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Finite element analysis of an arbitrary Lagrangian–Eulerian method for Stokes/parabolic moving interface problem with jump coefficients

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ABSTRACT

In this paper, a type of arbitrary Lagrangian–Eulerian (ALE) finite element method in the monolithic frame is developed for a linearized fluid–structure interaction (FSI) problem — an unsteady Stokes/parabolic interface problem with jump coefficients and moving interface, where, the corresponding mixed finite element approximation in both semi- and fully discrete scheme are developed and analyzed based upon one type of ALE formulation and a novel H^1-projection technique associated with a moving interface problem, and the stability and optimal convergence properties in the energy norm are obtained for both discretizations to approximate the solution of a transient Stokes/parabolic interface problem that is equipped with a low regularity. Numerical experiments further validate all theoretical results. The developed analytical approaches and numerical implementations can be similarly extended to a realistic FSI problem in the future.

1. Introduction

The interaction of a flexible structure with a flowing fluid in which it is submersed or by which it is surrounded gives rise to a rich variety of physical phenomena with applications in many fields of engineering and biology, e.g., the vibration of turbine blades impacted by the fluid flow, the floating parachute wafted by the air current, the flow of blood through arteries, and etc. These interactions basically comprise many applications of an important problem – fluid–structure interaction (FSI) problem – in hydrodynamics, aerodynamics and hemodynamics [1–4]. However, the study of FSI problems are often too complex to solve analytically and are therefore done using numerical methods. Towards numerical analyses for a realistic and complex FSI problem in the future, in this paper we consider to solve a simplified FSI model instead, which is represented by a type of linearized FSI problem — an unsteady Stokes/parabolic moving interface problem, where the fluid is modeled by Stokes equations in terms of fluid velocity and pressure, while the structure is modeled by a vector-valued parabolic equation in terms of the structure velocity.

Due to its high accuracy and practicality, the body-fitted mesh method has become the most reliable numerical approach for solving unsteady moving interface/boundary problems including FSI. The challenge is of course efficiently generating a moving mesh that adapts to the moving interface/boundary at all times, and to tackle that challenging

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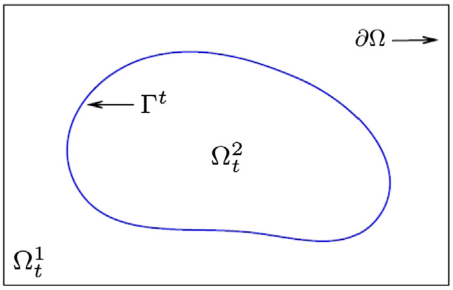

Fig. 1. Schematic domain Ω with the interface Γ^t between two subdomains Ω_t^1 and Ω_t^2.

problem, the arbitrary Lagrangian–Eulerian (ALE) method has been standing out for decades because of its high feasibility in practice and accurate interface tracking all the time, where, the generated moving ALE mesh on the interface continuously accommodates to the shared interface of materials on both sides, and therefore interface conditions of moving interface problems can be satisfied at all times, exactly. In summary, ALE methods take the domain motion into consideration then redescribe the moving interface/boundary problem.

In this paper we develop a type of ALE-finite element method for a linearized FSI problem between the unsteady Stokes equations and a vector-valued parabolic equation coupled over a moving interface with jump coefficients, moreover, we will analyze the optimal convergence property for both semi- and fully discrete scheme of this ALE method with respect to a realistically low regularity of real solution to the presented Stokes/parabolic interface problem, where, we will utilize a novel H^1-projection technique [5] that is associated with a moving interface problem. In the literature, a classical H^1-projection is adopted to carry out ALE-finite element analyses for single Stokes equations on a moving domain [6], and a limited sub-optimal convergence order is provided due to the effect of extra approximation error from the discrete ALE mapping. A novel H^1-projection that is introduced in [5] can derive an optimal convergence rate for one type of ALE-finite element method since it takes the full influence of the discrete ALE mapping into consideration. In this paper, we will apply this special H^1-projection to another type of ALE-finite element method for an unsteady Stokes/parabolic moving interface problem, and analyze its optimal convergence property for both semi- and full discretizations. In addition, the developed finite element analysis technique in this paper that utilizes the novel H^1-projection for a type of ALE method can be similarly extended to numerical analyses of a realistic FSI problem, which will be our next work in the future.

This paper is divided into six sections. In Section 2, we present the model description of an unsteady Stokes/parabolic interface problem, establish the ALE mapping and some standard definitions, followed by the ALE formulation of the model problem, then finish this section with a type of ALE weak form. Section 3 defines the semi-discrete ALE-finite element scheme, and the novel H^1-projection that is first introduced in [5]. Stability and error analyses are carried out for the semi-discrete scheme in this section as well. Section 4 begins with the derivation of the fully-discrete scheme, then analyzes its optimal error estimates by means of the H^1-projection. Numerical experiments are carried out in Section 5 to validate the theoretical results. We end the paper with a few concluding remarks in Section 6.

2. The model problem and its weak form in ALE description

2.1. Model description

Let $\Omega \subset \boldsymbol{R}^d$ $(d = 2, 3)$, and $T > 0$. Two subdomains, $\Omega_i^t := \Omega_i(t) \subset \Omega$ $(i = 1, 2)$ $(0 \le t \le T)$, satisfying $\overline{\Omega_1^t} \cup \overline{\Omega_2^t} = \overline{\Omega}$ and $\Omega_1^t \cap \Omega_2^t = \emptyset$, are separated by an interface: $\Gamma^t := \Gamma(t) = \partial\Omega_1^t \cap \partial\Omega_1^t$ that may move/deform along with $t \in (0, T]$, causing Ω_i^t $(i = 1, 2)$, which are termed as the current (Eulerian) domains with respect to $\boldsymbol{x}$, to change with $t \in (0, T]$, in contrast to their initial (reference/Lagrangian) domains, Ω_i^0 $(i = 1, 2)$ with respect to $\hat{\boldsymbol{x}}$. Here a *flow map* is defined from Ω_i^0 to Ω_i^t $(i = 1, 2)$ as: $\hat{\boldsymbol{x}}_i \mapsto \boldsymbol{x}_i(\hat{\boldsymbol{x}}_i, t)$ such that $\boldsymbol{x}_i(\hat{\boldsymbol{x}}_i, t) = \hat{\boldsymbol{x}}_i + \boldsymbol{s}_i(\hat{\boldsymbol{x}}_i, t)$, $\forall t \in (0, T]$, where $\boldsymbol{s}_i$ is the displacement field in the Lagrangian frame. An example of this type of domain configuration with an immersed subdomain is illustrated in Fig. 1.

In what follows, we set $\hat{\psi} = \hat{\psi}(\hat{\boldsymbol{x}}, t)$ which equals $\psi(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)$, and $\hat{\nabla} = \nabla_{\hat{\boldsymbol{x}}_i}$ $(i = 1, 2)$.

In the aforementioned domain Ω, we define the following unsteady Stokes/parabolic interface problem with respect to $\boldsymbol{u}_i \in (H^1 \cap L^\infty)(0, T; H^2(\Omega_1^t)^d \cup H^2(\Omega_2^t)^d)$ $(i = 1, 2)$ and $p_1 \in L^2(0, T; H^1(\Omega_1^t))$:

$$\begin{cases} \frac{\partial \boldsymbol{u}_1}{\partial t} - \nabla \cdot (\mu_1 \nabla \boldsymbol{u}_1) + \nabla p_1 = \boldsymbol{f}_1, & \text{in } \Omega_1^t \times (0, T] \\ \nabla \cdot \boldsymbol{u}_1 = 0, & \text{in } \Omega_1^t \times (0, T] \\ \boldsymbol{u}_1 = 0, & \text{on } \partial\Omega_1^t \backslash \Gamma^t \times (0, T] \\ \boldsymbol{u}_1(\boldsymbol{x}, 0) = \boldsymbol{u}_1^0, & \text{in } \Omega_1^0 \\ \frac{\partial \boldsymbol{u}_2}{\partial t} - \nabla \cdot (\mu_2 \nabla \boldsymbol{u}_2) = \boldsymbol{f}_2, & \text{in } \Omega_2^t \times (0, T] \\ \boldsymbol{u}_2 = 0, & \text{on } \partial\Omega_2^t \backslash \Gamma^t \times (0, T] \\ \boldsymbol{u}_2(\boldsymbol{x}, 0) = \boldsymbol{u}_2^0, & \text{in } \Omega_2^0 \\ \boldsymbol{u}_1 = \boldsymbol{u}_2, & \text{on } \Gamma^t \times [0, T] \\ (-p_1 I + \mu_1 \nabla \boldsymbol{u}_1)\boldsymbol{n}_1 + \mu_2 \nabla \boldsymbol{u}_2 \boldsymbol{n}_2 = \boldsymbol{\tau}, & \text{on } \Gamma^t \times [0, T] \end{cases} \quad (1)$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two jump coefficients, i.e., $\mu_1 \neq \mu_2$, in this paper we take both μ_1 and μ_2 are constant. And, $\boldsymbol{f}_i \in L^2(\Omega_i^t)$ $(i = 1, 2)$, $\boldsymbol{\tau} \in H^{1/2}(\Gamma^t)$.

2.2. ALE mapping

With the model problem in place, we now define the affine mapping that allows us to use the ALE description of the model problem. Assume $\exists \boldsymbol{X}_i^t \in H^1(0, T; W^{2,\infty}(\Omega_i^0)^d)$ $(i = 1, 2)$ such that $\forall t \in (0, T]$, the mapping:

$$\begin{aligned} \boldsymbol{X}_i^t : \quad & \Omega_i^0 \rightarrow \Omega_i^t \\ & \hat{\boldsymbol{x}}_i \rightarrow \boldsymbol{x}_i(\hat{\boldsymbol{x}}_i, t) \end{aligned}$$

is invertible such that $(\boldsymbol{X}_i^t)^{-1} \in W^{1,\infty}(\Omega_i^t)^d$, where $\hat{\boldsymbol{x}}_i \in \Omega_i^0$ is known as the reference coordinate variable. The domain velocity is then defined as

$$\boldsymbol{\omega}_i : \Omega_i^t \times (0, T] \rightarrow \mathbb{R}^d, \quad \boldsymbol{\omega}_i(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{X}_i^t(\hat{\boldsymbol{x}}, t)}{\partial t} \circ \left(\boldsymbol{X}_i^t\right)^{-1}, \text{ for } i = 1, 2.$$

With this domain velocity, we can define the ALE-time derivative which takes the domain velocity into account, as

$$\begin{aligned} \left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}} &: \Omega_i^t \times (0, T] \rightarrow \mathbb{R}^d \\ (\boldsymbol{x}, t) &\rightarrow \left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}_i}{\partial t}(\boldsymbol{x}, t) + (\boldsymbol{\omega}_i(\boldsymbol{x}, t) \cdot \nabla)\boldsymbol{u}_i(\boldsymbol{x}, t). \end{aligned} \quad (2)$$

Equipped with the domain velocity and ALE-time derivative, we can proceed to rewrite our problem using the ALE description as follows.

$$\begin{cases} \left.\frac{\partial \boldsymbol{u}_1}{\partial t}\right|_{\hat{\boldsymbol{x}}} - \nabla \cdot (\mu_1 \nabla \boldsymbol{u}_1) - (\boldsymbol{\omega}_2 \cdot \nabla)\boldsymbol{u}_1 + \nabla p_1 = \boldsymbol{f}_1, & \text{in } \Omega_1^t \times (0, T] \\ \nabla \cdot \boldsymbol{u}_1 = 0, & \text{in } \Omega_1^t \times (0, T] \\ \boldsymbol{u}_1 = 0, & \text{on } \partial\Omega_1^t \backslash \Gamma^t \times (0, T] \\ \boldsymbol{u}_1(\boldsymbol{x}, 0) = \boldsymbol{u}_1^0, & \text{in } \Omega_1^0 \\ \left.\frac{\partial \boldsymbol{u}_2}{\partial t}\right|_{\hat{\boldsymbol{x}}} - \nabla \cdot (\mu_2 \nabla \boldsymbol{u}_2) - (\boldsymbol{\omega}_2 \cdot \nabla)\boldsymbol{u}_2 = \boldsymbol{f}_2, & \text{in } \Omega_2^t \times (0, T] \\ \boldsymbol{u}_2 = 0, & \text{on } \partial\Omega_2^t \backslash \Gamma^t \times (0, T] \\ \boldsymbol{u}_2(\boldsymbol{x}, 0) = \boldsymbol{u}_2^0, & \text{in } \Omega_2^0 \\ \boldsymbol{\omega}_1 = \boldsymbol{\omega}_2, & \text{on } \Gamma^t \times [0, T] \\ \boldsymbol{u}_1 = \boldsymbol{u}_2, & \text{on } \Gamma^t \times [0, T] \\ (-p_1 I + \mu_1 \nabla \boldsymbol{u}_1)\boldsymbol{n}_1 + \mu_2 \nabla \boldsymbol{u}_2 \boldsymbol{n}_2 = \boldsymbol{\tau}, & \text{on } \Gamma^t \times [0, T] \end{cases} \quad (3)$$

2.3. The ALE weak form

To define the weak form of (3), we need to introduce the following functional spaces for $t \in [0, T]$:

$$\begin{aligned} \boldsymbol{U}_i^t &:= \{\boldsymbol{\psi}_i \in H^1(\Omega_i^t)^d \big| \boldsymbol{\psi}_i = \hat{\boldsymbol{\psi}}_i \circ (\boldsymbol{X}_i^t)^{-1}, \forall \hat{\boldsymbol{\psi}}_i \in H^1(\Omega_i^0)^d, \boldsymbol{\psi}_i = 0 \text{ on } \partial\Omega_t^i \backslash \Gamma^t\} \ (i = 1, 2), \\ \boldsymbol{U}^t &:= \{(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \in \boldsymbol{U}_1^t \times \boldsymbol{U}_2^t \big| \boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 \text{ on } \Gamma^t\}, \\ Q_1^t &:= L^2(\Omega_1^t), \quad Q_{1,0}^t := \{q_1 \in Q_1^t \big| \textstyle\int_{\Omega_1^t} q_1 d\boldsymbol{x}_1 = 0\}. \end{aligned}$$

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With these spaces we can now define the ALE weak form of (3) as follows: find $(\boldsymbol{u}_1, \boldsymbol{u}_2) \in (H^1 \cap L^\infty)(0, T; \boldsymbol{U}^t)$ and $p_1 \in L^2(0, T; Q^t_{1,0})$ such that

$$\sum_{i=1}^{2}\left[\frac{d}{dt}(\boldsymbol{u}_i, \boldsymbol{\psi}_i)_{\Omega_i^t} + (\mu_i \nabla \boldsymbol{u}_i, \nabla \boldsymbol{\psi}_i)_{\Omega_i^t} - ((\boldsymbol{\omega}_i \cdot \nabla)\boldsymbol{u}_i, \boldsymbol{\psi}_i)_{\Omega_i^t} - ((\nabla \cdot \boldsymbol{\omega}_i)\boldsymbol{u}_i, \boldsymbol{\psi}_i)_{\Omega_i^t}\right] \tag{4}$$
$$-(p_1, \nabla \cdot \boldsymbol{\psi}_1)_{\Omega_1^t} + (\nabla \cdot \boldsymbol{u}_1, q_1)_{\Omega_1^t} = \sum_{i=1}^{2}(\boldsymbol{f}_i, \boldsymbol{\psi}_i)_{\Omega_i^t} + \langle \boldsymbol{\tau}, \boldsymbol{\psi}_1 \rangle_{\Gamma^t}, \forall (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \in \boldsymbol{U}^t,\ q_1 \in Q_1^t,$$

where we employ the fact: $\frac{\partial \boldsymbol{\psi}_i}{\partial t} = \frac{\partial}{\partial t}\left(\hat{\boldsymbol{\psi}}_i \circ (\boldsymbol{X}_i^t)^{-1}\right) = 0,\ \forall \hat{\boldsymbol{\psi}}_i : \Omega_i^0 \to \mathbb{R}^d$, thus by the Reynold's transport theorem [7,8], we have

$$\frac{d}{dt}(\boldsymbol{u}_i, \boldsymbol{\psi}_i)_{\Omega_i^t} = \left(\left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}, \boldsymbol{\psi}_i\right)_{\Omega_i^t} + ((\nabla \cdot \boldsymbol{\omega}_i)\boldsymbol{u}_i, \boldsymbol{\psi}_i)_{\Omega_i^t},\ \text{for } i = 1, 2. \tag{5}$$

3. Semi-discrete finite element approximation

Denote the mesh size with $h\,(0 < h < 1)$, we construct the quasi-uniform triangulation $\mathcal{T}^0_{h,i}$ in Ω_i^0 $(i = 1, 2)$. We assume also no triangle of $\mathcal{T}^0_{h,i}$ has two edges on $\partial\Omega_i^0$ and that no triangle crosses the interface Γ^0, moreover, $\mathcal{T}_h^0 = \mathcal{T}^0_{h,1} \cup \mathcal{T}^0_{h,2}$ is conforming through the interface Γ^0.

