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GELFOND-MAHLER INEQUALITY FOR MULTIPOLYNOMIAL RESULTANTS

ALEKSANDRA GALA-JASKÓRZYŃSKA, KRZYSZTOF KURDYKA,
KATARZYNA RUDNICKA, AND STANISŁAW SPODZIEJA

ABSTRACT. We give a bound of the height of a multipolynomial resultant in terms of polynomial degrees, the resultant of which applies. Additionally we give a Gelfond-Mahler type bound of the height of homogeneous divisors of a homogeneous polynomial.

1. INTRODUCTION

Let $f \in \mathbb{Z}[u]$, where $u = (u_1, \dots, u_N)$ is a system of variables and $\mathbb{Z}$ is the ring of integers, be a nonzero polynomial of the form

$$f(u) = \sum_{|\nu| \leqslant d_f} a_\nu u^\nu, \tag{1}$$

where $a_\nu \in \mathbb{Z}$, $u^\nu = u_1^{\nu_1} \cdots u_N^{\nu_N}$ and $|\nu| = \nu_1 + \cdots + \nu_N$ for $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{N}^N$ and $\mathbb{N}$ denotes the set of nonnegative integers. By the *height* of the polynomial f we mean

$$H(f) := \max\{|a_\nu| : \nu \in \mathbb{N}^N,\, |\nu| \leq d_f\}.$$

Let $f_1, \dots, f_r \in \mathbb{Z}[u]$ be nonzero polynomials, and let d_j be the degree of $f = f_1 \cdots f_r$ with respect to u_j for $j = 1, \dots, N$.

A.P. Gelfond [3] obtained the following bound.

Theorem 1.1 (Gelfond)**.**

$$H(f_1) \cdots H(f_r) \leqslant 2^{d_1 + \cdots + d_N - k} \sqrt{(d_1 + 1) \cdots (d_N + 1)}\, H(f) \tag{2}$$

where k is the number of variables u_j that genuinely appear in f.

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K. Mahler [6] introduced a measure $M(f)$ of a polynomial $f \in \mathbb{C}[u]$ (currently called *Mahler measure*, see Section 2.1) and in [7] reproved (2) and proved the following

Theorem 1.2 (Mahler)**.** *Under notations of Theorem 1.1,*

$$H(f) \leqslant 2^{d_1+\cdots+d_N-k} M(f). \tag{3}$$

Moreover,

$$L_1(f_1)\cdots L_1(f_r) \leqslant 2^{d_1+\cdots+d_N} M(f) \leqslant 2^{d_1+\cdots+d_N} L_1(f), \tag{4}$$

where $L_1(f) := \sum_{|\nu|\leqslant d_f} |a_\nu|$ *is the* L_1-norm *of a polynomial* f *of the form* (1).

The aim of the article is to obtain a similar to the above-described estimates for the height, L_1-norms and Mahler's measures of a resultant for systems of homogeneous forms. More precisely let $d_0, \ldots, d_n$ be fixed positive integers and let $f_0, \ldots, f_n$ be a system of homogeneous polynomials in $x = (x_0, \ldots, x_n)$ with indeterminate coefficients of degrees $d_0, \ldots, d_n$ in x, respectively. By a resultant $\mathrm{Res}_{d_0,\ldots,d_n}$ we mean the unique irreducible polynomial in the coefficients of $f_0, \ldots, f_n$ with integral coefficients such that for any specializations $f_{0,a_0}, \ldots, f_{n,a_n}$ of $f_0, \ldots, f_n$, the value $\mathrm{Res}_{d_0,\ldots,d_n}(f_{0,a_0}, \ldots, f_{n,a_n})$ is equal to zero if and only if the polynomials $f_{0,a_0}, \ldots, f_{n,a_n}$ have a common nontrivial zero. For basic notations and properties of the resultants, see Section 3.1 and for more detailed description on the resultant see for instance [2]. The main result of this paper is Theorem 3.12 which says that:

$$M(\mathrm{Res}_{d_0,\ldots,d_n}) \leqslant (d_*+1)^{nK_n d_*^n},$$
$$H(\mathrm{Res}_{d_0,\ldots,d_n}) \leqslant (d_*+1)^{n(K_n+n+1)d_*^n - n(n+1)},$$
$$L_1(\mathrm{Res}_{d_0,\ldots,d_n}) \leqslant (d_*+1)^{n(K_n+n+1)d_*^n},$$

where $K_n = e^{n+1}/\sqrt{2\pi n}$ and $d_* = \max\{d_0, \ldots, d_n\}$. Moreover if $n \geqslant 2$ and $d_* \geqslant 4$ then we have the following estimates:

$$M(\mathrm{Res}_{d_0,\ldots,d_n}) \leqslant (d_*)^{nK_n d_*^n},$$
$$H(\mathrm{Res}_{d_0,\ldots,d_n}) \leqslant (d_*)^{n(K_n+n+1)d_*^n - n(n+1)},$$
$$L_1(\mathrm{Res}_{d_0,\ldots,d_n}) \leqslant (d_*)^{n(K_n+n+1)d_*^n}.$$

Note that the above estimates of $L_1(\mathrm{Res}_{d_0,\ldots,d_n})$ are not a direct consequences of the estimates of $H(\mathrm{Res}_{d_0,\ldots,d_n})$ (see Remark 3.13).

