NATURALLY GRADED 2-FILIFORM LEIBNIZ ALGEBRAS.

L.M. CAMACHO, J. R. GÓMEZ, A.J. GONZÁLEZ, AND B.A. OMIROV

Abstract. The Leibniz algebras appear as a generalization of the Lie algebras [8]. The classification of naturally graded p-filiform Lie algebras is known [3], [4], [5], [9]. In this work we deal with the classification of 2-filiform Leibniz algebras. The study of p-filiform Leibniz non Lie algebras is solved for $p = 0$ (trivial) and $p = 1$ [1]. In this work we get the classification of naturally graded non Lie 2-filiform Leibniz algebras.

1. Introduction

In this work we study the naturally graded 2-filiform Leibniz algebras. Since the filiform (1-filiform) Lie algebras have the maximal nilindex, Vergne studied them and obtained the classification of naturally graded [9]. Many authors have studied the complete classification for low dimensions. The lists up to dimension 8 can be found in [7] and the classification filiform up to dimension 11 in [6] The notion of p-filiform Lie (resp. Leibniz) algebras can be considered as a generalization of filiform Lie algebras.

The knowledge of naturally graded algebras of a certain family offers significant information about the complete family. The classification of 2-filiform Lie algebras and p-filiform has been obtained [5], [4].

In the case of Leibniz algebras only the classification of 0-filiform and 1-filiform algebras is known [1], [2]. In the present paper we get the classification of naturally graded 2-filiform Leibniz algebras.

Leibniz algebras are defined by the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

We have used the software *Mathematica* to study particular cases in concrete finite dimensions and later, by induction, the obtained results are generalized for arbitrary finite dimension.

Let $\mathcal{L}$ be a Leibniz algebra, we define the following sequence:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{n+1} = [\mathcal{L}^n, \mathcal{L}]$$

An algebra $\mathcal{L}$ is *nilpotent* if $\mathcal{L}^n = 0$ for some $n \in \mathbf{N}$.

For any element x of $\mathcal{L}$ we define R_x the operator of right multiplication as

$$R_x : \quad z \quad \rightarrow \quad [z, x], \ z \in \mathcal{L}$$

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Let us $x \in \mathcal{L} \backslash [\mathcal{L}, \mathcal{L}]$ and for the nilpotent operator R_x of right multiplication, define the decreasing sequence $C(x) = (n_1, n_2, ..., n_k)$ that consists of the dimensions of the Jordan blocks of the R_x. Endow the set of these sequences with the lexicographic order.

The sequence $C(\mathcal{L}) = max_{x \in \mathcal{L} \backslash [\mathcal{L}, \mathcal{L}]} C(x)$ is defined to be the *characteristic sequence* of the algebra $\mathcal{L}$.

Definition 1.1. A Leibniz algebra $\mathcal{L}$ is called p-filiform if $C(L) = (n-p, \underbrace{1, \ldots, 1}_{p})$, where $p \geq 0$.

Note that this definition when $p > 0$ agrees with the definition of p-filiform Lie algebras.

From now we will use the expression "graded algebra" instead of "naturally graded algebra".

Let $\mathcal{L}$ be a graded p-filiform n-dimensional Leibniz algebra, then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ such that $e_1 \in \mathcal{L} - \mathcal{L}^2$ and $C(e_1) = (n-p, \underbrace{1, \ldots, 1}_{p})$.

By definition of characteristic sequence the operator R_{e_1} in Jordan form has one block J_{n-p} of size $n-p$ and p block J_1 (where $J_1 = \{0\}$) of size one.

The possibilities for operator R_{e_1} are the follow:

$$\begin{pmatrix} J_{n-p} & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}, \quad \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_{n-p} & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}, \quad \cdots,$$

$$\begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_{n-p} \end{pmatrix}$$

It is easy to prove that when J_{n-p} is placed an a different position from the first are isomorphic cases. Thus, we have only the following possibilities of Jordan form of the matrix R_{e_1}:

$$\begin{pmatrix} J_{n-p} & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}, \quad \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_{n-p} & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}$$

Definition 1.2. A p-filiform Leibniz algebra $\mathcal{L}$ is called first type (type I) if the operator R_{e_1} has the form:

$$\begin{pmatrix} J_{n-p} & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}$$

and second type (type II) in the other case.

