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NEGATIVE CURVES ON SPECIAL RATIONAL SURFACES

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ABSTRACT. We study negative curves on surfaces obtained by blowing up special configurations of points in \mathbb{P}^2 . Our main results concern the following configurations: very general points on a cubic, 3-torsion points on an elliptic curve and nine Fermat points. As a consequence of our analysis, we also show that the Bounded Negativity Conjecture holds for the surfaces we consider. The note contains also some problems for future attention.

1. INTRODUCTION

Negative curves on algebraic surfaces are an object of classical interest. One of the most prominent achievements of the Italian School of algebraic geometry was Castelnuovo's Contractibility Criterion.

Definition 1.1 (Negative curve). *We say that a reduced and irreducible curve C on a smooth projective surface is negative, if its self-intersection number C^2 is less than zero.*

Example 1.2 (Exceptional divisor, (-1) -curves). Let X be a smooth projective surface and let $P \in X$ be a closed point. Let $f : \text{Bl}_P X \rightarrow X$ be the blow up of X at the point P . Then the exceptional divisor E of f (i.e., the set of points in $\text{Bl}_P X$ mapped by f to P) is a negative curve. More precisely, E is rational and $E^2 = -1$. By a slight abuse of language we will call such curves simply (-1) -curves.

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Castelnuovo's result asserts that the converse is also true; for example, see [10, Theorem V.5.7] or [1, Theorem III.4.1].

Theorem 1.3 (Castelnuovo's Contractibility Criterion). *Let Y be a smooth projective surface defined over an algebraically closed field. If C is a rational curve with $C^2 = -1$, then there exists a smooth projective surface X and a projective morphism $f : Y \rightarrow X$ contracting C to a point on X . In other words, Y is isomorphic to $\text{Bl}_P X$ for some point $P \in X$.*

The above result plays a pivotal role in the Enriques-Kodaira classification of surfaces.

Of course, there are other situations in which negative curves on algebraic surfaces appear.

Example 1.4. Let C be a smooth curve of genus $g(C) \geq 2$. Then the diagonal $\Delta \subset C \times C$ is a negative curve as its self-intersection is given by $\Delta^2 = 2 - 2g$.

It is quite curious that it is in general not known if for a general curve C , there are other negative curves on the surface $C \times C$, see [12]. It is in fact even more interesting, that there is a direct relation between this problem and the famous Nagata Conjecture, which was observed by Ciliberto and Kouvidakis [5].

There is also a connection between negative curves and the Nagata Conjecture on general blow ups of \mathbb{P}^2 . We recall the following conjecture about (-1) -curves which in fact implies the Nagata Conjecture; see [4, Lemma 2.4].

Conjecture 1.5 (Weak SHGH Conjecture). *Let $f : X \rightarrow \mathbb{P}^2$ be the blow up of the projective plane \mathbb{P}^2 in general points P_1, \dots, P_s . If $s \geq 10$, then the only negative curves on X are the (-1) -curves.*

On the other hand, it is well known that already a blow up of \mathbb{P}^2 in 9 general points carries *infinitely* many (-1) -curves.

One of the central and widely open problems concerning negative curves on algebraic surfaces asks whether on a fixed surface negativity is bounded. More precisely, we have the following conjecture (BNC in short). See [2] for an extended introduction to this problem.

Conjecture 1.6 (Bounded Negativity Conjecture). *Let X be a smooth projective surface. Then there exist a number τ such that*

$$C^2 \geq \tau$$

for any reduced and irreducible curve $C \subset X$.

If the Conjecture holds on a surface X , then we denote by $b(X)$ the largest number τ such that the Conjecture holds. It is known (see [2, Proposition 5.1]) that if the negativity of reduced and irreducible curves is bounded below, then the negativity of all reduced curves is also bounded below.

Conjecture 1.6 is known to fail in the positive characteristic; see [8, 2]. In fact Example 1.4 combined with the action of the Frobenius morphism provides a counterexample. In characteristic zero, Conjecture 1.6 is open in general. It is easy to prove BNC in some cases; see Remark 3.7 for an easy argument when the anti-canonical divisor of X is \mathbb{Q} -effective. However, in many other cases the conjecture is open. In particular the following question is open and answering it may lead to a better understanding of Conjecture 1.6.

Question 1.7. Let X, Y be smooth projective surfaces and suppose that X and Y are birational and Conjecture 1.6 holds for X . Does then Conjecture 1.6 hold for Y also?

