken and Tomasello (2009) said, writing about cognition and culture, we form "shared intentionality" (p. 469). For example, institutions like marriage, money, and government require social recognition to maintain their meaning/value. An example is "cognitive off-loading" (Pea, 1993, p. 48), the way in which we store information in our environment, which beckons us to an ecological theory of mind, and this leads to their next and final point.

The extended thesis is the view that the boundaries of cognition extend beyond the individual. The first two theses, as they noted, are not as controversial as the third, the extended thesis, which entails an ontological claim about what the mind is. As they noted, the reason to take seriously the extended thesis has been interest in dynamical systems, whereby the entire system has properties that cannot be accounted for by merely understanding its parts. Rather, we have to understand the mind as part of the environment, as one entire system.

In a collection of papers on situated cognition, Damon (1993) has pointed out that situated cognition has been discussed in three ways: (a) as practice-centered knowledge; (b) as a scientific approach that views thinking as actions; and (c) as viewing all knowledge as embedded in historical, cultural, social relations. He has raised the question of what role development plays in the social construction of knowledge.

Discussing situated cognition in relation to the cognitive science program, St. Julien (1997) characterized it as acting with competence, which she discussed as a generic web of social relations and artefacts that define the context of actions. In fact, in her paper, she avoided defining situated cognition and offered a narrative to exemplify it. Concluding that the cognitive scientists' approach to explaining cognition has been too narrow, she claimed we need to study it with "a fine-grained analysis of this connection [between the social world and individual]" (p. 266).

Engeström and Cole (1997), summing up a collection of papers in *Situated Cognition: Social, Semiotic, and Psychological Perspectives,* however, warned that defining situated cognition is a "Pandora's Box" (p. 301). Roughly, they characterized it as a practice-bound approach to cognition. They build their views upon the work of Luria

(1976), one of Vygotsky's students who emphasized the socio-historical nature of knowledge, which I shall briefly remark upon.

Luria (1976) contrasted graphic functionalism, where we are guided by the physical features of objects, from semantic-functionalism, where we pick things out based on abstract categories. According to him, the basic categories of mental life are produced by social history, and as our socio-economic conditions change so too do the categories of mental life.

Cole, writing a forward for Luria's (1976) English edition, observed that Vygotsky thought the Soviet psychology of his day was in crisis, caught between the behaviourists and cognitivists. He used the idea of historical mediation to link the stimulus and response. Cole (1995) cited several reasons for taking the sociocultural foundations of human nature as a starting point: (a) disenchantment with positivist social sciences; (b) erosion of support for Piagetian theories; (c) skepticism of terms used to study artificial intelligence; (d) despair at the fractionalization of psychology; (e) and search for viable alternatives to various kinds of social learning theory.

Cole cited the research of Lamb and Wozniak (1990), who wrote about the methodological aspects of sociocultural research, as a good example of investigators advocating a constructivist theoretical approach. As Cole (1995) noted, understanding behaviour through its genesis has long been a tenet of cultural-historical approaches. He suggested a "mesogenic" (1995, p. 194) approach that falls between the microgenetic scale (studying individual cognition) and the macrogenetic scale (e.g., studying the social transformations that influence cognition).

Core contributions to the situated cognition literature. The most influential work on situated learning has been done by Lave (1988, 1993a, 1993b, 1997; Lave & Wenger, 1991). Lave (1993b) has stressed that situated learning is a general theoretical perspective that interrelates accounts of activity, meaning, cognition, learning, and knowing, and the interdependence of the agent and the world. Whereas internalization is the explanation of what the individual and social world has in common, the situated perspective prompts us to emphasize social interaction.

Lave (1997) has distinguished between thinking of learning as acquisition and thinking of learning as the practice of understanding. Acquisition theorists presuppose that learning is a matter of transfer to particular contexts. The practice of understanding theorists view learning as tied to the socio-cultural context in which it occurs. Understanding-in-practice theory, she said, looks like a more powerful source of enculturation. She suggested that we need opportunities to practice mathematics in contexts where it is applied.

In fact, Lave (1988) summarized her work as attempts to view learning as extension of everyday practices. She is opposed to what she calls a "claustrophobic view of cognition" (1988, p. 1). Lave (1988) argued that cognition is distributed over the mind, body, activity, and culturally organized settings. And those activities take on different forms in various settings. She refered to the "person-acting" (p. 180). This concept underscores the notion of an embodied-self entailed with the world and relations that are not completely decomposable for analytic purposes into their elements. Her goal is to cross-cut mind-body dualism. Learning transfer, she said, is not the sources of continuity across domains but social reproduction is. She characterized this view as taking cognition outdoors: out of the laboratory and out of the head.

At a conference on the "context problem" Lave (1993a) discussed it as (a) conceptualizing the relationship between the person-acting and the social world, and (b) to provide an account of the social world of activity in relational terms. The point, according to Lave, is that we must study cognition in the lived-in world, not as if it happened just in the mind.

Lave and Wenger (1991) coined the term "legitimate peripheral participation" (p. 29), which is the idea that mastery involves full participation in the socio-cultural practices of a community. For them, learning is inseparable from social practices. According to them, there is a relational character to knowledge and learning; a negotiated character of meaning; and an engaged nature of learning activities, as learners struggle with problems. Even abstract knowledge, they noted, occurs in a specific context.

Legitimacy, then, takes the form of "belonging" (p. 35), and the legitimate periphery concerns the social structures involving power relations. The idea of peripheral participation brings to mind Vygotsky's ZPD, which Lave and Wenger (1991) noted has been discussed in terms of mastery, entry into a society, and the historical change of bodies of knowledge. Their aim is to emphasize the interdependence of: agents, the world, activity, meaning, cognition, learning, and knowledge.

As Lave and Wenger (1991) observed, taking the person as the basic unit results in a non-personal view of knowledge; conversely, taking the social context as fundamental personalizes knowledge. For example, the identity of learners is central to legitimate peripheral participation. The concept of a person links meaning to action, which has inspired situated studies of learning. Building on Lave's research, Cobb (1994) explained, "In general, socio-cultural accounts of psychological development use the individual's participation in culturally organized practices and face-to-face interactions as primary explanatory constructs" (p. 15).

A caveat: Do not lose sight of the individual. In a collection of papers on the emergence of mathematical meaning, Cobb and Bauerseld (1995) wanted to explore the cognitive and sociological perspectives of education in mathematics. Voigt (1995), contributing to Cobb and Bauerseld's (1995) collection, argued that the social and cultural dimensions are not peripheral to learning, but intrinsic. According to him, the interactionalist approach views mediation occurring between the individual, their cognitive level, and the collective, and their socialization into a pre-given culture.

Voigt's interactionalist perspective also echoes Perkins (1993), who observed that our understanding of thinking of learning has often led to an asymmetrical posture towards the person and the arena where it happens. Following on coattails of the situated movement, Perkins (1981) has emphasized getting concrete: placing problems within a context. Perkins (1981) stressed that creativity is not just doing something novel, but doing it with a purpose. As such, he draws upon Getzels' and Csizszentmihayi's (1976) work on problem-finding, who emphasized that this process involves selection (discussed as *challenge-identifying* in Chapter 5, "Discussion: Solidifying the Common Link").

Because of their attempt to explore the factors that motivate creativity in the arts—and since the general idea of linking problem-finding to creativity could just as easily apply to mathematics—it is worthwhile to consider the work of Getzels' and Csizszentmihayi's (1976) *The Creative Vision: A Longitudinal Study of Problem Finding* *in Art,* here. They studied several hundred artists, and from a subgroup of them, conducted the first longitudinal study of this population type. They also claimed to have provided the first analytic description of the creative process in a true-to-life setting.

According to Getzels and Csizszentmihayi (1976), the key to creative activity is *problem-finding,* that is, the way problems are envisaged, posed, formulated, and created. The creative aimed at what was original. For instance, in their study, each artist attempted to describe the source from which creative problems arose for them. The artists spoke of being motivated by values such as discovery of self, self-knowledge, understanding, and a quest for reality. Getzels and Csizszentmihayi identified six possible motivations: theoretical, related to the pursuit of truth; economic, seeking financial independence; aesthetic, finding meaning through art; social, related to interpersonal relations; political, dealing with interpersonal persuasion and power; and religious, related to supernatural goals.

According to Getzels and Csizszentmihayi (1976), those with social goals tended to go into teaching or applied fields. Only those with very high aesthetic commitments went into fine arts. Our values, they conjectured, provide a basis to pursue creative problem-finding. Their participants indicated that they were influenced by school experiences, but held a strong value system and extreme personality traits. That is, they would be the ones to persist in the face of failure (something, incidentally, that perplexed Bandura (1997)).

Specifically, Getzels and Csizszentmihayi (1976) hypothesized that there would be a positive relationship between problem-finding and the quality of art produced. They noted that aesthetic criteria are vague and subjective, but consistent and predictable. That is, when a group of people are exposed to an artwork the reactions, overall, tend to be consistent. They believed that they found evidence for their hypothesis: successful problem-finding leads to higher quality art.

As Getzels and Csizszentmihayi pointed out, even art occurs in a social context. For instance, identity is linked to the occupation, "being an artist," which must be legitimated by social institutions at some point. Problem-finding, said Getzels and Csizszentmihayi, may be at the basis of the creative vision.

We may wish to recall Newell's and Simon's (1972) *Problem Solving*, where they defined a *problem* as when we want something and do not know how to get it. According to them, a task environment includes the problem solver, task, and goals; the invariant features are those that remain constant (e.g., in playing tick-tack-toe, we must place marks on three lines to be successful). Yet, Newell and Simon (1972) also claimed that problem-solving occurs in a "problem space" (p. 789); and that effective problem-solving requires that information must be encoded into it. "Thus," Newell and Simon (1972) wrote, "the question remains of how the particular problem space used by the problem solver is determined" (p. 789).

Undertaking a critical analysis of current research and theory in psychology and the computational sciences, Wagam (2002) sought to elucidate problem-solving processes in everyday life and science; which we may wish to take note of in passing, since given Getzels' and Csizszentmihayi's (1976) definition of *problem-finding* (i.e., recall, the way problems are envisaged, posed, formulated, and created), it is reasonable to think that the cognitive processes involved there can overlap with problem-solving.

That is to say, the cognitive processes involved in problem-solving can likely help us illuminate the cognitive processes involved in problem-finding, and vice versa.

Wagman distinguished between representing problems synthetically or analytically. Synthetically, according to Wagman, we can: (a) restructure the configuration of a problem; (b) differentially emphasize different aspects of a problem; (c) interpret the problem at a higher level of abstraction; (d) apply a structural analogy to a problem; and (e) apply relevant knowledge. Analytically, we can: (a) decompose the problem into familiar components; (b) partition the problem into minimally small and equivalent units; (c) reduce the problem to known formalisms; and (d) apply knowledge to the sub-goal of the problem.

It is worth digressing to retell, because it is so striking, the famous example of how restructuring problems can be used to solve them, which brings to mind the mathematical discoveries of Poincaré.

> Then I [Poincaré] turned my attention to the study of some arithmetical questions apparently without much success and without a suspicion of any connection with my proceeding researches. Disgusted with my failure, I went to spend a few days at the seaside, and thought of something else. One morning, walking on the bluff, the idea came to me, with just the same characteristics of *brevity, suddenness, and immediate certainty,* that arithmetic transformations of indeterminate ternary quadratic forms were identical with those of non-Euclidean geometry. (cited in Wagman, 2002, p. 5)

What we can take away from this passage is that Poincaré's discoveries, striking as they are, occurred in a context — a problem space that has been internalized, to use the terminology of Newell and Simon (1972).

Returning to Perkins (1986), he has attempted to look at knowledge as design. He saw the giving and taking of knowledge as part of the universal human enterprise of transmitting knowledge. Even our artefacts, like tools embody knowledge (e.g., a knife), he claimed. Perkins (1986) suggested that knowledge *is* design in that it is not merely information. Contra Ryle: knowing something requires knowing how to do something. In a sense, Perkins conceded, "We are what we know," but added, "Our specific and general knowledge and know-how channels how we think and act" (p. 231).

For example, Turkle (2005), who explored human-machine relationships, looked at computers not as tools, not in terms of what they can do *for us*, but *to us*. She inverted the traditional aim of cognitive science researchers, interested in explaining our thinking using a computer model, by asking, this: will we ever think like machines?

As Perkins (1995) explained, we have tended to view the learner as a person-solo, a lone computer, as it were. Most cognitive science accounts of intelligence view it as an attribute of solitary individuals carrying out internal transformations of mental representations of symbols for goals, objects, and relations. We may wish to recall that in Newell's and Simon's (1972) investigation of problem-solving they excluded motivational and personality variables, the so-called "hot cognition" (1972, p. 8). In ending this work, however, they noted that semantics, the denotative aspect of symbol systems, has emerged as a central focus of information processing theorists. Yet it is precisely the semantic and denotative aspects of symbol use in information processing, including the motivational and personality variables that are likely mediated by—the socio-cultural context—the core invariable of any task environment.

Alternatively, and extending Newell's and Simon's (1972) work, the person-plus view prompts us to take into account the role of the surround (say, the use of a notebook to solve a problem through externalizing the thought process); as well as the residue left by thinking (i.e., sketches in the notebook). Basically, the person-plus is the surround and the social environment. The surround is treated as a participant in cognition, a vehicle of thought. Also, Perkins (1995) emphasized, thinking leaves residues in the surround (e.g., in the notebook).

Analyzing the person-plus model, Perkins (1995) uses four access characteristics of a system, common in the cognitive science literature: *knowledge*, of which there are different kinds, like factual or procedural knowledge; *representations* that concern how we pick up, store, and record knowledge. *Retrieval* deals with whether we can find the knowledge in question efficiently; and *construction* concerns assembling knowledge into new constructions.

As he pointed out, though computeresque, the four access characteristics can fruitfully be applied to understanding the flow of information in other domains (e.g., DNA replication) as well as the process of thinking and learning. *Knowledge* concerns the content-level of facts and procedures; it also deals with higher-order knowledge of problem solving strategies (e.g., styles of justification and explanation). *Representations* can include visual and mental models that are known to assist in understanding complex and novel concepts. *Retrieval* is an issue here because it may be inert, that is, only accessible under realistic conditions. Finally, *construction* concerns us, because, among other reasons, of the limits of short-term memory; thus creating a rich surround can serve as a useful aid to thinking and learning.

However, Perkins (1985, 1995) has written about, and criticised, the *fingertip effect*, the idea that when an opportunity is given to people they take it. As he pointed out, novices will not take advantage of opportunities just in virtue of them being there; when the cognitive burden is high, new approaches will be avoided; and, more generally, there may be no motivation to take them up.

Within a surround, Perkins (1995) distinguished the *with* effects of technology when it is at hand (e.g., doing mathematics); and the *of* effects as residual (e.g., thinking in a linear and systematic way). For example, technology like print media, once part of our surround, "not only informs us about practical activities like baking a cake from a recipe [*with* effects], but also broadens horizons and fosters combinations of work and play with data and ideas supplied in print [*of* effects]" (Perkins, 1985, p. 11). Thus, technology, so much part of our surround, has both intended and unintended consequences, effecting behaviour and cognition.

As Perkins (1993) pointed out, however, traditionally educationalists have focused on making us a person-solo. Admittedly, higher-order knowledge, according to Perkins, can inform the construction of understanding content knowledge and supporting executive functions, that is, orienting various functions of working memory to larger systemic goals and interpreting information related to them (Hogan, 2003); and one of the goals of pedagogy is indeed to internalize, and automate, these higher-order abilities. Nevertheless, active thinkers, he noted, form a rich surround to obtain their objectives.

Conversely, he pointed out, open-ended instruction can suffer from not having adequate executive or surround support to carry out tasks.

Perkins noted that at the center of the person-plus is the person. Summarizing the philosophical moral, however, he aimed at a sophisticated view that captures both our solo and plus aspects, a view that resonates with others (Donald, 2001, 2006; Martin & Sugarman, 1998, 1999a, 2002; Sugarman, 2008); Perkins (1993) likened us to a set of interactions and dependencies, a "union of involvements;" we are, he went on, the "the sum and swarm of participations" (p. 107).

According to Perkins, mature human beings, then, are never truly a person-solo. Observing that the 1960s reform movement in education, inspired by thinkers like Dewey (1938/1997), gave way to the 1970s "back to basics" battle cry because students were not understanding what they learned, he advocated pedagogy of understanding. That is, we should, according to Perkins, make it explicit in our teaching practice that part of why we learn something, and in a certain style, is because it, and doing so, has a social value—a value of which we are in part constituted by.

But taking a critical look at the notion of distributed cognition, Salomon (1993) suggested they must have some key sources, like individuals. Distribution, he noted, suggests sharing resources, authority, language, experience, tasks, and a cultural heritage.

However, Salomon (1993) considered the following philosophical puzzle: if we emphasize the distributed nature of cognition, an individual may be fully determined by the system, which seems counter-intuitive because we think we make decisions of our own volition. Conversely, Salomon champions the idea that we are free, but affected through the interactions we engage in, a view he considered akin to Bandura's (1997) theory of reciprocal determinism. Salomon (1993) emphasized that we should speak of the effects "with [so-and-so]" (p. 126) to capture the idea of interaction with the social environment.

In sum, according to Salomon (1993), the psychology of individual competencies and distributed cognitions should be accommodated within a single theoretical framework. Yet, coming full circle, he warns that no theory of individual cognition would be complete without taking into account the reciprocal interplay with situations of distributed cognitions. As Nickerson (1993) summarized when he wrote the concluding chapter to the collection *Distributed Cognition*, we need to recognize the distributed cognitions and the individual mind.

Similarly, Krishner and Whitson (1997b) noted the need to integrate both social and individualist (e.g., neurological and psychoanalytic) perspectives. That is, Krischner and Whitson (1997b) do not just want a one-sided, social approach to cognition, rather they agree with Lave's (1988) suggestion that we require a complex, multifocal strategy—which would include individual and biological factors.

Writing the final chapter of a collection on the sociocultural approach to the mind, del Río and Alvarez (1995) said that Vygotsky tried to find a way between viewing psychological processes in universal terms or taking a culturally relative approach according to which we are constituted by different cultural settings. Following Luria's (1976) Vygotskian lead, del Río and Alvarez (1995) suggested a cultural systemic approach according to which the external, cultural architecture is seen as where functional systems are constructed. As they remarked, the cognitive self is understood in terms of the mechanisms by which it is the receptacle of sense data. The pragmatic self, on the contrary, they said, is understood in terms of how we act in relation to our environment — and that environment includes other people.

So the moral of the situated movement, in brief, is that providing an account of our cognitive processes requires providing an account of our interpersonal relationships. The situated movement brings us full circle, from the Piagetian inspired constructivist reaction against behaviourists, unto the cognitive science models of mental processes, we once again return to incorporate the role of the social context in learning, in this case, in algebra. We need a concept-rich and socially-rich approach to learning in algebra that does not allow us to lose sight of the individual (Donald, 2001, 2006; Martin & Sugarman, 1998, 1999a, 2002). As we shall see in the ensuing section, I shall suggest the same holds true for the visual arts.

Visual Arts

"Could we ever know each other in the slightest without the arts?" –G. Roy (1909 – 1983) [From the Canadian twenty dollar bill (2009)].

In what follows, I consider the literature related to the visual arts and meaning-making in terms of its overtly neurological and cognitive science aspects, to its epistemological implications, as well as its praxis. My goal is to move from a discussion that is primarily focused by what goes on and into the skull at a semantic level, to what goes on in the skull at a neurobiological level, towards the role of interactions with other skulls when producing artworks.

Epistemology of art. According to Rogers (2001), a physician, educator, and artist, what he does is "very, very selfish" (p. 47). As he explained, taking his artistic work seriously makes it at single-minded, individual quest that requires vast amounts of time spent alone. Gilot (2001), a professional painter, adds however, in a contribution to the same collection, that the time alone to resolve inner conflict still requires that we are part of society.

Writing the introduction for *Cognition and the Arts,* Perkins and Leondard (1977) pointed out that the sort of psychological investigations they undertook were aimed at restructuring many different attitudes towards the arts: whereby we dichotomize the emotional with the cognitive, subjectivity with objectivity, and spontaneity with deliberation; and we think art concerns the first member of the dyads. Conversely, they said, the proponents of the cognitive approach, emphasize that all human activity occurs in relation to a knowledge base (e.g., prior knowledge); our reactions are ways of knowing, that is, functional; and cognition requires both knowing *how* and knowing *that* (discussed previously in "Aesthetics and Concept-Rich Learning").

Remarking on the work of the philosopher of art, Nelson Goodman (1968, 1972), Roupas (1977) noted that all art is symbolic in virtue of bearing information, which has been reiterated by others (Deacon, 2006). But, as Roupas also said, an object is only symbolic when it is interpreted. For Goodman, he went on, we can distinguish the syntactic (i.e., structural) from the semantic (i.e., the detonative) features of art. It is the rules by which we interpret artworks that Roupas calls a representation.

Allowing us to summarize the key idea, Lopes (1996) remarked that, for Goodman, pictures belong to systems which can be described as sets of designs referring to objects, types of objects, and properties. Goodman (1972) remarked that his views resonate with Cassier (1953-1957), the author of *Philosophy of Symbolic Forms.* Cassier, a Jewish neo-Kantian, provided a benchmark of a philosophy of culture. He emphasized that we develop our understanding of the world primarily through symbols, and this impacts our perception of reality. Goodman (1972) emphasized, similarly, that aesthetic experience is a process of "creation and re-creation" (p. 103), for example, when we view artworks. Goodman claimed that in art emotions function cognitively. Although Goodman's discussion of art is framed in terms overly influenced by the language philosophy of his day, with his concern with denotation, he has been very influential in promoting the relativist view.

As Korzenick (1977) explained, in a paper in Perkin's and Leondar's (1977) anthology, representational drawing is not intended as a replica of the world. We attempt to decontextualize content when we produce artworks: we separate ourselves from others; the medium of representation is different from what it represents; and we understand pictures apart from the context in which they occur. Specifically, we separate the medium from the artist's intentions; engage in *distancing,* taking the role of the other; and when we modify a statement or actions from which their intelligibility depends (e.g., drawing a man with wings), we need to account for that (e.g., perhaps we are symbolizing freedom). Commenting on the communicative basis of artworks, Korzenick (1977) wrote that:

> The artist must go where his audience is. His awareness of his community allows him to enter into dialogue with them.... Then he must carefully, with full knowledge of the vagrancies and cares of his audience, lead them where *he* wants them to go. (p. 205)

In explaining how the artist can be successful at communicating, Howard (1977) noted that we have traditionally understood the artistic process along the lines of one of two theories. For the follower of Athena Theory, artworks are the result of the artist's spontaneous imagination, inspiration, creativity and the like. For the proponent of Penelope Theory, the artist must hone their skills and abilities through practice to produce mature works of art. Howard observed that Plato had forged the three-way link between creation-inspiration-unreason, characteristic of Athena Theory. As Howard pointed out, also, to practise is to do so with some criteria of what constitutes mastery.

According to Somerville and Hartley (1986), who summarized a slew of research on art education for *Cognition and Instruction,* children follow certain cognitive rules when drawing. They noted that the goal of art education is allow one to develop their own style, understood in terms of their expression (intended meaning), form or technique, and subject matter. They distinguished between concepts that are *ill-defined* and *well-defined,* that is, where one defining characteristic is shared by all members; *style,* they said, is an ill-defined concept.

They observed that children's earliest drawings are parts, each represented by a line or shape. For example, the well-remarked upon tadpole drawings of humans by children have been understood, they said, as evidence of information-processing deficits or as visual concepts not to be confused with copies. Investigations into the cognitive and motor sources of children's drawings have shown that they evolve and retain a graphic schema for organizing parts of a drawing. Further, each child evolves their own way to merge rule-governed production, for example, strategies for perspective representation that were not devised till the end of the Renaissance and now accepted as realistic, with expressive qualities. They suggested that it is beneficial for learners to encounter typical examples of a concept; be exposed to a wide sampling of the concept; and that new examples should remain stable for a long period of time.

Donald (2001), having extensively written about evolution and consciousness, has extended his views (2006) to account for the phenomenon of art in a recent paper. Donald's (2006) account of art along evolutionary lines can be summarized in seven points. First, art is a type of *cognitive engineering*, because we intend to influence an audience; and this builds upon something we do all the time: reciprocally control the attention of others (e.g., when we communicate).

Second, art requires *distributed cognitions*, the linking of many minds; and artists reflect that network in their works. Third, art is *constructivist*; we strive to integrate perceptual and conceptual content in artworks. Fourth, art is *meta-cognitive*; it engages the artist, and hence the society, in self-reflection, because each individual reflects their social experiences. Fifth, art is technology-driven, which influences what and how it is represented (e.g., we would not have literature without the technology to inscribe, which for us are digital printing presses). Sixth, the technology of art can affect cognition (e.g., allowing us to develop some brain functions more than others). Finally, seventh, art always has a cognitive outcome.

Donald divided the history of art into three rough periods: mimetic, mythic, and theoretical. According to him, because art is a universal phenomenon and species-unique, we naturally inquire into its evolutionary role. He pointed out that what is remarkable about brain evolution in humans is not biological, though we have larger brains than other animals, but what it allows us to do: we are involved in the co-evolution of culture and biology. That is to say, we use culture to enrich, literally, our cognitive abilities. The most important drivers behind the experience of art, he said, are cultural not neurological.

According to Donald, each stage of art development reflects our cultural and cognitive development; and each stage is conservative, that is, including what went before it. At the mimetic stage, we engage in copying, for example, which he claimed lies at the origin of culture. As he put it, the most radical fact of brain evolution is culture itself, and that requires we better understand mimesis.

In a review of Gombrich's (1960/2000) *Art and Illusion: A Study in the Psychology of Pictorial Representations,* Goodman's (1972) wrote of this "important and illuminating book" (p. 84). Lopes (1996) in a philosophical reflection, observed that Gombrich, in that work, attempted to understand how pictures can rely on perception that is universal, and has also cultural components. It is this problematic tussle between the universal and particular aspects of art that occupies us as we turn to consider Gombrich's work.

Gombrich (1960/2000), with works like *Art and Illusion: A Study in the Psychology of Pictorial Representations,* is a well-known art theorist. Gombrich (1960/2000) asked the following question that guided his research: why do artists of different periods represent the world differently. Gombrich (1995), in his most popular work, *The Story of Art,* began by claiming, "There really is no such thing as art. There are only artists" (p. 15). His point in this passage was to emphasize that art depends on the time and place. According to him, art is made for humans by humans. That is, each piece is intended to communicate to a specific audience. For example, the majority of older pictures in the West had religious themes taken from the Bible or story of saints; others were of Greek mythology, for example, of wars between the gods.

As he noted, the ancients saw art as the attempt to master imitation *(memesis).* In modern art, according to Gombrich (1982), there was no longer a real world we were representing; rather, we were making up the world in new ways. However, in Gombrich's (1991) later writings, he seeks to thwart the threat of cultural relativism to aesthetic appreciation (though, admittedly, this is an argument that is not developed).

Nevertheless, Cunliffe (1998), capturing the prevailing view, attempted to read Gombrich as a social constructivist. As she puts it, for the social constructivist, cognition is an active process (i.e., we are not just receptacles of information). According to Cunliffe, for Gombrich, art is the result of a dynamic between schema and correction, trial and error, making and matching, as well as formula and experience. She claimed that art, for Gombrich, requires information processing in the zone of proximal development.

Gombrich's work, in the context of this thesis, is attractive because it focuses us upon the construction of meanings for the subject in terms of mental, physical, and socio-historical descriptions. Thus, we need not solve the philosophical debate about aesthetic relativism to appreciate the role the social context plays in shaping our tastes; and we are best advised to steer clear of it, lest we get bogged down in a philosophical quandary.

Further, the epistemological stance resonates with postmodernists too, who think, in brief, that bodies of knowledge are merely divergent and competing sets of narrative constructions. Hawkings (2002), in "Children's Drawings, Self-expression, Identity, and the Imagination" took the postmodern stand. Different arts are various ways, according to him, to construct meaning.

In any case, in one more postmodern venture, Greene (1997), a professor emeritus of philosophy and education at Teacher's College, Columbia University, sought to make explicit the process of meaning-making. Specifically, her aim was to make us cognizant of the meaning we attribute to history. Relying on Dewey, she claimed that drawing is a way to clarify meanings (because there are no ultimate truths). Relying on Clifford Geertz, an anthropological postmodernist, she noted that any representation is a collective formation. The diversity of representations, Greene held, lets us see the "endless" (1997, p. 344) complexity of the world.

In addition to cognition-in-the-social context, psychologists have also been concerned with interpersonal relationships on an individual level. In a study conducted in Slovene society, Cugmas (2004) hypothesized that works of art by young children expose their attachment styles, which are believed indicative of how much they trust other people. For example, children, he said, may be identified as avoidant or anxious by analyzing their artistic creations.

Kaiser (1996) noted that drawings in psychology were first used by psychoanalysts and now by a variety of therapists. She considered attachment styles as revealed by children's drawings of a bird's nest (BND). She claimed that attachment theory (i.e., basically the idea that early secure relationships lead to healthy emotional and social development) helps us understand both the psychological development of the healthy and the disturbed. For example, the presence of birds in the drawings, she explained, suggests secure attachments; and when nests have bottoms missing, insecure relationships are perceived by the artist.

In sum, our artworks illustrate developmental differences, depending on when they are composed; socio-cultural differences, relating to the technology and preoccupations of the society, as well as the anthropology of human evolution; and our personal preferences shaped by prior experience. As such, art is indeed a window into the self, which naturally leads us to at least consider the biological basis of art in a bit more depth.

The neurobiology of art. I shall discuss several prominent, neurobiological accounts of the visual arts, because they have exerted influence on what cognitive scientists think about the nature of mental processes, and as a prelude to considering some of the practical implications of art.

Stevens (2001), in a contribution to a collection of papers on creativity, has written that "much of art requires abstracting from reality" (p. 177). That is, he said, in a now familiar refrain, in art we do not aim to make literal copies of the world, but focus on certain features. According to him, creativity requires handling formal qualities, like line and colour, yet the brain does not allow us to do so any which way. As Stevens (2001) put it, "[T]he language of the arts reveals the nature of information-processing in the brain, and the arts in turn depend on these brain mechanisms to determine the ways art can abstract the world" (p. 189).

Yet, York (2004) of the Department of Neurology at the Såa Institute in California pointed out that modern neuropsychology "localizes sensory motor functioning, not mental ones" (p. 1). As York noted however, the artistic process is intrinsically mental. So, in the language of cognitive scientists, what we need is some account of the relationship between the physical level and the semantic level, which I turn to discuss next in relation to concept formation.

Art is sometimes considered as attempts to represent the way the mind puts together patches of sense data to represent the world (Zeki, 1998). Zeki (1999) of the

Wellcome Department of Cognitive Neurology, in London, noted that different aspects of visual processing (e.g., the recognition of colour, form, or motion) can be localized.

Zeki (1999) contended in *Inner Visions: An Exploration of Art and the Brain*, that the "aims of art are an extension of the function of the brain" (p. 1). He said that we see in order to gain knowledge of the world, which we do through the process of concept formation. He explained concepts as

> the brain's stored record formed from the many images it has seen, and to select from those images only that which is necessary for it to extract the essential qualities of objects and to discord 'the profusion of details and accidents. (Zeki, 1999, p. 42)

Zeki (1993) pointed out that though the brain is highly specialized, it is difficult to separate integration (i.e., the unity of brain functions) from categorization. As he put it, the processes of "seeing and understanding" (1993, p. 319) are not easily, if at all, distinguishable. The localization literature does allow us to identify what areas of the brain are necessary for certain functions, and yet there is still an entire integrated system of the brain that lies in the background of any one process. Since we do not yet know how specialized memory is, it is at least reasonable to think that a semi-global neural system is necessary to be able to see a house as such.

In a study of cases, Zeki (2004) wrote: "I try to show that we can trace the origins of their [Dante's, Michelangelo's, and Wagner's] art to a fundamental characteristic of the brain, namely its ability to form concepts" (2004, p. 13). Commenting on the ability to create abstract representations of things, he remarked, "Art is basically a by-product of this abstracting, concept-forming, knowledge-acquiring system of the brain and can only be understood biologically in that context" (Zeki, 2004, p. 16).

According to Zeki (1999), cubist works were attempts to resolve the neurological paradox between the reality of perception (the different ways of seeing an object and the single view of appearance of Platonic ideas). He thinks cubists failed. Some of Picasso's cubist works, he said, were intelligible only because of the title. Generally, Zeki (2006) held that we should attempt to understand aesthetic appreciation by studying the brain, which is not just passive but helps construct the world. For example, he attempted to explain ambiguity in art as our encounter with an object where multiple interpretations are possible. In sum, for him, there is continuity between brain functioning and art: it is a way to know the world.

In addition to Zeki (1998), Solso (2000) has written extensively on the neuroscience and cognitive basis of artistic production. He has discussed "meaning" as "filling in" (2000, p. 76) (e.g., we make inferences about what is behind objects, their motion, and the historical context). As he noted, even colours, like red, evoke particular emotions. Solo approached meaning-making in terms of how we render the environment intelligible, by perceiving an object, and our reaction to the entire socio-historical context in which we cognize.

For 20 years, Solso (1994, 2003) has been investigating the general principles of how the eye and brain allow us to perceive and interpret artworks. His hope has been that science will help us understand the experience of art; and art will help us understand the mind. Art, as he held, allows us to see into our minds.

Solso (2003) investigated visual art using three characteristics: the sensory or physical attributes of a piece of art (e.g., lines, colours, contours, shapes); psychological or semantic interpretation (its meaning, e.g., a pastoral scene); and what he dubs "Level 3 comprehension," that is, the way in which an artwork can touch us in a profound way, speaking to its beauty and philosophy.

Following the origins of the hominid brain, he claimed that artistic phenomena emerged with our ability to be able to image internally and represent externally. He pointed out that we represent using both bottom-up and top-down processing methods; that is, we put together artworks neurologically from sensory inputs, but also bring our past experience and knowledge to bear upon what we see to make sense of it. For instance, the eye that sees things only in 2D is interpreted by the brain as 3D. Recalling the work of Donald (2001, 2006), he noted that symbolic expression was first played out as gestures, play acting, and imitation—key features of art.

As Solso puts it, 120,000 years ago was a good year, marking advances in brain functioning, and 60,000 years ago we were able to engage in more complex abstract thinking than before. Slowly, then, with the co-evolution of brain and vascular system (which keeps the brain cool) we became better at being able to represent the external world. Through art we were able to represent attractive, non-present objects, the first overwhelming being vulvas. As we already saw with Gombrich's work, in the West we mastered mimetic art during the Renaissance. During the impressionist phase, we moved, according to Solso (2003) from "form to feeling, from the cortex to the heart" (p. 105). As Solso (2003) reminds us,

> Lest we become overwhelmed by the study of the neurology, synapses, blood flow, and the evolution of the conscious brain, it is essential that we do not forget that art, of all types, is one of life's most noble expressions. (p. 1)

That is to say, the Level 3 comprehension of artworks touches on an understanding that is at once intimately personal and socio-cultural, a virtual communiqué between the artist and audience about an understanding of some facet of our lives.

Head of the Department of Neurology at the University of Iowa and Adjunct Professor at The Salk Institute in La Jolla, California, Damasio (2001) also emphasized that we react emotionally to certain combinations of colours, certain tones and their combinations, certain shapes and their combinations. As he explained, an emotional response has a somatic basis, which can occur in two ways. First, being thrilled by a work of art, there are physiological changes in skin conductivity, heart rate, breathing rhythm, and so on, which he dubs the *body-loop*: certain features of the artwork impact the somatic cortices, pre-frontal cortex, and emotions. Second, having become habituated over time to certain aesthetic objects or emotional situations the pre-frontal cortex can lead the somatic cortices to assume a physical state, the *as-if-body-loop*: that is, the causal chain runs thus: the pre-frontal-cortex, emotions, and somatic cortices. In other words, there is a two-way signalling process between the pre-frontal cortex and the somatic cortices.

A painter by trade, Aiken (1998) was prompted to explore the biological basis for art when asked, "Why do you paint?" Since the production of art is such a persistent feature of human behaviour, and common in our history, she takes an ethological approach. That is, she views art as behaviour.

Art, Aiken contended, draws upon "biologically significant stimuli" (1998, p. 3), soliciting an emotional response through certain configurations of line (e.g., some lines can be calming or frightening), shape, colour and sound. As she pointed out, predator-prey relations are often the result of "fixed action patterns" (p. 6) (e.g., flight at the sight of a predator), which she takes to be the basic unit of behaviour.

Turning to art, emotional responses, according to her, can be the result of general associations (e.g., the American flag), specific personal associations (e.g., a song), universal reactions (e.g., line, shape, colour), or the knowledge that an artwork exhibits great technique or originality. With regard to the meaning or art, she distinguishes between (a) what it is about from (b) what it means, and (c) the emotional meaning, that is, the feeling associated with the message. She claimed that the values that result from viewing works of art are universal. Emotions, she pointed out, can be elicited through symbols, depiction of emotionally charged events, superior technical ability, or a particular combination of line, shape, and colour.

According to her, each unique solution requires the responder to be aware of the problem. Art provides stimuli for unconditioned reflexes, which leads her to ask: how reflexive behaviour can be flexible, and how our emotional response is constrained by evolutionary adaptation. Biology, she said, sets the parameters of our behaviour.

Aiken (1998) calls the relationship between the stimuli and response the "releaser-response package" (p. 49), which, according to her, is the origin of art. That is, the emotional response moving from the general to the particular case. In sum, we move along the following reflex arc—from a stimulus/situation to an appropriate nervous system response, unto adaptive appropriate behaviour. She hypothesized that the stimuli used to evoke emotional responses in art are associated with defence reactions (e.g., flight, freeze, and flee). Of course, however, emotional responses are not just hardwired into the brain; they are also context-dependent, relying in her language, upon the way the reflex arch can be inhibited. In other words, we react differently based on a plethora of contextual features in conjunction with our innate biologically ingrained dispositions. In fact, Livingstone (2002), a neurophysiologist interested in colour and luminance (or lightness), has wondered if better understanding of the biology of art will increase the use of effects and illusions in artworks. That is, Livingstone conjectured that the understanding of the underlying biological aspects of art will impact artistic behaviour.

Encapsulated in the idea of the releaser-response package, is that of the role of emotions, which has been a perennial point of concern for those interested in the study of arts and cognition. Hogan (2003) in a study of the arts from a generically cognitive science point of view, identified several psychological roles that the emotions play in viewing art: *eliciting conditions,* brings on an emotion; *phenomenological tone,* is the actual experience of the emotion; *expressive output,* is the reaction that the emotion prompts; and *attentional focus,* is the way in which emotions focus us on certain features of our environment.

Recently, Zaidel (2005) has discussed the human beginnings and biological origins of works of art, as well as the effects of brain damage on established artists. However, he also noted, in addition to the neuropsychological accounts, that art is made with a social anchor that communicates ideas, concepts, meanings, and emotions.

