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ON THE DUAL HESSE ARRANGEMENT

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Abstract. In the present note we investigate to which extent the configuration of 9 lines intersecting in triples in 12 points is determined by these incidences. We show that up to a projective automorphism there is exactly one such configuration in characteristic zero and one in characteristic 3. We pin down the geometric difference between these two realizations.

1. Introduction

In projective geometry, a point-line configuration consists of a finite set of points, and a finite arrangement of lines, such that each point is incident to the same number of lines and each line is incident to the same number of points. Their systematic study has been initiated by Theodor Reye in 1876 but they are a much more classical subject of study.

To a configuration there is assigned a symbol (p_γ, ℓ_π), where p is the number of points, ℓ is the number of lines, γ is the number of lines through each point and π is the number of points on each line. For example the famous Pappus configuration is denoted by $(9_3, 9_3)$, see Figure 1. One should bear in mind that the same symbol might be assigned to many non-isomorphic configurations. In the present note we are interested in the configuration $(12_3, 9_4)$. This is a remarkable configuration because there are no incidences among the 9 lines other than the 12 configuration points. One incarnation of this configuration, namely the dual Hesse configuration plays an important role in testing various properties in the theory of arrangements and recently also in commutative algebra (see e.g. the work of Dumnicki, Szemberg

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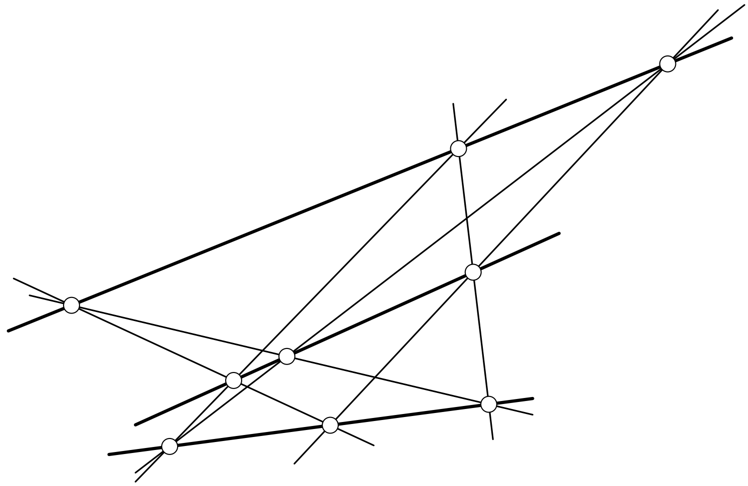

Figure 1. Pappus configuration

and Tutaj-Gasińska [3]) and algebraic geometry, more precisely in the theory of unexpected hypersurfaces (see e.g. the work of Bauer, Malara, Szemberg and Szpond [2]). All this has motivated our research. We want to find out to what extent the configuration is determined by the combinatorics involved.

2. Dual Hesse configuration

Ludwig Otto Hesse published in 1844 in Crelle's Journal an article [5] which contains, in particular, a description of a remarkable point-line configuration. The configuration consists of 9 points and 12 lines arranged so that there are 4 lines through every point and 3 points on every line, it is a $(9_4, 12_3)$ configuration. The incidences are indicated in Figure 2, which we borrowed from Wikipedia.

It is not possible to draw this configuration in the real plane without bending the lines. This follows from the celebrated Sylvester-Gallai Theorem. In fact, it seems that Hesse's discovery has prompted Sylvester to ask in [8] if there are non-trivial (i.e. not a pencil) point-line configurations in the *real* projective plane such that there are no intersection points among configuration lines where only 2 configuration lines meet.