As a special case of this question, one can ask whether Conjecture 1.6 holds for blow ups of \mathbb{P}^2 . Since the conjecture clearly holds for \mathbb{P}^2 , it is interesting to consider the blow ups of \mathbb{P}^2 . If the blown up points are general, then one has Conjecture 1.5 stated above. On the other hand, it is also interesting to study blow ups of \mathbb{P}^2 at special points.

In this paper, we consider some examples of such special rational surfaces and completely list all the negative curves on them. In particular, we focus on blow ups of \mathbb{P}^2 at certain points which lie on elliptic curves. Our main results classify negative curves on such surfaces; see Theorems 2.4, 3.3 and 3.6. As a consequence, we show that Conjecture 1.6 holds for such surfaces. Additionally we provide effective optimal values of the number $b(X)$.

2. VERY GENERAL POINTS ON A CUBIC

In this section we study negative curves on blow ups of \mathbb{P}^2 at an arbitrary number s of very general points on a plane curve of degree 3. This situation was studied in detail by Harbourne in [9]. Before stating our main result we need to recall some notation. For the first notion, see [6, Definition 5] or [7] where this property is called *adequate* rather than standard.

Definition 2.1 (Standard form). *Let P_1, \dots, P_s be points in \mathbb{P}^2 . Let Γ be a plane curve of degree d with $m_i := \text{mult}_{P_i} \Gamma$, for $i = 1, \dots, s$. We say that Γ is in the standard form if*

- *the multiplicities m_1, \dots, m_s form a weakly decreasing sequence and*
- *$d \geq m_1 + m_2 + m_3$.*

Gimigliano showed in [7, page 25] that if the points P_1, \dots, P_s are general in \mathbb{P}^2 , then any curve Γ can be brought to the standard form by a finite sequence of standard Cremona transformations.

Theorem 2.2 (Gimigliano). *Let P_1, \dots, P_s be general points in \mathbb{P}^2 . Let Γ be a curve of degree d passing through points P_1, \dots, P_s with multiplicities m_1, \dots, m_s . Then there exists a birational transformation σ of \mathbb{P}^2 and general points P'_1, \dots, P'_s*

and a curve Γ' of degree d' passing through P'_1, \dots, P'_s with multiplicities m'_1, \dots, m'_s such that

- Γ' is in a standard form;
- $\Gamma' = \sigma(\Gamma)$;
- $d^2 - \sum_{i=1}^s m_i^2 = (d')^2 - \sum_{i=1}^s (m'_i)^2$.

We recall also the following Lemma, which is modeled on [7, Lemma 3.2].

Lemma 2.3. *Let $d \geq m_1 \geq \dots \geq m_r \geq 0$ and $t \geq n_1 \geq \dots \geq n_r \geq 0$ be integers. Further assume that $d \geq m_1 + m_2$, $3d \geq m_1 + \dots + m_r$ and $t \geq n_1 + n_2 + n_3$. Then $dt \geq \sum_i m_i n_i$.*

Proof. We first note that if $m_3 = 0$, then the lemma follows easily. Indeed, $d \geq m_1 + m_2$, $t \geq n_1 + n_2 + n_3$ imply $dt \geq m_1 n_1 + m_2 n_2$.

We now induct on d . If at any point we have $m_3 = 0$, we are done by the above argument.

The base case is $d = 0$, which is easy.

Suppose the statement is true for $d - 1$. Given d, m_1, m_2, \dots, m_r satisfying the hypothesis, consider $d - 1, m_1 - 1, m_2 - 1, m_3 - 1, m_4, \dots, m_r$. Note that $m_3 > 0$.

Then the tuple $(d - 1, m_1 - 1, m_2 - 1, m_3 - 1, m_4, \dots, m_r)$ satisfies the hypothesis, after permuting the m_i if necessary. If $m_4 = d$, then $m_1 = m_2 = m_3 = m_4 = d$ and this violates $3d \geq m_1 + \dots + m_r$. So $m_i < d$ for all $i \geq 4$.

By induction hypothesis,

$$(d - 1)t \geq (m_1 - 1)n_1 + (m_2 - 1)n_2 + (m_3 - 1)n_3 + m_4 n_4 + \dots + m_r n_r$$

implies

$$dt - \sum_i m_i n_i \geq t - n_1 - n_2 - n_3 \geq 0.$$

□

Now we are in a position to prove our first result.