M. Sombra in [9], as a corollary from a study of the height of the mixed sparse resultant, gave an estimation of $H(\mathrm{Res}_{d,\ldots,d})$:

$$H(\mathrm{Res}_{d,\ldots,d}) \leqslant (d+1)^{n(n+1)!d^n}.$$

Since $K_n+n+1 = n+1+e^{n+1}/\sqrt{2\pi n} < (n+1)!$ for $n \geqslant 3$, so the estimation (26) is more explicit than the above for $n \geqslant 3$.

The paper is organized as follows. In Section 2 we collect basic notations concering the Mahler measure of a polynomial and we prove a Mahler type bounds for the height and the L_1-norm of multihomogeneous polynomials (see Lemma 2.2). The proof of Theorem 3.12 we give in Section 3. The crucial role in the proof plays an estimation of the L_1 norm of the Macaulay discriminant of a coefficients matrix for a powers of polynomials $f_0, \ldots, f_n$ (see Lemma 3.9).

Additionally, in Section 4 we give Corollaries 4.1 and 4.2 which are versions of Theorems 1.1 and 1.2 for the multihomogeneous and homogeneous polynomials cases.

2. Auxiliary results

2.1. **Notations.** Let $f \in \mathbb{C}[u]$, where $u = (u_1, \ldots, u_N)$ is a system of variables, be a nonzero polynomial of the form

$$f(u) = \sum_{|\nu| \leqslant d_f} a_\nu u^\nu, \tag{5}$$

where for $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{N}^N$ the coefficient a_ν is a complex number and we put $|\nu| = \nu_1 + \cdots + \nu_N$ and $u^\nu = u_1^{\nu_1} \cdots u_N^{\nu_N}$.

In this section I denotes the interval $[0,1]$ and i the imaginary unit (i.e., $i^2 = -1$). Let $\mathbf{e} : I^N \to \mathbb{C}^N$ be a mapping defined by

$$\mathbf{e}(\mathbf{t}) = (\exp(2\pi t_1 i), \ldots, \exp(2\pi t_N i)) \quad \text{for } \mathbf{t} = (t_1, \ldots, t_N) \in I^N.$$

For a complex polynomial $f \in \mathbb{C}[u]$, the number

$$M(f) = \exp\left(\int_{I^N} \log|f(\mathbf{e}(\mathbf{t}))| d\mathbf{t}\right)$$

is called the *Mahler measure* of f (see [7]). A significient property of the Mahler measure is the following (see [7]): for $f, g \in \mathbb{C}[u]$,

$$M(fg) = M(f)M(g). \tag{6}$$

Moreover, if $f \in \mathbb{Z}[u]$, $f \neq 0$, then (see for instance [8, Corollary 2]),

$$M(f) \geqslant 1. \tag{7}$$

By L_2-*norm* of a polynomial $f \in \mathbb{C}[u]$ we mean

$$L_2(f) = \left(\int_{I^N} |f(\mathbf{e}(\mathbf{t}))|^2 d\mathbf{t}\right)^{1/2}.$$

For a polynomial $f \in \mathbb{C}[u]$ of the form (5) we have

$$L_2(f) = \left(\sum_{|\nu| \leqslant d_f} |a_\nu|^2\right)^{1/2}, \tag{8}$$

By Jensen's inequality we obtain

$$M(f) \leqslant L_2(f). \tag{9}$$

2.2. **Mahler type inequalities for multihomogeneous polynomials.** By analogous argument as in [7] we obtain the following lemma.

Lemma 2.1. *Let $f \in \mathbb{C}[u]$, where $u = (u_1, \dots, u_N)$, be a homogeneous polynomial of degree $d_f > 0$ of the form*

$$f(u) = \sum_{|\nu|=d_f} a_\nu u^\nu.$$

Then there are homogeneous polynomials $f_{k_1,\dots,k_\ell} \in \mathbb{C}[u_{\ell+1}, \dots, u_N]$, with $\deg f_{k_1,\dots,k_\ell} = d_f - k_1 - \dots - k_\ell$ for $k_1 + \dots + k_\ell \leqslant d_f$, $\ell = 1, \dots, N$, such that

$$f(u_1, \dots, u_N) = \sum_{k_1=0}^{d_f} f_{k_1}(u_2, \dots, u_N) u_1^{k_1}$$
$$f_{k_1,\dots,k_{\ell-1}}(u_\ell, \dots, u_N) = \sum_{k_\ell=0}^{d_f-k_1-\dots-k_{\ell-1}} f_{k_1,\dots,k_\ell}(u_{\ell+1}, \dots, u_N) u_\ell^{k_\ell}.$$

