THE ROLE OF THE SADDLE-FOCI ON THE STRUCTURE OF A BYKOV ATTRACTING SET

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ABSTRACT. We consider a one-parameter family $(f_\lambda)_{\lambda \geqslant 0}$ of symmetric vector fields on the three-dimensional sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ whose flows exhibit a heteroclinic network between two saddle-foci inside a global attracting set. More precisely, when $\lambda = 0$, there is an attracting heteroclinic cycle between the two equilibria which is made of two 1-dimensional connections together with a 2-dimensional sphere which is both the stable manifold of one saddle-focus and the unstable manifold of the other. After slightly increasing the parameter while keeping the 1-dimensional connections unaltered, the two-dimensional invariant manifolds of the equilibria become transversal, and thereby create homoclinic and heteroclinic tangles. It is known that these newborn structures are the source of a countable union of topological horseshoes, which prompt the coexistence of infinitely many sinks and saddle-type invariant sets for many values of λ. We show that, for every small enough positive parameter λ, the stable and unstable manifolds of the equilibria and those infinitely many horseshoes are contained in the global attracting set of f_λ. Moreover, we prove that the horseshoes belong to the heteroclinic class of the equilibria. In addition, we verify that the set of chain-accessible points from either of the saddle-foci is chain-stable and contains the closure of the invariant manifolds of the two equilibria.

1. INTRODUCTION

This work starts with an autonomous $(\mathbb{Z}_2 \times \mathbb{Z}_2)$−equivariant vector field f_0 on the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ exhibiting an attracting heteroclinic network between two saddle-foci σ_1 and σ_2 which belong to a flow-invariant submanifold whose existence is a consequence of the symmetry. This network consists of a two-dimensional sphere connecting σ_2 and σ_1 and two one-dimensional connections from σ_1 to σ_2. See Figure 1 (left). Afterwards, f_0 is perturbed generating a smooth one-parameter family $(f_\lambda)_{\lambda \geqslant 0}$ of vector fields subject to several conditions we will summarize in Section 2. Although small, the perturbation breaks part of the symmetry. More precisely, when $\lambda > 0$, the one-dimensional connections persist while the two-dimensional invariant manifolds, which coincided in the beginning, now intersect transversely. Yet, the closeness of f_λ when $\lambda > 0$ to the symmetric context somehow simplifies our study due to constrains it induces on the geometry of the invariant manifolds of the equilibria, allowing

some control of their relative positions. The aim of this work was to understand how the configuration of these invariant manifolds enables hyperbolic saddle-type behavior to coexist with infinitely many sinks and chaotic attractors. This might explain, though partially, the origins of the chaos that computer simulations of these kind of systems very often disclose, and the role of the stable and unstable manifolds of both the equilibria in the bifurcation phenomena that are known to occur when the parameter λ is changed.
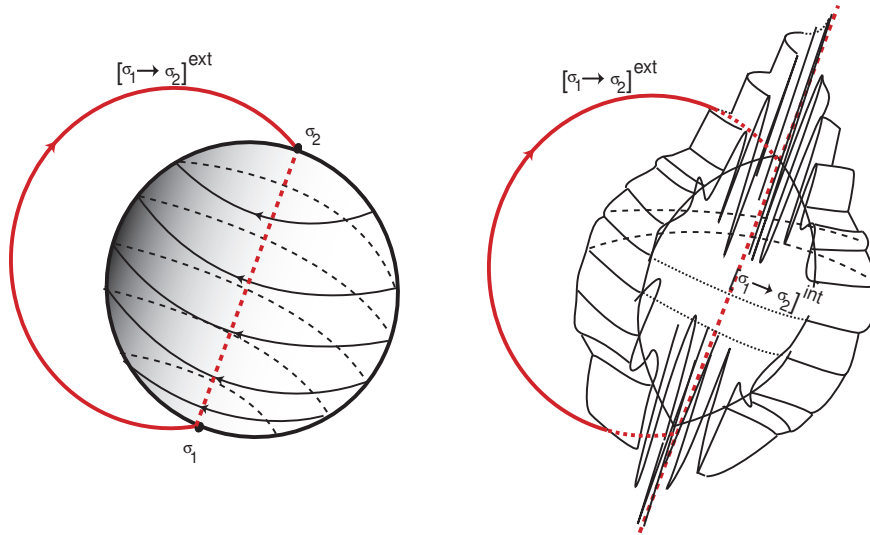

FIGURE 1. The invariant manifolds of the saddle-foci for $\lambda = 0$ (left) and $\lambda > 0$ small (right). The latter image was inspired by Figure 4 of [5].

We briefly recall in Section 3 the intervention, in the bifurcation process that $(f_\lambda)_{\lambda \geqslant 0}$ is undergoing, of both the symmetry assumptions and the complex nature of eigenvalues of the equilibria, as well as of the transversal intersection (when $\lambda > 0$) of the two-dimensional invariant manifolds of the equilibria. They are responsible for the appearance of infinitely many topological horseshoes and a profusion of homoclinic tangencies, resulting in the outbreak of Newhouse phenomena and the genesis of huge sets of sinks. See Figure 1 (right). This pattern agrees with the known models of quasi-stochastic attractors, found analytically or evinced through computer simulations (cf. [1]). Up to our knowledge, the two-dimensional invariant manifolds associated to the heteroclinic phenomena have been considered on previous references only locally, near an equilibrium or inside a tubular neighborhood of the cycle. In this work we focus on the behavior of the entire two-dimensional manifolds of the equilibria as organizers of the dynamics. However, in order to be able to compute the dynamical data, we have to restrict our analysis to a model of such a family of vector fields, whose properties will be specified in Section 6.

The paper is organized as follows. After introducing the model of the one-parameter family of vector fields we work with, and discussing its main properties, we relate the countable union of topological horseshoes that are known to appear in this setting with the heteroclinic class of the two equilibria, and elucidate their relevance in the bifurcation portrait. Noticing that the global attracting set of f_λ may be non transitive due to the presence of proper attractors, we describe in Section 10 a dynamically relevant chain-stable set. For the reader's convenience, we have compiled at the end of the paper a list of definitions in a short glossary.

2. The setting

Let $(f_\lambda)_{\lambda \in \mathbb{R}}$ be a one-parameter family of vector fields in $\mathfrak{X}^r(\mathbb{S}^3)$, $r \geqslant 3$, whose flows are given by the solutions $(\varphi_\lambda(t,x))_{t \in \mathbb{R},\, x \in \mathbb{S}^3}$ of the differential equations $\dot{x} = f_\lambda(x)$. We will enumerate the main assumptions concerning symmetry, chirality, the complex eigenvalues at the saddle-foci and the intersection of the two bi-dimensional invariant manifolds of these equilibria. We refer the reader to the Appendix A for precise definitions.

2.1. **Main assumptions.** We will request that the organizing center at $\lambda = 0$ satisfies the following assumptions:

(**P1**) The vector field f_0 is equivariant under the action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\mathbb{S}^3$ induced by the linear maps $\gamma_1(x_1, x_2, x_3, x_4) = (-x_1, -x_2, x_3, x_4)$ and $\gamma_2(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, x_4)$ on $\mathbb{R}^4$.

(**P2**) The set $\mathrm{Fix}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \{Q \in \mathbb{S}^3 : \gamma_1(Q) = \gamma_2(Q) = Q\}$ reduces to the equilibria $\sigma_1 = (0,0,0,1)$ and $\sigma_2 = (0,0,0,-1)$, which are hyperbolic saddle-foci and whose eigenvalues are respectively $-C_1 \pm \alpha_1 i$ and E_1, where $\alpha_1 \neq 0$ and $C_1 > E_1 > 0$, and $E_2 \pm \alpha_2 i$ and $-C_2$, where $\alpha_2 \neq 0$ and $C_2 > E_2 > 0$. We will assume that $\alpha_1 = \alpha_2 = 1$.

(**P3**) The flow-invariant circle $\mathrm{Fix}(\mathbb{Z}_2\langle \gamma_1 \rangle) = \{Q \in \mathbb{S}^3 : \gamma_1(Q) = Q\}$ consists of σ_1 and σ_2, and two heteroclinic trajectories $[\sigma_1 \to \sigma_2]^{ext}$ and $[\sigma_1 \to \sigma_2]^{int}$ from σ_1 to σ_2 whose union will be denoted by $[\sigma_1 \to \sigma_2]$.

(**P4**) The f_0−invariant sphere $\mathrm{Fix}(\mathbb{Z}_2\langle \gamma_2 \rangle) = \{Q \in \mathbb{S}^3 : \gamma_2(Q) = Q\}$ is made of σ_1 and σ_2 and a two-dimensional heteroclinic connection from σ_2 to σ_1.

(**P5**) The saddle-foci σ_1 and σ_2 have the same chirality (cf. Appendix A.7).

The sphere $\mathrm{Fix}(\mathbb{Z}_2\langle \gamma_2 \rangle)$ divides $\mathbb{S}^3$ into two connected components (inner and outer) and $[\sigma_1 \to \sigma_2] \setminus \{\sigma_1, \sigma_2\}$ has also two connected components, one on each connected component of $\mathbb{S}^3 \setminus \mathrm{Fix}(\mathbb{Z}_2\langle \gamma_2 \rangle)$. The set $[\sigma_1 \to \sigma_2]^{int}$ (respectively, $[\sigma_1 \to \sigma_2]^{out}$) is precisely the connection lying in the inner (respectively, outer) component of $\mathbb{S}^3 \setminus \mathrm{Fix}(\mathbb{Z}_2\langle \gamma_2 \rangle)$. The two equilibria, the two trajectories referred to in (P3) and the two-dimensional heteroclinic connection from σ_2 to σ_1 mentioned in (P4) build a *heteroclinic network* we will denote hereafter by Σ_0. It is illustrated in Figure 1 (left).

Denote by $\mathfrak{X}^r_{\mathrm{Byk}}(\mathbb{S}^3) \subset \mathfrak{X}^r(\mathbb{S}^3)$ the set, endowed with the C^r−Whitney topology, of $\langle \gamma_1 \rangle$−equivariant C^r vector fields on $\mathbb{S}^3$, $r \geqslant 3$, satisfying (P1)–(P5). Consider $f_0 \in \mathfrak{X}^r_{\mathrm{Byk}}(\mathbb{S}^3)$ embedded in a continuous one-parameter family of vector fields

$$\lambda \in \mathbb{R} \; \mapsto \; f_\lambda \equiv f(x, \lambda) : \mathbb{S}^3 \to T\mathbb{S}^3 \in \mathfrak{X}^r_{\mathrm{Byk}}(\mathbb{S}^3)$$

breaking the $\langle \gamma_2 \rangle$−equivariance at $\lambda = 0$ in a generic way. The next three assumptions are the properties we will also need to be valid for the family $(f_\lambda)_{\lambda \in \mathbb{R}}$.

(**P6**) For every $\lambda \in \mathbb{R}$, the map f_λ is a $\langle \gamma_1 \rangle$−equivariant C^r vector field, $r \geqslant 3$.