He claimed that there is a difference between meaning and aesthetics of works of art, because we can appreciate them even when we do not know what their intended purpose was. Perhaps, he conjectured, beauty was an "emergent property" (p. 54).

At any rate, to consider the neuropsychological components of art, according to Zaidel, we must investigate: (a) the sensory reality of the artist, (b) genetics and talent, (c) brain damage and alternations in health, (d) the cultural, educational, and intellectual atmosphere, (e) early hominid evolution, and (f) the biological roots of art.

According to him, the first art was human decoration, which perhaps spoke to our group membership or rank. We have not, he noted, however, been able to develop neuropsychological tests for artistic ability because there are no equivalences for *words* and *grammar* in art. In addition, the significance of works of art depends upon the context in which they are produced. Further, according to him, the language-art evolutionary relationship is far from clear.

The biological approach to the arts, in the cognitive science jargon, is concentrated on the way the brain encodes and processes information—the brain, as such, is the hardware. However, the common refrain in the context of this thesis, and the aspect I have focused upon, is the social context, providing the software, to follow the analogy, in which the semantic level comes to fruition: how art reflects our world. According to Damasio (2001), emotions are central to the commerce between us and the world. There is likely a basic releaser-response package, as Aiken (1998) suggested, at the basis of artistic behaviour. But how the package is configured will also be influenced, I emphasize, by the social context.

What we have been able to witness thus far from the discussion of the neurobiology of art is the to-and-fro between the brain, behaviour, and cognition. The following picture emerges: brain evolution in conjunction with general physiological evolution allows artistic representations; artistic communication, over time, in turn can influence the structure of the brain in an individual artist, and perhaps even at a group level, if certain qualities were so selected.

As we communicate with each other, through works of art, we experience cognitive change in the short term; and social changes in the longer-term, as well as perhaps—I go out on a limb here—even biological change down the road insofar as communication and understanding become a preeminent social value that would likely leave its genetic traces. But this sort of mind-evolution will depend a great deal on changes in the socio-cultural context, which, among other things, I turn to discuss next.

The praxis of art. The cognitive science of artworks, their meaning-making capacity, also includes the political or personal changes they can inspire.

Art as social transformation. Nicol, Moore, Zappa, Yusyp, and Sasges (2004) reflected upon art as creating figurative worlds where we can come together to "construct meanings" (p. 312). Identity, they claimed, is formed through social relations where we "attempt to make meanings" (Nicol et al., 2004, p. 312).

Considering the ethical realm, Garoian (1998), a former high school art teacher, explored his field as a vehicle to increase our appreciation of nature. He analyzed how art reveals our anthropological relationship to the earth. In the ancient pictures of bison on caves in Salon Noir in Naiux, he noted, there are no boundaries depicted because they were nomadic peoples. He catalogued five metaphors in art works: pictorial space is circumscribed (agrarian societies), perspective (when we survey land), sublime (when we valorize land), mapping (to simulate land on paper), and the machine (to construct a surrogate land).

Art as personal transformation. Glanville (1996), whose work bridges architecture and cybernetics, argued that meaning transfers are expressible through representations that do not involve the utterances of communication. He contended, "We give them [art works] meaning" (1996, p. 458). In stressing the private aspect of meanings, however, we are led to wonder if art is just a rhetoric aimed at convincing, cajoling, or amusing others? Let us turn to some examples of how art can be a matter of personal transformation.

Mullen, Buttignol, and Patrick (2005) provided the retelling of the story of Kal, a suicidal young boy. They suggested that using art works in therapy can increase empathy. They wrote,

> We propose that when artistic forms are exploited, meaning can be constructed and empathy enlisted that might otherwise prove illusive. (Mullen et al., 2005, p. 11)

For Mullen et al., retelling is not just a case study, but intended to invite "multiple realizations" (2005, p. 11). According to Mullen and associates (2005), who are concerned with affecting positive changes in individuals, art therapy is premised on thinking that mobilizing creativity can: (a) help construct a new outlook; (b) allow an expression that compliments language; and (c) aids us to drop our defences. They wrote,

> Most important of all, art-making, is the sense of fabricating something that bears a personal mark and can be seen by others, restores the refuges' self-respect. (2005, p. 440)

Also concerned about the meaning of our lives and art, Wertheim-Cahen, Van Dijk, Schouten, Roozen, and Drožđek (2004) contributed to a collection of papers *Broken Spirits: The Treatment of Traumatized Asylum seekers, Refugees, War and Torture Victim.* Psychological trauma, they said, is a destruction of meaning and fragmentation of identity. They found art useful to reduce stress, rebuild trust, and lessen isolation. Wertheim-Cahen et al. (2004) claimed that art allows a situation of trust, as we are vulnerable when we create art works, and the product increases self-esteem. Mathiasen and Alpert (1993), contributing to a volume, *Empathy and the Practice of Medicine: Beyond Pills and the Scalpel,* argued that the creation of art works can increase our humility by reminding us that pain and suffering are part of all our lives.

In studying how best to foster artistic works, Merry, Wei, and Roger (2006) in "What's Got Two Heads and No Nose? Young British and Chinese Children's Representations of Unreality" conducted a cross-cultural study in which they queried how Chinese children would cope with drawing non-existent people (e.g., people with several heads). Merry et al. (2006) found that a traditional nursery was more "uncertain and reluctant" (p. 9) to draw people that do not exist.

Educationalists, however, have often focused on individual, academic achievement, not psychological development as such. Smagornsky (1997), in "Artistic Composing as a Representational Process," considered how students' attribution of meaning to literary figures reveals their self-perceptions. He wrote:

> [B]roadening the means through which students represent their understandings symbolically can lead to greater competence in understanding academic materials, albeit through untraditional academic ways. (Smagorinsky, 1997, p. 104)

Within a constructivist pedagogical context, Petit (2002) claimed that art education provides the paradigm for a useful heuristic. According to Petit, constructivist environments: (a) allow multiple representations of reality; (b) knowledge construction is valued over its reproduction; (c) learning tasks take place in meaningful contexts; (d) there is a predetermined sequence of instructions (with little stress, the use of real world or case-based scenarios); (e) thoughtful reflection on experience is encouraged; and (f) like the classroom environment, support the construction of knowledge through social negotiation (not competition).

Kelehear and Heid (2002) have examined the theory of socio-cultural learning in mentoring applied to a multi-age classroom. Their socio-cultural theory is an amalgamation of many others. For example, looking to Bandura, they noted that children achieve more when they have a better self-perception regarding the task. Building confidence likely lets us think that we can accomplish things in a specific domain in the future. As they also said, according to Vygotsky, children perform at more advanced levels through social interaction.

They noted that Noddings (1984) introduced care as an important ingredient in the learning process. That is, if the teacher cares for the students that will increase their motivation, likely because it gives them new goals, often desired by the instructor. A few words are in order about what Noddings is on about and why her views are so groundbreaking for educationalists.

Noddings (1984) positions herself against traditional approaches to ethics. Whereas we have tended to concentrate on ethical reasoning to deduce correct actions, she noted that these approaches “move discussion beyond the sphere of actual human activity and the feeling that pervades such activity” (p. 1). Instead of focusing on principles, propositions, justification, fairness and justice, she takes as her point of departure human caring and the memory of caring as the foundation of the ethical response. She wrote,

> How good *I* can be is partly a function of how *you*—the other—receives and responds to me. Whatever virtue I exercise is completed and fulfilled in you. The primary aim of education must be nurturance of the ethical ideal. (p. 6)

She said that the ethics of care finds its foundation in relationships. And in relationships, according to her, we can find joy. She characterized her view as tending to concretization, not abstraction, because of the attention to the specifics of the relationship in question.

Noddings distinguished *aesthetic caring* that concerns things and ideas from *ethical caring,* which we have already discussed. One of her goals was to offer a phenomenology of caring. To this end, for example, she described the one-caring as acting with special regard to a particular person in a concrete situation. She said that caring involves “feeling with others” (p. 30) and increases vulnerability on the part of the one-caring. She also believes that attitude is important to caring. Caring, she further claimed, is something we can learn. For instance, she noted that women are not socialized into the role of mother, but a relationship. That is to say, we learn how to interact through appropriate experiences doing so. Sentiment, she said, is central to ethics and we cannot get above it by reasoning. The ethics of care, as we can anticipate and as Noddings (2002) claimed, cross-cuts both the private and public domains.

In my account of Noddings' thinking, I have paid short shrift to the way in which she links her views to women (e.g., that an ethics of care is feminine) and the emphasis on reciprocity (e.g., her scepticism with animal rights advocates' ethics). My purpose is to take her emphasis on caring, and relationships, to add some context to Kelehear's and Heid's (2002) allusion to her work and not to delve into the host of other issues she attempts to address based on her theory.

Mentorship, Kelehear and Heid (2002) concluded can increase self-perception; and collaboration often leads to care for others and tolerance. And care leads us to act on our concerns. Art learning, they said in their study, was not so much individual learning as interpersonal development.

Whether social or personal transformation is at issue, artworks have been more than a stimulus in the environment: They have been the vehicle by which the artist can impact an audience in a specific way, providing experiences of beauty, which are acts of meaning-making themselves.

The ways and purposes by which we generate meaning in the visual arts reflects the cornerstones of cognitive processes: replicating what we see around us, expressing the spirit (*Giest*) of our times, assembling sense data, piecing together our past experiences, communicating, and changing within a changing world (*Welt*). I have used a few German words here in parentheses because they capture, more than their English equivalents, the social nature of human activities. And it is the German tradition of philosophy, as we saw when we witnessed its impact on Vygotsky for instance, more than any other, which has exerted an influence on our turn towards the social within academia.

Looking back to the cognitive science model for the mind, we can see how the discussion of art has occurred at the biological, symbolic, and semantic levels. What we have seen, then, is that attempts to understand artworks have run the entire social cognitive gambit: they have dealt with, in the order that I roughly discussed them, the personal factors, including the biological aspects, and the social factors.

Further, however, as for instance Donald (2001, 2006) and Aikens (1998) pointed out, taken together, all three aspects work together; and they are not easily teased apart. Biology sets the parameters of what and how we perceive works of art—providing the cognitive tools that prompt us to create them in the first place. Culture influences, over the course of evolutionary time can leave their evolutionary traces. And our own unique experiences, within the social context, influence how we compose and react to works of art. Since I have focused upon the semantic and social level of description, I shall naturally concentrate on that in connecting algebra to the visual arts.

Some Resources for Connecting Algebra and Art with a Common Link

> "The aesthetics of natural science and mathematics is at one with the aesthetics of music and painting—both inhere in the discovery of a partially concealed pattern" (Simon, 1969/1996, p. 3).
>
> "Human cognition is always situated in a complex socio-cultural world and cannot be unaffected by it" (Hutchins, 1995, p. xiii).

The cognitive approach to learning, according to Perkins (1977), takes us beyond the arts, prompting us to consider what is held in common in how we learn across domains. Considering what is common in how we learn in algebra and the visual arts has been discussed by a historically-minded social critic (Mumford, 1926, 1934, 1952, 1963, 1966, 1986), a mathematics educator (Holt, 1971), a philosophical historian of mathematics (Gray, 2008), and cognitive psychologists (Perkins & Leondard, 1977) in terms of the role of imagination (Smolucha & Smolucha, 1986).

Even for thinkers as diverse as Tolstoy (1899), Fry (1947, 1956), Holt (1971), Arnheim (1965, 1969, 1986), and Csikszentmihalyi and Robinson (1990) there was agreement that art and science share something, but they would not agree on what. For Tolstoy, recall, it was the participation of science (including mathematics) and the arts, in their own ways, in goodness; for Fry it was the emotional response of unity that they shared. However, for Arnheim (1965, 1969, 1986), psychologically art and science both involved thinking and perception; because those two processes were, he claimed, inseparable. And for Csikszentmihalyi and Robinson (1990), we could say both realms of science and the arts allow us an aesthetic encounter along the lines of the flow experience described by Csikszentmihalyi (1990).

So for any of these several thinkers, though for different reasons, art and science were not to be separated by an unbridgeable divide. On the contrary. Art and science could be united by their ethical consequences, emotional components, or the psychological processes they depended upon.

In fact, it is interesting to note that Dewey (1934/1958, 1938/1997) thought that art is a type of communication, like Tolstoy, and emphasized the social context in which they are developed, which offered yet another avenue to connect art and science (Gray, 2008, Holt, 1971, Mumford 1986). In any case, the idea of an aesthetic connection between mathematics and the arts resonated with Sinclair (2006) and Hardy (1967). But we have little more of the connection between algebra and the visual arts, and then only in a cursory fashion, prompting us to inquire further.

Holt (1971), in *Mathematics in Art,* sought to unearth at least "tenuous" (p. 7), interconnections between mathematics and the arts. As he noted, we are apt to think that mathematics deals with eternal truth and art is subjective. However, he pointed out that even the truths of mathematics, like parallel lines never meet, has been overturned. That is, in this case, we found that parallel lines never meet in Euclidian space, but in some curved spaces they do. The Greek mathematicians' penchant for exactitude, according to him, has been overturned by a more uncertain world, which impacted our arts in various ways.

At the symposium *Higher Brain Function, Art and Science: An Interdisciplinary Examination of the Creative Process* in honour of the 20th Anniversary of the Given Biomedical Institute in Aspen, Colorado, Stent (2001) contributed a paper "Meaning in Art and Science." He noted, in the paper, that Snow (1959) had spoken of "two cultures [one of art and the other of science]" (cited in Stent, 2001, p. 31). But Stent argued against the antinomy of creativity in art and science (in which I include algebra), claiming this partition has no historical or philosophical merit. Rather, according to him, the divide between art and the sciences rests largely upon stereotypes of each field.

As Stent claimed, both art and science deal with the communication of truth. However, according to him, art occupies with an inner, subjective world; and science an, objective, outer world.

He claimed that we confuse the *works* in the two domains with the *content.* Basically, the idea is that works of art are categorically different from those of science—for example, a scientific paper—but what is referred to in them (e.g., the double-helix structure of DNA) is similar: both domains allow us to speak about the nature of reality in different ways. According to Stent (2001), works of art and the content of science are essentially different in some ways, and also "essentially the same" (p. 35) in other ways. For example, even art must harmonize with the creators' comprehension of reality and requires assent from an audience. The difference, according to Stent, is that science relies upon language to communicate its truths; and art relies on linguistic, tonal, and visual structures. Regardless of what we think of the specifics of Stent's claims, which go beyond our concerns here, we are prompted to ask: whether there are similarities in how we learn in the disparate domains of science and art.

Pfenninger and Shubik (2001), attempting to synthesize the papers collected together in *The Origins of Creativity,* noted that there are important points of contact between the arts and sciences. Scientists, they said, react emotionally to original ideas. According to them, creativity, regardless of the domain in which it is exercised, must be able to generate in one's brain (the association cortex) novel contexts and representations that elicit associations with symbols and principles of order. The ability to make connections between disparate ideas is innate in the brain or part of acquired dispositional representations that form one's culture or society. Also, we must be able to translate these representations into art and science; all of which, according to them, form part of the process of evolution (Donald, 2001, 2006).

Since part of the motivation for this thesis is that scant work has been carried out to explore the common link between algebra and the visual arts from a social cognitive point of view, it should come as little surprise, therefore, that there is no substantial literature on this topic. What I can offer in lieu of a literature review of the generically psychological connections between algebra and the visual arts is some thematic connections, considered next, and which culminate in an analysis-cum-rationale for this study.

Before making connections between the two fields that concern us, I turn to take stock by summarizing some key points from the literature review thus far, beginning with algebra, followed by the visual arts.

Recapitulation: Algebra. An embodied approach to mathematics is consistent with the aspirations of educators that have emphasized allowing students to reinvent its structures by engaging in hands-on problems in conjunction with others (Ben-Hur, 2006).

According to Tolstoy (1899), Fry (1947, 1956), Arnheim (1965, 1969, 1986), Hardy (1967), Csikszentmihalyi and Robinson (1990), and Sinclair (2006), at the very least, similar psychological processes, like pattern- or beauty-recognition, aspects of flow experiences, would be functioning in both the production of algebra and the production of works of visual art. Taking a recent example, Sinclair (2006) emphasized the aesthetic aspects of learning mathematics, recognizing patterns, cycles of repetition, and what fits, as a motivational factor to pursue the study of mathematics.

The mind, it has been argued, is not only embodied (Frie, 2007, 2008b; Johnson, 1986), but extended, a thesis emphasized by adherents of the situated movement and others (Frie, 2008a; Frie & Coburn, in press; Martin & Sugarman, 1997a, 1997b, 1999b, 2001a, 2001b, 2001c, 2003; Martin, Prupas, & Sugarman, 1998; Sugarman, 2008; Sugarman & Martin, in press; Sugarman, Martin, & Hickenbottom, 2009; Sugarman, Martin, & Thompson, 2003; Smith, 1999).

The situated approach to learning has impacted mathematics education in a positive way. Hatfield, Edwards, and Bitter (1997) have argued that we must take into account the cultural context to effectively teach mathematics. And educators working in an aboriginal context have done so, too. For example, Cajete (1994, 2000) has demonstrated the importance of integrating modern science into a Native-American context, both to protect the culture and to help it adapt to the modern world. So we have reason to think that the transformations occurring at the theoretical level are having important practical consequences that further validate the change in our understanding of cognition, from being an individual activity to one often embedded in a social enterprise.

In this vein, Bishop (1988) had emphasized the importance of mathematical enculturation. That is, according to him, we need to move away from a preoccupation with an individual's technique and towards meaning-making, which focuses us upon the role of interpersonal relations and the entire socio-cultural context.

Let us return to consider the process of generalizing, which also must be situated, and that is often dwelt upon in the literature. Algebra involves generalizing, which is part and parcel of what the mind does through conceptualization, and has been explored by Piaget (1956, 1959, 1964, 1970, 1985) at length. Suffice it to say that the picture that emerges is that we begin by actions on objects (e.g., counting pebbles backwards and forwards), which then take on a life of their own through the manipulation of symbols.

From the more contemporary, cognitive science point of view, Kaput (2008) also noted, we use symbols in algebra to generalize. And as Smith and Thompson (2008) noted, algebra arises from abstraction, but is also applied to specific cases, providing a basis for the recognition of a dialectical relationship between the concrete and abstract.

In fact, Lakoff and Núñez (1997) proposed that this new embodied and contextualized way of understanding the foundations of knowledge gives rise to a new philosophy of mathematics based in cognitive science. They build upon Johnson's (1986) work, in which he explored the bodily basis of generalizing behaviour, and which has, we may wish to note, been extended into a plethora of fields, like psychoanalysis (Frie, 2007, 2008b).

Fuson's (1988) account of meaning-making in mathematics can be discussed in terms of the role of memory, the integration of arithmetical habits of thought and action, as well as their combination at higher levels of abstraction. My social cognitive orientation is consistent with the notion that we—in fact, construct mathematical meanings the way Resnick and colleagues (1981) envision, moving from concrete cases to abstract principles, which are combined to produce new ones, like laws; as well as backwards, when we apply the rules (see "Early Math Strategy," 2008; "Leading Math Success," 2004).

Baroody's (1987) emphasized the cognitive aspects of mathematical learning dovetails with the emphasis in this proposed thesis that we must take into account what the learner purports to be doing. And the message we may take from Samara's and Clements's (2002, 2004) research is that working in the context in which we wish to, is beneficial to learners.

However, Ernest (1999) noted that the problem with constructivist views is that they emphasized subjectivity, not the socio-cultural context that is part of the learning process. But all mathematics and visual arts occur in a socio-cultural context, which I have discussed at length in this chapter, in "The Historical Background of the Situated Movement."

The historical basis for the situated movement impacts our understanding of learning in mathematics (Lerman, 2000; Resnick, 1993). Historically, the genesis of this movement is varied and diverse, which we briefly recount here. We have already seen that the general concern with situated cognition is linked to behaviourism (Ross & Nisbett, 1991), or the pragmatism of Dewey (1938/1997) (Sinclair, 2006; Wertsch, del Río, & Alvarez, 1995).

Sometimes the situated movement has been juxtaposed to Piaget's thought (Lubin & Forbes, 1984; Walkerdine, 1997; White & Siegel), while some other authors have taken Piaget's thought as a point of departure (Bandura, 1997; Bauersfeld, 1995; Glick, 1981; Karplus, 1981; Sigel, 1981; Youniss, 1981; Zimmerman, 1981). The situated movement's genesis has also been traced to Soviet psychology influenced by Marx and developed by Vygotsky (1978) (Lave & Wenger, 1991;Wertsch, 1993; Wertsch, del Río, & Alvarez, 1995); and as a means to extend Vygotsky's thought through emphasizing history by Luria (1976) (Alvarez & del Río, 1995; Cole, 1993; Engerström & Cole, 1997).

More contemporarily, the situated movement has been connected to the artificial intelligence concerns of Simon (1969/1996) (Pea, 1993), recent interest in ecological psychology (Neisser, 1993, 1999; Reed, 1996a, 1996b), or clinical psychiatry (Frie, 2007, 2008a; Martin & Sugarman, 1997a, 1999b, 2001a, 2001b, 2001c, 2003; Martin, Prupas, & Sugarman, 1998; Sugarman & Martin, in press). Further, there are accounts of the genesis of the situated cognition movement available that are generically historical (Clancey, 2009), and specifically philosophical (Gallagher, 2009), too.

Attempts have been made to define *situated cognition* (Robbins & Aydede, 2009; Damon, 1993; Krishner & Whitson, 1997; Resnick, 1993, Smith, 1993; Wertsch, 1993, 1995). The difficulties lie in specifying what we mean by *the situation* in which cognition occurs. The *situation* may be conceived as small as the context where a specific task takes place (e.g., Jane Doe's classroom); and as large as the entire socio-cultural context in which the task is carried out. In addition, some researchers have indicated the limits in defining *situated cognition* (Engeström & Cole, 1997; St. Julien, 1997). For a social cognitive theorist, the situation is writ small and large because both are distributed through the social context, and constructed through our collective, hence interpersonal, relations to the world. In a nutshell, the idea is that cognition is shaped as part of the social practices in which we engage.

The importance of the context has been discussed by such contemporary scholars as Lave (1997). Lave's research (1988, 1993a, 1993b, 1997) provided empirical support for the contextual and cultural basis for mathematics called for by mathematics educators. For example, Lave's and Wenger's (1991) notion of legitimate peripheral participation provides a theoretical vehicle to discuss the role of the social context in terms of membership in communities of practice. Cobb (1994), in fact, emphasized bringing together the cognitive-Piagetian and sociological-Vygotskian approaches to learning, yielding an approach to mathematics that is situated.

Contrary to GOFAI, there have been attempts to develop a distributed view of cognition that is stretched across the person; environment, both physical and social; and situations. For example, Pea (1993) argued that we need to think of intelligence as something accomplished not possessed—as an act of participation. One way in which we utilize the environment, according to Pea, is off-loading, storing information we need to perform tasks in our environment.

Over all, we have been urged to a take a more synthetic view of cognition: Perkins (1981) for his part looked at knowledge as design, emphasizing its *purpose*: we want to know so-and-so for a purpose in a social environment.

For instance, exploring the mechanism of creativity, we may wish to recall that Getzels and Csikszentmihalyi (1976) suggested that creative minds excel at being able to find a problem that is original. In addition, Wagman's (2002) distinction between synthetic and analytic approaches to problem-solving allowed us to add detail to Getzels' and Csikszentmihalyi's (1976) account of problem-finding, which we need not rehearse here. But what is original, and important for us to take away from this recapitulation is this: problem-finding/problem-solving (or challenge-identifying/challenge-overcoming) behaviour will depend, it is reasonable to think, both on the content and social context: for example, what is original today may not be so tomorrow, except in retrospect.

Perkins (1993), building upon the work of previous researchers then, criticized the person-solo view of knowledge, arguing that the person-plus situation (e.g., including the physical environment or social environment) has emergent characteristics that change an information-processing system (Newell & Simon, 1972; Turkle 2005). Perkins (1993) suggested, as well as others (Donald, 2001, 2006; Martin & Sugarman, 1998, 1999a, 2002; Sugarman, 2008), in fact, that we view the person as a series of involvements, interactions, and participations—with the surround that, in one way or another, includes other people and their artefacts. Perkins' work, and those whose research he draws upon, puts us in good stead to appreciate Lave's (1997) work encapsulated in the title of one of her co-authored books *Situated Learning: Legitimate Peripheral Participation* (Lave & Wenger,1997).

There are, of course, differences among all the authors cited in this section as sources for the idea of situated cognition. Suffice it to say that those interested in situated cognition today emphasize more the interaction between the individual and the social context, and are less positivistic in their thinking than their predecessors (Salomon, 1993). The main point to take away, however, is that learning in algebra is contextual: it serves certain functions, in a certain type of society, with members aspiring to certain roles therein; and all of this influences how we teach it or if we do so at all.

We can understand learning in algebra as an extension of the natural process of conceptual generalizing and the cognitive science attempt to explain this process. Yet, what has come to the fore for cognitive scientists is that the nature of generalizing, what concepts we construct and to what purposes, is shaped by the social context in which it occurs. So the behaviours we adopt, and our intentions for acquiring them, will bear not only a biological and cognitive stamp, but a social one.

Recapitulation: The visual arts. Even though Rogers (2001) perceptively observed, in an autobiographical piece, that his work was self-centered, Gilot (2001), also an artist, remarked that the works are produced in a social context. In other words, how we perceive our motivations need not be the entire account.

In fact, it has often been held that because of the ubiquity of art in our anthropological history, it is a gateway to understanding the mind (Donald, 2001, 2006; Hogan, 2003; Zaidel, 2005). Whether we consider the first works of art human decoration (Zaidel, 2005), vulvas (Solso, 2003), or architecture (Hegel, 1842-3/2001) or architectural decorations (Garoian, 1998), what is common is the social role it serves. We depict things important to us, and art is first inextricably linked to a social utility: group membership, fertility, or animals.

Korzenick (1977) emphasized a salient feature of artworks that allows us to set the stage for our discussion, the role of *distancing,* or being able to take the role of the other in producing artworks that allow successful communication. Goodman (1972) had exploited the conventional and a communicative aspect of art to emphasize that it involves us in culturally relative symbol systems.

Leading us to perhaps a similar conclusion as Goodman (1972), Howard (1977) pointed out that we have used two types of theories to explain the artistic process: Athena Theory, focusing us on spontaneity; and Penelope Theory, focusing us on practice. Regardless of how we reconcile these two broad theoretical approaches, what constitutes mastery is the site where the negotiation of meaning between artist and audience occurs. And the way we acquire mastery is in a social context. For example, Somerville and Hartley (1986) emphasize the importance of vicarious learning in acquiring artistic skills, specifically of being exposed to models of certain styles of artworks.

In his writings, and taking an evolutionary perspective, Donald (2001, 2006) has emphasized the role of culture in brain ontogeny and sought to extend his views to the arts. According to him, the arts reflect our cultural and cognitive development as a species; but lying at the basis of both art and culture is mimesis. Art, then, is for Donald, through-and-through, cultural.

We are caught between two poles: visual art also involves generalizing concepts drawn from the hurly-burly of our experiences, as well as making connections between them. According to Gombrich (e.g., 1960/2000) art is a form of communication that is time-and-place specific, although he would go on to attempt to avoid the relativist sorts of conclusions of Goodman (1972) that fall out of emphasizing the role of the social context. In any case, according to Cunliffe (1994), for Gombrich (1968), art is the result of a dynamic between prior knowledge and articulating new meanings, and is, overall, consistent with a postmodern epistemology.

The interplay between universal cognitive processes and culture is seen clearly when we examine the neurobiological literature on the arts. For example, abstraction, according to Zeki (2004), frees the brain (but not totally) from dependence on the memory system. We do not have to recall each specific house-like object we ever saw once we can represent the concept "house" to ourselves. In art, we form abstract types of events (e.g., Picasso's surrealist *Minotauromachy* (1935)), things (e.g., *Garçon à la Pipe* (1905)), and of course Picasso's analytical cubism where objects were reduced to their bare geometric forms (e.g., *L'Accordéoniste*, (1911)), which it is reasonable to think can affect our conceptual apparatus. Without the appropriate representational apparatus, however, we would not be able to identify or appreciate these artworks at all.

It is unlikely, however, that Picasso's intent was to attempt to depict visual processing, so he did not fail as Zeki claimed. Yet Zeki is on to something. Instead of going back to how the brain processes information, Picasso requires his audience to have a robust background of information to appreciate a work. Picasso's work is abstract in two senses, relying, first, upon a representation as well as, second, our ability to fill in meaning with our entire repertoire of socio-cultural understandings.

Similarly, Solso (2000, 2003), we may wish to recall, had investigated in some depth the relationship between the eye and the brain; for instance, how our brain transforms 2D images into 3D ones. From this research, Solso was led to also pay heed to what he dubbed "Level 3" meaning, or the semantic level.

Offering an account that bridges the biological-psychological divide, Aikens (1998) proposed the releaser-response package as the way in which we learn to react to stimuli; this package, we may wish to recall, includes both innate, biological, factors and social influences. Her account, in fact, tallies nicely with social cognitive theory because it draws upon the language of behaviourism and cognitive theorists: we hear of stimulus and response; the visceral emotions (Damasio, 2001; Goleman, 1995; Hogan, 2003; Perkins and Leonard, 1977; Pfenninger & Shubik, 2001) that make the package powerful; and the social context where they are shaped.

For instance, Hawking (2002) thought that we should view children's drawings as exposing their idea of power and meaning (e.g., their depiction of themselves in relation to others via the use of size and space). Understanding the meaning-making process, Greene (2007) also said, helps us understand our identities, agency, and mode of praxis; although we can question if art opens up endless possibilities for reinventing ourselves politically.

Reflecting a psychological approach, which typically occurs at the individual level, Cugmas (2004) found support for the hypothesis that children's drawings are a potential robust measure for insight into their social behaviour and represent various models of attachment to the teacher. Kaiser (1996) concluded that interpreting BND's (Bird Nest Drawings), can help us understand relationship and intimacy issues, promote recognition and insight, as well as assist in healing disturbed attachments.

Looking at things at a neurological level, art is sometimes considered as attempts to represent the way the mind assembles an intelligible world (Zeki, 1998). Solo (2000, 2003) approached meaning-making in terms of how we render the environment intelligible, both through the evolution of brain functioning, the eye, and attendant cultural advances related to the production of works of art, which has been explored by other researchers (Aiken, 1998; Goguen, 1999; Greene, 2005; Hogan, 2003; Livingstone, 2002; Lopes, 1997; Perkins, 1977; Salomon, 2003; Zaidel, 2005).

Returning to the everyday world we experience, art's ability to allow us to make meaning has been thought to hold consequences for how we act. Nicol, Moore, Zappa, Yusyp, and Sasges (2004) claimed that art, its making and appreciating, helps us form and solidify social relations. In fact, Garoian (1998) suggested that works of art can affect praxis, giving rise to an ecological consciousness about our links to the natural world.

For Mullen and associates (2005), who we may wish to recall considered the suicidal young boy Kal, art is a means of meaning-making, self-construction, as well as healing. In fact, for Wertheim-Cahen et al. (2004), art was thought to increase the self-esteem of traumatized refugee victims. According to Mathiasen and Alpert (1993), art can increase empathy, helping us become better people. According to Merry et al. (2006), progressive schools produced students that were less threatened with drawing non-existent people, offered more creative solutions, and further, this effect was cross-cultural.

Art has also been thought to be a heuristic to increase academic achievement. Smagornsky (1997) was interested in the ability of students to understand that events can be transformed from one symbol system to another. He thought that through artistic activity, reflection is heightened, which he claimed has positive benefits for understanding academic material.

According to Petit (2002), artistic problems can result in individual images and knowledge construction; conceptual learning; the use of social settings, like classroom critiques (e.g., thinking critically about the activities of peers); a record of human experiences; as well as facilitating us to emphasize collaborative learning. He concluded that the production of art works can assist in achieving academic goals.

Also, according to Kelehear and Heid (2002), art can shape our perceptions because it allows us to build relationships with others. Referring to the work of Noddings (1984), for example, they emphasize the interpersonal role of artworks as communication, and indeed, as acts of caring, perhaps heightening our sense of empathy and tolerance for diversity.

So art is far from a person-solo activity as we have often been apt to think; on the contrary, it relies upon a store of socially accumulated knowledge, and social relations, even impacting the way we perceive and engage in these relations. We change the practices we participate in; our practices in turn change us, our perceptions and behaviours, as we engage in them. This dictum, of the reciprocity between ourselves and the practices we engage in, reflects, once again, the social cognitive imperative that all three dimension—the cognitive, social, and behavioural—are intertwined. Having recapitulated both the salient points from the literature on algebra and the visual arts, we are now in a position to make connections between them.

Comparative Analysis: Meaning-Making in Algebra and the Visual Arts

> "Science and art as closely bound together as the heart and lungs, so that if one organ is vitiated the other cannot act rightly" (Tolstoy, 1899, p. 201).

I shall make connections between algebra and the visual arts along three thematic lines: generalizing, specifying, and aesthetics; the socio-cultural context; and the interaction between the individual and the social context—all of which gives rise to sub-themes outlined at the end of this chapter, in "Rationale."

Generalizing, specifying, and aesthetics. In our philosophical tradition, we are already apt to consider artistic practices as embodied and socially embedded. However, we have reason to believe that many of the psychological processes must be the same; for example, concepts are used in both algebra and the visual arts to make sense of a plethora of instances. In algebra, generalizations allow us to generate meaning from instances of arithmetic operations. How and what we generalize about, however, I emphasize, is determined in part by the society we live in, which in turn has already been influenced by interactions with the environment, both physical and social.

In the visual arts, depictions offer generalizations over human experiences, like the suffering in time of war; also we use metaphor, like an ocean to depict ideas of infinity, freedom, or isolation. In addition, art allows us to imagine the world in different ways with these metaphors. Admittedly, the symbols used in algebra are standardized and formal (e.g., a conventional notational system), whereas those of the visual arts are open to, and often demand, more interpretation and have a greater connotative dimension, too. Nevertheless, we can still ferret out a common cognitive component—like generalizing to make meaning—at play in both algebra and the visual arts.

And generalizing and specifying behaviour would sometimes be guided by aesthetics, playing a motivational role in algebraic and visual arts behaviours. We seek meaningful experiences that provide a sense of focus, exhilaration, integration, and self-acceptance. As we may wish to recall, the aesthetic experience has been likened, by Csikszentmihalyi and Robinson (1990), to being in the flow which could be experienced by practitioners in each field.

Further, suffice it to say, the cognitive science of algebra and the visual arts can be seen as the natural process of abstracting and generalizing to make meaning. A cognitive science account for the generalizing activity will likely require some overlap of cognitive processes in terms of the use of memory, retrieval, and processing of symbols. Also, in algebra and the visual arts we generate generalizations, applying them to our specific situation, hence making meaning in specific contexts.

That is to say, generalization does not form a one-way street, but part of an overall process that works in two directions: we induce, moving to principles of great generality; and we deduce, applying the said principles to specific cases. In psychology, we may wish to recall, it is commonplace to speak of bottom-up and top-down processing, which also captures generalizing and specifying/applying behaviours. Moreover we see both processes of generalizing and applying at work in both algebra and the visual arts. The reason the rules of algebra have value is that they allow applications in specific mathematical instances; the reason we learn how to manipulate perspective in a painting is so we can do so in certain creative circumstances.

In fact, what counts as creativity and originality in both algebra and the visual arts will likely involve the ability to excel at what Getzels and Csizszentmihayi (1976) called problem-finding (and I have called challenge-identifying) or our ability of detect what is novel, to make the connection between the discovery and audience—to communicate, in other words. What counts as an original problem requires abstracting from instances of what has gone before, hence building upon the past, and applying our ideas to a specific situation, perhaps a novel one.

And here is where we return to our overriding theme: the circumstances are shaped by a confluence of factors, biological, psychological, and social, already familiar to the social cognitive theorist. Art and mathematics are both universal and species-unique behaviours, though they differ by culture. But that is what is important: the social context in which these activities occur, how they do so, and to what end.

One thing that has become apparent is the importance of the emotions (Goleman, 1995; Pfenninger & Shubik, 2001) in shaping behaviour and cognition. In the case of art, it has been pointed out that the motivation is communication with other human beings, which is apt to be emotional. In the case of algebra, to the extent that it is a social activity, it cannot help but involve emotions: the motivation to want to pursue algebra as a course of study, the pleasure from understanding a concept or solving an algebraic problem, the creation of learning events that make algebra contextually relevant; these are some of the emotional facets of algebraic behaviour and cognition that have their twin in the arts. And the emotions evoked are often of an aesthetic sort: we find algebraic concepts, and their application, as well as of course, the visual arts, beautiful.

The role of the emotions in learning in the arts and algebra brings us to the key point of contact between the role of the socio-cultural context in learning in algebra and the visual arts. At the very least, I have argued that from the social cognitive perspective adopted as the framework for this study, the role of the socio-cultural context impacts both the personal and behavioural bases of learning in algebra and the visual arts. I turn next to consider how the situated movement impacts our understanding of learning in both algebra and the visual arts.

The socio-cultural context. So far, I have considered situated cognition in conjunction mostly with authors concerned with mathematics: for example, Lave's (1988) concern with arithmetic used in practical contexts, such as by Liberian tailors or in grocery shopping (Lave, Murtaugh, & de la Rocha, 1984); Resnick's (1993) account of situated cognition; Walkerdine's (1988) critique of rationality; as well as others. Yet the philosophical literature on situated cognition goes beyond a concern with mathematics.

It is reasonable to think that both the communities of practice of mathematicians and visual artists help shape what is developed there, as well as the artefacts they use: for example, in algebra we use certain notational systems and in the visual arts the medium we work in affects our creations. Moreover, these communities of practice are themselves embedded in historical traditions as well as the changing socio-cultural context of the society in which they work.

The context determines, in part, what symbols we use in algebra and what types of representations we create in the visual arts, even what medium we work in. The context determines, by way of the judgments of our peers, what gets to be deemed as algebra and fine art, too. The context, in addition, shapes what uses we put algebra and the arts to, both socially and personally. Our philosophy of mathematics, the arts, and learning in these domains, also shapes how we teach, which in turn is influenced by the cultural context.

Social and personal transformations. We have already been sufficiently warned that we must not lose sight of the individual when exploring the role of the social context in cognition (Bechtel, 2009, Donald, 2001, 2006; Damon, 1993; del Río & Alvarez, 1995; Krishner & Whitson, 1997b; Lave, 1988; Martin & Sugarman, 1998, 1999a, 2002; Nickerson, 1993; Perkins, 1993; Salomon, 1993; Smith & Conrey, 2009; Sutton, 2009; Voigt, 1995).

We may wish to recall that I am interested in the macro- and micro-levels as they propel us to pursue algebra and create works of art, which within a social context, can play a role in identity constructions of certain types (i.e., the mathematician and visual artist, respectively). Effecting social transformation can be considered as a possible motivator for pursuing algebra and the creation of art works. For example, mathematics has been thought beneficial to producing the enlightened, rational society; and art has been used to envision alternative notions of the body politic. Doing algebra, and creating and viewing works of art, can help us forge new equilibriums in situations of cognitive dissonance, too.