Moreover, for any $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{N}^N$, $|\nu| = d_f$, we have

$$|a_\nu| = |f_\nu| \leqslant \binom{d_f - \nu_1 - \ldots - \nu_{N-1}}{\nu_N} M(f_{\nu_1,\dots,\nu_{N-1}}),$$
$$M(f_{\nu_1}) \leqslant \binom{d_f}{\nu_1} M(f),$$
$$M(f_{\nu_1,\dots,\nu_\ell}) \leqslant \binom{d_f - \nu_1 - \dots - \nu_{\ell-1}}{\nu_\ell} M(f_{\nu_1,\dots,\nu_{\ell-1}}),\ 2 \leqslant \ell \leqslant N.$$

In particular,

$$|a_\nu| \leqslant \binom{d_f}{\nu_1}\binom{d_f - \nu_1}{\nu_2} \cdots \binom{d_f - \nu_1 - \dots - \nu_{N-1}}{\nu_N} M(f) \leqslant \binom{d_f}{\nu_1, \dots, \nu_N} M(f) \leqslant N^{d_f-1} M(f)$$

and so,

$$H(f) \leqslant N^{d_f-1} M(f),$$
$$L_1(f) \leqslant N^{d_f} M(f).$$

Let now $m, d_0, \dots, d_n$ be fixed positive integers, $n \in \mathbb{N}$, and let

$$u_{(m,j)} = (u_{m,j,\nu} : \nu \in \mathbb{N}^{m+1},\ |\nu| = d_j), \quad j = 0, \dots, n,$$

be systems of variables. In fact $u_{(m,j)}$ is a system of

$$N_{m,d_j} := \binom{d_j + m}{m}$$

variables.

From Lemma 2.1, by a similar method as in [7], we obtain the following Mahler type inequalities for multihomogeneous polynomials.

Lemma 2.2. *Let $f \in \mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ be a nonzero polynomial such that f is homogeneous as a polynomial in each system of variables $u_{(m,j)}$. Then for any polynomial $g \in \mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ which divides f in $\mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ and have degree e_j with respect to system $u_{(m,j)}$ for $j = 0, \dots, n$, we have*

$$H(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j-1}\right) M(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j-1}\right) M(f)$$

and

$$L_1(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j}\right) M(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j}\right) M(f).$$

Proof. For simplicity $u_{(m,j)}$ we denote by $u_{(j)}$ and N_{m,d_j} – by N_j for $j = 0, \dots, n$. Let $g \in \mathbb{Z}[u_{(0)}, \dots, u_{(n)}]$ be a divisor of f in $\mathbb{Z}[u_{(0)}, \dots, u_{(n)}]$ and let $g_1 = f/g$. By the assumptions, g is a homogeneous polynomial as a polynomial in each $u_{(j)}$ of some degree e_j for $j = 0, \dots, n$. Let

$$\mathscr{I} = \{\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathbb{N}^{N_0} \times \dots \times \mathbb{N}^{N_n} : |\eta^{(j)}| = e_j \quad \text{for } j = 0, \dots, n\}.$$

The polynomial g is of the form

$$g(u_{(0)}, \dots, u_{(n)}) = \sum_{\eta \in \mathscr{I}} C_\eta J_\eta,$$

where $C_\eta \in \mathbb{Z}$ and $J_\eta = u_{(0)}^{\eta^{(0)}} \cdots u_{(n)}^{\eta^{(n)}}$ for $\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathscr{I}$. So, we may write

$$g(u_{(0)}, \dots, u_{(n)}) = \sum_{|\eta^{(0)}| = e_0} g_{1,\eta^{(0)}}(u_{(1)}, \dots, u_{(n)}) u_{(0)}^{\eta^{(0)}},$$

where $g_{1,\eta^{(0)}} \in \mathbb{Z}[u_{(1)}, \dots, u_{(n)}]$ for $\eta^{(0)} \in \mathbb{N}^{N_0}$, $|\eta^{(0)}| = e_0$. By induction for $j = 1, \dots, n$ we may write

$$g_{j,\eta^{(j-1)}}(u_{(j)}, \dots, u_{(n)}) = \sum_{|\eta^{(j)}| = e_j} g_{j+1,\eta^{(j)}}(u_{(j+1)}, \dots, u_{(n)}) u_{(j)}^{\eta^{(j)}},$$

where $g_{j+1,\eta^{(j)}} \in \mathbb{Z}[u_{(j+1)}, \dots, u_{(n)}]$ for $\eta^{(j)} \in \mathbb{N}^{N_j}$, $|\eta^{(j)}| = e_j$. Then any coefficient C_η, $\eta \in \mathscr{I}$, is a coefficient of some polynomial $g_{n,\eta^{(n-1)}}$. Then applying $n+1$ times Lemma 2.1, we obtain

$$H(g) \leqslant N_0^{e_0-1} \cdots N_n^{e_n-1} M(g)$$

and

$$L_1(g) \leqslant N_0^{e_0} \cdots N_n^{e_n} M(g).$$

Since g_1 have integral coefficients, by (7) we have $M(g_1) \geqslant 1$. Then (6) gives the assertion. $\square$

3. Height of a multipolynomial resultant

3.1. **Basic notations on a multipolynomial resultant.** Recall some notations and facts concerning the resultant for several homogeneous polynomials (see [2], see also [1]).

In this section $x = (x_0, \dots, x_n)$ is a system of $n+1$ variables.