(**P7**) For $\lambda \neq 0$, the two-dimensional manifolds $W_\lambda^u(\sigma_2)$ and $W_\lambda^s(\sigma_1)$ intersect transversely (abbreviated to $W_\lambda^u(\sigma_2) \cap W_\lambda^s(\sigma_1)$) at two trajectories β_1 and β_2, called *primary links* or 0–*pulses.*

(**P8**) For every $\lambda \geqslant 0$, the transitions along the connections $[\sigma_1 \to \sigma_2]$ and $[\sigma_2 \to \sigma_1]$ are given, in local coordinates, by the Identity map and, up to high order terms in the variable y, the transformation

$$(x, y) \mapsto (x, y + \lambda \sin(x))$$

respectively. This assumption will be detailed in §5.2 and §6.

A few comments are in order. Since the equilibria lie on $\text{Fix}(\mathbb{Z}_2\langle\gamma_1\rangle)$ and are hyperbolic, they persist for small $\lambda > 0$, and still satisfy the properties (P2) and (P3). Besides, σ_1 and σ_2, the primary links and the connections $[\sigma_1 \to \sigma_2]^{ext}$ and $[\sigma_1 \to \sigma_2]^{int}$ form a heteroclinic network that we denote by $\Sigma_\lambda^\star$. In what follows, Γ will stand for a *Bykov cycle* in $\Sigma_\lambda^\star$ with the connections mentioned in (P3) and (P7). According to [18], the existence of the network $\Sigma_\lambda^\star$ implies the presence of a bigger network Σ_λ (not yet entirely understood) containing infinitely many copies of a Bykov cycle. More precisely, beyond the original primary links β_1 and β_2, there exist infinitely many subsidiary heteroclinic connections (transversal or not) turning around the original Bykov cycle. Since the complete description of Σ_λ is, at the moment, unreachable, we concentrate our attention on $\Sigma_\lambda^\star$.

The set of vector fields $f_0 \in \mathfrak{X}_{\text{Byk}}^r(\mathbb{S}^3)$, $r \geqslant 3$, which satisfy the conditions (P1)–(P5) contains a non-empty open subset of families $(f_\lambda)_{\lambda \in \mathbb{R}}$ for which the assumptions (P6)–(P8) hold (cf. [26]). In particular, there are such families of polynomial vector fields.

3. Overview

Let $(f_\lambda)_{\lambda \geqslant 0}$ be a one-parameter family of vector fields in $\mathfrak{X}_{Byk}^3(\mathbb{S}^3)$ satisfying (P1)–(P8). When $\lambda = 0$, the heteroclinic connections in the network Σ_0 are contained in fixed point subspaces satisfying the hypothesis (H1) of [17]. Due to the inequality $C_1 C_2 > E_1 E_2$, the stability criterion in [17] may be applied to Σ_0, and so there exists a three-dimensional normally hyperbolic invariant manifold where every solution eventually enters. Therefore, we may find an open neighborhood $\mathcal{U}^0$ of the heteroclinic network Σ_0 (whose closure is the set V^0) having its boundary transverse to the flow of the vector field f_0 and such that every solution starting in $\mathcal{U}^0$ remains in it for all positive time and is forward asymptotic to Σ_0. The global attracting set of f_0, namely

$$\mathcal{A}_0 = [\sigma_1 \to \sigma_2] \cup W_0^u(\sigma_2) = [\sigma_1 \to \sigma_2] \cup W_0^s(\sigma_1)$$

is made of σ_1, σ_2 and the two connections $[\sigma_1 \to \sigma_2]^{ext}$ and $[\sigma_1 \to \sigma_2]^{int}$, together with a sphere which is both the stable manifold of σ_1 and the unstable manifold of σ_2 (see the left part of Figure 1).

As the transverse property is robust, for small enough $\lambda > 0$, the neighborhood V^0 is also positively invariant by f_λ (that is, every solution of the vector field eventually enters this region and never leaves it in the future) and the flow of f_λ still has a three-dimensional normally hyperbolic invariant manifold transverse to the flow, inwardly oriented and containing the network Σ_λ. Due to the transversality of the flow to the boundary of some compact neighborhood V^0 of $\mathcal{A}_0$ such that $\varphi_\lambda(t, V^0)$ is contained in the interior of V^0 for all $t \geqslant 0$,

when $\lambda > 0$ is small enough the flow of f_λ still has a global attracting set, namely

$$\mathcal{A}_\lambda = \bigcap_{t \geqslant 0} \varphi\left(t, V^0\right). \tag{3.1}$$

We observe that, as the initial attracting set $\mathcal{A}_0$ is asymptotically stable and contains all the invariant manifolds of the equilibria, there exists $\lambda_0 > 0$ such that, for every $0 \leqslant \lambda \leqslant \lambda_0$, those invariant manifolds associated to f_λ are still in the interior of V^0. Using their flow invariance, we conclude that:

$$\overline{W_\lambda^s(\sigma_1) \cup W_\lambda^u(\sigma_1) \cup W_\lambda^s(\sigma_2) \cup W_\lambda^u(\sigma_2)} \subset \mathcal{A}_\lambda. \tag{3.2}$$

In what follows, $\mathcal{A}_\lambda$ will be called *a Bykov attracting set*, which may have infinitely many connected components.

In [2, 6, 18, 20], it was proved that there exists an open subset $\mathcal{C}$ of $\mathfrak{X}^3_{Byk}(\mathbb{S}^3)$ containing f_0 and such that the following properties hold for every family of vector fields $(f_\lambda)_\lambda$ subject to the conditions (P1)–(P8):

1. For $n \in \mathbb{N}$ and $\lambda > 0$, any tubular neighborhood of a Bykov cycle in $\Sigma_\lambda^\star$ contains infinitely many n–pulse (cf. Appendix A.8) heteroclinic connections $[\sigma_2 \to \sigma_1]$.

2. For any tubular neighborhood $\mathcal{T}$ of $\Sigma_\lambda^\star$ and every cross-section to the flow $S_q \subset \mathcal{T}$ at a point q in $[\sigma_2 \to \sigma_1]$, the first return map to S_q has a countable family of compact invariant sets $(\Lambda_{\lambda,m})_{m \in \mathbb{N}}$ in each of which the dynamics is conjugate to a full shift over a finite number of symbols. The union Λ_λ of these sets accumulates on $\Sigma_\lambda^\star$ while the number of symbols coding the first return map to S_q tends to infinity. Moreover, for every $m \in \mathbb{N}$, the topological horseshoe $\Lambda_{\lambda,m}$ is uniformly hyperbolic [2].

3. At any cross-section $S_q \subset \mathcal{T}$, the set of initial conditions in S_q that do not leave $\mathcal{T}$ by the flow, for all future and past times, is precisely Λ_λ. In particular, any tubular neighborhood $\mathcal{T}$ of a Bykov cycle Γ inside $\Sigma_\lambda^\star$ contains points not lying in Γ whose trajectories remain in $\mathcal{T}$ for all time.

4. There is a strictly decreasing sequence $(\lambda_i)_{i \in \mathbb{N}}$ of positive real numbers converging to 0 and such that, for any $\lambda > \lambda_i$, there are two 1–pulse heteroclinic connections for the flow of $\dot{x} = f_\lambda(x)$ that collapse into a 1–pulse heteroclinic tangency at $\lambda = \lambda_i$, and then disappear for $\lambda < \lambda_i$.

5. For each $i \in \mathbb{N}$, there is a sequence of parameter values $(\lambda_{ij})_{j \in \mathbb{N}}$ accumulating at λ_i for which the corresponding vector field has a k–pulse heteroclinic tangency for every $k \in \mathbb{N}$.

6. As λ decreases to zero, some of the saturated horseshoes associated to these heteroclinic tangencies disappear, while we witness the emergence of Newhouse phenomena and several k–pulses heteroclinic connections are destroyed through saddle-node type bifurcations.

In order to improve the readability of the paper, we have gathered in the following table the most important symbols used in the text.

Notation	Meaning
Σ_0	Initial attracting heteroclinic network
$\Sigma_\lambda^\star$	Network with four Bykov cycles
Σ_λ	Bigger network with infinitely many pulses
$\mathcal{A}_\lambda$	Global attracting set
Λ_λ	Union of countable many horseshoes

TABLE 1. Symbols used throughout the paper.

The transition from a simple steady state at $\lambda = 0$ to chaotic attractors when $\lambda > 0$ reminds of vector fields exhibiting instant chaos, as the ones presented in [13]. When λ changes, we witness the creation or destruction of invariant pieces of the non-wandering set, as well as the loss of the hyperbolic properties at some of them. We expect to see the family of flows $(f_\lambda)_\lambda$ undergoing saddle-node bifurcations inducing the destruction of horseshoes [9], cascades of period-doubling [28] and the unfolding of homoclinic tangencies [24]. Besides, some of these bifurcations may happen simultaneously. Yet, the complete bifurcation diagram is at the moment out of reach.

4. MAIN RESULTS

Let $(f_\lambda)_{\lambda \in \mathbb{R}}$ be a one-parameter family of vector fields in $\mathfrak{X}^3_{Byk}(\mathbb{S}^3)$ satisfying the conditions (P1)–(P8). For every small enough $\lambda > 0$, there exists a saddle-type set Λ_λ, invariant by the first-return map $\mathcal{P}_\lambda$ defined at an adequate cross-section S_λ, which is a countable union of horseshoes. Our first result, whose proof will be presented in Section 7, states that the saturation $\widetilde{\Lambda_\lambda}$ by the flow is dynamically linked to the equilibria (the operator tilde is defined in Appendix A.9). In what follows, the solution by the vector field of a fixed point in Λ_λ under the first return map to the cross section S_q is called 1-periodic orbit of $\widetilde{\Lambda_\lambda}$.

Theorem A. *Let $(f_\lambda)_\lambda$ be a one-parameter family of vector fields in $\mathfrak{X}^3_{Byk}(\mathbb{S}^3)$ satisfying (P1)–(P8). Consider a tubular neighborhood $\mathcal{T}$ of the Bykov network $\Sigma_\lambda^\star$. Then there exists $\lambda_0 > 0$ such that, for every $0 < \lambda \leqslant \lambda_0$, one has:*

(1) $\overline{\widetilde{\Lambda_\lambda}} \subset \overline{W_\lambda^u(\sigma_2) \cap W_\lambda^s(\sigma_1)}.$

(2) $\overline{W_\lambda^s(\widetilde{\Lambda_\lambda})} \subset \overline{W_\lambda^s(\sigma_1)}$ *and* $\overline{W_\lambda^u(\widetilde{\Lambda_\lambda})} \subset \overline{W_\lambda^u(\sigma_2)}.$

(3) *The Lyapunov exponents at the 1-periodic orbits of $\widetilde{\Lambda_\lambda}$ are uniformly bounded away from zero.*

Without much information about the global attracting set $\mathcal{A}_\lambda$, we decided to look for chain-accessible and chain-recurrent points. In spite of exploring approximate solutions instead of true trajectories, this strategy has the advantage of reasonably explaining some difficult aspects of the phase portrait of f_λ. Let V^0 be as in Section 3 and denote by $\mathcal{B}_\lambda$ the set

$$\mathcal{B}_\lambda = \{P \in V^0 : P \text{ is chain-accessible from } \sigma_2\}. \tag{4.1}$$

When $\lambda = 0$, one has $\mathcal{A}_0 = \mathcal{B}_0$. For $\lambda > 0$, there may exist either attracting periodic orbits or other more complex proper attractors, whose presence in the phase space of f_λ is hard to confirm, but the dynamical role of the set $\mathcal{B}_\lambda$ within the global attracting set is also worthwhile to unravel due to the following properties (to be proved in Section 10).

