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Intertemporal Choice of Fuzzy Soft Sets

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Abstract: This paper first merges two noteworthy aspects of choice. On the one hand, soft sets and fuzzy soft sets are popular models that have been largely applied to decision making problems, such as real estate valuation, medical diagnosis (glaucoma, prostate cancer, etc.), data mining, or international trade. They provide crisp or fuzzy parameterized descriptions of the universe of alternatives. On the other hand, in many decisions, costs and benefits occur at different points in time. This brings about intertemporal choices, which may involve an indefinitely large number of periods. However, the literature does not provide a model, let alone a solution, to the intertemporal problem when the alternatives are described by (fuzzy) parameterizations. In this paper, we propose a novel soft set inspired model that applies to the intertemporal framework, hence it fills an important gap in the development of fuzzy soft set theory. An algorithm allows the selection of the optimal option in intertemporal choice problems with an infinite time horizon. We illustrate its application with a numerical example involving alternative portfolios of projects that a public administration may undertake. This allows us to establish a pioneering intertemporal model of choice in the framework of extended fuzzy set theories.

Keywords: fuzzy soft set; intertemporal choice; comparison table; decision making

1. Introduction

The scientific contribution of this paper is setting up a novel framework for making decisions that stems from the first cross-fertilization of two features: (a) intertemporal aspects of choice; and (b) extended fuzzy set models. We also give a novel adjustable algorithm that prioritizes alternatives with the aforementioned features.

Decisions whose consequences extend across multiple time periods are called intertemporal choices. The entry "Intertemporal choice" in the Palgrave Dictionary of Economics states: "Most choices require decision-makers to trade-off costs and benefits at different points in time. Decisions with consequences in multiple time periods are referred to as intertemporal choices. Decisions about savings, work effort, education, nutrition, exercise, and health care are all intertemporal choices" [1]. Although its analysis is preeminent in the standard crisp literature, to the best of our knowledge, the problem of intertemporal choice has never been modeled when the data are imprecise, uncertain or subjective in the sense of the extended fuzzy set theories. In this paper, we first put forward a model that fills this important gap. To prove that it can be used to make decisions, we propose a flexible mechanism that provides a ranking of the alternatives that are characterized by these characteristics. We also present some examples that illustrate the application of our decision making procedure.

To achieve our goals, we have selected the successful setting of fuzzy soft sets. (Other models of imprecise knowledge would require an ad hoc analysis, which we postpone for subsequent investigations to avoid confusions.) The new model that arises (from the amalgamation of the intertemporal setting of choice and data in the form of fuzzy soft sets) is called intertemporal fuzzy soft sets. We put forward various equivalent and complementary definitions of this concept. Some are more convenient for the purpose of algebraic manipulations and intuitions. Some are better suited to describe the computational machinery that produces the results from which the decisions are achieved.

In relation with the latter issue, the gist of standard intertemporal problems is that the consequences of a decision span along an infinite number of periods. However, to decide among various alternatives, their consequences across time are summarized by an amount called their respective Net Present Values. In this fashion, the infinite expansion that characterizes an alternative is summarized by a unique number, for example through a discounted sum. By inspiration of this widely accepted position, we propose to condense the information of intertemporal fuzzy soft sets into fuzzy soft sets in order to make optimal decisions. The tool that we introduce to achieve this target is called a reduction mechanism. Reduction mechanisms can both indicate symmetry in the valuation of a reward irrespective of the period when it is obtained, or a preference for earlier rewards (i.e., violation of the symmetric treatment of the periods). We provide several noteworthy examples of both behaviors. Once this reduction to a fuzzy soft set has been performed, our decision can rely on widely accepted solutions stated for that setting.

Actually, the main reason for choosing fuzzy soft sets in our pioneering approach is that there is a fully-developed theory for fuzzy soft set based decision making. To further assess the importance of this setting, in the next section, we review some general background about soft computing models with a more explicit explanation about soft set based modelizations and their decision making. We also dwell on the fundamentals of the intertemporal problem of choice and its applications.

This paper is organized as follows. In Section 2, we give some general background about the main notions that are used in our research. A fully developed real example helps to clarify the application of the discounted utility aggregation model. Section 3 recalls some terminology and definitions. In Section 4, we define our new model of intertemporal fuzzy soft sets and we also offer alternative formalizations. In Section 5, we define the notion of a reduction mechanism, which we use to state the decision algorithm that prioritizes alternatives in the framework of intertemporal fuzzy soft sets. We also illustrate the model with a numerical application to the selection of alternative portfolios of public projects. Finally, we conclude in Section 6.

2. Background

In this section, we give some background about various pertinent topics. Firstly, we provide some basic knowledge about the theory and practice of intertemporal choices, inclusive of a fully developed real example. Secondly, we give a general overview of fuzzy sets and other related models of uncertain information. Finally, we focus on the specific characteristics of the framework where we develop our contribution, namely, fuzzy soft sets and their decision making.

2.1. Intertemporal Choice: Theory and Practice

Intertemporal choices are decisions whose consequences (costs and benefits) are distributed over time [2]. Decisions about investments, spending and savings are standard examples of monetary intertemporal choices. However, there are also non-monetary intertemporal choices such as decisions related with sustainability (environmental issues such as forestry [3], climate policy [4] or the use of energy-using durables [5]), health (diet, exercise, and addictions [6,7]), job search [8], or work effort [9].

Discounted utility theory is the normative theory for intertemporal choice, or choices between outcomes accruing at different points in time; usually between immediate and delayed outcomes [10]. Since its introduction by Samuelson [11] in 1937, the discounted utility (DU) model has dominated the economic analysis of intertemporal choice (e.g., the aforementioned [4,5,9]). DU model was completed by Koopmans [12] who clarified its logic and main assumptions. This model presumes that people evaluate the pleasures and pains resulting from a decision in a similar way that financial markets evaluate gains and losses spread out over time. Anyone prefers to get 1000 dollars now rather than 1000 dollars in a year. However, people behave differently if they have to choose between receiving 1000 dollars now or 1100 dollars in a year. To compare choices made in different moments of time, under DU, it is assumed that agents exponentially discount these costs and benefits according to how delayed they are in time [13]. Although there is experimental evidence showing that this is not always the case [2,14,15], a fact that prompted the appearance of other explanatory models such as hyperbolic discounting [16–18] or q-exponential discounting [19–21], the DU model is nevertheless used as the common tool for public policy in the evaluation of public projects. The model can be calibrated with suitable discount factors, for example, using a decreasing sequence of discount rates for projects with very long-term impacts to account for intergenerational equity [22]. The governments of the United Kingdom and France, in line with the proposals of several authors for long-term valuations [23], recommend the use of decreasing discount rates in public projects with long time horizons [24,25].