Theorem 2.4 (Very general points on a cubic). *Let D be an irreducible and reduced plane cubic and let P_1, \dots, P_s be very general points on D . Let $f : X \rightarrow \mathbb{P}^2$ be the blow up at P_1, \dots, P_s . If $C \subset X$ is any reduced and irreducible curve such that $C^2 < 0$, then*

- a) C is the proper transform of D , or
- b) C can be brought by a Cremona transformation to the proper transform of a line in \mathbb{P}^2 through any two of the points P_1, \dots, P_s , or
- c) C is an exceptional divisor of f .

Proof. Assume that C is a reduced and irreducible curve on X different from the curves mentioned in cases a), b) or c). Then $C = dH - m_1 E_1 - \dots - m_s E_s$, for

some $d \geq 1$ and $m_1, \dots, m_s \geq 0$. Here $H = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $E_i = f^{-1}(P_i)$ are the exceptional divisors of f .

Intersecting C with the proper transform of D we get

$$(2.1) \quad 3d \geq m_1 + \dots + m_r.$$

Let $\Gamma = f(C)$ be the image of C on \mathbb{P}^2 . Then Γ has a singularity of order at least m_i at p_i for $i = 1, \dots, s$. By Theorem 2.2, we can assume that Γ is in the standard form, so that

$$(2.2) \quad d \geq m_1 + m_2 + m_3 \quad \text{and} \quad m_1 \geq m_2 \geq \dots \geq m_s.$$

Now inequalities (2.1) and (2.2) allow us to use Lemma 2.3 with $t = d$ and $n_i = m_i$ for $i = 1, \dots, s$. We get

$$d^2 \geq m_1^2 + m_2^2 + \dots + m_r^2,$$

which is equivalent to $C^2 \geq 0$. This shows that the only negative curves on X are the curves listed in a), b) or c). \square

Corollary 2.5. *Let X be a surface as in Theorem 2.4 with $s > 0$. Then Conjecture 1.6 holds for X and we have*

$$b(X) = \min \{-1, 9 - s\}.$$

3. SPECIAL POINTS ON A CUBIC

In this section, we consider blow ups of \mathbb{P}^2 at 3-torsion points of an elliptic curve as well as the points of intersection of the Fermat arrangement. In order to consider these two cases, we deal first with the following numerical lemma which seems quite interesting in its own right.

Lemma 3.1. *Let m_1, \dots, m_9 be nonnegative real numbers satisfying the following 12 inequalities:*

$$(3.1) \quad m_1 + m_2 + m_3 \leq 1,$$

$$(3.2) \quad m_4 + m_5 + m_6 \leq 1,$$

$$(3.3) \quad m_7 + m_8 + m_9 \leq 1,$$

$$(3.4) \quad m_1 + m_4 + m_7 \leq 1,$$

$$(3.5) \quad m_2 + m_5 + m_8 \leq 1,$$

$$(3.6) \quad m_3 + m_6 + m_9 \leq 1,$$

$$(3.7) \quad m_1 + m_5 + m_9 \leq 1,$$

$$(3.8) \quad m_2 + m_6 + m_7 \leq 1,$$

$$(3.9) \quad m_3 + m_4 + m_8 \leq 1,$$

$$(3.10) \quad m_1 + m_6 + m_8 \leq 1,$$

$$(3.11) \quad m_2 + m_4 + m_9 \leq 1,$$

$$(3.12) \quad m_3 + m_5 + m_7 \leq 1.$$

Then $m_1^2 + \dots + m_9^2 \leq 1$.

Proof. Assume that the biggest number among m_1, \dots, m_9 is $m_1 = 1 - m$ for some $0 \leq m \leq 1$.

Consider the following four pairs of numbers

$$p_1 = (m_2, m_3), p_2 = (m_4, m_7), p_3 = (m_9, m_5), p_4 = (m_6, m_8).$$

These are pairs such that together with m_1 they occur in one of the 12 inequalities. In each pair one of the numbers is greater or equal than the other. Let us call this bigger number a *giant*. A simple check shows that there are always three pairs, such that their giants are subject to one of the 12 inequalities in the Lemma.

Without loss of generality, let p_1, p_2, p_3 be such pairs. Also without loss of generality, let m_2, m_4 and m_9 be the giants. Thus $m_2 + m_4 + m_9 \leq 1$. Assume that also m_6 is a giant.