The practice of algebra and the making of works of visual arts do not just teach us truths (algebraic or representative truths, respectively): they reflect, in their own ways, our society—and what kind of society we want. That is, these social practices, from an educational point of view, are opportunities to reflect on our interpersonal relations, where we fit in, and when feasible, our sense of care for our fellows.

Even though Cole (1995) cited finding alternatives to social learning theory as a motivation for the situated movement, it should be noted that the two are not incompatible. On the contrary, cognitive theory is naturally evoked when we speak of learning in a situation in terms of mental processes. Also, behaviourists have incorporated the social (Zimmerman, 1981) and situational dimensions (Bereiter, 1997) into their theory. Taken together, social cognitive theory (Bandura, 1997) I believe can provide a theoretical foundation for the situated movement. At the very least we should take away from this discussion that the social and cognitive dimensions of meaning-making in algebra and the visual arts are inevitably intertwined.

The term social cognitive theory, then, is to pay heed to the fact that, in the context of this proposed thesis, algebra and the visual arts arise through our own attempts to make meaning, both upwards between the relation of ideas, and downwards, connecting them to specific situations; through our cognitive interactions with the physical environment; and in a socio-cultural setting. The important point is to acknowledge how the interaction between the individual and the socio-cultural context leads to the production and maintenance of bodies of knowledge and social practices, and in turn, certain types of cognitions and behaviours, in a cycle that continues on.

Rationale

> "Only an alliance and, eventually, the merger between art and science into a single culture will satisfy this need [to overcome fragmented and reductionist approaches to the world]" (Pfenninger & Shubik, 2001, p. 215).

This literature review, then, provides the theoretical basis for the study, giving rise to the following questions, which compliment the "'Research Questions' (discussed in Chapter 1, "Introduction: A Philosophically Minded Prelude"):

1. Is it possible that aesthetics plays a role in the choice of a way of life as either an algebraist or visual artist, or the problems and projects chosen therein?
2. Does generalization and specialization have a role in the learning processes in algebra and the visual arts?
3. In what way could biology play a part in becoming an algebraist or visual artist? (The role of biology is taken up in Chapter 5, "Discussion: Solidifying the Common Link.")
4. What role does culture play in becoming an algebraist or visual artist?
5. How does interaction, specifically, algebraic or visual arts behaviour affect the personal and environmental dimensions?

The significance of this proposed thesis is that we may get a clearer idea of how social relations shape how we make meaning within the domains of algebra and the visual arts. The topic is useful to educationists interested in the psychology of learning as a theoretical rationale—and perhaps may serve as a guide—for the implementation of heuristics in algebra and the visual arts.

3. Methodology: The Case Study Approach

"Sometimes a lot of data can be meaningless; at other times one single piece of information can be very meaningful. It is true that a thousand days cannot prove you right, but one day can prove you wrong" (Taleb, 2007, pp. 56-57).

Case studies have been around for a long time and employed for a variety of purposes. In this chapter, I will explain what I understand by *case study*; and how and why I utilized this research methodology. Specifically, I shall explain why the case study method is an appropriate approach for exploring the social cognitive dimensions of meaning-making in algebra and the visual arts. I can anticipate my answer in advance: the case study approach allows me to answer the research questions (see Chapter 1, "Introduction: A Philosophically Minded Prelude," in "Research Questions") by yielding the *prima materia* needed for an analysis of participants' socio-cognitive processes (see Chapter 5, "Discussion: Solidifying the Common Link").

Historical Background

My aim here is to render the research design, that is, the logic that links the data collected, with the initial questions and conclusions drawn, explicit (Yin, 2003). Qualitative methodologies are often employed in the social sciences to capture the nuances of individual cases in great detail (Berg, 2004; Bogdan & Biklen, 2003; Byman & Teevan, 2005;

Creswell, 1993; Marshall & Rossman, 1995). In a standard text in the field, *Research Design: Qualitative, Quantitative, and Mixed Methods Approaches,* Creswell (2003) noted that qualitative investigators view social phenomena holistically and "research studied often appear as broad panoramic views rather than micro-analyses" (p. 182). According to him, the focus of qualitative studies is how participants perceive how they make sense of their lives.

My topic has the breath appropriate for a qualitative study as I consider the processes that lead to meaning-making in algebra and the visual arts at both the macro- and micro-levels. In fact, studying the role of the socio-cultural context in learning in algebra and the visual arts entails considering the interaction of social and cognitive realms.

As Creswell noted, qualitative studies can take various forms, studying individuals (e.g., through narrative or phenomenology); process, activities, events (e.g., through the case study method or grounded theory); or broad cultural behaviour (e.g., with ethnographic studies). My investigation employs the case study method, which is often used to understand the *how* of certain processes (Creswell, 2003). As will be discussed further in this section, however, in a case study we explore a process by way of the participants—who serve as the window into the issue, as it were.

The case study method was first employed in medical research in the 19th century and is still used to teach medicine (Yin, 2003). Drawing on the work of Jennifer Platt, Yin (2003) noted that case studies can be traced back to the life histories of the Chicago school of sociology. Although Merriam (1988) in her book *Case Study Research in Education*, noted that the case method (used for teaching) and case studies (used for research) do not "equate" (p. xii). For teaching, case studies have often been designed with certain pedagogical outcomes in mind, but research may be more open-ended, for example, not knowing what we will learn. However, it is edifying, I think, to consider both types of case studies together, as there are commonalities in both their genesis and content.

The case method, introduced by Edwin F. Gray in 1908, was made famous by its canonical use at the Harvard Business School (Copeland, 1954). Contributing to a collection on the use of the case method at Harvard, Dewing (1954) explained that learning can happen by telling or by grappling with problems that teach us how to deal with future ones. When a topic is complex and there is no easy solution, he said, the case method is the way to go. Even though there are many types of cases, Schoen and Sprague (1954) identified these commonalities: (a) a focus on experience; (b) a focus on the particular not general; and (c), a focus on the student's experience of material.

Yet, Harvard scholars Christensen and Zaleznick (1954) noted that the case method is not just a teaching tool, but more generally, a way to learn from our experiences (Towl, 1954). Donham, who contributed to the same volume, explained the rationale of the case method, noting that the world in the 20th century and onwards is in unprecedented flux. Arguing for a pedagogy that uses relevant examples and emphasizes collaboration, Donham wrote, "In this country [America], we take pride in that our college training is intellectual. Yet the lives we live are primarily emotional" (p. 210). He held that using cases helps connect our minds to active life.

The case method in research draws upon many of the benefits and rationales that motivate its use at Harvard Business School. Social science phenomena is messy and often not open to clear cut solutions at least not that are mutually exclusive of other approaches. Social scientists often are interested in affecting practice. Also, they must pay heed to a participant's interpretations of events, not just those of the researcher. And the case method allows us to anchor our theoretical constructions in human lives, taking account of the affective dimensions. In fact, the case method has an extensive history in both qualitative and quantitative educational research (Merrian, 1988; Scholz & Tietjie, 2002; Stake, 1995).

The case study method involves an in-depth exploration of a *bounded entity*, a concept coined by Louis Smith, one of the first educational ethnographic researchers (Creswell, 2005; Stake, 1995). An entity is bounded in terms of its time, place, or some physical limits. The case study is not used to study functions per se (e.g., voting) but entities that engage in them (e.g., voters) (Singleton & Straits, 1993). According to Stake (1995), an expert on these studies, cases are often used to study people and programs, not problems or relationships, which are included within them; also, they may be quantitative or qualitative. Cases are intended to yield detailed descriptions of the setting and an individual, their uniqueness, followed by an analysis of the data according to themes (Stake, 1995). As Stake (2006) put it, cases are nouns and seldom verbs.

Even though some researchers on situated cognition suggest studying actions (Cole, 1995; Rogoff, 1995; Wertsch, 1995), there is reason to think that we can also understand the interaction between the individual and the social context using the case method. As Yin (2003) explained, in his detailed discussion of case studies, they are used to explore the "how" and "why" of complex social phenomena. Yin noted that case studies can be characterized as being applied to holistic and meaningful characteristics of real life events, as well as when the boundary between the phenomenon and context is not well defined. In this thesis, the case studies were configured in qualitative terms where the phenomenon of meaning-making straddles, with no well-defined boundary, between both individual cognitive processes and the social context (see Chapter 1, "Research Questions").

Also, case studies can differ by being *exploratory*, where fieldwork precedes the research questions; *explanatory*, when causality is involved (e.g., with communities); and *descriptive*, where we begin with a theory (Berg, 2004). (A *descriptive case study* can also be called a *theoretical case study.*) According to Yin's (2003) taxonomy, there are six varieties: to explain, to describe, to illustrate, to explore, a meta-evaluation (a study of evaluations), or a mix of the aforementioned, all running the gambit from simple presentation to broad generalizations.

"At a minimum," in a reflection on the case method, Ragin (1992a) said, "most social scientists believe that their methods are powerful enough to overwhelm the uniqueness inherent in objects and events in the social world" (pp. 1-2). Watson (1992), contributing to the same volume, said, "Cases come wrapped in theories" (p. 122). That is to say, as Watson noted, a case "presumes a theory based on causal analogies" (p. 135). Summing up, "empirical research can be seen," as Ragin (1992b) also explained, "as culminating in theoretically structured descriptions" (p. 218). He said that the goal is to "use theory to make sense of evidence and use evidence to sharpen and refine theory" (Ragin, 1992b, p. 225).

What is a Case Study?

There are, however, different types of case studies, those where the aim is "intrinsic," to understand a unique bounded system; *instrumental,* when it is to shed light on some theme; *collective*, involving several individuals, where the goal is to provide insight into an issue (Stake, 1995). With multiple case studies, the members form what has been called a "quintain" (Stake, 2006, p. 4), that is, a group that is bound together by a common characteristic. We study the individual cases, in the collective investigation, to understand what is common among them better.

As Yin (2003) noted, the unit of analysis is usually indicated in the research questions, and this study is no exception. My goal, in summary, can be stated thus: the unit of analysis was how we make meaning in algebra and the visual arts. Embedded within the previous unit is another one: how we use social constructions—abstractions like notions of self-identity—to pursue meaning-making in algebra and the visual arts.

There is an overlap, however, between the case study method and narrative inquiry. In fact, Stake (1995) noted that writing a case report is often referred to as storytelling. Yin (2006), a specialist on the case study method, has pointed out that it began to wane in education in the late 1970s, through the 1980s, and was often not present in textbooks of the era.

A few words are in order to explore the relationship between the case study and narrative inquiry. Though it has a long history within the enlightened humanities, narrative inquiry has gained prominence with the "growing appreciation for rhetoric," said Gubrium and Holstein (1997, p. 89), who were considering the ideological underpinnings of various methodologies (Marshall & Rossman, 1995). It is reasonable to think that postmodern ideology, that is, the notion that we should not aim for universal truths but pay attention to the particulars of contexts, buttressed the rise of narrative inquiry in educational research.

First, the aim of the researcher using narrative inquiry is to provide detailed accounts of people's stories, which lends itself to a chronological presentation (Clandinin & Connelly, 2000). Second, it is often the researcher who rewrites the story with the participant's assistance, where a collaborative relationship is valued. Third, the researcher may interweave their personal story into the project. Finally, narrative inquiry may have a social justice orientation or allow adopting a related theoretical lens, for instance, some version of feminism.

There are important differences between the case study method described hitherto and narrative inquiry. Philosophically, this project takes as its point of departure the cognitive sciences, whose practitioners are naturalists and would likely be opposed to postmodernism. We can summarize the salient characteristics of a theoretical case study that distinguish it from narrative inquiry as follows: with theoretical case studies, themes that are used to interpret data are determined before the retelling of stories; the literature has a central place; the researcher has a privileged position as an interpreter of data; and the investigator's background is of marginal interest. Also, the purpose of a theoretical case study is not primarily to give participants a voice.

The characteristics of the case study method and narrative inquiry, however, can be complimentary, too. Narrative inquiry, insofar as the researcher merely relies upon the participants' interpretation of data, is sometimes considered "uncritical" (N. Dlamini, personal communication, March 27, 2008); and theoretical case studies are vulnerable to ignoring their understandings, which is well known in critiques of Freud.

The moral is this: it is advisable to emphasize the role of both the participant and researcher, which can be done within the context of a case study; there is no necessity to venture into narrative inquiry to take into account the participant's interpretation of experiences, for example (discussed further in "Interviews," when considering the issues of reliability and validity). In all qualitative research, interpretations are negotiated, according to Creswell (2003), with human data sources (discussed in "Interviews"). Positioning ourselves is also considered important in qualitative research generally (discussed in "Interviews") (Creswell, 2003). In a narrative inquiry, we have to deal with a few participants' stories in depth, and it is common practice in case studies to use less than four of them (Stake, 2006).

Why Case Studies?

I can say at least a word to justify the case approach I used that can be characterized as descriptive/theoretical, critical, and representative. The Indian neuroscientist, V. S. Ramachandran (2005), commented on the debate between qualitative and statistical methods in his discipline, which allows some measure of defence from sceptics of case studies. In brief and more generally, within the social sciences, qualitative research is sometimes looked upon with suspicion by quantitative researchers. Ramachandran noted that people have worried that qualitative studies will not allow us to generalize neuroscience results. Ramachanrdran (2005) wrote, "[B]ut this is non-sense. Most of the cases that have stood the test of time ... were initially discovered by a careful study of single cases [in the neurosciences] and I do not know of even one that was discovered by averaging results from large samples" (pp. xi-xii). I have chosen the methodology that has the greatest legacy of yielding results in the history of dynamic psychiatry—case studies.

A case study approach is appropriate, more specifically, because my orientation is theoretical, dealing with social cognitive theory. I explore the process of meaning-making through interviewing participants. Further, a case study approach allows me to deal with several of the different modalities of the social context I have identified holistically. For example, the case study approach allows us to explore how prior knowledge, family experiences, school experiences, or the role of social models, shape behaviour.

Interview Instrument

Interviews are a common technique used to collect qualitative data in the social sciences (Berg, 2004; Bogdan & Biklen, 2003; Bryman & Teevan, 2005; Creswell, 1993; Marshall & Rossman, 1995). My methodology has two dimensions. First, my research involved a three-phase semi-structured interview design, dealing with two practitioners in the domain of algebra, two in the visual arts (Ellis & Bochner, 2000). As much as possible, open-ended questions were employed, within the context of the general themes expressed in the research questions (see Chapter 1, "Research Questions") (Seidman, 2006). Because of the open-ended format, the interviews ranged in time from just over one hour to two and a half hours (exact times are given in Chapter 4, "Findings: Four Case Studies").

Siedman (2006), in his book on this methodological technique, advocated the three-phase interview structure, which I followed. The first part of the interview, Siedman said, concerns the participant's background; I usually asked *how* not *why* they came to where they are. They were asked, for instance, to reconstruct their past. Second, details of the participants' experiences were gathered; I asked them to talk about *what* they did on a given day, from waking to sleep. In the third and final phase of this interview process, the participant was asked to reflect on the meaning of their experiences, to connect their emotional and intellectual life, and to speculate on where they see themselves in the future. As Siedman noted, each stage of the interview process provides a foundation for the next and helps build trust.

Here are the interview questions I used, organized according to the three-phases.

I. Background

1. Preamble to the Interview.

- What is your name? Where were you born and raised? What is your occupation, rank, educational background?
- What would you like your students to achieve as a mathematics or visual arts educator and what goals would you set?

2. Socio-Cultural Background. After explaining what I mean by "social cultural background" to each participant, I asked the following: What social background are you from? What cultural backgrounds are you from? What "value" did mathematics or the visual arts have in your socio-cultural context? How would you describe your socio-cultural background?

3. Family and Friends. How were you supported (or not supported) as a learner by family, teachers, peers, and self?

4. School Experiences. This was discussed with the following questions:

- What was the nature of your "engagement" in mathematics or visual arts in different school settings (e.g., elementary, secondary, university settings)

- What are your memories of teaching practices (e.g., was it a lecture format, was there group work, were there any activities involving real life contexts) from different school settings (e.g., elementary, secondary, university settings)?
- Do you recall any mathematics or visual arts teachers that are memorable to you? Why do you consider them memorable?
- How did teachers generally act when students made mistakes or asked questions deemed "silly"?
- How did a mathematics or visual arts textbook—for example, their readability; presentation style concerning summary, clarity, organization, coverage of subject matter as adequate or inadequate; guiding working examples; and relevance and applicability to today's world—help you to understand in these domains?

5. Personal Reflections

- What "attitudes" and "feelings" do you have about mathematics or visual arts teaching and learning?
- What role did aesthetics (i.e., a sense of beauty) play in choosing your field, problems or solutions within it?
- What did you do to solve problems in your field when a mathematics or visual art learning was difficult?
- How did you feel when other students found mathematics or visual

arts easy or difficult?

II. Present

1. Values

- What still poses challenges to you (e.g., relating to what you want to know more about or what you did not fully understand when you were in school) in the domain of mathematics or the visual arts?
- What "kind" of mathematics or visual arts would you consider important to schools and the academia, industry, and today's society at large? What is important to you in this regard? What "purpose," social or personal, do you think the study of visual arts/mathematics fulfills?
- What type of problem do you find particularly interesting in your field? How do you tackle challenging mathematical or visual arts problems?

III. Future

1. Ideas About Learning.

- Are your ideas about learning drawn from family, friends, teachers, or textbooks?
- Are there any influences from formal theories of learning (e.g., behaviourism, constructivism, multiple intelligences, etc.)?
- Are your ideas of learning developed independently?
- What reflections do you have on your ideas about learning in terms of

how successful they have been? What are, in your opinion, their pros and cons?

- Why do you think you hold the ideas about learning you do?
- Have you ever modified or changed your ideas about learning?
- If you could change these ideas of learning, how would you?

2. Aspirations.

- What do you want to accomplish in your given field?
- How does it relate to your life overall, for example, the socio-cultural context of your upbringing and their values?
- How do you plan to achieve your future goals in your field?

In this thesis, I employed triangulation, which came from naval military science. In that context, triangulation is the procedure used by sailors to locate an object's exact location at sea, by using multiple reference points (Creswell, 2005). The idea, in our context, is that different data sources were used to build a coherent justification for the themes selected (Creswell, 2003). Yin (2003), further, distinguished between data triangulation, which involves using multiple sources of evidence, and method triangulation, which deals with using various facts that are part of the same study. The two forms of triangulation likely overlap in practice. For example, I triangulated data by relying on interviews as well as artefacts produced by the participants; which also entails method triangulation, that is, talking to participants and collecting some of their productions (e.g., syllabi, drawings, and research notes).

I can be a little more specific. Data triangulation: I took field notes and collected artefacts, for example, samples of participants' work in their respective fields, notebook sketches, or artworks, and even audio-visual materials, in order to provide alternative vantage points from which to generate data (Denzin, 1978; Statham, Richardson, & Cook, 1991; Yin, 2003). All of the data were first-hand: there is no reliance on third-party observations or analyses.

In addition, as St. Pierre (2005) has pointed out, we may obtain crystallization. Whereas triangulation focuses on one point, crystallization allows for many perspectives. Varying perspectives, specifically problems with data interpretation, is what has made qualitative research look weak, however. But considering various perspectives can also add rigor to the presentation.

Some authors like Guba and Lincoln (2000), who have been influential in educational studies, have attempted to politicize the relationship between quantitative and qualitative methods as competing ideologies. Quantitative studies have been sometimes considered more scientific by allowing consensus reached by the application of accepted methods involving measurement. Suffice it to say, however, that it is reasonable to think that both methods often work in tandem.

A perennial concern of researchers using the case study, for instance, is whether their findings can be generalized. Yin (2003), however, noted that we should distinguish generalizing to theory (analytic generalization), which case studies assist in, from enumerating frequency (statistical generalization). It is reasonable to think that the analytic variety extends theory or gives rise to hypotheses, which are in turn tested by statistical studies, but both varieties allow the development of theory (Stake, 1995). In fact, Yin classified cases along the lines of their specific method: to test theory (a critical case), one-of-a-kind (a unique case), to be informative about the average person (a representative case), to explore what was previously inaccessible to scientific scrutiny (a revelatory case), and to study the same case at different times (a longitudinal case).

Further, Stake (1995) drew upon the well-worn distinction between explanation and understanding. Some hermeneutic philosophers, most notably Hans George Gadamer (1960/1995), have held that the natural scientists seek explanation and the those in the humanities seek understanding. Thus, with detailed examination of individual cases, we can, it is often thought, be attentive to a host of variables that lie between the phenomenon being studied and the context that is often excluded by quantitative studies. We can increase understanding, but that is just a way of making explanation more robust and refined. As Stake (1995) noted, case studies can be used to elaborate and modify generalizations, too.

Data Interpretation

What I did. I conducted a theoretical case study (Bryman & Teevan, 2005; Marshall & Rossman, 1995; Yin, 2003). Social cognitive theory provided the theoretical lens through which to analyze the data. The layout of the findings can be gleaned by returning to the issue of different varieties of cases. Yin (2006) identified several types of case studies: the classic single case (discussed in Chapter 3, "Methodology: The Case Study Approach" in "Why Case Studies?"); a multiple case study, where each one constitutes a chapter followed by a cross-case analysis; or multiple ones where the cases are not detailed, and only a cross-case analysis is done. And, fitting the mold of my study, there also is the single or multiple case studies without narrative. I conducted a multiple case study in a classical sense, broadened with some cross-case analyses when deemed appropriate.

As Yin (2003) noted, the purpose of data analysis is to allow us to examine, tabulate, test, and recombine evidence, so as to investigate the validity of the initial proposition of the study. In the following two subsections, "Participants" and "Interviews," I shall explain some of the salient interpretive logistics of the cross-case data analysis.

Participants. Traditional methods of using interviews were employed to assess the implications of the research. The first part of the process involved inviting several participants to participate as interviewees, who were purposely chosen (Creswell, 2003). All candidates were from the province of Ontario, Canada.

The choice of four participants was based on minimum criteria of being a professional practitioner in one of the domains under investigation, determined by a study of their curriculum vitae in addition to other factors, such as providing a heterogeneous population, in terms of gender, cultural group, or socio-economic background; although, admittedly, that matters little in a qualitative study, and consequently did not play a significant role in their selection compared to their qualifications. In order to make this study relevant to high school pedagogy, one participant from each field was a university professor or professional equivalent (i.e., professional artist), while one participant from each field was a teacher at the secondary school level or thereabout.

I did not know three of the participants before the study, and had only seen one in a group setting. To the extent that some of the research, then, occurred in my "backyard," that is involving one participant I had interacted with, albeit in an insignificant way, multiple strategies were employed to ensure validity (discussed further on in this chapter).

Interviews. The second part of the process is the conducting of the interviews. I aimed to build a trusting relationship (Haverkamp, 2005) with the participants in my research. I followed Creswell's (2003) suggestion: interviewees were informed about the research project, its purpose, and potential benefits. I made plain to interviewees what prospective benefits there are to my research, such as understanding our cognitive processes better and allowing us to better design educational regimes for the development of algebraic and visual arts skills; written permission was requested to participate; verbatim transcripts were available to them; and even though participants were rendered anonymous by the use of pseudonyms, they had the final decisions about the presentation of any data that could be considered identifying.

In this regard, Ben, the first interviewee and a mathematician, asserted, "I do not care less.... You [Anoop] can post the interview [with me, Ben] on the Internet." With this statement in view, I maintained Ben's anonymity to retain the style in which social science studies are presented at this time; to ensure authenticity of answers from participants, I followed the rule of thumb that it is advisable to build some anonymity and confidentiality into the research design. However, I did, on occasion, refer to some identifying remarks of Ben's life when I think they added detail to the case and when there was not a semantically equivalent replacement for my analytic, synthetic, or sometimes aesthetic purposes.

More generally, in all instances where I referred to actual facts of a case, I have the participant's consent to do so, and they have approved a draft of my written description of their specific case (see Appendices A-F). Since I have given the participants pseudonyms, I do not disclose their actual names or signatures in this thesis. These are, however, available for instance to the Research Ethics Board of the University of Windsor.

I was sensitive to what the interviewees were giving up in terms of time and disclosure, which I acknowledged on forms of consent provided to them. Interviewees were also informed about their right to terminate the meetings at any time. All data obtained from the interviews were secured in my office in the Leonard and Dorothy Neal Educational Building that houses the Faculty of Education at the University of Windsor. Specifically, the data were locked in a filing cabinet; this is enclosed in a locked office, which further is locked within a secure area. Audio tapes were deleted and the only materials retained were what appear in the thesis, in terms of quotations from the interviews. I hold a Certificate of Completion (2006) for the Introductory Tutorial for the Tri-Council Policy Statement: Ethical Conduct for Research Involving Humans (TCPS). All ethical obligations were fulfilled, having, for instance, obtained approval of the Research Ethics Board of the University of Windsor and the participants in this research (see Appendices A-F).

Finally, an on-going process of coding data occurred (Creswell, 2003). I employed an interview protocol, providing a heading; opening statement; key questions; and, as required, reflective notes (Creswell, 2003). Interviews were recorded on a Dictaphone and, as required, transcribed. I listened, as Creswell (2003) advocated, to the tapes containing data; and studied the transcriptions for thoroughness in order to organize and prepare the data for analysis and get a sense of their general character.

In the write-up of the cases, and the emailed letters of consent (Appendices E-F), I made minor changes to some verbatim quotations and text for the purposes of readability. On occasion, I placed my own clarifications, with regard to a slang or technical term, in square brackets or parentheses. However, I tried to maintain much of the vernacular tone used by the participants. I also attempted to retain their sometimes off-beat colloquial expressions drawn from their field.

I organized the data into themes for interpretation. Data were coded, putting them into "chunks" (Creswell, 2003, p. 19). Taking a page out of the narrative inquirer's method-book, there was also an open-coding process, in addition to the preliminary themes, generating broad and specific ones to avoid redundancy (Creswell, 2005). The data were also categorized through axial coding by which emergent themes were selected and placed within the theoretical framework. I also used as emergent themes *in vivo* terms, which are employed by the participants themselves. The coding process, said Creswell (2003), allows "additional layers of complex analysis [yielding] complex themes and connections" (p. 194).

The study entailed both deductive and inductive methods (Creswell, 2003). A deductive process: I used a social cognitive conceptual framework to interpret data (see Chapter 5, Table 4, "The Social Cognitive Themes in This Study"). Also, an inductive process: the broad themes—namely, the cognitive, social, and biological dimensions—were refined once data were collected, for example, allowing for the emergence of sub-themes (see Chapter 5, Table 4, "The Social Cognitive Themes in This Study").

In fact, the interviews were organized along the lines of the research questions. The first topic concerns the factors that *motivated the participants* to work in their field of choice in terms of family; culture; social economic status (SES); temperament; and personal experiences, academic or otherwise. The second topic involves *what experiences participants had that shaped their pedagogical philosophy and learning style*, which includes their interaction with other experts or exemplars in their field. The third topic concerns *how participants have developed strategies* to *acquire specific skills*, related to lifestyle, practice, seeking out feedback or exemplars, making works public (e.g., through publication or exhibitions), and to be able to transfer abilities to new problems in their domain or a related one.

Reliability, noted Creswell (2003), has a specific meaning in a qualitative study, namely, the consistency of patterns and themes developed. Reliable findings are dependable, which require that we account for the context in which the study occurs.

Validity, in a qualitative study, involves the accuracy of the findings from the point of view of the researcher, participants, and audience. *External validity* deals with generalizing results, which goes beyond our ken. Related to qualitative research, *internal validity* (henceforth "validity") concerns the accuracy within a study. For qualitative researchers, the accuracy of findings can be evaluated by their credibility, namely, their believability; their transferability, or ability to generalize; and if they can be confirmed, that is, can they be corroborated by others.

The two matters are connected thus: reliable findings may be supported by valid data and vice versa, but a reciprocal relationship need not hold. First, reliable findings can be invalid. Consistent themes may be based on flawed data. Second, contrary to the conventions of quantitative research (Newman, 1991) unreliable findings could be valid in qualitative studies. A lack of pattern regularities (e.g., a set of anomalies in relation to previous studies) may be based on accurate data.

With the foregoing in view, we can consider reliability and validity in relation to the three-phase interview design. It is reasonable to think the interview design reliable because it allows access to the data needed to address the research problem and the questions raised (see Chapter 1, the sections, "Problem," and "Research Questions"). In short, the interview design facilitated the development of consistent themes and pattern regularities. I also checked the *face validity* of the interview instrument, that is, making sure it allowed me access to the data that would relate to my research questions, by having it securitized by several social scientists, which led to emendations and refinements to the wording.

I also believe that the case studies yielded valid findings because of the rigorous steps I took to ensure accuracy. Let me explain. To increase our confidence in the validity of the study, I followed Creswell's (2003) suggestions in this way: within two weeks after interviews occurred, I queried participants by email to double-check obscurities as I transcribed sections of the digital recordings; I, on occasion, member-checked, taking into account participants' comments on my interpretations. In addition, I presented negative or contrary evidence in the thesis (see Chapters 4 and 5); and engaged in peer-debriefing, using a third-party (e.g., my thesis committee and colleagues) to check that the case descriptions I produced are believable.

Furthermore, following Yin's (2003) lead, I paid attention to all the evidence, which was ardently collected; sought out alternative explanations and patterns; addressed the most significant aspects of the study; built upon my prior experience as a philosopher; as well as, attempted to check that the motivating question was not artificial by conducting a thorough study of the literature. Finally, I aimed to produce detailed descriptions so that anyone interested in transferring my work will have a solid framework for comparison (Creswell, 2003).

Self as Instrument

The purpose of this narrative is to disclose Gupta's background, as it relates to this study, so as to allow the reader to gauge some of the biases that he may unwittingly introduce into his interpretation of the data (Chapters 4 and 5, "Findings: Four Case Studies" and "Discussion: Solidifying the Common Link," respectively). In this narrative, then, the trajectory of his beliefs and knowledge about student learning, as well as pedagogy, specifically as it relates to algebra and the visual arts will be discussed. We may wish to note that this narrative is not intended to explain Gupta's interest in philosophy or the social sciences, other stories in their own right.

It is useful to note the setting: Gupta is from a Punjabi-Indian family that immigrated to Canada, from England in 1975. He was raised in Ontario, Canada. His father was an engineer and his mother holds a Master of Arts degree in English literature.

First to be discussed are Gupta's early mathematical experiences, followed by those in the visual arts. Third, his current beliefs about teaching and learning are considered, as well as, his future academic aspirations.

Early mathematics experiences. Gupta's first memory of mathematics was in high school. High school mathematics, he recalls, was organized around the textbook. The design principle was Euclidian, beginning with definitions of basic concepts. It was the usual thing, practice problems or elaborations. One textbook, he remembers, illustrated the fact that five sets of five are 25. The visual images like this helped him understand the concept of *multiplication.*

He had tutorials in mathematics at home. On Saturdays, while watching television, his father would call him into the kitchen. His father thought mathematics would be fun to do, and a way to spend time together. However, Gupta rarely got the answers under the intense time constraints. As he worked through problems, his father would often get agitated; Gupta became frightened.

When mathematical problems were tough, he would seek advice. Not looking to seek help from his father, he asked his mother. If he got stuck on a mathematics problem at home, it was a good enough excuse to stop working on it.

His achievement in mathematics was not remarkable, and though passing, it was unlikely, in retrospect, to be an indication of a career in science. Gupta recalls only one of his mathematics teachers. He was non-threatening. A teacher that was friendly was a blessing in a subject where he did not obtain sterling results.

Mathematics, Gupta thinks, is part of being literate in modern society, which includes its application in finance (paying our bills), applications (building a shed, bridge, or skyscraper), and understanding information (from the news or text media). But, in advanced civilizations, he believes, mathematics must also have a value unto itself.

Some remarks on attitudes about the arts. One of his earliest creative ventures in the visual arts, when young, was paint-by-numbers. He continues to paint, off and on. He also loves working with his hands, gardening and carpentry. His ideas about arts were shaped by two forces, one social and one personal.

Socially, we are apt, in the Western culture, to romanticize the lone wolf; and indeed this has impacted his learning strategies. Also, straddling, both the social and personal dimensions, Western cultures are more individualistically oriented, and solitary, than many traditional societies, like we would find in Punjab, for instance. Compounding matters, coming from a Punjabi family, Gupta was already the odd-man-out; he was different from his peers growing up in Canada.

In addition to living in Western society, Gupta has been accustomed to a solitary sort of work. Reading and writing are not team sports, for instance, because we cannot read a serious book with someone else, though of course we could form a reading group provided others wanted to study similar texts.

Not only was Gupta isolated when young, socially and personally, but he thought it a good thing, the sine-qua-non to achieve excellence. Practice. Practice. Practice. That is the way work must be done, so he thought. But becoming an artist was out of the question; and mathematics, for the time being, was already out of the picture, having left it behind in high school. Coming from an educated immigrant family, there was an expectation to pursue higher studies, however, and he did, in philosophy, which is discussed next.

Educational experiences in philosophy and educational psychology. At the university level, in 1989, his philosophical education began with a first-year course Mind and Truth, offered by Dr. Brook, an Oxonian and previously Director of the Institute for Cognitive Science at Carleton University. Transferring to McMaster University, he took a course on philosophical psychology with, Dr. Allen, a contemporary pragmatist, going on to be the teaching assistant for the course during 1995-1997. Gupta wrote his master's thesis under a Russell scholar, Dr. Griffin (Director of the Bertrand Russell Research Center, at McMaster University).

He went on to study at the University of Ottawa, where Gupta was briefly under the supervision of Dr. McCormick, a European philosopher, taking his graduate course in aesthetics (Rationality, 1997, University of Ottawa). Gupta studied under an Oxonian Wittgenstein scholar, Dr. Marion (Canadian Research Chair in philosophy of mathematics), who rejected the notion that abstract mathematical objects exist independently of the mind. In his first dissertation, "Benecerraf's Dilemma and Natural Realism in Arithmetic," Gupta (2002) argued (against his supervisor) that mathematical truths are eternal and empirically knowable. He was not done with mathematics.

Gupta sometimes freezes up when doing mathematics questions on the spot and prefers to work on them alone. It is only much later, after ten years of work in philosophy, that Gupta came to enjoy mathematics. Now, he often toys with simple algebraic problems for fun. Gupta is a philosopher and social scientist that finds peace, challenge, and substance in mathematics.

An attempt to find his place, and understand himself, overlapped with an on-going philosophical conversation he was engaged in as an academic writing about such topics as the nature of mathematical truth.

Gupta long thought, however, that to make headway in metaphysics, on matters of truth, he must first solve puzzles about the workings of the mind and learning. What is the mind? How do we, in fact, learn mathematics?

After his PhD in philosophy, Gupta enrolled in the Bachelor of Education program in the Faculty of Education, University of Windsor, which became the avenue to tackle puzzles about the nature of the mind. Gupta studied educational psychology and wrote narratives, stretching his skills into areas philosophers often do not broach when young.

During the MEd, also at the University of Windsor, Gupta studied the classics of twentieth century educational psychology, Skinner, Piaget, Vygotsky, and Bandura both as part of the course work and independently. Also, he learned how to write in the style of the social sciences, in terms of documentation and temperament. For example, he has been immeasurably helped by Dr. Maldonado in learning about the "humanity" of qualitative research, and of course Dr. Ezeife (discussed further in "Acknowledgements").

One reason Gupta entered the Joint PhD program in educational studies, specializing in cognition and learning, at the University of Windsor was to write this proposed dissertation. According to Gupta, writing this dissertation would allow him to synthesize several passions, related to hands-on activities and abstract pursuits; many of the ideas Gupta has about the mind; as well as skills related to doing research as a philosopher and social scientist.

Reflections on mathematics and the visual arts. Having assimilated the value of mathematics in Indian, Canadian, and British cultures, Gupta believed that mathematics is abstract and done in our heads. In fact, the founder of analytic philosophy, Gottleb Frege (1918-1919/1984), in "Thoughts," claimed that mathematical truths inhabit an independent reality—the third realm.

However, Gupta currently believes that mathematics must be taught in a way that utilizes visual and tactile objects, allowing for a deep understanding of concepts and an alternative way of thinking about mathematical problems. According to him, motivation, love and respect for the discipline must be a key to teaching mathematics and the visual arts.

Since, Gupta believes, mathematics and the visual arts are inherently interesting, the main goal of the teacher is to facilitate mathematical and visual arts experiences. Gupta also believes that sometimes rote learning and practice are important to mastery of mathematics and the visual arts, but not those alone.

His ideas about the visual arts, for instance, dovetail with what Gupta thinks about mathematics. He thinks both require practice, patience, and exposure to exemplars. Though both fields require individual commitment, he has become more convinced that there are important underlying socio-cultural currents that shape our interest and strategy within these fields—which have prompted him to capitalize upon social interactions to optimize learning in his pedagogical practice.

Teaching philosophy at the university level, educational psychology, and mathematics methods would be satisfying for him. His interest in mathematics and the visual arts are motivated by social, cultural, and educational experiences. A goal of his pedagogy, more generally, is to create experiences of discovery, and co-discovery amongst our peers, in conjunction with practice. Gupta is led, as was Frege (1918-1919/1984), to abstraction, and as a philosopher and social scientist, back to his own life—and those around him.

4. Findings: Four Case Studies

Case 1: The Finitist Mathematician, Ben

> "If it can be planned, it ain't research. Or at least, if it can be planned it's already half done, and hence it isn't the best of research. Thus my primary plan of mathematical research for the next few years is to *follow my nose*" ([emphasis in original]. "Ben," 2002).

Born in Kiryat Gat, a small town about 40 miles outside of Tel Aviv, Israel, in the 1960s, Ben's elementary and secondary education was in Israel, as well the Master's of Science degree he earned in mathematics. He earned a Doctor of Philosophy degree in mathematics from Princeton University in the early 1990s.

Ben is in his early 40s. He has taught mathematics at the Hebrew University in Jerusalem, the University of California, Harvard University, and now teaches at a large urban Canadian University. His specialization is quantum algebra and knot theory (part of the field of low dimensional topology, discussed further on in this case study).

He has published at least 37 academic papers, 10 of which he identifies as "most significant;" given 40 talks; and taught 12 different courses, ranging from the undergraduate level to the graduate level, most of which (not including calculus) deal explicitly with algebra. At present he supervises three doctoral students and one master's student. Altogether, he has supervised three doctoral students, four master's students, and seven undergraduate projects. On his Website, he lists 16 papers published by his students under his supervision. In addition, he is the editor of a mathematical journal and holds several numerous National Science and Engineering Research Council of Canada (NSERC) Discovery grants. I now turn to his case, largely drawn from my interview with him (which yielded 2 hours, 33 minutes, and 28 seconds of recorded raw data); and supplemented with information from his Web site that includes a vast range of documents (e.g., see Appendices G-J), such as his curriculum vitae, research statement, teaching philosophy, course syllabi, pictures, published papers, manuscripts, and so on.

Ben described Israel, in relation to his upbringing, as a developing, and now developed country. He grew up in a secular, Westernized, non-religious household. His parents immigrated there, his father in the late 1940s, and his mother in the late 1950s. Ben has two older sisters, one by 10 years, and the other by just 5 years, neither are mathematicians.

