Burgers' Equation and Some Applications

Mr. Khankham VONGSAVANG

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Abstract

In this thesis, I present Burgers' equation and some of its applications. I consider the inviscid and the viscid Burgers' equations and present different analytical methods for their study: the Method of Characteristic for the inviscid case, and the Cole-Hopf Transformation for the viscid one.

Two applications of Burgers' equations are given: one in simple models of Traffic Flow (which have been introduced independently by Lighthill-Whitham and Richards) and another in Coagulation theory (in which we use Laplace Transform to obtain Burgers' equations from the original coagulation integro-differential equation). In both applications we consider only analytical methods.

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Chapter 1

Introduction

The study of differential equations is an important field of mathematics, they are used as models describing phenomena arising in all the sciences. Meanwhile, modelling real life situations gives rise to complicated differential equations the mathematical analysis of which usually require some simplification. An example of such a simplification process is provided by the Burgers' equations.

Burgers' equations are the nonlinear partial differential equations (for short, PDEs)

$$u_t + uu_x = \varepsilon u_{xx}$$

for the unknown function $u = u(x,t)$ of variables $(x,t) \in \mathbb{R} \times (0,+\infty)$, where case $\varepsilon = 0$ is a paradigm of the so called hyperbolic pde type and case $\varepsilon > 0$ of the parabolic pde type. The mathematical analysis of those two types of equations is quite different: because while in the later solutions are regular functions, in the former physical solutions are commonly discontinuous functions.

The first time we noticed Burgers' equation in the literature ($\varepsilon > 0$) was in vol.6 of Forsyth book [9, (1906), page 100] related with a classification of linear pdes (Art. 206-207, pp. 97-102), the second one ($\varepsilon > 0$, $\varepsilon = 0$) was in Bateman article [3, (1915), page 165, left column] concerned with the motion of fluids, where the author tried to have a mathematical model ("Burgers' equation") developing discontinuities from the evolution of a continuous motion: a controversial issue enrolling mathematicians and physicists as d'Alembert, Stokes, Helmholtz, Kelvin, Kirchhoff, Rayleigh, Levi-Civita, Prandtl, Lorentz, Sommerfeld, von Mises, von Kármán, to cite a few. Then Burgers [6, (1948), page 181][1] used them ($\varepsilon > 0$, $\varepsilon = 0$) to study a

[1] Actually Burgers began that work with the communication [5, (1940), page 8] where he used already both equations.

system of equations simpler than that of hydrodynamics "which in a sense form a mathematical model of turbulence" (cf. p.172 of [6], last lines of its Introduction). And, even if in fact it does not model turbulence, the equation is named after him.

Burgers' equations arises in a number of unrelated applications, as in gas dynamics, nonlinear elasticity, shallow water theory, geometric optics, combustion theory, cancer medicine, petroleum engineering, irrigation systems, traffic or crowd panic. It appears often as a simplification of complex or more sophisticated models. A first example was shown by Lagerstrom-Cole-Trilling [13, (1949), page 151] in the context of the study of viscous compressible fluids at supersonic regime (Appendix B, pp.146-154), there the Burgers' equation is obtained as a limiting simplified form of the compressible Navier-Stokes momentum equation.

The main relevance of Burgers' equations stands in the fact that it is a fundamental equation to understand more general models and how to study the behaviour of phenomena where the effects of nonlinear transport and dissipation (as viscosity or diffusion) are conflicting as time goes by.

1.1 Motivation

Here, in order to better explain the ideas above, we will follow V.I. Arnold's words and example in his book [2, page vi]: we will "adhere to the principle of minimal generality, according to which every idea should first be clearly understood in the simplest situation".

In a straight line, imagine a collection of particles moving freely. Because no force is acting on the particles, by Newton's second law ("force = mass × acceleration") each particle has zero acceleration. And thus constant velocity. But the velocities of different particles can be different of each other.

Let $u(x,t)$ be the velocity of a particle which is in position x (over the straight line) at time t. Notice that, at a fixed time t, the different particles in their different positions x can have different velocities u and, at a fixed position x, as time t evolves there are different particles passing through that position x and having different velocities u. The function $u(\cdot,\cdot)$ is named a field of velocities.

Now, let us observe an arbitrary single particle. Consider initial time $t_0 = 0$. As at some position we can have a unique particle, we will *identify* (label) the particle with its position x_0 at initial time. Then, let $x = x(t; x_0)$ be the position at time t of the particle x_0, which we will abbreviate by $x = x(t)$. We have by Newton's law $x''(t) = 0$, so $x(t) = x_0 + \vec{v}\,t$ where $\vec{v} = u(x_0, 0) = u(x(t), t)$ for any $t \geq 0$ because the velocity of particle x_0 is constant.

Next we can resume V.I. Arnold's point of view to explain the meaning of the 'wave-particle duality' ([2], pp.2-3).

Particle description: given $u(x, 0) = u_0(x)$, the motion of our collection (physical system) of particles is full described by an *infinite* set of Ordinary Differential Equations (ODEs). One equation for each particle $x_0 \in \mathbb{R}$, concerning its position $x(t)$ along time,

$$\begin{cases} x'(t) = u_0(x_0), \quad t > 0 \\ x(0) = x_0 \end{cases} . \tag{1.1}$$

Equivalently, if we can assume that for each (x, t) there exist some line given by (1.1) such that $(x, t) = (x(t), t)$, we can describe the physical system otherwise. By the definition of our function u, for any time t, $u(x(t), t) = x'(t)$. Then

$$0 = x''(t) = \frac{d}{dt}\big(u(x(t), t)\big) = x'(t)\, u_x(x(t), t) + u_t(x(t), t) = uu_x + u_t, \text{ for any } x \in \mathbb{R} \text{ and } t > 0.$$

Wave description: given $u(x, 0) = u_0(x)$, the motion of our physical system is full described by a *single* pde, concerning the field of velocities $u(x, t)$ along position and time,

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R},\, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} , \tag{1.2}$$

where the pde (the Burgers' equation with $\varepsilon = 0$) is the Euler's momentum equation for the motion.

So (1.1) and (1.2) are two equivalent descriptions of the motion, the former using Newton's equation for particles and the later using Euler's equation for the velocity field ("momentum").

In (1.2), from the given initial datum $u_0(\cdot)$ (i.e., the condition at initial time $t = 0$) and using the pde we ask to know what is the evolution of $u(\cdot, \cdot)$ for time $t > 0$. We say that (1.2) is an *initial value problem*[2].

Coming back to our system of particles moving in a straight line, suppose now that density (or concentration) of particles is "large". Then particles begin to interact. This slow down our

[2]If instead of $x \in \mathbb{R}$ we were considering $x \in [a, b]$, then we should provide together with the *initial datum* also *boundary data* at $x = a$ and $x = b$: $u(a, t) = a(t)$ and $u(b, t) = b(t)$, for given functions $a(\cdot)$ and $b(\cdot)$. We would name that problem an *initial-boundary value problem.*

flow. We are observing an effect of viscosity which we need to take into account in the model (pde). For a motion with small variation on u_x, it can be done introducing a term proportional to u_{xx} in the equation:

$$\begin{cases} u_t + uu_x = \varepsilon\, u_{xx}, & x \in \mathbb{R},\, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} , \tag{1.3}$$

where $\varepsilon > 0$ is the viscosity coefficient (and the pde is the Burgers' viscid equation).

Both inviscid (1.2) and viscid (1.3) Burgers' equations rules phenomena changing with time and such equations are said evolutionary equations, in opposition to stationary equations (or elliptic pdes). Still, they are quite different. Actually (1.2) is the paradigm of a nonlinear hyperbolic problem and (1.3) that of a parabolic one. To study the inviscid and the viscid Burgers' equations we need use different methods, we will use the Method of Characteristics to study the first and the Cole-Hopf Transformation to study the second.

1.2 The purpose of the study

There are many interesting problems in Burgers' equation. Because of this reason, I would like to work on this equation, namely to:

1. Study the idea/concept and technique/method of how to solve Burgers' equation by the Method of Characteristics and the Cole-Hopf transformation.
2. Study the occurrence Burgers' equation in two different areas, namely on a Traffic model and on a Coagulation equation.

1.3 Scope of the study

1. Scope of knowledge

   In my study of Burgers' equation the focus will be on the role of changes of variables and its consequences either in the context of the Method of Characteristics and in the Cole-Hopf transformation. I shall need to study evolution equations (hyperbolic and parabolic PDE, as well as ODE) using tools from classical Mathematical Analysis, including results from vector calculus, multivariable Analysis, and Laplace transform.

   To illustrate some applications of Burgers' equation, I study the occurrence of Burgers' equation in a Traffic model and in the analysis of a Coagulation model.

2. Time duration since September, 2016 to June, 2017.

3. Place of research activity Department of mathematics, Faculty of Natural Sciences, National University of Laos, 2016 – 2017.

1.4 Related Research

1. Mikel Landajuela, in [14], review results on Burgers equation in its Viscous and non-Viscous versions. Some applications of this paradigmatic equation are given, including some traffic models, and several numerical schemes for solving these equations are presented.

2. The book of Mark Holmes, [10], has a well explained chapter on the application of Burgers' equation to several traffic flow situations, with reference to current research literature.

3. Robert Pego, in [17], presents several mathematical models of coarsening, among them the coagulation equations, and illustrate the use of several methods, including Laplace transforms, in their study.

1.5 Research Methodology

In this thesis we study Burgers' equations using some analytical methods that we briefly describe in the subsections below.

1.5.1 The Method of Characteristics

If we have to solve the initial value problem to the inviscid Burgers' equation (Burgers' equation, for shortness)

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases},$$

? we can reverse the argument above and reduce our problem to the solution of a system of odes.

To proceed we just need to get the (characteristic) lines defined by $x'(t) = u(x(t),t)$ supposing that u is a solution of Burgers' equation. Then, over these lines, Burgers' equation show us that

$$0 = u_t(x(t),t) + u(x(t),t)\, u_x(x(t),t) = u_t(x(t),t) + x'(t)\, u_x(x(t),t) = \frac{d}{dt}u(x(t),t),$$

meaning that over that lines u is constant, say $\forall t > 0 \;\; u(x(t), t) = u(x(0), 0) = u_0(x(0))$, using the initial datum. Thus, we conclude u is constant along the lines defined by $x'(t) = u(x(t), t)$. Those lines are straight lines. Now we 'see' the problem is solved, geometrically, provided that for each given point $(x, t) \in \mathbb{R} \times (0, +\infty)$ we can get one such straight line coming from some x_0 point in the x axis (so $x_0 = x(0)$).