Theorem B. *Under the hypotheses of Theorem A, there exists $\lambda_1 > 0$ such that, for every $0 < \lambda \leqslant \lambda_1$, the set $\mathcal{B}_\lambda$ is compact, forward invariant and chain-stable. Moreover, the union $\overline{W_\lambda^u(\sigma_2)} \cup \overline{W_\lambda^s(\sigma_1)}$ is contained in $\mathcal{A}_\lambda \cap \mathcal{B}_\lambda$.*

5. Transition maps

In this section we will analyze the dynamics near the Bykov cycles of $\Sigma_\lambda^\star$ through local maps, after selecting appropriate linearizing coordinates in cylindrical neighborhoods of the saddle-foci σ_1 and σ_2 (see Figure 2), as done in [18]. In those cylindrical coordinates (ρ, θ, z) the linearization of the dynamics at σ_1 and σ_2 is specified by the following equations:

$$\begin{cases} \dot{\rho} = -C_1\rho \\ \dot{\theta} = 1 \\ \dot{z} = E_1 z \end{cases} \quad \text{and} \quad \begin{cases} \dot{\rho} = E_2\rho \\ \dot{\theta} = 1 \\ \dot{z} = -C_2 z. \end{cases} \tag{5.1}$$
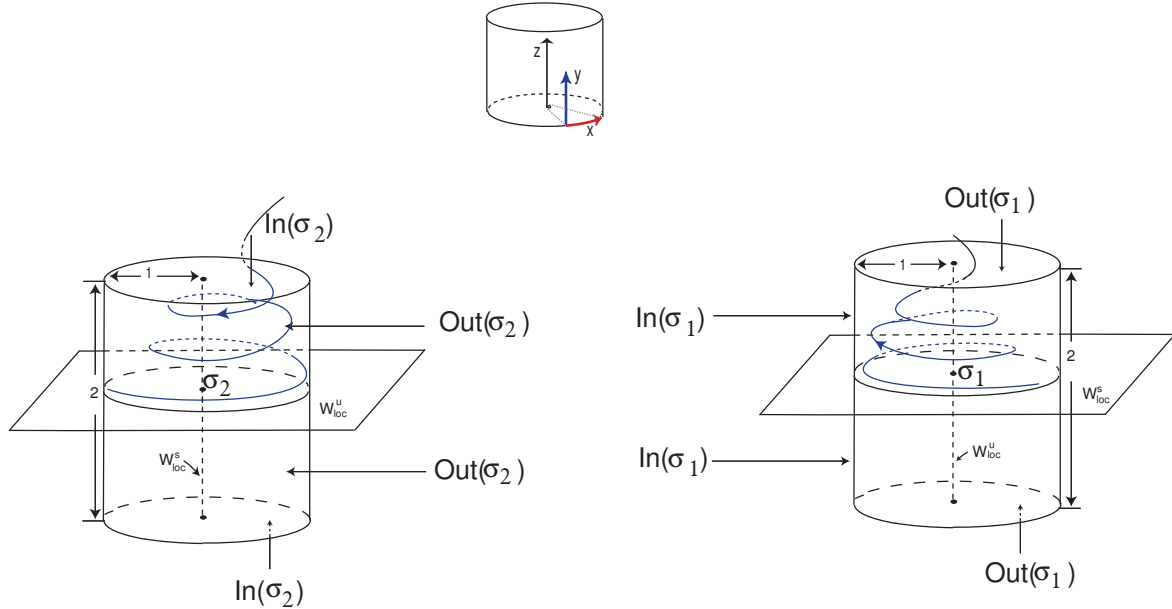

FIGURE 2. Local cylindrical coordinates in V_1 and V_2.

Assume that those cylindrical neighborhoods of σ_1 and σ_2, which we call V_1 and V_2, have base-radius $\varepsilon > 0$ and height 2ε. After a linear rescaling of the variables, we may suppose that $\varepsilon = 1$. The boundaries of V_1 and V_2 have three components: the cylinder wall, parameterized in cylindrical coordinates $(r, \varphi, z) \equiv (1, x, y)$ by $x \in [0, 2\pi[$ and $|y| \leqslant 1$; and two discs, the top and bottom of each cylinder, with polar parameterizations $(r, \varphi, \pm 1)$, where $0 \leqslant r \leqslant 1$ and $\varphi \in [0, 2\pi[$.

In V_1, we will use the following notation: $\text{In}(\sigma_1)$ is the subset of the cylinder wall of V_1 consisting of points that enter V_1 in positive time; $\text{Out}(\sigma_1)$ is the subset of the top and bottom of V_1 made of the points that leave V_1 in positive time; $\text{In}^+(\sigma_1)$ is the upper part of the cylinder, parameterized by (x, y) with $x \in [0, 2\pi)$ and $y \in (0, 1)$; dually, $\text{In}^-(\sigma_1)$ is its lower part, parameterized by (x, y) with $x \in [0, 2\pi)$ and $y \in (-1, 0)$. Similarly, we define the cross-sections for the linearization inside the neighborhood V_2 of σ_2. Observe that the flow is transverse to these cross-sections.

The local stable manifold of σ_1, say $W^s_{\lambda, \text{loc}}(\sigma_1)$, corresponds precisely to the disk in V_1 parameterized by $z = 0$. The local unstable manifold $W^u_{\lambda, \text{loc}}(\sigma_1)$ of σ_1 is the z-axis in V_1, intersecting the top and bottom disks of this cylinder at their centers. A similar description of the local invariant manifolds of σ_2 is valid in V_2.

5.1. **Local maps near the saddle-foci.** Integrating (5.1), we deduce that the trajectory of a point $(x, y) \in \text{In}^+(\sigma_1)$ leaves V_1 through $\text{Out}^+(\sigma_1)$ at the point

$$\Phi_1^+(x, y) = \left(y^{\delta_1}, \ -\frac{\log y}{E_1} + x\right) = (r, \varphi) \tag{5.2}$$

where $\delta_1 = \frac{C_1}{E_1} > 1$. Similarly, a point $(r, \varphi) \in \text{In}^+(\sigma_2) \setminus W^s_{\lambda, \text{loc}}(\sigma_2)$ leaves V_2 by $\text{Out}^+(\sigma_2)$ at

$$\Phi_2^+(r, \varphi) = \left(-\frac{\log r}{E_2} + \varphi, \ r^{\delta_2}\right) = (x, y) \tag{5.3}$$

where $\delta_2 = \frac{C_2}{E_2} > 1$. Analogous expressions define the local maps $\Phi_1^- : \text{In}^-(\sigma_1) \to \text{Out}^-(\sigma_1)$ and $\Phi_2^- : \text{In}^-(\sigma_2) \to \text{Out}^-(\sigma_2)$. Finally, the map

$$\Phi_1 : \text{In}^+(\sigma_1) \cup \text{In}^-(\sigma_1) \to \text{Out}(\sigma_1)$$

is just given by Φ_1^+ in $\text{In}^+(\sigma_1)$ and Φ_1^- in $\text{In}^-(\sigma_1)$. Similarly, we gather in Φ_2 both maps Φ_2^+ and Φ_2^-

5.2. **The transitions.** The coordinates on V_1 and V_2 are chosen so that $[\sigma_1 \to \sigma_2]$ connects points with $z > 0$ (resp. $z < 0$) in V_1 to points with $z > 0$ (resp. $z < 0$) in V_2. Points in $\text{Out}(\sigma_1) \setminus W^u_{\lambda, \text{loc}}(\sigma_1)$ are mapped into $\text{In}(\sigma_2)$ along a flow-box around each of the connections $[\sigma_1 \to \sigma_2]$. We will assume that the transition

$$\Psi_{1 \to 2}^\pm : \quad \text{Out}(\sigma_1) \quad \to \quad \text{In}(\sigma_2)$$

does not depend on λ and is the Identity map, a choice compatible with hypothesis (P5) due to the fact that the nodes σ_1 and σ_2 have the same chirality. Denote by η^+ the first hit map

$$\eta^+ = \Phi_2^+ \circ \Psi_{1 \to 2}^+ \circ \Phi_1^+ : \quad \text{In}^+(\sigma_1) \quad \to \quad \text{Out}^+(\sigma_2).$$

From (5.2) and (5.3) we infer that, in local coordinates, for $y > 0$ we have

$$\eta^+(x, y) = \left(x - K \log y \ (mod\, 2\pi), \ y^\delta\right) + o(y^\delta) \tag{5.4}$$

with

$$\delta = \delta_1 \delta_2 > 1 \qquad \text{and} \qquad K = \frac{C_1 + E_2}{E_1 E_2} > 0. \tag{5.5}$$

Analogously, we define the transition $\eta^- = \Phi_2^- \circ \Psi_{1 \to 2}^- \circ \Phi_1^-$ in $\operatorname{In}^-(\sigma_1)$. The global first hit map is precisely

$$\eta : \operatorname{In}^+(\sigma_1) \cup \operatorname{In}^-(\sigma_1) \to \operatorname{Out}(\sigma_2)$$

given by $\eta = \eta^+$ in $\operatorname{In}^+(\sigma_1)$ and $\eta = \eta^-$ in $\operatorname{In}^-(\sigma_1)$. When $\lambda \neq 0$, we have another transition map, namely

$$\Psi_{2 \to 1}^\lambda : \quad \operatorname{Out}(\sigma_2) \to \operatorname{In}(\sigma_1)$$

which depends on the parameter λ. Due to the mentioned higher order terms and the assumption (P7), its expression is only approximately known, and this is why we will have to resort to a model (cf. Section 6). To build such a model, let us examine the images by $\Psi_{2 \to 1}^\lambda$ of the two-dimensional invariant manifolds of σ_1 and σ_2 when $\lambda \neq 0$.