3.1. Discrete ALE mapping and semi-discrete scheme

We now consider the discrete ALE mapping of $\boldsymbol{X}_i^t$ by means of piecewise linear Lagrangian finite elements denoted by $\boldsymbol{X}^t_{h,i}$ and defined as

$$\begin{array}{rccc} \boldsymbol{X}^t_{h,i}: & \Omega_i^0 & \to & \Omega_i^t \\ & \hat{\boldsymbol{x}}_i & \to & \boldsymbol{x}_i(\hat{\boldsymbol{x}}_i, t) \end{array}$$

where $\boldsymbol{X}^t_{h,i}$ is smooth and invertible. Likewise, the discrete mesh velocity is defined as follows:

$$\boldsymbol{\omega}_{h,i} : \Omega_i^t \times (0, T] \to \mathbb{R}^d, \quad \boldsymbol{\omega}_{h,i}(\boldsymbol{x}, t) = \frac{d\boldsymbol{X}^t_{h,i}(\hat{\boldsymbol{x}}, t)}{dt} \circ \left(\boldsymbol{X}^t_{h,i}\right)^{-1},\ \ i = 1, 2,$$

which leads to the discrete ALE-time derivative:

$$\left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|^h_{\hat{\boldsymbol{x}}} : \Omega_i^t \times (0, T] \to \mathbb{R}^d$$
$$(\boldsymbol{x}, t) \to \left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|^h_{\hat{\boldsymbol{x}}} := \frac{\partial \boldsymbol{u}_i}{\partial t}(\boldsymbol{x}, t) + (\boldsymbol{\omega}_{h,i}(\boldsymbol{x}, t) \cdot \nabla)\boldsymbol{u}_i(\boldsymbol{x}, t).$$

We denote the image of $\mathcal{T}^0_{h,i}$ under this discrete mapping as $\mathcal{T}^t_{h,i}$ for $t \in (0, T]$ that is non-degenerate with time. Then, $\boldsymbol{X}^t_{h,i}$ $(i = 1, 2)$ represents a moving mesh that adapts to the moving interface/boundary. $\boldsymbol{X}^t_{h,i}$ $(i = 1, 2)$ can be arbitrarily defined, for instance, by the following harmonic mapping:

$$\begin{cases} -\Delta \boldsymbol{X}^t_{h,i} = 0, & \text{in } \hat{\Omega}^i, \\ \boldsymbol{X}^t_{h,i} = 0, & \text{on } \partial\hat{\Omega}^i \backslash \hat{\Gamma}, \\ \boldsymbol{X}^t_{h,i} = \boldsymbol{x}_\Gamma(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t), & \text{on } \hat{\Gamma}, \end{cases} \tag{6}$$

where $\boldsymbol{x}_\Gamma$ denotes a prescribed interface motion.

Referring to low regularity results of the solution to elliptic interface problem [9–12] and to Stokes interface problem [13–15] due to jump coefficients across the interface, the low regularity properties of the solution to the presented Stokes/parabolic interface problem (1) are assumed as follows

$$\begin{array}{l} \boldsymbol{u}_i \in L^\infty\left(0, T; H^2(\Omega_t^i)^d\right),\ \left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}} \in L^2(0; T; H^2(\Omega_t^i)^d) \quad (i = 1, 2), \\ p_1 \in L^\infty\left(0, T; H^1(\Omega_t^1)\right),\ \left.\frac{\partial p_1}{\partial t}\right|_{\hat{\boldsymbol{x}}} \in L^2(0; T; H^1(\Omega_t^1)). \end{array} \tag{7}$$

To account for the above low regularity assumption of the real solution $(\boldsymbol{u}_1, \boldsymbol{u}_2, p_1)$, we introduce the following discrete ALE finite element spaces using MINI-mixed finite element [16]

$$\begin{array}{rcl} \boldsymbol{U}_h^t &=& \{(\boldsymbol{\psi}_{h,1}, \boldsymbol{\psi}_{h,2}) \in \boldsymbol{U}^t \big| \boldsymbol{\psi}_{h,1}\big|_K \in P_b^1(K), \forall K \in \mathcal{T}^t_{h,1}, \boldsymbol{\psi}_{h,2}\big|_K \in P^1(K), \forall K \in \mathcal{T}^t_{h,2}\}, \\ Q_h^t &=& \{q_{h,1} \in Q^t_{1,0} \big| q_{h,1}\big|_K \in P^1(K), \forall K \in \mathcal{T}^t_{h,1}\} \end{array} \tag{8}$$

where $P^1(K)$ is the set of piecewise linear polynomials on the element K, while $P_b^1(K)$ denotes $P^1(K)$ enriched with bubble functions in each element K. Standard mixed finite element theory assures that the Stokes-MINI mixed finite element is stable and converges linearly for both velocity and pressure [17].

Then, the corresponding semi-discrete ALE finite element discretization is defined as follows: find $(\boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,2}) \in \boldsymbol{U}_h^t$, $p_{h,1} \in Q_h^t$ such that

$$\sum_{i=1}^{2}\left[\frac{d}{dt}(\boldsymbol{u}_{h,i}, \boldsymbol{\psi}_{h,i})_{\Omega_i^t} + (\mu_i \nabla \boldsymbol{u}_{h,i}, \nabla \boldsymbol{\psi}_{h,i})_{\Omega_i^t} - ((\boldsymbol{\omega}_{h,i}\cdot\nabla)\boldsymbol{u}_{h,i}, \boldsymbol{\psi}_{h,i})_{\Omega_i^t} - ((\nabla\cdot\boldsymbol{\omega}_{h,i})\boldsymbol{u}_{h,i}, \boldsymbol{\psi}_{h,i})_{\Omega_i^t}\right]$$
$$-(p_{h,1}, \nabla\cdot\boldsymbol{\psi}_{h,1})_{\Omega_1^t} + (\nabla\cdot\boldsymbol{u}_{h,1}, q_{h,1})_{\Omega_1^t} = \sum_{i=1}^{2}(\boldsymbol{f}_i, \boldsymbol{\psi}_{h,i})_{\Omega_i^t} + \langle\boldsymbol{\tau}, \boldsymbol{\psi}_{h,1}\rangle_{\Gamma^t}, \forall(\boldsymbol{\psi}_{h,1}, \boldsymbol{\psi}_{h,2}) \in \boldsymbol{U}_h^t,\ q_{h,1} \in Q_h^t. \tag{9}$$

The error analysis of the above semi-discrete scheme relies on a couple of assumptions about the discrete ALE mapping, $\boldsymbol{X}_{h,i}^t$ $(i = 1, 2)$. We assume that the following error estimates hold [6,18]

$$\|\boldsymbol{X}_i^t - \boldsymbol{X}_{h,i}^t\|_{0,\infty} + h\|\nabla(\boldsymbol{X}_i^t - \boldsymbol{X}_{h,i}^t)\|_{0,\infty} \le Ch^2|\ln h|\|\boldsymbol{X}_i^t\|_{2,\infty},$$
$$\|\boldsymbol{\omega}_i - \boldsymbol{\omega}_{h,i}\|_{0,\infty} + h\|\nabla(\boldsymbol{\omega}_i - \boldsymbol{\omega}_{h,i})\|_{0,\infty} \le Ch^2|\ln h|\|\boldsymbol{\omega}_i\|_{2,\infty}, \tag{10}$$

where we assume $\boldsymbol{\omega}_i \in W^{2,\infty}(\Omega_i^t)^d$.

3.2. Semi-discrete stability analysis

Theorem 3.1. *The following stability result holds for the semi-discrete scheme (9) for any $t \in (0, T]$:*

$$\sum_{i=1}^{2}(\|\boldsymbol{u}_{h,i}\|_{L^\infty(0,t;L^2(\Omega_i^t)^d)} + \|\boldsymbol{u}_{h,i}\|_{L^2(0,t;H^1(\Omega_i^t)^d)})$$
$$\le C(\sum_{i=1}^{2}(\|\boldsymbol{f}_i\|_{L^2(0,t;L^2(\Omega_i^t)^d)} + \|\boldsymbol{u}_i^0\|_{L^2(\Omega_i^0)^d}) + \|\boldsymbol{\tau}\|_{L^2(0,T;L^2(\Gamma^t)^d)}). \tag{11}$$

Proof. Let $\psi_{h,i} = \boldsymbol{u}_{h,i},\ q_{h,1} = p_{h,1}$ in (9), and use (5), yield:

$$\sum_{i=1}^{2}\left[\left(\left.\frac{\partial \boldsymbol{u}_{h,i}}{\partial t}\right|_{\hat{\boldsymbol{x}}}^{h}, \boldsymbol{u}_{h,i}\right)_{\Omega_i^t} + (\mu_i\nabla\boldsymbol{u}_{h,i}, \nabla\boldsymbol{u}_{h,i})_{\Omega_i^t} - ((\boldsymbol{\omega}_{h,i}\cdot\nabla)\boldsymbol{u}_{h,i}, \boldsymbol{u}_{h,i})_{\Omega_i^t}\right] = \sum_{i=1}^{2}(\boldsymbol{f}_i, \boldsymbol{u}_{h,i})_{\Omega_i^t} + \langle\boldsymbol{\tau}, \boldsymbol{u}_{h,1}\rangle_{\Gamma^t}.$$

By using the following identities

$$\left(\left.\frac{\partial \boldsymbol{u}_{h,i}}{\partial t}\right|_{\hat{\boldsymbol{x}}}^{h}, \boldsymbol{u}_{h,i}\right)_{\Omega_i^t} = \frac{1}{2}\left(\frac{d}{dt}\|\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t}^2 - (\boldsymbol{u}_{h,i}\nabla\cdot\boldsymbol{\omega}_{h,i}, \boldsymbol{u}_{h,i})\right),$$
$$(\mu_i\nabla\boldsymbol{u}_{h,i}, \nabla\boldsymbol{u}_{h,i})_{\Omega_i^t} = \mu_i\|\nabla\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t}^2,$$

and Poincaré inequality, we then have,

$$\sum_{i=1}^{2}\left[\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t}^2 + \|\boldsymbol{u}_{h,i}\|_{1,\Omega_i^t}^2\right]$$
$$\le \sum_{i=1}^{2}\left[(\boldsymbol{f}_i, \boldsymbol{u}_{h,i})_{\Omega_i^t} + \frac{1}{2}(\boldsymbol{u}_{h,i}\nabla\cdot\boldsymbol{\omega}_{h,i}, \boldsymbol{u}_{h,i})_{\Omega_i^t} + ((\boldsymbol{\omega}_{h,i}\cdot\nabla)\boldsymbol{u}_{h,i}, \boldsymbol{u}_{h,i})_{\Omega_i^t}\right] + \langle\boldsymbol{\tau}, \boldsymbol{u}_{h,1}\rangle_{\Gamma^t}.$$

Using the boundedness of $\|\boldsymbol{\omega}_{h,i}\|_{1,\infty}$ due to (10), Young's inequality with ϵ, the Cauchy–Schwarz inequality and the trace theorem, we have the following:

$$((\boldsymbol{\omega}_{h,i}\cdot\nabla)\boldsymbol{u}_{h,i}, \boldsymbol{u}_{h,i})_{\Omega_i^t} \le \|\boldsymbol{\omega}_{h,i}\|_{\infty,\Omega_i^t}\|\boldsymbol{u}_{h,i}\|_{1,\Omega_i^t}\|\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t} \le \epsilon\|\boldsymbol{u}_{h,i}\|_{1,\Omega_i^t}^2 + C\|\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t}^2,$$
$$(\boldsymbol{u}_{h,i}\nabla\cdot\boldsymbol{\omega}_{h,i}, \boldsymbol{u}_{h,i})_{\Omega_i^t} \le C\|\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t}^2,$$
$$(\boldsymbol{f}_i, \boldsymbol{u}_{h,i})_{\Omega_i^t} \le \|\boldsymbol{f}_i\|_{0,\Omega_i^t}\|\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t} \le C(\|\boldsymbol{f}_i\|_{0,\Omega_i^t}^2 + \|\boldsymbol{u}_{h,i}\|_{0,\Omega_i^t}^2),$$
$$\langle\boldsymbol{\tau}, \boldsymbol{u}_{h,1}\rangle_{\Gamma^t} \le \|\boldsymbol{\tau}\|_{0,\Gamma^t}\|\boldsymbol{u}_{h,1}\|_{0,\Gamma^t} \le C\|\boldsymbol{\tau}\|_{0,\Gamma^t}\|\boldsymbol{u}_{h,1}\|_{1,\Omega_1^t} \le C\|\boldsymbol{\tau}\|_{0,\Gamma^t}^2 + \epsilon\|\boldsymbol{u}_{h,1}\|_{1,\Omega_1^t}^2.$$

We choose $\epsilon = \frac{\mu_i}{4}$, leading to

$$\sum_{i=1}^{2}\left[\frac{d}{dt}\|\boldsymbol{u}_{h,i}\|^2_{0,\Omega_i^t} + \|\boldsymbol{u}_{h,i}\|^2_{1,\Omega_i^t}\right] \le C\left(\sum_{i=1}^{2}\left(\|\boldsymbol{f}_i\|^2_{0,\Omega_i^t} + \|\boldsymbol{u}_{h,i}\|^2_{0,\Omega_i^t}\right) + \|\boldsymbol{\tau}\|^2_{0,\Gamma^t}\right).$$

Integrating over time from 0 to t, then

$$\sum_{i=1}^{2}\|\boldsymbol{u}_{h,i}\|^2_{0,\Omega_i^t} + \sum_{i=1}^{2}\int_0^t \|\boldsymbol{u}_{h,i}\|^2_{1,\Omega_i^t}dt \tag{12}$$
$$\le C\left(\sum_{i=1}^{2}\left(\int_0^t(\|\boldsymbol{f}_i\|^2_{0,\Omega_i^t} + \|\boldsymbol{u}_{h,i}\|^2_{0,\Omega_i^t})dt + \|\boldsymbol{u}_i^0\|^2_{0,\Omega_i^0}\right) + \int_0^t \|\boldsymbol{\tau}\|^2_{0,\Gamma^t}dt\right).$$

Using Grönwall's inequality, we have the desired stability result. □

3.3. Semi-discrete error analysis

We begin by introducing the following H^1-projection associated with a moving interface problem [5].

Definition 3.2. Let $(\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2) \in \boldsymbol{U}_h^t$ and $\tilde{p}_1 \in Q_h^t$ be the H^1-projection of the solution to (1) such that

$$\sum_{i=1}^{2}\left[(\mu_i\nabla(\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i), \nabla\boldsymbol{\psi}_{h,i})_{\Omega_i^t} - ((\boldsymbol{\omega}_{h,i}\cdot\nabla)(\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i), \boldsymbol{\psi}_{h,i})_{\Omega_i^t} + \kappa((\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i), \boldsymbol{\psi}_{h,i})_{\Omega_i^t}\right]$$
$$-((p_1 - \tilde{p}_1), \nabla\cdot\boldsymbol{\psi}_{h,1})_{\Omega_1^t} + (q_{h,1}, \nabla\cdot(\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i))_{\Omega_1^t} = 0, \quad \forall(\boldsymbol{\psi}_{h,1}, \boldsymbol{\psi}_{h,2}) \in \boldsymbol{U}_h^t,\ q_{h,1} \in Q_h^t, \tag{13}$$

provided that $\boldsymbol{\omega}_{h,i}$ is given and $\|\boldsymbol{\omega}_{h,i}\|_{L^\infty(\Omega_i^t)} \le M_i\ (i = 1, 2)$, $\kappa = \max\left(\frac{M_1^2}{2\mu_1} + \frac{\mu_1}{2} + M_1, \frac{M_2^2}{2\mu_2} + \frac{\mu_2}{2} + M_2\right)$.

Then, we have the following error estimates for this particular H^1-projection using MINI-mixed finite elements.

