

Orlicz–Pettis Theorem through Summability Methods

Fernando León-Saavedra ¹ , María del Pilar Romero de la Rosa ^{2,*} and Antonio Sala ³

¹ Department of Mathematics, University of Cádiz, Facultad Ciencias Sociales y de la Comunicación, 11405 Jerez de la Frontera, Cádiz, Spain; fernando.leon@uca.es

² Department of Mathematics, University of Cádiz, CASEM, 11510 Puerto Real, Cadiz, Spain

³ Departamento de Matemáticas, University of Cádiz, Escuela Superior de Ingeniería, 11510 Puerto Real, Cadiz, Spain; antonio.sala@uca.es

* Correspondence: pilar.romero@uca.es

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Abstract: This paper unifies several versions of the Orlicz–Pettis theorem that incorporate summability methods. We show that a series is unconditionally convergent if and only if the series is weakly subseries convergent with respect to a regular linear summability method. This includes results using matrix summability, statistical convergence with respect to an ideal, and other variations of summability methods.

Keywords: summability method; ideal convergence; weakly unconditional Cauchy series; Orlicz–Pettis theorem

MSC: 40H05; 40A35

1. Introduction

Let X be a normed space. A linear summability method \mathcal{R} on X is a rule to assign limits to a sequence, that is, it is a linear map $\mathcal{R} : D_{\mathcal{R}} \subset X^{\mathbb{N}} \rightarrow X$. A summability method \mathcal{R} is said to be regular if, for each convergent sequence $(x_n)_n$ in X , that is, $\lim_{n \rightarrow \infty} x_n = x_0$, we have that $\mathcal{R}((x_n)_n) = x_0$.

The methods of summability were born at the beginning of the 20th century, with the development of the theory of Fourier Analysis. For example, statistical convergence and strong Cesàro convergence were introduced respectively by Zygmund [1] and Hardy [2], and both concepts were surprisingly connected thanks to the work of Connor [3] fifty years later. Since then, the Summability Theory has taken on a life of its own, with deep and beautiful results (see the recent monographs [4,5] for historical notes). Moreover, the theory has important applications on Applied Mathematics (see the recent monograph by Mursaleen [6]).

The Orlicz–Pettis Theorem is a classic result concerning a convergent series, so beautiful that it has attracted the interest of many mathematicians and it has been strengthened and generalized in many directions. An early survey is Kalton’s paper [7]. The reader can see in [8–19] recent results about the Orlicz–Pettis type Theorems.

Let us recall that a series $\sum_i x_i$ in a Banach space X is said to be unconditionally convergent (u.c) if, for each permutation of the natural numbers $\pi : \mathbb{N} \rightarrow \mathbb{N}$, we have that $\sum_i x_{\pi(i)}$ is convergent. A series $\sum_i x_i$ is weak-subseries convergent if, for any $M \subset \mathbb{N}$, there exists $x_M \in X$ such that the partial sums $S_n^M = \sum_{i=1}^n \chi_M(i)x_i$ converges weakly to x_M (here $\chi_M(\cdot)$ denotes the characteristic function on M). The classical Orlicz–Pettis Theorem states that a series $\sum_i x_i$ is unconditionally convergent if and only if $\sum_i x_i$ is weakly subseries convergent.

Diestel and Faires sharpened the classical result by Orlicz–Pettis in the following sense ([20] I.4.7). Let X be a Banach space that contains no copy of ℓ_{∞} and let Γ be a total subset of X^* . Then, a formal

series $\sum_n x_n$ in X such that every subseries is Γ -convergent, that is, for each subset $A \subset \mathbb{N}$, there exists $x_A \in X$, such that $\sum_n x^*(x_n) = x^*(x_A)$ for all $x^* \in \Gamma$; then, $\sum_n x_n$ is norm unconditionally convergent.

On the other hand, attempts have been made to replace weak convergence with other (weak) summability method. At this point, we find several results in the literature. For instance, the Orlicz–Pettis Theorem remains true if we replace the weak convergence by the weak statistical convergence (see [8]). It is also true for the weak-Cesàro convergence [21], for the weak A -convergence, A being an infinite matrix with non-negative entries [22], for the weak-statistical Cesàro convergence [9], and more recently for the w^p -strong Cesàro convergence [15]. In this note, we aim to unify all known results, obtaining an Orlicz–Pettis Theorem for a general summability method; of course, we need to place some limits on the summability method because the result fails for an arbitrarily summability method. Namely, the result is true for any linear regular summability method. It is surprising how we can weaken the weak convergence hypothesis in the Orlicz–Pettis result by almost any other weaker summability method. The paper is organized as follows: in Section 2, we will show a General Orlicz–Pettis Theorem for summability methods. Next, we will see how effectively our result unifies the known results, and we will see new applications.