Hesse construction works over complex numbers. Taking 9 inflection points of an elliptic curve C embedded into $\mathbb{P}^2$ as a smooth cubic curve (which is equivalent to taking 3–torsion points on C) one gets the Hesse configuration joining all pairs of these points. Because of the arithmetic properties of an elliptic curve, a line

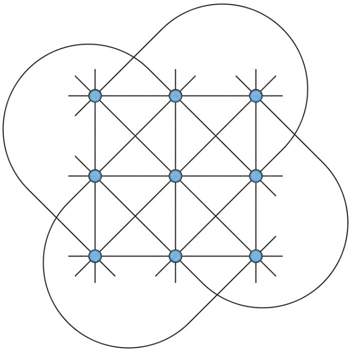

FIGURE 2. Hesse configuration

intersecting it in two 3–torsion points must go through a third 3–torsion point. Thus there are 12 such lines altogether. Another, very illuminating description comes from the *Hesse pencil*, which in appropriate coordinates can be written as

$$s\,xyz\;+\;t\,(x^3+y^3+z^3)\;=\;0. \tag{1}$$

There are exactly 4 singular members in this pencil, each of which splits into 3 lines, see the work of Artebani and Dolgachev [1] for a beautiful account on various aspects of this pencil.

Passing to the dual we obtain what is known as the *dual Hesse configuration*. Over complex numbers it can be given by linear factors of the polynomial

$$(x^3-y^3)\,(y^3-z^3)\,(z^3-x^3)\;=\;0. \tag{2}$$

In this description it is immediately recognized as a member of an infinite family of Fermat arrangements, see [9] for an extensive survey on this kind of arrangements. The dual Hesse configuration is of course of type $(9_3, 12_4)$. It is one of very few examples of point-line configurations where all intersection points between configuration lines belong to exactly 3 of these lines. The other examples are:

- trivial configuration: 3 lines meeting in a single point;
- the finite projective plane $\mathbb{P}^2(\mathbb{F}_3)$ defined in characteristic 3 with the whole pencil of lines through a single point removed.

The purpose of this note is to investigate to which extent incidences of the dual Hesse configuration determine it up to a projective automorphism. Our main results are the following.

Theorem 1. *Any $(12_3, 9_4)$ point-line configuration in the projective plane defined over a field $\mathbb{F}$ of characteristic zero is projectively equivalent to that given by linear factors of the Fermat equation (2).*
In particular, the field $\mathbb{F}$ must contain roots of unity of order 3.

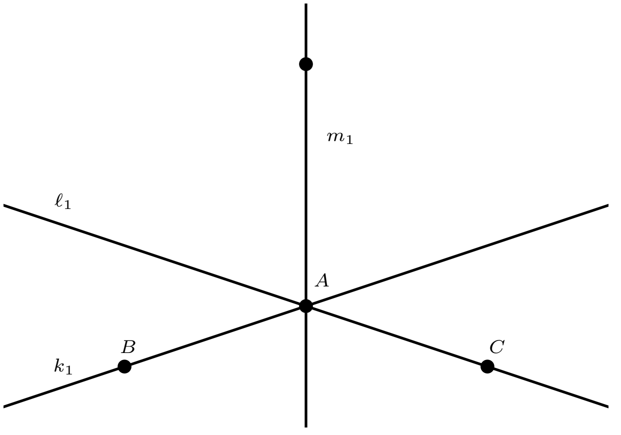

FIGURE 3. First step

As a by-product of our proof we obtain also the following complementary result.

Theorem 2. *Any* $(12_3, 9_4)$ *point-line configuration in the projective plane defined over a field* $\mathbb{F}$ *of positive characteristic is projectively equivalent to that obtained from* $\mathbb{P}^2(\mathbb{F}_3)$ *by removing the whole pencil of lines passing through a point.*
In particular, $\mathbb{F}$ *must be of characteristic* 3.

We prove these statements in the subsequent section.

3. COORDINATIZATION OF THE DUAL HESSE CONFIGURATION

In this section we use incidences of the $(12_3, 9_4)$ point-line configuration to recover step by step its coordinates in a conveniently chosen system of coordinates. This approach is motivated by Sturmfels [7]. It has been applied successfully recently in the study of parameter spaces of Böröczky arrangements, see [6] and [4].