Lemma 3.3 (*[5]*). *With the regularity assumption (7) holding for $((\boldsymbol{u}_1, \boldsymbol{u}_2), p_1)$ to (1), there exists a unique solution $((\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2), \tilde{p}_1) \in \boldsymbol{U}_h^t \times Q_h^t$ to (13), $\forall t \in [0, T]$, such that:*

$$\sum_{i=1}^{2}\|\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i\|_{0,\Omega_i^t} + h\left(\sum_{i=1}^{2}\|\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i\|_{1,\Omega_i^t} + \|p_1 - \tilde{p}_1\|_{0,\Omega_1^t}\right) \le h^2\left(\sum_{i=1}^{2}\|\boldsymbol{u}_i\|_{2,\Omega_i^t} + \|p_1\|_{1,\Omega_1^t}\right). \tag{14}$$

Lemma 3.4 (*[5]*). *With the same condition of Lemma 3.3, we have the following error estimate:*

$$\sum_{i=1}^{2}\left\|\frac{\partial\boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^h - \frac{\partial\tilde{\boldsymbol{u}}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^h\right\|_{1,\Omega_i^t} + \left\|\frac{\partial p_1}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^h - \frac{\partial\tilde{p}_1}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_i^t}$$
$$\le Ch|\ln h|\left(\sum_{i=1}^{2}\left[\|\boldsymbol{u}_i\|_{2,\Omega_i^t} + \left\|\frac{\partial\boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}\right\|_{2,\Omega_i^t}\right] + \|p_1\|_{1,\Omega_i^t} + \left\|\frac{\partial p_1}{\partial t}\Big|_{\hat{\boldsymbol{x}}}\right\|_{1,\Omega_i^t}\right), \tag{15}$$

$$\sum_{i=1}^{2}\left\|\frac{\partial\boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^h - \frac{\partial\tilde{\boldsymbol{u}}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_i^t}$$
$$\le Ch\left(\sum_{i=1}^{2}\left[\|\boldsymbol{u}_i\|_{2,\Omega_i^t} + \left\|\frac{\partial\boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}\right\|_{2,\Omega_i^t}\right] + \|p_1\|_{1,\Omega_i^t} + \left\|\frac{\partial p_1}{\partial t}\Big|_{\hat{\boldsymbol{x}}}\right\|_{1,\Omega_i^t}\right). \tag{16}$$

Applying the H^1-projection (13) to the ALE weak form (4), we then get the following ALE weak form with the projection:

$$\sum_{i=1}^{2}\left[\frac{d}{dt}(\boldsymbol{u}_i, \boldsymbol{\psi}_{h,i})_{\Omega_i^t} + (\mu_i\nabla\tilde{\boldsymbol{u}}_i, \nabla\boldsymbol{\psi}_{h,i})_{\Omega_i^t} - ((\boldsymbol{\omega}_{h,i}\cdot\nabla)\tilde{\boldsymbol{u}}_i, \boldsymbol{\psi}_{h,i})_{\Omega_1^t} - ((\nabla\cdot\boldsymbol{\omega}_{h,i})\boldsymbol{u}_i, \boldsymbol{\psi}_{h,i})_{\Omega_1^t}\right]$$
$$-(\tilde{p}_1, \nabla\cdot\boldsymbol{\psi}_{h,1})_{\Omega_1^t} + (q_{h,1}, \nabla\cdot\tilde{\boldsymbol{u}}_1)_{\Omega_1^t} = \sum_{i=1}^{2}\left[(\boldsymbol{f}_i, \boldsymbol{\psi}_{h,i})_{\Omega_i^t} + \kappa((\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i), \boldsymbol{\psi}_{h,i})_{\Omega_i^t}\right] + \langle\boldsymbol{\tau}, \boldsymbol{\psi}_{h,1}\rangle_{\Gamma^t}$$
$$\forall(\boldsymbol{\psi}_{h,1}, \boldsymbol{\psi}_{h,2}) \in \boldsymbol{U}_h^t,\ q_{h,1} \in Q_h^t. \tag{17}$$

We can now proceed to the main theorem of this section as follows.

Theorem 3.5. *Suppose* $(\boldsymbol{u}_1, p_1, \boldsymbol{u}_2)$ *is the solution to* (1) *satisfying the regularity properties* (7)*, and* $(\boldsymbol{u}_{h,1}, p_{h,1}, \boldsymbol{u}_{h,2})$ *is the solution to* (9)*, then we have the following error estimate:*

$$\sum_{i=1}^{2}\left(\|\boldsymbol{u}_i-\boldsymbol{u}_{h,i}\|_{L^\infty(0,T;(H^1(\Omega_i^t))^d)}+\left\|\left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h-\left.\frac{\partial \boldsymbol{u}_{h,i}}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right\|_{L^2(0,T;(L^2(\Omega_i^t))^d)}\right)+\|p_1-p_{h,1}\|_{L^2(0,T;L^2(\Omega_1^t))}$$
$$\leq Ch\left[\sum_{i=1}^{2}\left(\|\boldsymbol{u}_i\|_{L^\infty(0,T;(H^2(\Omega_i^t))^d)}+\left\|\left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,T;(H^2(\Omega_i^t))^d)}\right)\right.$$
$$\left.+\|p_1\|_{L^\infty(0,T;H^1(\Omega_1^t))}+\left\|\left.\frac{\partial p_1}{\partial t}\right|_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,T;H^1(\Omega_i^t))}\right]. \tag{18}$$

Proof. Subtracting (9) from (17), we get the error equation:

$$\sum_{i=1}^{2}\left[\frac{d}{dt}(\boldsymbol{u}_i-\boldsymbol{u}_{h,i},\boldsymbol{\psi}_{h,i})_{\Omega_i^t}+(\mu_i\nabla(\tilde{\boldsymbol{u}}_i-\boldsymbol{u}_{h,i}),\nabla\boldsymbol{\psi}_{h,i})_{\Omega_i^t}-((\boldsymbol{\omega}_{h,i}\cdot\nabla)(\tilde{\boldsymbol{u}}_i-\boldsymbol{u}_{h,i}),\boldsymbol{\psi}_{h,i})_{\Omega_i^t}\right.$$
$$\left.-((\nabla\cdot\boldsymbol{\omega}_{h,i})(\boldsymbol{u}_i-\boldsymbol{u}_{h,i}),\boldsymbol{\psi}_{h,i})_{\Omega_i^t}-(\tilde{p}_1-p_{h,1},\nabla\cdot\boldsymbol{\psi}_{h,1})_{\Omega_1^t}+(q_{h,1},\nabla\cdot(\tilde{\boldsymbol{u}}_1-\boldsymbol{u}_{h,1}))_{\Omega_1^t}\right]$$
$$=\sum_{i=1}^{2}\left[\kappa((\boldsymbol{u}_i-\tilde{\boldsymbol{u}}_i),\boldsymbol{\psi}_{h,i})_{\Omega_i^t}\right],\quad\forall(\boldsymbol{\psi}_{h,1},\boldsymbol{\psi}_{h,2})\in\boldsymbol{U}_h^t,\ q_{h,1}\in Q_h^t. \tag{19}$$

Picking new variables $\delta_i=\boldsymbol{u}_i-\tilde{\boldsymbol{u}}_i$, $\sigma_i=\tilde{\boldsymbol{u}}_i-\boldsymbol{u}_{h,i}$, $\phi=\tilde{p}_1-p_{h,1}$, and using (5), we can rewrite (19) as

$$\sum_{i=1}^{2}\left[\left(\left.\frac{\partial(\delta_i+\sigma_i)}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h,\boldsymbol{\psi}_{h,i}\right)_{\Omega_i^t}+\mu_i(\nabla\sigma_i,\nabla\boldsymbol{\psi}_{h,i})_{\Omega_i^t}\right]-(\phi,\nabla\cdot\boldsymbol{\psi}_{h,1})_{\Omega_1^t}+(q_{h,1},\nabla\cdot\sigma_1)_{\Omega_1^t}$$
$$=\sum_{i=1}^{2}\left[((\boldsymbol{\omega}_{h,i}\cdot\nabla)\sigma_i,\boldsymbol{\psi}_{h,i})_{\Omega_i^t}+\kappa(\delta_i,\boldsymbol{\psi}_{h,i})_{\Omega_i^t}\right]. \tag{20}$$

Choosing $\boldsymbol{\psi}_{h,i}=\sigma_i$, $q_{h,1}=\phi$, (20) becomes

$$\sum_{i=1}^{2}\left[\left(\left.\frac{\partial\delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h+\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h,\sigma_i\right)_{\Omega_i^t}+\mu_i(\nabla\sigma_i,\nabla\sigma_i)_{\Omega_i^t}\right]=\sum_{i=1}^{2}\left[((\boldsymbol{\omega}_{h,i}\cdot\nabla)\sigma_i,\sigma_i)_{\Omega_i^t}+\kappa(\delta_i,\sigma_i)_{\Omega_i^t}\right]. \tag{21}$$

Using Young's inequality with ϵ, the Cauchy–Schwarz inequality, applying Lemma 3.3 and the boundedness of $\|\boldsymbol{\omega}_{h,i}\|_{1,\infty}$ due to (10), we get the following estimates on the right hand side:

$$((\boldsymbol{\omega}_{h,i}\cdot\nabla)\sigma_i,\sigma_i)\leq\epsilon\|\nabla\sigma_i\|_{0,\Omega_i^t}^2+C\|\sigma_i\|_{0,\Omega_i^t}^2, \tag{22}$$
$$\kappa(\delta_i,\sigma_i)_{\Omega_i^t}\leq C\left(h^4\left(\|\boldsymbol{u}_1\|_{2,\Omega_1^t}+\|\boldsymbol{u}_2\|_{2,\Omega_2^t}+\|p_1\|_{1,\Omega_i^t}\right)^2+\|\sigma_i\|_{0,\Omega_i^t}^2\right). \tag{23}$$

For the left hand side terms, we note that

$$\left(\left.\frac{\partial\delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h+\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h,\sigma_i\right)_{\Omega_i^t}=\frac{1}{2}\frac{d}{dt}\|\sigma_i\|_{0,\Omega_i^t}^2-\frac{1}{2}((\nabla\cdot\boldsymbol{\omega}_{h,i})\sigma_i,\sigma_i)_{\Omega_i^t}+\left(\left.\frac{d\delta_i}{dt}\right|_{\hat{\boldsymbol{x}}}^h,\sigma_i\right)_{\Omega_i^t}, \tag{24}$$
$$\mu_i(\nabla\sigma_i,\nabla\sigma_i)_{\Omega_i^t}=\mu_i\|\nabla\sigma_i\|_{0,\Omega_i^t}^2. \tag{25}$$

Applying Lemma 3.4 as well as Young's inequality we get the following estimates:

$$\frac{1}{2}((\nabla\cdot\boldsymbol{\omega}_{h,i})\sigma_i,\sigma_i)_{\Omega_i^t}\leq C\|\sigma_i\|_{0,\Omega_i^t}^2, \tag{26}$$
$$\left(\left.\frac{d\delta_i}{dt}\right|_{\hat{\boldsymbol{x}}}^h,\sigma_i\right)_{\Omega_i^t}\leq C\left(h^2\left(\sum_{i=1}^{2}\left[\|\boldsymbol{u}_i\|_{2,\Omega_i^t}+\left\|\left.\frac{\partial\boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}\right\|_{2,\Omega_i^t}\right]+\|p_1\|_{1,\Omega_i^t}+\left\|\left.\frac{\partial p_1}{\partial t}\right|_{\hat{\boldsymbol{x}}}\right\|_{1,\Omega_i^t}\right)^2+\|\sigma_i\|_{0,\Omega_i^t}^2\right). \tag{27}$$

Apply the estimates obtained above, choose a sufficiently small ϵ, then integrate in time from 0 to t and apply Grönwall's inequality, yields

$$\begin{aligned}&\sum_{i=1}^{2}\left[\|\sigma_i\|_{0,\Omega_i^t}^2+\int_0^t\|\nabla\sigma_i\|_{0,\Omega_i^t}^2 ds\right]\\&\le\sum_{i=1}^{2}\left[\|\sigma_i^0\|_{0,\Omega_i^0}^2+Ch^2\int_0^t\left(\sum_{i=1}^{2}\left[\|\boldsymbol{u}_i\|_{2,\Omega_i^t}+\left\|\left.\frac{\partial \boldsymbol{u}_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}\right\|_{2,\Omega_i^t}\right]+\|p_1\|_{1,\Omega_i^t}+\left\|\left.\frac{\partial p_1}{\partial t}\right|_{\hat{\boldsymbol{x}}}\right\|_{1,\Omega_i^t}\right)^2 ds\right].\end{aligned}\tag{28}$$

In order to conduct the pressure's error estimate, we shall first estimate $\left\|\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_i^t}$. Let $\boldsymbol{\psi}_{h,1}=\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h$ and $q_{h,1}=\phi$ in (20), yields

$$\begin{aligned}&\sum_{i=1}^{2}\left[\left(\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}+\mu_i\left(\nabla\sigma_i,\nabla\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}\right]-\left(\phi,\nabla\cdot\left.\frac{\partial\sigma_1}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_1^t}+(\phi,\nabla\cdot\sigma_1)_{\Omega_1^t}\\&=\sum_{i=1}^{2}\left[-\left(\left.\frac{\partial\delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}+\left((\boldsymbol{\omega}_{h,i}\cdot\nabla)\sigma_i,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}+\kappa\left(\delta_i,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}\right].\end{aligned}\tag{29}$$

Applying the same formulations in [6, Lemma 3.2], we have

$$\begin{aligned}\left(\mu_i\nabla\sigma_i,\nabla\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}&=\frac{1}{2}\left[\frac{d}{dt}(\mu_i\nabla\sigma_i,\nabla\sigma_i)_{\Omega_i^t}-\left(\mu_i\nabla\cdot\boldsymbol{\omega}_{h,i}\nabla\sigma_i,\nabla\sigma_i\right)_{\Omega_i^t}\right.\\&\qquad\left.+\left(\mu_i\left(\nabla\boldsymbol{\omega}_{h,i}+\nabla\boldsymbol{\omega}_{h,i}^T\right)\nabla\sigma_i,\nabla\sigma_i\right)_{\Omega_i^t}\right],\\\left(\phi,\nabla\cdot\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_1^t}&=-\left(\phi,\nabla\cdot\boldsymbol{\omega}_{h,1}\nabla\cdot\sigma_i\right)_{\Omega_1^t}+\left(\phi,\nabla\boldsymbol{\omega}_{h,1}:\nabla\sigma_1^T\right)_{\Omega_1^t}.\end{aligned}$$

Then (29) can be rewritten as

$$\begin{aligned}&\sum_{i=1}^{2}\left[\left(\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}+\frac{1}{2}\frac{d}{dt}(\mu_i\nabla\sigma_i,\nabla\sigma_i)_{\Omega_i^t}\right]=\sum_{i=1}^{2}\left[-\left(\left.\frac{\partial\delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}\right.\\&+\left((\boldsymbol{\omega}_{h,i}\cdot\nabla)\sigma_i,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}+\left(\frac{1}{2}\mu_i\nabla\cdot\boldsymbol{\omega}_{h,i}\nabla\sigma_i,\nabla\sigma_i\right)_{\Omega_i^t}+\kappa\left(\delta_i,\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)_{\Omega_i^t}\\&\left.-\frac{1}{2}\left(\mu_i\left(\nabla\boldsymbol{\omega}_{h,i}+\nabla\boldsymbol{\omega}_{h,i}^T\right)\nabla\sigma_i,\nabla\sigma_i\right)_{\Omega_i^t}\right]+\left(\phi,\nabla\cdot\boldsymbol{\omega}_{h,1}\nabla\cdot\sigma_i\right)_{\Omega_1^t}-\left(\phi,\nabla\boldsymbol{\omega}_{h,1}:\nabla\sigma_1^T\right)_{\Omega_1^t}\\&=\sum_{k=1}^{7}T_k,\end{aligned}\tag{30}$$

where, by the Cauchy–Schwarz inequality and Young's inequality with ϵ, we have

$$T_1\le C\sum_{i=1}^{2}\left(\left\|\left.\frac{\partial\delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_i^t}^2+\epsilon\left\|\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_i^t}^2\right),\tag{31}$$

$$T_2\le C\sum_{i=1}^{2}\left(\|\sigma_i\|_{1,\Omega_i^t}^2+\epsilon\left\|\left.\frac{\partial\sigma_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right\|_{0,\Omega_i^t}^2\right),\tag{32}$$

$$T_3+T_5\le C\sum_{i=1}^{2}\|\sigma_i\|_{1,\Omega_i^t}^2,\tag{33}$$

$$T_4 \le C\sum_{i=1}^{2}\left(\|\delta_i\|^2_{0,\Omega_i^t}+\epsilon\left\|\frac{\partial\sigma_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{0,\Omega_i^t}\right), \tag{34}$$

$$T_6+T_7 \le C\|\sigma_1\|^2_{1,\Omega_1^t}+\epsilon_\phi\|\phi\|^2_{0,\Omega_1^t}. \tag{35}$$

Pick a sufficiently small ϵ, results

$$\sum_{i=1}^{2}\left[\left\|\frac{\partial\sigma_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{0,\Omega_i^t}+\frac{d}{dt}\|\nabla\sigma_i\|^2_{0,\Omega_i^t}\right]\le C\sum_{i=1}^{2}\left(\left\|\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{0,\Omega_i^t}+\|\sigma_i\|^2_{1,\Omega_1^t}+\|\delta_i\|^2_{0,\Omega_t^i}\right)+\epsilon_\phi\|\phi\|^2_{0,\Omega_1^t}. \tag{36}$$

By the inf-sup condition [19, Lemma 2], we have

$$\begin{aligned}\gamma\|\phi\|_{0,\Omega_t^1} &\le \sup_{(\boldsymbol{\psi}_{h,1},\boldsymbol{\psi}_{h,2})\in \boldsymbol{V}^0_{h,t}}\frac{(\phi,\nabla\cdot\boldsymbol{\psi}_{h,1})_{\Omega_1^t}}{\|(\boldsymbol{\psi}_{h,1},\boldsymbol{\psi}_{h,2})\|_1}\\ &\le \sum_{i=1}^{2}\left(\left\|\frac{\partial\sigma_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^t}+\left\|\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^t}\right)+C\sum_{i=1}^{2}\left(\|\sigma_i\|_{1,\Omega_i^t}+\|\delta_i\|_{0,\Omega_i^t}\right),\end{aligned} \tag{37}$$

where we apply (19) and the Cauchy–Schwarz inequality. Substitute (37) into (36), leads to

$$\sum_{i=1}^{2}\left[\left\|\frac{\partial\sigma_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{0,\Omega_i^t}+\frac{d}{dt}\|\nabla\sigma_i\|^2_{0,\Omega_i^t}\right]\le C\sum_{i=1}^{2}\left(\left\|\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{0,\Omega_i^t}+\|\sigma_i\|^2_{1,\Omega_1^t}+\|\delta_i\|^2_{0,\Omega_t^i}\right)+\frac{\epsilon_\phi}{\gamma}\sum_{i=1}^{2}\left\|\frac{\partial\sigma_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^t}. \tag{38}$$

Take $\epsilon_\phi=\frac{\gamma}{2}$, integrate (38) in time from 0 to t, then apply Grönwall's inequality, and take $\boldsymbol{u}_{h,i}(0)=\tilde{\boldsymbol{u}}_i(0)$, yield

$$\sum_{i=1}^{2}\left[\left\|\frac{\partial\sigma_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,t;(L^2(\Omega_i^t))^d)}+\|\sigma_i\|_{L^\infty(0,t;(H^1(\Omega_i^t))^d)}\right]\le C\sum_{i=1}^{2}\left(\left\|\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,t;(L^2(\Omega_i^t))^d)}+\|\delta_i\|_{L^2(0,t;(L^2(\Omega_t^i))^d)}\right). \tag{39}$$

In addition, integrating (37) in time from 0 to t, taking $p_{h,1}(0)=\tilde{p}_1(0)$ and combining with (39), we obtain

$$\|\phi\|_{L^2(0,t;L^2(\Omega_1^t))}\le C\sum_{i=1}^{2}\left(\left\|\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,t;(L^2(\Omega_i^t))^d)}+\|\delta_i\|_{L^2(0,t;(L^2(\Omega_t^i))^d)}\right). \tag{40}$$

Adding (40) to (39), applying Lemma 3.3 as well as the triangular inequality, we obtain the desired convergence result. □

Remark 3.6. Theorem 3.5 shows an optimal (first-order) error estimate for $\|\boldsymbol{u}_i-\boldsymbol{u}_{h,i}\|_{L^2(0,T;H^1(\Omega_i^t)^d)}$ with respect to the low regularity assumption of the solution $\boldsymbol{u}_i\in L^\infty\left(0,T;H^2(\Omega_t^i)^d\right)$, $i=1,2$, which can be considered as a remarkable improvement over the classical H^1-projection technique that is used in [6] for Stokes equations in a moving domain, where the convergence order in energy norm is only suboptimal, i.e., $O(h|\ln h|)$ [6, Theorem 2.1].