His father was, in his words, "a low-level administrator." Both his parents were open-minded about science. His parents were happy he was reading math, "but," as he put it, "they could not help me because I was already ahead of them as a child." Ben recalls, however, that in order to keep him silent his father would have him digits of the phone numbers in the classified to sum. Also, when Ben was in Grade 7 or 8, his parents bought him a "nice calculator, a TI-58, but the point is that it was unusual at the time." Ben said that he "really, really wanted it, and it was a significant expense." As he explained, his father not only incurred the expense but it was not easy to get; he had to wait for relatives to visit him from America. "So on some level," Ben said, "I got encouraged."

He grew up in a household that was much more connected to the State than the situation in Canada, and somewhat richer than many around him. He described himself as lower middle-class. Adding some detail, however, he explained that in his town: "I was part of the upper ten percent of the population, well-to-do; however, if I was to go to Tel Aviv, I drop down to the lower middle class." So, as he pointed out, his social economic status "is relative to the scale; at the time I was not aware that Israel was also a speck [like my small town was within the state]."

He was "secular Israeli," which he noted, "means quite a lot." His family was part of the Zionist movement, committed to the Jewish nationalist ideology, to which he no longer relates. "Secular Israel still distinguishes itself from secular Canada," he emphasized. After I suggested that the Jewish identity is strong because of the history of the Holocaust, he replied, "Your guess is absolutely right." As he described the secular Jew: "It means no praying in the morning, no praying no matter how many times a day, but still the Jewish identity was very strong. But here I am describing my socio-cultural background, not me [specifically]."

When asked about the value of mathematics in his socio-cultural background, he said, "Zero, I mean you know the town I grew up in, there was no one I can remember with interesting math. Zero is a good approximation."

Ben was supported in his interest in mathematics in different ways. He first pointed out that a lot depended on the year and the scale. "Basically, no one encouraged me to do mathematics, or science more generally," as he put it, "but there was openness to it. I was extremely lucky because the schools I went to were open; they did not block it." "I was a bit of an extreme student," he went on, "some [grades] were 100 and some below 50, and I had a fair number of both. When you looked at my grade point average it was about 75, but that is an average between many 100s and 50s." The low grades related to things he had no interest in and subsequently spent little time on. The 100s were in mathematics where he also spent little time, because, he said, "I did not need to; so I passed with 100s without spending the time."

As Ben recalled,

> The bad thing I could say about school was that it was extremely boring. I felt that I was serving a jail sentence. But to their [my teachers'] credit, they recognized it [that it was a jail sentence] and they let me graduate two years early. So basically they reduced my sentence. And that was exactly how I felt about it: that I was doing jail time and getting a reduction of my sentence. But I have to acknowledge that many more people suffer their jail sentence all the way to the end and thus suffer much more than me.

Thus, he finished high school at 16, not 18 as was common, even though his GPA was spotty. "University was not at all jail, it was already fun," he remarked however. Teachers, he pointed out, encouraged him in an unusual way: "They allowed me to part from them early, which is a significant point [laughter]".

Starting in a summer camp at the Weitzmann Institute, between Grades 11 and 12, he first met mathematicians: "You know people who do real math, really think about math, not like high school ... teachers that [just] translate the content of the book," as Ben said.

Up to the end of high school, the vast majority of students, he said, "could not care less, mathematics, or science in general, was not conversation topic." But, "the closer group to me, my own class, had no interest in mathematical conversation, yet they respected my interest in mathematics, science in general, or computers—" as he explained," at the time it was computers." However, the students above him and different classes did stigmatize him. However, he added, "There was an even nearer group, I did have a few friends that were scientifically-minded." In sixth grade, he participated in science clubs, taking him to Be'er Sheva, the nearest town to his own. "The very few that were close to me all have PhDs now, and by the way, that is quite unusual, in my small town. My class has the most PhDs than those before or after."

In university Ben said that he was not "exceptional in any way. I had lots of peers and to share mathematics with." Incidentally, his wife is also a mathematician, who he met in the first year of university. It was computers and mathematics that interested him. And computers, at the time, required travelling to the nearest town. And he wrote computer programs that were unusual for his age, beginning in Grade 7.

There was also an interest in electronics. "I learned little from my teachers. But they let me do my own thing.... The class would do their own thing and I would sit on the side and play." In school there was relatively little interaction with math, but they allowed me to participate outside of school. "Already in sixth grade," he pointed out, "I was sent to this computer enrichment club [at a local university], and then there was the summer camp, and the Israeli math Olympiad, which was not in school but at the time of school."

In university, people were very open—letting Ben take whatever classes he wanted. I queried, "What about Princeton?" He firmly remarked, "No, Princeton is a different story. I was done with classes ... and the expectation is that you start doing research. And research is a different thing, at different times, for different people, and in different moods." He went on, "There is a component of extreme frustration," he reflected. "Once it dawns on you that what you are going to do at the best is going to be extremely insignificant; it's going to be totally worthless at some level. Many of the people try, fail. Many people that succeed get a PhD, do not get a job. It's [thesis work is] going extremely slow; your advisor does not appreciate you, the usual."

Ben's supervisor was a significant figure, but there were many others. I probed, "do you have your own style." He replied, "Well, I definitely stole everything, but it was not just from him [my supervisor]." In any case, he noted, more generally, that no learning is "independent."

The teaching style, up to the high school level, was "irrelevant," said Ben. "The teaching style was happening outside of me; I was daydreaming inside of me." "If I got A's it was because it was easy. I can tell you what there was [in my classrooms] because I would occasionally peak out [from my daydreaming]. The teacher comes to board, homework, [and] the usual." "University was the usual, big lectures early on, smaller classes later on, with little student participation in the lower classes, a bit more in the higher classes," he added. "I spent most of my time daydreaming, but I did study for examinations."

At the university he began to have relationships with professors. Up to university, there were no teachers that were memorable to him, with the exception of one that was a master of chess; "without the chess, it would not be memorable.... In university it was completely different." There were perhaps 20 teachers that were memorable in university to Ben, being too long a list to start to go through. One example he gives is of an assistant professor without tenure—the first real mathematician he met from whom he learned "interesting math."

The high school books were read "hardly ... never dwelled upon." But at "university I started reading books.... These types of questions [Anoop is asking (see Chapter 3, "Methodology: The Case Study Approach," in "Interview Instrument")] sound like they are asked by the Ministry of Education.... But the books I am talking about are not written by the Ministry of [explicative], but live mathematicians; and there was nothing you can do about that. Some of it was good. Some of it was bad. Some of it was well-written and some not. I am not going to talk to you about what had wider margins or more pictures." So, in sum, up to high school it was, as he said, "a ten year jail sentence."

Ben tells this anecdote, when asked if his ideas about learning were developed independently. His wife's advisor, a very distinguished mathematician, said that one of his favourite activities is to write book reviews. "He has complete freedom to say what he likes," according to Ben. "Because he [his wife's supervisor] was asked to do the review, hardly anyone reviews his work. He can start with two pages with whatever philosophical rambling he wants, then he takes half page to connect it to what is to be reviewed, then half a page of review and his job is done. But he got the freedom to speak as he wants." This story, though sort of off-track, does reinforce Ben's penchant for freedom.

Having two kids, Ben can now reflect upon mathematics teaching and learning from the perspective of a parent. One of his kids is in Grade 11 and is quite happy and enjoys being near the university his father teaches in, attending his lectures occasionally. The Grade 8 child is more exceptional in mathematics than Ben was, by a margin; he is totally exempt from mathematics in school and his experience is much closer to his father's, according to Ben.

Even at university, Ben said,

> It was not that every course was great. There were lots of lousy courses and lousy professors. But now they were individuals and I still expect them to be that way. If high school was extremely boring it was because the syllabus was set by the Ministry of Education.... There is some room for the individuality of teachers, but it is so rigid. But at the university level, I still feel the class is much more the responsibility of the professor. I mean at the high school level the teacher is even told what words to use. At the university that is not the case.

Ben clarified that his experience is perhaps unique—professors tend to like top-level students. "So if you are a top-level student you are having a great time. On the whole if you are low-level student your experience is completely different." As an aside, and providing an analogy, he pointed out that his experience of the city he teaches in is completely different from the people that live in the suburbs. So it is a matter of perspective: "If you are a C-level student you see a rigid system that does not care about you ... well, I don't know what they [C-level students] see, I'd have to ask."

Once Ben was reading an advanced research article and knocked on a famous professor's door: "Do you have a minute, I have a question?" This met with, from the professor's side, "I am sorry, my office hours are.... Perhaps a TA could." Persisting, Ben pressed that he "only had a question about this particular sentence." When the professor noticed it was one of his own research articles, he pronounced, "Oh, I'm sorry," and went on to spend half an hour with him. "This still happens all the time," Ben observed.

Beauty is something that appears in Ben's description of his work repeatedly. For example, he has written about [emphasis mine]:

Beautiful techniques.

The big dream of categorizing all of quantum algebra is too big for me. I remain convinced that when fully drawn, *the picture will be beautiful.*

Behind every worthwhile argument there's *a beautiful picture* that fits together with everything else.

A *beautiful picture* always emerges.

And again, he spoke of the:

The beauty and depth of the theory.

So when asked about the role of beauty, Ben proclaims, "large," in his choice of mathematics and the problems he sought within it. "I tend to move away things that come out ugly," as he put it. "Large computations with no fun story." Like the epigram of this case indicates, Ben values "surprises" in his research.

Another feature of Ben's work is the persistent use of pictures. "I often think in pictures and like to tell my stories in pictures." As he put it, "The working mathematician fears complicated words but loves pictures and diagrams." In fact on his website he has an "Image Gallery" that includes pictures of knotted objects, symmetry, Jerusalem, Plants (like ferns), places, miscellaneous items, and symmetrical tiling. He conceded that the younger generation is more apt to use pictures because of their ease with technology, and that he probably uses more pictures than many mathematicians of his generation in his publications.

When faced with difficult problems in his field, he uses a variety of strategies, too many to comment on. As he observed: "If I knew how to solve my problems, I would solve them. I write [to] myself [see "Appendix I: Samples of Questions from Ben's Notebooks"], I draw pictures [see "Appendix G: Samples of Pictures from Ben's Notebooks"]. My notebooks are public, except when they relate to another person." In fact, as we know from Ben's notebooks, he engaged in experimentation, using trial-and-error (see "Appendix H: Samples of Experimentation from Ben's Notebooks").

When other students find mathematics difficult, Ben's response ran the gauntlet from "disrespect, to those people that do not try and have no effort, to admiration to those because of their ability or amazing ability to work." Even as a teacher, Ben harbours this range of feelings depending on the students. "In a class of 100 students, there will be a few who will be crooks, who never wanted to open their minds, and more hopefully some that genuinely want to learn, a few quiet ones you can ever understand [smirks], and a few loud ones. There is no single answer."

Ben emphasized, "I am still a student, at a professor status." Thinking to himself, he said, "Perhaps I should put a list somewhere of things I don't understand. I don't understand quantum mechanics, but no one does." Ben found equilibrium between his interests and needs.

> The practical need is to produce research. I cannot sit down all day and satisfy my own curiosity. Or maybe I can. But by definition, the fact that I have tenure proves I have not done that. You see, the system rejects people that are motivated by their own curiosity; it is called natural selection. Sorry, I have some cynical side. But it is

true. But if you only want to solve the big problems of the universe you are likely not to get tenure, because you are likely not to solve them.

The two most interesting questions in mathematics, according to Ben, are the foundations of physics questions, and the foundations of computer science questions, "and I [Ben] have nothing to do with them." Ben described his interests as centering around three points. First, there is the research on algebraic knot theory. Basically, he is interested in finding algebraic expressions for knots, their invariants; and similarities between such descriptions and quantum field theory.

Though he concedes that algebraic knot theory does not yet exist, his goal is to provide a unifying theory that would benefit algebra for the study of quantum groups and provide new methods for knot theorists. Further, and more speculatively, he believes there will be interesting connections found between knot theory and quantum field theory.

As Ben explained in his agenda for a first-year course on algebra he teaches thus: "To appreciate that the simplest is also the most fundamental." In focusing on the basics, like algebra, he insisted that number theory is "just related to everything," including knot theory. In fact, Leopold Kronecker (1823-1891), the German mathematician and logician, famously said , "God made the integers and all else is the work of man." Ben wrote, modifying this, "God created knots, all else in topology is the work of mortals."

Adding a bit more, as he explained his idea for knot theory,

1. A map taking knots to be algebraic entities; such a map may be useful to tell different knots apart.

2. A collection of rules of the general nature of "if two knots are related in such and

> such a way, their corresponding algebraic entities are related in such and such a way. Such rules may allow us, say, to tell how far a knot is from the unknot or how far two knots are from each other.

Second, and related to the first point, he is interested in knot invariants, which I will not elaborate on it because it is a technical idea that takes us beyond our purposes here.

Finally, he contributes to a website to integrate computer computation into an integral part of pure mathematics. He believes that "computers are extremely valuable tools for mathematical research. "So much of what we do is computable, and actually computing it very often leads to new conjectures and insights [in mathematics]," he said. "Thus, in my opinion, one of the biggest challenges in mathematics, perhaps the biggest challenge, is to turn computation from theory to practice.... And gradually we all need to learn to appreciate computational rigour."

As he pointed out, mathematics existed before computers. Parts of mathematics have nothing to do with computers and it "would be silly to program on a computer [some mathematics]." Ben explained,

> Yet another part of mathematics by definition is about computation. So, mathematics came because people wanted to compute certain things. You had a field three miles wide and you want to know [etc.].... So, mathematics has been about computation from the start ... and remains related to computation. However, because of various historical reasons, it is not computed. And a lot of the mathematical literature does not make the separation between what part is theory and what parts have practical value.... There is no tradition of computing what is computable.

Some of this separation between mathematics and computation, he explained, has to do with the nature of learning. "My teachers did not have computers, so they only computed what they could do by hand.... We tend to give the courses we took." "There is no infrastructure" he also said. "Or I am lying, there is some, but it is lacking. Some of the reason is sociological. You get credit for new things. Making the mathematical infrastructure for the mathematics developed 100 years ago will unlikely get you credit."

As Ben pointed out, the current programs for working on mathematics, like Mathematica™ and Maple are not adequate: "Being commercial, these programs are closed; we cannot inspect, verify, or modify their internals.... It's as if Cauchy and Legendre had the copyrights on ε and δ, and the rest of us had to pay them fees to use their notation and could use it only as prescribed by the original authors."

Clarifying the nature of his research, however, he said, "the most interesting questions in math are not the most interesting questions in the world, [for example] how to bridge the gap between rich and poor." According to Ben, the usefulness of mathematics gives rise to several cynical answers.

To begin with, he has pointed out that the most useful mathematics in everyday life is addition and subtraction, multiplication somewhat (e.g., "If I had 5 candles that cost $1.29 and then I...."). He interjected, for instance, that carpenters (e.g., "You have to build a three by five gadget, how much wood do you need?"), and craftsmen in general, use mathematics a lot more than him in daily life.

He observed that the typical person never uses powers in their life. He is unusual, he remarked, because he used powers to calculate his mortgage, when most people would consult a table. However, powers "I have only used twice in my life," he added.

He concluded, "Math is less and less useful the deeper you go. And that is a fact of life." He went on, then, smiling,

> The math you learn in elementary school is absolutely and truly useful. The math you learn in high school is rarely very useful; the math you learn in university is even less useful; and the math you learn in graduate school is totally useless. And the math you do research on, God knows why people are doing it.

However, he added, "To a large extent the math that is taught is relevant.... Calculus really is significant; I admit it is not as useful as addition and subtraction, but it is the next best thing."

The cynical answer, to why mathematics is important, is that "it pays my bills, the more I do the higher my salary becomes.... There is also some curiosity. I really want to know the answers to the questions I ask. That may be delusional. To work on something you have to convince yourself that it is valuable or that it is interesting, regardless of the truth. Maybe that is what makes a good scientist, to be able to cheat oneself into thinking that one's work is important. But I have cheated myself; I do think my work interesting." As he observed, "I am extremely good at convincing myself that the things I write I care about." He added, "Sometimes the answers are technical.... I want to know what happens if I add 16 to each side. Now I am making fun."

Between financial cynicism and pure curiosity, there is, according to Ben, cynicism at other levels. "I want to impress my peers." "I honestly do not know what is true, what is right or what's wrong [between the cynical and optimistic accounts of my motivation to do mathematics]."

Here is a most cynical answer Ben mustered at the social level:

> Science is a conspiracy, the way university professors convince the government that there is value in what we do and they should invest in that; that the future of the country depends on that. And if you don't do that [invest in math and science] the other side will do that and have bigger bombs.

In the context of the cold war, it was the Russians, Ben observed, now it is the Japanese and the Koreans; "they'll have faster computers." "To tell you the truth," Ben added, "the gap between the math done to build bigger bombs or faster computers is quite large. So maybe you can point to one or two percent of research that contributes to such causes, but the vast majority doesn't."

There is another side to this conspiracy, according to Ben: "It is a pyramid scheme. The professors are paid well and having good lives; and the students want to be there. It's a classical pyramid scheme. They pay our salaries because we promise them a dream."

There is truth to some cynical views, according to Ben, but "if we do not push forward [doing math research] we degenerate.... Basically, we have a small number of books [consequently] that becomes [like] a Bible, and we teach less and less well, and with less and less understanding." Further, according to Ben, "If every math teacher has not had contact with research, and that needs a huge infrastructure [i.e., teachers need exposure to people doing mathematics research, which requires a lot of mathematicians in universities], we will start losing what we have. It is not for me to put the weight between these views and the cynical views of before." There is also the case of "what may be useful 50 or 100 years from now." Ben concluded, "This is a relatively cheap expense, all this math research is less expense than sending a man to the moon, and it is not clear sending the man to the moon achieved anything more."

Ben's ideas about learning are not based on studying formal theories of learning: "I never had taken a teaching class, no formal training [in teaching]." "The only way I know something is if I add to it," Ben observed.

> The only way I understand something in mathematics, is if I do something further, something more than the textbook. The only way I ever learn something is by being creative. And this creativity can take many forms. Sometimes it can be writing a computer program, finding analogies, using it in another context, sometimes it is sitting for an hour and thinking about it, till a mental image of the subject appears, and that was not in the textbook. And it is very hard if you have not experienced it.... I will only understand it once I have my own image of it.

However, according to him, "This contradicts our system of rewarding for achievement.... Every week new material is covered and for a reason.... If we left time for students to reflect they would use it [reflection time] to go partying.... It is not something the university can give you, it is in you ... there is no way to encourage it [reflection], unless it is perhaps a graduate student you meet regularly."

"Most students do not know what it means to be creative," said Ben. "And I should say only I discovered it relatively late. The idea of doing more than reviewing what was in the blackboard came to me in second year of university." He goes on, "If I tell my students to reflect, it would be a 95% failure; as a student in the faculty of education, you [Anoop] would have to categorize it as a failure."

Ben said his ideas about learning have evolved, but there has never been a revolution. As he pointed out, "You know, there is a completely different teaching philosophy that goes, you memorize everything; and there is a very strong case in favour of it. Pick a book." I query, "Do you want me to get it?" "No." Looking around his office, I blurt out, "*Abstract Algebra.*" Ben quips, "No, that's too easy." So, I suggest "*Quantum Field and Strings.*" "Okay," he continued. He explained, then, that if he had memorized *Quantum Field and Strings*, he could think about it while riding the bus or seeing a cloud formation in the sky and it will remind him "of page 422." "The idea is that you will grow into understanding," he explained. "I know people who this [memorization] works. For me, I am weak at that. I am not good at memorizing, I add my creativity as a learner. For some people this [memorization] is the right way to learn."

So, said Ben, "I try to entertain my students, having given up on trying to educate them. Sorry, now you are seeing my cynical side. When you teach a class, there is an aspect of circus, and comedy. There is an aspect of being a stand-up comedy type. I am not saying I am good at that; most of the time I fail."

More seriously, in his "Teaching Philosophy," he aims to teach to a student like himself:

> I am a horrible student and horrible listener.... It's very difficult for me to pay attention ... if the purpose is not absolutely clear to me.... I assume there is no purpose and fall asleep. If the route to the goal seems too tedious, I'm quick to lose my patience. I sense right away if the speaker doesn't fully understand what they are

> talking about.... If the language is not clear, I don't listen. If her writing is not readable, I don't bother reading.... It all gets worse if I don't have sympathy for the speaker as a person; if I don't get the signal that (s)he really wants me to listen (ME; not just somebody else in the room). And I'm shallow—jokes, anecdotes and (well performed) technological slights of hand grab my attention and make me sympathetic ... finally, I forget everything I have heard very quickly unless there is a clear and meaningful moral to it and the logical structure leading to it is simple, motivated, clear and free of unnecessary complications. When I lecture or teach I try to satisfy myself, assuming I am in the audience.... I always try to tell my students about the big picture and where to find it (but only few of them take notice).

From these excerpts, we can glean what Ben values in teaching: transparency of purpose; expertise, clear language, spoken and written, humanity, and meaningfulness to larger contexts—the big picture. I inquired, "What would you like your students to achieve as a mathematics educator?" He said, "There is no single goal for every student." He conjectured, "Happiness!" He went on, "For some of them it would be career advancement, for some of them it would be to satisfy their curiosity, for some of them to do research like I am doing, but I really can't say."

I turned to ask about the relationship between this work and the socio-cultural context of his upbringing. Ben referred to the famous last lecture by Randy Pausch, a professor of computer science at Carnegie Mellon, diagnosed with cancer, who spoke about achieving his childhood dreams. Pausch spoke of how he wanted to visit Disney Land, and how he eventually did and was also able to contribute to that place professionally.

Ben's sister pointed out to him, he recalled, however, that there is something disturbing about his lecture because it makes you think that you should try to achieve your childhood dreams. "What about your dreams when you are 30?" Ben said. "In some sense it is childish to want to achieve your childhood dreams. When I was three, I wanted to be astronaut; now does it makes it sense I still want to risk my life for no purpose whatsoever." So he concluded that his upbringing is irrelevant to his current research. He hopes to achieve his research goals by "working hard, get lucky, and it would be helpful to have fewer administrative duties."

Ben described himself as a "philosophical extreme in math.... I do not believe what I can't calculate [see "Appendix J: An Example of Philosophical Remarks from Ben's Notebooks"]." As he pointed out, "There is a whole spectrum of disbelief." He remarked:

a) It is valuable if you can compute them [the numbers] quickly

b) Less valuable if you compute them less quickly

c) Much less valuable if you can theoretically compute them (but no computer could actually do so)

d) No value if you can never compute them

e) No [significant mathematical] value if they are formal consequences of a system

As he pointed out, relating to (e), Gödel's incompleteness theorem (i.e., the idea that any finite formal system like Peano Number Theory gives rise to statements that are not provable within in it) is not very interesting mathematics, but makes up for its mathematical value because of its philosophical value. He aligns himself, accordingly, with finitism and intuitionism, the idea, in slogan form and as he put it, is: "if you can program it, it is finite."

Case 2: The Pragmatic Mathematical Educator, Alfred

> "I would not let it [a mathematics problem] go till I knew why I was doing it.... Why does it work" ("Alfred," 2010).

Born in Southern Ontario, Canada, in the 1940s, Alfred returned, after his education, to teach in the city of his birth. Alfred holds a bachelor's degree in mathematics from a university in Ontario earned in the 1960s. He also holds a master's of education degree, specializing in mathematics education, from an American University.

He had pursued a master's degree in mathematics in the United States but changed into education because his primary goal was to be a teacher. Also, he had been admitted to several PhD programs in mathematics education, an achievement in its own right, but did not complete because his goal, as he realized, was to become a teacher—not a researcher.

With over 47 years teaching experience, Alfred has taught mathematics at every level in the school system, from elementary to secondary. Being in his late 60s, he is retired as a secondary school mathematics teacher. He is currently a sessional instructor in a faculty of education in Ontario, teaching mathematics methods.

In fact, by private arrangement, he still drops in to teach mathematics students where he used to be employed. And he participates as a tutor for students involved in the Ontario Mathematics Olympiad. He also lobbied the government successfully for the inclusion of teacher's input into curriculum design, and between 2000 and 2003, contributed to the Ontario Ministry of Education curriculum guides for mathematics, Grades 1-8, 9, as well as calculus. He has written several textbooks for math education, too. In addition, he was a program consultant for teachers in his school board focusing on mathematics education.

Alfred is the recipient of the Descartes Medal from the University of Waterloo for his contribution to mathematics education; past president of the Ontario Association of Mathematics Education (OAME), and now life-time member of that organization. He has presented, and continues to, at many academic conferences, like the National Council of Teaching Mathematics (NCTM); delivered over 300 workshops on teaching mathematics; and he is a previous editor of a mathematics education journal. I now turn to his case, drawn almost exclusively from my interview with him (which yielded 2 hours, 19 minutes, and 48 seconds of recorded raw data).

Knowing he wanted to be a teacher from early on, Alfred first pursued the idea of being a physical education teacher. However, finding that teaching physical education did not offer much intellectual challenge, he came to focus on teaching mathematics.

He identified himself as middle-class. His father was trained as a draftsman and later as estimator for a construction company. His mother was a floral designer. As an only child, he was the focus of attention; and "never did without," as he put it. He considered himself Canadian. He noted that one side of his family is Italian and the other English, but as he observed, "that goes far back." His parents were not church goers, and have changed denominations within the Christian religion; he is now Presbyterian, a sect of Protestantism. He regarded mathematics as not important in his socio-cultural background.

However, his parents were always supportive of what Alfred wanted to do. They knew he wanted to be a teacher and supported him. His father, Alfred noted, was more supportive than his mother, only because he was more able to be so because of his own education. Yet, Alfred never considered his interest in mathematics as related to his father's work as a draftsman. Alfred's wife is a mathematics teacher at the secondary school level, and he has four children, one daughter and three sons: his daughter teaches Grade 7; of the sons, one is in law and teaches Grade 5, one a chef, and another a university student. He also has two granddaughters, one aged five, and one aged seven.

As Alfred recalled, teachers were more custodial than humanistic in his student days. As he remarked, "I had a lot of crummy teachers.... I feel I was cheated by some of them." I suggested, "You are a survivor." He retorted, "Ya, despite them [I succeeded]." He even recalled, going to an extreme example of egregious behaviour that one of high school teachers would come to school drunk.

However, he has fond recollections of his Grade 11 math teacher that was, he said, "nice, liked kids. He [the teacher] liked being there [in the classroom]." Also, he has positive recollections of a math teacher when he was in Grade 13 that had, in his words, "a good reputation." At the university level, he recalled two mathematics teachers that stand out, in logic and in the theory of equations, respectively. "They were more human," he observed. "They interacted. You know in university they just lectured ... they [my good teachers] were very human. We used to go to the pub."

He felt he was not greatly affected by peers in his pursuit of a career as a mathematics educator. Being self-willed, he knew he wanted to be a teacher: "I would not let anyone get in the way of that," he observed. He described himself as the quickest kid doing the math at the elementary level in his classes, though there would be often some minor mistakes in his work. At the secondary level, his focus changed to sports, and consequently his mathematics marks were average because of his new interest. In fact, his first job was to teach mathematics and physical education. In university, his mathematics marks improved because he knew where he wanted to go; he wanted to be a math teacher. "I had a strong sense that I did not want to disappoint my parents," he observed.

In Grade 8, he remembered mathematics being interactive, as he said, "Socratic stuff." At the university level, it was mostly lectures and assignments. "There was no interaction, particularly," he reflected. Explaining his stint as a master's student in mathematics, he said he thought to himself: "What in the world am I doing here?" He recalls an interview in his master's program being asked, "Do you know this theorem, Stokes Theorem, or something, and I [Albert] thought, why should I remember that?" He subsequently shifted into education. He regarded himself as an average math student.

He does not remember teachers reacting negatively when students asked a silly question. Most of the textbooks he remembered had practice questions, with very few pictures. "Most textbooks were like this [pointing to my interview instruments]," he observed. "There were very few pictures."

He did as much mathematics as was necessary to be successful. At the university level, he started to find mathematics more interesting. But mathematics was an afterthought.

Aesthetics did not he said "in any way shape or form" have a role in his choosing to be math teacher. However, he noted that when he started to teach he began to see the beauty of math, as he said, for example, "like proofs." "Like when you see a student do a problem in a different way," he also asserted. I added, "surprise." He quipped, in agreement, "Uh-ha."

When mathematics problems were hard he sometimes talked to others. At the university level, he has a "vivid recollection," in relation to the epigram of this case, of when he decided he wanted to understand a mathematical problem, namely, what mathematics he needed to apply to a specific question. "I am not just going to apply rules," he explained. "I wanted to know why it [a proof] was going to happen."

He regarded numeracy as important to mathematics in our society, as well as patterning in algebra. According to him, calculus is important for a select group of people, namely people going on to higher education, like engineers. More importantly, at the secondary level, he explained:

> Students need to know how to use a calculator in appropriate ways. Because there are inappropriate ways to use it [a calculator]. I graduated from mathematics with no concept of the use of math. If we have no reason to do it. Let's not do it.

He thought that a mathematics education builds logical thinking and reasoning ability. We are training people to become better citizens, to be more aware. I pressed, "Do you think mathematics makes you smarter?" He added, "I would not say smarter, but more aware of the world around you."

Most of his ideas about learning are drawn internally, from what he wanted to do and from his experience. He cited no theories of learning as being influential. However, he said his learning is complex, due to the fact that he knows his field well already. In fact, I pointed out that, when he entered the teaching profession in the 1960s, in Ontario, we did not need a bachelor's of education degree. I queried that Alfred learned by teaching, by making mistakes. He concurred, saying that he considered that his ideas have been developed, to a large extent, independently by trial and error.

However, he attended many conferences and always found them very useful. "Either you find something new or two ideas that are not," he said. "I would always learn from seeing someone I respected."

He considered his idea of learning "right up there where they should be." He tries, that is, to teach mathematics in a positive way, basically, using active learning, constructivism, and being human. "You can't teach kids," he said, "without engaging them." He recalled that when he was teaching students that the sum of the odd numbers is a perfect square (e.g., $1 + 3 + 5 = 3^2$, $1 + 3 + 5 + 7 = 4^2$), one student asked what happens if we add the even numbers? "That is where you jump up and down," he added, enthusiastically.

> But most kids don't get to that point [of posing novel questions]. It was a learning moment. Some teachers without math background will say, 'We don't have time, and throw it [questions like this] away.' It gives kids ownership of the curriculum. We never give kids ownership of the curriculum.

He added, "It is so important to be well-liked. To know your students. If you know them, they will perform for you."

From these comments we get insight into Alfred's teaching philosophy. As he pointed out, at the primary-junior level, his goal was to make students feel comfortable, because they have little math background and tend to be, as he put it, "math phobic." At the secondary level, he emphasized: "don't tell them [students] anything." That is, he emphasized letting students discover mathematics for themselves, as opposed to lecturing them. "It is easy to present material, it is not easy to interact with kids," he observed.

For example, his strategy is to turn students' questions around. "I don't tell kids anything.... You are trying to encourage them [students] to see what they can do," he said.

> Let students show you what they can do. I don't want to be the sage on the stage. I want them to do it. I should be directing. As I tell them [my pre-service teachers], if you are going home tired, you are doing it wrong.

When I pressed, if it was possible to teach mathematics, all of it, in an engaged way, without lecturing, Alfred said we could do that at least 90% of the time—perhaps even all of it. I suggested the idea of the area of circle ($A = \pi r^2$) or circumference ($2\pi r$) would have to be taught using a lecture method. However, he showed me how the area of a circle could be taught breaking the circle into pie pieces arranged into a rectangular shape, that is, by allowing students to experiment—no lecturing is required.

"Where does it make sense to do this [mathematics]?" he asked. He still begins, in his teaching practice, with a context in which mathematics is applied. In fact, he thinks that mathematics textbooks should start with a situation where mathematics can be done. He gave the example of the recent 2010 earthquake in Haiti, and reference to the Richter scale; and insurance, all cases where mathematics can be applied and explored. "But there are things we do that we don't know why we do them," he observed. "When do we use fractions?.... We could eliminate transformation geometry and probability theory [based on the fact that, for instance, probability theory ends in Grade 8, unless you go into data management you never use it]." As he emphasized, "There is no application [to some of the mathematics we teach]." Pointedly, he said, in a different context, "I spend no time that will not help them [pre-service teachers] be better in the classroom. If it is not beneficial, I don't do it."

Alfred has always been focused on all students, at all grade levels. However, he teaches for the mean in any one classroom. As he said, it is not possible to teach every single kid. He pointed out we now have integrated students with a variety of levels into the classroom, which he was sceptical about. In any case, he observed that one teacher he thought was terrible; years later, a student identified that teacher as the best they ever had. Upon reflection, he said, "We all do the best we can." That is, according to him, different people will connect with different students.

He remarked that the best way to learn math is to teach it. He reflected that he once had to tutor a colleague in logic, and he got really good at it. As he observed, "He [my student] would ask me a lot of questions. I had to teach him. But I was always learning [teaching him]."

He believed that students should be able to choose how many practice questions they want to do. "There is no point doing 25 questions if you get it," as he said. Also, as he pointed out, no one person creates everything. Thus, it is important to learn from, and build upon other people's ideas. Of course, this is the value of going to conferences, and in the classroom, collaboration. He noted that he is a big supporter of collaboration in solving mathematics problems. So he wanted to build, in his teaching practice, not only a mathematical environment but a mathematical community.

At present, he noted, he is interested in the history of mathematics. As he said, "Kids need to know that this [mathematics] did not just drop down from the sky." Adding, "My favourite mathematician is [Basal] Pascal." I pointed out the story of Galois, the famous French mathematician that died in a joust over a girl. Alfred added, "Ya, he [Galois] wrote down all the mathematics he knew the night before he died; and he still lives on in infamy in functional analysis."

Given that he is at the end of his career, he did not identify any more problems that interest him in mathematics. He is interested in teaching teachers. And he still teaches mathematics in the secondary school system. As he pointed out, he feels that it gives him credibility as a teacher in a faculty of education by being in the school system. He is currently trying to write a Grade 7-8 activity booklet for teaching mathematics.

Case 3: The Joyous Artist, Molly

"I have loved art all my life. I paint because it is so much fun" ("Molly," 2009).

Molly has lived her entire life in a major metropolitan city in Canada where she was born. She was educated at the Ontario College of Art in the 1970s and, subsequently, at a large urban Teacher's college in Canada. She also holds a Bachelor of Arts degree from the University of British Columbia, which she earned over a decade later. She is a professional artist, working in oil, acrylic, and watercolour. She is a self-described educator; environmentalist; artist, continuing to have exhibitions of her work; and identifies herself with the city she lives in. Also, she has been elected president of the alumni association of a large college which focuses on art and design.

She is also an educator. Currently, she works as a Grade 7/8 ESL teacher in one of Canada's most culturally diverse middle schools, and she has been with her district school board for 32 years. Molly taught Kindergarten to Grade 10 and earned additional qualifications in special education and ESL. In fact, she is a team leader for a family of six schools with a focus on ESL; that is to say, she has consultant responsibilities for a major school board in Canada.

She has been awarded grants from TD Canada Trust Friends of the Environment Foundation to support work using art and nature to present environmental issues to refugee students new to Canada. She was nominated twice for the Canadian Network for Environmental Education and Communication award for excellence in environmental education. Her community involvements include involvement with Out of the Cold outreach program, which provides food to those in need; fundraising for the Red Door Family Shelter; and volunteering for the Children's Breakfast Club. Her case is largely drawn from my interview with her (which yielded 1 hour, 9 minutes, and 28 seconds recorded raw data), and her Web site, as well as information published about her in an educational magazine. I turn to consider her case next.

Molly's father died when she was eight years old. He was a musician, studying classical violin, but according to her, there was no direct influence on her art by her father because of his early death. Molly's mother raised her and a sister. Her sister is seven years older, not an artist, as Molly put it, "more athletic." Molly has a husband, a self-taught musician who plays guitar; five children, whom are musicians, but one has also gone into medical school. The family has a dog.

Her mother was a war bride from Wales, her father a Canadian solider. They came to Canada after the Second World War. She was raised in, what she described as "an Anglo-Saxon setting." I challenged her to consider if she would paint beyond the Canadian context, say Wales. Her mother, in fact, already told her "Wales is better [than Canada]." But she persisted in her interest in Canadian landscape. I interjected, "We don't have trifle in Canada [laughter, both of us]." "Well you can make it," she noted. "I just think that this country is so fabulous." I pressed, "So your roots hit rock bottom in Canada." She calmly replied, "Probably."

There was no extended family, due to the fact that all her relatives were in Wales, and there was not enough money to visit or for them to come to Canada. So she first met her relatives when she was in her twenties. When she got married, she went to Wales with her husband to meet her relatives.

"We were not aware of how little we had," she said, "certainly, less than middle class." She was raised in the Anglican Church. She identified herself as being from English stock, but her mother was "quite staunch about being Welsh." But because, according to Molly, she was from a lower class, she had the opportunity to be in a very mixed setting. "I had many friends," as she explained, "that were Jewish, or black, or from communities that were German, Italian, and Ukrainian."

She started taking art classes when ten years old. "I bugged my mother to get me art lessons. I saw that there was a lady, a friend of ours that was giving lessons," she recalled. At the elementary level, she had art on Friday afternoons, "my favourite day," she observed. "I loved anything to do with getting your hands gooped [sic] up." In high school her classes ran on a rotary system; and later on she just took as many art classes as possible. She won an award at the end of junior high school, which encouraged her to continue in art. In high school, there were opportunities to choose elective subjects and she consistently chose art classes.

When she was in Grade 1, she recalled the following story. She had a babysitter that had a daughter that was one year ahead of Molly in school, and (the babysitter was) very competitive. When Molly brought home her painting that filled the whole page, she got an A +; but the babysitter's daughter's painting was just a tiny drawing located in one corner of the page, and she got a C, or a lower grade. "The mom [the babysitter] immediately tried to justify that her daughter's was just as good as hers [Molly's], just smaller." She reflected, "Gee, I thought mine was pretty good, but hers was just as good; and the teacher liked mine because it was big. Not because it was good but because it was big. So she sort of demoted me." This anecdote perhaps provides a clue to Molly's recognition of some of the emotional aspects of artistic activity.

One teacher in particular, in Grades 7 and 8, encouraged her to work on sculptures. "I could see I was painting in 3D," she said. "I could recognize my style.... It belongs to you because there is something about yourself that goes into it. I remember having that awareness that we put our signature into what we do." The influence of sculpture made her think more carefully about perspective; the different vantage points an object could be represented. "I might look for a different pose [because of working with sculptures]," she remarked, for example.

She had art as a major in high school, going on to the Ontario College of Art (OCA). In fact, as she recalled, she was accepted into a major university's program in art history. However, her scholarship was given to a man, because of a stipulation relating to gender in the 1960s, specifically in connection with this award. Being denied the scholarship eventually led her to OCA, where she was awarded a scholarship.