Let $d_0, \dots, d_n$ be fixed positive integers and let $u_{(0)}, \dots, u_{(n)}$ be systems of variables of the form

$$u_{(j)} = (u_{j,\nu} : \nu \in \mathbb{N}^{n+1},\ |\nu| = d_j), \quad j = 0, \dots, n, \tag{10}$$

In fact $u_{(m,j)}$ is a system of

$$N_{d_j} := \binom{d_j + n}{n} \tag{11}$$

variables.

Let $f_0, \dots, f_n \in \mathbb{C}[u_{(0)}, \dots, u_{(n)}, x]$ be homogeneous polynomials in x of degrees $d_0, \dots, d_n$, respectively of the forms

$$f_j(u_{(0)}, \dots, u_{(n)}, x) = \sum_{\substack{\nu \in \mathbb{N}^{n+1} \\ |\nu| = d_j}} u_{j,\nu} x^\nu, \quad j = 0, \dots, n.$$

In fact $f_j \in \mathbb{Z}[u_{(j)}, x]$.

For any $a_j = (a_{j,\nu} : \nu \in \mathbb{N}^{n+1},\ |\nu| = d_j) \in \mathbb{C}^{N_{d_j}}$ by, $f_{j,a_j} \in \mathbb{C}[x]$ we denote the *specialization* of f_j, i.e., the polynomial $f_{j,a_j}(x) = f_j(a_j, x)$.

Fact 3.1 ([2], Chapter 13). *There exists a unique polynomial* $P_{d_0,\dots,d_n} \in \mathbb{Z}[u_{(0)}, \dots, u_{(n)}]$ *such that:*

(i) *For any* $a_0 \in \mathbb{C}^{N_{d_0}}, \dots, a_n \in \mathbb{C}^{N_{d_n}}$

$P_{d_0,\dots,d_n}(a_0, \dots, a_n) = 0 \;\Leftrightarrow\; f_{0,a_0}, \dots, f_{n,a_n}$ *have a common nontrivial zero.*

(ii) *For* $a_0 \in \mathbb{C}^{N_{d_0}}, \dots, a_n \in \mathbb{C}^{N_{d_n}}$ *such that* $f_{0,a_0} = x_0^{d_0}, \dots, f_{n,a_n} = x_n^{d_n}$,

$$P_{d_0,\dots,d_n}(a_0, \dots, a_n) = 1.$$

(iii) $P_{d_0,\dots,d_n}$ *is irreducible in* $\mathbb{C}[u_{(0)}, \dots, u_{(n)}]$.

The polynomial $P_{d_0,\dots,d_n}$ in Fact 3.1 is called *resultant* or *multipolynomial resultant* and denoted by $\mathrm{Res}_{d_0,\dots,d_n}$ or shortly by Res. We will also write $\mathrm{Res}(f_{0,a_0}, \dots, f_{n,a_n})$ instead of $\mathrm{Res}(a_0, \dots, a_n)$.

Fact 3.2 ([2], Proposition 1.1 in Chapter 13). *For any* $j = 0, \dots, n$ *the resultant* $\mathrm{Res}_{d_0,\dots,d_n}$ *is a homogeneous polynomial in* $u_{(j)}$ *of degree* $d_0 \cdots d_{j-1} d_{j+1} \cdots d_n$.

Set

$$\delta = d_0 + \cdots + d_n - n,$$

and let

$$S_j = \{\nu = (\nu_0, \ldots, \nu_n) \in \mathbb{N}^{n+1} : |\nu| = \delta,\ \nu_0 < d_0, \ldots, \\ \nu_{j-1} < d_{j-1}, \nu_j \geqslant d_j\} \quad \text{for } j = 0, \ldots, n.$$

Fact 3.3. *The sets $S_0, \ldots, S_n$ are mutually disjoint and*

$$\{\nu \in \mathbb{N}^{n+1} : |\nu| = \delta\} = S_0 \cup \cdots \cup S_n. \tag{12}$$

Consider the following system of equations

$$\begin{cases} \frac{x^\nu}{x_0^{d_0}} f_0(u_{(0)}, x) = 0 \quad \text{for } \nu \in S_0 \\ \vdots \\ \frac{x^\nu}{x_n^{d_n}} f_n(u_{(n)}, x) = 0 \quad \text{for } \nu \in S_n. \end{cases} \tag{13}$$

Any of the above equation is homegenous of degree δ and depends on

$$N_{d_0,\ldots,d_n} = \binom{d_0 + \cdots + d_n}{n}$$

monomials of degree δ. Let's arrange these monomials in a sequence $J_1, \ldots, J_N$. Then (13) one can consider as a system of N linear equations with N indeterminates $J_1, \ldots, J_N$. Denote by $\mathscr{D}_{d_0,\ldots,d_n}$ the matrix of this system of equations and by $D_{d_0,\ldots,d_n}$ – the determinat of $\mathscr{D}_{d_0,\ldots,d_n}$. From Fact 3.3 and the definition of $D_{d_0,\ldots,d_n}$ we easily obtain the following fact.