In real practice, and provided that the DU model is adopted for the evaluation of intertemporal projects, the expected cash-flows of a project are always discounted to obtain its Net Present Value (NPV) at instant 0 (time of evaluation), see for example [26]. To that purpose, a well-known formula is applied which requires using an appropriate discount rate. For private projects, the weighted average cost of capital or a required profitability are usually used as a discount rate. For public projects, the social time preference rate is employed to calculate the discount rate, which is then called *social discount rate* (SDR). The social time preference rate is a rate used for discounting future benefits and costs, and it is based on comparisons of utility across different points in time or different generations [24]. The SDR "is used by society to give relative weight to social consumption or income accruing at different points in time" [3].

Since we are applying our intertemporal model of choice to concrete examples, we need to fix an appropriate discount rate. Our choice is not arbitrary. On the contrary, we take advantage of the fact that, for EU funded projects, the European Commission [26] recommends the use of the exponential discounting model and a constant 5% European social discount rate for the Cohesion Fund eligible countries and 3% for the others (countries non-eligible for the Cohesion Fund). Therefore, 5% is our reference rate unless otherwise stated.

According to the previous discussion, discounted utility computations made for choices in the present and at various moments in the future $(x_0, x_1, \ldots, x_T)$ adopt the form

$$\sum_{t=0}^{T} \beta^t u(x_t) \tag{1}$$

Here, x_t is the choice made at moment t, whose utility at that moment is $u(x_t)$; β^t is the discount factor for a time period of t periods, usually years (for example, $\beta = \frac{1}{1+0.05}$ under our assumption for the reference rate); choices are made along periods $t = 0$ (the present), $t = 1, 2, \ldots, T$; and T may be $+\infty$. Put otherwise, if we want to assess the temporal sequence $(x_0, x_1, \ldots, x_T)$ and the utility u gives us the degree of satisfaction of each choice x_t, which is of course $u(x_t)$, then $\sum_{t=0}^{T} \beta^t u(x_t)$ gives the discounted utility of this temporal sequence. The β parameter accounts for the fact that people prefer to enjoy utility as soon as possible.

Let us now give a real example that illustrates the application of our reference model for intertemporal choice.

Example 1. *On 7 November 2017, the Spanish infrastructure operator Ferrovial published a press release (see https://www.ferrovial.com/en/press-room/press_releases/500-million-euro-2-124-perpetual-hybrid-bond/, retrieved 18 August 2018.) It stated: "Taking advantage of a favorable market environment with low interest rates, Ferrovial today successfully priced a 500 million euro perpetual hybrid bond. The issue pays a* 2.124% *annual coupon until 14 May 2023. Subsequently, it will pay a fixed coupon equal to the applicable swap rate plus a spread of* 2.127% *until 14 May 2043 and of* 2.877% *thereafter. The swap rate will be updated every five years."*

Therefore, Ferrovial perpetual bonds, with a face value of 100 *euros, will pay a* 2.124% *annual coupon during the first six years,* 2.127% *(supposing a swap rate of* 0%*) the following 20 years, and a* 2.877% *thereafter. Table 1 expresses this intertemporal situation and gives the computations that produce the NPV at time 0 of such a bond when we assume a discount rate of* 3%*, hence* $\beta = (1 + 0.03)^{-1} = 0.97087$.

Table 1. Detailed computations of the NPV, assuming a discount rate of 3%, in the real Example 1. It is the sum of the present values at the right column of the table.

Period	Interest Rate	Cash-Flows Annual Coupon	Discount Factor	Present Value
1	2.124%	2.124 = 100 × 0.02124	$0.97087 = (1+0.03)^{-1} = \beta^1$	2.0621 = 2.124 × 0.97087
2	2.124%	2.124	$0.94260 = (1+0.03)^{-2} = \beta^2$	2.0021 = 2.124 × 0.94260
3	2.124%	2.124	$0.91514 = \beta^3$	1.9438
4	2.124%	2.124	$0.88849 = \beta^4$	1.8871
5	2.124%	2.124	$0.86261 = \beta^5$	1.8322
6	2.124%	2.124	$0.83748 = \beta^6$	1.7788
7	2.127%	2.127 = 100 × 0.02127	$0.81309 = \beta^7$	1.7294
8	2.127%	2.127	$0.78941 = \beta^8$	1.6791
⋮	⋮	⋮	⋮	⋮
26	2.127%	2.127	$0.46369 = \beta^{26}$	0.9863
27 and onwards	2.877%	2.877 = 100 × 0.02877		$44.4683 = 2.877\dfrac{\beta^{26}}{0.03}$
NPV				82.4761

2.2. A Concise Presentation of Fuzzy Sets and Related Notions

Since Zadeh [27] laid the foundations of fuzzy set theory, whose main feature is the introduction of partial membership degrees, many authors produced a large amount of literature on their advantages and potential applications in decision making. Mardani et al. [28] gave a summary of articles about fuzzy multi-criteria decision making from the period 1994–2014. Other classical references for the fundamentals of decision making in fuzzy set theory include Tanino [29] and Fodor and Roubens [30].

When imprecise individual or collective knowledge cannot be faithfully represented by fuzzy sets, extensions of this concept and multiple variations offer more suitable models. Atanassov [31,32] presented the idea of intuitionistic fuzzy sets. Afterwards, Chen et al. [33] or Wei [34] produced intuitionistic fuzzy multi-attribute group decision making methods, and De Miguel et al. [35] applied interval-valued Atanassov intuitionistic fuzzy sets in multi-expert decision making. Pythagorean fuzzy sets are surveyed in Peng and Selvachandran [36], and interval-valued Pythagorean fuzzy sets were studied by Peng and Yang [37] and Peng [38]. Hesitancy was first merged with fuzzy sets by Torra [39] (for more information, a good source is Rodríguez et al. [40]; see also Alcantud and Torra [41] for the first decomposition theorems and extension principles in the framework of hesitant fuzzy sets).