4. Fully discrete finite element approximation

Let $\Delta t>0$ be the time step and $t^n=n\Delta t$ for $n=0,\dots,N$ such that $t^N\le T$ and $t^{N+1}>T$, and $\varphi^n=\varphi(\boldsymbol{x}(\hat{\boldsymbol{x}},t^n),t^n)$. We introduce the following notation to account for the backward Euler scheme that is used to discretize the temporal derivative $\frac{d}{dt}\left(\varphi_{h,i},\psi_{h,i}\right)_{\Omega_t^i}$:

$$\partial_t(\varphi_{h,i},\psi_{h,i})^{n+\frac{1}{2}}=\frac{(\varphi^{n+1}_{h,i},\psi^{n+1}_{h,i})_{\Omega_i^{n+1}}-(\varphi^n_{h,i},\psi^{n+1}_{h,i}\circ\boldsymbol{X}_i^{n,n+1})_{\Omega_i^n}}{\Delta t}.$$

where $\boldsymbol{X}_i^{n,n+1}=\boldsymbol{X}^{n+1}_{h,i}\circ(\boldsymbol{X}^n_{h,i})^{-1}$ for $i=1,2$, We further let $\boldsymbol{J}_i^t$ and $\boldsymbol{J}^t_{h,i}$ denote the determinant of Jacobian matrix of the continuous and discrete ALE mapping, respectively, defined as

$$\boldsymbol{J}_i^t:=\det(\boldsymbol{F}_i^t)=\det\left(\frac{\partial\boldsymbol{X}_i^t(\hat{\boldsymbol{x}})}{\partial\hat{\boldsymbol{x}}}\right),\quad \boldsymbol{J}^t_{h,i}:=\det(\boldsymbol{F}^t_{h,i})=\det\left(\frac{\partial\boldsymbol{X}^t_{h,i}(\hat{\boldsymbol{x}})}{\partial\hat{\boldsymbol{x}}}\right),\quad i=1,2.$$

Define $\boldsymbol{U}_h^{t_{n+1}}$ and $Q_h^{t_{n+1}}$ as $\boldsymbol{U}_h^{n+1}$ and Q_h^{n+1}, respectively. The fully discrete ALE-finite element approximation can now be defined as follows: find $(\boldsymbol{u}_{h,1}^{n+1}, \boldsymbol{u}_{h,2}^{n+1}) \in \boldsymbol{U}_h^{n+1}$, $p_{h,1}^{n+1} \in Q_h^{n+1}$ for $n = 0, \ldots, N-1$ such that:

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\ \partial_t(\boldsymbol{u}_{h,i}, \boldsymbol{\psi}_{h,i})^{n+\frac{1}{2}} + \mu_i(\nabla \boldsymbol{u}_{h,i}^{n+1}, \nabla \boldsymbol{\psi}_{h,i}^{n+1})_{\Omega_i^{n+1}} - ((\nabla\cdot\boldsymbol{\omega}_{h,i}^{n+1})\boldsymbol{u}_{h,i}^{n+1}, \boldsymbol{\psi}_{h,i}^{n+1})_{\Omega_i^{n+1}} \\
&- ((\boldsymbol{\omega}_{h,i}^{n+1}\cdot\nabla)\boldsymbol{u}_{h,i}^{n+1}, \boldsymbol{\psi}_{h,i}^{n+1})_{\Omega_i^{n+1}}\ \Bigg] - (p_{h,1}^{n+1}, \nabla\cdot\boldsymbol{\psi}_{h,1}^{n+1})_{\Omega_1^{n+1}} + (\nabla\cdot\boldsymbol{u}_{h,1}^{n+1}, q_{h,1}^{n+1})_{\Omega_1^{n+1}} \\
&= \sum_{i=1}^{2}\Big[(\boldsymbol{f}_i^{n+1}, \boldsymbol{\psi}_{h,i}^{n+1})_{\Omega_i^{n+1}}\Big] + \left\langle \boldsymbol{\tau}^{n+1}, \boldsymbol{\psi}_{h,1}^{n+1}\right\rangle_{\Gamma^{n+1}}, \quad \forall(\boldsymbol{\psi}_{h,1}, \boldsymbol{\psi}_{h,2}) \in \boldsymbol{U}_h^{n+1},\ q_{h,1} \in Q_h^{n+1}.
\end{aligned} \tag{41}$$

In the following, we first introduce a few lemmas which will allow us to perform the required error analysis for the fully discrete scheme (41).

Lemma 4.1 ([6,20]). *Let* $\varphi_{h,i}^{n+1} \in \boldsymbol{U}_h^{n+1}$, *then*

$$\|\varphi_{h,i}^{n+1}\circ\boldsymbol{X}_i^{n,n+1}\|_{0,\Omega_i^n}^2 = \|\varphi_{h,i}^{n+1}\|_{0,\Omega_i^{n+1}}^2 - \int_{t^n}^{t^{n+1}}\left(\int_{\Omega_i^t}|\varphi_{h,i}^{n+1}\circ\boldsymbol{X}_i^{t,n+1}|^2\nabla\cdot\boldsymbol{\omega}_{h,i}dx\right)dt, \tag{42}$$

$$\|\varphi_{h,i}^{n+1}\circ\boldsymbol{X}_i^{n,n+1}\|_{0,\Omega_i^n}^2 \le \left(1 + \Delta t\sup_{t\in[t_n,t_{n+1}]}\|\nabla\cdot\boldsymbol{\omega}_{h,i}\boldsymbol{J}_{h,i}^t\|_{\infty,\Omega_i^t}\|(\boldsymbol{J}_{h,i}^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}}\right)\|\varphi_{h,i}^{n+1}\|_{0,\Omega_i^{n+1}}^2. \tag{43}$$

Lemma 4.2 ([20]). *$\forall\varphi_i \in H^2(0,T;H^1(\Omega_i^t))$ where Ω_i^t is mapped from Ω_i^0 by the discrete ALE mapping $\boldsymbol{X}_{h,i}^t$. Then*

$$\frac{\varphi_i(\boldsymbol{x}^{n+1},t_{n+1})-\varphi_i(\boldsymbol{x}^n,t^n)}{\Delta t} = \left(\frac{\partial\varphi_i}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \frac{\Delta t}{2}\left[\left(\frac{\partial^2\varphi_i}{\partial t^2}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \boldsymbol{\omega}_{h,i}^{n+1}(\nabla\boldsymbol{\omega}_{h,i})^{n+1}(\nabla\varphi_i)^{n+1}\right] + O((\Delta t)^2), \tag{44}$$

where, $\boldsymbol{\omega}_{h,i} = \frac{\partial\boldsymbol{x}}{\partial t}$ denotes the moving mesh velocity on account of the discrete ALE mapping.

Lemma 4.3 ([6]). *There exists C_1 and C_2 depending on the discrete ALE mapping $\boldsymbol{X}_{h,i}^t$ $(i = 1,2)$ and $h_0 > 0$ such that for $i = 1,2$,*

$$\|\boldsymbol{J}_{h,i}^t\|_{L^\infty(\Omega_i^0)} \le C_1, \quad \|(\boldsymbol{J}_{h,i}^t)^{-1}\|_{L^\infty(\Omega_i^t)} \le C_2, \quad \forall t\in[0,T],\ \forall h\in(0,h_0).$$

And,

$$\|\boldsymbol{J}_{h,i}^t - \boldsymbol{J}_{h,i}^n\|_{L^\infty(\Omega_i^0)} \le C\Delta t, \quad \forall t\in[t^n,t^{n+1}].$$

4.1. Fully discrete stability analysis

Theorem 4.4. *Suppose $(\boldsymbol{u}_{h,1}^{n+1}, p_{h,1}^{n+1}, \boldsymbol{u}_{h,2}^{n+1})$ is the solution to (41) for $n = 0,1,\ldots,N-1$. Then we have the following stability result*

$$\begin{aligned}
\sum_{i=1}^{2}\left[\|\boldsymbol{u}_{h,i}^N\|_{0,\Omega_i^N} + \left(\Delta t\sum_{n=1}^{N}\|\boldsymbol{u}_{h,i}^n\|_{1,\Omega_i^n}^2\right)^{1/2}\right] \le \sum_{i=1}^{2}\|\boldsymbol{u}_{h,i}^0\|_{0,\hat{\Omega}_i} + C\Bigg[\sum_{i=1}^{2}\left(\Delta t\sum_{n=1}^{N}\|\boldsymbol{f}_i^n\|_{0,\Omega_i^n}^2\right)^{1/2} \\
+ \left(\Delta t\sum_{n=1}^{N}\|\boldsymbol{\tau}^n\|_{0,\Gamma^n}^2\right)^{1/2}\Bigg].
\end{aligned} \tag{45}$$

Proof. Let $\boldsymbol{\psi}_{h,i} = \boldsymbol{u}_{h,i}$ $(i = 1,2)$, $q_{h,1}^{n+1} = p_{h,1}^{n+1}$ in (41), yields

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\ \partial_t(\boldsymbol{u}_{h,i}, \boldsymbol{u}_{h,i})^{n+\frac{1}{2}} + \mu_i(\nabla\boldsymbol{u}_{h,i}^{n+1}, \nabla\boldsymbol{u}_{h,i}^{n+1})_{\Omega_i^{n+1}} - ((\nabla\cdot\boldsymbol{\omega}_{h,i}^{n+1})\boldsymbol{u}_{h,i}^{n+1}, \boldsymbol{u}_{h,i}^{n+1})_{\Omega_i^{n+1}} \\
&- ((\boldsymbol{\omega}_{h,i}^{n+1}\cdot\nabla)\boldsymbol{u}_{h,i}^{n+1}, \boldsymbol{u}_{h,i}^{n+1})_{\Omega_i^{n+1}}\ \Bigg] = \sum_{i=1}^{2}(\boldsymbol{f}_i^{n+1}, \boldsymbol{u}_{h,i}^{n+1})_{\Omega_i^{n+1}} + \left\langle\boldsymbol{\tau}^{n+1}, \boldsymbol{u}_{h,1}^{n+1}\right\rangle_{\Gamma^{n+1}}.
\end{aligned} \tag{46}$$

Then, by the identity $-ab = \frac{(a-b)^2-a^2-b^2}{2}$ and (42), we have

$$\begin{aligned}
\partial_t(\boldsymbol{u}_{h,i}, \boldsymbol{u}_{h,i})^{n+\frac{1}{2}} &= \frac{\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{u}_{h,i}^{n}\|^2_{0,\Omega_i^{n}}}{2\Delta t} + \frac{\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{u}_{h,i}^{n+1}\circ \boldsymbol{X}_i^{n,n+1}\|^2_{0,\Omega_i^{n}}}{2\Delta t} \\
&\quad + \frac{\|\boldsymbol{u}_{h,i}^{n} - \boldsymbol{u}_{h,i}^{n+1}\circ \boldsymbol{X}_i^{n,n+1}\|^2_{0,\Omega_i^{n}}}{2\Delta t} \\
&\geq \frac{\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{u}_{h,i}^{n}\|^2_{0,\Omega_i^{n}}}{2\Delta t} + \frac{\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{u}_{h,i}^{n+1}\circ \boldsymbol{X}_i^{n,n+1}\|^2_{0,\Omega_i^{n}}}{2\Delta t} \\
&\geq \frac{\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{u}_{h,i}^{n}\|^2_{0,\Omega_i^{n}}}{2\Delta t} + \frac{1}{2\Delta t}\int_{t^n}^{t^{n+1}}\left(\int_{\Omega_i^t} |\boldsymbol{u}_{h,i}^{n+1}\circ \boldsymbol{X}_i^{t,n+1}|^2 \nabla\cdot\boldsymbol{\omega}_{h,i} dx\right) dt \\
&\geq \frac{\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{u}_{h,i}^{n}\|^2_{0,\Omega_i^{n}}}{2\Delta t} - C\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}}.
\end{aligned} \tag{47}$$

Other terms in (46) are estimated as follows.

$$\begin{aligned}
&\left(\mu_i \nabla \boldsymbol{u}_{h,i}^{n+1}, \nabla \boldsymbol{u}_{h,i}^{n+1}\right)_{\Omega_i^{n+1}} - ((\nabla\cdot\boldsymbol{\omega}_{h,i}^{n+1})\boldsymbol{u}_{h,i}^{n+1}, \boldsymbol{u}_{h,i}^{n+1})_{\Omega_i^{n+1}} - \left((\boldsymbol{\omega}_{h,i}^{n+1}\cdot\nabla)\,\boldsymbol{u}_{h,i}^{n+1}, \boldsymbol{u}_{h,i}^{n+1}\right)_{\Omega_i^{n+1}} \\
&\geq \mu_i\|\nabla \boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{\omega}_{h,i}^{n+1}\|_{W^{1,\infty}(\Omega_i^{n+1})}\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \frac{\mu_i}{4}\|\nabla \boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \frac{\|\boldsymbol{\omega}_{h,i}^{n+1}\|_{L^\infty(\Omega_i^{n+1})}}{\mu_i}\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} \\
&\geq \frac{3\mu_i}{4}\|\nabla \boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - C\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}}, \\
&(\boldsymbol{f}_i^{n+1}, \boldsymbol{u}_{h,i}^{n+1})_{\Omega_i^{n+1}} \leq C\left(\|\boldsymbol{f}_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + \|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}}\right), \\
&\left\langle \boldsymbol{\tau}^{n+1}, \boldsymbol{u}_{h,1}^{n+1}\right\rangle_{\Gamma^{n+1}} \leq C\|\boldsymbol{\tau}^{n+1}\|^2_{0,\Gamma^{n+1}} + \frac{1}{4}\|\boldsymbol{u}_{h,1}^{n+1}\|^2_{1,\Omega_i^{n+1}}.
\end{aligned}$$

Combine all the above estimations, yields

$$\begin{aligned}
\sum_{i=1}^{2}\left[\frac{\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\boldsymbol{u}_{h,i}^{n}\|^2_{0,\Omega_i^{n}}}{\Delta t} + \|\nabla \boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}}\right] &\leq C\sum_{i=1}^{2}\left[\|\boldsymbol{u}_{h,i}^{n+1}\|^2_{0,\Omega_i^{n+1}} + \|\boldsymbol{f}_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right] \\
&\quad + C\|\boldsymbol{\tau}^{n+1}\|^2_{0,\Gamma^{n+1}}.
\end{aligned} \tag{48}$$

Sum over the time step n from 0 to $N-1$, leads to

$$\sum_{i=1}^{2}\left[\frac{\|\boldsymbol{u}_{h,i}^{N}\|^2_{0,\Omega_i^{N}} - \|\boldsymbol{u}_{h,i}^{0}\|^2_{0,\hat{\Omega}_i}}{\Delta t} + \sum_{n=1}^{N}\|\nabla \boldsymbol{u}_{h,i}^{n}\|^2_{0,\Omega_i^{n}}\right] \leq C\sum_{n=1}^{N}\left(\sum_{i=1}^{2}\left[\|\boldsymbol{u}_{h,i}^{n}\|^2_{0,\Omega_i^{n}} + \|\boldsymbol{f}_i^{n}\|^2_{0,\Omega_i^{n}}\right] + \|\boldsymbol{\tau}^{n}\|^2_{0,\Gamma_n}\right).$$

Multiply both sides by Δt and apply the discrete Grönwall's inequality, results in (45). □

4.2. Fully discrete error analysis

We describe the convergence theorem of the fully discrete ALE-finite element method as follows.