Moreover we see that, if the initial datum $u_0(\cdot)$ is not a nondecreasing function, then there exist $x_1 < x_2$ such that $u_0(x_1) > u_0(x_1)$, and the straight lines going up from these points x_1 and x_2 in the x axis will intersect in some point, say (x_*, t_*). By the previous construction we should have $u_0(x_1) = u(x_*, t_*) = u_0(x_2)$, which for a given initial datum u_0 is in general not possible. This means that at some finite time the solutions of our hyperbolic initial value problem will in general develop discontinuities (shocks), despite how many regular the initial datum could be.

In Chapter 2 we will develop in full generality a theory of this Method of Characteristics.

1.5.2 The Cole-Hopf Transformation

For the viscid Burgers' equation

$$\begin{cases} u_t + uu_x - \varepsilon\, u_{xx} = 0, & \varepsilon > 0, \quad x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1.4}$$

the solutions do not develop discontinuities. Actually, the viscosity term $\varepsilon\, u_{xx}$ introduces in the equation a dissipative effect which provides extra regularity to the solutions. This should be not surprising if we consider that using a change of variables, the so called *Cole-Hopf Transformation*, the viscid Burgers' equation is converted into the Heat equation where the dissipative term models the heat diffusion. We will study the Cole-Hopf Transform in Chapter 3.

?

1.6 Two applications of Burgers' Equations

Perhaps the most elementary example of application of both types of Burgers' equations is given by some simple models of Traffic Flow which have been introduced independently by Lighthill-Whitham [15] and Richards [18]. We will have a glimpse of Traffic Flow in Chapter 4.

Another example of the appearance of Burgers' equation is in coagulation theory. Smoluchowski's coagulation equation is an integro-differential equation modelling the growth of clusters by coagulating reactions in which a cluster of size x and another of size y coagulate at a rate $a(x, y)$ to give one of size $x + y$. When the rates are of the so called solvable cases, i.e., when $a(x, y) = 1,\ x + y$ or xy, the application of Laplace transform to the coagulation equations results in ordinary or partial differential equations [16, 17]. It so happens that in the last two cases the pde satisfied by the Laplace transform is a Burger's equation. We shall present an introductory bird's eye view of these results in Chapter 4.

Chapter 2

The Method of Characteristics

2.1 The Advection Equation

Let us see the role of *characteristic lines* in the simplest case possible, the initial value problem for the advection equation ($a \in \mathbb{R}$ is a constant, $u_0(\cdot)$ is the given initial datum, at $t = 0$)

$$\begin{cases} u_t + au_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} . \tag{2.1}$$

Using the standard inner product we can rewrite the pde, the (linear) advection or transport equation, as

$$(u_x, u_t) \cdot (a, 1) = 0 \iff \nabla u \perp \vec{a}\,, \tag{2.2}$$

where $\nabla u(x,t) = (u_x(x,t), u_t(x,t))$ is the gradient of u at (x,t) and $\vec{a} = (a,1)$. Now, if for $c \in \mathbb{R}$ we define the c-Level curve of a function u by

$$L_c(u) = \{(x,t) \in \mathbb{R} \times [0, +\infty) : \;\; u(x,t) = c\}\,,$$

remember from calculus that at regular points of the function u we have

$$\nabla u(x,t) \perp L_c(u)(x,t)^1. \tag{2.3}$$

From (2.2) and (2.3), because we are in the plane, we conclude that *each* level curve $L_c(u)$ is parallel to $\vec{a}$ at *each* of its points (x,t). Thus the $L_c(u)$ curves are straight lines and moreover they are all parallel to each other, having the common direction of $\vec{a} = (a,1)$, $L_c(u) \,//\, \vec{a}$. They

[1] We mean: if (x,t) is a point over the line $L_c(u)$, then the gradient vector of u at this point (x,t), $\nabla u(x,t)$, is perpendicular to the line $L_c(u)$ in that point. Also, we say that $\vec{v}$ is perpendicular (or parallel) to $L_c(u)$ at (x,t), notation $\vec{v} \perp L_c(u)(x,t)$ (or $\vec{v} \,//\, L_c(u)$), if $(x,t) \in L_c(u)$ and the vector $\vec{v}$ is perpendicular (or parallel) to the tangent line or tangent vectors of $L_c(u)$ at (x,t).

are a bundle of straight lines covering the half-plan $t \geq 0$ (with slope $1/a$).

For an arbitrary (x, t) where $t > 0$, let $(x_0, 0)$ be the point of intersection with the x-axis of the straight level line passing by (x, t). This straight line is described by the equation $x_0 = x - at$ (see Fig. 2.1) and because it is a level curve $L_c(u)$ (see Fig. 2.2),

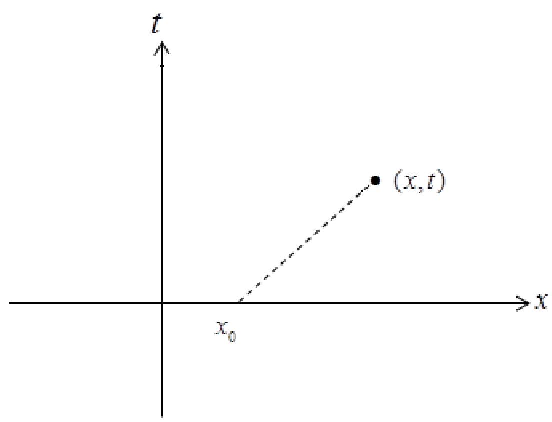

Figure 2.1: Abscissa for a given (x, t)

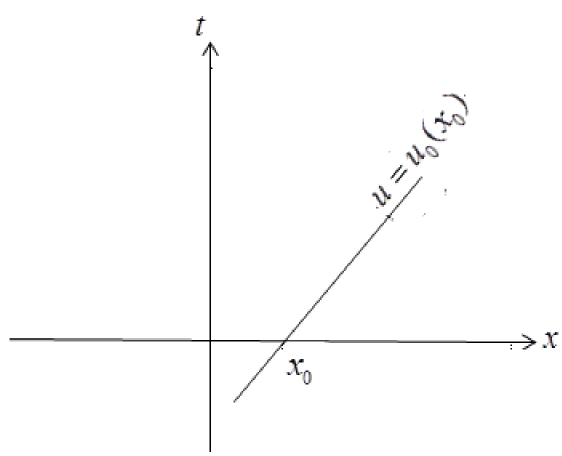

Figure 2.2: Characteristic line $(a > 0)$

using the initial condition $u(x_0, 0) = u_0(x_0)$:

(2.4) $$u(x, t) = c = u(x_0, 0) = u_0(x_0) = u_0(x - at).$$

It is now easy to verify that this u defined by $u(x,t) = u_0(x - at)$ is the solution of problem (2.1), if the given initial datum u_0 is regular enough (if it has first derivative).

Particle description: the bundle of those straight level lines is carrying up (into $t > 0$) the initial signal, u_0. They are characteristic curves (Fig. 2.2). They move at speed $|a|$ to the right if $a > 0$ or to the left if $a < 0$, cf. Fig. 2.3.

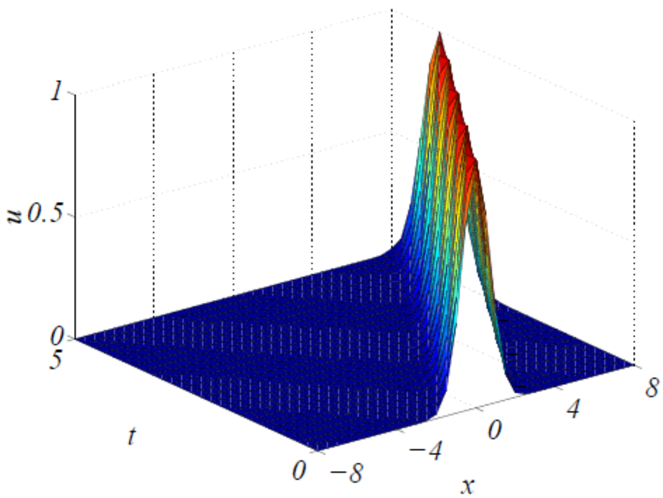

Figure 2.3: Plot of the solution $u(x,t)$ with $a = 2$ and $u_0(x) = e^{-x^2}$

Wave description: if we fix the time $t = t_1$ in the solution $u(x,t)$, then the graphic of $u(x,t_1) = u_0(x - at_1)$ is the graphic of u_0 translated of at_1 to the right (supposing $a > 0$). Our solution $u(x,t)$ describes an initial wave (graphic of u_0) moving with velocity a (Fig. 2.3).

In this example because the advection equation is linear the characteristic lines moves parallel and the initial wave keep its form along its evolution (with time). For nonlinear equations, as in the case of the inviscid Burgers' equation, this is no more true and the characteristic lines, at finite time, can provide "contradictory information": the initial wave (graphic of u_0) along its evolution is deforming until a critical time where its graphic is no more the graphic of a function (at that time, in some x_1 the u should assume more than one value), see Section 2.4.

Meanwhile in the following two sections we will present the general theory of the Method of Characteristics, first for linear equations (actually for semi-linear equations the theory is similar) and then for nonlinear equations of quasi-linear type.

2.2 Linear Equations

We will consider the following first-order linear equation

$$a(x,t)u_x + b(x,t)u_t = c(x,t)\,. \tag{2.5}$$

Suppose we can find a solution $u(x,t)$. Consider the graph of this function given by

$$S \equiv \{(x,t,u(x,t)) : x \in \mathbb{R}, t \in \mathbb{R}_+\}\,.$$

At each point (x,t) we can write (2.5) equivalently as the perpendicularity relationship:

$$(a(x,t), b(x,t), c(x,t)) \,.\, (u_x(x,t), u_y(x,t), -1) = 0, \tag{2.6}$$

where the dot "$\cdot$" represents the standard inner product in $\mathbb{R}^3$. But, recall from calculus, a normal vector to the surface S at any of its points $(x,y,u(x,y))$ is given by $N(x,t) = (u_x(x,t), u_t(x,t), -1)$, where $(u_x(x,t), u_t(x,t)) \equiv \nabla u(x,t)$ is the gradient of u at (x,t). Therefore (2.6) tell us that the vector $(a(x,t), b(x,t), c(x,t))$ lies in the tangent plane to S at the point determined by (x,t). Consequently, to find a solution to (2.5), we will look for a surface

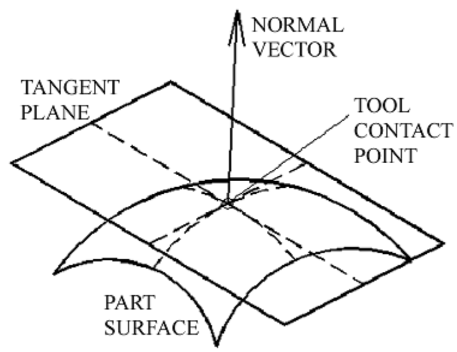

Figure 2.4: The normal vector and the tangent plane at a point of surface S.

S such that at each point (x,t,u) on S the vector $(a(x,t), b(x,t), c(x,t))$ lies in the tangent plane to S. How do we construct such a surface?