Inequality $m_2 + m_3 \leq m$ implies that

$$m_2^2 + m_3^2 = (m_2 + m_3)^2 - 2m_2m_3 \leq m(m_2 + m_3) - 2m_2m_3.$$

Observe also that

$$(m_2 + m_3)^2 - 4m_2m_3 \leq m(m_2 - m_3).$$

Analogous inequalities hold for pairs p_2, p_3 and p_4 . Therefore

$$\begin{aligned} & m_2^2 + m_3^2 + m_4^2 + m_7^2 + m_5^2 + m_9^2 \leq \\ & \leq m(m_2 + m_4 + m_9 + m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9 \leq \\ & \leq m + [m(m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9]. \end{aligned}$$

But we have also

$$\begin{aligned} & m_2^2 + m_3^2 + m_4^2 + m_7^2 + m_5^2 + m_9^2 = \\ & = (m_2 + m_3)^2 + (m_4 + m_7)^2 + (m_5 + m_9)^2 - 2m_2m_3 - 2m_4m_7 - 2m_5m_9 = \\ & = (m_2 + m_3)^2 - 4m_2m_3 + (m_4 + m_7)^2 - 4m_4m_7 + \\ & \quad + (m_5 + m_9)^2 - 4m_5m_9 + 2m_2m_3 + 2m_4m_7 + 2m_5m_9 \leq \\ & \leq m(m_2 - m_3) + m(m_4 - m_7) + m(m_9 - m_5) + 2m_2m_3 + 2m_4m_7 + 2m_5m_9 \leq \\ & \leq m - [m(m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9], \end{aligned}$$

which obviously gives

$$m_2^2 + m_3^2 + m_4^2 + m_7^2 + m_5^2 + m_9^2 \leq m.$$

Since

$$m_6^2 + m_8^2 \leq m_6^2 + m_6 m_8 \leq m_6(m_6 + m_8) \leq (1 - m)m,$$

we get that the sum of all nine squares is bounded by

$$(1 - m)^2 + m + (1 - m)m = 1. \quad \square$$

If we think of numbers m_1, \dots, m_9 as arranged in a 3×3 matrix

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix},$$

then the inequalities in the Lemma 3.1 are obtained considering the horizontal, vertical triples and the triples determined by the condition that there is exactly one element m_i in every column and every row of the matrix (so determined by permutation matrices). Bounding sums of only such triples allows us to bound the sum of squares of all entries in the matrix. It is natural to wonder, if this phenomena extends to higher dimensional matrices. One possible extension is formulated as the next question.

Problem 3.2. Let $M = (m_{ij})_{i,j=1\dots k}$ be a matrix whose entries are non-negative real numbers. Assume that all the horizontal, vertical and permutational k -tuples of entries in the matrix M are bounded by 1. Is it true then that the sum of squares of all entries of M is also bounded by 1?

3.1. Torsion points. We now consider a blow up of \mathbb{P}^2 at 9 points which are torsion points of order 3 on an elliptic curve embedded as a smooth cubic.

Theorem 3.3 (3-torsion points on an elliptic curve). *Let D be a smooth plane cubic and let P_1, \dots, P_9 be the flexes of D . Let $f : X \rightarrow \mathbb{P}^2$ be the blow up of \mathbb{P}^2 at P_1, \dots, P_9 . If C is a negative curve on X , then*

- a) C is the proper transform of a line passing through two (hence three) of the points P_1, \dots, P_9 , or
- b) C is an exceptional divisor of f .

Proof. It is well known that there is a group law on D such that the flexes are 3-torsion points. Since any line passing through two of the torsion points automatically meets D in a third torsion point, there are altogether 12 such lines. The torsion points form a subgroup of D which is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. We can pick this isomorphism so that

$$\begin{aligned} P_1 &= (0, 0), P_2 = (1, 0), P_3 = (2, 0), \\ P_4 &= (0, 1), P_5 = (1, 1), P_6 = (2, 1), \\ P_7 &= (0, 2), P_8 = (1, 2), P_9 = (2, 2). \end{aligned}$$

This implies that the following triples of points are collinear:

$$\begin{aligned} (P_1, P_2, P_3), (P_4, P_5, P_6), (P_7, P_8, P_9), (P_1, P_4, P_7), \\ (P_2, P_5, P_8), (P_3, P_6, P_9), (P_1, P_5, P_9), (P_2, P_6, P_7), \end{aligned}$$

$$(P_3, P_4, P_8), (P_1, P_6, P_8), (P_2, P_4, P_9), (P_3, P_5, P_7).$$

Let C be a reduced and irreducible curve on X different from the exceptional divisors of f and the proper transforms of lines through the torsion points. Then C is of the form

$$C = dH - k_1E_1 - \dots - k_9E_9,$$

where E_1, \dots, E_9 are the exceptional divisors of f and $k_1, \dots, k_9 \geq 0$ and $d > 0$ is the degree of the image $f(C)$ in \mathbb{P}^2 .