2. Main Results

Let ρ be a linear summability method, that is, a subset $D_\rho \subset \mathbb{R}^{\mathbb{N}}$ and a linear function $\rho : D_\rho \rightarrow \mathbb{R}$, which assigns a unique real number $\rho((x_n))$ to a sequence $(x_n) \in D_\rho$. In addition, ρ is said to be regular, if, for every convergent sequence $\lim_n x_n = x_0$, the sequence (x_n) , ρ -converge to the same limit. A summability method ρ induces a weak summability method \mathcal{R} in X as follows: a sequence $(x_n) \in X^{\mathbb{N}}$ is \mathcal{R} -convergent to $x_0 \in X$ if and only if $f(x_n)$ is ρ -convergent to $f(x_0)$ for all $f \in X^*$. Let us observe that, in general, the convergence method \mathcal{R} could be degenerate, that is, $D_{\mathcal{R}} = \emptyset$. However, if ρ is regular, then $D_{\mathcal{R}}$ is non-empty; moreover, \mathcal{R} is also regular.

Proposition 1. *If ρ is regular, then \mathcal{R} is regular.*

Proof. Let us suppose that $\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0$; then, for each $f \in X^*$, $\lim_{n \rightarrow \infty} |f(x_n) - f(x_0)| = 0$, since ρ is regular $f(x_n) \xrightarrow{\rho} f(x_0)$. Therefore, $x_n \xrightarrow{\mathcal{R}} x_0$ as desired. \square

Theorem 1. *Let X be a real Banach space, ρ a regular summability method on \mathbb{R} and \mathcal{R} the summability method induced by ρ . Then, a series $\sum_i x_i$ is unconditionally convergent if and only $\sum_i x_i$ is \mathcal{R} -subseries convergent in X .*

Proof. Let $\sum_i x_i$ be an unconditionally convergent series, and let $M \subset \mathbb{N}$. By applying the classical Orlicz–Pettis Theorem, we obtain that there exists $x_M \in X$ such that the sequence $S_n^M = \sum_{i=1}^n \chi_M(i)x_i$ weakly converges to x_M , that is, for each $f \in X^*$, we have that the sequence $\sum_{i=1}^n \chi_M(i)f(x_i) \xrightarrow{|\cdot|} f(x_M)$. Since ρ is regular, we have that $\sum_{i=1}^n \chi_M(i)f(x_i) \xrightarrow{\rho} f(x_M)$ for each $f \in X^*$, that is, $S_n^M = \sum_{i=1}^n \chi_M(i)x_i$ \mathcal{R} -converges to x_M , as desired.

Now, let us suppose that, for any $M \subset \mathbb{N}$, there exists x_M such that $S_n^M = \sum_{i=1}^n \chi_M(i)x_i$ \mathcal{R} -converges to x_M . First of all, we will prove that $\sum_i x_i$ is a weakly unconditionally Cauchy series. If not, let us argue by contradiction, so let us suppose that there exists $f \in X^*$ such that $\sum_i |f(x_i)| = +\infty$. Let us consider the following subsets $M = \{i \in \mathbb{N} : f(x_i) \geq 0\}$ and $N = \{i \in \mathbb{N} : f(x_i) < 0\}$. In addition, let us define the sequence

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \in M \\ -1 & \text{if } i \in N, \end{cases}$$

then $\sum_{i=1}^{\infty} \varepsilon_i f(x_i) = +\infty$, hence the sequence $\sum_{i=1}^n \varepsilon_i x_i$ does not \mathcal{R} converge to any $L \in X$. On the other hand, by hypothesis, given $M, N \subset \mathbb{N}$, there exists that $x_M, x_N \in X$ such that $\sum_{i=1}^n \chi_M(i)x_i \xrightarrow{\mathcal{R}} x_M$ and $\sum_{i=1}^n \chi_N(i)x_i \xrightarrow{\mathcal{R}} x_N$. Therefore,

$$\sum_{i=1}^n \varepsilon_i f(x_i) = \sum_{i=1}^n \chi_M(i)f(x_i) - \sum_{i=1}^n \chi_N(i)f(x_i) \xrightarrow{\rho} f(x_M) - f(x_N) = f(x_M - x_N),$$

a contradiction. Therefore, for any $f \in X^*$, we have $\sum_i |f(x_i)| < \infty$.

Now, let us show that, given $M \subset \mathbb{N}$, there exists $x_M \in X$ such that $\sum_{i=1}^n \chi_M(i)x_i$ weakly converges to x_M . Let $f \in X^*$, since $\sum_i |f(x_i)| < \infty$, we deduce that the series $\sum_{i=1}^n \chi_M(i)f(x_i)$ is convergent to some $\lambda_{M,f} \in \mathbb{R}$, and hence ρ -convergent to $\lambda_{M,f}$. On the other hand, by hypothesis, there exists $x_M \in X$ such that $\sum_{i=1}^n \chi_M(i)x_i \xrightarrow{\mathcal{R}} x_M$, that is, for each $f \in X^*$, we have that $\sum_{i=1}^n \chi_M(i)f(x_i) \xrightarrow{\rho} f(x_M)$. Therefore, $\lambda_{M,f} = f(x_M)$. Hence, we obtain that, for any $f \in X^*$, the sequence $\sum_{i=1}^n \chi_M(i)f(x_i)$ converges to $f(x_M)$, that is, $\sum_{i=1}^n \chi_M(i)x_i$ weakly converges to x_M . Thus, by applying the classical Orlicz–Pettis Theorem, we obtain that the series $\sum_i x_i$ is unconditionally convergent as desired. \square