From a different position, rough set theory was established by Pawlak [42] and, in his first formulation, an equivalence binary relation is the source of granulation of the set of alternatives.

It is at this junction that a different, parameterized description of the alternatives made its appearance. The idea produced soft sets, extensions and hybrid models. Since they are a benchmark in our paper, we proceed to describe them succinctly in the next subsection.

2.3. Soft Sets, Extensions and Hybrid Models

The theory of soft sets originates with the seminal paper [43]. Feng and Zhou ([44], Section 1) cleverly described soft set theory in the following terms: it "is considered as a new mathematical tool for dealing with uncertainties which is free from the inadequacy of parameter tools. In soft set theory, the problem of setting the membership function simply does not arise as in fuzzy set theory, which makes the theory convenient and easy to use in practice." Its relevancy to decision making in various fields was already pointed out in [43], which also explained that the models by fuzzy sets and soft sets are linked to each other. Further relationships are proven in [45–48]. The early works of Maji et al. [49] and Aktaş and Çağman [50] among others expanded the basic theory of soft sets. Khameneh and Kılıçman [51] systematically reviewed multi-attribute decision-making based on soft set theory. Zhan and Alcantud [52] recently summarized parameter reduction of soft sets, a thriving area that allows for approaches in extended models.

Indeed, suitable extensions of soft sets come up from the incorporation of ideas such as the aforementioned fuzziness and hesitancy. Fuzzy soft sets were designed by Maji, Biswas and Roy [53]. Parameter reduction in the context of fuzzy soft sets was developed, e.g., by Khameneh and Kılıçman [54]. Wang, Li and Chen [55] produced hesitant fuzzy soft sets by adding up hesitancy to the latter concept. Because data collection and measurement often produce errors or are restricted, some studies (e.g., [56–58]) are concerned with another extended form of soft sets called incomplete soft sets, while other [59,60] are concerned with the natural extension called incomplete fuzzy soft sets.

Let us now consider fuzzy soft set based decision making. The performance of the pioneering analysis by Roy and Maji [61] has been improved by [62,63]. Feng et al. [64] put forward a flexible method based on level soft sets. They explained that [65] challenges the position in [61] by claiming that the criterion for making a decision should use scores instead of fuzzy choice values, a point of view that found little support among other scholars. Liu et al. [66] gave another methodology for fuzzy soft set based decision-making based on an ideal solution. Feng and Guo [67] intended to effectively resolve the natural group decision-making problem in the context of fuzzy soft sets. To achieve their goal, they first design another adjustable method for solving fuzzy soft set based decision-making problems.

Hybrid models combine the spirit of soft sets with other methodologies. Peng, Dai and Yuan ([68] and the references therein) contributed to interval-valued fuzzy soft decision making. Park, Kwun and Son [69] gave an approach to decision making problems based on generalized intuitionistic fuzzy soft sets (see also [70] for the operations in that framework). Feng et al. [48] used soft approximation spaces instead of binary relations in rough set theory. Zhan and Wang [71] built five new different types of soft coverings based rough sets and investigated relationships between soft rough sets and soft covering based rough sets. Zhan and Alcantud [72] designed a soft rough covering by means of soft neighborhoods, which they utilized to improve decision making in a multi-criteria group environment. Ma et al. [73] presented an updated summary of decision making methodologies based on two classes of hybrid soft set models. Fatimah et al. [74] studied the decision-making implications of probabilistic and dual probabilistic soft sets.

Now, we proceed to state the formal definitions that we shall use in the remaining of this paper.

3. Definitions: Soft Sets and Fuzzy Soft Sets

In soft set theory and their extensions, we start with U, which is a set of alternatives (or a universe of objects), and E, which is a universal set of attributes, parameters or characteristics. We let $\mathcal{P}(U)$ denote the set of parts of U, i.e., the set formed by all the subsets of U.

Definition 1 (Molodtsov [43])**.** *A soft set over U is a pair (F, A) with $A \subseteq E$ and $F : A \longrightarrow \mathcal{P}(U)$.*

A soft set over U is a parameterized family of subsets of U, and the set A contains the relevant parameters. For every $e \in A$, the subset $F(e)$ is the set of e-approximate elements, or also, the subset of U approximated by e. For example, if $U = \{s_1, s_2, s_3\}$ is a universe of shirts and A contains the parameter e that describes "blue color" and the parameter e' that describes "silk fabric" then $F(e) = \{s_1\}$ means that the only shirt with blue color is s_1 and $F(e') = \{s_1, s_3\}$ means that exactly s_1 and s_3 have silk fabric.

To model more general situations, the following notion is subsequently proposed and investigated in [53]:

Definition 2 (Maji, Biswas and Roy [53])**.** *Let $\mathbf{FS}(U)$ denote the set of all fuzzy sets on U. The pair (F, A) is a fuzzy soft set (FSS) over U when $A \subseteq E$ and $F : A \longrightarrow \mathbf{FS}(U)$.*

Needless to say, soft sets are an example of fuzzy soft sets. However, fuzzy soft sets are better suited to model subjectively perceived properties, since partial memberships are designed to express such subjectivity.

In the standard instance with finite U and A, both soft and fuzzy soft sets can be displayed in the form of a table, where rows correspond to the alternatives in U, columns correspond to the attributes in A, and there is a number from $[0, 1]$ in each cell. Of course, the cells of these matrices contain either 0 or 1 when the fuzzy soft set is a soft set.