Fact 3.4. *For $a_j \in \mathbb{C}^{N_{d_j}}$ such that $f_{j,a_j}(x) = x_j^{d_j}$, $j = 0, \ldots, n$, we have*

$$|D_{d_0,\ldots,d_n}(a_0, \ldots, a_n)| = 1,$$

In particular, $D_{d_0,\ldots,d_n} \neq 0$.

Proof. Indeed, by Fact 3.3, for the assumed specializations f_{j,a_j}, $j = 0, \ldots, n$, the matrix $\mathscr{D}_{d_0,\ldots,d_n}(f_{0,a_0}, \ldots, f_{n,a_n})$ have in any row and any column exactly one nonzero entry equal to 1. □

From the definition of $D_{d_0,\ldots,d_n}$ we see that $D_{d_0,\ldots,d_n}$ is a homogeneous polynomoal in $u_{(j)}$ of degree equal to the number of elements $\#S_j$ of S_j and the total degree equal to $N_{d_0,\ldots,d_n}$. Moreover, we have the following Macaulay result [5, Theorem 6] (see also [4] and [2, Theorem 1.5 in Chapter 13] for Caley determinantal formula).

Fact 3.5. *The polynomial $D_{d_0,\ldots,d_n}$ is divisible by $\mathrm{Res}_{d_0,\ldots,d_n}$ in $\mathbb{Z}[u_{(0)}, \ldots, u_{(n)}]$.*

Put

$$d_* = \max\{d_0, \ldots, d_n\}.$$

From the definition of the polynomial $D_{d_0,\ldots,d_n}$ we obtain

Lemma 3.6. $L_1(D_{d_0,\dots,d_n}) \leqslant N_{d_0}^{\#S_0} \cdots N_{d_n}^{\#S_n} \leqslant \binom{d_*+n}{n}^{\binom{(n+1)d_*}{n}}$.

Proof. Let $D = D_{d_0,\dots,d_n}$ and $N_j = N_{d_j}$. Monomials of D are of the form

$$J_\eta = C_\eta u_{(0)}^{\eta^{(0)}} \cdots u_{(n)}^{\eta^{(n)}},$$

where $C_\eta \in \mathbb{Z}$ for $\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathbb{N}^{N_0} \times \dots \times \mathbb{N}^{N_n}$ and $|\eta^{(j)}| = \#S_j$ for $j = 0, \dots, n$. Let $\eta^{(j)} = (\eta_{j,1}, \dots, \eta_{j,N_j})$. Then from definition of D,

$$|C_\eta| \leqslant \prod_{j=0}^{n} \binom{\#S_j}{\eta_{j,1}} \binom{\#S_j - \eta_{j,1}}{\eta_{j,2}} \cdots \binom{\#S_j - \eta_{j,1} - \cdots - \eta_{j,N_j-1}}{\eta_{j,N_j}}$$
$$= \prod_{j=0}^{n} \binom{\#S_j}{\eta_{j,1}, \dots, \eta_{j,N_j}},$$

so

$$L_1(D) \leqslant \sum_{\substack{\eta=(\eta^{(0)},\dots,\eta^{(n)}) \in \mathbb{N}^{N_0+\cdots+N_n} \\ |\eta^{(k)}|=\#S_k \text{ for } k=0,\dots,n}} \prod_{j=0}^{n} \binom{\#S_j}{\eta_{j,1}, \dots, \eta_{j,N_j}} \leqslant N_0^{\#S_0} \cdots N_n^{\#S_n},$$

which gives the first inequality in the assertion. Since $N_j \leqslant \binom{d_*+n}{n}$ and $\#S_0 + \cdots + \#S_n = N_{d_0,\dots,d_n} \leqslant \binom{(n+1)d_*}{n}$, then we obtain the second inequality in the assertion. □

3.2. **Multipolynomial resultant for powers of polynomials.** Take any $k \in \mathbb{Z}$, $k > 0$. The resultant $\mathrm{Res}_{kd_0,\dots,kd_n}$ and the discriminant $D_{kd_0,\dots,kd_n}$ are polynomials with integer coefficients in a system of variables $w_k = (w_{(k,0)}, \dots, w_{(k,n)})$, where

$$w_{(k,j)} = \left(w_{k,j,\nu} : \nu \in \mathbb{N}^{n+1},\ |\nu| = kd_j\right), \tag{14}$$

is a system of indeterminate coefficients of the polynomial

$$F_{k,j}(w_{(k,j)}, x) = \sum_{\substack{\nu \in \mathbb{N}^{n+1} \\ |\nu|=kd_j}} w_{k,j,\nu} x^\nu, \quad j = 0, \dots, n.$$

In fact $w_{(k,j)}$ is a system of

$$N_{kd_j} := \binom{kd_j + n}{n} \tag{15}$$

variables. From Fact 3.2 we have that $\mathrm{Res}_{kd_0,\dots,kd_n}$ is homogeneous in any system of variables $w_{(k,j)}$ of degree

$$e_{k,j} = k^n d_0 \cdots d_{j-1} d_{j+1} \cdots d_n, \quad j = 0, \dots, n.$$

The polynomial $D_{kd_0,\dots,kd_n}$ is also homogeneous in any system of variables $w_{(k,j)}$. Let $s_{k,j}$ be the degree of $D_{kd_0,\dots,kd_n}$ with respect to $w_{(k,j)}$, $j = 0, \dots, n$. Obviously

$$s_{k,0} + \cdots + s_{k,n} = \binom{k(d_0 + \cdots + d_n)}{n}. \tag{16}$$

Let

$$\mathscr{I}_k = \{\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathbb{N}^{N_{kd_0}} \times \dots \times \mathbb{N}^{N_{kd_n}} : |\eta^{(j)}| = s_{k,j} \quad \text{for } j = 0, \dots, n\}.$$