At OCA, she specialized in drawing and painting, the fine arts. "I think art was something that gave me an identity. And I was good at it," she reflected. "At OCA, everyone was very good. I recall it was the first time I met people that were thinking about art. That excited me. That got me interested in art history at a much deeper level."

At the OCA, there were classes that were in the lecture and seminar formats, particularly art history and literature classes. She went on,

Primarily it [her fine arts program] was an applied program. So the assignments were something you would work on for an entire term. There would be a class in colour; there would be a class in drawing, costume drawing, figure drawing, and those kinds of things. And you could attend or you could do your work elsewhere.

Art was serious business at OCA, now being surrounded by artists.

I noted that I was hesitant to ask her about textbooks, because I thought the question perhaps better applied to mathematics. However, Molly said,

> I don't mind that question [about textbooks]. Terminology, and language, and being able to articulate what you are doing. Communication is huge. Right now, if I am submitting something to an art show, part of the requirement is to have an artist's statement.... Without a textbook it is hard to articulate those things. But they could be formulas in a textbook that get you going.

She went on, "And a lot of people learn from a textbook; they don't paint, but they get it. She studied with a colour expert at OCA that left an impression on her. "He was so brilliant and so gentle" she said, smiling. "The way he was able to analyze a painting and was able to help you find your way to the end of it."

There was not a great deal of homework assistance. As she said, "Basically, we were on our own. And I was a reasonably conscientious student." "I really felt I was being supported. If I wanted to buy, if I wanted materials for birthdays or Christmas gifts, canvases, that is something they [my mother was] was always happy to provide." She observed, "Family never inhibited me. They were open to the idea [of me doing art] but I was self-directed."

During "school I used art to make my way," she observed. "In drama class I would paint the scenery. If there ever was an opportunity to do something with a paintbrush, I was on-deck with that."

With the death of her father when she was young, Molly observed, she thinks her mother welcomed the fact that she could keep herself occupied with "a small pad of paper." "When one loses a parent at a young age, it does go away," she said. "He [my father] was ill for a long time. It does impact your life when you are small. And we are talking of the 1950s, so my mother was a single mother." "The balance goes out of the family," she added.

Her peers recognized that she had a little more ability than others. And others liked being around an artist. So she feels she was supported by them. She was impressed, when meeting artists, that "it could be a career. You could get up and draw [in the morning]." From friends, then, she learned that "art could be a business, you could sustain yourself. And [also] don't make excuses, just do it ... the more you do it, the more comfortable it feels. And it is good for you, it is therapeutic," she said. "Artists are very "generous, they do not usually feel threatened you are going to steal an idea."

Art education was useful for her. "I think that art can be used across the curriculum and can lighten things up," she said. "As a teacher, I can use drawing to link things.... It just adds another dimension to things." She likes her students to find new ways to express themselves, "leaving the technical part of it to the end. We do experimental work in the beginning. And there is an aspect of self-esteem building these." "I want them to develop the ability to bring it on whenever they want to."

She had written of her own artistic endeavours:

> My paintings [see "Appendix K: Samples of Molly's Artwork"] are about the culture that creates Canada and the atmosphere that seeps from our landscape. Its colour, weight, density, patterns, rhythms, and motion inspire my work. My paintings are intended to indentify specific elements in the natural world; some change quickly like shadows, others over centuries. I use colour and light to melt impressionism with realism so that the viewers may draw from their personal experiences with nature to complete the paintings.

As she added, she has always been interested in Canadian history, making reference to the Group of Seven. "I enjoy the knowledge that is attached to any subject," she commented. I pointed out that she had spoken, for instance, about her interest in art history. Molly mused, "That may be the educator side of me coming out. It is interesting to speak about something you are passionate about."

As she observed, the Jesuits that came to Canada were, in addition to missionaries, artists "that used their sketchbooks to record flora and fauna. Very quickly their religious paintings were influenced by the landscape. And if you look back at Canadian history, it very strikingly evolves, away from traditional art that represents figures of the Bible stories to immersion in the climate and environment of Canada.

As she contended, we use our senses a lot in Canada because of the changing of the seasons and the environment; and that has a bearing on the art produced here. "I wanted to know as much as I could, she observed. "I wanted to travel.... If you go north you see another type of art. So my attitude is the more experiences you can gather the more interesting your art can become." I queried that art changes not only based on place but time. Molly added, "For sure, there is a timeline. But I have to say art overcomes, and the landscape overcomes.... It will help you think in a more of a truthful way.... I absolutely feei it cleanses your mind to be involved with nature and to feel where the beauty comes from [i.e., nature]."

According to Molly, "The purpose of art is to express a view of something, beauty." She deemed "beauty" as "very important" in her choice to be an artist. "I am interested in having beauty around, colour, and being able to gather things that are lovely by nature," she said. "My feeling is that human beings want to be surrounded by beautiful things. It is just part of how we function."

"I realize art can also put in our face the ugliness in the world, and we have to look at it, yes, it's [the ugliness is] there" she also recognized. "But mine is going with what is beautiful, capitalizing on that." She explained, then, "There is joyousness to being able to create.... Some art you can look art and it wants to tell you the tragedies of life. For me, it is the opposite. It is the feeling of being alive." "Basically," she said, "I think art has given me that mindset ... to being open to beauty ... that there is always beauty, it is not always apparent but it is there."

When confronting a difficult problem, she will on occasion walk away. However, "Many of my canvases are layered. I don't have a problem starting over.... I am not afraid of it [the art] not being good. If I don't like it, I'll change it to something else." "I like to work on my own, but I will bring in other people, even though I may not take their advice," she said. "I'll ask is it too big, is it too small?" When I asked whether she ever poses questions to herself, Molly quipped, "Ya, after a few glasses of wine."

Explaining her motivation to paint, and expanding on the epigram for this case, she said, "I often feel a wave of energy take over my time and I quickly pay attention to the instinctive desire to create. I always have a canvas, brushes and paint ready no matter what time of the day or night." "Being part of a community of artists is also extremely important to me" she said. "It's the like-mindedness. They [artists] may have a different point of view. And there is no age block [between artists] ... there is a process where you need to learn some of the technical things. But basically the creative spirit rules."

According to Molly, there is "snobbery" in art, and she feels that interferes with "how people look at and value art, not mine, but I know that it is out there." Molly thought that most students, who struggle in art class, do so because they are afraid of getting it wrong. "They are looking for approval" she said. "If you give that up and say what I am doing is fun for me [some problems disappear]. And at the end of the day, if you have ten paintings, one will have value to someone else." She added, "It surprises me that the ones [my artworks] people like, I did not think much of."

"You were at some point able to paint more for yourself," I queried. "It is true [I paint to please myself]," Molly added quickly. She changed styles recently. "I get not bored, but I like to be energized by a new topic" she observed. "I actually require a change, to recalibrate myself." She has increased her use of the computer, for instance, which marks a change in learning style from the paintbrush and canvas.

In one case, she was looking at gravitational waves, and did a series of paintings based on a graph, leading to six canvases, "I love them, I talk about them all the time, I have got gravitational waves, behaviourism, the theory of group selection, the big bang theory, I got so wrapped up in science [as an inspiration for this series of paintings]."

In today's world, Molly thinks one cannot undervalue art in a school situation. "Everyone is capable of doing something and appreciating something," according to her. "We could do a lot to decorate the subway ... the advertisements have a lot of psychological impact and are done by art designers." "My understanding is," she said, "lower income people are more likely seeing art as something you would own and have in your house, I mean we [the affluent] have prints, or an art decorator would choose a piece of art for you because it matches the scheme [of your dwelling]," as she noted, "which is different than living with a piece of art." I asked, then, "So you think art could be more integrated into our lives without compromising art?" "That is a fair statement," she replied.

Technology, she contended, "is just another way of going [doing art]. When we painted with egg tempera in the 1500s, and then oil came along, they [artists] did not just throw away egg tempera. Now we have another [medium].... Younger people are very capable of working electronically." So I asked if she sees art as transcending the computer programs used to create it? "Yes," she affirmed.

A challenge she is working with now is helping artists become technology-based, so they can market themselves. And it is a challenge to help people find careers. "Lots of people end up graduating with an art background, but they have to do something to support them; the art may become a hobby."

The recent series she produced is called *Angles in the Air*, because she said, "there is good in everyone." I noted, now there is a "religious theme in her recent work." "I am spiritual," she remarked. "I guess everyone says that. I like the tradition. I know the Bible, all the stories ... but not so much organized religion."

Dwelling on the change from landscape to people, I prodded her to talk further. "I was always good at figure drawing," she said. "I wanted to use the gesture [in figure drawings] and introduce the colour. And a few [of the paintings] took off ... in the last few years." Lately, she has been doing action-oriented portraitures, "related to movement and related to colours." "But you can take figures and think of them in terms of landscape," Molly added.

She explained the sizes of her artwork are based on practical matters: "I am out in a canoe, or out hiking, or out somewhere. And the sketching is about being in that moment at that place. In a studio I can choose to have a much larger canvas. I like the freshness of things that are done on sight." I queried, so the atmosphere was important?" "Absolutely," Molly stressed.

There are no real influences of theories of learning in her style, according to her. "I know how not to learn" she said, however. She explained the way to learn thus:

> Not listening to other people, who limit you. [For example] if they [other people] tell you they don't care for something.... You want to be independent but aware of where you want to go, or what you want to do with it [your art]. If you are interested

in selling, then you can paint something on a formula basis that is strictly perhaps a money-making vehicle. But if you are interested in painting and having a great time you don't think of money.

I asked if Molly thought trying to make money could lead to great art. "I think you can produce art people will enjoy," she remarked, thoughtfully. "Whether it is great art is a matter of time. It has to survive a long, long time."

According to her, she can learn from a book, a class, but remarked:

[M]ore successfully by doing.... You don't need to take a class with expensive equipment.... Get a book from the library and practice. And with anything else, with time you can let the book go.... Well, I am self-motivated. I would try to encourage that [self-motivation] in other people. Don't wait for someone to help you. You can problem solve with a library book.

She thinks people learn because they think it will be fun, combined with a social element. Molly says, "Learning in group situations where people support each other [is also very good]." "For a show [art exhibition], there is a social component, your friends come out," she remarked, for example, "and you know, you have your ego totally messaged for a while and then on you go." "So there is an egocentric aspect to artistic production," I asked. "I think it is a positive, why not!" she asserted.

She only "wishes she could make more time for art." Because of her busy schedule as a mother, wife, artist, educator, and dog owner, she strives to make time for herself. As she put it,

I would like to just keep doing what I am doing. And help other artists, whether it is

> inspirational, or whether it is to help them get financing, there are Canada Council Grants.... I am putting myself in a position where I have access to information and so people will trust me with things. So I can use my art and identity to offer assistance to other people.

She expresses and amplifies her joyousness through art and "helping other people."

Citing Nelson Mandela, she advised, "Don't be small. God did not give you talent so you could hide your light.... You can't waste your gifts; you can't waste your time." She wants to find new ways to make herself available to help others. She likes the idea of there being actions and consequences related to work, for instance, participating in the administration of a major art school. "Walk the talk," she counselled. "I can give you an example. Last night I was here [at my college of art and design] to thank artists that had donated a piece of art.... That's fun and a direct result of having donated a piece of art."

Case 4: The Caring Visual Arts Educator, Zöe

> "Art is in itself a very powerful means of communication. It transcends the need for language" ("Zöe," 2009).

Born in Faja de Baixo, located on a small Island, Sao Miguel, in the Azores, Zöe's mother tongue was Portuguese, the language of her land. She holds a Bachelor of Arts degree, with a major in the visual arts and a minor in French; a Bachelor of Education degree; and some additional qualifications, all of which are from Canadian universities.

She has taught primarily at three different schools over the course of 24 years (about 400 students a year), ranging from Grades 1-8. Presently, she teaches Grades 7 and 8, with a focus on the visual arts, French, the language arts, and math. In addition, she has taught visual arts methods in a faculty of education in Ontario. Further, she has taught at art camps, as well as related evening classes for ages 8-15, as well as adults.

She has been nominated for the Michael J. Farrell Award for contribution to the promotion of arts in her community, presented workshops on teaching the visual arts, and won a design contest for a logo for the Canadian Federation of Teachers. Zöe is perhaps best described as an amateur artist (i.e., she does not, as of yet, generally sell her art works). She works in oil, acrylic, and sculpture. Her paintings run the gamut, including organic abstract, portraitures, floral scenes, and landscapes.

Taking additional qualifications, at various Canadian universities in French and special education, it is interesting to note that she speaks four languages fluently: Portuguese, Spanish, French, English, and some Italian. Her case is largely drawn from my interview with her (which yielded 1 hours, 27 minutes, and 45 seconds of recorded raw data), supplemented by her curriculum vitae, course syllabi, and one previous interview conducted by a journalist for a magazine, all of which she was kind enough to provide me with before our meeting. I turn to consider her case next.

Zöe has two older brothers, one by four years and one by two, the latter had some interest in art. She described her two brothers as her "best friends ... my roots." Her mother was a homemaker. Her father was an artist, and musician, playing violin, guitar, mandolin, and many other stringed instruments. He was a businessman who owned a shoe store where he built his own shoes; he was also a "philosopher," said Zöe, "talking long hours with the town priest and friends about philosophy and religion;" and a smoker.

Her father died of an illness two weeks before her thirteenth birthday, prompting her and her oldest brother and mother to immigrate to Canada in the 1970s. Also involved in the decision to migrate was political unrest in the Azores; specifically, due to the conscription which was common in Europe at the time, her mother did not want her youngest son, about to be 18, to join the army.

She identified herself as middle-class, although with the death of her father, her mother had to work hard in a factory to make ends meet. She is Roman Catholic, but not practicing. Her cultural background, as she put it, was "very European, very Victorian, hypocritical ... everything was about what other people think ... so worried about the community." She has a husband, who is not an artist; a daughter, who is artistically inclined, and a son who is interested in computers.

According to her, art had little value in her socio-cultural context. She remembered going to an art Gallery in the Azores, when young, which was mostly folk art. She found it fascinating. Also, she recalled that when she was 13, her brother's girlfriend did a painting, and she was very impressed, "awed by the talent."

When I queried whether her father was the main influence on her interest in art. "Absolutely," she said. "He [my father] encouraged me to draw." She went on, "I think I wanted to be like him. And he and I were awful lot alike, kindred souls." She also drew with her oldest brother, on occasion.

However, there were not many art classes before high school. At this time, her teachers, she recalled, influenced her on what not to do: "Hit a child for getting the wrong answer," she said. In fact, a Grade 10 guidance counsellor had given her an intelligence test, an oral test administered in English (not her mother tongue) and basically told her she was dull—and that she should get married and have some children. She went home and cried. This led her to quit school (which you could do at 15 years of age in her community), only coming back through correspondence classes. She wondered, "How many kids did he do this to?"

Most of her education, then, was in Canada. But it was not an easy transition. She observed,

> Here I am, strange, curly-haired. So let's pick on [Zöe] and see what she says ... that was a time when you really need to be accepted. You feel lost. The kids are picking on you and the teachers don't make you feel comfortable. And you sit in the back of the room and you just get lost in the cracks.

So there were linguistic and cultural challenges in her new setting. "I never felt I belonged here [in Canada]," she explained. "[I] had a terrible time in high school. Was bullied, mocked—strange kid, strange accent ... just strange. Self-esteem disappeared. I spent most of my time drawing." "To this day, I am still a loner by choice," she said.

She wanted to keep proving to herself she could do art, however. "There is no greater competitor than your own self," as she put it. Most of the art was done on her own. In high school, in Grade 10, she had a very supportive, "kind" art teacher; it was not so much the technical training he provided, say, in drawing, but just the opportunity to do art. In fact, she recalled, he helped her get a job working with stained glass. She commented,

> And that really built my confidence. He [my Grade 10 art teacher] trusted me. He thought I was good enough to be the one recommended for this job. Oh, I think that was the first door that opened for me. Someone that believed in me. Someone that

> respected me that I had some sort of talent.

This gave her confidence.

At the university level, the professors were "kind" and she liked that, and remembers at least a dozen of them. According to her, each was good in their way, but all passionate about their work. In her previous culture, teachers were held up [on a pedestal] and intimidating. However, at the university level, professors were like, as she said, "friends." So, she remarked,

> The fear was gone ... the fact I always felt shunned and I did not want to have that feeling again. I felt welcomed, and I felt wanted, and I felt important.... I felt like I belonged...and I had rights as a student, and I did not feel I had that before.

The university environment was "safe, secure, and comfortable," she noted. "And that is how I teach." "You can't teach anyone till the first three needs [of safety, security, and comfort] are met.... I live by that." In fact, at the front of her classroom are the words "Accept, Respect, Tolerate." She advised, "You don't have to walk hand in hand, skipping down the hall. But, accept, respect, and tolerate [each other]."

Teachers, in art class, almost always treated questions with respect. She felt she could learn by herself. With textbooks concerning the visual arts she mostly liked the pictures, and would focus on them.

Her early experiences in school were traditional, teacher-centered (i.e., the idea that students are like empty vessels to be filled with knowledge), and authoritarian. For instance, the teacher could smack her desk with the ruler or apply knuckles to her head. Even in art class there was not a great deal of moving around, she said. Even later when she was allowed to move around, she still stayed put out of habit and thinking it respectful.

At the university level, however, school was "free and open and wonderful and student-oriented (i.e., the idea that who students are impacts curricula and its delivery). There was no one in charge. Of course the professor was there. But it was all fun and camaraderie. I thought it was lovely. What a difference [from earlier school experiences]."

"I have come to the conclusion," she said, "that it [artistic production] is 5% skill and 95% passion. It truly is.... Everyone can be an artist. I mean being a great artist depends upon what you bring to it. It is a lot like math. I do use a lot of math examples. You learn what a horizon line is. You learn what a vanishing line is." She observed that once students feel there is a method to producing art they feel very confident. "It is all formulas. You learn how to mix colours, primary colours, secondary colours, you get a new colour," she said.

According to her, also, art is therapeutic. "You are lost in the moment," she observed. "You are one with the piece of paper and the pencil or brush or whatever. And it is amazing to see what is coming out of your body." "It's like meditation or yoga," she noted.

Echoing the epigram for this case, "Art is a language on its own," she remarked. "One does not need to know different languages to understand art. There is a message in itself; in the visual.... It is an international, universal language." She went on to explain, "It is not about the language; it's about the people. It [language] is a way of thinking." In fact, the importance of inclusion as a perennial theme in her artworks and teaching practice is epitomized by her design of a logo for the Canadian Teacher's Federation, which depicts a diversity of people from different social, ethnic, and learning backgrounds within an apple, which symbolizes the potential for growth (see "Appendix L: Sample of Zöe's Artwork").

She considered beauty as very important in choosing to pursue a life surrounded by art. As she noted, growing up in the Azores, she comes from a beautiful place. "Colours are very important to me," she said. "I need aesthetically pleasing things around me. Look around me. I could not function in a little tiny room with bare walls, windows. I need pretty things around me [at her classroom wall, which are plastered with artworks]. I need visuals." In her artworks the message is very important, as she put it, "the guiding idea. Every viewer will see something different. But there is some emotion I can bring out of you [see "Appendix L: Sample of Zöe's Artwork"]." Specifically, she added, "It [beauty] is the most important thing."

When problems were difficult, she "kept doing it over and over. Persistence. Practice." As she said, "It's like sports. Like music. Practice ... [for example] I tell my students, you can dress me in the best hockey gear, the most expensive in the world. [She noted she teaches in a town where hockey is important to many students].... But do you think I'll be able to put the puck in the net? It is the fact I have not had any practice. But if someone shows you." However, "I [alone] can't show you the correct procedure. You have to practice [too]."

According to Zöe, "Ideally, I would love nothing better than to have my students develop a real passion for the arts, continue practicing and growing with their skill and become artists themselves. Realistically we all know that is not the case." But, according to her, everyone will become an art appreciator. So she sees her role to instil the knowledge of art history, about the different styles. "Adults should be able to appreciate the difference, and like certain art over others," she said. "They should be able to explain the reason why. Why there is impressionism. Realism. Monet, Degas [both impressionistic painters] ... or whatever." So there are many interdisciplinary connections in her teaching of art. "Kids need to be well-rounded," she asserted. "They [courses] are all connected.

"I think it is crucial to see if it is something they want, need, to pursue." I think "art is for personal growth, personal satisfaction ... If you are incredibly strong ... it [being an artist] is hard work and perseverance. But most for kids I teach I just want them to find out what they are capable of, for growth, for personal satisfaction."

"I think it all depends upon your confidence, or lack of; your self-esteem, or lack of; your pushing yourself to see how far you can go. Having high expectations of yourself." I pressed, "So you think that more self-confidence would result in more willingness to experiment?" "It all comes down to confidence," she explained. "And that is my number one priority with my students. Once they have that, they can achieve amazing things. But if they feel [I can't do it] ... get over that poor little me nonsense. You can draw a straight line."

She quoted the adage, "You don't have to feel safe to feel unafraid." As she explained, if you have confidence you will take risks. "Those self-help books," she added, "positive thinking, that is true. Positive thinking will take you places." I noticed on her classroom wall this quotation from Albert Einstein, which speaks to her idea: "Those who have not made a mistake have never tried anything new." She explained, "I keep telling kids, there is no such thing as mistakes ... only opportunities for improvement. If you don't like it start over again." She went on, "It bothers me [when they students say 'I can't do it'].

They learned that somewhere. [Paraphrasing Picasso] Every child thinks they are an artist. The problem is remaining an artist when you grow.... Someone or something makes them feeling negative. It's the doubt. My job is to take that doubt away."

I asked however, where, if at all, criticism comes in. She noted that confidence and self-esteem are important. "You can't be telling them everything is wonderful when it is not." Art, she said, firmly, "is not one of those filler subjects." And I add, "Tough." "Absolutely," she agreed. She agreed that criticism is important and part and parcel of having high expectations of students. She sighted the theory of multiple intelligences as an "eye opener," the idea that there are different ways to learn. The curriculum guidelines have also been helpful, providing a logical structure. Also the workshops [she has given]. She noted that several teachers delivering the same subject all were doing something different. She commented, "You are going to have your own style. You are going to do things the way you feel comfortable." "I'm funny, I'm goofy. I better make it [teaching] interesting." Also, she said, "You can't teach well if you are not confident about what you teach."

"Everything I have learned, I have learned in the classroom [as a teacher]. I have learned about me [for example]," she said. And she went on,

> All of them [my students] compete with themselves. They want to please me. They want to get a good mark. And they want to prove to themselves they can do it. Most of them will rise [to my expectations].... It goes back to being comfortable and safe. If you feel comfortable and safe you can tackle the more challenging aspects [of your work].

So, once again, the psychological aspects of learning loomed large in her thought about teaching and learning.

On the negative side, she wished she had been more formal in her teaching and covered more of her teaching. On the positive side, however, she is proud of the thousands of students she has taught. "I have really built a good reputation in the community. Not my own, the school's. It has nothing to do with me. I am just the vessel." She added, "I am very proud of what I do." However, she said, "What I do and who I am are separate." When I asked for clarification, she said, "Well, deep down we are all that frightened little kid inside. But I am secure, I am confident [in what I do as a teacher]."

"Overall, I think I am very kind, accepting, respecting of others," she said. As she reflected, given the difficulties she has had adjusting to Canada, having been "ripped," in her words, from her home in the Azores, and to be able to "succeed [in Canada]," she believed have shaped her teaching philosophy in the visual arts. She used to think only the elite could go into higher education. But now she thinks we all can. I interject, "and luck." "No," she said firmly. "There is no luck. It is what you make of it [opportunities in life and academics]." "I would like to believe I am providing this [self-esteem increasing] opportunity to my students" she said of her teaching philosophy, and went on to explain. "Someone once said, '20 years from now, your students will not remember what you taught them, but they will always remember how you made them feel.' That's ultimately my goal."

Looking ahead, she would like to write a book, her memoirs, focusing on how self-esteem and confidence is the key to success. She has had local art exhibitions, so feels she has "accomplished that," as she said. She is also teaching visual arts methods to teachers.

Zöe's very difficult experiences, of not fitting in as an immigrant, have resulted in a positive outcome—her teaching philosophy that is aimed at making students feel like they belong. "I am absolutely happy. I love what I do. I am able to share my passion with students. I am able to create beautiful things everyday.... I adore this profession. It is the most rewarding profession, being an art teacher," she observed. "I believe every child can learn.... I believe in the universality of art. I believe that everyone can feel [the beauty of art]. At my age, I still love what I do." I added, "I hope I am where you are [when a veteran in my field]." She replied, "You will be, it is all in the attitude."

5. Discussion: Solidifying the Common Link

"Possibly, the connection between mathematics and art lies in the idea of a common culture" (Holt, 1971, p. 15).

In this chapter, the connections between algebra and the visual arts are discussed around several themes that have occupied us throughout this study, which are made plain, in Table 4, "The Themes of Analysis in This Study" (below), and reiterated in "Concluding Remarks on the Data Interpretation." Subsequently, the limitations of this thesis are outlined. Lastly, the philosophical and pedagogical significance of the study are discussed.

Data Interpretation: Analysis of the Four Cases

Table 4.

The Themes of Analysis in This Study

General Themes	**Specific Themes**
I. The Personal Dimension The personal dimension includes what goes on at the semantic level and processes involved therein.	**A. Aesthetics** This concerns the sense of beauty and its role in judgment.

	B. Generalizing and specifying *Generalizing* is the inductive process of reasoning from experience with similar instances, *a,* to conceptual stereotypes, *x*. Inversely, *specifying* is the use of a broadly deductive process or applying knowledge of types, *x,* to specific cases, *a*.
	C. Cognitive and affective domains This includes such things as problem solving strategies; and emotions, like surprise, curiosity, puzzlement, fulfillment, and joy.
	D. Biological-based dispositions and talents Aspects of thought and behaviour that could be possibly considered the result of physiology, like genetics, providing some limits to human possibilities.
II. The Environmental Dimension The environmental dimension includes what goes on at the semantic level in relation to other people.	**A. Culture** The "supra-social" (Bishop, 1988, p. 14), or relations that may go beyond, and shape, the ones we actually encounter.

	B. Family and friends Included here is the early family setting of guardians and siblings, if applicable, and tangentially the participants' present family setting. In addition, *peers* are those that acted in the capacity of age- or level-based associates.
	C. Teachers and school experiences Included here are teachers in institutional settings as well as tutors. *School experiences* are those that occur through institutions, be they traditional school settings, camps, or institutes.
III. The Behavioural Dimension Behaviour impacts the personal and social dimensions, and may affect what we will do in the future.	**A. Interactions** Behaviour potentially impacts our cognition (e.g., our values), socio-cultural context (e.g., spawning social change), and could result in its automation and consequent increased likelihood of reoccurrence.

As pointed out in Chapter 3 "Methodology: The Case Study Approach," in "Participants," two sources were used to generate the themes of the ensuing data analysis. The general themes (Table 4, Column 1) were drawn from Bandara's (1986,1997) social cognitive theory; however, for the purposes of my analysis. Some specific themes (Table 4, Column 2) emerged from the literature review (see Chapter 2, "Literature Review: Identifying the Common Link," in "Rationale"). Some sub-themes like *surprise* were discussed in the ensuing data analysis (in "Aesthetics" and "Cognitive and Affective Domains"), or *technology* (in "Culture"). With the themes at hand from Table 4 "The Themes of Social Cognitive Analysis in This Study," we now turn to the data analysis.

The Personal Dimension

Aesthetics. Writing about the connection between mathematics and beauty, Holt (1971) and Sinclair (2006), interpreting the work of Dewey (1938/1997), have observed that in both areas, practitioners enjoyed a sense of fit. Recall, in flow experiences, which seems to typify the aesthetic experience, Csikszentmihalyi and Robinson (1990) pointed out that there is an autotelic experience where people do the said activity for its own reward. Also, according to them, like other types of flow experiences, the aesthetics experience is personally meaningful, and becomes more complex with time. We witness all these aspects—a sense of fit, flow, being self-motivated (i.e., working for the pleasure involved in the task, not an external reward; discussed further in "Culture"), and increasing levels of creative sophistication—when we consider the role of aesthetics in the participants' chosen field.

For Ben, the first participant interviewed and a mathematician, aesthetics played a "large" role. He found beauty in the connection of mathematical ideas, or their application to the world around him. In fact, he had, just as Hardy (1967) before him, extolled the uselessness of pure mathematics—it was largely about beauty.

Ben revels in moments of "surprise," when he discovers something. Intellectual delight was the main driver behind his choice of career and the problems he pursues within his field. Based on even a cursory look at Ben's notebooks, we can see that he must be quite immersed in his mathematics (see Appendices G-J). However, according to him, he came to creativity quite late in his training, about the second year of university. So, for Ben, we have a probable example of his mathematical aesthetic sense becoming more complex and personally meaningful over time.

Molly, the second participant interviewed and a visual artist, is much like Ben when we consider the importance of aesthetics. For Molly, art was all about "expressing beauty," which she conjectured came from nature. Her art was a tribute to the natural world in Canada, as she persistently painted landscapes, only recently beginning to experiment with portraitures. Art offered moments of immersion in her work, a connection to the land, and a sense of belonging to it. Recall, for Molly, her art was an on-going expression of joy. Once again, as Molly progressed as an artist, she became more sophisticated. The art became more meaningful, reaching the level of connoisseur of her own work and approach to it.

Being more pragmatic than Ben or Molly, Alfred, the third participant interviewed and a math educator, had categorically denied aesthetics had any role in his desire to pursue a career in mathematics education. Negatively, he came to mathematics education after not finding enough challenge in physical education. Positively, and upon reflection, he did see beauty in mathematics—but this was when he became a math teacher.

Coming from the Azores, Zöe, the final participant interviewed and a visual arts educator, noted that beauty was always important to her. For her, beauty was, in addition to the idea she was trying to convey in a piece of art, “the most important thing.” Like Alfred, who had focused on teaching, she was driven by more practical considerations and different goals—like teaching.

But what both Alfred and Molly want their students to achieve also speaks to aesthetics. Alfred wanted them to investigate mathematical ideas independently to have “ownership,” of the curriculum, as he put it. And it is a precisely aesthetics experience that likely captures that inner motivation that contributes to life-long learning and a sense of self-attainment. In fact, Ben seems like the type of self-motivated math student Alfred would have been proud to have mentored.

Even though, upon first blush, aesthetics would seem more the purview of visual artists like Molly than mathematicians like Ben, psychologically, it played a crucial role, based upon the introspective reports I drew upon, for both the math and visual arts groups, and was at least recognized by Alfred as a facet of mathematical activity, if not occupying a central role in his development.

Generalizing and specifying. Kaput, Blanton, and Moreno (2008) have argued that generalizing is how we create a symbolized object, like a number concept. Specifying directly takes us in the opposite direction: we apply the said symbolic objects to specific cases (“Early Math Strategy,” 2008; “Leading Math Success,” 2004). It is natural to consider both processes together because they are involved in the very nature of algebra (Kaput, 1987a, 1987b, 2008; Kaput, Blanton, & Moreno, 2008; Krshner & Awtry, 2004; Mason, 2008; Shoenfeld, 2008; Smith & Thompson, 2008).

We wish to note, however, that these processes can occur in two ways: We can generalize and specify in relation to physical objects; and we can generalize and specify about mathematical objects, for example, categorizing different types of equations. Indeed, we can categorize different types of numbers depending upon how the five laws of algebra discussed in the "Introduction" apply to them (these are sometimes called the algebraic numbers), and for which types there are exceptions, for example, real numbers (e.g., $\sqrt{2}$, π, e), complex numbers (e.g., $\sqrt{2}$ + Oi), or imaginary numbers (e.g., $\sqrt{-1}$).

Ben's research begins at the symbolic level, with a good deal of advanced mathematics. However, even here, with the tools of abstract algebra in hand, he aimed to come up with generalizations about the nature of different types of knots. That is, he is both generalizing, seeking similarities between knots, and specifying, applying algebra to physical objects. He also is interested in the categorization of quantum algebra, which is an example of generalizing about symbolic objects, namely mathematical objects.

Alfred's pedagogical strategy is to motivate the type of investigative behaviour that Ben epitomizes. He does not want to "tell them [students] anything." That is, he wants students to discover the rules of mathematics for themselves through experimentation, exploration, collaboration, problems solving, and so on; students should come up with their own generalizations. Of course, once students have understood certain procedures, they can apply them to a number of different cases.

Generalizing and specifying take on a specific form in the visual arts. Aiken (1998) argued that artists rely upon the way we react to certain colours, shapes, configurations, and indeed images, to evoke responses in their audiences. Also, Korzenick (1977) had noted that in viewing artworks, we engage in distancing, that is, we take the role of the other, all the while realizing the work is not real. In fact all of our trained responses are themselves generalizations, whether innate or learned. Knowledge of these general meanings (e.g., of a red, dilapidated barn) is applied to specific contexts, in this or that painting, with a particular audience in mind to elicit certain effects.

Molly created depictions of the Canadian landscape. There were certain generalized features of the Canadian landscape that came to typify her style, and indeed the mood of her paintings; we can see the influence of the Group of Seven on her work, which she referred to (see "Appendix K: Some Samples of Molly's Artworks"). Her artworks, we can assume, evoke special meaning for those familiar with the Canadian landscape or the art history of Canada. So her work is context-dependent both in terms of the location and time of the work, and the audience it speaks to. And it is reasonable to think that the more she generalizes and specifies, as her experience grows, the more complex her approach to her work becomes.

Also, generalizing and specifying behaviour probably operate on a micro-level related to learning how to wield a paint brush, mix paints, prepare a canvas, or depict perspective, lighting, texture, and so on. However, I have not endeavoured to study behaviour in detail here because it takes us away from the broader socio-cultural themes that are my focus, which I discuss in "The Social Dimension." In any case, and to exemplify the role of generalizing and specifying as it relates to specific skills, Molly recalled that studying with a sculptor improved her sense of perspective in her paintings.

We find the use of generalizing and specifying behaviour in Zöe's work, but with more of a focus on ideas than in Molly's. Zöe used an apple as an image of growth in her work of the Canadian Teacher's Federation (see "Appendix L: Sample of Zöe's Artwork"); iconic symbols, like a woman in a wheelchair to indicate inclusion of people with different abilities; as well as people of different skin tone to indicate the diversity of the human community. That is, she relied on our knowledge of certain stereotypes and our emotional associations with them to convey her overriding theme of inclusion, acceptance, and growth.

Generalizing and specifying behaviour is integral to the learning process, for both the algebra and visual arts groups. Our participants studied the work and technique of others, like teachers and peers, to assimilate their generalized accomplishments; came up with their own generalizations through practice and good luck; and applied their mental constructions to specific and novel cases.

In the title to this main section, "The Cognitive Dimension," I do not distinguish between the cognitive and affective domains. Arnheim (1969) sought to blur the lines between what counts as cognitive and what counts as perception. As he argued, all perception involves thinking. And as Solso (2000) argued, perception is emotional. When we first see something, we have a feeling about it, as interesting, threatening, attractive, and so on; all of these judgements, notice, often evoke feelings and physiological responses (Aiken, 1998; Damasio, 2001, Goleman, 1995; Hogan, 2003).

If we put together the psychological arguments of the type of Arnheim (1969), and neurobiological ones like that of Solso (2000), we have anachronistic support for the views of Tolstoy (1899) and Fry (1956) who emphasized the role of emotions in aesthetic appreciation; and more recently Perkins and Leonard (1977), who challenged the notion that the emotions are solely the purview of the arts. In fact, Pfenninger and Shubik (2001) pointed out that scientists react emotionally to ideas. We would think, for instance, that the experience of *surprise* would be an intellectual and emotional event. Cognitively, in a moment of surprise, perhaps, we make connections between two disparate ideas, and emotionally, we feel elated having done so.

Almost universally, emotions played a big role in early educational memories of Ben, Alfred, Molly, and Zöe. In Ben's case, we saw clear moments of emotion linked to aesthetic reactions to mathematics. He avoids "ugly" mathematics that has "no fun story." And he remembers an early mathematics teacher because he played chess; this personal bit of knowledge has stuck with him—not the mathematics he learned from the chess-playing math teacher.

In a similar vein, Molly painted "because it was so much fun." I have dubbed her the "joyous artist," because, in her own words, she described her pursuit of beauty in an emotional way. She finds joy in everything from the visceral act of applying colour to a canvas with a paintbrush, to participating in the community of artists, to more subtlety, communing with nature, which Garoian (1998) suggested is increased through creating works of art.

Alfred recalled some of his bad teachers, who did not care, and ones he liked because of the way they made him feel: "they wanted to be there," as he remarked. Good teachers' interest in their subject matter, we can infer, was infectious. And being interested in something is an emotional state as much as a cognitive one.

Alfred's first goal, recall, for primary/junior pre-service teachers was to overcome a phobia of math. He wanted them to feel comfortable. As is well known, people often convey a lot through gesture; so when a teacher feels uncomfortable with mathematics this attitude will likely be conveyed to students. Changing attitudes towards math, then, is naturally a priority, for Alfred.

Zöe, also has vivid memories of the teacher that made her feel left out, and now she emphasizes the importance of belonging in her classroom. As she put it, "Accept, respect, tolerate."

Further, Greene (1997), as well as Nicol, Moore, Zappa, Yusyp, and Sasges (2004), have discussed the way artworks are used to make meanings and shape identity. Molly's identity as a Canadian is solidified through her paintings of the landscapes of this country. In fact, as she noted, this fascination with Canada also resonates with her interest in art history. Her thoughts and feelings about herself are caught up in her art, her sense of place, and celebration of life, both local, here in Canada, and mystical, speaking to universal truth through aesthetic encounters with nature.

Sticking with the affective theme, it has been emphasized that creating works of art can be therapeutic, for example, contributing to the healing of emotional traumas (Mullen, Buttignol & Patrick, 2005;Wertheim-Cahen, Van Dijk, Schouten, Roozen, & Drožđek, 2004). And Zöe's artwork, at least the example I have explored (see "Appendix L: Sample of Zöe's Artwork"), probably relates to her early experience of being excluded, converting this experience, through art, into a positive.

Zöe's visual arts pedagogy is rich in ideas. As she put it, the idea in a piece of art is very important to her, right up there with the beauty of an artwork. The importance of ideas in her artwork obviously speaks to the cognitive domain. She wants to communicate a certain message, such as the idea of inclusion in a classroom. However, at the same time, her entire idea of making students feel like they belong speaks to how important emotions are to her.

It is interesting to note that both Molly's and Zöe's father died when they were young, which is still, as far as I could tell from their demeanour when they discussed the matter, an emotional issue for both of them. We can at least conjecture that some of Molly's loss of her father plays into her quest to anchor her works in the Canadian landscape, though admittedly a good portion of her motivation is rooted in joy. Also, the loss of Zöe's father perhaps further exacerbated her sense of exclusion in Canada when a teenager, thus at least potentially contributing to the development of her inclusive pedagogy.

Suffice it to say, insofar as Ben, Alfred, Molly, and Zöe experience flow in their work (discussed in "Aesthetics"), we can conjecture they all derive some therapeutic value from it. Their work is therapeutic in different ways that merit an investigation in its own right, thus taking us beyond the scope of this thesis. We can, however, at least provide some examples of how our participants' work could be therapeutic by considering what needs it fulfills.