To illustrate these ideas and motivate the subsequent decisional analysis, let us put forward an example in terms of an object recognition problem:

Example 2. *A collection of objects* $U = \{o_1, ..., o_6\}$ *is characterized in terms of a space of attributes which is denoted as* $A = \{p_1, \ldots, p_7\}$. *Here, the attributes represent the relevant combinations of characteristics. The fuzzy soft set that describes the objects is* (F, A), *which is given by the tabular representation in Table 2. For illustration, number* 0.650 *at the junction of Row* o_1 *and Column* p_1 *means that the degree of membership of* o_1 *to the objects that verify characteristic* p_1 *is* 0.650.

Table 2. Tabular representation of the fuzzy soft set (F, A) in Example 2.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
o_1	0.650	0.150	0.064	0.216	0.048	0.054	0.405
o_2	0.144	0.720	0.360	0.045	0.036	0.020	0.175
o_3	0.120	0.084	0.180	0.350	0.096	0.021	0.294
o_4	0.504	0.192	0.108	0.090	0.048	0.620	0.280
o_5	0.084	0.245	0.036	0.096	0.270	0.200	0.320
o_6	0.216	0.315	0.042	0.108	0.224	0.126	0.410

Fuzzy Soft Set Based Decision Making

Soft set based decision making relies on [44,75,76]. However, since the appearance of the seminal [61], there have been many remarkable approaches to decision making in the framework of fuzzy soft sets. The most successful contributions include [61–67]. We do not describe them all in detail here. For our purposes, it should suffice to know their general features and relative advantages, which are summarized in Table 3. It compares various noteworthy criteria with respect to their main characteristics.

Table 3. A critical summary of the main fuzzy soft set based decision making procedures.

Ref.	Aggregation	Methodology	Solution	Other Issues
[61]	Min operator	Scores from a comparison matrix	Unique	Many ties Loss of information [63] reformulates algorithm
[65]	Not discussed	Fuzzy choice values	Unique	It is too controversial
[64]	Not discussed	Choice value of level soft set	Not unique	Ties multiply Richness at the cost of indeterminacy Additional inputs needed (e.g., threshold fuzzy set)
[62]	Product operator	Scores from alternative comparison matrix	Unique	Improved power of discrimination
[66]	Not discussed	Similarity measure and substitutable	Unique	Application of subjective weights Low time complexity
[67]	Not discussed	Distance measure for Group Decision Making	Not unique	Two methods for obtaining appropriate experts' weights

As explained in Section 1, ultimately we need to make choices from a fuzzy soft set to solve the intertemporal choice problem for fuzzy soft sets that we present in Section 4. Therefore, it is convenient to be familiar with the machinery of at least one such procedure. For the sake of clarity, to apply fuzzy soft sets in decision making practice, we focus on the proposal by Alcantud [62], who stated a feasible algorithmic solution to solve problems in the format of Example 2.

To make this paper self-contained, we recall that the application of Alcantud's algorithm proceeds as follows (afterwards Example 3 illustrates the application of Algorithm 1 below to a concrete situation):

Algorithm 1 (Alcantud [62])

Input: a fuzzy soft set (F, A), which we place in the form of a table. Its cell (i, j) is represented by t_{ij}

1: For every attribute j, let M_j denote the maximum membership value of the alternatives, i.e., $M_j = \max_{i=1,...,k} t_{ij}$ for each $j = 1, ..., q$.

Produce a $k \times k$ comparison matrix $A = (a_{ij})_{k \times k}$ as follows: for every i, j, a_{ij} is the sum of the non-negative values in the following finite sequence:

$$\frac{t_{i1} - t_{j1}}{M_1}, \frac{t_{i2} - t_{j2}}{M_2}, ..., \frac{t_{iq} - t_{jq}}{M_q}.$$

We can display this matrix as a comparison table.

2: For each $i = 1, ..., k$, calculate R_i as the sum of the elements in row i of A, and T_i as the sum of the elements in column i of A. For every $i = 1, ..., k$, calculate the score $S_i = R_i - T_i$ of object i.

3: The result of the decision is any object o_k such that $S_k = \max_{i=1,...,k} S_i$.

Example 3. *Let us assume that we have the input data of Example 2. Its comparison table is given in Table 4, as computed by Algorithm 1 above. Then, Table 5 shows its associated scores.*

As a result, one concludes that o_4 should be selected when we consider the input data of Example 2 and we adhere to Algorithm 1.

Table 4. Comparison table of the fuzzy soft set (F, A) in Example 2 using Algorithm 1.

	o_1	o_2	o_3	o_4	o_5	o_6
o_1	0	1.93	1.23	0.89	1.5	1.04
o_2	1.61	0	1.42	1.43	1.65	1.45
o_3	0.88	1.39	0	1.15	1.18	1.07
o_4	1.09	1.95	1.71	0	1.52	1.42
o_5	1.19	1.66	1.22	1.01	0	0.29
o_6	1.01	1.73	1.39	1.19	0.57	0

Table 5. Score table of the fuzzy soft set (F, A), derived from its Comparison table by Algorithm 1.

	Row-Sum (R_i)	Column-Sum (T_i)	Score (S_i)
o_1	6.58	5.79	0.79
o_2	7.57	8.65	−1.08
o_3	5.68	6.97	−1.29
o_4	7.7	5.68	2.02
o_5	5.37	6.43	−1.06
o_6	5.9	5.27	0.63

4. A New Model: Intertemporal Fuzzy Soft Sets

Thus far, the literature has dealt with alternatives with a very simple structure: for each characteristic, we know the degree of membership of the alternative to the set of elements that verify the characteristic. However, it is not difficult to find examples where the performance of the options is far more complex. Particularly, in this paper, we are concerned with an intertemporal setting. Indeed, in the general framework of project selection (e.g., solar energy projects [77], environmental impact assessment [78], financial portfolio [79], etc.), each alternative has a performance along an indefinite number of periods, typically years. Therefore, if we want to decide which of a list of projects should be selected, we face an infinite number of fuzzy soft sets (one that represents each possible period). Neither of the existing approaches to fuzzy soft set based decision making can deal with this potentially infinite structure.

To tackle this new problem, now we proceed to formalize our model. It accounts for the intertemporal setting that we have motivated. Afterwards, we interpret the formal statement of the model in terms of tables when the number of alternatives and attributes is finite, which facilitates their computational manipulation. In the next section, we give a procedure for making decisions in this novel framework. In addition, in that section, an example illustrates the decision making algorithm in a situation motivated by public projects evaluation. For comparison, recall that Example 1 shows a recent, real situation in a financial environment with crisp data.