We start by looking for curves which lies in S. We want the vector $(a(x,t), b(x,t), c(x,t))$ to lie in the tangent plane to our surface S at each point (x,t,u) on the surface. Therefore, let's start by constructing a curve $C = \{(x(s), t(s), u(s))\}$ parametrized by s such that at each point on the curve C the vector $(a(x(s),t(s)), b(x(s),t(s)), c(x(s),t(s)))$ is the tangent to the curve. In particular, the curve C will satisfy the following system of odes

$$\begin{cases} \frac{dx}{ds} = a(x(s),t(s)) \\ \frac{dt}{ds} = b(x(s),t(s)) \\ \frac{du}{ds} = c(x(s),t(s)) \end{cases} , \tag{2.7}$$

because the vector $\left(\frac{dx}{ds}, \frac{dt}{ds}, \frac{du}{ds}\right)$ is also tangent to the curve and it lies in the tangent plane. Such a curve C is known as an *integral curve* for the vector field $(a(x,t), b(x,t), c(x,t))$.

For a pde of the form (2.5) with associated vector field $V = (a(x,t), b(x,t), c(x,t))$, we look for its integral curves defined by (2.7), called the *characteristic curves* for (2.5), where the equations in (2.7) are known as the characteristic equations for (2.5).

Once we have found the characteristic curves the goal is to construct a solution of (2.5) by forming a surface S as a union of these characteristic curves. A surface S for which the vector field $V = (a(x,t), b(x,t), c(x,t))$ lies in the tangent plane to S at each point (x,t,u) on S is known as an integral surface for V.

In effect, introducing these characteristic equations, we have reduced our partial differential equation to a system of ordinary differential equations. We can use ode theory to solve the system of characteristic equations, then piece together these characteristic curves to form a surface. Such a surface will provide us with a solution to the pde.

Partial differential equations such as (2.5) are usually provided with additional conditions, usually named *initial* or *boundary* conditions, which consists in known values of the solution $u(x,t)$ over a 1-dimensional manifold (i.e, a curve) Γ in the domain of u (a subset of $\mathbb{R}^2$).

Consider the case $\Gamma = \{\gamma(r) = (\gamma_1(r), \gamma_2(r)) : \ r \in \mathbb{R}\}$, where γ is some given differentiable parametrization of Γ. Let us prescribe our data in Γ by imposing that at each point (x,t) of Γ the value of $u(x,t)$ is given by a prescribed function u_0: $u|_\Gamma = u_0$.

Let $\overline{\Gamma} = \{(\gamma_1(r), \gamma_2(r), u_0(\gamma_1(r), \gamma_2(r))\}$ be the curve in $\mathbb{R}^3$ determined by Γ and the prescribed data u_0. Now, if the solution to (2.5) must satisfy $u|_\Gamma = u_0$ we need to construct characteristic curves emanating from $\overline{\Gamma}$, that is, looking for the solution of the characteristic system

$$(2.8) \qquad \begin{cases} \frac{dx}{ds} = a(x(r,s), t(r,s)) \\ \frac{dt}{ds} = b(x(r,s), t(r,s)) \\ \frac{du}{ds} = c(x(r,s), t(r,s)) \end{cases}$$

satisfying the initial condition at $s = 0$

$$\begin{cases} x(r,0) = \gamma_1(r) \\ t(r,0) = \gamma_2(r) \\ u(r,0) = u_0(\gamma_1(r), \gamma_2(r)) \end{cases}$$

Figure (2.5) provides the geometric interpretation of what has just been said.

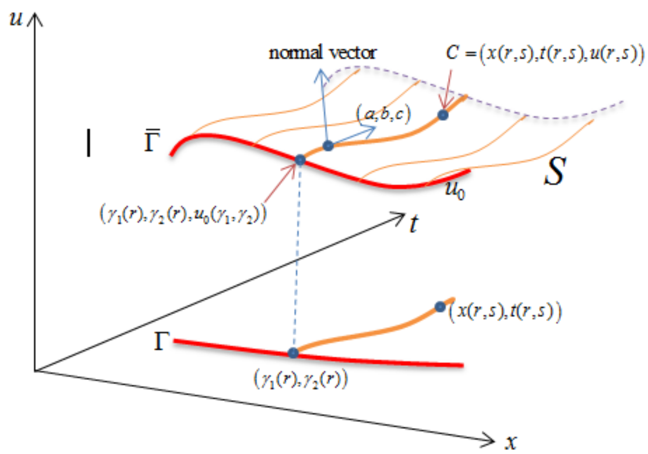

Figure 2.5: Relation between the boundary line Γ, the boundary condition u_0, and the integral curve C and integral surface S of (2.5) satisfying $u|_\Gamma = u_0$

By the theory of odes the system (2.8) has a unique solution

$$\begin{cases} x = x(r,s) \\ t = t(r,s) \\ u = u(r,s) \end{cases}$$

which satisfies the initial condition.

In defining the characteristic equations, we defined $u(r,s) = u(x(r,s), t(r,s))$. If we can find some function H such that $(r,s) = H(x,t)$, then we will have found the unique solution of (2.5)

$$\tilde{u}(x,t) = u(H(x,t))\,.$$

A condition that guarantee the existence of such a function H as above is the inverse function theorem (in dimension 2) that we now state (the proof can be found in [12, page 140])

Theorem 2.2.1 *(Inverse Function Theorem) Assume $G : U \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ is a C^1 function and $JG(\vec{x}_0) \neq 0$ where $JG(\vec{x}_0)$ is the Jacobian of G at the point $\vec{x}_0$. Then there exists an open set $V \subset U$ with $\vec{x}_0 \in V$ and an open set $W \subset \mathbb{R}^n$, with $\vec{z}_0 = G(\vec{x}_0) \in W$ such that*

$$G : V \longrightarrow W$$

is one to one and onto, and the inverse function is C^1

$$H = G^{-1} : W \longrightarrow V\,.$$

2.3 Quasi-Linear Equations

The general form of quasi-linear pdes is

$$a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} = c(x,t,u) \tag{2.9}$$

where a, b, c are functions of x, t, u. The initial condition $u(x, 0)$ is specified at $t = 0$

$$u(x,0) = f(x)\,. \tag{2.10}$$

The geometry of the problem was already explained in the linear case, the difference is that now the functions a, b, c depend on u, the solution itself.

We represent the characteristic curve parametrically by

$$x = x(r,s), \quad t = t(r,s), \quad u = u(r,s),$$

where r labels the points where we start on the initial curve. So, the parameter s tells us how far along the characteristic curve we are from that start point, while r governs the evolution along the initial curve.

From (2.9), at each point (x, t) a particular tangent vector to the solution surface $z = u(x, t)$ is

$$(a(x,t,u), b(x,t,u), c(x,t,u))\,.$$

Given any curve in the variable s, $x = x(r, s), t = t(r, s), u = u(x, s)$, with parameter r (r acts as a label only), its tangent vector is given by

$$\left(\frac{\partial x}{\partial s}, \frac{\partial t}{\partial s}, \frac{\partial u}{\partial s}\right).$$

For a general curve on the surface $z = u(x, t)$, the tangent vector (a, b, c) will be in general different than the tangent vector at $(x, t, u(x, t))$. However, we choose our curves with variables (s, r) such that $(x = x(r, s), t = t(r, s), u = u(r, s))$ have tangent equal to (a, b, c):

$$\begin{cases} x_s = a(x,t,u) \\ t_s = b(x,t,u) \\ u_s = c(x,t,u) \end{cases} \tag{2.11}$$

where (a, b, c) depends on (x, t, u). We have written partial derivatives to denote differentiation with respect to s, since x, t, u are functions of both r and s. However, since only derivatives in s are present in (2.11), these equations are odes. This has greatly simplified our task: we have reduced the solution of the pde to solve a system of odes.

Now, we need fix the initial values at $s = 0$ for the odes (2.11). We are free to choose the value of t, we take $t(r, 0) = 0$. Since x changes with r, we choose r to denote the initial value of $x(r, s)$ along the $x-$axis (when $t = 0$) in the space time domain. Thus the initial values (at $s = 0$) are

$$\begin{cases} x(r,0) = r \\ t(r,0) = 0 \\ u(r,0) = f(r) \end{cases} . \tag{2.12}$$

Using the odes theory, we are able to find a unique (local) solution to (2.11). As long as we can invert the function $G(r, s) = (x(r, s), t(r, s))$, we can find a solution of (2.9) given by

$$u(x, t) = z(r(x, t), s(x, t)) \, .$$

For this we can use, again, the inverse function theorem (5.0.2).

2.4 The Inviscid Burgers' Equation

Consider Burgers' equation (1.2) with initial value problem $\phi(x)$. From (1.2) notice that we can write this equation, in standard form as

$$u_t + \left[\frac{u^2}{2}\right]_x = 0$$

In trying to solve this equation using the method of characteristics, our characteristic equations are given by

$$\begin{cases} \dfrac{dt}{ds} = 1 \\ \dfrac{dx}{ds} = z \\ \dfrac{dz}{ds} = 0 \end{cases}$$

with initial conditions

$$\begin{cases} t(r,0) = 0 \\ x(r,0) = r \\ z(r,0) = \phi(r) \end{cases}$$

we see that the solution is given by

$$\begin{cases} t = s \\ x = \phi(r)s + r \\ z = \phi(r) \end{cases}$$

From these solutions, we arrive at an implicit solution for (1.2) as

$$u = \phi(x - ut)$$

We consider the projected characteristic curve, which are given by $x = \phi(r)s + r$ and $t = s$, which implies $x = \phi(r)t + r$. Now suppose the initial data ϕ satisfies the following. Suppose there is an $r_1 < r_2$ such that $\phi(r_1) > \phi(r_2)$. The projected characteristic curves interest at some point (x_0, t_0). What does this mean? Look again at the characteristic equation. In particular, the solution u satisfies $\frac{du}{ds} = 0$. This mean u is constant along characteristic curves. Therefore, $u(x_0, t_0) = u(r_1, 0) = \phi(r_1)$. But, also $u(x_0, t_0) = u(r_2, 0) = \phi(r_2)$. But by assumption $\phi(r_1) > \phi(r_2)$. Therefore, we get a contradiction! We get a singularity formation at some time t. For example, consider (1.2) with initial data $\phi(x) = e^{-x^2}$. For $x > 0$,

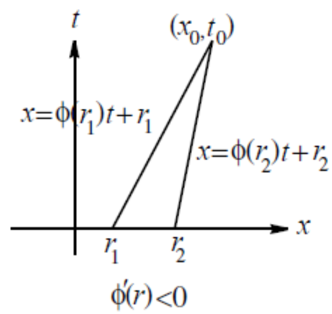

Figure 2.6: Plot the characteristic curve intersect at some time t

$\phi'(x) < 0$. Therefore, $\phi(r_1) > \phi(r_2)$ for $0 < r_1 < r_2$. Consequently, as described above, the projected characteristics will cross. What does the solution look like? As show in Figure 2.7 below, the taller part of the wave will overtake the shorter part of the wave, causing the wave to break. At a time T the wave breaks, the "solution"' u will cease to be a function, taking on multiple values, thus leading to a singularity in the solution. Of course, a function can't take on multiple values. Consequently, the wave pictured below can't be a solution of (1.2) after the time T when the wave breaks. if $\phi'(x) \geq 0$, projected characteristic curves will not intersect, so there will be no conflict in defining u. For example, consider (1.2) with the initial data $\phi(x) = 5\arctan(x)$. In the movie below, we see that the wave do not break and we have a smooth solution.