For understanding how our participants work could be therapeutic, we turn now to consider Maslow's (1970) hierarchy of needs. In the works of our professional mathematician and an artist, we see the actualization of higher-order needs. That is, Ben and Molly achieve what their pedagogical colleagues, both Alfred and Zöe, want them to—to personalize knowledge, to become life-long learners, to find satisfaction from their work, and to self-actualize through their work.

As Ben noted, cynically, perhaps we have to create lies about the importance of our work to do it. But he admitted he had successfully cheated himself. As he said of his curiosity, "I really do want answers to the questions I ask."

Similarly, Molly really does relish in the joy of depicting the Canadian landscape visually, like the tall pines of Northern Ontario; intellectually, in her self-conception as a Canadian; and indeed emotionally, from the sheer pleasure of being able to see, and feel, and think coherently. That is, she makes meaning of her world by re-telling it to herself through paintings.

Zöe, like Alfred, is intent on making students feel safe, attending to their lower-order needs, as well as at least laying the foundation for their self-actualization propensities. Of course, Alfred and Zöe likely self-actualize, that is, meet their higher-order needs, through teaching, since that involves helping others, namely, students, and participating in the community of educators (likely this is more important to Alfred than Zöe, who identified herself as a "loner to this day").

I have wanted, like the authors I have often cited in this subsection, to blur the distinction between the cognitive and affective domains, since that is one fault line upon which algebra and the visual arts have been separated. Yet we can identify two ways that emotions function within algebra and the visual arts. First, I have discussed emotions relating to participants' experiences in relation to an external cause. Participants like, do not like, this or that teacher, institution, or family member. Second, we can identify the emotions involved by engaging in algebraic or visual arts activity. Participants like, or do not like, such-and-such type of problem or activity.

That is to say, some emotions are clearly caused by agents in the external environment. Some emotions are internally generated: participants sometimes find intellectual, affective, and physical pleasure in doing what they do as mathematicians, artists, or educators in these fields. Their work, whether algebraic of artistic, is emotional and embodied in many different ways, some of the biological dynamics of which we shall consider next.

Biological-based dispositions and talents. Although ostensibly outside the scope of this study, the inclusion of a consideration of the biological aspect of human behaviour is as follows. Biology may be used to explain, even if by way of conjecture, the behaviour of the four participants that just goes beyond both their personal and social experiences discussed in this study. Pinker (2002) in his book *The Blank Slate: The Modern Denial of Human Nature* has argued convincingly, if not conclusively, for instance by reviewing studies of identical twins raised in different environments, that biology plays more of a determining factor in shaping human behaviour than we, especially social scientists care to admit.

The case of Ben is most striking when discussing biology, because (as discussed in Chapter 5, the section "The Environmental Dimension") he had little direct support for mathematics activity from his parents, teachers, or institutions, but still scored often perfect marks in this subject with little effort. As Ben noted, his parents were supportive of his interest in mathematics, computers, and science, but could not help him because of the technical nature of the subject matter. His parents facilitated his interest however, buying him a calculator, allowing him to travel outside his town to attend camps that focused on mathematics or computers.

In fact, he said that the value of mathematics in his socio-cultural environment was "zero," and further that his present aspirations in mathematics bear no obvious relation to his childhood dreams. Of course, these are his own assertions, and he may not see the entire picture: nevertheless, we are left to explain why he found mathematics interesting and excelled at it. For that, it is at least reasonable to assume a biological factor in predisposing Ben towards mathematics.

We can also at least see some similarities among Ben and Alfred, Molly, and Zöe on this score. The fact that Alfred sought more challenge than he found in physical education speaks to perhaps innate capacities: he needed something more intellectually demanding. And Molly's interest in art began at a young age, when she asked her mother to enrol her in art lessons. She identified herself as having "talent," as an artist compared to her peers.

Zöe is the case where we have a clear line between her upbringing and her interest in the arts; her father was artistic and encouraged her. In her own words, she wanted to "be like him." Even with exposure to artistic behaviour, however, she had the most interest in the arts of her two brothers; some biological components may be involved in that.

But, of course, environmental influence is not something we escape easily. As I suggested in Chapter 2, "Literature Review: Identifying the Common Link," "The Neurobiology of Art," culture can even influence biological change over the long term (Donald, 2001, 2006). Hence, in this thesis I have stressed the interaction of the environmental and personal dimensions—that is, the confluence of environmental and personal factors—which resonates in different ways, with proponents of various stripes within the situated movement. As Godfrey-Smith (1996) noted, Dewey (1938/1997) was an interactional empiricist. We are the result of the influence of the environment. We are the consequence of our attempt to render the environment stable. And we are shaped by the to-and-fro between stability and precariousness.

In fact, according to Cobb and Bauerseld (1995), Voigt (1995), Perkins (1981, 1993, 1995), Salomon (1993), del Río and Alvarez (1995), Krishner and Whitson (1997b), Martin and Sugarman (1998, 1999a, 2002), and Donald (2001, 2006), we have to understand social practices as being dynamic and interactional: the individual presses against the norms of social practices and vice versa (Simon, 1969/1996; Turkle, 2005).

As we have already discussed, in "Culture," this dynamic, interactional process between the individual, which would include her biology and broader social context, occurred in all of our participants. In fact, as noted in that subsection, our participants, even when not obviously influenced by their social environment as a key factor to pursue their passions, did influence others—which supports the interactionalist thesis, and leads to the discussion of the socio-cultural context next.

The Environmental Dimension

Culture. Kaput (2008) argued that algebra is a cultural artefact because it involves symbols that are usually socially constructed. Similarly, Bishop (1988) beckoned us to consider the notion of a mathematical anthropology, and others (Cajete, 1994, 2000; Hatfield, Edwards, & Bitter, 1997) prompt us in a similar vein. That is, Bishop (1988) saw mathematics as rooted in a land, a people, and a time. Mathematics was like any other technology that functioned differently in various societies. So Bishop (1988) emphasized that we need to think about mathematical enculturation—how can we best bring students to use a mathematics that is useful and meaningful to them?

Further, drawing on the thought of Vygotsky (1978), there have been attempts to emphasize the role of cultural artefacts in our social practices (Lave & Wenger, 1991;Wertsch, 1993; Wertsch, del Río, & Alvarez, 1995); and his thought has also been extended by considering the historical dimensions of technology by Luria (1976) (Alvarez & del Río, 1995; Cole, 1993; Engerström & Cole, 1997).

In his writings, and taking an evolutionary perspective, Donald (2001, 2006) contended that art, too, is through-and-through, cultural—it involves other people. Art involves *cognitive engineering*, influencing an audience; *distributed cognitions*, the linking of many minds; is *constructivist*, as we strive to integrate perceptual and conceptual content in artworks; is *meta-cognitive*, engaging us in self-reflection; is technology-driven, influencing what and how it is represented, which can affect cognition; and finally, always has a cognitive outcome.

The notion that our epistemic practices must be understood within cultural and technological contexts is consistent with the work of Pea (1993), who discussed *off-loading*, that is, storing information in the surround; Perkins (1985, 1993, 1995) that explored the role of the socio-cultural environment in learning; and with proponents of the situated movement (Robbins & Aydede, 2009; Smith, 1999), too. Taking our cue from adherents of situated learning, then, when analyzing the four case studies, we must discuss the general issue of the usefulness of mathematics and the visual arts alongside the details of the cultural and technological contexts in which they are found, which I turn to next.

The idea of the usefulness of mathematics led to lively remarks from both Ben and Alfred, as we have already seen in our discussion of aesthetics. For instance, Ben observed that much pure mathematics is useless or at least has no obvious application today. He thus emphasized the importance of elementary mathematics, calculus, and computers. Ben had an interest in computers early on, which took him to the next town, Be'er Sheva, to work with them. He even wrote a computer program in Grade 7.

As he explained, what mathematics looks like today has changed. The computer has allowed more pictures in academic papers and new types of mathematics. Technology in the form of the computer, according to Ben, provides "perhaps the greatest challenge" because it holds the possibility of new mathematical discoveries. As he pointed out, computers allow us to make new and interesting mathematical conjectures that previously would have been impossible. Computers also allow us to carry out computations that would otherwise have been unfeasible to do by hand, with a paper and pencil, or an abacus.

Technology has also impacted art: what we depict—what we paint with, where we display our artworks, and what role they have in our society. For example, Molly contended that technology "is just another way of going [doing art]. When we painted with egg tempera in the 1500s, and then oil came along, they [artists] did not just throw away egg tempera. Now we have another [medium].... Younger people are very capable of working electronically." In fact, one of Molly's goals is to help young artists become more adept with the new technologies, so that they can find careers in art and design. She encouraged the integration of art in the advertising economy, because it adds beauty to our world, like the subway, and provides employment for artists.

Ben and Alfred generally agreed on exactly what is useful and what not. Alfred thought that some mathematics is not useful, like fractions. And he also thought that it is important to properly teach how to use technology, like the calculator, in the math class. However, they both were adamant that mathematics that is taught should be useful in some sense, either by application or simply for its sheer beauty. And Zöe, asserting something Molly would likely agree with, felt that even though most of her students would not end up with careers in the visual arts, they should be able to appreciate art and that requires some familiarity with art history.

The idea of the role of culture in the development of mathematics and the visual arts resonates with the work of Mumford (1986), Holt (1971), and Gray (2008). The ethos of the times, according to these thinkers leaves its mark in both mathematics and the arts. In fact, in the modern age, according to them, mathematics had a profound effect on how we think about perspective, perception, and the nature of reality, all of which affected how our world was depicted by, for instance, by adherents of the cubist movement. In fact, Getzels and Csizszentmihayi (1976), expanding Newell's and Simon's (1972) work, pointed out that only certain problems will be considered significant within a field; and, by finding new, significant problems, we also change the nature of the practice we are engaged in.

What counts as beautiful mathematics is at least partly culturally determined by providing a context for it to arise and for its evaluation. At the most general level, if mathematical technologies, like calculators and computers, were not part of Ben's environment, he could not be influenced by them; if there were no math classes or teachers or textbooks, no matter how "boring," he would have little opportunity to do mathematics; and if there were no institutions, like math camps held at the Weitzmann Institute, contests, and universities with mathematics departments, he could not have a career in mathematics.

What counts as art is also culturally determined providing a context for it to arise and for its evaluation. If Molly did not have access to art classes, leading to advanced studies in the visual arts, she would not have been able to develop her talents easily. She also had access to a variety of technologies, from paints and canvases, to more recently, digital technologies to develop her penchant for creativity in this field. However, she also emphasized the need to take the initiative, like she does, and just paint.

This interactional pattern between the self and the social realm holds true for Alfred, too. Alfred had exposure to sports and mathematics; he excelled at the first, and chose the second as a career path. All the while he was navigating within various social institutions with the aim of being a teacher. With Zöe we also decipher this interactional pattern between the self and the social realm because of the clear links between her interests now in the visual arts pedagogy of caring and her past—for example, early experiences with her father and feeling alienated in Canada as a teenage student.

Lave's and Wenger's (1991) idea of legitimate peripheral participation—the idea that mastery involves full participation in the socio-cultural practices of a community—is an attempt to explain how we become acculturated into the mathematics of our society. Lave, we recall, had studied the use of the mathematics of Liberian tailors, which is one example how mathematics is used in a specific social context.

If mathematics is socio-culturally bound and not floating free in Plato's heaven, there are radical consequences for pedagogy. For instance, we can emphasize that learning is not just a matter of assimilating information, but rather relying on the environment through, what Pea (1993) called, off-loading. This could mean intelligently using books, notebooks, technologies, and other people, that is, develop relationships, handy to get specific jobs done in a task environment (Perkins, 1985, 1993, 1995).

A few definitions are helpful to have on-hand. Expert learners often are characterized by high levels of self-regulation, which Woolfolk, Winne, and Perry (2006) defined as follows:

> A view of learning as *skills* and *will* applied to analyzing learning tasks, setting goals, and planning how to do the task, applying skills, and especially making adjustments about how learning is carried out. ([emphasis mine], p. 543)

Part and parcel of self-regulation is meta-cognition, which can become automated (P. Winne, personal communication, May 7, 2010a, May 7, 2010b). That is, expert learners monitor themselves, analyze their behaviour, and adjust it to achieve their goals—as well as amend their goals, as desired.

Self-regulation is apparent in many of the case studies. Ben employed a variety of strategies to overcome mathematical challenges. In fact, based on the epigram for his case in Chapter 4, he has learned that research needs to be an open-ended process, unplanned to a large extent, in order to be "the best of research." This is an example of meta-cognition (being aware of these cognitive processes) and self-regulation (adopting certain strategies to identify and overcome challenges). That is to say, Ben's recognition of the serendipitous nature of inquiry shaped his research strategy.

He used interactions with colleagues, students, and perhaps even in his interview with me, to better understand mathematics and his involvement in it. In fact, he is married to a mathematician, which we can also imagine as professionally helpful. He experiments in his mathematical notebooks by drawing pictures, experimenting, and posing questions (see Appendices G-J). His Web site contains various examples of where we can apply mathematics to the world. And of course, there is on-going involvement with programming computers to do mathematics for specific ad hoc tasks related to his research, which he presents at national and international conferences.

The cultural context was also explicit in the work of Molly: her artworks were depictions of the Canadian landscape. She made meaning of her environment, background, and identity, through her artistic engagements. Many of her progressive ideas about education and social justice work also reflect the Western, liberal values that took on a prominent form in the 1960's counterculture movement—the time she was a student at the Ontario College of Art.

Even though Zöe's inclusive educational model is rooted in her own traumatic experience of being a stranger in Canada, she too, appears to have embraced a progressive, liberal education model; this is evidenced by the painting she did for the Canadian Federation of Teachers (see "Appendix L: An Example of Zöe's Artwork").

Molly cited her involvement with other artists as one of the great pleasures in what she does. Community is very important to her. She uses her interaction with artists as a sounding board for ideas, in conjunction with her own repertoire of artistic skills. In fact, she even gets ideas from interactions in her community. For example, she took great pride in a series of paintings she did that was inspired by the work of astrophysicists and physicists. And this marked a turn away from a preoccupation with painting landscapes. So not only does she influence those who are under her tutelage, but she is influenced from without, too.

Alfred and Zöe both profit from their teaching, as well as conferences and workshops they either attend or give. Even for the consummate mathematical researcher, like Ben, his career has taken him into university teaching, allowing interactions with students, conferences, and the precious time needed away from the hustle and bustle of daily life to enter the arena of pure mathematics. We see a similar pattern with Molly, who became an educator for financial reasons, because it facilitated her artistic activity, and since she likes to help other artists, too.

It is interesting to note that three of the four participants married people with similar interests. Ben's wife is a mathematician; Alfred's wife is a math educator; and Molly's husband is a musician. We need not draw any general moral from this choice of marriage partner, because so many other variables would be at play.

In any case, and further, Ben has at least one son that has an interest in mathematics; and all members of Molly's family, including her husband, are inclined towards the arts. Alfred has at least one daughter who has taken up the teaching profession; and Zöe has a daughter who is artistically inclined.

In sum, anyway that we dissect the four cases, we can find the mark of culture on the activities of the math and art participants studied here. Our four participants have been influenced by the technology of their time; the institutions, both formal and informal; the modes of interaction that were available to them; and the aspirations, values, and spirit of their brethren in the times in which they live.

In turn, they continue to make valuable contributions to their culture, as Ben adds to the storehouse of mathematical truths, contributing to an entirely new field, knot theory; and Alfred's penchant for wanting his teaching to be relevant to students' lives sends the proverbial ripples far and wide. Molly contributes to her culture by rendering depictions of the Canadian landscape and making connections to design technologies for young artists; and Zöe by seeking a more inclusive pedagogy to engage young learners. This list does not even include detailed accounts of all the students, colleagues, and friends they have influenced. An all-inclusive list would be too vast and unnecessary to discuss in a case study focusing more on participants' background influences, which I turn to discuss in the next subsection, "Family and Friends," rather than on who they personally impact, by their pursuit of mathematics and the visual arts.

So the participants became a factor in the thinking, feeling, and behaviour of others, through teaching in classrooms, conference attendance, exhibitions in the case of artists, workshops given, and serendipitous meetings at dinner tables, coffee shops, bars, on the Internet, and just by their being-here, which is the interactional thesis writ small, and writ large (discussed further in "The Biological Dimension").

Family and friends. Perkins (1993) had discussed a revised notion of selfhood: from the person-solo view to the person-plus view, according to which we are best understood as the result of a swarm of participations. Invariably our first interpersonal relations are going to be with a primary caregiver, often in the form of family, specifically, a mother. As we become more independent, we also become more socially sophisticated, broadening our network of dependencies. We have friends, associates, students, mentors, neighbours, enemies, and so on.

The overwhelming commonality between our four participants is that the family was supportive of their career aspirations and the interests that motivated them. Ben's parents supported him, recall, buying him a "nice calculator, a TI-58." Ben said that he "really, really wanted it and it was a significant expense." He even noted that he had to wait for relatives to bring it from America because it was not readily available in Israel at the time. This singular memory is telling. It contains the density and intensity, so to speak, of his perception of his parents' care for him and his interests. Even though, as he noted, his math quickly became too advanced for his parents to tutor him in, they were happy he was reading mathematics.

We encounter a similar story with Alfred: "I had a strong sense that I did not want to disappoint my parents," he observed. His parents were pleased he wanted to be a teacher, but there is little connection between his specific interest in mathematics, that he could pinpoint, and his father's activity as a draftsman.

Turning to our arts cases, Molly also felt supported by her family. She was enrolled in an art class when young, at her explicit request. Her mother always supported her. "I really felt I was being supported. If I wanted to buy something, if I wanted materials for birthdays or Christmas gifts, canvases, that is something they [my mother was] always happy to provide," As she put it. "Family never inhibited me. They were open to the idea [of me doing art] but I was self-motivated."

In the case of Zöe, there was more of a direct influence on her desire to pursue the arts. "He [my father] encouraged me to draw," she observed. "I think I wanted to be like him. And he and I were awful lot alike, kindred souls." She also drew with her oldest brother, on occasion.

Given the relative uniqueness of the participants' activities, in mathematics and the visual arts, peers were not an important influence at a young age. As Ben explained, only his closer circle of peers could discuss mathematics and science with him, and by and large he was respected. As he got older, he developed more friends who shared his interest in mathematics. In university, he noted, "I had lots of peers to share mathematics with."

Alfred described himself as "self-willed." He probably had more peer support, when young, when engaged in sports, which involved group activity. Once he turned to mathematics, he pursued it on his own, and then branching out to interact with others in university.

Similarly, Molly was the stand-out, in this case an artist, among her peers and was supported by being appreciated. As she got older, and developed friendships with other artists, she learned that art could be a career.

Also, too, though Zöe was the odd-person-out as the "strange," girl with the "funny" accent in Canada, and "to this day a loner by choice," she has fond memories of university, where she developed friendships with other artists and art educators. So Ben's, Alfred's, Molly's, and Zöe's narrative with relation to peers' influence run on similar tracks.

In sum, we find that common familial and friendship narratives develop. All the participants feel supported by family and peers, except for Zöe whose father was an artist-craftsman, not in any technical way. With peers, there was general acceptance of their talents and interests when young that became more specific as they progressed to the university level. In other words, at the university or college level the participants were able to form friendships with people that shared their passion for mathematics, the arts, or educating in these fields.

Teachers and school experiences. It is when we turn to consider teachers and school experiences that we find some of the stark, even shocking, recollections, by the participants. And even here, like familial and peer relations, there is a great deal of coherence to their experiences, though sometimes for different reasons.

Ben had described his early experience in school as a "jail sentence" because it was "extremely boring." He said, commenting on being able to graduate two years early, "And that is exactly how I felt about it: that I am doing jail time and getting a reduction of my sentence." The teachers, as he put it, did the usual thing, going up to the black board, using textbooks, and assigning homework. His only early recollection of a math teacher he liked was one that played chess; it was this human fact, something beyond the mathematics that remained memorable.

The experience of boredom changed at the university level that was, as he said, "fun." Upon further consideration, he suggested that what made university a better experience than high school was, first, that he could study mathematics to the exclusion of subjects that did not interest him; and second, his teachers designed the curriculum themselves, thus imparting more personality into it.

Alfred sometimes felt "cheated" by his teachers. As he observed, when he was a student in the 1960's, education was done in the traditional British way, where discipline and authority figured highly. It was a teacher-centred and curriculum-centered education. Using commonplace metaphors, Alfred was a vessel to be filled with knowledge, the proverbial *tabula rasa*. He had some positive recollections of math teachers because they look for what made teachers memorable, whether in math or visual arts was that they were human, for example, having some personal interest like chess they would share with students; they liked their job, that is, being in the classroom and around students; they had a passion for their subject matter, helping students progress on an individual basis; they were kind and caring. So what stands out, surprisingly perhaps, is not the technical expertise of the participants' teachers, as great mathematicians or exemplary visual artists, though they may have been those things as well, but rather their demeanour, values, personality, and the encouragement they provided.

The Behavioural Dimension

Interactions. In the Cartesian tradition, it has been commonplace to hold that we first make decisions in our head before we act. However, at least some of what we think and feel is based on what we have done. In "Biological-Based dispositions and Talents," I emphasized there is an interaction between behaviour and the personal and social dimensions (Cobb & Bauerseld, 1995; del Río & Alvarez, 1995; Dewey, 1938/1997; Donald, 2001, 2006; Godfrey-Smith, 1996; Krishner & Whitson, 1997b; Martin & Sugarman, 1998, 1999a, 2002; Perkins, 1981, 1993, 1995; Salomon 1993; Simon, 1969/1996; Turkle, 2005; Voigt, 1995).

We may wish to recall that the notion of behaviour affecting the cognitive domain is consistent with Piaget's (1955, 1975) explanation of the development of logico-mathematical thinking (Clark et al., 1997; Csikszentmihalyi & Robinson, 1990; Karplus, 1981), and other empirical accounts of learning the exact sciences (Jenkins, 2008; Kitcher, 1983; Lakoff & Núñez, 1997, 2001; Mill, 1843/1876), for example, by generalizing and specializing (Kaput, 1987a, 1987b, 2008; Kaput, Blanton, & Moreno, 2008; Korzenick, 1977; Krshner & Awtry, 2004; Mason, 2008; Shoenfeld, 2008; Smith & Thompson, 2008) (discussed in "Aesthetics" and "Generalizing and Specializing"). Further, behaviours that stretch over long periods of our lives could enhance self-actualization, which according to Maslow (1970) is the highest need we can fulfill.

It is reasonable to think that by doing mathematics, for Ben and Alfred, or the visual arts, in the cases of Molly and Zöe, they came to appreciate it more then their initial propensities allowed for. Ben does mathematics incessantly (e.g., see Appendices A-J). Molly paints and holds exhibitions of her work, now helping younger artists. Zöe took on a life in the arts at a young age, and now continues in this field as, primarily, an educator. Alfred has now over 47 years of teaching experience in the subject, and still plods on.

As I discussed in "Cognitive and Affective Domains, the fact that Ben, Alfred, Molly, and Zöe have spent their lives in their field makes it reasonable to think that what they do is an important part of their self-identity as part of the group of mathematicians, artists, or educators in these fields. In "Biological-Based dispositions and Talents," also, I have pointed out how in various ways their behaviour potentially influences the socio-cultural context, and the thinking and actions of those around them.

Concluding Remarks on the Data Interpretation

What I have shown thus far, is that there are commonalities between cognition and learning, broadly conceived, in the participants studied. There are often personal similarities related to the role of aesthetics, generalizing and specializing, and cognition and affect, biology and the interaction between the individual and their environment. There are environmental similarities in the role of culture, family and friends, and teachers and school experiences. And there are also similarities in the role of behaviour, like habit.

Of course, there are differences regarding how much any one factor plays in a participant's life, and I have not determined those. But that is how it should be. For the social cognitive theorist conducting a qualitative study, we are merely attempting to explicate the confluence of factors that shape how we think and act in rich detail—thick descriptions as anthropologists say.

We may well wonder why I have spoken of "the common link," the socio-cultural context, when I have considered also the cognitive dimension—hence common links. The socio-cultural context, I have attempted to emphasize pervades all cognitive acts, such as aesthetic appreciation, generalizing and specifying, as well as the emotions, biology; and behaviour. To follow the analogy inherent in the idea of *the link*, we can think of the personal and behavioural dimensions as threads constituting a single link. That is to say, I have attempted to emphasize specifically the role of the social context, even when considering the personal and behavioural dimensions, and I now turn to consider the limitations of this study in order to refine the scope of the broad similarities I have discussed.

Limitations

> "[T]here are in reality few average persons, or at least they are average in different ways" (Csikszentmihalyi & Robinson, 1990, p. 186).

As was anticipated before fieldwork was conducted (A. Ezeife, personal communication, May 20, 2009), several limitations of this study became apparent as data were collected. There were limitations arising from the selection of participants, for example, how representative they would be of the population of algebraists or visual artists. It would have been nice to have a more balanced population in terms of gender.

However, after placing several requests for interviews, I also had to consider the participants' expertise and experience in the fields in question, as well as how feasible it was to meet— for example, who was willing to participate in this study? At the time that this study occurred, I was not able to interview female mathematicians and math educators; or male visual artists and visual art educators. What does this mean?

As I have already noted (e.g., Ragin, 1992a, discussed further in Chapter 3, "Methodology: The Case Study Approach"), case studies are often thought to generalize to larger populations because the processes in question are not unique to the participants. However, any results from a case study would require scientific verification by the employment of quantitative methods to ensure they can be generalized to their respective populations, which would take into account the issue of gender more than I could, or intended to do, in this study.

The interview instrument itself posed significant limitations. My interview instrument did not always elicit the sort of detailed responses needed to address the key questions in this study. For example, Ben made no connections, or provided no data, that allowed me to make links between his early socio-cultural background and his interest in mathematics. Also, there could be other aspects of the social context, like participants' relations with their colleagues, which I may have missed or not dwelt upon enough. Though, my goal was not to offer a psychological analysis of the participants; rather, I sought out pattern regularities that could be the basis for such inquiries as well as other ones.

The participants attempted to answer all of my questions. However, there is the issue of concern question about the veracity of their claims. That is, I have assumed in large measure that participants are being truthful in their responses to the interview questions, and not merely trying to just please me. This is why I assured participants, even when it was not a priority for them, of their right to confidentiality as far as possible. And if they did answer the questions truthfully, and their answers are consistent with other materials I drew upon, like samples of their work: course syllabi, research and teaching statements, other published interviews, and so on, I cannot be sure that my interpretations of their responses are accurate.

However, by considering alternative interpretations of data (e.g., in Chapter 5, in "Biological-Based Dispositions and Talents"), member checking, and taking into account third-party feedback about my interpretations (namely, those of my thesis committee), I have aimed to at least attempt to reduce my theoretical bias.

There are specific topics I have only been able to touch upon; hence, there is the need for further research, and I point some of them out now. It would be interesting to study further the role of "flow experiences" as a motivational factor in our work. Also, it would be worthwhile exploring to what extent, if at all, work can be considered therapeutic, and specifically what aspects of our work can make it so. My contention, based on the research undertaken here, is that interpersonal relations and the socio-cultural context will play major roles in both cases.

Admittedly, however, implications for practice from studying the interaction between the individual and the social context in relation to, for example, motivation, career choice, career progress, career and epistemic aspirations, retirement objectives, as well as identity formation, are issues of vast scholarly debates unto themselves: I have not intended to resolve any specific debate about the motivation, its mechanisms, or the trajectory of participants' abilities in this study.

Yet it would be interesting to see what findings will materialize from my study in terms of making a contribution to the discourse in the literature with regard to motivation and related issues. Motivation, related to career choice, career progress, career and epistemic aspirations, retirement objectives, as well as identity formation, may be influenced by personal factors, environmental factors, and the personal-biological aspects I have discussed in Chapter 5.

Also, I have not calculated the role of any one factor, as is commonplace in a quantitative study. For example, we could consider the roles played by participants' socio-cultural background, family and friends, school experiences, individual volition, their attitudes, the influence of theories of learning, and their aspirations. We could, then, consider how identity may be a factor that could have a role on the motivation to pursue algebra or the visual arts based on an individual's socio-cultural background and experiences. My aims have been more modest. I have only wanted to explore some of, for instance, the social factors, like the role of culture, friends and family, teachers and school experiences that propel us to pursue algebra and the visual arts, not quantify how much effect any one has.

Whether philosophical or scientific, there are likely always loose ends when exploring the intricacies of the human experience. Specifically, I have not explored any one theme in this study conceptually in depth, like *aesthetics,* because this study is a work in the social sciences and not philosophy. Other philosophers may wish to do so.

When I discussed the recommendations for pedagogical practice, in Chapter 5, in "Practical Significance: For the Pedagogy of Algebra and the Visual Arts," I have not compared the strand of algebra ("The Ontario Curriculum," 2007) and the visual arts ("The Ontario Curriculum," 2009), which went beyond my purposes of indicating that a *common link* already exists between them. Others may wish to expose the links in the curriculum documents in greater depth than I did, as follow-up research.

Though I explained in Chapter 3, "Methodology: The Case Study Approach," in "Interviews," what *reliability* and *validity* mean in this study, they likely do not carry over into the recommendations (discussed in Chapter 5, in "Practical Significance: For the Pedagogy of Algebra and the Visual Arts"). At best, my recommendations are reliable in the sense they are consistent with the themes developed in the data analysis. The recommendations are valid insofar as they have been corroborated by anecdotal evidence (sometimes indicated in parentheses), previous research, or are already commonplace in the pedagogical practices of algebra ("The Ontario Curriculum," 2007) or the visual arts ("The Ontario Curriculum," 2009).

However, quantitative studies are needed to attempt to establish if any one recommendation, such as increasing transparency (discussed in "Cognitive and Affective Domain Outcomes"), yields a positive learning outcome generally in algebra or the visual arts, and in any one social context. (In fact, going back and forth between qualitative and quantitative research studies is perhaps the ideal way to make headway on determining what practices yield the best learning outcomes.)

Further, there is an emerging literature on applying cognitive science to education. In "Practical Significance: For the Pedagogy of Algebra and the Visual Arts," I have sometimes indicated Reif's (2008) book *Applying Cognitive Science to Education and Other Complex Domains* in parentheses, but did not discuss it in the literature review, because it only came to my attention in April 2010. Also, his focus was science and mathematics pedagogy, so it perhaps best fit with my discussion of recommendations for practice. Nevertheless, follow-up research should consider the cutting-edge advances on connecting cognitive science to algebra and the visual arts in the classroom; and discussing it in more depth than I was able to do in this thesis.

Finally, though I only focus on algebra and the visual arts, my results may be extendable to other fields of learning. Researchers would have to carry out further case studies to see how far my results could be generalized across mathematics and the arts, respectively, as well as other fields.

The specific implications of this study for other fields may also require quantitative analysis, and further qualitative study to further scrutinize any new results like statistical outliers, opening up avenues for further research by other scholars. Limitations in this study often arose from the specific situation in which this research was conducted, but I now turn to consider the significance of what has been achieved.

Two Types of Significance

Qualitative research often takes the form of a context (a problem: e.g., in this thesis, the historical split between mathematics and the visual arts), actions (the data: e.g., four case studies), and consequences (the data analysis and outcomes for practice: answering the question: why should we care?). The significance of a qualitative study lies in connecting the initial problem, through the data, to practical consequences. We are now in a position to discuss the theoretical and practical significance of this study, which I consider in turn.

Theoretical Significance: Philosophical and Psychological

In the Introduction, I began by a philosophical discussion of Plato's dividing line. I pointed out that in the platonic tradition, mathematics and the arts have been separated ontologically, epistemologically, and ethically: their practitioners, according to Plato, aimed at different types of objects (mathematical objects were more real than artistic images); different ways of coming to know their respective sets of objects (using reason in mathematics and the senses in the arts); and had a different evaluation within the social body (where mathematics was good, providing a rung for our lateral assent to the forms, and the arts bad, distracting us with illusions).

However, in this study we have positioned ourselves against the platonic configuration in several ways, and with reason. First, Johnson (1986) argued that all mental activity is embodied, along with others from different fields, like clinical psychiatry (Frie, 2007, 2008b; Martin & Sugarman, 1997b, 1999b; Martin, Prupas, & Sugarman, 1998). Second, Smith (1999) and others (Frie, 2008a; Martin & Sugarman, 1997a, 1997b, 1999b, 2001a, 2001b, 2001c, 2003; Sugarman, 2008; Sugarman & Martin, in press; Sugarman, Martin, & Hickenbottom, 2009; Sugarman, Martin, & Thompson, 2003) have noted that knowledge is not only embodied, but extended, that is, knowledge is contingent upon matrices of interpersonal relations.

Finally, there has been an ontological revaluation of the everyday world for cognitive scientists (Winograd & Flores, 1987) and others (Frie, 2008a, Frie & Coburn, in press; Martin & Sugarman, 1997a, 1997b, 1999b, 2001a, 2001b, 2001c, 2003; Martin, Prupas, & Sugarman, 1998; Sugarman, 2008; Sugarman & Martin, in press; Sugarman, Martin, & Hickenbottom, 2009; Sugarman, Martin, & Thompson, 2003) concerned about the nature of the mind along the embodied and extended lines we have discussed.

All this philosophical debate would be of little interest to us here, if it did not have specific counterparts in the literature on mathematics and the visual arts, and it does. There has been philosophical and scientific work done on an empirical account of mathematics (Jenkins, 2008; Kitcher, 1983; Lakoff & Núñez, 1997, 2001; Mill, 1843/1876; Piaget 1964, 1970, 1975). That is, there is a basis to think that mathematics does involve our interactions with the physical world and our interrelations to each other; it is historical, social, and situated.

Similarly, there has been a wealth of research on the role of the social context in the visual arts (Arnheim, 1965, 1969, 1986; Csikszentmihalyi & Robinson, 1990; Fry, 1947, 1956; Greene, 1997; Tolstoy, 1899) that suggests the visual arts are also an interpersonal enterprise in multifarious ways (Cassier, 1953-1957; Cugmas, 2004; Cunliffe, 1998; Deacon, 2006; Donald, 2006; Gilot, 2001; Gombrich, 1960/2000, 1982, 1991, 1995; Goodman, 1968, 1972; Greene, 1997; Hawkings, 2002; Howard, 1977; Kaiser, 1966; Korzenick, 1977; Lopes, 1996; Perkins & Leonard, 1977; Roupas, 1977; Somerville & Hartley, 1986), which I discussed in Chapter 2, "Literature Review: Identifying the Common Link," in "The Socio-Cultural Context," and in Chapter 5, in "Culture."

Our artworks are culturally specific. The cultural context in which artworks are produced are affected by and possibly affect our cognitive development; interpersonal relations; the anthropology of human evolution; and our personal preferences (discussed in Chapter 2, "Literature Review: Identifying the Common Link," in "The Socio-Cultural Context" and "A Caveat: Do Not Lose Sight of the Individual").

What makes this thesis interesting, in general, is the rich detail in which I have studied the connections between algebra and the visual arts. That is to say, I have gone beyond mere philosophical debate, interesting as that is, and looked at how exactly the four participants' activities in algebra and the visual arts are embodied, situated, and personally meaningful for them. It is the detailed empirical exploration of these broad philosophical themes—which we can sum with the phrase *being situated*—that makes this study significant, in spite of some of the limitations I have already discussed (see "Limitations").

The upshot, writ large, is that there are similarities in how the mind ticks regardless of whether we are working in algebra or the visual arts. Similar factors, I have argued, relating to cognition, like prior knowledge; and the social world—for example, family experiences, school experiences, the role of social models, and culture—as well as biology—impinge upon us, shaping how we go about solving the problems, the ones we choose, our career path, and our identity.

In fact, considering the role of aesthetics and identity as a motivator to pursue algebra or the visual arts allows us to potentially explain what remained anomalous for Bandura (1997): namely, some people have positive self-efficacy beliefs about their ability at p and have persisted in doing p despite having faced continual failure in p (discussed further in "Teacher and School Experiences Outcomes"). We are, I emphasize, in a position to corroborate and enlarge a more robust social cognitive theory by taking into account the role of aesthetics and identity as a motivator of human behaviour and thinking (discussed further in "Teacher and School Experiences Outcomes").

Practical Significance: For the Pedagogy of Algebra and the Visual Arts

> "All of the arts [and mathematics] communicate through complex symbols—verbal, visual, and aural—and help students understand aspects of life in different ways" ("The Ontario Curriculum," 2009, p. 3).

It is edifying to note that Bishop (1992) had argued that the purpose of educational research, as opposed to many other types of inquiry in the academy, is to influence practice. That is to say, in the context of this thesis, we are to make contributions to theory with an eye to practice.

I have pointed to several documents to illustrate the current currency of my recommendations: *Leading Math Success* (2004); *The Ontario Curriculum Grades 11 - 12: Mathematics* (2007) (henceforth "The Ontario Curriculum," 2007); and *The Ontario Curriculum Grades 11 12: The Arts* (2009) (henceforth "The Ontario Curriculum," 2009). My purpose is not to offer an in depth comparison of the algebra or visual arts strand in the aforementioned curriculum documents. My goal is only to indicate the pedagogical common link between them (e.g., "The Ontario Curriculum," 2007, 2009) (see "Limitations").

What is novel in this work is the link between the problem that motivated this study—namely, what is common to learning in algebra and the visual arts?—and the recommendations for teaching in these domains. On the one hand, we have a theoretical, social cognitive basis (discussed in "Theoretical Significance: Philosophical and

Psychological") to consider a pedagogical regime that is collaborative, personally meaningful, and situated.

On the other hand, we have specific recommendations, soon to be discussed, that emerge from the social cognitive (Bandura, 1986, 1997) theoretical framework employed in this study (see Table 2, "The Themes of Social Cognitive Analysis in This Study," Column 1, "General Themes"); pattern regularities (see Table 2, Column 2, "Specific Themes"); anecdotal evidence drawn from the case studies; current literature on educational practices; and curriculum documents (e.g., "The Ontario Curriculum," 2007, 2009). And some of the recommendations are somewhat novel, like the role of aesthetics in learning. However, it is important to note that all recommendations to be discussed must be considered programmatic (discussed further in "Limitations").

Taken together, I submit, social cognitive theory has been corroborated and enlarged (discussed in "Theoretical Significance: Philosophical and Psychological"). In so doing, I have utilized a social cognitive theoretical framework to couch specific, programmatic recommendations (see "Limitations"). I think that social cognitive theory and the recommendations to be discussed will indicate reciprocal justification. At the very least, social cognitive theory is consistent with the recommendations to be considered.

Further, similar recommendations apply, as I shall suggest, as much to the pedagogy of algebra as the visual arts—my overall approach, in sum, is novel because, among other reasons already considered, it unities theory and practice across disparate domains. Indeed, my aim is to make the common link theoretically and practically; having remarked on the first (discussed in "Theoretical Significance: Philosophical and

Psychological"), I turn now to consider praxis. Drawn from my interviews with participants, we are given a tool to refine the pedagogy of algebra with some of the best practices of the teaching of the visual arts and vice versa, all the while keeping within view the situated nature of learning.