4.1. The Structure of Intertemporal Fuzzy Soft Sets

Molodtsov's notion of parameterized descriptions of the universe has already been combined with features that do not pertain to the original formulation of soft sets. Here, we propose a model where for each of a possibly infinite number of periods, each attribute produces a possibly different fuzzy parameterization of the universe.

The statement of the model is simple but powerful:

Definition 3. *An intertemporal fuzzy soft set (ItFSS) over U is a sequence $\mathbb{F} = \{(F_i, A)\}_{i\in\mathbb{N}}$ of fuzzy soft sets over the common universe U.*

In Section 4.2, we specify two alternative formulations of this definition, that are amenable for calculations with computers.

The basic idea is that the fuzzy parameterization of the universe is allowed to vary with time. According to this model, a fixed set of attributes is given, and then for each period i a parameterized description of the common universe produces a fuzzy soft set (F_i, A) which accounts for the situation at that period. To emphasize differences, on occasions, we refer to these standard FSSs as *static* FSSs.

Remark 1. *When the sequence in Definition 3 is constant (in other words, we use $\{(F_i, A)\}_{i\in\mathbb{N}}$ with $F_i = F$ for each i), then we identify the ItFSS with a standard FSS. Therefore, our model of ItFSSs is a natural extension of FSSs in Definition 2.*

Henceforth, we assume that the common attributes in A are all positive (otherwise, we can proceed as in [80] to convert the input into this format). Therefore, the higher is the membership degree, the better. The decision problem that arises consists of determining an optimal alternative from the list U. We propose a flexible solution in Section 5.

The next motivating example illustrates the structure of our model above. Observe that in this simplified statement it is possible to display the information pertaining to an intertemporal fuzzy soft set in one table, even though it concerns an infinite number of periods:

Example 4. *A civil project (e.g., building a bridge or a dam) will have a long-term impact on the population of a certain geographical area. There are two targets that should be achieved: environmental effects (e_1) and economic development (e_2). Let $A = \{e_1, e_2\}$. There are two possible projects, namely, p_1 and p_2. Their respective yearly effects are captured by the ItFSS $\mathbb{G} = \{(G_i, A)\}_{i\in\mathbb{N}}$ over U shown in Table 6, where $U = \{p_1, p_2\}$.*

It is not difficult to check that, in every period i, the positive effects of p_1 exceed the effects of p_2 at any of the two target attributes:

$$0.3 + \frac{1}{i+1} > 0.2 + \frac{1}{i+2} \text{ for each } i$$

$$0.4 + \frac{1}{i+1} > 0.3 + \frac{1}{i+2} \text{ for each } i$$

Therefore, project p_1 should be selected.

Table 6. A tabular representation of the intertemporal fuzzy soft set $\mathbb{G} = \{(G_i, A)\}_{i \in \mathbb{N}}$ in Example 4.

(G_i, A)	e_1	e_2
p_1	$0.3 + \frac{1}{i+1}$	$0.4 + \frac{1}{i+1}$
p_2	$0.2 + \frac{1}{i+2}$	$0.3 + \frac{1}{i+2}$

The streamlined Example above is very simple for two reasons. Firstly, even though there are an infinite number of different FSSs for the infinite periods, the ItFSS can be represented by one parametric table, which is often not the case. Secondly, we do not need to use the theoretical contribution of Section 5 because the decision about which project should be selected is trivial: there is a sort of domination of the first project over the second in all attributes and for all periods that makes the decision obvious. Now, we proceed to give a different example where the modelling power of ItFSSs is more apparent. We can also infer the need for a formal analysis of decision making in that context, hence the next example motivates Section 5.

Example 5. *A new regulation will have long-term effects on the population of a country. There are three groups that are potentially affected in terms of satisfaction: students* (e_1)*, working class* (e_2) *and retired people* (e_3)*. There are two possibilities, namely, passing the law* (p_1) *and retaining the current regulation* (p_2)*. Their respective yearly effects on the satisfaction across groups are captured by Table 7, which embodies an ItFSS* $\mathbb{F} = \{(F_i, A)\}_{i \in \mathbb{N}}$ *over* $U = \{p_1, p_2\}$ *where* $A = \{e_1, e_2, e_3\}$*.*

We read, for example, that in year 1 *the students' degree of membership to being satisfied with the new regulation is* 0.3*, and it is* 0.2 *under the current law. For the working class, the respective degrees of satisfaction are* 0.4 *and* 0.6*. For retired people, the respective degrees of satisfaction are* 0.7 *and* 0.8*.*

From the third year onwards, the students' degree of membership to being satisfied with the new regulation is 0.6*, and it is* 0.3 *under the current law. For the working class, the respective degrees of satisfaction are* 0.6 *and* 0.4*. For retired people, the respective degrees of satisfaction are* 0.8 *and* 0.6*.*

Table 7. Tabular representation of the intertemporal fuzzy soft set in Example 5.

(F_1, A)	e_1	e_2	e_3
p_1	0.3	0.4	0.7
p_2	0.2	0.6	0.8
(F_2, A)	e_1	e_2	e_3
p_1	0.4	0.4	0.7
p_2	0.3	0.5	0.7
(F_i, A)	e_1	e_2	e_3
p_1	0.6	0.6	0.8
p_2	0.3	0.4	0.6
$\forall i > 2$			

Example 5 shows the intrinsic difficulty of dealing with these problems. The snapshots at different moments can vary substantially from each other, and of course they are not always as obvious as the situation of Example 4. Additionally, the "attributes" can have different weights, for example because they represent characteristics of groups with different proportions in the society.

For the purpose of favoring implementability, now we proceed to state an equivalent formulation of our intertemporal model. It allows us to work with a tabular format that is amenable for computations.

4.2. An Alternative Representation of Intertemporal Fuzzy Soft Sets

In practical terms, when both $U = \{o_1, \ldots, o_m\}$ and $A = \{e_1, \ldots, e_n\}$ are finite, we can represent the information that describes an ItFSS in a table where the cells are either finite or infinite sequences of membership degrees. Table 8 gives the general form of such a representation.