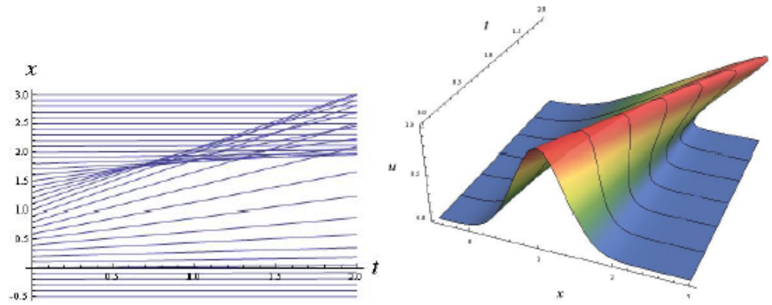

Figure 2.7: Break up of the solution (1.2) after time T when the wave breaks (from [14]).

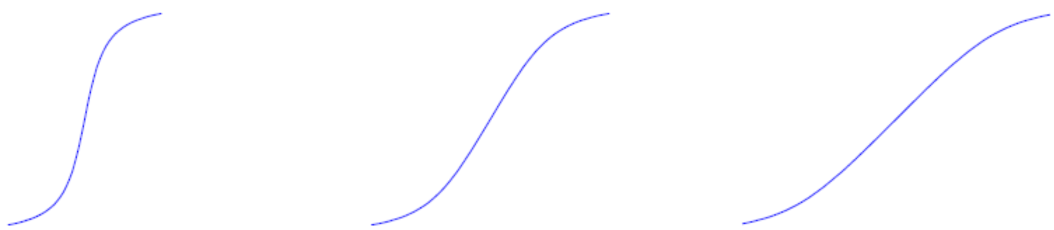

Figure 2.8: Solution smooth when the projected characteristic curve will not intersect

In the case when the projected characteristic curves do not intersect, we will not have a conflict in defining our solution u. However, it is possible that we will not have enough information to define u everywhere and in this case additional conditions (entropy condition) need to be given.

Chapter 3

The Cole-Hopf Transformation

In this chapter we will consider the initial value problem for the viscous Burgers' equation (i.e., the Burgers' equation with a ε-viscosity term in the right-hand side):

$$\begin{cases} u_t + uu_x = \varepsilon u_{xx}, & \varepsilon > 0, \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x), & \qquad\quad\; x \in \mathbb{R} \end{cases} . \tag{3.1}$$

It is no longer an hyperbolic equation we are considering and the Method of Characteristics is no more applicable. It is a parabolic equation and an important model in the study of this type of nonlinear pdes (cf. the Introduction). Rather surprisingly, it can be solved analytically. In fact, via the so called *Cole-Hopf transformation* (a change of the dependent variable) the nonlinear viscous Burgers' equation is converted into the linear heat (or diffusion) equation and to solve (3.1) it is enough to solve the linear heat initial value problem

$$\begin{cases} v_t = \varepsilon v_{xx}, & x \in \mathbb{R}, t > 0 \\ v(x,0) = v_0(x), & x \in \mathbb{R} \end{cases} , \tag{3.2}$$

which can then be solved by a variety of methods such as change of variables or Fourier transform.

Now, as $\varepsilon \searrow 0$ the viscous Burgers' equations limit is the inviscid Burgers' equation. And it is natural to wait the solutions of (3.1) converge, as $\varepsilon \searrow 0$, to the solution of the inviscid Burgers' problem. This is the *vanishing viscosity method* to solve the inviscid problem (see Whitham [21]). This was the idea of Bateman [3, page 165, left column] in 1915 and this is a source of todays research if we consider not just dissipation but also dispersion effects in the models (see Bedjaoui-Correia-Mammeri [4]). We will not consider this issue in this thesis.

In this chapter we start in section 3.1 by studying the Cole-Hopf transformation, independently discovered by Cole [7] and Hopf [11] (while we will follow Salsa book [19]) and in section 3.2 we will solve the resulting heat equation by using dimensionless variables that transform the problem into an ode. Then in section 3.3 we state the solution of problem (3.1).

3.1 The Cole-Hopf Transformation

We can rewrite the viscous Burgers' equation in the conservative (or divergence) form:

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2 - \varepsilon u_x \right) = 0 \iff \mathrm{Curl}(-u, \frac{1}{2}u^2 - \varepsilon u_x) = 0\,.$$

The planar vector field $(-u, \frac{1}{2}u^2 - \varepsilon u_x)$ is *Curl-free* and therefore, there exist some scalar potential $\psi = \psi(x,t)$ such that

$$\psi_x = -u\,, \quad \psi_t = \frac{1}{2}u^2 - \varepsilon u_x\,.$$

From the first equation it follows that $\psi_{xx} = -u_x$ and, substituting in the second equation, we conclude that ψ must verify the equation

$$\psi_t = \frac{1}{2}\psi_x^2 + \varepsilon \psi_{xx}\,. \tag{3.3}$$

Next we try to get rid of the quadratic term by an adequate change of variable letting $\psi = g(v)$, with g to get be chosen. We have

$$\psi_t = g'(v)v_t\,, \qquad \psi_x = g'(v)v_x\,, \qquad \psi_{xx} = g''(v)(v_x)^2 + g'(v)v_{xx}$$

and substituting these into (3.3) we find

$$\begin{aligned} & g'(v)v_t = \frac{1}{2}\left(g'(v)v_x\right)^2 + \varepsilon\left(g''(v)(v_x)^2 + g'(v)v_{xx}\right) \iff \\ & g'(v)(v_t - \varepsilon v_{xx}) = \left[\frac{1}{2}(g'(v))^2 + \varepsilon g''(v)\right](v_x)^2 \iff \\ & v_t - \varepsilon v_{xx} = 0\,, \end{aligned}$$

if we choose g such that $\frac{1}{2}(g'(v))^2 + \varepsilon g''(v) = 0$ which is an ode of separable type with solution $g(v) = 2\varepsilon \log(v)$ (where $g'(v) \neq 0$). Thus, our change of variable is $\psi = 2\varepsilon \log(v)$.

We resume. If we know how to solve the heat equation

$$v_t - \varepsilon v_{xx} = 0\,,$$

then we know how to solve the viscid Burgers' equation (as $u = -\psi_x$):

$$u = -2\varepsilon \left(\log(v)\right)_x = -2\varepsilon \frac{v_x}{v}. \tag{3.4}$$

Moreover, using the (3.1) initial datum and from (3.4) we get the equivalent initial datum $v_0(x)$ for the heat equation:

$$u_0(x) = -2\varepsilon \frac{d}{dx}\left(\log(v_0(x))\right) \iff v_0(x) = \exp\left(\int_0^x -\frac{1}{2\varepsilon}\, u_0(z)dz \right). \tag{3.5}$$

Conclusion: to solve (3.1) it is sufficient to solve (3.2) with the particular initial datum v_0 of the form given in (3.5).

Finally we want to remark that this method is useful when $\varepsilon > 0$, just to solve the parabolic problem. To solve the hyperbolic problem, for $\varepsilon = 0$, we must use some other method (as the Method of Characteristics). Still, we want to point out that in the limit as $\varepsilon \searrow 0^+$ the solutions of problem (3.2) converge to the solution of problem (3.1), see [21].

3.2 The Heat Equation

In this section we are going to solve the initial value problem (3.2). There are various methods to solve the linear equation $v_t - \varepsilon v_{xx} = 0$ on the real line $\mathbb{R}$. To keep with the main underlying tool on this thesis, which is the use of transformation of variables, we will analyse the heat (or diffusion) equation using adequate changes of variables.

The linear homogeneous diffusion equation has simple but revealing physical properties and there are special particular solutions which can be used to generate many other solutions. Those special solutions are constructed identifying particular natural/physical changes of variable that leave the equation itself unchanged and because of this these changes of variable are called *invariant transformations*.

Changing scales: let $u = u(x,t)$ be a solution in the half-space $\mathbb{R} \times (0, +\infty)$ of

$$u_t - \varepsilon u_{xx} = 0 \tag{3.6}$$

and consider the linear transformations defined by

$$x \mapsto ax, \quad t \mapsto bt, \quad u \mapsto cu \qquad (a, b, c > 0)\,,$$

which represents dilation-contractions of the graph of u, changes in the scale of each axis. Then, let us check for which values of a, b, c

$$u^*(x,t) = c\, u(ax, bt)$$

is still a solution of (3.6): $u^*_t(x,t) = cb\, u_t(ax, bt)$ and $u^*_{xx}(x,t) = ca^2\, u_{xx}(ax, bt)$, so we get

$$u^*_t - \varepsilon u^*_{xx} = cb\, u_t - ca^2\, \varepsilon u_{xx} = 0$$

if $b = a^2$. This relation suggests the name of *parabolic dilation* for the transformation

$$x \mapsto ax, \quad t \mapsto a^2 t \qquad (a > 0).$$

We notice that under such transformations the ratio $\frac{x}{\sqrt{\varepsilon t}}$ remains unchanged:

$$\frac{ax}{\sqrt{\varepsilon(a^2 t)}} = \frac{ax}{|a|\sqrt{\varepsilon t}} = \frac{x}{\sqrt{\varepsilon t}} \tag{3.7}$$

and because of the physical dimensions of the parameter ε this ratio is dimensionless, see [19].