Denote by $P_1 = (p_1, 0)$ and $P_2 = (p_2, 0)$, with $0 \leqslant p_1 < p_2 \leqslant 2\pi$, the two points where the connections $[\sigma_2 \to \sigma_1]$ intersect $\operatorname{Out}(\sigma_2)$, and by $Q_1 = (q_1, 0)$ and $Q_2 = (q_2, 0)$, with $0 \leqslant q_2 < q_1 \leqslant 2\pi$, the two points where $[\sigma_2 \to \sigma_1]$ meets $\operatorname{In}(\sigma_1)$ (seen Figure 3). Notice that, for each $j = 1, 2$, the points P_j and Q_j are on the same trajectory. As the manifolds $W_0^u(\sigma_2)$ and $W_0^s(\sigma_1)$ coincide when $\lambda = 0$, we may assume that, for small $\lambda > 0$, the manifold $W_{\lambda, \text{loc}}^s(\sigma_1)$ (respectively $W_{\lambda, \text{loc}}^u(\sigma_2)$) intersects the wall $\operatorname{Out}(\sigma_2)$ of the cylinder V_2 (respectively the wall $\operatorname{In}(\sigma_1)$ of the cylinder V_1) on a closed curve. This is precisely the expected unfolding from the coincidence when $\lambda = 0$ of the manifolds $W_0^s(\sigma_1)$ and $W_0^u(\sigma_2)$, as established in [8].

We will assume that, for $\lambda > 0$ the intersection $W_\lambda^u(\sigma_2) \cap \operatorname{In}(\sigma_1)$ is sinusoidal in shape. Nevertheless, any smooth $2\pi-$periodic function would work (cf. Section 7 of [7]). So, for small $\lambda > 0$, these curves are such that $W_{\lambda, \text{loc}}^s(\sigma_1) \cap \operatorname{Out}(\sigma_2)$ is the graph of a map $y = \xi_s(x, \lambda)$, with $\xi_s(p_j, \lambda) = 0$ for $j = 1, 2$; $W_{\lambda, \text{loc}}^u(\sigma_2) \cap \operatorname{In}(\sigma_1)$ is the graph of $y = \xi_u(x, \lambda)$, with $\xi_u(q_j, \lambda) = 0$ for $j = 1, 2$; the maximum value of $\xi_u(x, \lambda)$ is attained at some point $(x_1(\lambda), y_1(\lambda))$ with $q_2 < x_1(\lambda) < q_1$; $\xi_s(x, 0) \equiv 0 \equiv \xi_u(x, 0)$; $\xi_s'(p_1, \lambda) > 0$, $\xi_s'(p_2, \lambda) < 0$, $\xi_u'(q_1, \lambda) < 0$ and $\xi_u'(q_2, \lambda) > 0$. Moreover, we have $\xi_u''(x) < 0$ in $]q_2, q_1[$. Check this technical details in Figure 3.

6. A MODEL FOR THE TRANSITION

In order to deal with an workable expression of the first return map, we have chosen a simple transition map from $\operatorname{Out}(\sigma_2)$ to $\operatorname{In}(\sigma_1)$. This is the main specificity of the model we will consider.

6.1. **The model.** In what follows, we will assume that

$$\xi_u(x, \lambda) \quad = \quad \lambda \sin x \tag{6.1}$$

and we get

$$\Psi_{2 \to 1}^\lambda(x, y) \quad = \quad (x, \, y + \lambda \sin x)$$
$$\eta^+(x, y) \quad = \quad \left(x - K \log y \ (mod \, 2\pi), \, y^\delta \right).$$

Let

$$\mathcal{P}_\lambda := \Psi_{2 \to 1}^\lambda \circ \eta = \Psi_{2 \to 1}^\lambda \circ \Phi_2 \circ \Psi_{1 \to 2} \circ \Phi_1 \tag{6.2}$$
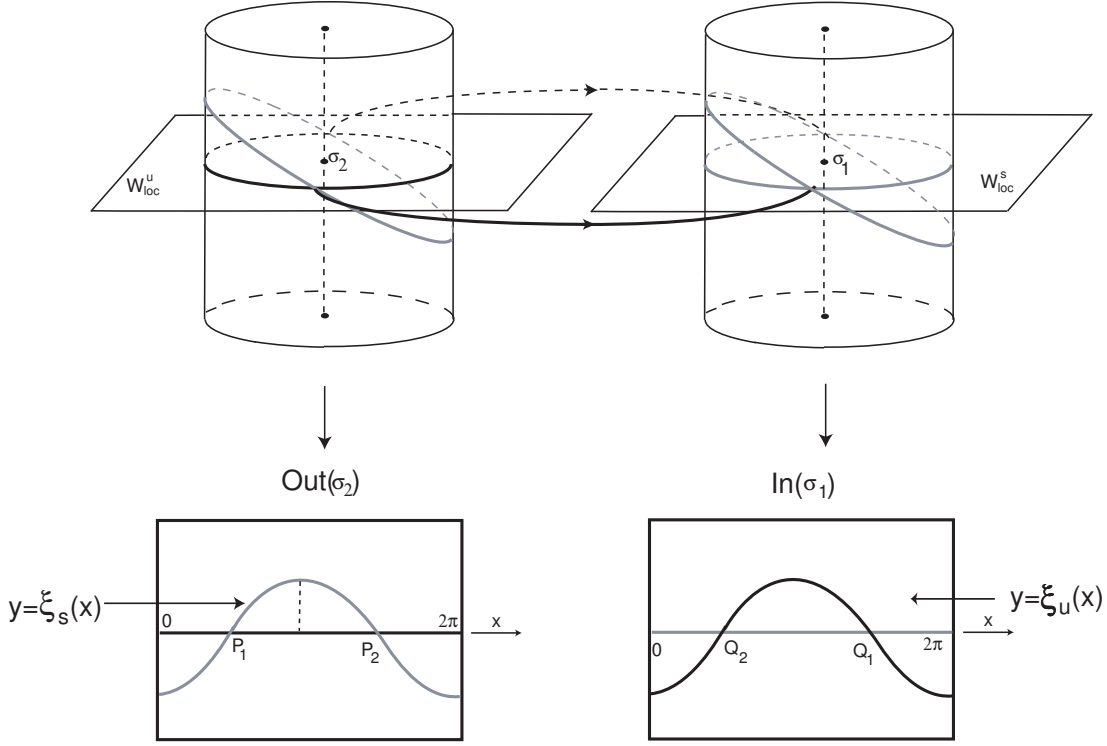
FIGURE 3. For $\lambda \gtrsim 0$, $W^s_\lambda(\sigma_1)$ intersects the wall $\mathrm{Out}(\sigma_2)$ of the cylinder V_2 and $W^u_\lambda(\sigma_2)$ intersects the wall $\mathrm{In}(\sigma_1)$ of the cylinder V_1 on closed curves given, in local coordinates, by the graph of a 2π−periodic function of sinusoidal type. In this work we assume that this sinusoidal curve is the graph of the sine map.

be the first return map to $\mathrm{In}(\sigma_1)$. Its domain $\mathcal{D}$ is the set of initial conditions $(x, y) \in \mathrm{In}(\sigma_1)$ whose trajectory returns to $\mathrm{In}(\sigma_1)$. It follows from (6.1) that, when $y > 0$, the first return map at (x, y), say $\mathcal{P}_\lambda(x, y) = (\mathcal{P}_{\lambda,1}(x, y), \mathcal{P}_{\lambda,2}(x, y))$, is given by

$$\mathcal{P}_\lambda(x, y) = \left(x - K \log y \ (mod\, 2\pi), \ y^\delta + \lambda \sin(x - K \log y) \right). \tag{6.3}$$

Besides,

$$D\Psi^\lambda_{2\to 1}(x, y) = \begin{pmatrix} 1 & 0 \\ \frac{\partial \xi_u}{\partial x} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda \cos(x) & 1 \end{pmatrix}.$$

Therefore, as

$$D\eta(x, y) = \begin{pmatrix} 1 & \dfrac{-K}{y} \\ 0 & \delta y^{\delta-1} \end{pmatrix}$$

we get

$$\det D\mathcal{P}_\lambda(x, y) = \det D\Psi^\lambda_{2\to 1}\left(\eta(x, y)\right) \times \det D\eta(x, y) = \delta y^{\delta-1}.$$

Thus, for sufficiently small $y > 0$, the first return map $\mathcal{P}_\lambda$ is contracting. After appropriate adaptations, we obtain similar expressions for $y < 0$.

6.2. First return time.

One of the advantages of considering the previous model is that we will be able to explicitly compute the time of flight around the heteroclinic cycle of a periodic solution that turns once around Σ_λ. A trajectory whose initial condition is $(1, \theta, z) \in \text{In}^+(\sigma_1)$ arrives at $\text{Out}^+(\sigma_1)$ after a period of time equal to

$$- \frac{\log z}{E_1}. \tag{6.4}$$

Consequently, given $(x, y) \in \mathcal{D} \subset \text{In}^+(\sigma_1)$, if $t_1(x, y) \in \mathbb{R}^+$ denotes the time needed for its first return to $\text{In}(\sigma_1)$, then

$$t_1(x, y) = \left| \frac{1}{E_1} \log y \right| + \left| \frac{1}{E_2} \frac{C_1}{E_1} \log y \right| = \left| \frac{E_2 + C_1}{E_1 E_2} \log y \right| = -K \log y. \tag{6.5}$$

Analogous formulas for $(x, y) \in \text{In}^-(\sigma_1)$.

7. Proof of Theorem A

Recall that we have fixed a tubular neighborhood $\mathcal{T}$ of $\Sigma_\lambda^\star$ where we may find a countable union of topological horseshoes $\left(\Lambda_{\lambda,m} \right)_{m \in \mathbb{N}}$, denoted by Λ_λ, whose saturation $\widetilde{\Lambda_\lambda}$ by the flow does not leave $\mathcal{T}$ (cf. [2, 27]). For each $m \in \mathbb{N}$, the set $\Lambda_{\lambda,m}$ is contained in a rectangle $[-\tau, \tau] \times [a_m, b_m]$ inside $\text{In}(\sigma_1)$, for sequences $(a_m)_{m \in \mathbb{N}}$ and $(b_m)_{m \in \mathbb{N}}$ explicitly given in [25]. Having fixed $\lambda > 0$, in this section we focus our attention on the dynamics inside two rectangles centered at the two connections β_1 and β_2, whose existence is guaranteed by (P7).

7.1. Linking $\widetilde{\Lambda_\lambda}$ with $W_\lambda^s(\sigma_1)$ and $W_\lambda^u(\sigma_2)$.

We start showing that $\widetilde{\Lambda_\lambda}$ is contained in the closure of the intersections of the two-dimensional manifolds $W_\lambda^u(\sigma_2)$ and $W_\lambda^s(\sigma_1)$, the so called heteroclinic class of σ_1 and σ_2. Let $p \in \Lambda_\lambda$.