Table 8. The tabular representation of our novel intertemporal model for fuzzy soft sets.

	e_1	e_2	$\ldots$	e_n
o_1	$(u^1_{11}, u^2_{11}, \ldots, u^t_{11}, \ldots)$	$(u^1_{12}, u^2_{12}, \ldots, u^t_{12}, \ldots)$	$\ldots$	$(u^1_{1n}, u^2_{1n}, \ldots, u^t_{1n}, \ldots)$
$\vdots$				
o_m	$(u^1_{m1}, u^2_{m1}, \ldots, u^t_{m1}, \ldots)$	$(u^1_{m2}, u^2_{m2}, \ldots, u^t_{m2}, \ldots)$	$\ldots$	$(u^1_{mn}, u^2_{mn}, \ldots, u^t_{mn}, \ldots)$

Let us analyze this alternative description. To that purpose, the set of infinite sequences of numbers from $[0, 1]$ (or infinite utility streams [81–83]) is denoted by $\mathcal{S}$. Our intertemporal model of fuzzy soft sets over U can also be defined by $\bar{F} : A \longrightarrow \mathbf{S}(U)$ where $\mathbf{S}(U)$ represents the mappings $U \longrightarrow \mathcal{S}$. Consequently, for each attribute, we capture the degree of membership of any alternative in each moment of time.

Indeed, in Table 8, we can define $\bar{F}(e_j)(o_i) = (u^1_{ij}, u^2_{ij}, \ldots, u^t_{ij}, \ldots) \in \mathcal{S}$, hence u^t_{ij} means the degree of membership of alternative o_i to the fuzzy set of elements that verify attribute e_j in period t. Conversely, every $\bar{F} : A \longrightarrow \mathbf{S}(U)$ produces a table with the structure of Table 8 under the finiteness restriction for both A and U.

We can also swap between the tabular form and the notation of Definition 3.

From the aforementioned tabular representation of $\bar{F} : A \longrightarrow \mathbf{S}(U)$, we can define the corresponding ItFSS over U as the sequence $\mathbb{F} = \{(F_i, A)\}_{i \in \mathbb{N}}$ where the tabular form of each FSS (F_i, A) is described in Table 9.

Table 9. The tabular representation of the fuzzy soft set (F_i, A) corresponding to moment i in the ItFSS $\mathbb{F} = \{(F_i, A)\}_{i \in \mathbb{N}}$ represented by Table 8.

	e_1	e_2	$\ldots$	e_n
o_1	u^i_{11}	u^i_{12}	$\ldots$	u^i_{1n}
$\vdots$				
o_m	u^i_{m1}	u^i_{m2}	$\ldots$	u^i_{mn}

Conversely, an ItFSS given by the construction in Section 4.1 can be trivially transformed into the tabular form presented in this subsection. We do this in Table 10 for the case of $\mathbb{G}$ in Example 4.

Table 10. The tabular representation of $\mathbb{G} = \{(G_i, A)\}_{i \in \mathbb{N}}$ in Example 4.

$\mathbb{G}$	e_1	e_2
p_1	$(0.3 + \frac{1}{2}, 0.3 + \frac{1}{3}, 0.3 + \frac{1}{4}, \ldots)$	$(0.4 + \frac{1}{2}, 0.4 + \frac{1}{3}, 0.4 + \frac{1}{4}, \ldots)$
p_2	$(0.2 + \frac{1}{3}, 0.2 + \frac{1}{4}, 0.2 + \frac{1}{5}, \ldots)$	$(0.3 + \frac{1}{3}, 0.3 + \frac{1}{4}, 0.3 + \frac{1}{5}, \ldots)$

5. Choices from Intertemporal Fuzzy Soft Sets

When it comes to prioritizing alternatives in the framework of intertemporal fuzzy soft sets, the most natural course of action is to associate a FSS with our original ItFSS and then apply Algorithm 1 (or any other existing proposal, see Table 3) to the latter fuzzy soft set.

To implement this solution, now we proceed to describe some procedures that from each ItFSS produce a static or standard fuzzy soft set (cf., Section 5.1). Afterwards, we show how we can integrate these procedures with decision making based on FSSs to design an intertemporal fuzzy soft set based decision making procedure (cf., Section 5.2).

5.1. Static FSSs Associated with an Intertemporal FSS

In this subsection, we formalize some methods that associate a static FSS (Definition 2) with any intertemporal FSS (Definition 3). We refer to these methods as *reduction mechanisms*.

The simplest reduction mechanisms act cell-by-cell on the tabular representation. For instance, one can pinpoint the evaluation at a distinguished moment (e.g., the first period); or, in the case of finite time horizon, their lowest or highest evaluation, their (either arithmetic or geometric) average, etc. Under a genuine infinity of periods, we can use the natural modifications by infimum, supremum or discounted sums for the same purpose.

Let us now formalize a few explicit reduction mechanisms. We fix $\mathbb{F} = \{(F_i, A)\}_{i\in\mathbb{N}}$, an intertemporal fuzzy soft set over U. We can obtain a *reduced* FSS associated with $\mathbb{F}$ by the application of one of the following expressions:

1. The pessimistic FSS associated with $\mathbb{F}$ is (F^p, A) such that the fuzzy parameterization $F^p : A \longrightarrow \mathbf{FS}(U)$ verifies that for each $a \in A$, $F^p(A) = \inf\{F_i(a) : i \in \mathbb{N}\}$.
2. The optimistic FSS associated with $\mathbb{F}$ is (F^o, A) such that the fuzzy parameterization $F^o : A \longrightarrow \mathbf{FS}(U)$ verifies that for each $a \in A$, $F^o(A) = \sup\{F_i(a) : i \in \mathbb{N}\}$.
3. Let $\delta \in [0, 1)$ be a factor. The δ-discounted FSS associated with $\mathbb{F}$ is (F^δ, A), where the fuzzy parameterization $F^\delta : A \longrightarrow \mathbf{FS}(U)$ verifies that for each $a \in A$,

$$F^\delta(A) = \frac{1-\delta}{\delta} \sum_{i=1}^{\infty} \delta^i F_i(a). \tag{2}$$

Observe that these definitions are correct: the only non-trivial case is justified in the following auxiliary result.