Let $\mathbb{F}$ be an arbitrary field with sufficiently many elements so that $\mathbb{P}^2(\mathbb{F})$ has at least 12 points. Since all points in the configuration have the same properties, we pick one, call it A and we assume that $A = (1 : 1 : 1)$. Then we name the lines passing through A as k_1, ℓ_1 and m_1. Since any configuration line contains 4 configuration points, we pick $B = (1 : 0 : 0)$ on k_1. On ℓ_1 we pick a point C which is not connected by a configuration line to the point B. It is possible, because there are only 3 lines passing through B and they intersect ℓ_1 in A and two other points. We take C to be the remaining point and we set its coordinates to $(0 : 1 : 0)$. Thus the lines k_1, ℓ_1 have equations $y - z = 0$ and $x - z = 0$ respectively. The choices made so far are depicted in Figure 3.

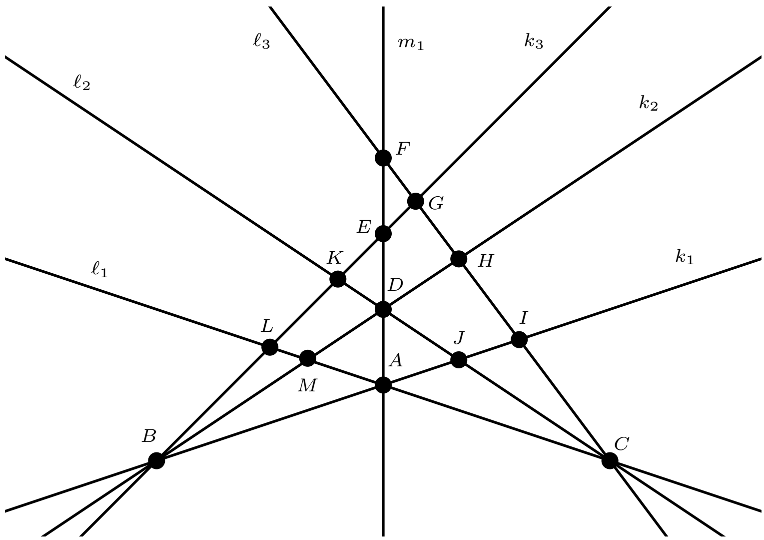

FIGURE 4. Case 1

Let k_2, k_3 be the configuration lines passing through B distinct from k_1, and similarly let ℓ_2, ℓ_3 be the configuration lines passing through C distinct from ℓ_1. Then we have two possibilities on how these lines meet the line m_1.

Case 1. We assume that k_2, k_3, ℓ_2, ℓ_3 intersect m_1 in 3 distinct points. Renumbering the lines if necessary, we may assume that these are k_2 and ℓ_2 which meet m_1 in the same point, which we call D. This situation is indicated in Figure 4.

Then A, B, C, D and intersection points between lines

$$E = k_3 \cap m_1,\ F = \ell_3 \cap m_1,\ G = k_3 \cap \ell_3,\ H = k_2 \cap \ell_3, I = k_1 \cap \ell_3,$$

$$J = k_1 \cap \ell_2,\ K = k_3 \cap \ell_2,\ L = k_3 \cap \ell_1,\ M = k_2 \cap \ell_1$$

must be all mutually distinct (otherwise some 2 distinct lines would intersect in 2 distinct points). But then there would be already 13 points in the configuration. A contradiction.

Case 2. Thus we are left with the case in which the lines k_2, ℓ_2 and k_3, ℓ_3 intersect m_1 pairwise in the same point. Let $E = k_2 \cap \ell_2$ and $F = k_1 \cap \ell_2$. In this situation let D be the point on m_1 not connected neither to A, nor to B by a configuration line.

We have now again two possibilities depending on whether the points B, C, D are collinear or not.

Subcase 2.1. The points B, C, D are collinear.