Conservation (of mass or energy): again, let $u = u(x,t)$ be a solution of (3.6) in $\mathbb{R} \times (0,+\infty)$. We just saw that the functions

$$u^*(x,t) = c\, u(ax, a^2 t) \quad (a, c > 0)$$

are also solutions of (3.6) in $\mathbb{R} \times (0,+\infty)$. Now, suppose u satisfies the condition, for some $q \in \mathbb{R}$ (e.g., for $q = \int_{\mathbb{R}} u_0(x)dx$),

$$\forall t > 0 \qquad \int_{\mathbb{R}} u(x,t)dx = q, \tag{3.8}$$

which stands for conservation of the integral measurement along time. If, for instance u represents concentration of a substance (density of mass), equation (3.8) states that the total mass remains equal to q for every time t. Or if u is temperature, (3.8) says that the system keep its total internal energy constant along time. Then we ask for which a, c the solution u^* still satisfies (3.8)?

$$\int_{\mathbb{R}} u^*(x,t)dx = c \int_{\mathbb{R}} u(ax, a^2 t)dx$$

and letting $y = ax$, $dy = a\, dx$, we have

$$\int_{\mathbb{R}} u^*(x,t)dx = ca^{-1} \int_{\mathbb{R}} u(y, a^2 t)dy = ca^{-1} q\,.$$

Thus, for (3.8) to be satisfied we must have $c = a$. In conclusion, if $u = u(x,t)$ is a solution of (3.6) in $\mathbb{R} \times (0,+\infty)$ satisfying (3.8), the same is true for

$$u^*(x,t) = a\, u(ax, a^2 t) \quad (a > 0). \tag{3.9}$$

Think about our solution u of problem (3.2) as the concentration of a substance of initial total mass $q = \int_{\mathbb{R}} u_0(x)dx$ in a diffusion process which conserves the mass of that substance (during the process we do not loose mass, e.g., often by some change of phase as precipitation or evaporation we can loose mass). Then the relevance of (3.7) relies on the following argument. Fix, arbitrarily, a point $(x,t) \in \mathbb{R} \times (0,+\infty)$. In (3.9) take $a = \left(\sqrt{\varepsilon t}\right)^{-1}$, we have

$$u^*(x,t) = \frac{1}{\sqrt{\varepsilon t}} u\left(\frac{x}{\sqrt{\varepsilon t}}, \frac{1}{\varepsilon}\right)$$

which means that (as ε is fixed) u is a function of a single variable, say $U_\varepsilon(\xi)$, with $\xi = \frac{x}{\sqrt{\varepsilon t}}$. This variable ξ is dimensionless, meaning that $\sqrt{\varepsilon t}$ has length dimension. So $\frac{q}{\sqrt{\varepsilon t}}$ has concentration dimension and in

$$q\, u^*(x,t) = \frac{q}{\sqrt{\varepsilon t}}\, U_\varepsilon\left(\frac{x}{\sqrt{\varepsilon t}}\right) \tag{3.10}$$

neither the function U_ε nor its variable ξ have physical dimensions. We are now working with mathematical scale free solutions of the diffusion equation and of course, because the equation is linear, the expression in (3.10) is still a solution. We want to determine $U_\varepsilon = U_\varepsilon(\xi)$.

U_ε is related to the solution u^* by (3.10). So because u^* is a concentration we require $U_\varepsilon \geq 0$ and the total mass conservation yields, after the change of variable $x \mapsto \sqrt{\varepsilon t}$,

$$q = \int_{\mathbb{R}} u^*(x,t)dx = \frac{1}{\sqrt{\varepsilon t}} \int_{\mathbb{R}} U_\varepsilon \left(\frac{x}{\sqrt{\varepsilon t}} \right) dx = \int_{\mathbb{R}} U_\varepsilon \left(\xi \right) d\xi. \tag{3.11}$$

We keep the required mass conservation. Finally, we translate for U_ε what means that u^* is a solution of (3.6):

$$\begin{aligned}
u^*(x,t) &= \frac{1}{\sqrt{\varepsilon t}}\, U_\varepsilon(\xi) \quad \text{where } \xi = \frac{x}{\sqrt{\varepsilon t}}, \\
u_t^*(x,t) &= -\frac{1}{2t\sqrt{\varepsilon t}}\, U_\varepsilon(\xi) - \frac{x}{\sqrt{\varepsilon t}} \frac{1}{2t\sqrt{\varepsilon t}}\, U_\varepsilon'(\xi), \\
u_x^*(x,t) &= \frac{1}{\varepsilon t}\, U_\varepsilon'(\xi), \\
u_{xx}^*(x,t) &= \frac{1}{\varepsilon t\sqrt{\varepsilon t}}\, U_\varepsilon''(\xi), \\
u_t^* - \varepsilon u_{xx}^* &= -\frac{1}{t\sqrt{\varepsilon t}} \left(U_\varepsilon''(\xi) + \frac{1}{2}\xi U_\varepsilon'(\xi) + \frac{1}{2} U_\varepsilon(\xi) \right),
\end{aligned}$$

so U_ε must be a solution of the ode in $\mathbb{R}$

$$U_\varepsilon''(\xi) + \frac{1}{2}\xi U_\varepsilon'(\xi) + \frac{1}{2} U_\varepsilon(\xi) = 0. \tag{3.12}$$

Since $U_\varepsilon \geq 0$, (3.11) implies that

$$\lim_{\xi \to \pm\infty} U_\varepsilon(\xi) = 0$$

On the other hand, because (3.12) is invariant with respect to the change of variables $\xi \mapsto -\xi$, we look for even solutions $U_\varepsilon(-\xi) = U_\varepsilon(\xi)$. Then we can restrict ourselves to $\xi \geq 0$, with boundary conditions

$$U_\varepsilon'(0) = 0 = \lim_{\xi \to +\infty} U_\varepsilon(\xi). \tag{3.13}$$

To solve (3.12) observe that it can be written in the form

$$\frac{d}{d\xi} \left[U_\varepsilon'(\xi) + \frac{1}{2}\xi U_\varepsilon(\xi) \right] = 0$$

equivalent to

$$U_\varepsilon'(\xi) + \frac{1}{2}\xi U_\varepsilon(\xi) = c \quad (c \in \mathbb{R}). \tag{3.14}$$

Letting $\xi = 0$ in (3.14) and recalling (3.13) we deduce that $c = 0$ and therefore we have to solve the first order linear homogeneous ode

$$U'_\varepsilon(\xi) + \frac{1}{2}\xi\, U_\varepsilon(\xi) = 0$$

which has solution

$$U_\varepsilon(\xi) = c_0\, e^{-\frac{\xi^2}{4}} \quad (c_0 \in \mathbb{R}).$$

This function is even, positive, integrable and vanishes at infinity, it only remains to choose c_0 in order to ensure (3.11):

$$q = \int_{\mathbb{R}} U_\varepsilon(\xi)\, d\xi = c_0 \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}}\, d\xi$$

doing the change of variable $\xi = 2z$, we have

$$q = 2c_0 \int_{\mathbb{R}} e^{-z^2} dz = 2c_0\sqrt{\pi}$$

and thus $c_0 = \frac{q}{\sqrt{4\pi}}$.

Going back, we have found the following solution of (3.6)

$$u^*(x,t) = \frac{q}{\sqrt{4\pi\varepsilon t}}\, e^{-\frac{x^2}{4\varepsilon t}}, \quad x \in \mathbb{R}, t > 0. \tag{3.15}$$

The function

$$\Gamma_D(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\, e^{-\frac{x^2}{4\varepsilon t}}$$

is called the fundamental solution of equation (3.6) because it is proved [19] that the solution of problem (3.2) is given by

$$v(x,t) = \int_{\mathbb{R}} v_0(y)\, \Gamma_D(x-y,t) dy.$$

3.3 The Viscous Burgers' Equation

The initial value problem (3.2) has then the unique smooth integrable solution in the half-plane $t > 0$

$$v(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{+\infty} v_0(y)\, e^{-\frac{(x-y)^2}{4\varepsilon t}} dy.$$

Consequently, using (3.4) and (3.5), the problem (3.1) has a unique smooth solution in the half-plane $t > 0$ given by

$$u(x,t) = \frac{\int\limits_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left(-\frac{1}{2\varepsilon}\int\limits_0^y u_0(z)dz\right) \exp\left(-\frac{(x-y)^2}{4\varepsilon t}\right) dy}{\int\limits_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\varepsilon}\int\limits_0^y u_0(z)dz\right) \exp\left(-\frac{(x-y)^2}{4\varepsilon t}\right) dy}$$

which is continuous up to $t = 0$ at any continuity point of u_0, provided that (see [19])

$$\frac{1}{x^2}\int_0^x u_0(z)dz \to 0 \quad \text{as } |x| \to \infty. \tag{3.16}$$

Chapter 4

Applications

To exemplify the role of Burgers' equation in the mathematical modeling of different phenomena, in this chapter I will study two applications: in section 4.1 I present a model of traffic flow and we will see that the density of flowing objects obeys a Burgers' equation; in section 4.2 a model for the coagulation of particles will be presented and we will conclude that, under certain assumptions, the (modified) Laplace transform of those equations will result in a Burgers' equation.

In both applications that we shall consider, a full mathematical theory is still the subject of current mathematical research (see [8, 10, 14, 16, 17]).

4.1 A Traffic Model

There are many different situations in which we can speak about traffic flow. The simplest situations correspond to the flow of some entity along a one dimensional. The entity can be some macroscopic object, such as cars (Figure 4.1), or some microscopic or submicroscopic objects, such as cellules (Figure 4.2) or molecules (Figure 4.3).

Figure 4.1: Aerial view of cars moving on a highway [10, page 205].

Figure 4.2: Red blood cells flowing in an arteriol [10, page 206].

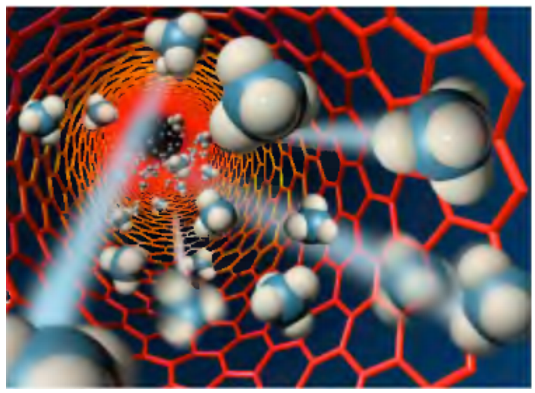

Figure 4.3: Computer illustration of methane molecules flowing though a carbon nanotube [10, page 206]

Although the underlying physics of each of these is quite different they all involve the movement of some entity along a one dimensional path. This simplicity is explicitly used when developing a mathematical model for the motion of these objects. In what follows we will consider the case of traffic flow of cars in a (single lane) street, but all can also be applied to other systems, such as the one dimensional motion of blood cells and molecules.

Several general modeling hypothesis are assumed to get from a real life situation, such as those in the figures 4.1–4.3, to a mathematical model of it. In this section we assume that the objects are numerous enough that it is not necessary to keep track of each one individually and we can use an averaged value.

4.1.1 Density

One of the most important variables in the model we shall present is the traffic density $\rho(x,t)$, that is, the number of cars per unit length at the position x and time t. In practice, to measure ρ at time $t = t_0$ and at the position $x = x_0$ along the highway we selects a small spatial interval of length $2\Delta x$ around that position, $x_0 - \Delta x < x < x_0 + \Delta x$, and then counts the number of cars within this interval. If Δx is small enough so that only cars in the immediate vicinity of x_0 are used to determine the density at this point, we have

$$\rho(x_0, t_0) \approx \frac{\text{number of cars from } x_0 - \Delta x \text{ to } x_0 + \Delta x \text{ at } t = t_0}{2\Delta x} \tag{4.1}$$

However, note that Δx cannot be so small that it is on the order of the length of individual cars (and the spacing between them), otherwise the above average would not make sense. In the continuum viewpoint, the cars are distributed smoothly over the entire x-axis, and the value of $\rho(x_0, t_0)$ is the limit of the right-hand side of (4.1) as $\Delta x \longrightarrow 0$.