Remark 1. *The following example was pointed out by one of the referees. Let us consider the following linear summability method: ρ , a sequence $(x_n) \subset \mathcal{R}$ is said to be ρ -convergent to x_0 if $\lim_{n \rightarrow \infty} \frac{x_n}{n^2} = x_0$. Then, it is clearly in the realm of bounded sequences $\rho|_{\ell^\infty} = 0$. Thus, ρ is not regular. Now, let us consider on ℓ^2 the summability method induced by ρ , which we denote by \mathcal{R} . For every $M \subset \mathbb{N}$, we have that $\sum_{i=1}^n \chi_M(i)e_i \xrightarrow{\mathcal{R}} 0 = x_M$. However, $\sum_{i=1}^n \chi_M(i)e_i$ is not norm convergent to 0. The argument of the proof breaks down if we can't guarantee that $\lambda_{M,f} = f(x_M)$. This fact highlights the importance of regularity in the proof of the above result.*

Now, let us see some applications of Theorem 1. We will say that $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is a non-trivial ideal if

1. $\mathcal{I} \neq \emptyset$ and $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$.
2. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
3. If $A \subset B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
4. Additionally, we say that \mathcal{I} is regular (or admissible) if it contains all finite subsets.

A non-trivial regular ideal \mathcal{I} defines a regular summability method on any metric space. We will say that a sequence $(x_n) \subset \mathbb{R}$ is \mathcal{I} -convergent to $L \in \mathbb{R}$ (in short $L = \mathcal{I} - \lim_{n \rightarrow \infty} x_n$) if, for any $\varepsilon > 0$, the subset

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| > \varepsilon\} \in \mathcal{I}.$$

Thus, given a Banach space X , the \mathcal{I} -convergence defines a weakly summability method in X ; we will say that a sequence $(x_n) \subset X$ is weakly- \mathcal{I} convergent to x_0 if and only if, for any $f \in X^*$, we have $f(x_n) \xrightarrow{\mathcal{I}} f(x_0)$.

Corollary 1. *Let \mathcal{I} be a non-trivial ideal. Then, a series $\sum_i x_i$ is a real Banach space; X is unconditionally convergent if and only if $\sum_i x_i$ is subseries weakly- \mathcal{I} convergent.*

In particular, if we consider the ideal \mathcal{I}_d of all subsets in \mathbb{N} with zero density, the ideal convergence induced by \mathcal{I}_d (which is non trivial and regular) is the statistical convergence. Therefore, the above Corollary is also true for the weak-statistical convergence [8].

Now, let us consider a regular matrix summability method induced by an infinite matrix $A = (a_{ij})$, which is defined as follows. A sequence $(x_n) \in \mathbb{R}^{\mathbb{N}}$ is A -summable to L if $\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj}x_j = L$. A matrix A is regular if the usual convergence implies the A -convergence, and the limits are preserved. Now, if X is a Banach space, then the matrix A also induces a summability method on a Banach space X ; we say that a sequence $(x_n) \in X^{\mathbb{N}}$ is A -convergent to $x_0 \in X$ if $\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj}x_j = x_0$. The matrix

A also induces a weakly convergence, a sequence $(x_n) \in X^{\mathbb{N}}$ is weakly A -convergent to $x_0 \in X$ if, for any $f \in X^*$, we have that $f(x_n)$ is A -convergent to $f(x_0)$. Applying Theorem 1, we get:

Corollary 2. *Let A be a regular matrix. Then, a series $\sum_i x_i$ is unconditionally convergent if and only if $\sum_i x_i$ is subseries weak- A -convergent.*

Thus, we obtain the results in [22]. In particular, if A is the Cesàro matrix, we obtain the results in [21].

Let us consider the following summability method ρ : we will say that a sequence (x_n) is Cesàro, statistically convergent to L , if the Cesàro means $(x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3} \dots)$ is statistically convergent to L . Given a Banach space X , the summability method ρ induces a weakly summability method in X . Namely, we say that a sequence $(x_n) \in X^{\mathbb{N}}$ is weakly statistically Cesàro convergent to $x_0 \in X$ if, for every $f \in X^*$, the sequence $f(x_n)$ is Cesàro statistically convergent to $f(x_0)$.

As a consequence, we obtain the results in [9].

Corollary 3. *Let A be a regular matrix. Then, a series $\sum_i x_i$ is unconditionally convergent if and only if $\sum_i x_i$ is subseries weakly statistically Cesàro-convergent.*

In brief, Theorem 1 not only unifies the known results, but also can be widely applied to several summability methods obtaining new versions of the Orlicz–Pettis theorem. For instance, it applies for the Erdős–Ulam convergence, w^p -Cesàro convergence, f -statistical convergence, etc. (see [15,23]).

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