Lemma 1. *The δ-discounted FSS is well-defined, i.e., it is a FSS.*

Proof. We just need to observe that, because each (F_i, A) is a FSS, $\sum_{i=1}^{\infty} \delta^i F_i(a)$ is bounded above by $\sum_{i=1}^{\infty} \delta^i = \frac{\delta}{1-\delta}$. Hence, $\sum_{i=1}^{\infty} \delta^i F_i(a)$ converges for each $a \in A$, and $\frac{1-\delta}{\delta} \sum_{i=1}^{\infty} \delta^i F_i(a) \leqslant 1$. Obviously, $\frac{1-\delta}{\delta} \sum_{i=1}^{\infty} \delta^i F_i(a) \geqslant 0$. □

The first two reduction mechanisms are symmetric, in the standard sense: when $\mathbb{F}_\sigma = \{(F_{\sigma(i)}, A)\}_{i\in\mathbb{N}}$ is the ItFSS derived from a permutation of the periods $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$, one has $F^p_\sigma = F^p$ and $F^o_\sigma = F^o$. However, the δ-discounted reduction mechanism violates the symmetric treatment of the periods, i.e., the statement $F^\delta_\sigma = F^\delta$ is in general false in the aforementioned conditions.

However, the δ-discounted reduction mechanism has an important advantage over the pessimistic and optimistic reduction mechanisms. The pessimistic and optimistic reduction mechanisms produce a considerable loss of information because they discard the data about the degrees of membership that are not minimal and maximal respectively, whereas the δ-discounted reduction mechanism uses all the information available to produce the reduced FSS.

Our next remark insists on the importance of this mechanism in decision making.

Remark 2. *The deduction of the δ-discounted FSS is motivated by a successful solution to the problem of aggregating intergenerational utilities [81–83]. In Section 2.1, we explained that according to the popular DU model [11,12,84], decision-makers evaluate the alternatives on the basis of the weighted addition of utilities, these weights being discount factors based on temporal delays. In our benchmark case, these delays can extend to infinity.*

Due to the aforementioned advantages, henceforth we adopt the δ-discounted reduction mechanism as the standard mechanism for transforming an ItFSS into a FSS in practical problems.

5.2. Decision Making in Intertemporal FSSs

We are ready to put forward a procedure for ranking a finite list of alternatives when the decision-making problem is characterized by an intertemporal FSS. It consists of three basic steps.

Put shortly, the algorithm suggests to reduce the ItFSS to a FSS (Step 1) and then order the alternatives according to standard decision-making in this framework (Step 2). The ordering in the reduced FSS carries forward in the ItFSS for which it is a natural representation (Step 3), which solves our problem. The next subsection illustrates how this can be put into practice in a concrete example of project appraisal.

5.3. An Example of Decision Making in the Framework of ItFSSs

In this section, we develop an example that serves two purposes. Firstly, we describe the structure of the problem in a practical fashion that is different from the description in Section 4.2. Secondly, it illustrates the application of Algorithm 2.

Algorithm 2 Algorithm for decision making

Inputs: An intertemporal table of fuzzy soft sets (in the notation of Table 8 or, otherwise, see Section 5.3). A reduction mechanism (e.g., from Section 5.1). A fuzzy soft set decision making procedure (e.g., from Table 3).

1: Apply the selected reduction mechanism to the ItFSS in order to obtain a (reduced) FSS.
2: Rank the alternatives in this FSS by the decision making procedure that we have singled out.
3: Any object that is at the top of the ranking in the previous step is an optimal choice of the intertemporal statement of the problem.

Example 6. *For the convenience of presentation, we are going to evaluate two alternative portfolios of projects that a public administration may undertake. Portfolios 1 and 2 are parameterized in terms of four attributes along an infinite number of periods, and each of these characteristics can also be regarded as a project on its own (e.g. bike lanes, urban parks, sports facilities, and sewage treatment plants). The objective of this evaluation is to choose the best alternative.*

For each project, a value for its social suitability in each period is assigned. To simplify, we consider projects whose utilities follow the following patterns: increasing and then constant; decreasing and then constant; decreasing, then increasing and finally constant; and constant.

Tables 11 and 12, respectively, describe these two portfolios P_1 and P_2. At their bottoms they contain additional values whose meaning we explain below.

These tables jointly define an ItFSS that we have displayed in Table 13, albeit in incomplete form due to obvious restrictions: we cannot display the infinite digits that appear at each cell. According to Step 1 of Algorithm 2, with this element we associate a standard FSS that we denote as (S, P), by the application of the DU reduction mechanism with a 0.05 discount rate, hence $\delta = \frac{1}{1.05} \approx 0.952381$. The suitability of this rate has been argued in Section 2.1. Let us denote $\mathbf{T} = \frac{1-\delta}{\delta} = 0.05$. The reduced fuzzy soft set (S, P) is displayed in Table 14. The results of the computations that produce it already appear at the bottom of Tables 11 and 12, and now we explain how we have obtained them (see Equation (2), and Remark 2 for inspiration). In the case of Portfolio 1,

$$\mathbf{A}_1 = 13.085321 = 0.05\delta + 0.10\delta^2 + 0.15\delta^3 + 0.20\delta^4 + \ldots$$

$$\mathbf{A}_2 = 10.60642701 = 1\delta + 0.95\delta^2 + 0.90\delta^3 + 0.85\delta^4 + \ldots$$

$$\mathbf{A}_3 = 14 = 0.70\delta + 0.70\delta^2 + 0.70\delta^3 + 0.70\delta^4 + \ldots$$

$$\mathbf{A}_4 = 18.3258941 = 0.05\delta + 0.10\delta^2 + 0.15\delta^3 + 0.20\delta^4 + \ldots + 0.70\delta^9 + 0.75\delta^{10} + \ldots$$

Therefore, in the reduced FSS (S, P)*, Equation (2) states that the degrees of membership for Portfolio 1 are* $\mathbf{T} * \mathbf{A}_i$*. We proceed similarly to obtain the degrees of membership for Portfolio 2.*

We now apply Algorithm 1 to the FSS (S, P)*. Table 15 gives the necessary computations.*

From the last computation in Table 15, we conclude that the first portfolio should be selected.