To illustrate how density is determined we follow [10, pages 207-8]: suppose the simple case where cars all have length $\ell > 0$, and they are spaced a constant distance $d \geq 0$ from each other (see Figure 4.4).

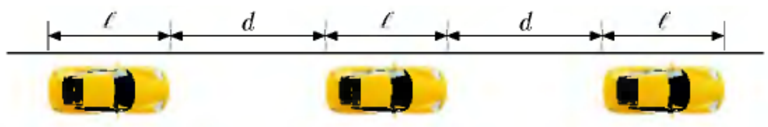

Figure 4.4: A queue of cars all the same length and evenly spaced (from [10, page 208]).

Given an interval $2\Delta x$ along the highway then the number of cars in this interval is, approximately, $2\Delta x/(\ell + d)$. Inserting this into (4.1) we find that

$$\rho = \frac{1}{\ell + d}$$

One conclusion that comes from this formula is that there is a maximum density: since the distance between cars is non negative and finite, $0 \leq d < \infty$, then $\ell + d > \ell$ and so $\rho = \frac{1}{\ell+d} < \frac{1}{\ell} =: \rho_M$. Note that ρ_M has a natural and easy interpretation: is the density that corresponds to all cars being aligned without any space between them (a situation that, naturally, in practical terms can only occur if the cars are not moving – see discussion in page 30 below).

4.1.2 Flux

The second variable we need is the flux $J(x, t)$, which has the dimensions of cars per unit time. To measure J at $x = x_0$ and $t = t_0$ we select a small time interval $t_0 - \Delta t < t < t_0 + \Delta t$ and counts the net number of cars that pass through $x = x_0$ during this time period, using the convention that a car moving to the right is counted as $+1$, while one moving to the left is counted as -1. Again here the underlying assumption is that Δt is small enough that only cars that are passing x_0 at, or near, $t = t_0$ are used to determine the flux at t_0. At the same time, from an experimental point of view, Δt cannot be so small that no cars are able to pass this location during this time interval. Under these conditions the flux J satisfies

$$J(x_0, t_0) \approx \frac{\text{net number of cars that pass } x_0 \text{ from } t = t_0 - \Delta t \text{ to } t = t_0 + \Delta t}{2\Delta t}. \tag{4.2}$$

As above, in the continuum viewpoint cars are distributed smoothly over the entire t-axis and the value of $J(x_0, t_0)$ is the limit of the right hand side of (4.2) as $\Delta t \longrightarrow 0$.

Returning to the above simple case of evenly spaced equal cars in a highway, if we now add the assumption that all cars are moving with a constant positive velocity v, then cars that start out a distance $2v\Delta t$ from x_0 will pass x_0 in the time interval from $t_0 - \Delta t$ to $t_0 + \Delta t$. The corresponding number of cars is, approximately, $2v\Delta t/(l + d)$. Inserting this into (4.2) yields

$$J = \frac{v}{l + d} = \rho v \tag{4.3}$$

This is a relation between density, velocity and flux quite basic for the model as we shall see now.

4.1.3 Balance Law

To derive an equation for the density of cars we will use what is known as the "control volume argument", already used before. The balance law for the cars within the highway interval $I := [x_0 - \Delta x, x_0 + \Delta x]$ is

$$\begin{aligned} &\Big\{\text{number of cars in } I \text{ at } t = t_0 + \Delta t\Big\} - \Big\{\text{number of cars in } I \text{ at } t = t_0 - \Delta t\Big\}, = \\ &= \Big\{\text{number of cars that cross } x_0 - \Delta x \text{ in the time interval } [t_0 - \Delta t, t_0 + \Delta t]\Big\} - \\ &- \Big\{\text{number of cars that cross } x_0 + \Delta x \text{ in the time interval } [t_0 - \Delta t, t_0 + \Delta t]\Big\} \end{aligned}$$

Rewriting this using (4.1) and (4.2) yields

$$\Delta x\Big(\rho(x_0, t_0 + \Delta t) - \rho(x_0, t_0 - \Delta t)\Big) = \Delta t\Big(J(x_0 - \Delta x, t_0) - J(x_0 + \Delta x, t_0)\Big)$$

Now, assuming the functions are sufficiently smooth, we can expand ρ and J about (x_0, t_0) using Taylor's theorem. Doing this

$$\begin{aligned} &\Delta x\Big(\rho + \rho_t \Delta t + \frac{1}{2}\rho_{tt}(\Delta t)^2 + \frac{1}{6}\rho_{ttt}(\Delta t)^3 + ... - \rho + \rho_t \Delta t - \frac{1}{2}\rho_{tt}(\Delta t)^2 + \frac{1}{6}\rho_{ttt}(\Delta t)^3 + ...\Big) \\ &= \Delta t\Big(J - J_x \Delta x + \frac{1}{2}J_{xx}(\Delta x)^2 - \frac{1}{6}J_{xxx}(\Delta x)^3 + ... \\ &\qquad\qquad - J - J_x \Delta x - \frac{1}{2}J_{xx}(\Delta x)^2 - \frac{1}{6}J_{xxx}(\Delta x)^3 + ...\Big), \end{aligned}$$

where ρ and J are evaluated at (x_0, t_0). Collecting the terms in the above equation we obtain

$$\rho_t + O((\Delta t)^2) = -J_x + O((\Delta x)^2),$$

and letting $\Delta x \longrightarrow 0$ and $\Delta t \longrightarrow 0$ we conclude that $\rho_t = -J_x$ or

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{4.4}$$

Equation (4.4) is our balance law. Assuming the relation between velocity and flux $J = \rho v$ arrived at in the previous subsection we can rewrite (4.4) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(v\rho) = 0. \tag{4.5}$$

Clearly, in order to study this equation one needs to assume the initial density is known, that is

$$\rho(x, 0) = f(x) \tag{4.6}$$

where f is given (eventually known from observation, or from the historical record).

Observe, however, that (4.5) has two unknown quantities: the density ρ and the velocity v. In order to study the mathematical model (4.5)-(4.6) we need to know how these two quantities are related. A relation of the type $v = v(\rho)$ is called a constitutive law and is not a relation that can be obtained from Mathematics alone: it is a relation that expresses the real-life problem under consideration and must be obtained, depending on the cases, either from observational measurements, from experiments, or from more fundamental (physical) theories.

In the case of cars flowing in a highway the constitutive law $v = v(\rho)$ has been studied by several authors and some experimental data is presented and discussed in [10, Section 5.4]. An example of these data is presented in Figure 4.5

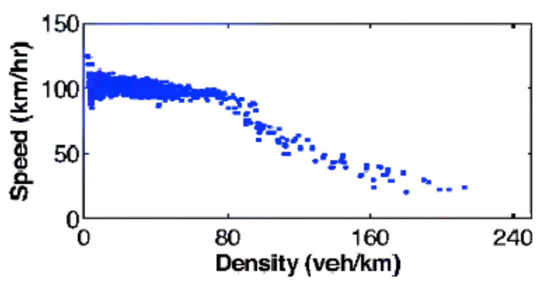

Figure 4.5: Experimental results of v as function of ρ for cars in a freeway near Amsterdam (from [10, Figure 5.6]).

Observe that there are essentially two different regimes: if the density is low (in the example in Figure 4.5 below 80 cars/kilometer) the velocity is constant (and equal to the maximum velocity allowed by law); in the other hand, if the density is high the velocity decreases monotonically with the increase of density, in an almost linear way that seems to attain velocity zero around a density[1] of 220 cars/kilometer. We now briefly consider these two regimes.

4.1.4 Constant velocity model

From (4.4), the simplest assumption, valid, as we saw, for low car densities, is that v is constant in terms of its dependence on ρ, in other words, $v = a$. This assumption results in the advection equation

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0 \tag{4.7}$$

already considered in Chapter 2. As we saw there, this is an example of a linear first order partial differential equations that can be completely solved using the Method of Characteristics. There is nothing more really interesting that can be presently be added about this model.

4.1.5 Greenshields model

A much more interesting regime is the high density one referred to above. One of the most widely used constitutive laws in traffic flow studies tries to capture this behaviour in the simplest possible way. It is the so called Greenshields model [10, Section 5.4.2], given by

$$v(\rho) = v_M \left(1 - \frac{\rho}{\rho_M}\right) \tag{4.8}$$

where v_M and ρ_M is the maximum velocity and density, respectively.

It turns out that in order to reflect that drivers will reduce their speed to account for an increasing density ahead, so we should suppose that J is function of the density gradient as well

$$J = \rho v - D\frac{\partial \rho}{\partial x} \tag{4.9}$$

where D is constant, if we put (4.8) and (4.9) into (4.4), we get that

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\frac{v_M}{\rho_M}\left(\rho_M - \rho\right)\rho\right] = D\frac{\partial^2 \rho}{\partial x^2}$$

[1] Observe that this density means that there is one car in about every 4,5 meters, which means that there will be no space between the cars: we are at a situation of maximum possible car density and so the traffic flow must have come to a halt, as already pointed out above in page 27.

or

$$\frac{\partial \rho}{\partial t} + a(\rho)\frac{\partial \rho}{\partial x} = D\frac{\partial^2 \rho}{\partial x^2} \tag{4.10}$$

where $a(\rho) = v_M\left(1 - \frac{2\rho}{\rho_M}\right)$.

The function $a(\rho)$ is known as the wave velocity and equation (4.10) is a non-linear conservation equation for ρ. In the case when $a(\rho) = \rho$, then equation (4.10) become to visid Burgers' Equation, where it can be solved by Cole-Hopf transform, as we already consider in Chapter 3.

4.1.6 An application to a traffic flow situation

In previous section we considered about the velocity is constant and in this section we will consider the situation when velocity is some function depend on density ρ, assuming that

$$v = F(\rho) \tag{4.11}$$

From (4.4) the general formula for flux is

$$J = \rho v = \rho F(\rho)$$

Assuming that F is a smooth function of ρ, using chain rule, it follows that

$$\frac{\partial J}{\partial x} = J'(\rho)\frac{\partial \rho}{\partial x}$$

substitute to (4.4), we have

$$\frac{\partial \rho}{\partial t} + a(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{4.12}$$

where $a(\rho) = J'(\rho)$ or equivalently

$$a(\rho) = F(\rho) + \rho F'(\rho) \tag{4.13}$$

The equation above is written resembles the constant velocity version in (4.7), one significant difference is that the wave velocity $a(\rho)$ can depend on the unknown ρ, and if this happen then (4.12) is non-linear. Generally non-linear Partial Differential Equations are very difficult to solve. However, the case of velocity is $a(\rho) = \rho$ the equation (4.12) is Burgers' Equation and we can consider the solution by Method of Characteristic as we already considered in Chapter 2.