Personal Outcomes

Aesthetics outcomes. Learning should be meaningful, and this takes on many forms, the first deals with aesthetics. Since aesthetics encompasses internal motivation, it is likely to be a boon for encouraging life-long learning and motivation, already recognized goals for the pedagogy of algebra ("The Ontario Curriculum," 2007; Reif, 2008) and the visual arts ("The Ontario Curriculum," 2009). Students will likely work harder, overcoming even great odds, when they see beauty in the products they produce; they need to see exemplars of the ends they are aiming at (Perkins, 1986). We could do a better job to share our aesthetic experiences in algebra and the visual arts with students. Indeed, when we first encounter the possibility of beauty, we strive to achieve it repeatedly (as we saw in Molly's persistent attempts to represent the Canadian landscape).

One goal of education, for the consummate learner, is personal satisfaction. With producing works of beauty we probably take pride in them, wanting to show them off. Taking pride in our work, and what counts as beautiful, also conveys an important axiological lesson for students. Self-satisfaction speaks to us of the end goals of knowledge, I believe.

Generalizing and specifying outcomes. These two types of cognitive acts are often part of active learning (Dewey, 1934/1958), and consistent with Bandura's idea of *enactive learning,* that is, gaining knowledge through personally experiencing the consequence of actions (Woolfolk, Winne, & Perry, 2006). Thus, it is useful to keep in mind a definition of active learning, which is gleaned from Cameron's (1999) book on the subject:

> Active learning comprises many ideas, but basically it requires students *participate* in the learning process. Active learning asks that students *use* content knowledge, not just acquire it. ([emphasis in original], p. 9)

Generalizing: working in a certain context can lead to new, principled understandings. Also specifying: in using content knowledge, students may be applying it in a certain context ("Leading Math Success," 2004; Reif, 2008) (discussed in Chapter 2, in "Aesthetics, Generalizing, and Specializing"). With the definition of active learning in hand, we can now go into more detail about how generalizing and specifying could function in the pedagogy of algebra or the visual arts.

Being able to have opportunities to practice has long been part of the algebraic education; we do practice problems. Having a better understanding of the social cognitive nature of algebra, however, we should facilitate opportunities to make meaning, inductively; and deductively applying algebraic rules to specific real life, and relevant, circumstances (see "Early Math Strategy," 2008; Reif, 2008). (Recall, relevance was important in Alfred's pedagogical practice.) In the visual arts, there is a challenge to make this discipline relevant to the market place. So some training in the visual arts can be geared more, than it perhaps has been, towards social needs, like marketing and design.

Exemplars are often considered important in teaching to provide scaffolding for students, which are generalizations. In both algebra and the visual arts, it is important to have exposure to exemplars that provide access to generalizations—which we can creatively apply in new ways.

In fact, according to Getzels and Csizszentmihayi (1976), and in my terms, the key to creative activity is *challenge-identifying,* that is, the way challenges are envisaged, posed, formulated, and created. Through generalization we can find new challenges, looking for commonalities among particular instances of so-and-so. We can overcome our challenges with principles that set out the relationships within the new-found categories. (Notice, Ben attempted to generalize by categorizing different types of knots, which we would potentially be the basis of a new field, "algebraic knot theory.") Through specifying we can consider how we apply an understanding to new contexts, which can also yield novel products. (Recall, Molly considered portraitures to be like landscapes.) So, we can both find challenges and overcome them though generalizing and specifying—in cycles that spur each other on.

Cognitive and affective outcomes. To ensure that work in algebra ("The Ontario Curriculum," 2007; Reif, 2008) and the visual arts ("The Ontario Curriculum," 2009) is personally meaningful, students should know why they are doing it, which can potentially be achieved by doing any one of several things. First, algebra and the visual arts should be fun (discussed in Chapter 5, in "Aesthetics"). (Recall both Ben and Molly made reference to "fun" as a reason to do algebra or the visual arts.)

Second, students likely need opportunities to, collaboratively or individually, investigate (i.e., to find, choose, or explore a challenge) and experiment (i.e., to try different ways to overcome a challenge) in algebra or the visual arts (Reif, 2008). An active learning pedagogy is an asset to meaning-making in these domains because it encourages challenge-identifying and challenge-overcoming (e.g., discussed in "Generalizing and Specifying," and "Teachers and School Experiences Outcomes").

Students need opportunities to personalize knowledge, that is, to construct truths in their own way. (Recall that Ben contended, "The only way I know something is if I add to it;" and Alfred concurred from his point of view as a math educator: "I don't tell students anything.") Also, personalization, as discussed, likely brings with it the ownership of knowledge. (And Alfred wanted students to have "ownership" of the curriculum.)

In addition, Perkins (1986) emphasized the role of design in knowledge—the idea that knowledge entails goals, which leads to these: the next set of points relate to *transparency*, knowing *why* we are learning something, or *why* in a specific way. Transparency likely reduces student anxiety, and in what follows I offer some ideas about how to achieve it. Students would profit from knowing how their disciplines fit into society. Also, students should know the role algebra and the visual arts have in their discipline and related ones. When feasible, it is advisable to make explicit the rationale of the heuristics employed (discussed further in "Teacher and School Experiences Outcomes").

Finally, we have traditionally given little attention to the emotions in learning. At least part of making the pedagogy of learning in algebra and the visual arts meaningful is to humanize it. In addition to being transparent, we can reduce student anxiety, by using non-threatening language. For example, we can speak instead of *criticisms,* of *opportunities for improvement*; instead of *problems,* of *challenges.*

We can reduce student anxiety (a core goal of Zöe's pedagogy) by creating networks of support. We need to share how we came to our field; how we overcome challenges in it; and how we learn in it (discussed further in "Teacher and School Experiences Outcomes"). We need to discuss how our work fits into our lives. When desirable, we need to make our experiences of attempts, false starts, and successes transparent for students. (Ben is an excellent example of modeling transparency since almost all of his mathematical notebooks are made public.) Also, we, of course, must allow students time to share with one another, for instance, by critiquing, commenting on, or expanding the work of their peers.

Biological-based dispositions and talents outcomes. As I believe I demonstrated in the related section, "Biological-Based Individual Dispositions and Talents," even though the biological dimension is outside the scope of this study, it does potentially allow us to refine our analysis of the data, and in this case, our recommendations for pedagogical practices.

In this context, "one size does *not* fit all." Students have different learning styles, abilities, and interests. Part of the reason to get to know our students, to recognize their talents and weaknesses, is to encourage them, for example, through individualized educational plans or extra-curricular work. This may entail placing students in the appropriate group settings, where they can feel challenged or help others. At the same time, we must be careful, of course, in labelling students because, when negative, this can have a devastating effect on their sense of self-efficacy—and because we can be wrong, too!

Given that secondary school algebra and the visual arts are intended to give exposure to these areas—allowing us to find out how much, and in what way, if at all, we want to purse these quests, as well as to encourage us to do so—it is important we pay heed to what students bring to the learning experiences. We can only cut against the grain for so long: we must build upon who students are and where they come from, which I turn to discuss next.

Environmental Outcomes

Cultural outcomes. Work in algebra and the visual arts should be culturally appropriate (Bishop, 1988; Cajete, 1994, 2000; Hatfield, Edwards, & Bitter, 1997). That is, these domains should involve the integration of the language, artefacts, and values of the culture in which they taught, when feasible and desirable. Something as simple as using names of students, say Chinese names instead of Anglo-Saxon ones, in a setting with members from the related community, can make a surprisingly big difference to learning outcomes, I think. At the very least, it gets students on side and makes them feel they belong (a perennial theme in Zöe's pedagogy).

In algebra and the visual arts, we are well advised to integrate more history into the teaching of these subjects: this allows us ready opportunities to connect to students' cultural backgrounds. Mathematics and the visual arts, after all, have been pillars, in different forms, in all human societies (Bishop, 1988; Hatfield, Edwards, & Bitter, 1997; "Leading Math Success," 2004). (Here we recall how important art and mathematics history was to Molly and Alfred, respectively.) We need to overcome the Eurocentric perception, common during the ninetieth century that the Western world has the monopoly on mathematics ("The Ontario Curriculum," 2007) and the visual arts ("The Ontario Curriculum," 2009), a view both historically spurious and pedagogically misleading. And the right place to begin to challenge the Eurocentric stance is the classroom.

We also need to continue to integrate computers and digital technologies into algebra ("Leading Math Success," 2004; "The Ontario Curriculum," 2007; Reif, 2008) and visual arts education ("The Ontario Curriculum," 2009). (Recall Ben's emphasis on the importance of computers to mathematical advancement, and Molly's conviction about the significance of new digital technologies for visual artists.)

As has become apparent through the case studies, the nature of mathematics and the visual arts is changing because of new technologies, blurring the disciplinary lines, on occasion, between computer science and mathematics, on the one side; and the fine arts and design, on the other. Such interdisciplinary enterprises—perhaps even giving rise to new, hybrid disciplines in the future—we can hope, will not only lead to original contributions in each field, but also improve the marketability of students who pursue studies in algebra ("The Ontario Curriculum," 2007; Reif, 2008) and the visual arts ("The Ontario Curriculum," 2009).

Family and friends outcomes. From time to time, educators emphasize the role of parents in the education of their children, and with good reasons. Important as the teacher is, we would be blind not to notice the impact on one's family background, like socio-economic status on learning. This is often attested to by psychometricians. Working with parents through regular meetings, workshops, and camps are some of the ways that family can become more supportive of their child's efforts in algebra and the visual arts. In fact, what is good for students holds good for parents, too: they need to understand why algebra and the visual arts are important, since it stands to reason that the more aware parents are, the more they are likely to help their children who are pursuing these fields.

At least in the classroom, we strive to build "learning communities," in algebra ("The Ontario Curriculum," 2007; Reif, 2008) and the visual arts ("The Ontario Curriculum," 2009). And there is value in doing so. Students need to work together, ask questions, challenge assumptions, and ultimately teach one another. Here the visual arts classroom is a good model. (There is a reason that a physics laboratory at the Massachusetts Institute of Technology is held in an art studio, for example: namely, I surmise, to encourage creativity and collaboration.) In the classroom, we need to encourage an environment of creativity, reasonable competition, interactive activity, experimentation, and movement—real communities do not sit in rows writing examinations: they do things together, whether on the World Wide Web or at powwows. Moreover, student's investigations should be, as much as possible, tailored to their level in heterogeneous groups, where they are pushed to go further in the zone of proximal development.

Teachers and school experiences. We may wish to recall that social cognitive theorists take into account both *enactive learning*, gaining knowledge through personally experiencing the consequence of actions; and *vicarious learning*, knowledge gained by observing others (Woolfolk, Winne, & Perry, 2006). I shall discuss some outcomes of each of these types of learning in turn.

We should encourage enactive learning. The teacher should take on, as much as possible, the role of facilitator of student learning. Suffice it to say that we should encourage active learning (discussed in Chapter 5, in "Generalizing and Specifying Outcomes," and further on with examples in this subsection).

I acknowledge that it is often difficult and time-consuming to design activities where students "learn for themselves." Admittedly, it takes discipline and trust on the part of teachers not just to pour their knowledge into students, as it were. But the pay-off is huge because enactive learning is likely to produce self-motivated, self-regulated, life-long learners ("Leading Math Success," 2004; "The Ontario Curriculum," 2009; Reif, 2008).

Also, when we find something pleasurable and do it for an extended period of time, it likely becomes part of our identity. And the role of aesthetics and identity potentially allows us to explain something that escaped Bandura. Bandura (1997) could not explain why some people have positive self-efficacy beliefs in their fields and persisted even when they faced continual "failure [negative reinforcements]" (p. 73), as is common among many luminaries throughout history. I conjecture the following. Some people, the rare ones perhaps, persist in a field where they have little success because they find it meaningful (discussed in Chapter 5 in "Aesthetics Outcomes") and have assimilated the said activity into their identity—for example, as algebraists or visual artists.

In the case of self-motivated behaviour, goals are internally driven. However, we may wish to note in passing that *self-regulation* and *self-motivation* can be antinomies (P. Winne, personal communication, April 30, 2010); in other words, they may also be found in the same individual. Though ostensibly successful, we can consider the case of Grigori Perelman, the Russian mathematician, to illustrate an extreme example of self-motivated, self-regulated behaviour. He won the Field's Medal in 2006, the most prestigious prize in mathematics worth one and a half million United States dollars for his work on the Poincaré conjecture (for an explanation of this conjecture see "Grigori Perelman," 2010); Perelman never showed up to collect the award, being the first person ever to decline it. Perelman explained,

> Everybody understood that the [my] proof [of the Poincaré conjecture] is correct, then no other recognition is needed.... I'm not interested in money or fame. I don't want to be on display like an animal in a zoo. I'm not a hero of mathematics. I'm not even very successful; that is why I don't want to have everybody looking at me. (cited in "Grigori Perelman," 2010)

Perelman had previously turned down a prestigious prize of the European Mathematical Society, saying he felt the prize committee was unqualified to assess his work, even positively. More recently Perelman refused the Clay Mathematics Institute's Millennium Prize, worth one million United States dollars for his proof of several conjectures, including the Poincaré conjecture. "I have got everything I need" (cited in "Grigori Perelman," 2010), Perelman said.

From my recent Internet search on *Wikipedia*, as of 2006, Perelman was jobless, lived with his mother in Saint Petersburg, and played table tennis against the wall. To use Ben's language, Perelman had successfully "cheated himself." (Though to be fair we all likely believe something to be of existential value.) Perelman seems to be self-motivated, having little interest in financial rewards. Also, in the jargon of educational psychology, Perelman self-regulated by building a surround conducive to achieving his goals, and as we can see, he has been successful, regardless of his own self-estimation (which seems spurious and may be even disingenuous).

We may wish to recall that Rogers (2001) held that his work as an artist was "selfish" (p. 47), which speaks to the idea of self-motivation taken to one possible extreme. It is not necessary that self-motivated behaviour be selfish, but that is a potential outcome when we are internally driven. Further, it is reasonable to think that self-motivation sometimes typifies behaviour that satiates, sublimated lower-order needs (Maslow, 1971), as we seek safety or belonging in ideals like the search for truth, quest for justice, or pursuit of beauty.

In fact, we can think of many artists, like Picasso (Gupta, in press-b) who displayed extreme, self-motivated, selfish, and self-regulated behaviour. Geniuses of this ilk are, I believe, in it for the beauty. So we should not take the issue of beauty lightly, since it can potentially direct behaviour in dramatic ways. Of course, educators are probably aiming to produce personalities more rounded than Perelman or Picasso.

In any case, social cognitive theory provides a clue to why beauty can be such a powerful motivator: in these cases of genius, for instance, doing mathematics or creating works of art gives pleasure, that is, is rewarding. (Notice, pleasure is relative and can be acquired for this or that stimulus; after all, some find chilly peppers pleasurable, even though they cause discomfort.) And for this group of geniuses the habituation to aesthetic pleasures becomes likely calcified as part and parcel of their ideals and identities.

Returning to our discussion of recommendations, to be a good facilitator the teacher must, first and foremost, be likeable: if a student does like the teacher as a person, it is possible that this may lead them to avoid the subject taught (by the teacher). And becoming likeable is possible, which leads us naturally to discuss vicarious learning.

Much learning is done through modeling; it is, at least initially, externally motivated. And it is edifying to notice that the reasons can be found in the three broad themes that provide the structure of the argument in this thesis: socially, in group settings there is usually a hierarchy, and we often look to those with authority to guide our thinking and behaviour; cognitively, we likely have social belonging needs as elaborated by Maslow (1970) (discussed in the "Introduction: A Philosophically Minded Prelude;" see details surrounding Table 2, "Maslow's Hierarchy of Needs"); and biologically, the aforementioned needs probably have been engrained through the process of natural selection or breeding.

The upshot is that constructivists, who emphasize enactive learning, often overlook the importance of vicarious learning. So we need to provide, in addition to opportunities for active learning, proper models as teachers, set high expectations, and provide exemplars in algebra and the visual arts (discussed in Chapter 5, in "Cognitive and Affective Domains"). Notice, vicarious learning allows us opportunities to increase transparency because it provides students models to know what their goal (Perkins, 1986) is, or at least could be, in algebra and the visual arts.

We should encourage self-motivation (discussed in Chapter 5, in "Aesthetics Outcomes") and self-regulation, allowing students some latitude to determine the course and pace of their study. Some of this is now old hat: allowing students to choose their courses, an idea that was first developed at Harvard University. And within a course, too, there is ample room for students to direct their research by choosing a topic, leading potentially to higher levels of self-motivation and self-regulation.

Also, feedback on students' work is a staple of the educational experience. Piccinin (2003, p. 25), who specializes on feedback and interpersonal relations said that feedback should serve four goals: (a) promote learning; (b) increase motivation; (c) enhance self-esteem; and (d) lead to a deepening relationship between giver and receiver. All of these four points are consistent with the findings of this thesis. I shall consider these four points collectively, dealing broadly with issues of motivation, intrapersonal and interpersonal relations, since all three are somehow related, I believe, as I turn to offer suggestions for pedagogical practice.

We know from the many studies that have been done to corroborate social cognitive theory is that people's perceptions of their abilities, their self-efficacy beliefs, affect their outcomes. So we need to provide opportunities for students to experience success in algebra ("Leading Math Success," 2004; "The Ontario Curriculum," 2007; Reif, 2008) and the visual arts ("The Ontario Curriculum," 2009). In addition, since much communication is non-verbal, for example conveyed by the way we dress, our posture, facial expressions, and so, it is important for teachers to convey positive attitudes about algebra ("Leading Math Success," 2004) and the visual arts ("The Ontario Curriculum," 2009).

Returning to the issue of evaluation, as much as possible if it should be a learning event, not merely to gauge how high we can jump, as it were. That is, evaluation should achieve Piccinin's (2003) four goals. Ironically (and sarcastically), when we often look back at our own education, there are no jobs that require someone who is good at writing examinations! This fact alone makes us rethink if the work we make students do is actually useful, meaningful, and encouraging.

Novel types of assessment in algebra ("Leading Math Success," 2004; "The Ontario Curriculum," 2007; Reif, 2008) and visual arts ("The Ontario Curriculum," 2009) pedagogy, in relationship to the traditional examination, may include: competitions, for example, using games; holistic assessment, like a portfolio, or process-folio, where we document our progress; the inclusion of generalizing and specializing behaviours, by having students work with specific situations, such as a mock-conference or exhibition, where students showcase their work, be it in algebra or visual arts; using group assessment activities that build upon peer motivation; and having types of evaluation that mirror the work done in class, for example, using manipulatives in algebra, or design projects related to revitalizing the students' campus or city in the visual arts.

Perhaps having students have input into how they are assessed is also a good idea, in some cases. And of course, we should always be thinking of how to make the assessment in algebra and the visual arts culturally appropriate—which requires knowing the student and her cultural background, a point I will return to shortly.

Even when we do require examinations, we can offer variations on the usual picture that is conjured up in our minds of rows of desks. For example, Perkin's (1995) emphasis on the surround and Pea's (1993) discussion of off-loading, support the idea of considering open-book examinations, which allows us to learn to work effectively with resources in our surround. For the same reason, providing questions in advance can be good, too, which encourages collaboration.

In fact, group assessment allows us to capitalize on the social nature of learning I have emphasized, for instance, "The Social Dimension." Of course, we can make assessments purposeful by requiring a demonstration of skills *in situ*. And lastly, assessment must be, when desirable and possible, tailored to the individual learner—this stands out quite clearly from the case of Ben. Oral examinations, for instance in the case of Zöe when she came to Canada, are also a good option. My list of recommendations for assessment is neither exhaustive nor specific, related to any one task or age level; however, it is just intended to show that, at the very least, many types of progressive assessment techniques, broadly conceived, are consistent with the data analysis.

However and more generally, in addition to making assessment a less threatening experience and a more purposeful event than it has often been, we also have to know when, and how to offer stiff criticism, too. And to know when to be encouraging and know when critical, we are well advised to follow the motto: *know thyself and know thy student*. By knowing ourselves, we are better aware of when we feel threatened and comfortable; this is an asset in dealing with diverse populations with different learning styles.

We must make sure that our limits do not become those of our students. There is more than one way to solve an algebraic problem; and more than one way to depict a floral scene. And by knowing our students, we will be in a better position to tailor algebraic ("Leading Math Success," 2004) or visual arts ("The Ontario Curriculum," 2009) curriculum to their personal interests and needs, and to implement culturally appropriate curricula.

The ideas I have discussed as recommendations in this subsection can be summed up by attending to lower-order needs like safety (Maslow, 1970): we must create a comfortable learning environment where students in algebra ("Leading Math Success," 2004; Reif, 2008) and the visual arts ("The Ontario Curriculum," 2009) feel free to take risks (recall how important this was to Zöe's pedagogy and demonstrated in Ben's (see Appendices G-J)).

Behavioural Outcomes

Interactions outcomes. In "Interactions," I suggested that algebraic and visual arts behaviour can affect our future actions and thinking, those of our peers, and even the nature of a socio-cultural context. Acting is a priority. (Recall, Molly advocated taking the initiative and painting.) As we can infer from a common adage, once habits and identity are established, for good or ill, they can be difficulty to change. In "Generalizing and Specifying Outcomes," I suggested students need opportunities to practice algebra and visual arts tasks to develop proficiency in these fields. Yet, what has concerned mathematics educators is the reliance on rote learning and drills. That is to say, we want students to be self-motivated, self-regulated life-long learners.

However, also in "Generalizing and Specifying Outcomes" and "Teachers and School Experiences Outcomes," I have hinted how we can achieve these lofty goals. For instance, I suggested we can encourage self-motivation by employing vicarious learning. In "Teachers and School Experiences Outcomes," in addition to modeling practice, I have suggested we should model identity. Finally, also in the same section, I suggested that in order to develop self-efficacy beliefs (and an identity as a mathematician or visual artist), students can likely benefit from the opportunity to experience success. To learn, we must encourage students to act.

Final Remarks

At the end of the “Introduction: A Philosophically Minded Prelude,” I referred to Fodor’s (1982) sceptical remark about introductions as being either superfluous or premature. Perhaps the idea of comparing learning in algebra and the visual arts, as proposed in the introduction, is too ambitious. On the contrary. Learning in algebra and the visual arts can be linked. I have suggested several detailed lines, housed under the broad themes of the personal dimension, environmental dimension, and behavioural dimension, along which we can connect algebra and the visual arts. Indeed, some of the connections like the role of aesthetics and fun have been dramatic.

Returning to the original question—what can we learn about the mind?—the following comes to the fore. The mind is extended in this: at the level of description of mental phenomena, cognition is distributed throughout our, broadly understood, social relations and practices. The mind can be likened to a drop within a socio-cultural stream that includes a past, present, and perceived possibilities—for the individual and collective, in an ongoing cycle of turbulence and harmony.

Ideally, understanding the role of our experiences and environmental factors in influencing cognition and behaviour, will allow us to better understand who we are—and who we want to be. Algebra and the visual arts art are part of two persistent human activities, where we use symbols to make meaning for ourselves and between ourselves. So in the algebraic and visual arts classroom, we invite students to participate in a mini-society embedded in the broader culture.

To follow Perkins' (1993) language, we are prompted to ask, what sort of bee in what sort of hive? We negotiate, both teacher and student, between the universal aims of our society, discipline, and institution—and the particular needs of every student trying to fit into a world that is constantly changing. When we strike the right balance, and make learning in algebra and the visual arts personally meaningful, along some of the lines I have suggested, we go to the place, with our feet, where we belong; and, in our heads, where find ourselves. This sense of social belonging and cognitive harmony has often been called, by Platonists no less, beautiful and ideal.

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Appendix A: Ethics Approval

Today's Date: November 26, 2009
Principal Investigator: Dr. Anoop Gupta
Department/School: Education
REB Number: 09-216
Research Project Title: The Common Link: An Exploration of the Social Cognitive Dimensions of Meaning-Making in Algebra and the Visual Arts Using a Case Study Approach
Clearance Date: November 26, 2009
Project End Date: January 31, 2011

Progress Report Due: November 26, 2010
Final Report Due: January 31, 2011

This is to inform you that the University of Windsor Research Ethics Board (REB), which is organized and operated according to the *Tri-Council Policy Statement* and the University of Windsor *Guidelines for Research Involving Human Subjects*, has granted approval to your research project on the date noted above. This approval is valid only until the Project End Date.

A Progress Report or Final Report is due by the date noted above. The REB may ask for monitoring information at some time during the project's approval period.

During the course of the research, no deviations from, or changes to, the protocol or consent form may be initiated without prior written approval from the REB. Minor change(s) in ongoing studies will be considered when submitted on the Request to Revise form.

Investigators must also report promptly to the REB:
a) changes increasing the risk to the participant(s) and/or affecting significantly the conduct of the study;
b) all adverse and unexpected experiences or events that are both serious and unexpected;
c) new information that may adversely affect the safety of the subjects or the conduct of the study.

Forms for submissions, notifications, or changes are available on the REB website: www.uwindsor.ca/reb. If your data is going to be used for another project, it is necessary to submit another application to the REB.

We wish you every success in your research.

Pierre Boulos, Ph.D.
Chair, Research Ethics Board

cc: Dr. Anthony Ezeife, Education
Mark Curran, Research Ethics Coordinator

This is an official document. Please retain the original in your files.

Appendix B: Consent to Participate in Research

University of Windsor

Consent to Participate in Research

Title of Study: *The Common Link: An Exploration of the Social Cognitive Dimensions of Meaning-Making in Algebra and the Visual Arts Using a Case Study Approach*

You are asked to participate in a research study conducted by Dr. Anoop Gupta, from the Faculty of Education at the University of Windsor. The results of this research will contribute to the completion of my PhD dissertation in Cognition and Learning.

I completed my first PhD in the philosophy of mathematics at the University of Ottawa in 2002, where I explored the empirical basis of arithmetic and, at the same time, defended realism for that domain. In my dissertation in the social sciences, I am interested in studying how we learn mathematics and, using aspects of cognitive science and social cognitive theory as a generic theoretical lens. In my current research, I aim to highlight the common socio-cultural basis of learning in algebra and the visual arts, thus throwing into question a long held divide between the two cultures.

If you have any questions or concerns about the research, please feel to contact me:

Telephone: 519.253.3000 x3808

Res.: 519.258.9061

Email: gupta03@uwindsor.ca

My PhD supervisor:

Dr. Anthony N. Ezeife

Telephone: 519.253.3000 x 2890

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Purpose of the Study

The purpose of this investigation is threefold.

(a) To explore the role of various modalities of the social context in meaning-making in algebra and visual arts from a social cognitive point of view.

(b) To offer suggestions on how we can develop pedagogical regimes for algebra and the visual arts that build upon the way we make meaning in various social contexts.

(c) To consider some factors that contribute to identity formation (e.g., prior knowledge, family and educational experiences), in the design and differentiation of curricula for students in algebra and the visual arts.

Procedures

Volunteers in this study would be asked to do the following things:

Subjects will be asked to meet for an interview and answer questions detailed on the interview instrument (attached), which relates generally to their learning in their specific field. Subjects will be asked to share any sample of their works they wish to discuss as it relates to the questions in the interview. Once the interviews are completed, and I have had time to work with the data, I will provide them with brief user friendly research summary of the initial results. Subjects will also be asked to look over the case study that pertains to them, to seek their feedback on my interpretation of their responses.

Potential Risks and Discomforts

This study poses no serious risks because it deals with only four adults from non-vulnerable populations in interviews. I foresee, thus, no reasonable potential for physical or social harm. However, it is always possible that a participant could become upset retelling say, a traumatic learning experience. To minimize risks, all participants are informed of their right to withdraw from the study at any time; and to not be included in the study once the interviews have been completed.

Potential Benefits to Subjects and/or Society

By participating in this study, subjects may gain a better sense of the learning process in their respective fields: how participant's best learn. Also, this study would fulfill the lacuna in the literature on cognitive science, by building a bridge between the way we understand learning in algebra and the visual arts. We need a theoretical rationale to promote a curriculum that includes identifiable modalities, like prior knowledge, family and educational experiences, and identity, of meaning-making.

The issue of identity, in fact, has become important to teachers as they attempt to instruct students that are male and female; and increasingly, are from many different socio-cultural backgrounds. It is important to understand the socio-cultural background of students and make reference to it in designing curricula in algebra and the visual arts. This study will help us do that.

Payment for Participation

A Tim Horton's or Chapter's Gift Card valued at ten dollars.

Confidentiality

Any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission. To ensure confidentiality your name and other identifying features of the data will be altered.

All materials will be secured, locked up, in my office. Specifically, the data will be locked in a filing cabinet; this is enclosed in a locked office, which further is locked within a secure area. Audio tapes will be deleted and only material to be retained will be what appears in the thesis, in terms of quotations from the interviews.

Participation and Withdrawal

You can choose whether to be in this study or not. If you volunteer to be in this study, you may withdraw at any time without consequences of any kind. You may also refuse to answer any questions you don't want to answer and still remain in the study. The investigator may withdraw you from this research if circumstances arise which warrant doing so.

Feedback of the Results of this Study to the Subjects

1. Subjects will be provided a brief user-friendly research summary of initial results.

2. Subjects will have the opportunity to review my case studies once typed.

3. Subjects will be provided relevant parts of the thesis, when completed (tentatively in January 2011).

Date when results are available: _______01.01.11_________________________

Subsequent Use of the Data

This data will not be used in subsequent studies.

Rights of the Research Subjects

You may withdraw your consent at any time and discontinue participation without penalty. If you have questions regarding your rights as a research subject, contact:

Research Ethics Coordinator, University of Windsor, Windsor, Ontario, N9B 3P4; Telephone: 519-253-3000, ext. 3948; e-mail: ethics@uwindsor.ca

Signature of the Research Subject/Legal Representative

I understand the information provided for the study *The Common Link: An Exploration of the Social Cognitive Dimensions of Meaning-Making in Algebra and the Visual Arts Using a Case Study Approach* as described herein. My questions have been answered to my satisfaction, and I agree to participate in this study. I have been given a copy of this form.

____

Name of Subject

______________________________ ____________

Signature of Subject Date

Signature of the Investigator

These are the terms under which I will conduct research.

______________________________ ____________

Signature of Investigator Date

Revised April 2009

Appendix C: Letter of Information for Consent to Participate in Research

University of Windsor

Letter of Information For Consent To Participate in Research

Title of Study: *The Common Link: An Exploration of the Social Cognitive Dimensions of Meaning-Making in Algebra and the Visual Arts Using a Case Study Approach*

You are asked to participate in a research study conducted by Dr. Anoop Gupta, from the Faculty of Education at the University of Windsor. The results of this research will contribute to the completion of my PhD dissertation in Cognition and Learning.

I completed my first PhD in the philosophy of mathematics at the University of Ottawa in 2002, where I explored the empirical basis of arithmetic and, at the same time, defended realism for that domain. In my current dissertation in the social sciences, I am interested in studying how we learn mathematics, using aspects of cognitive science and social cognitive theory as a generic theoretical lens. In this research, I aim to highlight the common socio-cultural basis between learning in algebra and the visual arts, thus throwing into question a long held divide between the two cultures.

If you have any questions or concerns about the research, please feel to contact me:

Telephone: 519.253.3000 x3808

Res.: 519.258.9061

Email: gupta03@uwindsor.ca

My PhD supervisor:

Dr. Anthony N. Ezeife

Telephone: 519.253.3000 x 2890

Email: aezeife@uwindsor.ca

Purpose of the Study

The purpose of this investigation is threefold.

(a) To explore the role of various modalities of the social context in meaning-making in algebra and visual arts from a social cognitive point of view.

(b) To offer suggestions on how we can develop pedagogical regimes for algebra and the visual arts that build upon the way we make meaning in various social contexts.

(c) To consider some factors that contribute to identity formation (e.g., prior knowledge, family and educational experiences), in the design and differentiation of curricula for students in algebra and the visual arts.

Procedures

Volunteers in this study would be asked to do the following things:

Subjects will be asked to meet for an interview and answer questions detailed on the interview instrument (attached), which relates generally to their learning in their specific field. Subjects will be asked to share any sample of their works they wish to discuss as it relates to the questions in the interview. Once the interviews are completed, and I have had time to work with the data, I will provide them with brief user friendly research summary of the initial results. Subjects will also be asked to look over the case study that pertains to them, to seek their feedback on my interpretation of their responses.

Potential Risks and Discomforts

This study poses no serious risks because it deals with only four adults from non-vulnerable populations in interviews. I foresee, thus, no reasonable potential for physical or social harm. However, it is always possible that a participant could become upset retelling say, a traumatic learning experience. To minimize risks, all participants are informed of their right to withdraw from the study at any time; and to not be included in the study once the interviews have been completed.

Potential Benefits to Subjects and/or to Society

By participating in this study, subjects may gain a better sense of the learning process in their respective fields; how participants best learn. Also, this study would fill the lacuna in the literature on cognitive science, by building a bridge between the way we understand learning in algebra and the visual arts. We need a theoretical rationale to promote a curriculum that includes identifiable modalities, like prior knowledge, family and educational experiences, and identity, of meaning-making.

The issue of identity, in fact, has become important to teachers as they attempt to instruct students that are male and female; and increasingly, are from many different socio-cultural backgrounds. It is important to understand the socio-cultural background of students and make reference to it in designing curricula in algebra and the visual arts. This study will help us do that.

Payment for Participation

Tim Horton's or Chapter's gift card valued at ten dollars.

Confidentiality

Any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission. To ensure confidentiality your name and other identifying features of the data will be altered.

All materials will be secured, locked up, in my office. Specifically, the data will be locked in a filing cabinet; this is enclosed in a locked office, which further is locked within a secure area. Audio tapes will be deleted and only material to be retained will be what appears in the thesis, in terms of quotations from the interviews.

Participation and Withdrawal

You can choose whether to be in this study or not. If you volunteer to be in this study, you may withdraw at any time without consequences of any kind. You may also refuse to answer any questions you don't want to answer and still remain in the study. The investigator may withdraw you from this research if circumstances arise which warrant doing so.

Feedback of the Results of this Study to the Subjects

1. Subjects will be provided a brief user-friendly research summary of initial results.
2. Subjects will have the opportunity to review my case studies once typed.
3. Subjects will be provided relevant parts of the thesis, when completed (tentatively in January 2011).

Date when results are available: ________________01.01.11________________

Subsequent Use of Data

This data may be used in subsequent studies.

Rights of the Research Subjects

You may withdraw your consent at any time and discontinue participation without penalty. If you have questions regarding your rights as a research subject, contact:

Research Ethics Coordinator, University of Windsor, Windsor, Ontario N9B 3P4;

Telephone: 519-253-3000, ext. 3948; e-mail: ethics@uwindsor.ca

Signature of the Investigator

These are the terms under which I will conduct research.

_______________________________ _______________

Signature of Investigator Date

Revised April 2009

Appendix D: Consent for Audio Taping

University of Windsor

Consent for Audio Taping

Research Subject Name:

Title of the Project: *The Common Link: An Exploration of the Social Cognitive Dimensions of Meaning-Making in Algebra and the Visual Arts Using a Case Study Approach*

I consent to the audio-taping of interviews.

I understand these are voluntary procedures and that I am free to withdraw at any time by requesting that the taping be stopped. I also understand that my name will not be revealed to anyone and that taping will be kept confidential. Tapes are filed by number only and store in a locked cabinet.

I understand that confidentiality will be respected and that the audio tape will be for professional use only.

____________________________ ____________________

Signature of Parent or Guardian Date

Or

____________________________ ____________________

Research Subject Date

Appendix E: Permission to Reproduce Molly's Artwork

From: [Molly <Anonymous>]

To: Anoop Gupta <gupta03@uwindsor.ca>

Cc:

Bcc: Anoop Gupta/gupta03/University of Windsor

Date: Tuesday, March 16, 2010 03:08PM

Subject: Re: Copy of your case and query about pictures

History: This message has been replied to.

I am truly honored. Thank you Anoop. *Use whatever paintings in any way you choose. I have no problem with it.*

Best regards,

Molly

Appendix F: Permission to Reproduce Zöe's Artwork

From: Gene Lewis <glewis@etfo.org>

To: Anoop Gupta <gupta03@uwindsor.ca>

Cc:

Bcc: Anoop Gupta/gupta03/University of Windsor

Date: Thursday, March 25, 2010 11:20AM

Subject: RE: Copy of image included, thanks

For Follow up: Normal Priority.

History: This message has been replied to.

Anoop,

Yes, you are approved to use the image on a one time basis.

Thanks,

Gene

Gene Lewis, General Secretary, Elementary Teachers' Federation of Ontario,

Suite 1000, 480 University Avenue, Toronto ON M5G 1V2 Canada

Telephone: 416-962-3836; Toll-free: 1-888-838-3836; Fax: 416-642-2424; Email:

glewis@etfo.org; Web site: www.etfo.ca

From: [Zöe <Anonymous>]

To: Anoop Gupta <gupta03@uwindsor.ca>

Cc:

Bcc: Anoop Gupta/gupta03/University of Windsor

Date: Saturday, March 20, 2010 03:29PM

Subject: Re: Postscript on picture

History: This message has been replied to.

Hi again,

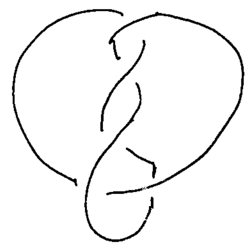
In as far as the visuals, I am flattered, thank you. The other issue regarding goggling for other pictures, hence finding out who " Zöe" is: again, *I don't have a problem with it [using my artwork in the dissertation].* I have been honest and forthright. Nothing to hide. I am proud of my profession, my life, and what I do.

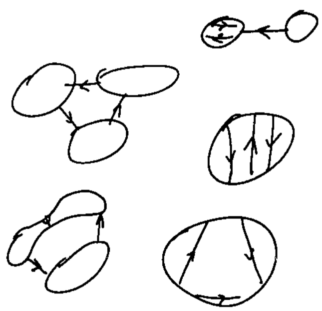
Take care,

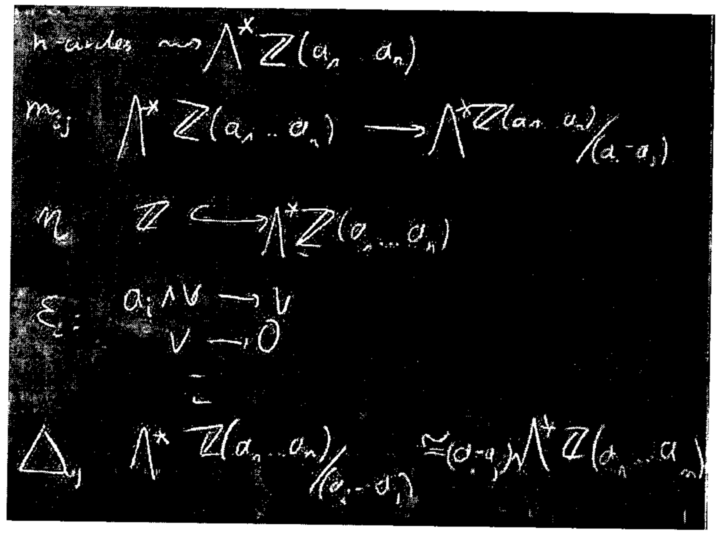
Zöe

P.S., I have no secrets.