In this situation we can assume after change of coordinates if necessary that $D = (1 : 1 : 0)$. Let $E = (a : a : 1)$ be a point on the line m_1 distinct from A and

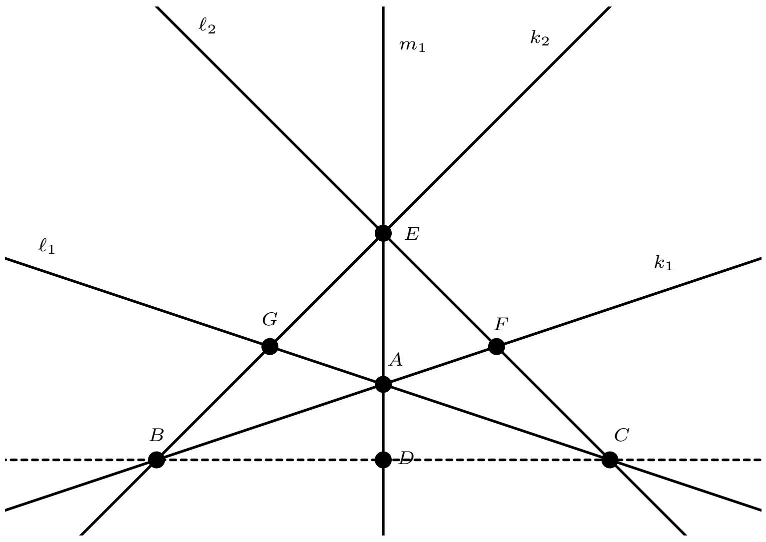

FIGURE 5. Case 2.1 first step of the construction

D, so that $a \neq 1$ is an element of $\mathbb{F}$. Then we have

$$k_2 : \; y - az = 0 \;\text{ and }\; \ell_2 : \; x - az = 0,$$

for the lines BE and CE, respectively. Then we compute

$$F = k_1 \cap \ell_2 = (a : 1 : 1), \;\text{ and }\; G = k_2 \cap \ell_1 = (1 : a : 1).$$

The situation so far is depicted in Figure 5.

Next we choose another point $H = (b : b : 1)$ on the line m_1. In order to keep points mutually distinct, we require now $b \neq 1$ and $b \neq a$. Then we have

$$k_3 : \; y - bz = 0 \;\text{ and }\; \ell_3 : \; x - bz = 0$$

for the lines joining B and H, and C and H respectively. This determines all other configuration points:

$$I = k_1 \cap \ell_3 = (b : 1 : 1), \; J = k_3 \cap \ell_1 = (1 : b : 1),$$

$$K = k_3 \cap \ell_2 = (a : b : 1), \; L = k_2 \cap \ell_3 = (b : a : 1).$$

The configuration is indicated in Figure 6.

We need still to determine the lines m_2 and m_3. Let m_2 be the line joining D and I. Then we have

$$m_2 : \; -x + y + (b - 1)z = 0.$$

Since $K = m_2 \cap \ell_2$, we obtain $a = -1$.

Then it must be $m_2 \cap \ell_1$ equal either G or J. In the first case we get $b = 3$, in the second $b = 1$, which is a contradiction with the assumption $b \neq a$. Hence m_2

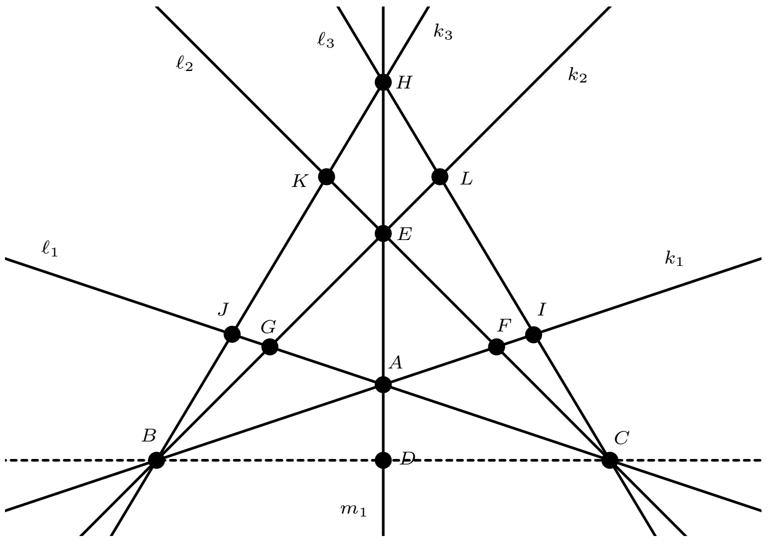

FIGURE 6. Case 2.1 second step of the construction

is the line through D, I, K and G. Consequently m_3 is the line through D and J. Thus we have

$$m_3:\ x - y + (b-1)z = 0.$$

Since L is also a point on m_3 we obtain $2b = a + 1 = 0$. It is not possible that $\mathbb{F}$ is of characteristic 2, because otherwise it would be $b = 1$, which is excluded by our assumptions. Thus we can divide by 2 and we have now $b = 3$ and $b = 0$. This is possible only in characteristic 3. Assuming this characteristic, we complete the construction by verifying that the condition $F \in m_3$ is satisfied.