Example: This example we will show a very simple situation of a traffic model with a given initial condition. We consider the constitutive law $v(\rho) = \frac{1}{2}\rho$, giving rise to the following equation:

$$\begin{cases} \rho_t + \rho\rho_x = 0 \\ \rho(x,0) = \rho_0(x) \end{cases} \tag{4.14}$$

where the initial condition $\rho_0(x)$ is $C^1(\mathbb{R})$, monotonic decreasing, $\rho_0 \in [0,1]$, and satisfies

$$\rho_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 1. \end{cases} \tag{4.15}$$

An example of this could be obtained by taking $\rho_0(x) = 1 - x$ for x between 0 and 1 and mollifying the resulting function.

The situation modelled by (4.14)-(4.15) can be thought of as a high density group of cars moving fast in a street that is blocked by a car which stopped at some point ($x = 1$).

By method of characteristics in chapter 2 we get the solution in implicit form

$$\rho(x,t) = \rho_0(x_0) = \psi(x - \rho t)$$

and we can see the solution in the Figure 4.6.

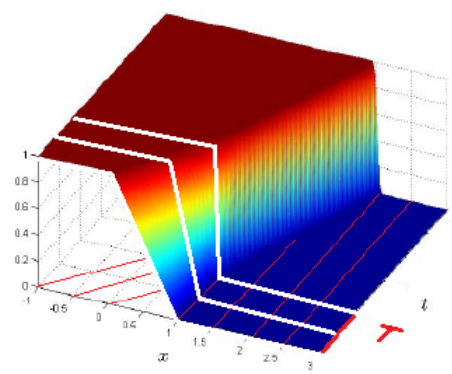

Figure 4.6: Example of the plot of the solution of (4.14) with an initial condition (4.15) as described in the text, (from [14]).

Let us see what is occurring in the $\rho - x$ plane. Figure 4.7 shows that, when time t increasing from $t_0 = 0$ to t_1 ($t_1 > t_0$), cars in the portion of the street between 0 and 1 ($x \in [0,1]$) are getting closer together (for any given x in this region, the density $\rho(t,x)$ either stays the same or increases, when t changes from t_0 to t_1) and at time T all cars at $x < 1$ are moving forward with velocity $v = \frac{1}{2}$ and the car at $x = 1$ is stopped, so there will be an accident (and the occurrence of a mathematical shock).

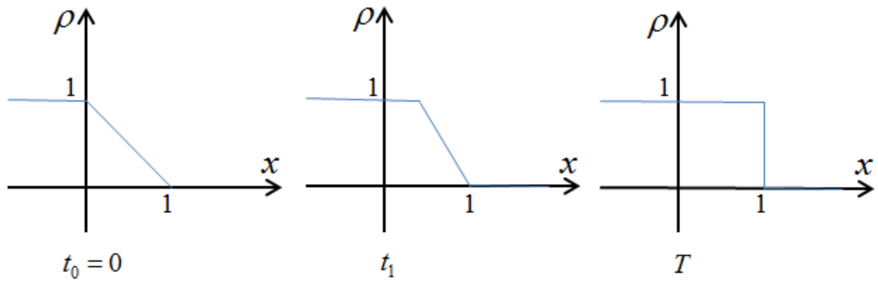

Figure 4.7: Plot moving of the density when time increasing

4.2 Coagulation equations

4.2.1 Introducing the mathematical model

In this section we will arrive at Burgers' equation in a completely different setting.

Consider a large container with a very large number of particles whose sizes are indexed by the positive real variable $x \in \mathbb{R}^+$. Assume that those particles can undergo coagulation reactions in which a particle of size x and a particle of size y can get together to form a particle of size $x + y$, or, schematically,

$$(x) + (y) \longrightarrow (x + y).$$

Let $c(x,t)$ be the density, at time t, of particles of size x (usually called "x-clusters", or simply "clusters"), and assume the number of particles is so large that this function can be considered smooth, then the differential equation modeling these coagulation reactions is the following continuous version of Smoluchowski's coagulation equations [8]:

$$\text{(4.16)}\quad \frac{\partial c}{\partial t}(x,t) = \frac{1}{2}\int_0^x a(x-y,y)c(x-y,t)c(y,t)dy - c(x,t)\int_0^\infty a(y,t)c(y,t)dy, \quad (x,t) \in \mathbb{R}^{\nvDash+}$$

where $a(x,y)$ is the rate of the reaction $(x)+(y) \to (x+y)$, which, from physical considerations, must be symmetric, $a(x,y) = a(y,x)$, and non negative, $a(x,y) \geq 0$. The nonlinear integro-differential equation (4.16) is to be suplemented by an initial condition

$$\text{(4.17)}\qquad c(x,0) = c_0(x),$$

with $c_0(x)$ some given initial distribution of particles.

In the studies of (4.16) the following reformulation of the equation, leading to a kind of weak version of the equation, is of fundamental importance. To avoid unessential technical difficulties we will deduce it under the assumption that $c(x,t)$ is sufficiently regular. We shall also consider a smooth and compactly supported function $\phi : \mathbb{R}^+ \to \mathbb{R}$. Multiplying equation

(4.16) by $\phi(x)$ and integrating the result in x between 0 and ∞ we can write

$$\frac{d}{dt}\int_0^\infty \phi(x)c(x,t)dx = \frac{1}{2}\int_0^\infty\int_0^x \phi(x)a(x-y,y)c(x-y,t)c(y,t)dydx - \\ -\int_0^\infty\int_0^\infty \phi(x)a(y,t)c(x,t)c(y,t)dydx. \tag{4.18}$$

Our goal is to rearrange the right-hand side terms so that we obtain the following "weak" version of this coagulation equation:

$$\frac{d}{dt}\int_0^\infty \phi(x)c(x,t)dx = \frac{1}{2}\int_0^\infty\int_0^\infty \big(\phi(x+y)-\phi(x)-\phi(y)\big)a(x,y)c(x,t)c(y,t)dxdy. \tag{4.19}$$

- Let us start by rearranging the first double integral in the right-hand side of (4.18). The region of integration is the conic "triangular" region of $\mathbb{R}^{2+}$ bounded by the x-axis and the line $y = x$ (see Figure 4.8).

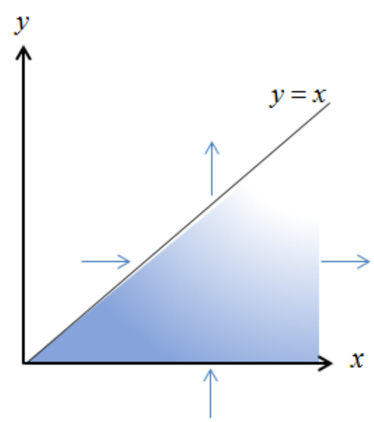

Figure 4.8: Region of first double integral in (4.18).

Changing the order of integration we obtain

$$\frac{1}{2}\int_0^\infty\int_y^\infty \phi(x)a(x-y,y)c(x-y,t)c(y,t)dxdy,$$

and now changing variables $(x,y)\mapsto(z,y)$ where $z = x-y$, we get

$$\frac{1}{2}\int_0^\infty\int_0^\infty \phi(z+y)a(z,y)c(z,t)c(y,t)dzdy. \tag{4.20}$$

- Now, for the second double integral in the right-hand side of (4.18), write it as

$$\int_0^\infty \int_0^\infty \phi(x)a(y,t)c(x,t)c(y,t)dydx = \frac{1}{2}\int_0^\infty \int_0^\infty \phi(x)a(x,y)c(x,t)c(y,t)dxdy + \\ + \frac{1}{2}\int_0^\infty \int_0^\infty \phi(x)a(x,y)c(x,t)c(y,t)dxdy$$

  and now, performing a change of notation $x \leftrightarrow y$ *only on the second* of these integrals and remembering the symmetry assumption on the rate coefficient $a(x,y)$, we can re-write the right-hand side as

$$\frac{1}{2}\int_0^\infty \int_0^\infty \big(\phi(x)+\phi(y)\big)a(x,y)c(x,t)c(y,t)dxdy \tag{4.21}$$

Hence, substituting (4.20)-(4.21) into (4.18) we get (4.19).

The usual practice in partial differential equations is that, once a weak version of the equation is obtained under convenient assumptions on the regularity of the functions involved, it is the weak version that will become the object of study by its own, independently of the assumptions made for its deduction, thus allowing for the consideration of much larges classes of solutions to be considered.

This principle is also used in coagulation equations and thus it is the equation (4.19) that is the object of analysis in most of the mathematical studies (see examples in [8]), even when the considered test function ϕ is no longer smooth and of compact support.

In the next section we will consider the coagulation equation with the so-called "solvable" reaction coefficients $a(x,y) = 1$, $a(x,y) = x + y$ and $a(x,y) = xy$, by exploiting (4.19) with $\phi(x)$ being a function related to the kernel function in the Laplace transform. We shall see that in two of these three cases we will arrive at a Burgers' equation.

4.2.2 Application of Laplace Transforms

Analysis of the solvable case $a(x,y) = 1$

From the three solvable coefficients[2] referred above, we will start with the simplest one, $a(x,y) = 1$, which will already show us the approach to be used in all others.

Consider $\phi(x) = 1 - e^{-xs}$ and let

$$\psi(s,t) := \int_0^\infty \phi(x)c(x,t)dx. \tag{4.22}$$

[2]Those three cases of $a(x,y)$ are called solvable exactly because the application of Laplace transforms to (4.19) results in an ordinary differential equation, or in a partial differential equation – Burgers' equation – that can then be, in principle, explicitly solved.

Observe that $\psi(\cdot,t)$ is very similar to the Laplace transform of $c(\cdot,t)$, which would have been obtained using $\phi(x) = e^{-xs}$ instead. The present use of this "modified" Laplace transform (called "de-singularized" in the mathematical literature – see [16, 17]) is more convenient for applications to coagulation studies. Having present that, for $c(\cdot,t)$ integrable in $\mathbb{R}^+$, $\psi(s,t) = \int_0^\infty c(x,t)dx - (Lc(\cdot,t))(s)$, where Lc represents the usual Laplace transform of c, properties of ψ can be easily deduced from those of L.