For $i = 1, \dots, 9$, let $m_i = \frac{k_i}{d}$. Since C is different from proper transforms of the 12 lines distinguished above, taking the intersection product of C with the 12 lines, and dividing by d , we obtain exactly the 12 inequalities in Lemma 3.1. The conclusion of Lemma 3.1 implies then that

$$C^2 = d^2 - \sum_{i=1}^9 m_i^2 \geq 0,$$

which finishes our argument. \square

Corollary 3.4. *Let X be a surface as in Theorem 3.3. Then Conjecture 1.6 holds for X and we have*

$$b(X) = -2.$$

Of course, there is no reason to restrict to 3-torsion points. In particular there is the following natural question, which we hope to come back to in the near future.

Problem 3.5. For $m \geq 4$, decide whether the Bounded Negativity Conjecture holds on the blow ups of \mathbb{P}^2 at all the m -torsion points of an elliptic curve embedded as a smooth cubic.

3.2. Fermat configuration of points. The 9 points and 12 lines considered in the above subsection form the famous Hesse arrangement of lines; see [11]. Any such arrangement is projectively equivalent to that obtained from the flex points of the Fermat cubic $x^3 + y^3 + z^3 = 0$ and the lines determined by their pairs. Explicitly in coordinates we have then

$$\begin{aligned} P_1 &= (1 : \varepsilon : 0), P_2 = (1 : \varepsilon^2 : 0), P_3 = (1 : 1 : 0), \\ P_4 &= (1 : 0 : \varepsilon), P_5 = (1 : 0 : \varepsilon^2), P_6 = (1 : 0 : 1), \\ P_7 &= (0 : 1 : \varepsilon), P_8 = (0 : 1 : \varepsilon^2), P_9 = (0 : 1 : 1) \end{aligned}$$

for the points and

$x = 0, y = 0, z = 0, x + y + z = 0, x + y + \varepsilon z = 0, x + y + \varepsilon^2 z = 0, x + \varepsilon y + z = 0,$
 $x + \varepsilon^2 y + z = 0, x + \varepsilon y + \varepsilon z = 0, x + \varepsilon y + \varepsilon^2 z = 0, x + \varepsilon^2 y + \varepsilon z = 0, x + \varepsilon^2 y + \varepsilon^2 z = 0$
 for the lines, where ε is a primitive root of unity of order 3.

Passing to the dual plane, we obtain an arrangement of 9 lines defined by the linear factors of the Fermat polynomial

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0.$$

These lines intersect in triples in 12 points, which are dual to the lines of the Hesse arrangement. The resulting dual Hesse configuration has the type $(9_4, 12_3)$ and it belongs to a much bigger family of Fermat arrangements; see [14]. Figure 1 is an attempt to visualize this arrangement (which cannot be drawn in the real plane due to the famous Sylvester-Gallai Theorem; for instance, see [13]).

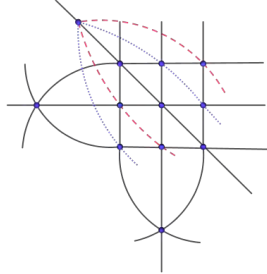


FIGURE 1. Fermat configuration of points

It is convenient to order the 9 intersection points in the affine part in the following way:

$$\begin{aligned} Q_1 &= (\varepsilon : \varepsilon : 1), & Q_2 &= (1 : \varepsilon : 1), & Q_3 &= (\varepsilon^2 : \varepsilon : 1), \\ Q_4 &= (\varepsilon : 1 : 1), & Q_5 &= (1 : 1 : 1), & Q_6 &= (\varepsilon^2 : 1 : 1), \\ Q_7 &= (\varepsilon : \varepsilon^2 : 1), & Q_8 &= (1 : \varepsilon^2 : 1), & Q_9 &= (\varepsilon^2 : \varepsilon^2 : 1). \end{aligned}$$

With this notation established, we have the following result.