Appendix G: Samples of Pictures from Ben's Notebooks







from BBS/Putyra

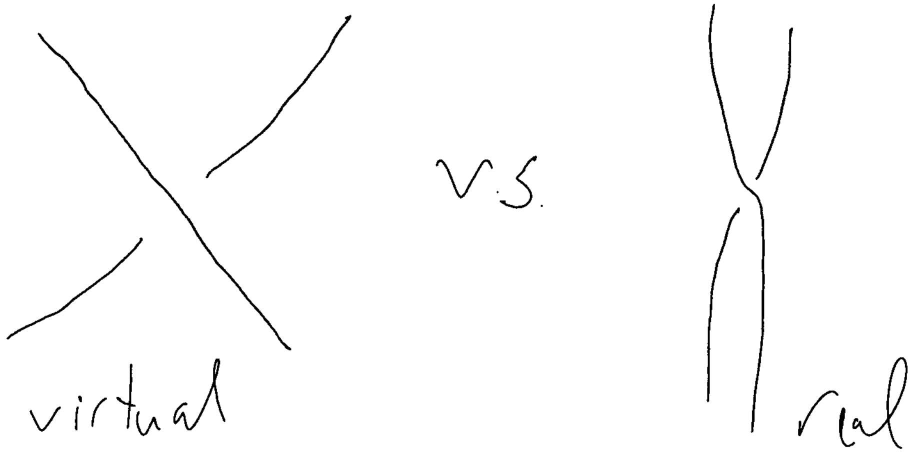
$$m: \quad \begin{aligned} 1\cdot 1 &\longrightarrow \alpha_1 1 \\ 1\cdot X &\longrightarrow \alpha_2 X \\ X\cdot 1 &\longrightarrow \alpha_3 X \\ X\cdot X &\longrightarrow 0 \end{aligned}$$

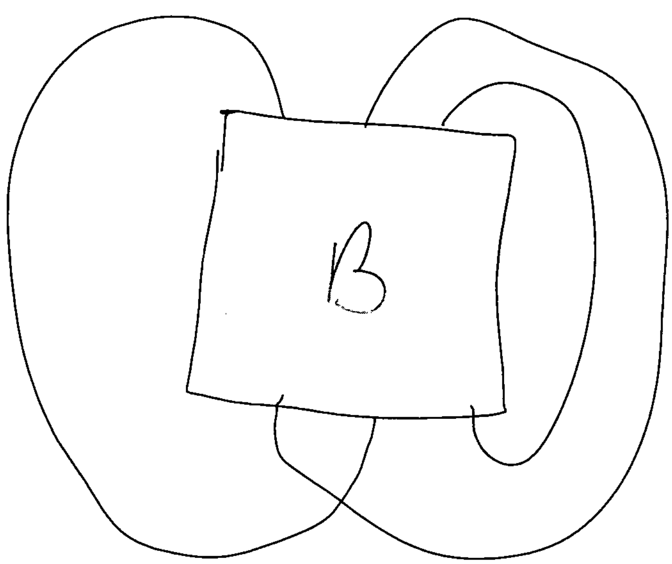
$$\Delta: \quad \begin{aligned} 1 &\longrightarrow \alpha_4\, 1\otimes X + \alpha_5\, X\otimes 1 \\ X &\longrightarrow \alpha_6\, X\otimes X \end{aligned}$$

The AMR idea

April-08-08
7:28 PM

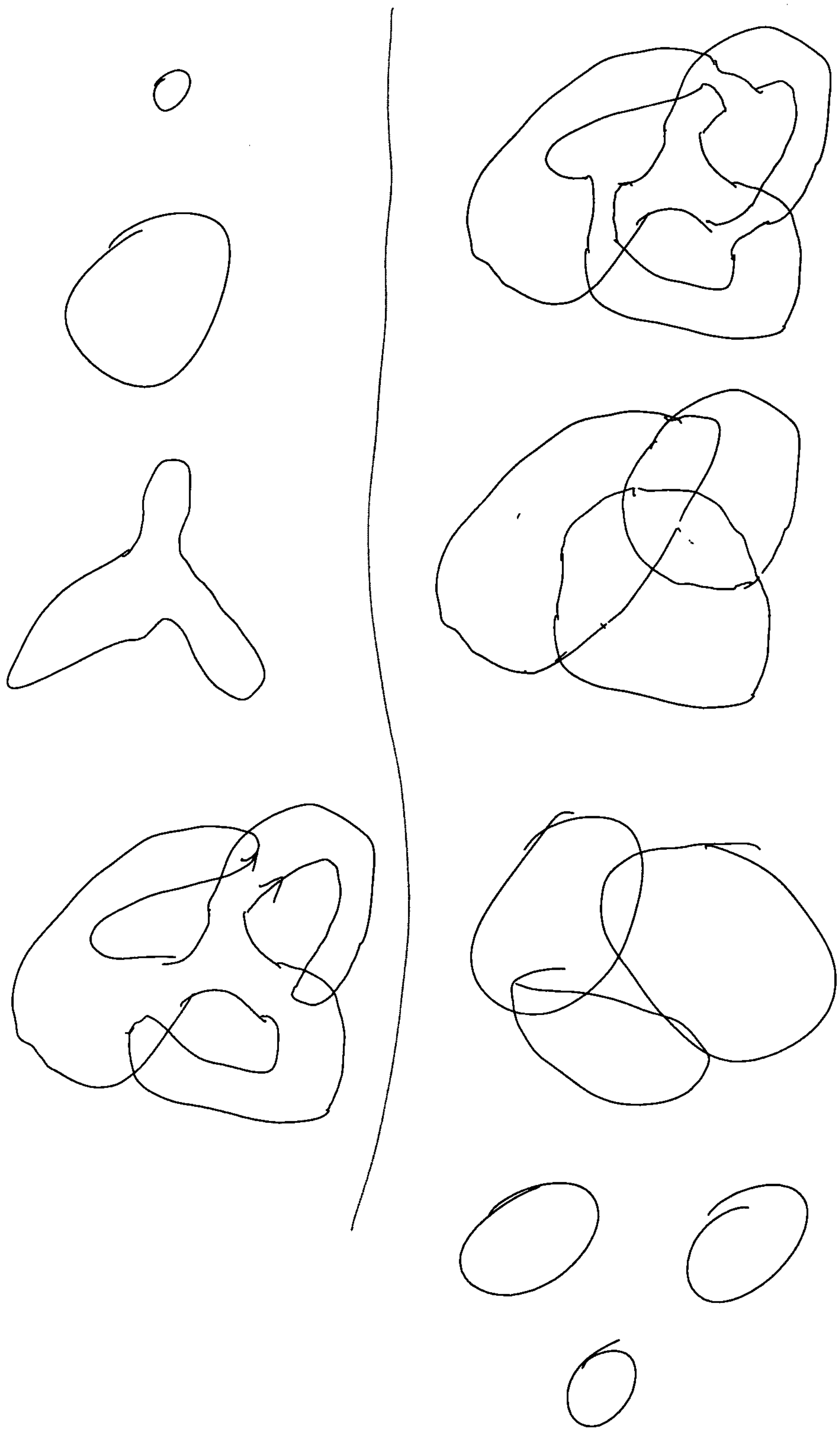
"Tangles with highlighting"





Boy's Surface

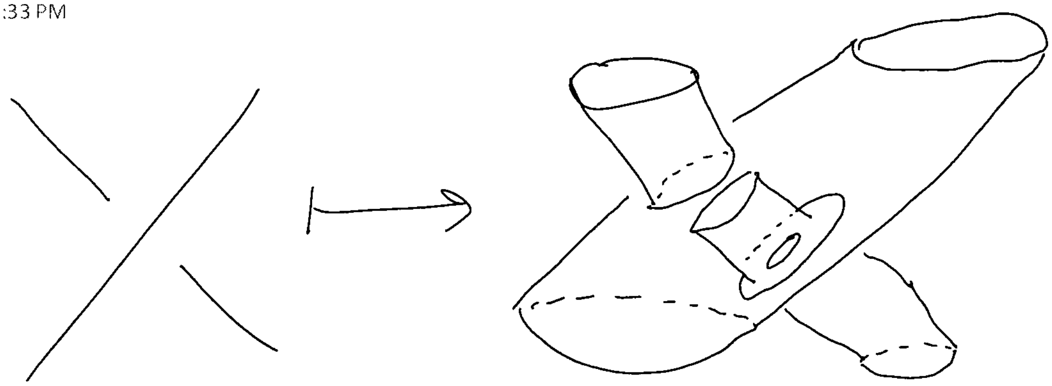
May-07-08
2:33 PM



$\chi =$
$1 - 3 + 3$
$= 1$

Carter's Picture of a Xing in 4D

May-19-08
1:33 PM

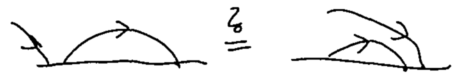


Too Much Agenda for July 4th

349

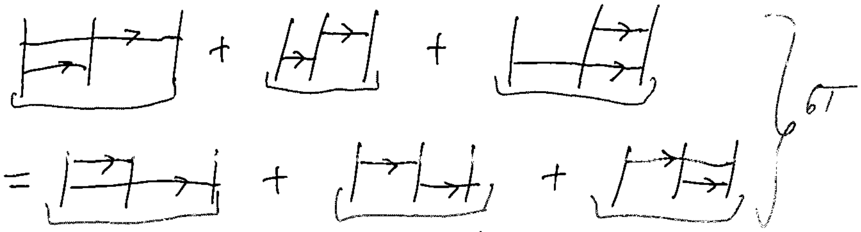
- ☺ Figure out the weight system formula underlying the determinant formula for w-Alexander. } also, use it to recover the global formula.
- ☺ Decide if it has an immanant extension.
- ☺ Decide if w-Alexander has an immanant extension.
- ☺ Finally understand what the Infinitesimal Alexander Module means, and how its A and W sectors globalize.

Does W_{alex} satisfy a 2T?



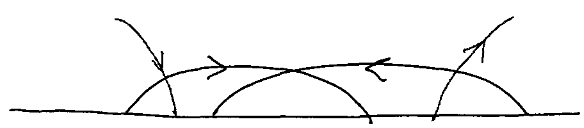
But who says $\vec{W}_{alex}$ & $\vec{W}_{gl}$ remain related?

Probably yes; in W_{gl} red & blue cancel independently; mod out also by TC and 2T is left:

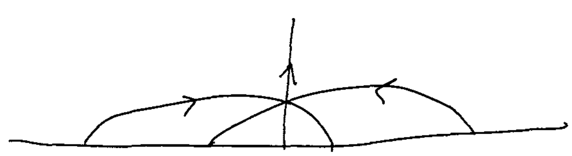


See 2009-01 / 6T For gl(N)

Q: What's $\mathcal{A}^w/2T$ (and $\mathcal{A}^v/3T$?)



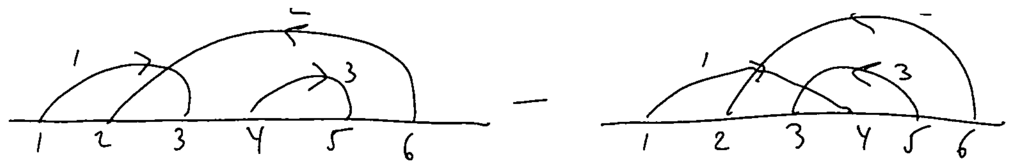
... cannot naively untrap



Bug or problem: ?

In[48]:= Wa[Diag[ar[1, 3], ar[4, 5], ar[6, 2]] - Diag[ar[1, 4], ar[5, 3], ar[6, 2]]]

Out[48]= -1



$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ There is at least a D_L, D_R, ω_1 issue!

Hilbert's 13th revisited

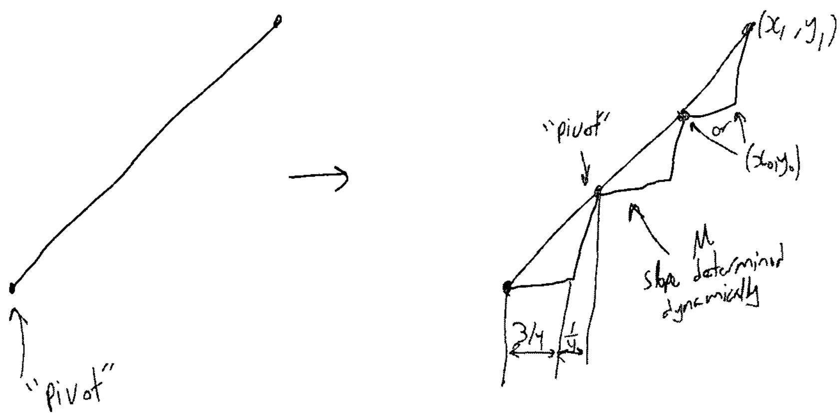
Proposed Title. "Dessert: Hilbert's 13th Problem, in Full Colour."

Abstract. To end a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnol'd solution of Hilbert's 13th problem with lots of computer-generated rainbow-painted 3D pictures.

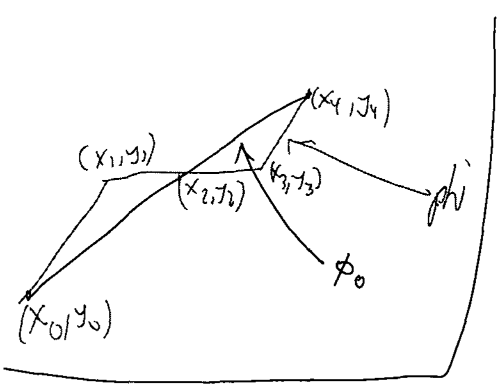
In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnol'd showed him silly (ok, it took 60-70 years, so it is a bit tricky) by showing that *any* continuous function f of any finite number of variables is such a finite composition of continuous functions of one variable and several instances of the binary function "+" (addition). For f(x,y)=xy, this may be xy=exp(log x + log y). For f(x,y,z)=x^y/z, this may be exp(exp(log y + log log x) + (-log z)). What might it be for the real part (say) of the Riemann zeta function?

The only original material in this talk will be the pictures. The math was known in the 60s. It was the first seminar lecture I ever gave, back as an undergraduate in Tel Aviv in 1983.

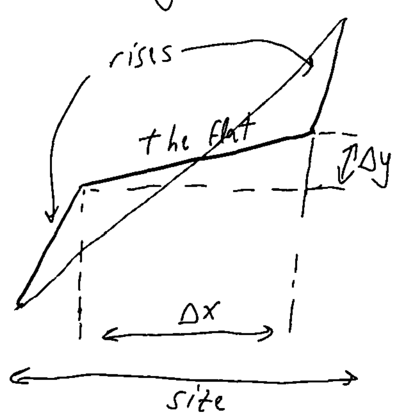
Scheme: $\phi_{\ldots 3,4}$ comes from $\phi_{\ldots,3}$ via



Fold; See Mathematica / Hilbert13th - Failure1.nb



Drawing for handout:



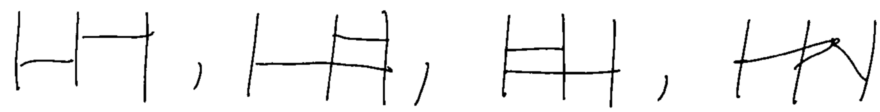
slope $= \frac{\Delta y}{\Delta x}$

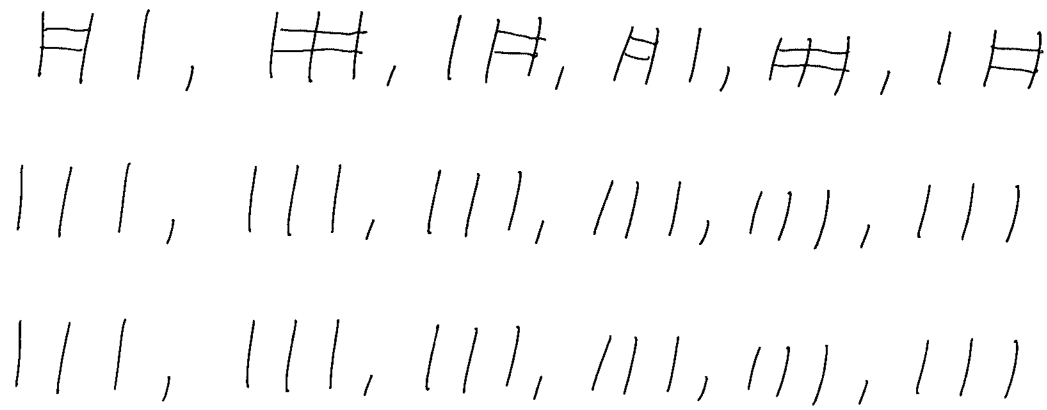
fill factor $= \frac{\Delta x}{\text{size}}$

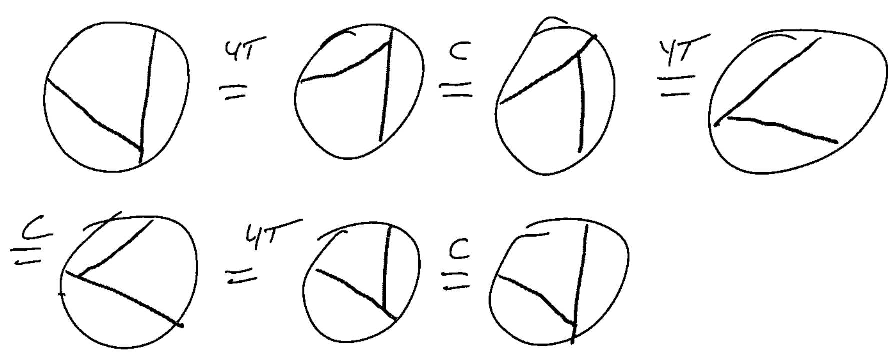
A Political statement.

Bad mouthing: I hate power point!

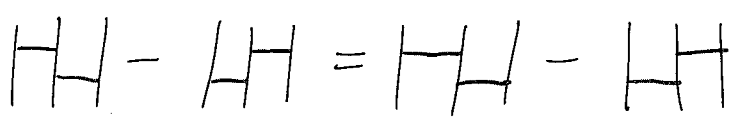
counting only non-deg ones, $\dim \mathcal{G}_2\mathcal{A}_3^{hor} = 4$:

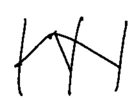


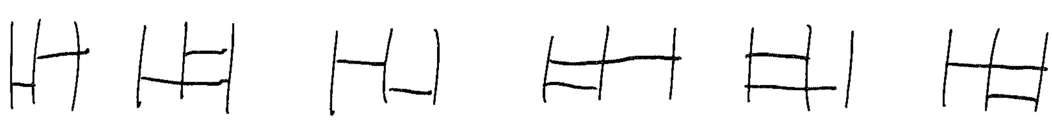




This means:

 etc.

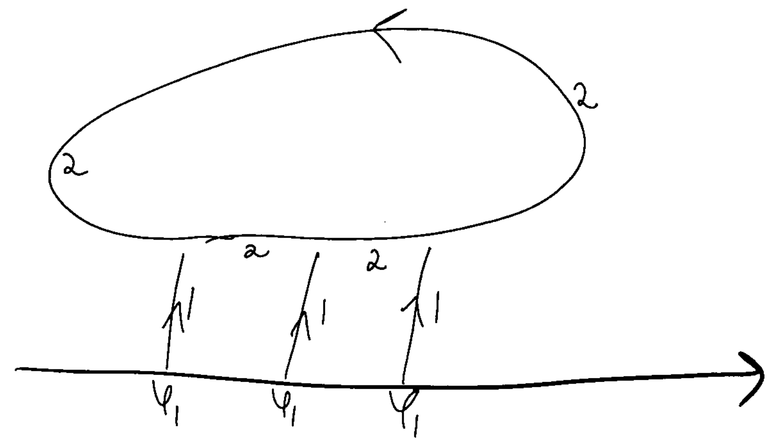
So $\dim \mathcal{G}_2\mathcal{A}_3^{hor,\,col,\,no\text{-}deg} = 7$:  and



Something might still be gained by labeling skeleton arcs.

The 2D Lie Algebra

April-29-09
9:15 AM



$$[x_1, x_2] = x_2$$

$$([x_1, \varphi_1])(x_1) = 0 \qquad ([x_1, \varphi_1])(x_2) := -\varphi_1([x_1, x_2]) = -\varphi_1(x_2) = 0$$

$$\Longrightarrow [x_1, \varphi_1] = 0$$

Likewise, $[x_1, \varphi_2] = -\varphi_2$

$$[x_2\ \varphi_1] = 0$$

$$[x_2\ \varphi_2] = \varphi_1 :$$

$$([x_2 \varphi_2])(x_1) = \varphi_2([x_1, x_2]) = \varphi_2(x_2) = 1$$

Appendix H: Samples of Experimentation in Ben's Notebooks

Siddarth's quandle

April-18-08
8:33 AM

$Q = \{1,2,3\}$ $\quad 2 \uparrow 3 = 1$, $3 \uparrow 2 = 3$

$(a \uparrow b) \uparrow c = (a \uparrow c) \uparrow (b \uparrow c)$

if $a = 1$; Follows from the unit axioms.

$b = 1$; -//-

$c = 1$; -//-

So we only need to check this for $a,b,c \in \{2,3\}$.

222 ✓ no inverses!

223 $1=1$

232 $1=1$

233 $1=1$

322 $3=3$

323 $3=3$

332 $3=3$

333 $3=3$

Comparison of Strategies

December-14-08
11:24 AM

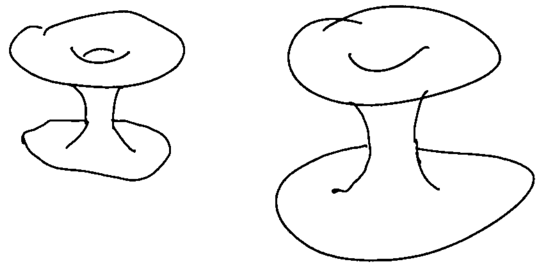
Log / BCH	Scatter and Glue
Almost the first thing that comes to mind.	Definitely not the first thing you would consider
$Z = \exp L$ illusion of simplicity	$Z = G^{-1}(\Gamma)$ Looks mysterious
All you need is L	All you need is Γ, but without S you cannot in practice compute anything
Needs BCH Two options: "All strands at once" - BCH/[[L,L],[L,L]] is not sufficient ☹ "BCH stand by strand" - may work.	Can be used to derive BCH
No consistancy condition required.	I don't know how to write the consistency that is required between S and G. Thus when solving equations, the unknown remains Z or L and cannot be replaced by G. $\Rightarrow$ The $L \to G$ function must be explicitly computable?!
Composition is highly non-linear, involves multiple BCH's, and I don't really understand how to implement it.	Composition is reasonably clear.
Make sence only in Δ-resorting	Makes sense in all internal

	be explicitly computable!
Composition is highly non-linear, involves multiple BCH's, and I don't really understand how to implement it.	Composition is reasonably clear.
Make sense only in Δ-respecting internal quotients.	Makes sense in <u>all</u> internal quotients, including ones in which disconnected relations are allowed. <u>Question</u>: can we use this, say, for the Jones quotient of $\mathcal{A}$ (classical F.T., no u)?

<u>Conclusion</u> For now, scatter and flow wins, though only because of my present difficulties working with BCH. Once I overcome these, Log/BCH may win.

Appendix I: Samples of Questions in Ben's Notebooks

Read Lemma 3.6 of
Ozsvath-Szabo's arXiv:math.SG/0101206



Could it be that there is an Alek-Tor
F in $\vec{A}_2^{hor}$?

In fact, does $\vec{A}_n^{hor}$ inject into $\vec{A}^{wt}$?
(Does it surject? (No, for divergence reasons))

Is the Fock free calculus related to projectivization?

Is there a relation between anonymization and equivariant homology?

Endomorphisms of Free Groups

March-03-08
7:28 PM

Question: Is it true that every quandle-like automorphism of the free group on n letters comes from a virtual braids?

Question Can I describe the category of free groups and homomorphisms between them using generators and relations?

— Unlikely — this seems too closely related to finding "optimal presentations" for finitely presented groups.

Combinatorial Algebra???

December 09-08
5:54 PM

"Combinatorial Algebra" needs a name?

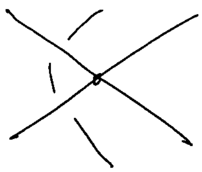
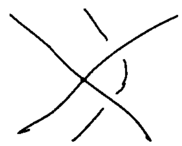
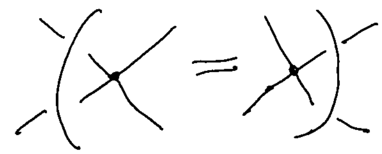
(or is it already named in this sentence?)

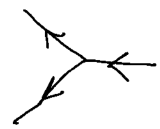
The science of posing and solving equations in spaces of diagrams/graphs mod "local" relations.
(which tend to represent "universal" formulas)

probably not — google already knows of "combinatorial algebra", as something else.

Graphical Algebra?
Pictorial Algebra?
Universal Algebra?
Formal Algebra?
Graphormal Algebra?
Diagrammatic Algebra?
Birdtracks Algebra?

These are virtual knots modulo just one of the naive relations:

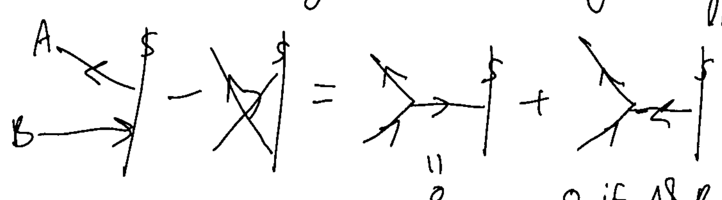
 $\neq$  but 

In $\vec{A}^t$, this becomes  $= 0$ (call the quotient $\vec{A}^{cc}$)

It is also (expectedly) invisible to the upper fundamental group.
It also reads "co-commutative Lie bialgebras".

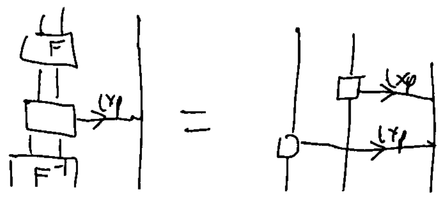
Given all this, perhaps I should forget all about automorphisms of free groups?

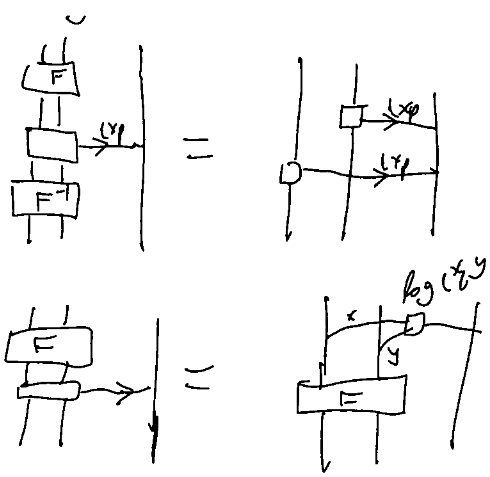
only  is present, and its back legs commute along a skeleton line.

Also, 

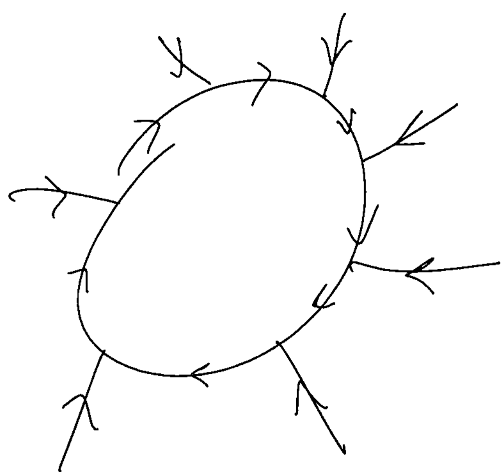
0 if A&B belong to the same component.

$F: x+y \mapsto \log e^x e^y$ in $\vec{A}^{cc}$:





Question: Do wheels as below vanish?



Random

July-02-09
11:09 AM

Is there a relationship between the many bases of the IAM for tangles and cluster algebras?

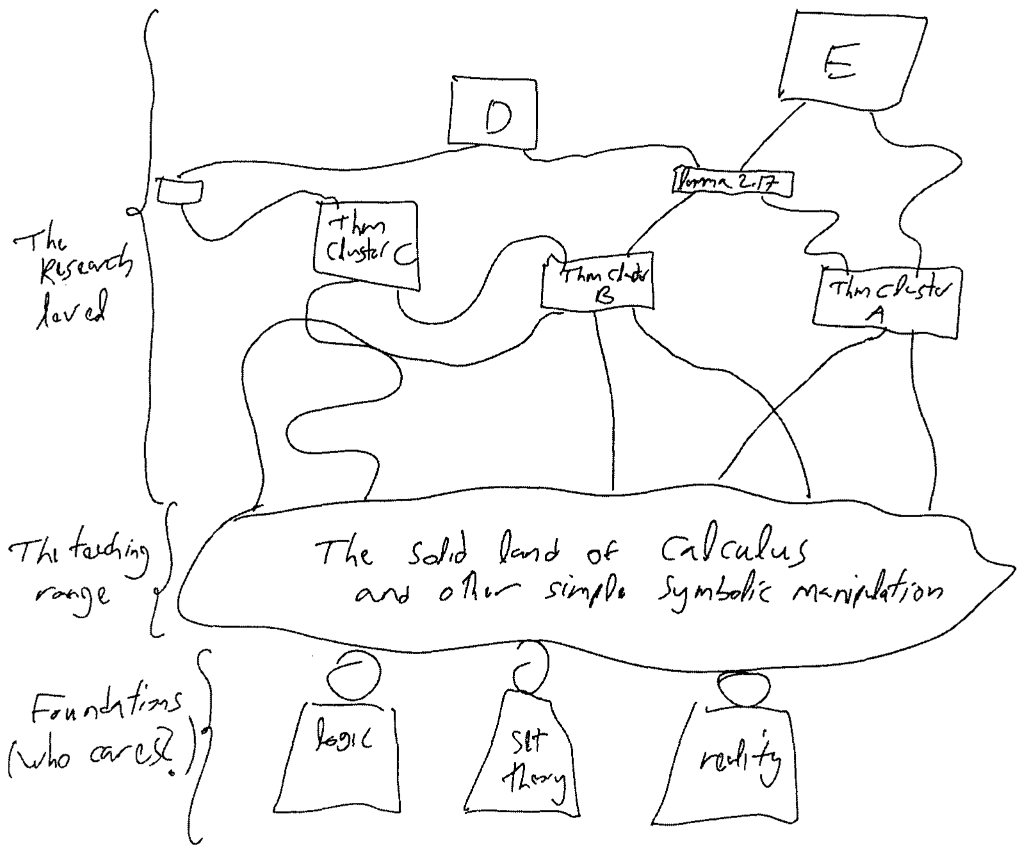
Is there an immanent formula for the Alexander polynomial, to generalize the immanent formula for its weight system?

Random

October-12-09
9:26 PM

Is every algebraic structure the projectivisation of a set-theoretic structure?

Appendix J: An Example of Philosophical Remarks in Ben's Notebooks



Research mathematics is happily standing on a web of infallible infinitly strong totally rigid and never corroding xarcon-nanotubes of argument threads; any cluster of mathematical knowledge might be totally dependent on several of those long and winding nanotubes, yet it is so solid it can support arbitrarily many other nano-tubes and theorem clusters.

This completely ignores the nature of humans!

Climbing to the top [where feeding occurrs] requires years of treking and gymnastics.

In practice, most of us cheat here and there, so we are no longer in the unique field of human thought in which anybody can trace everything (s)he uses to the basic principles. We often don't even know the length of thread that we have skipped.

Due to the hasty nature of our climbing, many sections of nano-tube and even some knowledge clusters remain unvisited and unmaintained. Can we really trust their infinite strength? Remember, the were created by human with career concerns, are refereed by other humans with other things on their mind.

Yet we give credit only to the first proof, we don't care about "cosmetic improvements", shortcuts and alternative paths up, or about "experimental verifications". We tend to write things up before we have fully digested them, hence our nanotube are too long and winding and our dependencies to many, and we defend our territory against improvements and re-writes. We only talk about our latest.

(Aside: there is a similar problem with "library inclusions" in computer science, which leads to "bloating").

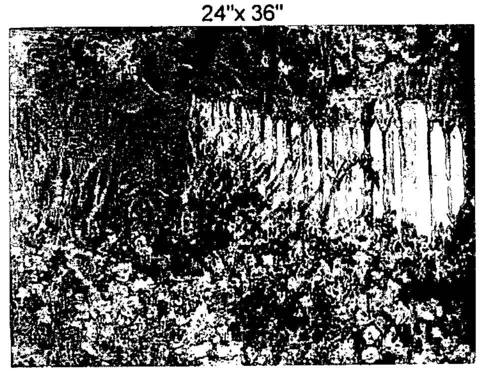
Appendix K: Samples of Molly's Artwork

Portage Point
24"x 30"

Crimson Shield
24"x 36"

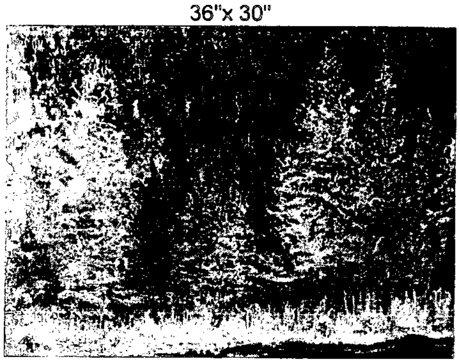
Blueberry Island

36"x 30"

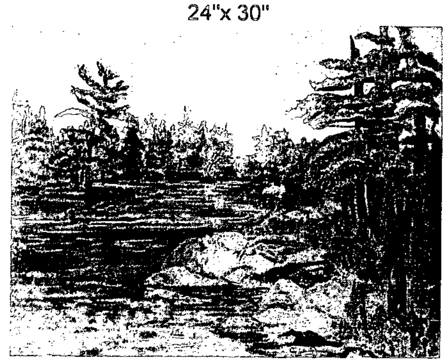


Forgotten Garden

24"x 30"



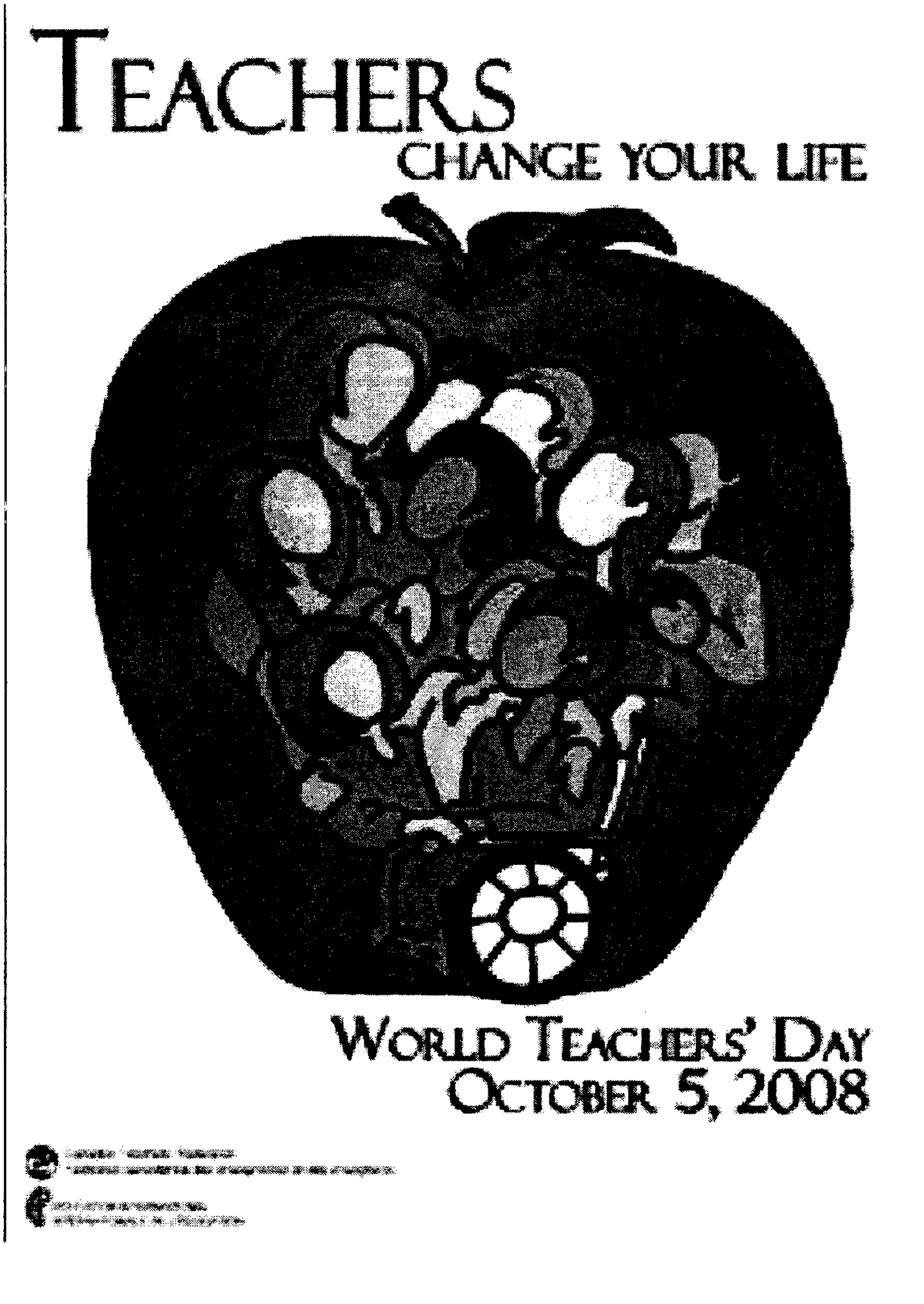
Tamaracks at Algonquin
24"x 30"



White Pines on Rock Lake
24"x 30"

Algonquin Wetlands
24"x 30"

Appendix L: Sample of Zöe's Artwork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WORLD TEACHERS' DAY
OCTOBER 5, 2008

Vita Auctoris

After the 1969-summer of love, Gupta was born in Poole, England, and is of Punjabi descent. He has citizenship in three countries: British, by birth; Canadian, by happenstance (permanent resident, 1975; Canadian citizen, 1997); and Indian, by pleasure (Indian Overseas Citizen, 2006).

Gupta holds a BA and MA in philosophy from McMaster University, in Canada, as well as a PhD specializing in the philosophy of mathematics, from the University of Ottawa, Canada. From the University of Windsor Gupta holds a BEd and MEd (in administration). He anticipates being a graduate of Windsor's Joint PhD in educational studies, specializing in cognition and learning.