1.1. **Naturally graded filiform and 2-filiform Lie algebras.** Naturally graded p-filiform Lie algebras are known for all $p > 0$, [4], [5], [9].

Examples of filiform Lie algebras are $\mathcal{L}_n$, Q_n defined as follows:

$$\mathcal{L}_n \quad (n \geq 3) : \{ \, [X_0, X_i] = X_{i+1} \quad 1 \leq i \leq n-2.$$

$$\mathcal{Q}_n \quad (n \geq 6,\ n \text{ even}) : \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-2, \\ [X_i, X_{n-1-i}] = (-1)^{i-1} X_{n-1} & 1 \leq i \leq \frac{n-2}{2}. \end{cases}$$

Provided examples of 2-filiform Lie algebras.

$\mathcal{L}(n, r) \quad \left(n \geq 5,\ 3 \leq r \leq 2 \left\lfloor \frac{n-1}{2} \right\rfloor - 1,\ r \text{ odd}\right)$:

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3, \\ [X_i, X_{r-i}] = (-1)^{i-1} Y & 1 \leq i \leq \frac{r-1}{2}. \end{cases}$$

$\mathcal{Q}(n, r) \quad (n \geq 7,\ n \text{ odd};\ 3 \leq r \leq n-4,\ r \text{ odd})$:

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3, \\ [X_i, X_{r-i}] = (-1)^{i-1} Y & 1 \leq i \leq \frac{r-1}{2}, \\ [X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-2} & 1 \leq i \leq \frac{n-3}{2}. \end{cases}$$

$\tau(n, n-4) \ \ (n \text{ odd},\ n \geq 7)$:

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3, \\ [X_i, X_{n-4-i}] = (-1)^{i-1} (X_{n-4} + Y) & 1 \leq i \leq \frac{n-5}{2}, \\ [X_i, X_{n-3-i}] = (-1)^{i-1} \frac{(n-3-2i)}{2} X_{n-3} & 1 \leq i \leq \frac{n-5}{2}, \\ [X_i, X_{n-2-i}] = (-1)^{i} (i-1) \frac{(n-3-i)}{2} X_{n-2} & 2 \leq i \leq \frac{n-3}{2}, \\ [X_i, Y] = \frac{(5-n)}{2} X_{n-4+i} & 1 \leq i \leq 2. \end{cases}$$

$\tau(n, n-3) \ \ (n \text{ even},\ n \geq 6)$:

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3, \\ [X_i, X_{n-3-i}] = (-1)^{i-1} (X_{n-3} + Y) & 1 \leq i \leq \frac{n-4}{2}, \\ [X_i, X_{n-2-i}] = (-1)^{i-1} \frac{(n-2-2i)}{2} X_{n-2} & 1 \leq i \leq \frac{n-4}{2}, \\ [X_1, Y] = \frac{(4-n)}{2} X_{n-2}. \end{cases}$$

2. Naturally graded p-filiform Leibniz algebra

It is easy to see that a Leibniz algebra of type I is not a Lie algebra.

Let $\mathcal{L}$ be an n-dimensional p-filiform Leibniz algebra. We define a natural gradation of $\mathcal{L}$ as follows. Take $\mathcal{L}_1 = \mathcal{L}$, $\mathcal{L}_i = \mathcal{L}^i/\mathcal{L}^{i+1},\ 2 \leq i \leq n-p$. It is clear that $\mathcal{L} \simeq$
$_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{n-p}$, where $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ and $\mathcal{L}_{i+1} = [\mathcal{L}_i, \mathcal{L}_1]$ for all i.