From the construction of Λ_λ (cf. [27, §2.3]), there exist a sequence $(s_n)_{n \in \mathbb{Z}}$ of integers, a sequence of horizontal strips $(H_{s_0 s_1 \ldots s_n})_{n \in \mathbb{N}}$ and a sequence of vertical strips $(V_{s_0 s_1 \ldots s_n})_{n \in \mathbb{N}}$ such that the diameter of the sequence of intersections $\left(V_{s_0 s_{-1} \ldots s_{-n}} \cap H_{s_0 s_1 \ldots s_n} \right)_{n \in \mathbb{N}}$ goes to zero and, for every $n \in \mathbb{N}_0$,

$$p \in V_{s_0 s_{-1} \ldots s_{-n}} \cap H_{s_0 s_1 \ldots s_n}.$$

Therefore, given $\varepsilon > 0$, there is $\ell \in \mathbb{N}$ large enough such that the diameter of the intersection $V_{s_0 s_{-1} \ldots s_{-\ell}} \cap H_{s_0 s_1 \ldots s_\ell}$ is smaller than ε. Moreover, ℓ may be chosen sufficiently large so that, in $V_{s_0 s_{-1} \ldots s_{-\ell}}$, the invariant sets $W_{\lambda,\text{loc}}^u(\sigma_2)$ and $W_{\lambda,\text{loc}}^s(\sigma_1)$ are vertical and horizontal fibers, respectively, which intersect transversely. Thus, $p \in \overline{W_\lambda^u(\sigma_2) \cap W_\lambda^s(\sigma_1)}$. Therefore $\widetilde{\Lambda_\lambda} \subset \overline{W_\lambda^u(\sigma_2) \cap W_\lambda^s(\sigma_1)}$.

7.2. Linking σ_1 and σ_2 with $\widetilde{\Lambda_\lambda}$.

Firstly, let us verify the inclusion $\overline{W_\lambda^u(\widetilde{\Lambda_\lambda})} \subset \overline{W_\lambda^u(\sigma_2)}$. Let P be an accumulation point of $W_\lambda^u(\widetilde{\Lambda_\lambda})$. Again, by the construction of Λ_λ, there is a sequence $(V_n)_n$ of vertical rectangles whose vertical boundaries are contained in $W_\lambda^u(\sigma_2)$, their widths approaches 0 as n goes to $+\infty$ and $\Lambda_\lambda \subset V_n$ for every $n \in \mathbb{N}$. By the continuity, in the adequate topologies, of the return map $\mathcal{P}_\lambda$ and of the compact parts of the invariant manifolds involved, for n sufficiently large the vertical boundary of V_n is arbitrarily close to $W_\lambda^u(\widetilde{\Lambda_\lambda})$. Thus, P is an accumulation point of $W_\lambda^u(\sigma_2)$.

7.3. **The fixed points of $\mathcal{P}_\lambda$.** Recall from (6.3) that the first return map in $\mathcal{D} \cap \mathrm{In}^+(\sigma_1)$ of the model is given by

$$\mathcal{P}_\lambda(x,y) = \left(x - K\log y \pmod{2\pi}, \ y^\delta + \lambda\sin(x - K\log(y))\right)$$

with $\delta > 1$, $K > 0$ and $\mathcal{D}$ as in (6.2). So, its fixed points in $\mathcal{D} \cap \mathrm{In}^+(\sigma_1)$, say $p_\ell = (x_\ell, y_\ell) \in \mathrm{In}^+(\sigma_1)$ with $\ell \in \mathbb{N}$, are solutions of the system of equations

$$\begin{cases} x - K\log(y) = x + 2\ell\pi \\[2mm] y^\delta + \lambda\sin(x - K\log(y)) = y. \end{cases}$$

Thus,

$$y_\ell = \exp\left(\frac{-2\ell\pi}{K}\right) \qquad \text{and} \qquad \sin(x_\ell) = \frac{y_\ell - y_\ell^\delta}{\lambda} = \frac{\exp\left(\frac{-2\ell\pi}{K}\right) - \exp\left(\frac{-2\ell\delta\pi}{K}\right)}{\lambda},$$

provided that $\left|\frac{y_\ell - y_\ell^\delta}{\lambda}\right| \leqslant 1$ or, equivalently, while it is true that

$$\lambda \geqslant \exp\left(\frac{-2\ell\pi}{K}\right) - \exp\left(\frac{-2\ell\delta\pi}{K}\right). \tag{7.1}$$

Consequently, as soon as λ becomes smaller that $\exp\left(\frac{-2\ell\pi}{K}\right) - \exp\left(\frac{-2\ell\delta\pi}{K}\right)$, the fixed point p_ℓ disappears. As $\Lambda_{\lambda,\ell}$ is contained in the homoclinic class $\overline{W_\lambda^s(p_{\lambda,\ell}) \cap W_\lambda^u(p_{\lambda,\ell})}$ of the fixed point $p_{\lambda,\ell}$, when the fixed point $p_{\lambda,\ell}$ disappears this horseshoe no longer exists.

Lemma 7.1 ([2]). *Given $\lambda > 0$, the derivative $D\mathcal{P}_\lambda$ at $p_\ell = (x_\ell, y_\ell) \in \mathcal{D} \cap \mathrm{In}^+(\sigma_1)$ has real eigenvalues $\mu_s(p_\ell)$ and $\mu_u(p_\ell)$ and eigenvectors $V_s(p_\ell)$ and $V_u(p_\ell)$ such that:*

(1) $\lim_{\ell \to +\infty} \mu_s(p_\ell) = 0^-$ *and* $\lim_{\ell \to +\infty} \mu_u(p_\ell) = -\infty$.

(2) $\lim_{\ell \to +\infty} \frac{V_s(p_\ell)}{\|V_s(p_\ell)\|} = \langle 1, 0\rangle$ *and* $\lim_{\ell \to +\infty} \frac{V_u(p_\ell)}{\|V_u(p_\ell)\|} = \langle 1, \lambda\rangle$.

The argument to prove this lemma is essentially the same as the proof of Theorem 6 of [2]. We include the explicit expressions for the trace and the determinant for future use. Having fixed $\lambda > 0$, the derivative of $\mathcal{P}_\lambda$ in $\mathcal{D} \cap \mathrm{In}^+(\sigma_1)$ is given by

$$D\mathcal{P}_\lambda(x,y) = \begin{pmatrix} 1 & -\frac{K}{y} \\[2mm] \lambda\cos(x - K\log(y)) & \delta y^{\delta-1} - \frac{\lambda K}{y}\cos(x - K\log(y)) \end{pmatrix}. \tag{7.2}$$

Consequently, the determinant of $D\mathcal{P}_\lambda(x,y)$ is positive and equal to $\delta y^{\delta-1}$ and its trace is given by $1 + \delta y^{\delta-1} - \frac{\lambda K}{y}\cos(x - K\log(y))$. At the fixed point $p_\ell = (x_\ell, y_\ell)$ we have $\det D\mathcal{P}_\lambda(p_\ell) > 0$ and $\lim_{\ell \to +\infty} \det D\mathcal{P}_\lambda(p_\ell) = 0$. In addition, the trace of $D\mathcal{P}_\lambda(p_\ell)$ may be rewritten as

$$\begin{aligned} \text{Trace } D\mathcal{P}_\lambda(p_\ell) &= 1 + \delta\exp\left(\frac{-2\pi(\delta-1)\ell}{K}\right) - \lambda K\exp\left(\frac{2\pi\ell}{K}\right)\cos(x_\ell) \\[2mm] &= 1 + \delta\exp\left(\frac{-2\pi(\delta-1)\ell}{K}\right) - \lambda K\exp\left(\frac{2\pi\ell}{K}\right)\sqrt{1 - \sin^2(x_\ell)} = \\[2mm] &= 1 + \delta\exp\left(\frac{-2\pi(\delta-1)\ell}{K}\right) - \\[2mm] &\quad - K\exp\left(\frac{2\pi\ell}{K}\right)\sqrt{\lambda^2 - \left[\exp\left(\frac{-2\ell\pi}{K}\right) - \exp\left(\frac{-2\ell\delta\pi}{K}\right)\right]^2}. \end{aligned}$$

Thus $\lim_{\ell \to +\infty} \text{Trace } D\mathcal{P}_\lambda(p_\ell) = -\infty$ if $\lambda > \exp\left(\frac{-2\ell\pi}{K}\right) - \exp\left(\frac{-2\ell\delta\pi}{K}\right)$.

Denote by det and trace the determinant and the trace of $D\mathcal{P}_\lambda(p_\ell)$, respectively. Given $\lambda > 0$ and $\ell \in \mathbb{N}$ such that

$$\lambda^2 - \left[\exp\left(\frac{-2\ell\pi}{K}\right) - \exp\left(\frac{-2\ell\delta\pi}{K}\right)\right]^2 > 0 \qquad \text{and} \qquad \text{trace}^2 - 4\det \geqslant 0,$$

the eigenvalues of $D\mathcal{P}_\lambda(p_\ell)$ satisfy the inequalities

$$\mu_u = \frac{\text{trace} - \sqrt{\text{trace}^2 - 4\det}}{2} \leqslant \frac{\text{trace}}{2} \qquad \text{and} \qquad \mu_s = \frac{\text{trace} + \sqrt{\text{trace}^2 - 4\det}}{2} < 0.$$

Since $\mu_s \times \mu_u = \det$ and $\mu_s + \mu_u = \text{trace}$, we conclude that $\lim_{\ell \to +\infty} \mu_s(p_\ell) = 0^-$ and $\lim_{\ell \to +\infty} \mu_u(p_\ell) = -\infty$.

8. THE EVOLUTION OF THE EIGENVALUES $\mu_s(p_\ell)$ AND $\mu_u(p_\ell)$

The values of $\mu_s(p_\ell)$ and $\mu_u(p_\ell)$ change with both ℓ and λ as depicted in Figures 4 and 5. The images on the left were obtained using Maple and the expressions of the trace and determinant of the derivative of the first return map; the image at the top right is a scheme of what we conjecture to be the global bifurcations diagram.

The analysis of these eigenvalues shows that, for each $\ell \in \mathbb{N}$, there are parameters $a_\ell < b_\ell < c_\ell < d_\ell$ in $[0,1]$ at which relevant dynamical changes occur. For $\lambda < a_\ell$, there is no fixed point p_ℓ. At $\lambda = a_\ell$, the fixed point p_ℓ is created through a saddle-node bifurcation yielding a sink and a saddle. In the interval $]b_\ell, c_\ell[$, the eigenvalues of the sink p_ℓ are complex, so it is a stable focus. At $\lambda = d_\ell$, the sink p_ℓ experiences another bifurcation, becoming a saddle. As expected (cf. [28]), cascades of saddle-nodes and period-doubling bifurcations happen after the parameter d_ℓ; and afterwards the stable and unstable manifolds of the fixed point p_ℓ unfold a tangency, thus creating the horseshoe $\Lambda_{\lambda,\ell}$ to which p_ℓ belongs.