Then $D_{kd_0,\dots,kd_n}$ one can write

$$D_{kd_0,\dots,kd_n} = \sum_{\eta \in \mathscr{I}_k} C_\eta J_\eta, \tag{17}$$

where $C_\eta \in \mathbb{Z}$ for $\eta \in \mathscr{I}_k$ and

$$J_\eta = w_{(k,0)}^{\eta^{(0)}} \cdots w_{(k,n)}^{\eta^{(n)}} \quad \text{for } \eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathscr{I}_k. \tag{18}$$

Since

$$f_j^k = \sum_{\substack{\nu \in \mathbb{N}^{n+1} \\ |\nu| = kd_j}} x^\nu \sum_{\substack{\nu^1, \dots, \nu^k \in \mathbb{N}^{n+1} \\ \nu^1 + \dots + \nu^k = \nu \\ |\nu^1| = \dots = |\nu^k| = d_j}} u_{j,\nu^1} \cdots u_{j,\nu^k}, \quad j = 0, \dots, n,$$

then we may define a mapping

$$W_k = (W_{(k,0)}, \dots, W_{(k,n)}) : \mathbb{C}^{N_{d_0}} \times \dots \times \mathbb{C}^{N_{d_n}} \to \mathbb{C}^{N_{kd_0}} \times \dots \times \mathbb{C}^{N_{kd_n}},$$

by $W_{(k,j)} = (W_{k,j,\nu} : \nu \in \mathbb{N}^{n+1}, |\nu| = kd_j)$ for $j = 0, \dots, n$, and

$$W_{k,j,\nu}(u_{(j)}) = \sum_{\substack{\nu^1, \dots, \nu^k \in \mathbb{N}^{n+1} \\ \nu^1 + \dots + \nu^k = \nu \\ |\nu^1| = \dots = |\nu^k| = d_j}} u_{j,\nu^1} \cdots u_{j,\nu^k} \quad \text{for } \nu \in \mathbb{N}^{n+1},\ |\nu| = kd_j.$$

In other words, $W_{(k,j)}$ is a system of coefficients of f_j^k as a polynomial in x. So for any positive integer k we may define

$$R_k = \mathrm{Res}_{kd_0,\dots,kd_n}(f_0^k, \dots, f_n^k),$$
$$D_k = D_{kd_0,\dots,kd_n}(f_0^k, \dots, f_n^k).$$

More precisely,

$$R_k = \mathrm{Res}_{kd_0,\dots,kd_n} \circ W_k\,,$$
$$D_k = D_{kd_0,\dots,kd_n} \circ W_k\,.$$

Then from (17) and (18) we have

$$D_k = \sum_{\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathscr{I}_k} C_\eta W_{(k,0)}^{\eta^{(0)}} \cdots W_{(k,n)}^{\eta^{(n)}}. \tag{19}$$

From Fact 3.4 we have

Fact 3.7. *For any positive integer k we have $D_k \neq 0$.*

From [2, Proposition 1.3 in Chapter 13] and [1, Theorem 3.2], we immediately obtain

Fact 3.8. *For any positive integer k we have*

$$\operatorname{Res}_{kd_0,\dots,kd_n}(f_0^k,\dots,f_n^k) = \operatorname{Res}_{d_0,\dots,d_n}(f_0,\dots,f_n)^{k^{n+1}}.$$

Recall that $d_* = \max\{d_0,\dots,d_n\}$. Put

$$N_{*,k} = \binom{kd_*+n}{n}, \quad N_k^* = \binom{(n+1)kd_*}{n}, \quad k\in\mathbb{Z},\ k>0.$$

Lemma 3.9. $L_1(D_k) \leqslant (N_{*,1})^{kN_k^*}\, L_1(D_{kd_0,\dots,kd_n})$.

Proof. Indeed, for any $j=0,\dots,n$ and any $\nu\in\mathbb{N}^{n+1}$, $|\nu| = kd_j$ the polynomial $W_{k,j,\nu}$ consists of at most $(N_{*,1})^k$ monomials with coefficients equal to 1, i.e., $(N_{*,1})^k$ is not smaller that

$$\#\{(\nu^1,\dots,\nu^k)\in(\mathbb{N}^{n+1})^k : \nu^1+\dots+\nu^k=\nu,\ |\nu^1|=\dots=|\nu^k|=d_j\}$$

for $j=0,\dots,n$. So from (19) we easily see that

$$L_1(D_k) \leqslant \sum_{\eta=(\eta^{(0)},\dots,\eta^{(n)})\in\mathscr{I}_k} |C_\eta|\,(N_{*,1})^{k|\eta^{(0)}|}\cdots(N_{*,1})^{k|\eta^{(n)}|}.$$