Subcase 2.2. The points B, C, D are not collinear.

In this situation we can assume after change of coordinates if necessary that $D = (0 : 0 : 1)$. We proceed as in the Subcase 2.1 and obtain the same coordinates of points and lines passing through B and C. We collect and present them below for convenience.

$$A = (1 : 1 : 1),\ B = (1 : 0 : 0),\ C = (0 : 1 : 0),\ D = (0 : 0 : 1),$$

$$E = (a : a : 1),\ F = (a : 1 : 1),\ G = (1 : a : 1),\ H = (b : b : 1),$$

$$I = (b : 1 : 1),\ J = (1 : b : 1),\ K = (a : b : 1),\ L = (b : a : 1),$$

with $a, b \in \mathbb{F}$ such that $a \neq 1 \neq b \neq a$.

$$k_1:\ y - z = 0,\ k_2:\ y - az = 0,\ k_3:\ y - bz = 0,$$

$$\ell_1:\ x - z = 0,\ \ell_2:\ x - az = 0,\ \ell_3:\ x - bz.$$

The incidences above are depicted in Figure 7.

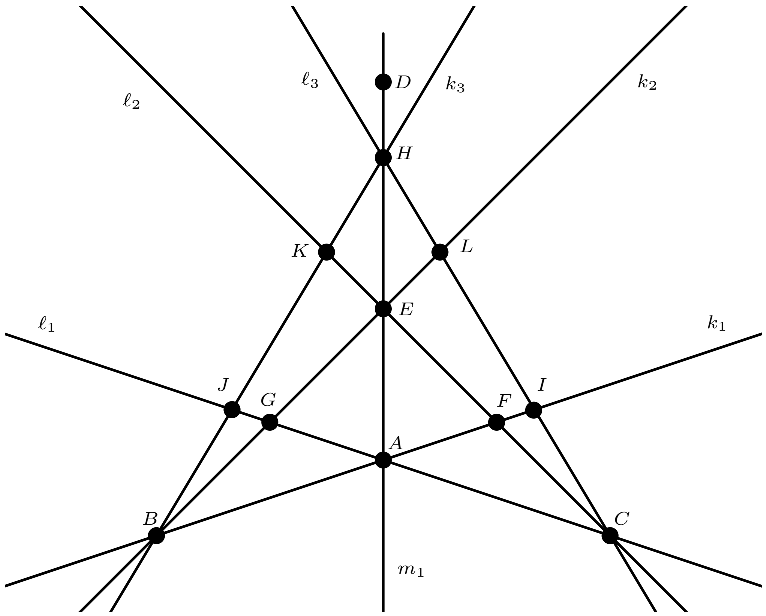

FIGURE 7. Case 2.2 the construction

The only difference compared to Subcase 2.1 occurs in lines m_2 and m_3 passing through D. We focus now on these lines. Let m_2 be the line determined by D and I. Then

$$m_2 : \ x - by = 0.$$

Checking the conditions for points F, G, K and L to lie on m_2, we get an immediate contradiction for F and L, so that it must be G and K on m_2. The incidence conditions are then

$$\begin{cases} 1 - ab & = & 0 \\ a - b^2 & = & 0 \end{cases}$$

and we see that it must be $b^3 = 1$. Since b cannot be equal 1, it must be a primitive root of 1 of order 3. Then $a = b^2$ is the other primitive order 3 root of 1.

It remains to check that the line m_3 determined by D and J with equation

$$m_3 : \ bx - y = 0$$

contains points F and L. As this is elementary, we are done with the proof of Theorems 1 and 2.

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