The stable and unstable eigenspaces of p_ℓ are the solutions of the linear equations

$$D\mathcal{P}_\lambda(p_\ell)\begin{pmatrix} v_1^s \\ v_2^s \end{pmatrix} = \mu_s(p_\ell)\begin{pmatrix} v_1^s \\ v_2^s \end{pmatrix} \qquad \text{and} \qquad D\mathcal{P}_\lambda(p_\ell)\begin{pmatrix} v_1^u \\ v_2^u \end{pmatrix} = \mu_u(p_\ell)\begin{pmatrix} v_1^u \\ v_2^u \end{pmatrix}.$$

Thus, concerning the eigenspace of $\mu_s(p_\ell)$, if we choose $v_1^s = 1$ then we get

$$\lim_{\ell \to +\infty} v_2^s(p_\ell) = \lim_{\ell \to +\infty} \frac{1 - \mu_s(p_\ell)}{K} y_\ell = 0,$$

which means that the one-dimensional stable direction tends to be horizontal as y_ℓ approaches 0. Similarly, if $v_1^u = 1$ then

$$\begin{aligned}
v_2^u(p_\ell) &= \frac{1 - \mu_u(p_\ell)}{K} y_\ell = \frac{1}{K} y_\ell - \frac{\text{trace} - \mu_s(p_\ell)}{K} y_\ell = \left(\frac{1}{K} + \frac{\mu_s(p_\ell)}{K}\right) y_\ell - \frac{\text{trace}}{K} y_\ell = \\
&= \left(\frac{1}{K} + \frac{\mu_s(p_\ell)}{K}\right) y_\ell - \frac{1}{K}\left[1 + \delta y_\ell^{\delta-1} - \frac{\lambda K}{y_\ell}\cos(x_\ell - K\log(y_\ell))\right] y_\ell = \\
&= \left(\frac{1}{K} + \frac{\mu_s(p_\ell)}{K}\right) y_\ell - \left(\frac{1 + \delta y_\ell^{\delta-1}}{K}\right) y_\ell + \lambda\sqrt{1 - \left(\frac{y_\ell - y_\ell^\delta}{\lambda}\right)^2}.
\end{aligned}$$

Consequently, $\lim_{\ell \to +\infty} v_2^u(p_\ell) = \lambda$, meaning that the unstable one-dimensional space tends to the direction spanned by $(1, \lambda)$, which is precisely the slope of the tangent to the graph of ξ_u at the primary connection.
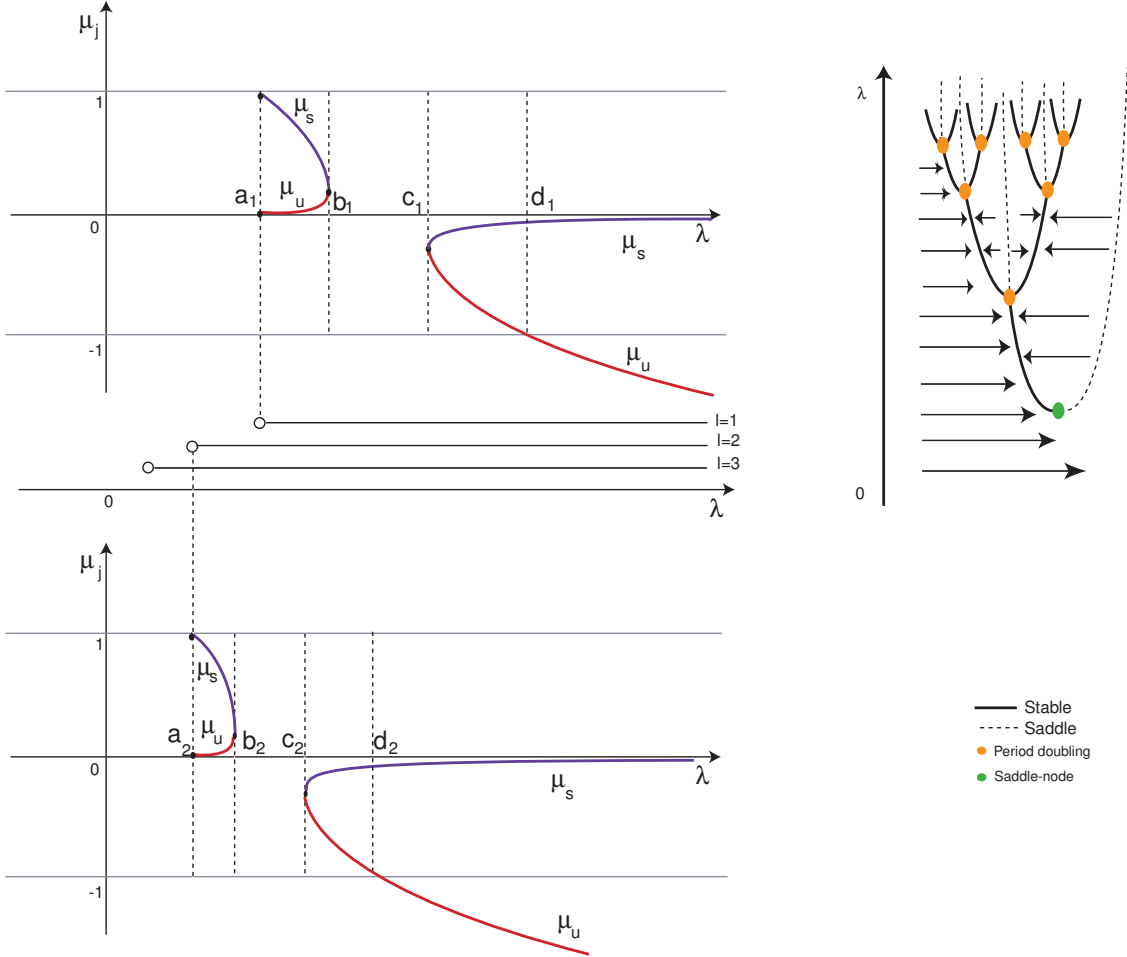
FIGURE 4. Left: Graphs of the eigenvalue-maps as functions of λ for $\ell = 1, 2$, $\delta = 4$ and $K = 0.2$. Right: A component of the global bifurcation diagram of $(f_\lambda)_\lambda$.

9. LYAPUNOV EXPONENTS OF $\mathcal{P}_\lambda$ AT p_ℓ

The Lyapunov exponents along the orbit of a fixed point of $\mathcal{P}_\lambda$ depend on the time needed for the trajectory to return to the cross section, whose main components are the sojourns spent near σ_1 and σ_2. Using the computations of Section 8, we may now estimate them.

The Lyapunov exponents at a fixed point $p_\ell \in \mathcal{D} \cap \mathrm{In}^+(\sigma_1)$ of $\mathcal{P}_\lambda$ are approximately given by

$$\chi_s(p_\ell) \simeq \frac{\log |\mu_s(p_\ell)|}{t_1(p_\ell)} \quad \text{and} \quad \chi_u(p_\ell) \simeq \frac{\log |\mu_u(p_\ell)|}{t_1(p_\ell)}$$

where $t_1(p_\ell)$ is the time of first return of (p_ℓ) to the cross section by the iteration of $\mathcal{P}_\lambda$. As $t_1(p_\ell) = -K \log y_\ell$ (see Subsection 6.2), $\delta > 1$ and $K > 0$ (cf. (5.5)), we will use the calculations of the previous subsection to estimate these exponents as follows. Firstly, using
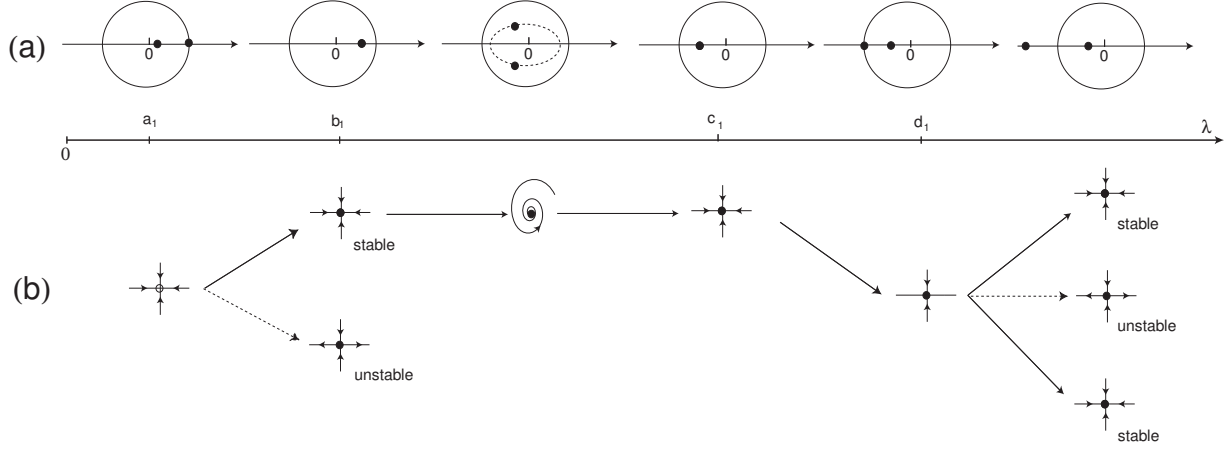
FIGURE 5. Lamerey diagram for the parameters of Figure 4 and $\ell = 1$. (a) Evolution of the eigenvalues within the unit circle. (b) Corresponding changes in the nature of the fixed point p_1.

(6.5), we get

$$\chi_s(p_\ell) + \chi_u(p_\ell) \simeq \frac{\log|\mu_s(p_\ell) \times \mu_u(p_\ell)|}{-K \log y_\ell} = \frac{\log\left(\delta\, y_\ell^{\delta-1}\right)}{-K \log y_\ell} = \frac{\log \delta}{-K \log y_\ell} + \frac{\delta - 1}{-K}$$

and so

$$\lim_{\ell \to +\infty} \left(\chi_s(p_\ell) + \chi_u(p_\ell)\right) = -\frac{\delta - 1}{K}. \tag{9.1}$$

On the other hand, we have

$$
\begin{aligned}
\frac{\log|\mu_s(p_\ell) + \mu_u(p_\ell)|}{-K \log y_\ell} &= \frac{\log\left(|\mu_u(p_\ell)|\left|1 + \frac{\mu_s(p_\ell)}{\mu_u(p_\ell)}\right|\right)}{-K \log y_\ell} = \frac{\log|\mu_u(p_\ell)|}{-K \log y_\ell} + \frac{\log\left|1 + \frac{\mu_s(p_\ell)}{\mu_u(p_\ell)}\right|}{-K \log y_\ell} \\
&\simeq \chi_u(p_\ell) + \frac{\log\left|1 + \frac{\mu_s(p_\ell)}{\mu_u(p_\ell)}\right|}{-K \log y_\ell}
\end{aligned}
$$

where

$$\lim_{\ell \to +\infty} \frac{\log\left|1 + \frac{\mu_s(p_\ell)}{\mu_u(p_\ell)}\right|}{-K \log y_\ell} = 0.$$