Theorem 4.5. *Suppose $(\boldsymbol{u}_1, p_1, \boldsymbol{u}_2)$ is the solution to (4) satisfying the regularity properties (7), and $(\boldsymbol{u}_{h,1}^{n+1}, p_{h,1}^{n+1}, \boldsymbol{u}_{h,2}^{n+1})$ is the solution to (41) for $n = 0, 1, \ldots, N-1$. Then we have the following error estimate:*

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\|\boldsymbol{u}_i^N - \boldsymbol{u}_{h,i}^N\|_{1,\Omega_i^N} + \left(\Delta t\sum_{n=1}^{N}\left\|\partial_t\left(\boldsymbol{u}_i^n - \boldsymbol{u}_{h,i}^n\right)\right\|^2_{\Omega_i^n}\right)^{\frac{1}{2}} + \left(\Delta t\sum_{n=1}^{N}\|p_1^n - p_{h,1}^n\|^2_{0,\Omega_1^n}\right)^{\frac{1}{2}}\right] \\
&\leq C(h+\Delta t)\left[\sum_{i=1}^{2}\left(\|\boldsymbol{u}_i\|_{L^\infty(0,T;H^2(\Omega_i^t))} + \left\|\frac{\partial \boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,T;H^2(\Omega_i^t))} + \left\|\frac{\partial^2 \boldsymbol{u}_i}{\partial t^2}\Big|_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,T;L^2(\Omega_i^t))}\right)\right. \\
&\left.+\|p_1\|_{L^\infty(0,T;H^1(\Omega_1^t))} + \left\|\frac{\partial p_1}{\partial t}\Big|_{\hat{\boldsymbol{x}}}\right\|_{L^2(0,T;H^1(\Omega_1^t))}\right], \text{ where, } \partial_t\boldsymbol{\phi}_i^n = \frac{\boldsymbol{\phi}_i^n - \boldsymbol{\phi}_i^{n-1}\circ \boldsymbol{X}_{n,n-1}^i}{\Delta t}.
\end{aligned} \tag{49}$$

Proof. Let (17) take values at t^{n+1} and add $\partial_t(\boldsymbol{u}_i, \boldsymbol{\psi}_{h,i})^{n+\frac{1}{2}}$ to both sides of the equation, then subtract (41) from this equation, yields the following error equation:

$$\begin{aligned}
&\sum_{i=1}^{2}\Big[\Big(\partial_t(\boldsymbol{u}_i, \boldsymbol{\psi}_{h,i})^{n+\frac{1}{2}} - \partial_t(\boldsymbol{u}_{h,i}, \boldsymbol{\psi}_{h,i})^{n+\frac{1}{2}}\Big) + \mu_i\Big(\nabla(\tilde{\boldsymbol{u}}_i^{n+1} - \boldsymbol{u}_{h.i}^{n+1}), \nabla\boldsymbol{\psi}_{h,i}^{n+1}\Big)_{\Omega_i^{n+1}}\Big]\\
&-\big((\tilde{p}_1^{n+1} - p_{h,1}^{n+1}), \nabla\cdot\boldsymbol{\psi}_{h,1}^{n+1}\big)_{\Omega_1^{n+1}} + \Big(q_{h,1}^{n+1}, \nabla\cdot(\tilde{\boldsymbol{u}}_1^{n+1} - \boldsymbol{u}_{h,1}^{n+1})\Big)_{\Omega_1^{n+1}}\\
&= \sum_{i=1}^{2}\Big[\kappa\Big((\boldsymbol{u}_i^{n+1} - \tilde{\boldsymbol{u}}_i^{n+1}), \boldsymbol{\psi}_{h,i}^{n+1}\Big)_{\Omega_i^{n+1}} + \Big((\boldsymbol{\omega}_{h,i}^{n+1}\cdot\nabla)(\tilde{\boldsymbol{u}}_i^{n+1} - \boldsymbol{u}_{h.i}^{n+1}), \boldsymbol{\psi}_{h,i}^{n+1}\Big)_{\Omega_i^{n+1}}\\
&+ \big((\nabla\cdot\boldsymbol{\omega}_{h,i})(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_{h.i}^{n+1}), \boldsymbol{\psi}_{h,i}^{n+1}\big)_{\Omega_i^{n+1}} - \Big(\frac{d}{dt}(\boldsymbol{u}_i^{n+1}, \boldsymbol{\psi}_{h,i}^{n+1})_{\Omega_i^{n+1}} - \partial_t(\boldsymbol{u}_i, \boldsymbol{\psi}_{h,i})^{n+\frac{1}{2}}\Big)\Big],\\
&\forall(\boldsymbol{\psi}_{h,1}^{n+1}, \boldsymbol{\psi}_{h,2}^{n+1}) \in \boldsymbol{U}_h^{n+1},\ q_{h,1}^{n+1} \in Q_h^{n+1}.
\end{aligned} \tag{50}$$

For the simplicity, we rename terms of (50) on both sides from left to right, as:

$$\sum_{i=1}^{2}\left(\sum_{j=1}^{2} L_j^i\right) + L_3 + L_4 = \sum_{i=1}^{2}\left(\sum_{j=1}^{4} R_j^i\right).$$

Pick new variables $\delta_i^{n+1} = \boldsymbol{u}_i^{n+1} - \tilde{\boldsymbol{u}}_i^{n+1}$, $\sigma_i^{n+1} = \tilde{\boldsymbol{u}}_i^{n+1} - \boldsymbol{u}_{h,i}^{n+1}$, $\phi^{n+1} = \tilde{p}_1^{n+1} - p_{h,i}^{n+1}$, and choose $\boldsymbol{\psi}_{h,i} = \sigma_i,\ q_{h,1} = \phi$ in (50), then apply Poincaré inequality, Cauchy–Schwarz inequality and Young's inequality with ϵ, leads to the following error estimates for each term.

$$\begin{aligned}
L_1^i &= \partial_t(\sigma_i, \sigma_i)^{n+\frac{1}{2}} + \partial_t(\delta_i, \sigma_i)^{n+\frac{1}{2}} = G_1 + G_2,\\
L_2^i &= \mu_i\|\nabla\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}} \geq C\|\sigma_i^{n+1}\|_{1,\Omega_i^{n+1}},\\
L_3 + L_4 &= 0,\\
R_1^i &\leq C\left(\|\delta_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + \|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right),\\
R_2^i &\leq \epsilon\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + C\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}},\\
R_3^i &\leq C\left(\|\delta_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + \|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right).
\end{aligned}$$

There are three terms, $G_1,\ G_2$ and R_4^i remaining for further error analysis. We start first with G_1 by applying (42):

$$\begin{aligned}
G_1 &= \partial_t(\sigma_i, \sigma_i)^{n+\frac{1}{2}} = \frac{1}{\Delta t}\Big[(\sigma_i^{n+1}, \sigma_i^{n+1})_{\Omega_i^{n+1}} - (\sigma_i^n, \sigma_i^{n+1}\circ\boldsymbol{X}_i^{n,n+1})_{\Omega_i^n}\Big],\\
&\geq \frac{1}{2\Delta t}\Big[\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\sigma_i^n\|^2_{0,\Omega_i^n} + \int_{t^n}^{t^{n+1}}\Big(\int_{\Omega_i^t}|\sigma_i^{n+1}\circ\boldsymbol{X}_i^{t,n+1}|^2\nabla\cdot\boldsymbol{\omega}_{h,i}dx\Big)dt\Big],
\end{aligned} \tag{51}$$

where the last term, which will be moved to the right hand side of (50), satisfies the following inequality due to (43) and Lemma 4.3:

$$\begin{aligned}
&\frac{1}{2\Delta t}\int_{t^n}^{t^{n+1}}\Big(\int_{\Omega_i^t}|\sigma_i^{n+1}\circ\boldsymbol{X}_i^{t,n+1}|^2\nabla\cdot\boldsymbol{\omega}_{h,i}dx\Big)dt\\
&\leq \frac{1}{2\Delta t}\sup_{t\in(t^n,t^{n+1})}\|\boldsymbol{J}_{h,i}^t(\nabla\cdot\boldsymbol{\omega}_{h,i})\|_{\infty,\Omega_i^0}\|(\boldsymbol{J}_{h,i}^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}}\int_{t^n}^{t^{n+1}}\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}dt\\
&\leq C\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}.
\end{aligned} \tag{52}$$

We move G_2 to the right hand side of (50) and reformulate $-G_2$ as follows.

$$\begin{aligned}
-G_2 &= -\partial_t(\delta_i, \sigma_i)^{n+\frac{1}{2}} = \frac{1}{\Delta t}\Big[(\delta_i^n, \sigma_i^{n+1}\circ\boldsymbol{X}_i^{n,n+1})_{\Omega_i^n} - (\delta_i^{n+1}, \sigma_i^{n+1})_{\Omega_i^{n+1}}\Big]\\
&= \frac{1}{\Delta t}\Big[(\delta_i^n\circ\boldsymbol{X}_i^{n+1,n}\frac{\boldsymbol{J}_{h,i}^n}{\boldsymbol{J}_{h,i}^{n+1}}, \sigma_i^{n+1})_{\Omega_i^{n+1}} - (\delta_i^{n+1}, \sigma_i^{n+1})_{\Omega_i^{n+1}}\Big] = -\left(\frac{\delta_i^{n+1} - \delta_i^n\circ\boldsymbol{X}_i^{n+1,n}\frac{\boldsymbol{J}_{h,i}^n}{\boldsymbol{J}_{h,i}^{n+1}}}{\Delta t}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}}
\end{aligned}$$

$$= -\left(\frac{\delta_i^{n+1} - \delta_i^n \circ \boldsymbol{X}_i^{n+1,n}}{\Delta t}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}} - \left(\frac{\delta_i^n \circ \boldsymbol{X}_i^{n+1,n} - \delta_i^n \circ \boldsymbol{X}_i^{n+1,n} \frac{\boldsymbol{J}_{h,i}^n}{\boldsymbol{J}_{h,i}^{n+1}}}{\Delta t}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}}$$
$$= H_1 + H_2. \tag{53}$$

We apply Lemma 4.2 to H_1 term as follows:

$$\begin{aligned} H_1 &= -\left(\left(\left.\frac{\partial \delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \frac{\Delta t}{2}\left[\left(\left.\frac{\partial^2 \delta_i}{\partial t^2}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \boldsymbol{\omega}_{h,i}^{n+1}(\nabla \boldsymbol{\omega}_{h,i})^{n+1}(\nabla \delta_i)^{n+1}\right] + O((\Delta t)^2), \sigma_i^{n+1}\right)_{\Omega_i^{n+1}} \\ &\le C\left(\left\|\left(\left.\frac{\partial \delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_i^{n+1}}^2 + (\Delta t)^2\beta^2 + \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2\right) + O((\Delta t)^4), \end{aligned}$$

where $\beta = \left\|\left(\left.\frac{\partial^2 \delta_i}{\partial t^2}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1} - \boldsymbol{\omega}_{h,i}^{n+1}(\nabla \boldsymbol{\omega}_{h,i})^{n+1}(\nabla \delta_i)^{n+1}\right\|_{0,\Omega_i^{n+1}} \le C$ based on Lemmas 3.3 and 3.4.

Term H_2 is handled based upon Lemmas 4.1 and 4.3, as

$$\begin{aligned} H_2 &= -\left(\frac{\left(\frac{\boldsymbol{J}_{h,i}^{n+1} - \boldsymbol{J}_{h,i}^n}{\boldsymbol{J}_{h,i}^{n+1}}\right)(\delta_i^n \circ \boldsymbol{X}_i^{n+1,n})}{\Delta t}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}} \le C\|\delta_i^n \circ \boldsymbol{X}_i^{n+1,n}\|_{0,\Omega_i^{n+1}}\|\sigma^{n+1}\|_{0,\Omega_i^{n+1}} \\ &\le C((1+\Delta t)\|\delta_i^n\|_{0,\Omega_i^n}^2 + \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2), \end{aligned}$$

Therefore,

$$-G_2 \le C\left(\|\delta_i^n\|_{0,\Omega_i^n}^2 + \left\|\left(\left.\frac{\partial \delta_i}{\partial t}\right|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|_{0,\Omega_i^{n+1}}^2 + (\Delta t)^2 + \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2\right). \tag{54}$$

To estimate R_4^i, let $\hat{\boldsymbol{u}}_i = \boldsymbol{u}_i(\hat{t})$ and first consider the Taylor expansion of $\frac{d}{dt}(\boldsymbol{u}_i, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{t,n+1})_{\Omega_i^t}$ at t^{n+1} as follows.

$$\begin{aligned} \left.\frac{d}{dt}(\boldsymbol{u}_i, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{t,n+1})_{\Omega_i^t}\right|_{t^{n+1}} = \frac{1}{\Delta t}\Big[&(\boldsymbol{u}_i^{n+1}, \sigma_i^{n+1})_{\Omega_i^{n+1}} - (\boldsymbol{u}_i^n, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{n,n+1})_{\Omega_i^n} \\ &+ \int_{t^n}^{t^{n+1}} (\hat{t} - t^n)\frac{d^2}{d\hat{t}^2}(\hat{\boldsymbol{u}}_i, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{\hat{t},n+1})_{\Omega_i^{\hat{t}}} d\hat{t}\Big]. \end{aligned} \tag{55}$$

Apply (5) to the second order temporal derivative in the last term of (55), yields

$$\begin{aligned} \frac{d^2}{d\hat{t}^2}\left(\hat{\boldsymbol{u}}_i, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} &= \left(\left.\frac{\partial^2 \hat{\boldsymbol{u}}_i}{\partial \hat{t}^2}\right|_{\hat{\boldsymbol{x}}}, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} + \left(\left.\frac{\partial \hat{\boldsymbol{u}}_i}{\partial \hat{t}}\right|_{\hat{\boldsymbol{x}}}(\nabla \cdot \boldsymbol{\omega}_{h,i}), \sigma_i^{n+1} \circ \boldsymbol{X}_i^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} \\ &+ \left(\left.\frac{\partial \hat{\boldsymbol{u}}_i}{\partial \hat{t}}\right|_{\hat{\boldsymbol{x}}}(\nabla \cdot \boldsymbol{\omega}_{h,i}) + \hat{\boldsymbol{u}}_i \left.\frac{\partial(\nabla \cdot \boldsymbol{\omega}_{h,i})}{\partial \hat{t}}\right|_{\hat{\boldsymbol{x}}}, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} + \left(\hat{\boldsymbol{u}}_i(\nabla \cdot \boldsymbol{\omega}_{h,i})^2, \sigma_i^{n+1} \circ \boldsymbol{X}_i^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} \\ &\le \Bigg[\left\|\left.\frac{\partial^2 \hat{\boldsymbol{u}}_i}{\partial \hat{t}^2}\right|_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^{\hat{t}}} + 2\|\nabla \cdot \boldsymbol{\omega}_{h,i}\|_{\infty,\Omega_i^{\hat{t}}}\left\|\left.\frac{\partial \hat{\boldsymbol{u}}_i}{\partial \hat{t}}\right|_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^{\hat{t}}} + \|\hat{\boldsymbol{u}}_i\|_{0,\Omega_i^{\hat{t}}}\left\|\left.\frac{\partial(\nabla \cdot \boldsymbol{\omega}_{h,i})}{d\hat{t}}\right|_{\hat{\boldsymbol{x}}}\right\|_{\infty,\Omega_i^{\hat{t}}} \\ &+ \|\nabla \cdot \boldsymbol{\omega}_{h,i}\|_{\infty,\Omega_i^{\hat{t}}}^2\|\hat{\boldsymbol{u}}_i\|_{0,\Omega_i^{\hat{t}}}\Bigg]\|\sigma_i^{n+1} \circ \boldsymbol{X}_i^{\hat{t},n+1}\|_{0,\Omega_i^{\hat{t}}}. \end{aligned} \tag{56}$$

We introduce the following notation:

$$\begin{aligned} \mathcal{R}(\hat{t}) &= \left\|\left.\frac{\partial^2 \hat{\boldsymbol{u}}_i}{\partial \hat{t}^2}\right|_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^{\hat{t}}} + 2\|\nabla \cdot \boldsymbol{\omega}_{h,i}\|_{\infty,\Omega_i^{\hat{t}}}\left\|\left.\frac{\partial \hat{\boldsymbol{u}}_i}{\partial \hat{t}}\right|_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^{\hat{t}}} + \|\hat{\boldsymbol{u}}_i\|_{0,\Omega_i^{\hat{t}}}\left\|\left.\frac{\partial(\nabla \cdot \boldsymbol{\omega}_{h,i})}{d\hat{t}}\right|_{\hat{\boldsymbol{x}}}\right\|_{\infty,\Omega_i^{\hat{t}}} + \|\nabla \cdot \boldsymbol{\omega}_{h,i}\|_{\infty,\Omega_i^{\hat{t}}}^2\|\hat{\boldsymbol{u}}_i\|_{0,\Omega_i^{\hat{t}}} \\ &\le C\left[\left\|\left.\frac{\partial^2 \hat{\boldsymbol{u}}_i}{\partial \hat{t}^2}\right|_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^{\hat{t}}} + \left\|\left.\frac{\partial \hat{\boldsymbol{u}}_i}{\partial \hat{t}}\right|_{\hat{\boldsymbol{x}}}\right\|_{0,\Omega_i^{\hat{t}}} + \|\hat{\boldsymbol{u}}_i\|_{0,\Omega_i^{\hat{t}}}\right], \end{aligned} \tag{57}$$

where, we apply again the boundedness of the discrete mesh velocity, i.e., $\|\boldsymbol{\omega}_{h,i}\|_{H^1(0,T;W^{1,\infty}(\Omega_t^i)^d)} \le C,\ i = 1, 2.$