Then (16) easily gives the assertion. □

3.3. **Height of a multipolynomial resultant.** From Lemmas 2.2, 3.6 and 3.9 and Fact 3.8 we have

Lemma 3.10. *For any $k\in\mathbb{Z}$, $k>0$ we have*

$$M(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{N_k^*/k^n}\,(N_{*,k})^{N_k^*/k^{n+1}}, \tag{20}$$

$$H(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n-n-1}\,M(\operatorname{Res}_{d_0,\dots,d_n}), \tag{21}$$

$$L_1(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n}\,M(\operatorname{Res}_{d_0,\dots,d_n}). \tag{22}$$

Proof. Let $e_j = d_0\cdots d_{j-1}d_{j+1}\cdots d_n$ for $j=0,\dots,n$. By Lemma 2.2 and (6) we obtain

$$H(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant \left(\prod_{j=0}^n (N_{d_j})^{e_j-1}\right) M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}},$$

$$L_1(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant \left(\prod_{j=0}^n (N_{d_j})^{e_j}\right) M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}}.$$

Since $e_0+\dots+e_n \leqslant (n+1)d_*^n$, then from the above we have

$$H(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n-n-1}\,M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}},$$

$$L_1(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n}\,M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}}.$$

This, together with Fact 3.8, gives (21) and (22).

From Fact 3.8 we also have $M(\mathrm{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}} = M(R_k)^{1/k^{n+1}}$, and since $M(R_k) \leqslant M(D_k)$ (by (7) and Facts 3.5 and 3.7), so (9) gives

$$M(\mathrm{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}} \leqslant L_2(D_k)^{1/k^{n+1}}. \tag{23}$$

By Lemma 3.9 we have

$$L_1(D_k) \leqslant (N_{*,1})^{kN_k^*} L_1(D_{kd_0,\dots,kd_n}). \tag{24}$$

Since

$$N_{kd_j} \leqslant N_{*,k}, \quad \text{for } j = 0,\dots,n,$$
$$N_{kd_0,\dots,kd_n} \leqslant N_k^*,$$

so, from Lemma 3.6 we obtain

$$L_1(D_{kd_0,\dots,kd_n}) \leqslant (N_{*,k})^{N_k^*} \quad \text{for } k > 0.$$

Since $L_2(D_k) \leqslant L_1(D_k)$ then (23) and (24) gives (20). □

In general $N_k^* \leqslant (n+1)!(kd_*)^n$. It turns out that asymptotically this number has better properties.

Lemma 3.11.

$$\lim_{k\to\infty} \frac{N_k^*}{k^n} = \frac{(n+1)^n d_*^n}{n!} < \frac{e^{n+1}}{\sqrt{2\pi n}} d_*^n.$$

Proof. Indeed,

$$\frac{N_k^*}{k^n} = \frac{\prod_{j=1}^n [(n+1)kd_* - n + j]}{n!k^n},$$

so,

$$\lim_{k\to\infty} \frac{N_k^*}{k^n} = \frac{(n+1)^n d_*^n}{n!} = \left(\frac{n+1}{n}\right)^n \frac{n^n}{n!} d_*^n < e\frac{n^n}{n!} d_*^n.$$

Since from Stirling formula,

$$\frac{n^n}{n!} \leqslant \frac{e^{n-1/(12n+1)}}{\sqrt{2\pi n}},$$

then we obtain the assertion. □

Lemmas 3.10 and 3.11 gives the main result of this paper.

Theorem 3.12. *Let* $d_* = \max\{d_0,\dots,d_n\}$ *and* $K_n = e^{n+1}/\sqrt{2\pi n}$, $n > 0$. *Then*

$$M(\mathrm{Res}_{d_0,\dots,d_n}) \leqslant (d_*+1)^{nK_n d_*^n}, \tag{25}$$

$$H(\mathrm{Res}_{d_0,\dots,d_n}) \leqslant (d_*+1)^{n(K_n+n+1)d_*^n - n(n+1)}, \tag{26}$$

$$L_1(\mathrm{Res}_{d_0,\dots,d_n}) \leqslant (d_*+1)^{n(K_n+n+1)d_*^n}. \tag{27}$$

Moreover, if $n \geqslant 2$ and $d_* \geqslant 4$, then

$$(28)\qquad \begin{aligned} M(\mathrm{Res}_{d_0,\dots,d_n}) &\leqslant (d_*)^{nK_n d_*^n},\\ H(\mathrm{Res}_{d_0,\dots,d_n}) &\leqslant (d_*)^{n(K_n+n+1)d_*^n - n(n+1)},\\ L_1(\mathrm{Res}_{d_0,\dots,d_n}) &\leqslant (d_*)^{n(K_n+n+1)d_*^n}. \end{aligned}$$