Moreover,

$$\lim_{\ell \to +\infty} \frac{\log |\mu_s(p_\ell) + \mu_u(p_\ell)|}{-K \log y_\ell} = \lim_{\ell \to +\infty} \frac{\log |\text{trace } D\mathcal{P}_\lambda(p_\ell)|}{-K \log y_\ell} =$$

$$= \lim_{\ell \to +\infty} \frac{\log \left(-1 - \delta y_\ell^{\delta-1} + \frac{\lambda K}{y_\ell} \cos(x_\ell - K \log y_\ell)\right)}{-K \log y_\ell} =$$

$$= \lim_{\ell \to +\infty} \frac{\log \left(-1 - \delta y_\ell^{\delta-1} + \frac{\lambda K}{y_\ell} \sqrt{1 - \left(\frac{y_\ell - y_\ell^\delta}{\lambda}\right)^2}\right)}{-K \log y_\ell} =$$

$$= \lim_{\ell \to +\infty} \frac{\log \left(\frac{-y_\ell - \delta y_\ell^\delta + K\sqrt{\lambda^2 - (y_\ell - y_\ell^\delta)^2}}{y_\ell}\right)}{-K \log y_\ell} =$$

$$= \lim_{\ell \to +\infty} \frac{-\log y_\ell}{-K \log y_\ell} + \lim_{\ell \to +\infty} \frac{\log \left(-y_\ell - \delta y_\ell^\delta + K\sqrt{\lambda^2 - (y_\ell - y_\ell^\delta)^2}\right)}{-K \log y_\ell} = \frac{1}{K}.$$

Thus,

$$\lim_{\ell \to +\infty} \chi_u(p_\ell) = \frac{1}{K} > 0$$

and so, by (9.1),

$$\lim_{\ell \to +\infty} \chi_s(p_\ell) = -\frac{\delta}{K} < 0.$$

10. Proof of Theorem B

Let V^0 be as in Section 3 and $\mathcal{B}_\lambda = \{P \in V^0 : P \text{ is chain-accessible from } \sigma_2\}$. We first observe that $\mathcal{B}_\lambda$ coincides with $\mathcal{C}_\lambda = \{P \in V^0 : P \text{ is chain-accessible from } \sigma_1\}$. This is a general property of any heteroclinic network linking two equilibria and a straightforward consequence of the following two facts. The equilibrium σ_2 is chain-accessible from σ_1 because, given $\varepsilon > 0$ and $\tau > 0$, any point of $[\sigma_1 \to \sigma_2] \setminus \{\sigma_1\}$ at a distance from σ_1 smaller than ε is an (ε, τ)–trajectory connecting σ_1 with σ_2. Thus, $\mathcal{B}_\lambda \subset \mathcal{C}_\lambda$. Conversely, σ_1 is chain-accessible from σ_2 due to the fact that, given $\varepsilon > 0$ and $\tau > 0$, any point $P_1 \neq \sigma_2$ of a 0–pulse at a distance from σ_2 smaller than ε is an (ε, τ)–trajectory connecting σ_2 with σ_1. So $\mathcal{C}_\lambda \subset \mathcal{B}_\lambda$.

We proceed showing that $\mathcal{B}_\lambda$ is forward invariant by the flow, closed and chain-stable.

10.1. **Forward invariant.** Given $\lambda \geqslant 0$, $P \in \mathcal{B}_\lambda$ and $t \geqslant 0$, if $\{P_1, \cdots, P_k\}$ is an (ε, τ)–trajectory connecting σ_2 to P, then $\{P_1, \cdots, P_k, P\}$ is an $(\varepsilon, \min\{\tau, t\})$–trajectory connecting σ_2 to $\varphi_\lambda(t, P)$. Thus, $\varphi_\lambda(t, \mathcal{B}_\lambda) \subset \mathcal{B}_\lambda$ for every $t \geqslant 0$; that is, $\mathcal{B}_\lambda$ is forward invariant.

10.2. **Closed.** We now verify that accessibility from σ_2 (and more generally from any point) is a closed property. Let Q be a point in $\overline{\mathcal{B}_\lambda}$ and $\varepsilon > 0$ arbitrarily small. Take the open ball $B_{\frac{\varepsilon}{2}}(Q)$ centered at Q with radius $\frac{\varepsilon}{2}$ and $P \in \mathcal{B}_\lambda \cap B_{\frac{\varepsilon}{2}}(Q)$. By definition of $\mathcal{B}_\lambda$, there are $\tau > 0$ and an $(\frac{\varepsilon}{2}, \tau)$–trajectory $\{P_1, \cdots, P_k\}$ connecting σ_2 to P. Therefore, the distance between P_k and Q is smaller than ε, and so $\{P_1, \cdots, P_k\}$ is an $(\frac{\varepsilon}{2}, \tau)$–trajectory connecting σ_2 to Q as well. Thus $Q \in \mathcal{B}_\lambda$ and $\mathcal{B}_\lambda$ is closed. Notice also that, if $P \in \mathcal{B}_\lambda$ and $\{P_1, \cdots, P_k\}$ is an (ε, τ)–trajectory connecting σ_2 to P, then any P_j is, by definition, chain-accessible from σ_2. Thus, $P_j \in \mathcal{B}_\lambda$ for every $1 \leqslant j \leqslant k$.

10.3. Chain-stable. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a decreasing sequence of positive real numbers converging to 0 and define $\mathcal{U}_j$ as the set of points $P \in V^0$ that are chain-accessible from σ_2 by an (ε_j, τ)–trajectory for some $\tau > 0$. Notice that each $\mathcal{U}_j$ is open. Indeed, given $P \in \mathcal{U}_j$, let $\{P_1, \cdots, P_k\}$ an (ε_j, τ)–trajectory connecting σ_2 to P. Then the open ball $B_\rho(P)$ centered at P with radius $\rho = \frac{\varepsilon_j - \text{distance}\,(P, P_k)}{2} > 0$ is contained in $\mathcal{U}_j$ since, for every $Q \in B_\rho(P)$, we have

$$\text{distance}\,(Q, P_k) \;\leqslant\; \text{distance}\,(Q, P) + \text{distance}\,(P, P_k) < \rho + \text{distance}\,(P, P_k)$$
$$< \;\frac{\varepsilon_j + \varepsilon_j}{2} = \varepsilon_j.$$

Therefore, $\{P_1, \cdots, P_k\}$ is an (ε_j, τ)–trajectory also connecting σ_2 to Q. Moreover, every (ε_j, τ)–trajectory connecting two points of $\mathcal{B}_\lambda$ is contained in $\mathcal{U}_j$. Hence $\mathcal{B}_\lambda = \bigcap_j \mathcal{U}_j$, and $(\mathcal{U}_j)_{j\in\mathbb{N}}$ is a family of nested neighborhoods of $\mathcal{B}_\lambda$. Now, given an open neighborhood $\mathcal{V}$ of $\mathcal{B}_\lambda$, there exists $j \in \mathbb{N}$ such that $\mathcal{U}_j \subset \mathcal{V}$. So every (ε_j, τ)–trajectory connecting two points of $\mathcal{B}_\lambda$ is contained in $\mathcal{V}$. Thus $\mathcal{B}_\lambda$ is chain-stable.

10.4. Relevant subsets of $\mathcal{B}_\lambda$. We are left to show that $\overline{W_\lambda^u(\sigma_2) \cup W_\lambda^s(\sigma_1)} \subset \mathcal{A}_\lambda \cap \mathcal{B}_\lambda$. We start recalling that both manifolds $W_\lambda^s(\sigma_1)$ and $W_\lambda^u(\sigma_2)$ are contained in $\mathcal{A}_\lambda$ (see (3.2)). It is also immediate to verify that the manifold $W_\lambda^u(\sigma_2)$, and so its closure, is contained in $\mathcal{B}_\lambda$. Indeed, take $P \in W_\lambda^u(\sigma_2)$, $\varepsilon > 0$ and $t > 0$ such that $\varphi_\lambda(-t, P)$ is at a distance smaller than ε from σ_2. Then the (ε, t)–trajectory $\{\varphi_\lambda(-t, P)\}$ connects σ_2 to P; thus, P belongs to $\mathcal{B}_\lambda$. Observe now that $W^s(\sigma_1) \subset \mathcal{C}_\lambda = \mathcal{B}_\lambda$, hence $\overline{W_\lambda^s(\sigma_1)} \subset \mathcal{B}_\lambda$.

10.5. Chain-recurrence. For $\lambda = 0$ the attracting set $\mathcal{A}_0$ is a chain-recurrent class and coincides with $\mathcal{B}_0$. When $\lambda > 0$, by [14, Lemma 3.2 and Remark 4.6] or [3, Theorem A], the set $\mathcal{B}_\lambda$ is a chain-recurrent class unless it contains proper attractors. We observe that, if $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ are chain-recurrent classes, one has $\mathcal{A}_\lambda = \mathcal{B}_\lambda$. Indeed, on the one hand, $\mathcal{B}_\lambda \subset \mathcal{A}_\lambda$ since we are assuming that both $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ are chain-recurrent classes, $\mathcal{A}_\lambda$ is an attracting set and $\mathcal{A}_\lambda \cap \mathcal{B}_\lambda \neq \emptyset$. On the other hand, as σ_2 belongs to $\mathcal{A}_\lambda$ and this set is a chain-recurrent class, any of its points is chain-accessible from σ_2; therefore $\mathcal{A}_\lambda \subset \mathcal{B}_\lambda$.

However, we do not know whether $\mathcal{A}_\lambda = \mathcal{B}_\lambda$ when $\lambda > 0$, and it seems unlikely to find λ whose $\mathcal{A}_\lambda$ does not contain proper attractors (though we do know that there are many parameters for which the flow exhibits proper attractors (cf. items (4)–(6) of Section 3). Yet, it is easy to conclude that the subset of $\mathcal{A}_\lambda$ whose elements are accumulation points of trajectories starting at initial conditions in $\mathcal{B}_\lambda$ is contained in $\mathcal{B}_\lambda$. Indeed, consider $\varepsilon > 0$ and $P \in \mathcal{A}_\lambda$ for which there exists $Q \in \mathcal{B}_\lambda$ whose ω-limit contains P. Take $\tau > 0$ and an (ε, τ)-trajectory $\{P_1, \cdots, P_k\}$ connecting σ_2 to Q. As P is in the ω-limit of Q, there is $t > \tau$ such that the distance between P and $\varphi_\lambda(t, Q)$ is smaller than ε. Therefore, the set $\{P_1, \cdots, P_k, Q\}$ is an (ε, τ)-trajectory connecting σ_2 to P. So P belongs to $\mathcal{B}_\lambda$.

10.6. Final remark. For $\lambda = 0$, the manifold $W_0^s(\sigma_1) \equiv W_0^u(\sigma_2)$ is chain-accessible from any point of V^0. In fact, as $\mathcal{A}_0 = [\sigma_1 \to \sigma_2] \cup W_0^s(\sigma_1)$, given $\varepsilon > 0$, $\tau > 0$ and $P \in V^0$, we may find $R \in V^0 \setminus [\sigma_1 \to \sigma_2]$ at a distance from P smaller than ε and whose ω–limit by the flow of f_0 contains $W_0^s(\sigma_1)$. Thus, there is $t > \tau$ such that the distance from $\varphi_\lambda(t, R)$ to $W_0^s(\sigma_1)$ is smaller than ε (that is, we may find $Q \in W_0^s(\sigma_1)$ such that the distance from $\varphi_\lambda(t, R)$ to Q is smaller than ε). Therefore, $W_0^s(\sigma_1)$ is (ε, τ)–chain-accessible from P through the (ε, τ)–trajectory $\{R\}$ that connects P to Q. As ε is arbitrary, this proves that $W_0^s(\sigma_1)$ is chain-accessible from any $P \in V^0$.