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Then, we obtain the following inequality from (55)–(57), the Cauchy–Schwarz inequality and Lemma 4.3:

$$
\begin{aligned}
R_4^i &= -\left(\frac{d}{dt}(\boldsymbol{u}_i^{n+1}, \sigma_i^{n+1})_{\Omega_i^{n+1}} - \partial_t(\boldsymbol{u}_i, \sigma_i)^{n+\frac{1}{2}}\right) \le \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (\hat{t}-t^n)\frac{d^2}{d\hat{t}^2}(\hat{\boldsymbol{u}}_i, \sigma_i^{n+1}\circ \boldsymbol{X}_i^{\hat{t},n+1})_{\Omega_i^{\hat{t}}} d\hat{t} \\
&\le \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (\hat{t}-t^n)\mathcal{R}(\hat{t})\|\sigma_i^{n+1}\circ \boldsymbol{X}_i^{\hat{t},n+1}\|_{0,\Omega_i^{\hat{t}}} d\hat{t} \\
&\le \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (\hat{t}-t^n)\mathcal{R}(\hat{t})\|\boldsymbol{J}_{h,i}^{\hat{t}}\|^{\frac{1}{2}}_{\infty,\Omega_i^{\hat{t}}} \|(\boldsymbol{J}_{h,i}^{n+1})^{-1}\|^{\frac{1}{2}}_{\infty,\Omega_i^{n+1}} \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}} d\hat{t} \\
&\le \frac{1}{\Delta t}\left(\int_{t^n}^{t^{n+1}} \|\boldsymbol{J}_{h,i}^{\hat{t}}\|_{\infty,\Omega_i^{\hat{t}}} \|(\boldsymbol{J}_{h,i}^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}} \mathcal{R}^2(\hat{t}) d\hat{t}\right)^{\frac{1}{2}} \left(\int_{t^n}^{t^{n+1}} (\hat{t}-t^n)^2 \|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} d\hat{t}\right)^{\frac{1}{2}} \\
&\le C\left(\frac{\Delta t}{3}\right)^{\frac{1}{2}} \sup_{\hat{t}\in(t^n,t^{n+1})} \|\boldsymbol{J}_{h,i}^{\hat{t}}\|_{\infty,\Omega_i^{\hat{t}}} \|(\boldsymbol{J}_{h,i}^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}} \left(\int_{t^n}^{t^{n+1}} \mathcal{R}^2(\hat{t}) d\hat{t}\right)^{\frac{1}{2}} \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}} \\
&\le C\left(\Delta t \int_{t^n}^{t^{n+1}} \mathcal{R}^2(\hat{t}) d\hat{t} + \|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right).
\end{aligned}
\tag{58}
$$

Combining all bounds, take sufficiently small ϵ, and multiplying all terms by Δt, we have

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\sigma_i^n\|^2_{0,\Omega_i^n} + \Delta t\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right] \\
&\le C\Delta t \sum_{i=1}^{2}\left[\left\|\left(\frac{\partial\delta_i}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|^2_{0,\Omega_i^{n+1}} + \|\delta_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + \|\delta_i^n\|^2_{0,\Omega_i^n} + (\Delta t)^2 \right. \\
&\left. + \|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)\int_{t^n}^{t^{n+1}} \mathcal{R}^2(\hat{t}) d\hat{t}\right].
\end{aligned}
\tag{59}
$$

Sum over n from 0 to $N-1$ on both sides, and apply the telescoping technique, yield

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\|\sigma_i^N\|^2_{0,\Omega_i^N} - \|\sigma_i^0\|^2_{0,\Omega_i^0} + \Delta t\sum_{n=0}^{N-1}\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right] \\
&\le C\sum_{i=1}^{2}\left[\Delta t\sum_{n=0}^{N}\|\delta_i^n\|^2_{0,\Omega_i^n} + \Delta t\sum_{n=0}^{N-1}\left\|\left(\frac{\partial\delta_i}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|^2_{0,\Omega_i^{n+1}} + (\Delta t)^3 \right. \\
&\left. + \Delta t\sum_{n=0}^{N-1}\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)^2\int_0^T \mathcal{R}^2(\hat{t}) d\hat{t}\right].
\end{aligned}
\tag{60}
$$

Apply the discrete Grönwall's inequality and use (57), results

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\|\sigma_i^N\|^2_{0,\Omega_i^N} - \|\sigma_i^0\|^2_{0,\Omega_i^0} + \Delta t\sum_{n=0}^{N-1}\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right] \\
&\le C\sum_{i=1}^{2}\left[\Delta t\sum_{n=0}^{N}\|\delta_i^n\|^2_{0,\Omega_i^n} + \Delta t\sum_{n=0}^{N-1}\left\|\left(\frac{\partial\delta_i}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}\right\|^2_{0,\Omega_i^{n+1}}\right] + (\Delta t)^2.
\end{aligned}
\tag{61}
$$

After applying Lemmas 3.3 and 3.4, Poincaré inequality as well as choosing $\boldsymbol{u}_{h,i}^0 = \tilde{\boldsymbol{u}}_i^0$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\|\sigma_i^N\|^2_{0,\Omega_i^N} + \Delta t\sum_{n=0}^{N-1}\|\sigma_i^{n+1}\|^2_{1,\Omega_i^{n+1}}\right] \\
&\le C(h^2+\Delta t^2)\Delta t\sum_{n=0}^{N}\left[\sum_{i=1}^{2}\left(\|\boldsymbol{u}_i^n\|^2_{2,\Omega_i^n} + \left\|\left(\frac{\partial\boldsymbol{u}_i}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}\right)^n\right\|^2_{2,\Omega_i^n}\right) + \|p_1^n\|^2_{1,\Omega_1^n} + \left\|\left(\frac{\partial p_1}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}\right)^{n+1}\right\|^2_{1,\Omega_1^{n+1}}\right].
\end{aligned}
\tag{62}
$$

Next, we estimate the pressure's error starting with $\|\phi^{n+1}\|_{0,\Omega_1^{n+1}}$. The discrete *inf-sup* condition [19, Corollary 1] and (50) lead to

$$
\begin{aligned}
\gamma\|\phi^{n+1}\|_{0,\Omega_1^{n+1}} &\le \sup_{\left(\boldsymbol{\psi}_{h,1}^{n+1},\boldsymbol{\psi}_{h,2}^{n+1}\right)\in \boldsymbol{V}_{h,n+1}^0} \frac{\left(\nabla\cdot\boldsymbol{\psi}_{h,1}^{n+1},\phi^{n+1}\right)_{\Omega_1^{n+1}}}{\left\|\left(\boldsymbol{\psi}_{h,1}^{n+1},\boldsymbol{\psi}_{h,2}^{n+1}\right)\right\|_1}\\
&\le \sum_{i=1}^{2}\left(\left\|\frac{\sigma_i^{n+1}-\sigma_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}} + \left\|\left(\frac{\partial \boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1} - \frac{\boldsymbol{u}_i^{n+1}-\boldsymbol{u}_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}}\right.\\
&\quad\left. + \left\|\frac{\delta_i^{n+1}-\delta_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}}\right) + C\sum_{i=1}^{2}\left(\|\sigma_i^{n+1}\|_{1,\Omega_i^{n+1}}+\|\delta_i^{n+1}\|_{0,\Omega_i^{n+1}}\right).
\end{aligned}
\tag{63}
$$

We first estimate the second term on the right hand side of (63).

$$
\left\|\left(\frac{\partial \boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1} - \frac{\boldsymbol{u}_i^{n+1}-\boldsymbol{u}_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}}^2 = \int_{\hat{\Omega}^i}\left(\left(\frac{\partial \hat{\boldsymbol{u}}_i}{\partial t}\right)^{n+1} - \frac{\hat{\boldsymbol{u}}_i^{n+1}-\hat{\boldsymbol{u}}_i^n}{\Delta t}\right)^2 \boldsymbol{J}_{n+1}^i d\hat{\boldsymbol{x}}_i,
$$

where, by the Taylor's expansion, $\hat{\boldsymbol{u}}_i^n = \hat{\boldsymbol{u}}_i^{n+1} - \Delta t\left(\frac{\partial \hat{\boldsymbol{u}}_i}{\partial t}\right)^{n+1} + \int_{t_n}^{t_{n+1}}(\tilde{t}-t_n)\frac{\partial^2\hat{\boldsymbol{u}}_i(\tilde{t})}{\partial \tilde{t}^2}d\tilde{t}$, then

$$
\begin{aligned}
&\left\|\left(\frac{\partial \boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1} - \frac{\boldsymbol{u}_i^{n+1}-\boldsymbol{u}_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}}^2 = \frac{1}{(\Delta t)^2}\int_{\hat{\Omega}_i}\left(\int_{t_n}^{t_{n+1}}(\tilde{t}-t_n)\frac{\partial^2\hat{\boldsymbol{u}}_i(\tilde{t})}{\partial\tilde{t}^2}d\tilde{t}\right)^2 \boldsymbol{J}_{n+1}^i d\hat{\boldsymbol{x}}_i\\
&\le \frac{1}{(\Delta t)^2}\int_{\hat{\Omega}_i}\int_{t_n}^{t_{n+1}}(\tilde{t}-t_n)^2 d\tilde{t}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2\hat{\boldsymbol{u}}_i(\tilde{t})}{\partial\tilde{t}^2}\right)^2 d\tilde{t}\boldsymbol{J}_{n+1}^i d\hat{\boldsymbol{x}}_i\\
&= \frac{\Delta t}{3}\int_{t_n}^{t_{n+1}}\int_{\hat{\Omega}_i}\left(\frac{\partial^2\hat{\boldsymbol{u}}_i(\tilde{t})}{\partial\tilde{t}^2}\right)^2 \boldsymbol{J}_{n+1}^i d\hat{\boldsymbol{x}}_i d\tilde{t} = \frac{\Delta t}{3}\int_{t_n}^{t_{n+1}}\int_{\Omega_{\tilde{t}}^i}\left(\frac{\partial^2\boldsymbol{u}_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^2 \frac{\boldsymbol{J}_{n+1}^i}{\boldsymbol{J}_{\tilde{t}}^i}d\boldsymbol{x}(\tilde{t})d\tilde{t}\\
&\le \frac{\Delta t}{3}\sup_{\tilde{t}\in(t_n,t_{n+1})}\|(\boldsymbol{J}_{\tilde{t}}^i)^{-1}\|_{\infty,\Omega_{\tilde{t}}^i}\|\boldsymbol{J}_{n+1}^i\|_{\infty,\Omega_i^{n+1}}\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^2\boldsymbol{u}_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{0,\Omega_{\tilde{t}}^i}^2 d\tilde{t}\\
&\le C\Delta t\left\|\frac{\partial^2\boldsymbol{u}_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2(t_n,t_{n+1};L^2(\Omega_{\tilde{t}}^i))}^2,
\end{aligned}
\tag{64}
$$

resulting in

$$
\left\|\left(\frac{\partial \boldsymbol{u}_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1} - \frac{\boldsymbol{u}_i^{n+1}-\boldsymbol{u}_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}} \le C(\Delta t)^{\frac{1}{2}}\left\|\frac{\partial^2\boldsymbol{u}_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2(t_n,t_{n+1};L^2(\Omega_{\tilde{t}}^i))}.
\tag{65}
$$

Similarly,

$$
\left\|\left(\frac{\partial \delta_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1} - \frac{\delta_i^{n+1}-\delta_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}} \le C(\Delta t)^{\frac{1}{2}}\left\|\frac{\partial^2\delta_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2(t_n,t_{n+1};L^2(\Omega_{\tilde{t}}^i))},
$$

where, we introduce $\delta_i(\tilde{t}) = \boldsymbol{u}_i(\tilde{t}) - \tilde{\boldsymbol{u}}_i^{\tilde{t}}$ and $\tilde{\boldsymbol{u}}_i^{\tilde{t}} = \tilde{\boldsymbol{u}}_i^n\circ\boldsymbol{X}_{\tilde{t},n} + \frac{\tilde{t}-t_n}{\Delta t}\left(\tilde{\boldsymbol{u}}_i^{n+1}\circ\boldsymbol{X}_{\tilde{t},n+1} - \tilde{\boldsymbol{u}}_i^n\circ\boldsymbol{X}_{\tilde{t},n}\right)$. Thus we have

$$
\frac{\partial^2\delta_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h} = \frac{\partial^2\left(\boldsymbol{u}_i(\tilde{t}) - \boldsymbol{u}_i^{P,\tilde{t}}\right)}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h} = \frac{\partial^2\boldsymbol{u}_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h}.
$$

Then the third term on the right hand side of (63) is estimated as

$$
\left\|\frac{\delta_i^{n+1}-\delta_i^n\circ \boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}} \le C(\Delta t)^{\frac{1}{2}}\left\|\frac{\partial^2\boldsymbol{u}_i(\tilde{t})}{\partial\tilde{t}^2}\Big|_{\hat{\boldsymbol{x}}}^{h}\right\|_{L^2(t_n,t_{n+1};L^2(\Omega_{\tilde{t}}^i))} + \left\|\left(\frac{\partial \delta_i}{\partial t}\Big|_{\hat{\boldsymbol{x}}}^{h}\right)^{n+1}\right\|_{0,\Omega_i^{n+1}}.
\tag{66}
$$

To find the estimation for $\left\|\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{0,\Omega_i^{n+1}}$, we take $\boldsymbol{\psi}_{h,i}^{n+1}=\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}$ in (50), then

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\left\|\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right\|_{\Omega_i^{n+1}}^2+\left(\mu_i\nabla\sigma_i^{n+1},\nabla\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_i^{n+1}}\right.\\
&-\left(\left(\boldsymbol{\omega}_{h,i}\cdot\nabla\right)\sigma_i^{n+1},\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_i^{n+1}}\\
&\left.-\kappa\left(\delta_i^{n+1},\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_i^{n+1}}\right]-\left(\phi^{n+1},\nabla\cdot\frac{\sigma_1^{n+1}-\sigma_1^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_1^{n+1}}\\
&=-\sum_{i=1}^{2}\left[\left(\left(\frac{\partial\boldsymbol{u}_i}{\partial t}\bigg|_{\hat{\boldsymbol{x}}}^h\right)^{n+1}-\frac{\boldsymbol{u}_i^{n+1}-\boldsymbol{u}_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t},\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_i^{n+1}}\right.\\
&\left.+\left(\frac{\delta_i^{n+1}-\delta_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t},\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_i^{n+1}}\right].
\end{aligned}\tag{67}$$

By a similar argument for deriving (48), we can attain

$$\left(\mu_i\nabla\sigma_i^{n+1},\nabla\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_i^{n+1}}\geq\frac{\|\nabla\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2-\|\nabla\sigma_i^n\|_{0,\Omega_i^n}^2}{2\Delta t}-C\|\nabla\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2.\tag{68}$$

As for the term $\left(\phi^{n+1},\nabla\cdot\frac{\sigma_i^{n+1}-\sigma_i^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_1^{n+1}}$, we define

$$\sigma_i^t=\sigma_i^n\circ\boldsymbol{X}_{h,i,n}+\frac{t-t_n}{\Delta t}\left(\sigma_i^{n+1}\circ\boldsymbol{X}_{h,i,n+1}-\sigma_i^n\circ\boldsymbol{X}_{h,i,n}\right),\quad\forall t\in[t_n,t_{n+1}],$$

then

$$\begin{aligned}
&\left(\phi^{n+1},\nabla\cdot\frac{\sigma_1^{n+1}-\sigma_1^n\circ\boldsymbol{X}_{n+1,n}^i}{\Delta t}\right)_{\Omega_1^{n+1}}\\
&=\frac{1}{\Delta t}\left(\phi^{n+1},\nabla\cdot\sigma_1^{n+1}\right)_{\Omega_1^{n+1}}-\frac{1}{\Delta t}\left(\phi^{n+1},\nabla\cdot\sigma_1^n\circ\boldsymbol{X}_{n+1,n}^i\right)_{\Omega_1^{n+1}}\\
&=\frac{1}{\Delta t}\left(\phi^{n+1},\nabla\cdot\sigma_1^{n+1}\right)_{\Omega_1^{n+1}}-\frac{1}{\Delta t}\left(\phi^{n+1}\circ\boldsymbol{X}_{n,n+1}^i,\frac{\boldsymbol{J}_{n+1}^1}{\boldsymbol{J}_n^1}\left(\frac{\partial\boldsymbol{x}_1^{n+1}}{\partial\boldsymbol{x}_1^n}\right)^{-T}:\nabla_{\boldsymbol{x}_1^n}\sigma_1^n\right)_{\Omega_n^1}\\
&=\frac{1}{\Delta t}\left(\phi^{n+1},\nabla_{\boldsymbol{x}_1^{n+1}}\cdot\sigma_1^{n+1}\right)_{\Omega_1^{n+1}}-\frac{1}{\Delta t}\left(\phi^{n+1}\circ\boldsymbol{X}_{n,n+1}^i,\nabla_{\boldsymbol{x}_1^n}\cdot\sigma_1^n\right)_{\Omega_n^1}\\
&\quad+\frac{1}{\Delta t}\left(\phi^{n+1}\circ\boldsymbol{X}_{n,n+1}^i,\nabla_{\boldsymbol{x}_1^n}\cdot\sigma_1^n\right)_{\Omega_n^1}-\frac{1}{\Delta t}\left(\phi^{n+1}\circ\boldsymbol{X}_{n,n+1}^i,\frac{\boldsymbol{J}_{n+1}^1}{\boldsymbol{J}_n^1}\left(\frac{\partial\boldsymbol{x}_1^{n+1}}{\partial\boldsymbol{x}_1^n}\right)^{-T}:\nabla_{\boldsymbol{x}_1^n}\sigma_1^n\right)_{\Omega_n^1}.
\end{aligned}\tag{69}$$