Theorem 3.6 (Fermat points). *Let $f : X \rightarrow \mathbb{P}^2$ be the blow up of \mathbb{P}^2 at Q_1, \dots, Q_9 . If C is a negative curve on X , then*

- a) C is the proper transform of a line passing through two or three of the points Q_1, \dots, Q_9 , or
- b) C is an exceptional divisor of f .

Proof. The proof of Theorem 3.3 works with very few adjustments.

Let us assume, to begin with, that C is a negative curve on X , distinct from the curves listed in the theorem. Then

$$C = dH - k_1E_1 - \dots - k_9E_9,$$

for some $d > 0$ and $k_1, \dots, k_9 \geq 0$. We can also assume that d is the smallest number for which such a negative curve exists. As before, we set

$$m_i = \frac{k_i}{d} \text{ for } i = 1, \dots, 9.$$

Then the inequalities (3.1) to (3.9) follow from the fact that C intersects the 9 lines in the arrangement non-negatively.

If one of the remaining inequalities (3.10), (3.11) or (3.12) fails, then we perform a standard Cremona transformation based on the points involved in the failing inequality. For example, if (3.10) fails, we make Cremona based on points Q_1, Q_6 and Q_8 . Note that these points are *not* collinear in the set-up of our Theorem.

Since C is assumed not to be a line through any two of these points, its image C' under Cremona is a curve of strictly lower degree, negative on the blow up of \mathbb{P}^2 at the 9 points. The points Q_1, \dots, Q_9 remain unchanged by the Cremona because, as already remarked, all dual Hesse arrangements are projectively equivalent, see [16]. Then C' is again a negative curve on X of degree strictly lower than d , which contradicts our choice of C such that $C \cdot H$ is minimal.

Hence, we can assume that the inequalities (3.10), (3.11) and (3.12) are also satisfied. Then we conclude exactly as in the proof of Theorem 3.3. \square

Remark 3.7. If we are interested only in the bounded negativity property on X , the assertion follows from the fact, that $-K_X$ is \mathbb{Q} -effective. Indeed, if $C \subset X$ is a reduced and irreducible curve, from the genus formula we get

$$1 + \frac{C \cdot (C + K_X)}{2} = g(C) \geq 0,$$

so

$$C^2 \geq -2 - CK_X.$$

The bounded negativity follows from the fact that $-CK_X$ may be negative only in finite number of cases.

Having classified all the negative curves on the blow up of \mathbb{P}^2 at the 9 Fermat points, it is natural to wonder about the negative curves on blow ups of \mathbb{P}^2 arising from the other Fermat configurations. Note that the argument given in Remark 3.7 is no longer valid, since $-K_X$ is not nef nor effective. So it will be interesting to ask whether BNC holds for such surfaces. We pose the following problem.

Problem 3.8. For a positive integer m , let $Z(m)$ be the set of all points of the form

$$(1 : \varepsilon^\alpha : \varepsilon^\beta),$$

where ε is a primitive root of unity of order m and $1 \leq \alpha, \beta \leq m$. Let $f_m : X(m) \rightarrow \mathbb{P}^2$ be the blow up of \mathbb{P}^2 at all the points of $Z(m)$. Is the negativity bounded on $X(m)$? If so, what is the value of $b(X(m))$?

We end this note by the following remark which discusses bounded negativity for blow ups of \mathbb{P}^2 at 10 points.

Remark 3.9. Let X denote a blow up of \mathbb{P}^2 at 10 points. As mentioned before, if the blown up points are general, then Conjecture 1.5 predicts that the only negative curves on X are (-1) -curves. This is an open question. On the other hand, let us consider a couple of examples of special points.

Let X be obtained by blowing up the 10 nodes of an irreducible and reduced rational nodal sextic. Such surfaces are called *Coble surfaces* (these are smooth rational surfaces X such that $|-K_X| = \emptyset$, but $|-2K_X| \neq \emptyset$). Then it is known that BNC holds for X . In fact, we have $C^2 \geq -4$ for every irreducible and reduced curve $C \subset X$; see [3, Section 3.2].

Now let X be the blow up of 10 double points of intersection of 5 general lines in \mathbb{P}^2 . Then $-K_X$ is a big divisor and by [15, Theorem 1], X is a *Mori dream space*. For such surfaces, the submonoid of the Picard group generated by the effective classes is finitely generated. Hence BNC holds for X ([8, Proposition I.2.5]).

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