For $\lambda > 0$, this property may not be valid due to the presence of sinks or other proper attractors in $\mathcal{A}_\lambda$. Meanwhile, we claim that:

Claim: *If either $W_\lambda^s(\sigma_1)$ or σ_2 is chain-accessible from a point in V^0, then so is every element of $\mathcal{B}_\lambda$.*

Proof. Consider $X \in V^0$, $P \in \mathcal{B}_\lambda$ and assume that $W_\lambda^s(\sigma_1)$ is chain-accessible from X (the argument is similar if we replace $W_\lambda^s(\sigma_1)$ by σ_2). Given $\varepsilon > 0$ arbitrarily small, there exist $S_1 \in W_\lambda^s(\sigma_1)$, $\tau > 0$ and an $(\frac{\varepsilon}{2}, \tau)$–trajectory $\{X_1, \cdots, X_m\}$ connecting X to S_1. Iterate now S_1 a time $t_1 > \tau$ so that the distance from $\varphi_\lambda(t_1, S_1)$ to σ_1 is smaller than $\frac{\varepsilon}{2}$. Thus, the set $\{X_1, \cdots, X_m, S_1\}$ is an $(\frac{\varepsilon}{2}, \tau)$–trajectory connecting X to σ_1. Afterwards, take a point $S_2 \in [\sigma_1 \to \sigma_2]$ at a distance smaller than $\frac{\varepsilon}{2}$ from σ_1; notice that the points $\{X_1, \cdots, X_m, S_1\}$ also form an (ε, τ)–trajectory connecting X to S_2. If we now iterate S_2 a time $t_2 > \tau$, we reach a point $\varphi_\lambda(t_2, S_2)$ which is at a distance smaller than $\frac{\varepsilon}{2}$ from σ_2. Moreover, as $P \in \mathcal{B}_\lambda$, we may find an $(\frac{\varepsilon}{2}, \tau)$–trajectory $\{P_1, \cdots, P_k\}$ connecting σ_2 to P. Therefore, the points $\{X_1, \cdots, X_m, S_2, P_1, \cdots, P_k\}$ produce an (ε, τ)–trajectory connecting X to P. This proves that $\mathcal{B}_\lambda$ is chain-accessible from X. $\square$

Appendix A. Glossary

A.1. **Hyperbolicity.** Let M be a smooth compact Riemannian manifold, and consider a diffeomorphism $h : M \to M$ and a compact h-invariant set $\mathcal{K} \subset \mathbb{S}^3$. We say that $\mathcal{K}$ is *uniformly hyperbolic* if there are constants $C > 0$ and $0 < \mu < 1$ such that, for every $x \in \mathcal{K}$, there is a splitting of the tangent space $T_x M = E_x^s \oplus E_x^u$ satisfying, for every $n \in \mathbb{N}$,

$$\|Dh^n(v)\| \leqslant C\mu^n \|v\| \quad \forall v \in E_x^s \qquad \text{and} \qquad \|Dh^{-n}(v)\| \leqslant C\mu^n \|v\| \quad \forall v \in E_x^u.$$

A.2. **Symmetry.** Given a group $\mathcal{G}$ of endomorphisms of $\mathbb{S}^3$, we say that a one-parameter family of vector fields $(f_\lambda)_{\lambda \in \mathbb{R}}$ is symmetric it it satisfies the equivariance assumption

$$f_\lambda(\gamma x) = \gamma f_\lambda(x) \qquad \forall x \in \mathbb{S}^3, \quad \forall \gamma \in \mathcal{G} \text{ and } \forall \lambda \in \mathbb{R}.$$

A.3. **Attracting set.** A subset A of a topological space $\mathcal{M}$ is said to be *attracting* by a flow φ if there exists an open set $U \subset \mathcal{M}$ satisfying

$$\varphi(t, U) \subset U \qquad \forall t \geqslant 0 \text{ and } \bigcap_{t \in \mathbb{R}^+} \varphi(t, U) = A.$$

Its basin of attraction, denoted by $\mathcal{B}(A)$, is the set of points in $\mathcal{M}$ whose orbits have $\omega-$limit in A. We say that A is *asymptotically stable* (or, equivalently, that A is a *global attracting set*) if $\mathcal{B}(A) = \mathcal{M}$. An attracting set is said to be *quasi-stochastic* if it encloses periodic trajectories with different Morse indices (the *Morse index* of a hyperbolic equilibrium is the dimension of its unstable manifold), structurally unstable cycles, sinks and saddle-type invariant sets.

A.4. **Heteroclinic phenomena.** Suppose that σ_1 and σ_2 are two hyperbolic saddle-foci of a vector field f with different Morse indices. We say that there is a *heteroclinic cycle* associated to σ_1 and σ_2 if $W^u(\sigma_1) \cap W^s(\sigma_2) \neq \emptyset$ and $W^u(\sigma_2) \cap W^s(\sigma_1) \neq \emptyset$. For $i, j \in \{1, 2\}$, the non-empty intersection of $W^u(\sigma_i)$ with $W^s(\sigma_j)$ is called a *heteroclinic connection* between σ_i and σ_j, and will be denoted by $[\sigma_i \to \sigma_j]$. Although heteroclinic cycles involving equilibria are not a generic feature within differential equations, they may be structurally stable within families of vector fields which are equivariant under the action of a compact Lie group $\mathcal{G} \subset \mathbf{O}(n)$, due

to the existence of flow-invariant subspaces [12]. A *heteroclinic network* is a connected finite union of heteroclinic cycles.

A.5. **Bykov cycle.** A heteroclinic cycle between two hyperbolic saddle-foci of a vector field f with different Morse indices, where one of the connections is transverse (and so stable under small perturbations) while the other is structurally unstable, is called a *Bykov cycle*. A *Bykov network* is a connected union of heteroclinic cycles, not necessarily in finite number.

A.6. **Tubular neighborhood of a Bykov cycle.** Given a Bykov cycle associated to σ_1 and σ_2, let V_1, V_2 be two small and disjoint cylindrical neighborhoods of σ_1 and σ_2, respectively. Consider two local sections transverse to the cycle at two points p_1 and p_2 in the connections $[\sigma_1 \to \sigma_2]$ and $[\sigma_2 \to \sigma_1]$, respectively, with $p_1, p_2 \notin V_1 \cup V_2$. Saturating the cross sections by the flow, we obtain two flow-invariant tubes joining V_1 and V_2 which contain the connections in their interior. The union of those tubes with V_1 and V_2 is called a *tubular neighborhood* of the Bykov cycle.

A.7. **Chirality.** There are two different possibilities for the geometry of a flow around a Bykov cycle Γ, depending on the direction the trajectories turn around the one-dimensional heteroclinic connection between σ_1 and σ_2. Let V_1 and V_2 be small disjoint neighborhoods of σ_1 and σ_2 with disjoint boundaries ∂V_1 and ∂V_2, respectively. Trajectories starting at $\text{In}(\sigma_1) \setminus W^s(\sigma_1)$ enter V_1 in positive time, then follow the connection from σ_1 to σ_2, enter V_2, finally leaving V_2 at ∂V_2. Let γ be a piece of trajectory like the one just described from $\text{In}(\sigma_1) \setminus W^s(\sigma_1)$ to ∂V_2. Now join its starting point to its end point by a line segment (see Figure 1 of [19]), forming a closed curve that we call the *loop of γ*. The Bykov cycle Γ and the loop of γ are disjoint closed sets. We say that the two saddle-foci σ_1 and σ_2 in Γ have the same *chirality* if the loop of every such a trajectory is linked to Γ, in the sense that these two sets cannot be disconnected by an isotopy in $\mathbb{R}^4$.

A.8. **Pulse.** Let V_1, V_2 be two small and disjoint neighborhoods of σ_1 and σ_2, respectively, and take $n \in \mathbb{N}$. A one-dimensional heteroclinic connection from σ_2 to σ_1 which, after leaving V_2, enters and leaves both V_1 and V_2 precisely n times is called an *n–pulse heteroclinic connection* with respect to V_1 and V_2, or simply an *n–pulse*. A 0–pulse is a one-dimensional heteroclinic connection from σ_2 to σ_1 which, after leaving V_2, enters V_1 and afterwards stays in this neighborhood.

A.9. **Saturated set and the operator tilde.** Let $\mathcal{S}$ be a cross-section to a flow φ and assume that $\mathcal{S}$ contains a compact set $\mathcal{K}$ invariant by the first return map $\mathcal{R}$ to $\mathcal{S}$. Then the *saturation of $\mathcal{K}$*, we denote by $\widetilde{\mathcal{K}}$, is the flow-invariant set formed by the φ-trajectories of points of $\mathcal{K}$, that is, $\left\{ \varphi(t, x) : t \in \mathbb{R}, \ x \in \mathcal{K} \right\}$.

A.10. **Chain-accessible point.** Given $\varepsilon > 0$ and $\tau > 0$, an (ε, τ)-*trajectory* of a flow φ is a finite set $\{P_1, \cdots, P_k\}$ such that, for all $i \in \{1, \ldots, k-1\}$, the point P_{i+1} is at a distance strictly smaller than ε from $\varphi(t, P_i)$ for some $t > \tau$. A point Q is said to be (ε, τ)-*chain-accessible* from a point P if there exists an (ε, τ)-trajectory $\{P_1, P_2, \cdots, P_k\}$ *connecting* P with Q; in particular, P and Q are at a distance strictly smaller than ε from P_1 and P_k, respectively. A point Q is said to be *chain-accessible* from a point P if, for any $\varepsilon > 0$, there exists $\tau > 0$ such that Q is (ε, τ)–chain-accessible from P.

A.11. **Chain-recurrent class.** A point P is called *chain-recurrent* for a flow φ if it is chain-accessible from $\varphi(t, P)$ for any $t \in \mathbb{R}$. More generally, a set $\mathcal{C}$ is *chain-accessible* from a point P if it contains a point chain-accessible from P. A compact invariant set $\mathcal{C}$ is a *chain-recurrent class* if, given any two points in $\mathcal{C}$ and $\varepsilon > 0$, we may find an (ε, τ)-trajectory connecting them entirely contained in $\mathcal{C}$. We observe that, by definition, every point of a chain-recurrent class is chain-recurrent. A set $\mathcal{C}$ is called *chain-stable* if, for any neighborhood $\mathcal{V}$ of $\mathcal{C}$, there exists an $\varepsilon > 0$ such that every (ε, τ)-trajectory connecting any two points of $\mathcal{C}$ is contained in $\mathcal{V}$.

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