Table 11. Portfolio 1.

Period	Attribute 1	Attribute 2	Attribute 3	Attribute 4
1	0.05	1.00	0.70	1.00
2	0.10	0.95	0.70	0.95
3	0.15	0.90	0.70	0.90
4	0.20	0.85	0.70	0.85
5	0.25	0.80	0.70	0.80
6	0.30	0.75	0.70	0.75
7	0.35	0.70	0.70	0.70
8	0.40	0.65	0.70	0.65
9	0.45	0.60	0.70	0.70
10	0.50	0.55	0.70	0.75
11	0.55	0.50	0.70	0.80
12	0.60	0.45	0.70	0.85
13	0.65	0.40	0.70	0.90
14	0.70	0.35	0.70	0.95
15	0.75	0.35	0.70	1.00
16	0.80	0.35	0.70	1.00
17	0.85	0.35	0.70	1.00
18	0.90	0.35	0.70	1.00
19	0.95	0.35	0.70	1.00
20	1.00	0.35	0.70	1.00
21	1.00	0.35	0.70	1.00
22	1.00	0.35	0.70	1.00
23 and onwards	1.00	0.35	0.70	1.00
$\mathbf{A}_i$	13.085321	10.60642701	14	18.3258941
$\mathbf{A}_i$***T**	0.65427	0.530321351	0.7	0.91629471

Table 12. Portfolio 2.

Period	Attribute 1	Attribute 2	Attribute 3	Attribute 4
1	0.00	1.00	0.60	1.00
2	0.10	0.94	0.60	0.90
3	0.20	0.88	0.60	0.80
4	0.30	0.82	0.60	0.70
5	0.40	0.76	0.60	0.60
6	0.50	0.70	0.60	0.50
7	0.60	0.64	0.60	0.40
8	0.70	0.58	0.60	0.30
9	0.80	0.52	0.60	0.40
10	0.90	0.52	0.60	0.50
11	1.00	0.52	0.60	0.60
12	1.00	0.52	0.60	0.70
13	1.00	0.52	0.60	0.80
14	1.00	0.52	0.60	0.90
15	1.00	0.52	0.60	1.00
16	1.00	0.52	0.60	1.00
17	1.00	0.52	0.60	1.00
18	1.00	0.52	0.60	1.00
19	1.00	0.52	0.60	1.00
20	1.00	0.52	0.60	1.00
21	1.00	0.52	0.60	1.00
22	1.00	0.52	0.60	1.00
23 and onwards	1.00	0.52	0.60	1.00
A$_i$	15.44	12.24	12.00	16.65
A$_i$*T	0.772173493	0.612207234	0.6	0.832589415

Table 13. A (necessarily incomplete) tabular representation of the intertemporal fuzzy soft set in Example 6. Each cell actually contains an infinite sequence.

	Attribute 1	Attribute 2	Attribute 3	Attribute 4
P_1	$(0.05, 0.10, 0.15, 0.20, 0.25, \ldots)$	$(1.00, 0.95, 0.90, 0.85, 0.80, \ldots)$	$(0.70, 0.70, 0.70, 0.70, \ldots)$	$(1.00, 0.95, 0.90, 0.85, 0.80, \ldots)$
P_2	$(0.00, 0.10, 0.20, 0.30, 0.40, \ldots)$	$(1.00, 0.94, 0.88, 0.82, 0, 76, \ldots)$	$(0.60, 0.60, 0.60, 0.60, \ldots)$	$(1.00, 0.90, 0.80, 0.70, 0.60, \ldots)$

Table 14. Tabular representation of the reduced fuzzy soft set (S, P) in Example 6.

	Attribute 1	Attribute 2	Attribute 3	Attribute 4
P_1	0.65	0.53	0.70	0.92
P_2	0.77	0.61	0.60	0.83

Table 15. Computing the Comparison table and scores of the reduced fuzzy soft set (S, P) through Algorithm 1.

	Attribute 1	Attribute 2	Attribute 3	Attribute 4
Diff. memberships P_1 vs. P_2	−0.12	−0.08	0.10	0.08
Diff. memberships P_2 vs. P_1	0.12	0.08	−0.10	−0.08
M_j	0.77	0.61	0.70	0.92

	P_1	P_2
P_1	0	0.29
P_2	0.23	0

	Row-Sum (R_i)	Column-Sum (T_i)	Score (S_i)
P_1	0.29	0.23	0.06
P_2	0.23	0.29	−0.06

6. Discussion and Concluding Remarks

We have designed a pioneering framework for making choices in soft computing models. For the first time in this broad area, we have considered the situation where the consequences of a decision extend along an unlimited number of periods, such as a financial investment or a social project. Existing models universally refer to a finite framework, hence they are incapable of dealing with these practical issues. We have set the grounds for a correct extension to this critical aspect of decision making.

In this paper, our reference model for uncertainty has been fuzzy soft sets, which allows for fuzzy parameterized description of the alternatives in terms of a list of attributes. We have opted for working with this environment because fuzzy soft sets are especially amenable for decision making, with plenty of interesting approaches in the literature. Future research should expand the scope of the intertemporal analysis that we have founded to other frameworks such as incomplete fuzzy soft sets, rough sets, hesitant fuzzy sets, or hesitant fuzzy soft sets among many others. Whatever the selected format for the input data, when choices extend along an infinite number of periods, the fundamental roadmap for making decisions has been established in this paper.

Obviously, it may also be possible to approach the exact problem that we have described in this paper by alternative methodologies to improve the performance of our proposal, or make it more faithfully adapted to the circumstances of the problem under inspection.

Overall, we believe that the intertemporal modelization may become a thriving area of research in the extended theories of fuzziness, vagueness and uncertainty.

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Abbreviations

The following abbreviations are used in this manuscript:

DU	Discounted Utility
FSS	Fuzzy soft set
NPV	Net Present Value
ItFSS	Intertemporal fuzzy soft set

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