Proof. From Lemma 3.10 for nay $k \in \mathbb{Z}$, $k > 0$ we have

$$\begin{aligned} M(\mathrm{Res}_{d_0,\dots,d_n}) &\leqslant (N_{*,1})^{N_k^*/k^n} (N_{*,k})^{N_k^*/k^{n+1}},\\ H(\mathrm{Res}_{d_0,\dots,d_n}) &\leqslant (N_{*,1})^{(n+1)d_*^n - n - 1} (N_{*,1})^{N_k^*/k^n} (N_{*,k})^{N_k^*/k^{n+1}},\\ L_1(\mathrm{Res}_{d_0,\dots,d_n}) &\leqslant (N_{*,1})^{(n+1)d_*^n} (N_{*,1})^{N_k^*/k^n} (N_{*,k})^{N_k^*/k^{n+1}}. \end{aligned}$$

Since $1 \leqslant N_{*,k} \leqslant (kd_* + 1)^n$, then

$$(29)\qquad \lim_{k\to\infty} (N_{*,k})^{1/k} = 1,$$

so passing to the limit as $k \to \infty$ in the above inequalities, by Lemma 3.11, we obtain (25), (26) and (27).

Since for $n \geqslant 2$ and $d_* \geqslant 4$ we have $N_{*,1} \leqslant d_*^n$ then we obtain the second part of the assertion (28). □

Remark 3.13. *The estimation* (27) *of $L_1(\mathrm{Res}_{d_0,\dots,d_n})$ is not a direct consequence of the estimation* (26) *of the height $H(\mathrm{Res}_{d_0,\dots,d_n})$ because the number of coefficients of $\mathrm{Res}_{d_0,\dots,d_n}$ can be bigger than $(d_* + 1)^{n(n+1)}$. The number of coefficients of the resultant can be estimated by*

$$E_{d_0,\dots,d_n} := \prod_{j=0}^{n} \binom{\binom{d_j+n}{n} + d_0 \cdots d_{j-1} d_{j+1} \cdots d_n}{d_0 \cdots d_{j-1} d_{j+1} \cdots d_n} \leqslant (d_* + 1)^{n(n+1)d_*^n}.$$

4. Gelfond-Mahler type inequalities for homogeneous polynomials

As a corollaries from Lemma 2.2 we obtain the following Gelfond-Mahler type theorems.

Corollary 4.1. *Let $f \in \mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ be a nonzero polynomial such that f is homogeneous of degree $s_j > 0$ as a polynomial in each system of variables $u_{(m,j)}$. Then for any polynomials $f_1, \dots, f_k \in \mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ such that $f = f_1 \cdots f_k$ we have*

$$(30)\qquad \begin{aligned} H(f_1)\cdots H(f_k) &\leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j-1}\right) M(f)\\ &\leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j-1}\right)\left(\prod_{j=0}^{n} \sqrt{N_{m,d_j}+1}^{\,s_j}\right) H(f) \end{aligned}$$

and

$$L_1(f_1)\cdots L_1(f_k) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j}\right) M(f) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j}\right) L_1(f). \tag{31}$$

Proof. The left hand inequalities in (30) and (31) immediately follows from Lemma 2.2, because $M(f_1)\cdots M(f_k) = M(f)$ from (6). Since the polynomial f is homogeneous with respoct to $u_{(m,j)}$ of degree s_j, $j = 0,\ldots,n$, then from (9) we have

$$M(f) \leqslant \left(\prod_{j=0}^{n} \sqrt{\binom{s_j + N_{m,d_j}}{N_{m,d_j}}}\right) H(f) \leqslant \left(\prod_{j=0}^{n} \sqrt{N_{m,d_j}+1}^{\,s_j}\right) H(f).$$

This gives the right hand inequalities in (30) and (31) and ends the proof. □

Applying Corollary 4.1 for $n = 0$, $d_0 = 1$ and $m = N - 1$ and a homogenisation $f^*(x_0,\ldots,x_m) := x_0^{\deg f} f(x_1/x_0.\ldots,x_m/x_0)$ of a polynomial $f \in \mathbb{Z}[x_1,\ldots,x_m]$ we obtain the following corollary.

Corollary 4.2. *Let $f \in \mathbb{Z}[x_1,\ldots,x_m]$ be a nonzero polynomial of degree $s > 0$. Then for any polynomials $f_1,\ldots,f_k \in \mathbb{Z}[x_1,\ldots,x_m]$ such that $f = f_1\cdots f_k$ we have*

$$H(f_1)\cdots H(f_k) \leqslant (N+1)^{s-1} M(f^*) \leqslant (N+1)^{s-1}\sqrt{N+2}^{\,s} H(f)$$

and

$$L_1(f_1)\cdots L_1(f_k) \leqslant (N+1)^s M(f^*) \leqslant (N+1)^s L_1(f).$$

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(Aleksandra Gala-Jaskórzyńska) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

E-mail address: agjaskorzynska@math.uni.lodz.pl

(Krzysztof Kurdyka) Laboratoire de Mathematiques (LAMA) Universié Savoie Mont Blanc, UMR-5127 de CNRS 73-376 Le Bourget-du-Lac cedex FRANCE

E-mail address: Krzysztof.Kurdyka@univ-savoie.fr

(Katarzyna Rudnicka) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

E-mail address: kkuta@math.uni.lodz.pl

(Stanisław Spodzieja) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

E-mail address: stanislaw.spodzieja@wmii.uni.lodz.pl