Since $\phi^{n+1}=\sum_{i=1}^{\mathcal{N}}a^i(t^{n+1})\hat{\varphi}_i\circ(\boldsymbol{X}_{h,1})^{-1}$, where $\mathcal{N}$ is the number of degree of freedom, $a^i(t)$ is independent of the spatial variables, and $\hat{\varphi}_i\circ(\boldsymbol{X}_{h,1})^{-1}$ are shape functions defined in Ω_t^1. Thanks to the divergence-free condition, we have

$$\left(\phi^{n+1},\nabla_{\boldsymbol{x}_1^{n+1}}\cdot\sigma_1^{n+1}\right)_{\Omega_1^{n+1}}=\left(\phi^{n+1}\circ\boldsymbol{X}_{n,n+1}^i,\nabla_{\boldsymbol{x}_1^n}\cdot\sigma_1^n\right)_{\Omega_n^1}=0.\tag{70}$$

Then,

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\phi^{n+1}\circ \boldsymbol{X}^i_{n,n+1}, \nabla_{\boldsymbol{x}_1^n}\cdot\sigma_1^n\right)_{\Omega_n^1} - \frac{1}{\Delta t}\left(\phi^{n+1}\circ \boldsymbol{X}^i_{n,n+1}, \frac{\boldsymbol{J}^1_{n+1}}{\boldsymbol{J}^1_n}\left(\frac{\partial \boldsymbol{x}_1^{n+1}}{\partial \boldsymbol{x}_1^n}\right)^{-T} : \nabla_{\boldsymbol{x}_1^n}\sigma_1^n\right)_{\Omega_n^1}\\
&= -\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\phi^{n+1}\circ \boldsymbol{X}^i_{n,n+1}, \frac{d}{dt}\left(\frac{\boldsymbol{J}^1_t}{\boldsymbol{J}^1_n}\left(\frac{\partial \boldsymbol{x}_1(t)}{\partial \boldsymbol{x}_1^n}\right)^{-T}\right) : \nabla_{\boldsymbol{x}_1^n}\sigma_1^n\right)_{\Omega_n^1} dt\\
&= -\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\phi^{n+1}\circ \boldsymbol{X}^i_{n,n+1}, \left(\frac{\boldsymbol{J}^1_t\nabla\cdot\boldsymbol{\omega}_{h,i}}{\boldsymbol{J}^1_n}\left(\frac{\partial \boldsymbol{x}_1(t)}{\partial \boldsymbol{x}_1^n}\right)^{-T}\right.\right.\\
&\qquad \left.\left. -\frac{\boldsymbol{J}^1_t}{\boldsymbol{J}^1_n}\left(\frac{\partial \boldsymbol{x}_1(t)}{\partial \boldsymbol{x}_1^n}\right)^{-T}\left(\frac{\partial \boldsymbol{\omega}_{h,1}(t)}{\partial \boldsymbol{x}_1^n}\right)^{T}\left(\frac{\partial \boldsymbol{x}_1(t)}{\partial \boldsymbol{x}_1^n}\right)^{-T}\right) : \nabla_{\boldsymbol{x}_1^n}\sigma_1^n\right)_{\Omega_n^1} dt\\
&\leq \frac{C}{\Delta t}\int_{t_n}^{t_{n+1}}\|\phi^{n+1}\circ \boldsymbol{X}^i_{n,n+1}\|_{0,\Omega_n^1}\|\nabla_{\boldsymbol{x}_1^n}\sigma_1^n\|_{0,\Omega_n^1}dt \leq C\|\phi^{n+1}\|_{0,\Omega_1^{n+1}}\|\nabla_{\boldsymbol{x}_1^n}\sigma_1^n\|_{0,\Omega_n^1}\\
&\leq \frac{\gamma^2}{2}\|\phi^{n+1}\|^2_{0,\Omega_1^{n+1}} + C\|\nabla_{\boldsymbol{x}_1^n}\sigma_1^n\|^2_{0,\Omega_n^1},
\end{aligned} \tag{71}$$

where the boundedness of $\boldsymbol{J}_t^1$, F_i^t and $\boldsymbol{\omega}_{h,i}$ are applied.

Summarize (67)–(71), multiply both sides of (67) by Δt, apply Young's inequality with ϵ, yield

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\Delta t\left\|\frac{\sigma_i^{n+1}-\sigma_i^n\circ \boldsymbol{X}^i_{n+1,n}}{\Delta t}\right\|^2_{\Omega_i^{n+1}} + \|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\nabla\sigma_i^n\|^2_{0,\Omega_i^n}\right)\\
&\leq \epsilon\sum_{i=1}^{2}\Delta t\left\|\frac{\sigma_i^{n+1}-\sigma_i^n\circ \boldsymbol{X}^i_{n+1,n}}{\Delta t}\right\|^2_{\Omega_i^{n+1}} + C\sum_{i=1}^{2}\left(\Delta t\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + \Delta t\|\delta_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right.\\
&\left. +(\Delta t)^2\left\|\frac{\partial^2\boldsymbol{u}_i(\tilde t)}{\partial\tilde t^2}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{L^2(t_n,t_{n+1};L^2(\Omega_i^{\tilde t}))} + \Delta t\left\|\left(\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right)^{n+1}\right\|^2_{0,\Omega_i^{n+1}}\right) + \frac{\Delta t\gamma^2}{2}\|\phi^{n+1}\|^2_{0,\Omega_1^{n+1}}.
\end{aligned} \tag{72}$$

Take a sufficiently small ϵ in (72), sum over the time step n from 0 to $N-1$, apply the discrete Grönwall's inequality, and take $\boldsymbol{u}_{h,i}(0) = \boldsymbol{u}_i^P(0)$, result

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\Delta t\sum_{n=0}^{N-1}\left\|\frac{\sigma_i^{n+1}-\sigma_i^n\circ \boldsymbol{X}^i_{n+1,n}}{\Delta t}\right\|^2_{\Omega^i_{n+1}} + \|\nabla\sigma_i^N\|^2_{0,\Omega^i_N}\right)\\
&\leq C\sum_{i=1}^{2}\left(\Delta t\sum_{n=1}^{N}\|\delta_i^n\|^2_{0,\Omega_i^n} + (\Delta t)^2\left\|\frac{\partial^2\boldsymbol{u}_i(\tilde t)}{\partial\tilde t^2}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{L^2(0,t_{n+1};L^2(\Omega^i_{\tilde t}))}\right.\\
&\left. +\Delta t\sum_{n=1}^{N}\left\|\left(\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right)^n\right\|^2_{0,\Omega^i_n}\right) + \frac{\Delta t\gamma^2}{2}\sum_{n=1}^{N}\|\phi^n\|^2_{0,\Omega_n^1}.
\end{aligned} \tag{73}$$

Square both sides of (63), multiply by $\frac{\Delta t}{2}$ and sum over the time step n from 0 to $N-1$, then we obtain the error estimate of the last term on the right hand side of (73). Substitute it into (73) and apply the discrete Grönwall's inequality, yield

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\Delta t\sum_{n=0}^{N-1}\left\|\frac{\sigma_i^{n+1}-\sigma_i^n\circ \boldsymbol{X}^i_{n+1,n}}{\Delta t}\right\|^2_{\Omega^i_{n+1}} + \|\sigma_i^N\|^2_{1,\Omega^i_N}\right)\\
&\leq C\sum_{i=1}^{2}\left(\Delta t\sum_{n=1}^{N}\|\delta_i^n\|^2_{0,\Omega_i^n} + (\Delta t)^2\left\|\frac{\partial^2\boldsymbol{u}_i}{\partial\tilde t^2}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{L^2(0,T;L^2(\Omega^i_t))} + \Delta t\sum_{n=1}^{N}\left\|\left(\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right)^n\right\|^2_{0,\Omega^i_n}\right).
\end{aligned} \tag{74}$$

Further, due to (63), we have

$$\Delta t\sum_{n=1}^{N}\|\phi^n\|^2_{0,\Omega_n^1} \leq C\sum_{i=1}^{2}\left[\Delta t\sum_{n=1}^{N}\left(\|\delta_i^n\|^2_{0,\Omega_i^n} + \left\|\left(\frac{\partial\delta_i}{\partial t}\Big|^h_{\hat{\boldsymbol{x}}}\right)^n\right\|^2_{0,\Omega^i_n} + \|\sigma_i^n\|^2_{1,\Omega_i^n}\right) + (\Delta t)^2\left\|\frac{\partial^2\boldsymbol{u}_i}{\partial\tilde t^2}\Big|^h_{\hat{\boldsymbol{x}}}\right\|^2_{L^2(0,T;L^2(\Omega^i_t))}\right]. \tag{75}$$

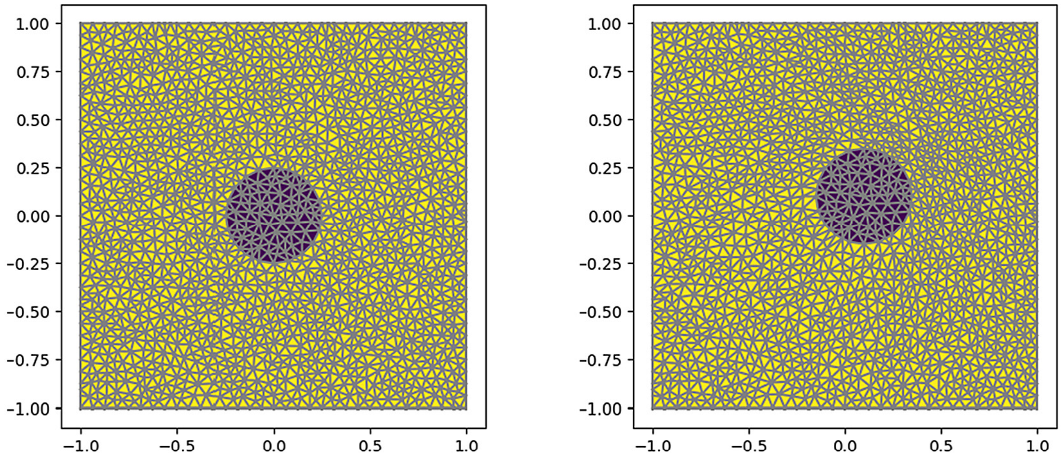

Fig. 2. Example 1: Initial (left) and terminal (right) subdomains and meshes with $\boldsymbol{\omega} = 0.1$ and $h = \frac{1}{16}$.

Add (75) to (74) then apply the discrete Grönwall's inequality, and the triangular inequality, the estimation (49) is then finally proved. □

Remark 4.6. Similar with Remark 3.6, Theorem 4.5 shows an optimal (first-order) error estimate for the fully discrete scheme in the discrete energy norm of velocity in $L^2(H^1)$ with respect to a low solution regularity, which is another improvement over the classical H^1-projection technique that is used in [6] for Stokes equations on a moving domain, where the convergence order in energy norm is only suboptimal [6, Theorem 2.3].

5. Numerical experiments

5.1. Translation without deformation

We consider an numerical example with a less smooth real solution in two-dimensional case, i.e., the velocity $\boldsymbol{u} = (u_1, u_2)^T \in (H^2 \cap L^\infty)(0, T; (H^2(\Omega_t^1 \cup \Omega_t^2))^2 \cap (H^1(\Omega))^2)$ and the pressure $p \in (H^1 \cap L^\infty)(0, T; H^1(\Omega_t^1))$, which are given as the following smooth functions:

$$\begin{aligned} u_1 &= (y - \boldsymbol{\omega} t)\left((x - \boldsymbol{\omega} t)^2 + (y - \boldsymbol{\omega} t)^2 - 0.0625\right) t/\beta, \\ u_2 &= -(x - \boldsymbol{\omega} t)\left((x - \boldsymbol{\omega} t)^2 + (y - \boldsymbol{\omega} t)^2 - 0.0625\right) t/\beta, \\ p &= (\pi \cos(2\pi(x - \boldsymbol{\omega} t))\cos(2\pi(y - \boldsymbol{\omega} t)) + 0.080716)\sin(t), \end{aligned} \tag{76}$$

by properly choosing $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ and $\boldsymbol{\tau}$ to satisfy the 2D Stokes/parabolic interface problem (1), where $\beta = \beta_i(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega_t^i$ $(i = 1, 2)$ are chosen as piecewise constants, $\boldsymbol{x} = (x, y)^T \in \bar{\hat{\Omega}} = [-1, 1] \times [-1, 1]$ that immerses the initial subdomains $\hat{\Omega}^2 = \{(x, y)\,|x^2 + y^2 \le 0.0625\}$, $\hat{\Omega}^1 = \Omega \backslash \bar{\hat{\Omega}}^2$, and $t \in [0, 1]$ with $T = 1$. Then the interface $\Gamma_t = \partial\Omega_t^2$ satisfies the equation of a circle:

$$(x - \boldsymbol{\omega} t)^2 + (y - \boldsymbol{\omega} t)^2 = 0.0625, \quad \forall t \in [0, T],$$

where, $\boldsymbol{\omega}$ is a prescribed moving velocity of Γ_t. By defining the real solution $\boldsymbol{u}$ and the interface Γ_t this way, we know $\nabla \boldsymbol{u} \in (L^2(\Omega))^4$, only, leading to $\boldsymbol{u} \in (H^1(\Omega))^2$. In addition, the interface motion, $\boldsymbol{x}_\Gamma$, is defined as $\boldsymbol{x}_\Gamma = \boldsymbol{\omega} t + \hat{\boldsymbol{x}}_\Gamma$, $\forall \hat{\boldsymbol{x}}_\Gamma \in \hat{\Gamma} = \partial\hat{\Omega}^2$, $\forall t \in [0, 1]$, according to which, we solve the discrete ALE mapping $\boldsymbol{X}_{h,i}^t$ on $\hat{\Omega}^i$ for the moving meshes $\mathcal{T}_{h,i}^t$ $(i = 1, 2)$, $\forall t \in [0, 1]$. In the following numerical experiments, we pick $\boldsymbol{\omega} = 0.1$. The initial and terminal domains and meshes are shown in Fig. 2, respectively.

We employ the fully discrete ALE finite element approximation (41) using the finite element spaces defined in (8), i.e., the MINI mixed element as a stable Stokes-pair, to solve the above Stokes/parabolic interface problem for $((\boldsymbol{u}_1, \boldsymbol{u}_2), p_1)$ with a grid doubling as well as an appropriate time step size Δt that is proportional to h^2, then to investigate the numerical convergence rate in terms of both h and Δt. With different ratios of the jump coefficients β_1 and β_2, we obtain the following convergence performances illustrated in Tables 1–3, where, we denote $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega_T^1 \cup \Omega_T^2)}$ by $e_{u,1}$, $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega_T^1 \cup \Omega_T^2)}$ by $e_{u,0}$, and $\|p - p_h\|_{L^2(\Omega_T^1)}$ by $e_{p,0}$, and, the convergence "*rate*" is calculated by $\log_2\left(\frac{e_{k,2h}}{e_{k,h}}\right)$ for $\boldsymbol{u}$ or "p". Figs. 3–5 illustrate convergence histories of each case via a log–log plot. From them we can see that the convergence rates of both the velocity in H^1-norm and the pressure in L^2-norm are of the first order. Additionally, velocity errors in L^2-norm even have the second order of convergence rate, and, all numerical convergence rates are independent of the jump ratios. It means all convergence rates are optimal regarding the adopted MINI element, Theorem 4.5 is thus validated for a Stokes/parabolic interface problem with a globally low solution regularity.