Let us substitute ϕ into (4.19) to get

$$\frac{\partial}{\partial t}\psi(s,t) = -\frac{1}{2}\int_0^\infty\int_0^\infty \big(e^{-(z+y)s} - e^{-zs} - e^{-ys} + 1\big)c(z,t)c(y,t)dzdy.$$

Writing the right-hand side in expanded form

$$\begin{aligned}\frac{\partial}{\partial t}\psi(s,t) = &- \frac{1}{2}\Big(\int_0^\infty e^{-zs}c(z,t)dz \int_0^\infty e^{-ys}c(y,t)dy - \int_0^\infty e^{-zs}c(z,t)dz \int_0^\infty c(y,t)dy - \\ &- \int_0^\infty e^{-ys}c(y,t)dy \int_0^\infty c(z,t)dz + \int_0^\infty c(z,t)dz \int_0^\infty c(y,t)dy\Big) \\ = &-\frac{1}{2}\left(\Big(\int_0^\infty e^{-us}c(u,t)du\Big)^2 - 2\int_0^\infty e^{-us}c(u,t)du\Big)\int_0^\infty c(u,t)du + \Big(\int_0^\infty c(u,t)du\Big)^2\right) \\ = &-\frac{1}{2}\left(\int_0^\infty e^{-us}c(u,t)du - \int_0^\infty c(u,t)du\right)^2 \\ = &-\frac{1}{2}\left(\int_0^\infty (1-e^{-us})c(u,t)du\right)^2 \\ = &-\frac{1}{2}\psi(s,t)^2\end{aligned}$$

Hence, we obtain the following ordinary differential equation[3] for the (modified) Laplace transform of the cluster distribution function $c(x,t)$ when the coagulation coefficients are $a(x,y) = 1$:

$$\psi_t = -\frac{1}{2}\psi^2, \tag{4.23}$$

with the corresponding initial condition obtained from the initial condition (4.17) for the coagulation equation:

$$\psi_0 := \int_0^\infty e^{-us}c_0(u)du.$$

[3]Although ψ is a function of two variable, in (4.23) one of them, s, acts as a mere parameter.

Analysis of the solvable case $a(x, y) = x + y$

Consider now the coagulation kernel $a(x, y) = x + y$, and let ϕ and ψ be as above in section 4.2.2. Doing exactly the same computations as before we arrive at

$$\frac{\partial}{\partial t}\psi(s,t) = - \left(-\Big(\int_0^\infty ue^{-us}c(u,t)du\Big)\Big(\int_0^\infty (1-e^{-us})c(u,t)du\Big) + \right.$$
$$\left. + \int_0^\infty uc(u,t)du \int_0^\infty (1-e^{-us})c(u,t)du \right),$$

and observing that $\psi_s(t,s) = \int_0^\infty ue^{-us}c(u,t)du$ we can finally obtain the following partial differential equation for ψ:

$$\psi_t - \psi\psi_s = -\rho\psi \tag{4.24}$$

where $\rho = \int_0^\infty uc(u,t)du$. This quantity has the physical interpretation of the total density of the system of clusters and it has been proved that, for the coagulation kernel under consideration, it is a constant quantity independent of the time t, [8], so that we can use the initial condition (4.17) to write $\rho = \int_0^\infty uc_0(u)du$.

Again, the corresponding initial condition is obtained in the natural way from the initial condition (4.17) for the coagulation equation:

$$\psi(s,0) = \int_0^\infty e^{-us}c_0(u)du.$$

Observe that (4.24) is a Burgers' partial differential equation that can be solved by the Method of Characteristics

Analysis of the solvable case $a(x, y) = xy$

Finally, let us consider $a(x,t) = xy$ and take $\phi(x) = \big(1 - e^{-xs}\big)x$, being $\psi(s,t)$ still defined by (4.22). Again the exact same computations as in the above to sections leads to

$$\frac{\partial}{\partial t}\psi(s,t) = -\frac{1}{2}\left(2\int_0^\infty u^2e^{-us}c(u,t)du\Big(\int_0^\infty ue^{-us}c(u,t) - \int_0^\infty uc(u,t)du\Big)\right)$$

and since in the present case we have $\psi_s(t,s) = \int\limits_0^\infty u^2 e^{-us} c(u,t)du$, we can use this and the definition of ψ to write the following partial differential equation for ϕ:

$$\psi_t - \psi\psi_s = 0, \tag{4.25}$$

with the initial condition obtained as before.

Final Remarks

Having obtained the ordinary differential equation (4.23) or the Burgers' equations (4.24) and (4.25) for the (modified) Laplace transforms of the solution of the coagulation equation (4.19), we could now be expected to present the way the solutions to (4.23), (4.24), or (4.25) (in the last two cases obtained, for instance, by the Method of Characteristics) can be used to gain knowledge about the solutions to (4.19). This, however, is a very difficult process since in most cases the solutions to (4.23), (4.24), or (4.25) are not explicitly known, and so to translate the behaviour of those not explicitly solutions to information about the behaviour of the coagulation solutions is a very hard problem which is the subject of current mathematical research. Furthermore, it is in fact far removed from the main topic of study or this thesis, which was Burgers' equation.

Chapter 5

Conclusion

In this thesis, I studied Burgers' equation and some of its applications. In the first chapter, I introduced the occurrence of Burgers' equation and gave the motivation to understand the model. Burgers' equation can be classified in two case: the inviscid and the viscid equations. In the study of both these cases, I used analytical methods: the Method of Characteristics in the first case, and the Cole-Hopf transformation in the second.

In chapter 2, to understand the Method of Characteristics, I introduced it by considering the simplest example of an advection equation, and then I studied the Method of Characteristics for linear, semi-linear, and quasi-linear equations in general form. The Method of Characteristics allow the reduction of the partial differential equation to a system of ODEs; using ODE theory to study the system of characteristic equation we conclude the existence of a system of characteristic curves, and then, recurring to tools from Real Analysis, such as the implicit and inverse function theorems, these characteristic curves are pieced together to get a solution of the PDE. In the last section of this chapter I considered inviscid Burgers' equation and solve it by using the Method of Characteristics. I also apply this method in chapter 4 to a Burgers' equation modeling traffic flow.

In chapter 3, I solved viscid Burgers' equation by using the Cole-Hopf transformation, in this method we converted non-linear viscous Burgers' equation into a linear heat equation, then solve the heat equation by noticing the existence of a dimensionless variable that allow for the transformation of the heat equation on the real line into a boundary value problem for a second order ODE. After getting the solution of the ODE, we transform it back to the solution of heat equation and then to the solution of viscid Burgers' equation.

In chapter 4, I studied two applications: in the first section I presented a model of traffic flow, where I explained the important variables in modeling traffic flow as density and flux, exhibit the relation between density, velocity and flux, and use a balance law to obtain an equation modeling the flow of traffic. Then I illustrate the model in the simplest case of velocity say: constant velocity model, and also in a more realistic case: the Greenshields' model. Finally I gave an example of a simple situation of a traffic model, by considering a linear relation between velocity and car density, with given initial condition. To solve the example I used the Method of Characteristics. For the second application, I studied an aspect related to the coagulation equations, namely, taking the continuous version of Smoluchowski's coagulation equation, the goal of this section was to show the appearance of Burgers' equation in an apparently surprising situation unrelated to conservation laws. We consider coagulation equation with the "solvable" reaction coefficients and apply (modified) Laplace transforms. The result of this will be that the Laplace transform will satisfy either an ODE , or a Burgers' equation, depending on the "solvable" case under consideration.

Appendix

In this appendix we collect a few standard definition and results used in previous chapter [1, 20]

Definition 5.0.1 *If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are vectors in $2-$space, then the dot product of u and v is written as $u \cdot v$ and defined as*

$$u \cdot v = u_1 v_1 + u_2 v_2$$

Theorem 5.0.2 *If u and v are non-zero vector in 2-space or 3-space, and θ is the angle between them, then*

$$u \cdot v = \|u\| \|v\| cos\theta$$

and if u and v are orthogonal if and only if $u \cdot v = 0$

Theorem 5.0.3 *Assume that $f(x, y)$ has continuous first-order partial derivatives in an open disk center at (x_0, y_0) and that $\nabla f(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0)$ is perpendicular to the level curve of f though (x_0, y_0)*

Proof 5.0.1 *Let*

$$f(x, y) = c \tag{5.1}$$

be the level curve though (x_0, y_0), this level curve can be represented parametrically by the equation

$$x = x(t)$$
$$y = y(t)$$

So, that the level curve has a non-zero tangent vector at (x_0, y_0). More precisely $x'(t_0)i + y'(t_0)j \neq 0$

Where to is the value of the parameter corresponding to (x_0, y_0).
Differentiating (3.3) with respect to t, and applying the Chain rule yields

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0$$

Substituting $t = t_0$, and using the fact that $x(t_0) = x_0$ and $y(t_0) = y_0$, we obtain

$$f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = 0$$

Which can be written as

$$\nabla f(x_0, y_0) \cdot (x'(t_0)i + y'(t_0)j) = 0$$

This equation tells us that $\nabla f(x_0, y_0)$ *is perpendicular to the tangent vector* $x'(t_0)i + y'(t_0)j$ *and, therefore, is perpendicular to the level curve though* (x_0, y_0)*, as shown in figure 5.1*

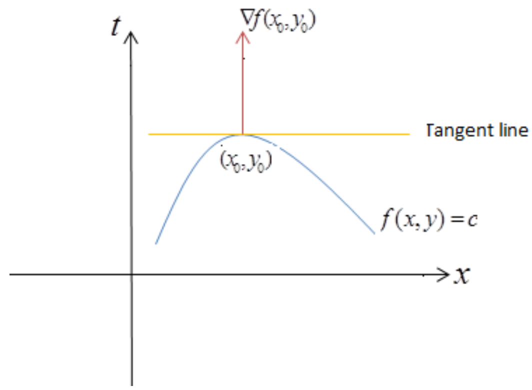

Figure 5.1: Gradient perpendicular to tangent line

Definition 5.0.4 *Let* f *be a real-valued function of the real variable* t*, defined for* $t > 0$*. Consider the function* F *defined by*

$$F(s) = \int_0^\infty e^{-st} f(t)dt \tag{5.2}$$

For all values of s *for which this integral exists. The function* f *is call the Laplace transform of the function* f*. We denote the Laplace transform* F *of* f *by* $\mathcal{L}\{f(t)\}$*.*

Theorem 5.0.5 *(Linearity) Let* f_1 *and* f_2 *be functions whose Laplace transform exists and let* c_1 *and* c_2 *be constant then,*

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

Definition 5.0.6 *A function* f *is said to be of exponential order if there exists a constant* α *and positive constants* t_0 *and* M *such that*

$$|f(t)| < Me^{\alpha t}$$

for all $t > t_0$ *at which* $f(t)$ *is defined. More explicitly, if* f *is of exponential order corresponding to some definite constant* α*, then we say that* f *is of exponential order* $e^{\alpha t}$

Theorem 5.0.7 *(Differentiation) Let* f *be a real function that is continuous for* $t \geq 0$*, and of exponential order* $e^{\alpha t}$ *and suppose* f' *is piecewise continuous in every finite closed interval* $0 \leq t \leq b$*, then* $\mathcal{L}\{f'\}$ *exists for* $s > \alpha$ *and* $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$*.*

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