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Table 1
Convergence performance of the case: $\beta_1 = 1, \beta_2 = 1$ in Example 1.

h	$e_{u,1}$	Rate	$e_{u,0}$	Rate	$e_{p,0}$	Rate
$\frac{1}{4}$	0.516986836		0.022440886		0.730733279	
$\frac{1}{8}$	0.231354028	1.16	0.00575924	1.96	0.20185467	1.86
$\frac{1}{16}$	0.116959946	0.98	0.001469527	1.97	0.064543055	1.64
$\frac{1}{32}$	0.058159269	1.01	0.00036935	1.99	0.024825937	1.38
$\frac{1}{64}$	0.029157309	1.00	9.30883E−05	1.99	0.011351926	1.13

Table 2
Convergence performance of the case: $\beta_1 = 1000, \beta_2 = 1$ in Example 1.

h	$e_{u,1}$	Rate	$e_{u,0}$	Rate	$e_{p,0}$	Rate
$\frac{1}{4}$	0.033954753		0.001951829		0.726987967	
$\frac{1}{8}$	0.014174702	1.26	0.000705712	1.47	0.201460044	1.85
$\frac{1}{16}$	0.006115952	1.21	0.00017009	2.05	0.069488858	1.54
$\frac{1}{32}$	0.002919325	1.07	4.18E−05	2.02	0.024796179	1.49
$\frac{1}{64}$	0.00136905	1.09	1.02E−05	2.03	0.011347748	1.13

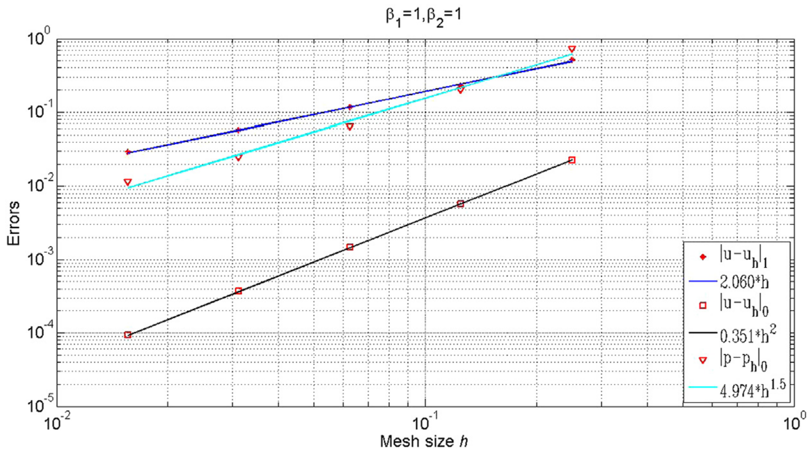

Fig. 3. Convergence history of the case: $\beta_1 = 1, \beta_2 = 1$ in Example 1.

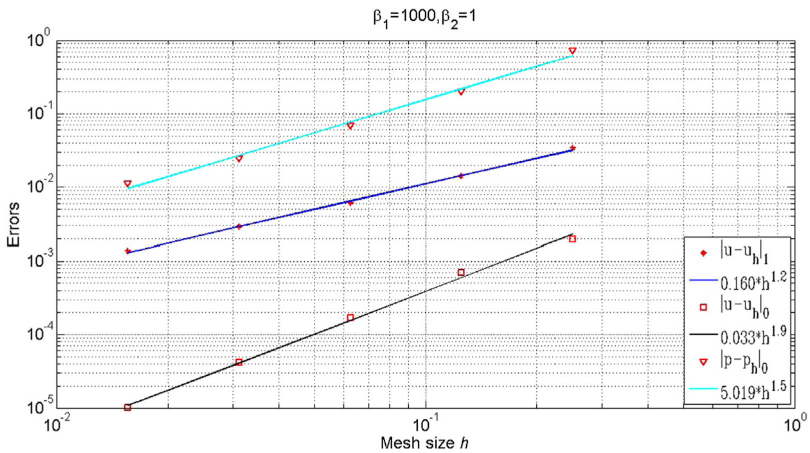

Fig. 4. Convergence history of the case: $\beta_1 = 1000, \beta_2 = 1$ in Example 1.

Table 3
Convergence performance of the case: $\beta_1 = 1, \beta_2 = 0.001$ in Example 1.

h	$e_{u,1}$	Rate	$e_{u,0}$	Rate	$e_{p,0}$	Rate
$\frac{1}{4}$	49.74042054		1.649159774		0.72878861	
$\frac{1}{8}$	15.71558051	1.66	0.492912811	1.74	0.204922747	1.83
$\frac{1}{16}$	6.973264726	1.17	0.100711781	2.29	0.064533986	1.67
$\frac{1}{32}$	3.061419158	1.19	0.027343689	1.88	0.024829483	1.38
$\frac{1}{64}$	1.383859114	1.15	0.006258234	2.13	0.011351034	1.13

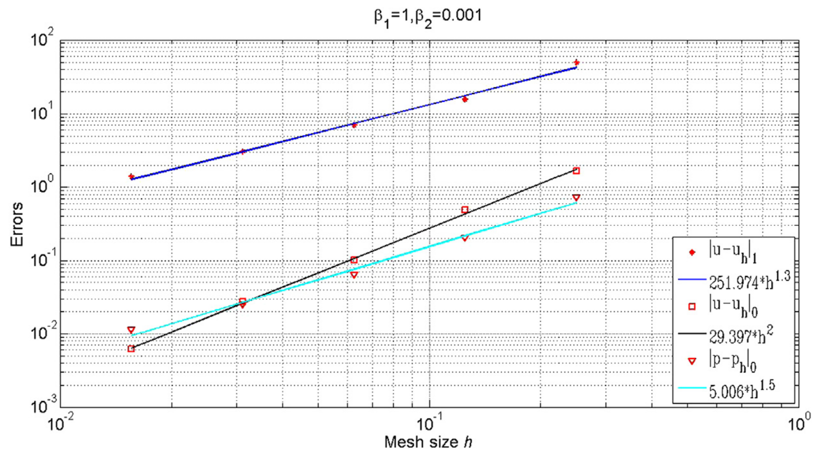

Fig. 5. Convergence history of the case: $\beta_1 = 1, \beta_2 = 0.001$ in Example 1.

5.2. Translation with deformation

In this example, we consider that the immersed subdomain Ω_t^2 conducts a translational motion combining with a deformation. Let the velocity $\boldsymbol{u} = (u_1, u_2)^T \in (H^2 \cap L^\infty)(0, T; (H^2(\Omega_t^1 \cup \Omega_t^2))^2 \cap (H^1(\Omega))^2)$ and the pressure $p \in (H^1 \cap L^\infty)(0, T; H^1(\Omega_t^1))$ be given as the following smooth functions:

$$\begin{aligned} u_1 &= (y - \omega t)\left((x - \omega t)^2(1 + \tfrac{t}{5})^2 + (y - \omega t)^2(1 + \tfrac{t}{5})^{-2} - 0.0625\right) t/\beta, \\ u_2 &= -(x - \omega t)\left((x - \omega t)^2 \left(1 + \tfrac{t}{5}\right)^6 + (y - \omega t)^2 \left(1 + \tfrac{t}{5}\right)^2 - 0.0625 \left(1 + \tfrac{t}{5}\right)^4\right) t/\beta, \\ p &= (x - \omega t)^2 \left(1 + \tfrac{t}{5}\right)^2 + (y - \omega t)^2 \left(1 + \tfrac{t}{5}\right)^{-2} - 0.0625, \end{aligned} \tag{77}$$

by properly choosing $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ and $\boldsymbol{\tau}$ to satisfy the 2D Stokes/parabolic interface problem (1), where again, $\beta = \beta_i(\boldsymbol{x})$ $(i = 1, 2)$ are chosen as piecewise constants across the interface. We adopt the same setup for $\bar{\Omega}$, initial subdomains $\hat{\Omega}^2$ and $\hat{\Omega}^1$, and the time interval $[0, 1]$ as shown in Section 5.1, but a different interface motion, $\boldsymbol{x}_\Gamma$, whose shape satisfies the following equation of an ellipse with a fixed area:

$$(x - \omega t)^2 \left(1 + \frac{t}{5}\right)^2 + (y - \omega t)^2 \left(1 + \frac{t}{5}\right)^{-2} = 0.0625, \quad \forall t \in [0, T], \tag{78}$$

and combines with a translational motion, i.e., $\boldsymbol{x}_\Gamma = \boldsymbol{\omega} t + \boldsymbol{x}_{\text{ellipse}}$, where $\boldsymbol{x}_{\text{ellipse}}$ is given in (78) for all $t \in [0, 1]$, and $\boldsymbol{\omega}$ is a prescribed translational velocity. It is easy to see that $\boldsymbol{u}$ still belongs to $(H^1(\Omega))^2 \cap (H^2(\Omega_t^1 \cup \Omega_t^2))^2$. In the following numerical experiments, we solve the discrete ALE mapping $\boldsymbol{X}_{h,i}^t$ on $\hat{\Omega}^i$ for the moving meshes $\mathcal{T}_{h,i}^t$ $(i = 1, 2)$, $\forall t \in [0, 1]$ with a picked $\boldsymbol{\omega} = 0.1$. Fig. 6 shows the initial and terminal domains and the obtained meshes from the discrete ALE mapping, respectively. On these translational and deforming meshes, we carry out the same ALE finite element computations as done in Section 5.1 for the presented Stokes/parabolic interface problem, and obtain very similar convergent results as shown in Tables 4–6 and Figs. 7–9, i.e., first-order convergence performances are obtained for both the velocity in H^1-norm and the pressure in L^2-norm without any dependence on the jump ratios, which is in accordance with the optimal convergence property of the adopted MINI element. Theorem 4.5 is then validated again for the case of translational and deformable interface motion.

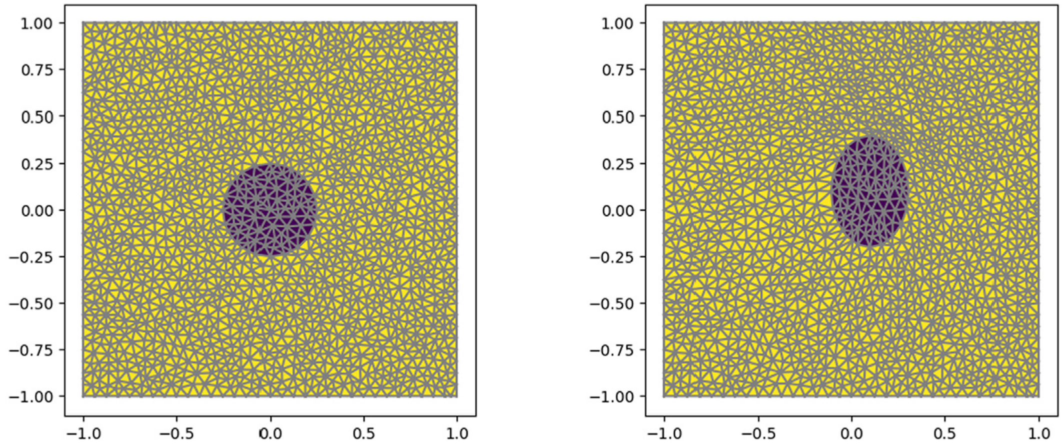

Fig. 6. Example 2: Initial (left) and terminal (right) subdomains and meshes with $\omega = 0.1$ and $h = \frac{1}{16}$.

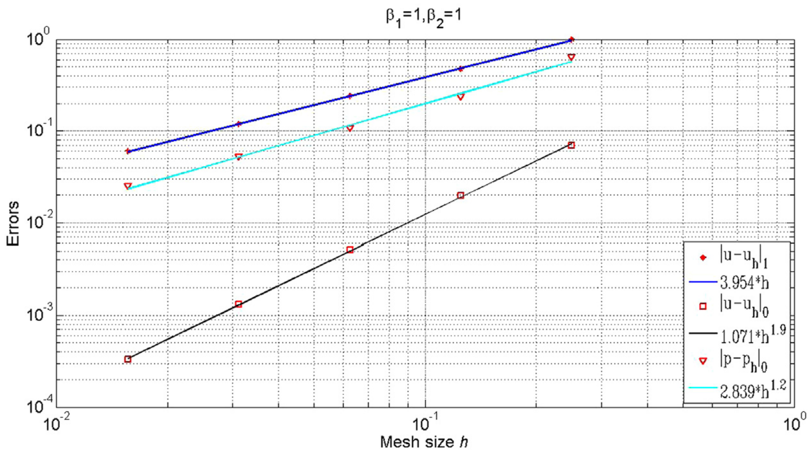

Fig. 7. Convergence history of the case: $\beta_1 = 1, \beta_2 = 1$ in Example 2.

Table 4
Convergence performance of the case: $\beta_1 = 1, \beta_2 = 1$ in Example 2.

h	$e_{u,1}$	Rate	$e_{u,0}$	Rate	$e_{p,0}$	Rate
$\frac{1}{4}$	0.996301753		0.069596115		0.642449907	
$\frac{1}{8}$	0.472722052	1.08	0.019931697	1.80	0.236682718	1.44
$\frac{1}{16}$	0.24105556	0.97	0.005159317	1.95	0.108150064	1.13
$\frac{1}{32}$	0.119485413	1.01	0.001306648	1.98	0.052168477	1.05
$\frac{1}{64}$	0.06003909	0.99	0.000331003	1.98	0.0253667	1.04

Table 5
Convergence performance of the case: $\beta_1 = 1000, \beta_2 = 1$ in Example 2.

h	$e_{u,1}$	Rate	$e_{u,0}$	Rate	$e_{p,0}$	Rate
$\frac{1}{4}$	0.08103879		0.003368715		0.591390759	
$\frac{1}{8}$	0.021046922	1.95	0.001043729	1.69	0.218702698	1.44
$\frac{1}{16}$	0.009594669	1.13	0.000250221	2.06	0.106047448	1.04
$\frac{1}{32}$	0.004978054	0.95	6.24E−05	2.00	0.051738145	1.04
$\frac{1}{64}$	0.002030196	1.29	1.51E−05	2.04	0.02531953	1.03

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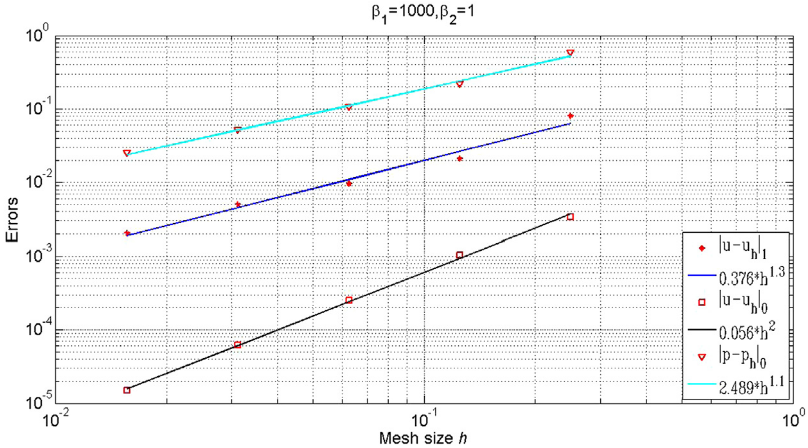

Fig. 8. Convergence history of the case: $\beta_1 = 1000$, $\beta_2 = 1$ in Example 2.

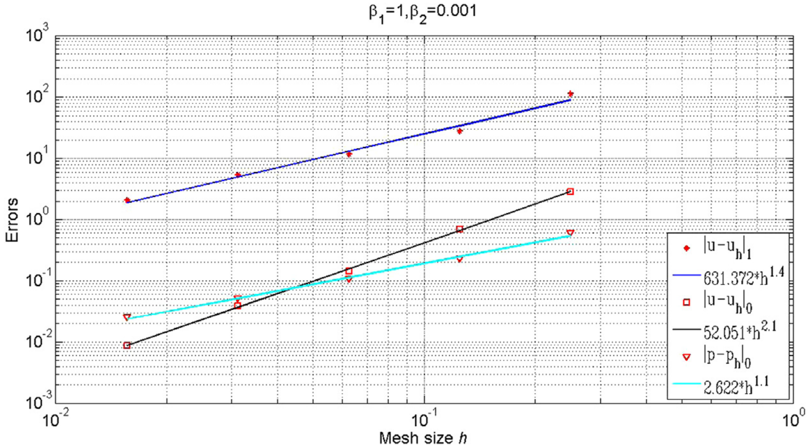

Fig. 9. Convergence history of the case: $\beta_1 = 1$, $\beta_2 = 0.001$ in Example 2.

Table 6
Convergence performance of the case: $\beta_1 = 1$, $\beta_2 = 0.001$ in Example 2.

h	$e_{u,1}$	Rate	$e_{u,0}$	Rate	$e_{p,0}$	Rate
$\frac{1}{4}$	114.103521		2.887720121		0.604718796	
$\frac{1}{8}$	27.75581716	2.04	0.697766225	2.05	0.229827914	1.40
$\frac{1}{16}$	11.64688374	1.25	0.144909544	2.27	0.106618476	1.11
$\frac{1}{32}$	5.370819847	1.12	0.039041095	1.89	0.051933702	1.04
$\frac{1}{64}$	2.075388731	1.37	0.008718433	2.16	0.025332275	1.04

6. Conclusions

In this paper, the Stokes/parabolic interface problem and its ALE-finite element analyses provide a foundation for more complex fluid–structure interaction problems' ALE finite element analysis with an optimal convergence rate on account of a lower solution regularity, realistically. In particular, we develop both semi- and fully discrete ALE-finite element approximations to a unsteady Stokes/parabolic moving interface problem in MINI-mixed finite element spaces, utilize a novel H^1-projection technique that is associated with a moving interface problem to analyze their stability and optimal error estimates, and obtain the convergence date of $O(h)$ for the semi-discrete scheme according to a low solution regularity. Moreover, we specifically discretize the moving temporal domain generated by ALE mapping using the implicit backward Euler scheme, defining the fully discrete ALE finite element approximation. Through additional error analyses with respect to the time step size Δt, and using the specific H^1-projection, we also obtain an optimal convergence order of $O(h + \Delta t)$ in energy norm for the fully discrete scheme, which is consistent with the spatial convergence rate of the

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semi-discrete scheme, also consistent with the temporal convergence (first) order of backward Euler-type time difference scheme. The error analysis techniques using a novel H^1-projection developed in this paper can be similarly extended to a realistic FSI problem in the future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Rihui Lan: Conceptualization, Formal analysis, Software, Validation. **Michael J. Ramirez:** Resources, Data curation, Writing - original draft. **Pengtao Sun:** Methodology, Supervision, Writing - review & editing, Funding acquisition, Project administration, Visualization.

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