Let $\mathcal{L}$ be a graded p-filiform Leibniz algebra of the first type. Then there exists a basis $\{e_1, e_2, \ldots, e_{n-p}, f_1, \ldots, f_p\}$ such that

$$\begin{array}{ll} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-p-1 \\ [f_j, e_1] = 0, & 1 \leq j \leq p. \end{array}$$

From this multiplication we have:

$$< e_1 >\subseteq \mathcal{L}_1, \quad < e_2 >\subseteq \mathcal{L}_2, \quad < e_3 >\subseteq \mathcal{L}_3, \dots, < e_{n-p} >\subseteq \mathcal{L}_{n-p}$$

but we do not know about the places of the elements $\{f_1, f_2, \dots, f_p\}$.

Let denote by $r_1, r_2, \dots, r_p$ the places of elements $f_1,\ f_2,\ \dots,\ f_p$ in natural gradation correspondingly, that is, $f_i \in \mathcal{L}_{r_i}$ with $1 \leq i \leq p$. Further the law of a Leibniz algebra of type I with the set $\{r_1, r_2, \dots, r_p\}$ will be denoted by $\mu_{(I,r_1,\dots,r_p)}$.

We can suppose that $1 \leq r_1 \leq r_2 \leq \dots \leq r_p \leq n-p$.

Theorem 2.1. *Let $\mathcal{L}$ be a graded p-filiform Leibniz algebra of type I. Then $r_s \leq s$ for any $s \in \{1, 2, \dots, p\}$.*

Proof:

Note $r_1 = 1$. In fact, if $r_1 > 1$, then the algebra $\mathcal{L}$ is one generated and by [[1], lemma 1] it is nul-filiform Leibniz algebra, and hence $C(\mathcal{L}) = (n, 0)$, that is, we obtain contradiction with condition $C(L) = (n-p, 1, 1, \dots, 1)$.

Let us prove that $r_2 \leq 2$. Suppose otherwise, that is, $r_2 > 2$. Then

$$\mathcal{L}_1 =< e_1, e_{n-p+1} >$$

$$\mathcal{L}_2 =< e_2 >$$

$$\mathcal{L}_{r_2} = [\mathcal{L}_{r_2-1}, \mathcal{L}_1] =< [< e_{r_2-1} >, < e_1, f_1 >] >=< e_{r_2}, [e_{r_2-1}, f_1] >$$

Consider the multiplication:

$$[e_{r_2-1}, f_1] = [[e_{r_2-2}, e_1], f_1] = [e_{r_2-2}, [e_1, f_1]] + [[e_{r_2-2}, f_1], e_1]$$

Since the multiplication $[e_1, f_1] \in \mathcal{L}_2 =< e_2 >\subseteq Z(\mathcal{L})$, then the first item is equal to zero. It is evident that the second item belongs to the linear span $< e_{r_2} >$. So, $f_2 \notin \mathcal{L}_{r_2}$ and we obtain contradiction method, hence $r_2 \leq 2$.

Let us suppose that the condition of the theorem is true for any value less than s. We prove that $r_s \leq s$. We shall prove it by contradiction, that is, suppose that $r_s > s$.

If $r_s > s$ we prove the following embedding:

$$[e_{r_s-r_t}, f_t] \subseteq < e_{r_s} >, \qquad 1 \leq t \leq s-1$$

We shall prove it by descending induction by t.

Let us prove it for $t = s-1$. Consider the multiplication:

$$\begin{aligned} [e_{r_s-r_{s-1}}, f_{s-1}] &= [[e_{r_s-r_{s-1}-1}, e_1], f_{s-1}] = [e_{r_s-r_{s-1}-1}, [e_1, f_{s-1}]]+ \\ &+ [[e_{r_s-r_{s-1}-1}, f_{s-1}], e_1] \end{aligned}$$

Since $r_s > s$, we have $[e_1, f_{s-1}] \in \mathcal{L}_{r_{s-1}+1} =< e_{r_{s-1}+1} >\in Z(\mathcal{L})$, that is, $[e_{r_s-r_{s-1}-1}, [e_1, f_{s-1}]] = 0$. From the multiplication on the right side on e_1 we have

$$[[e_{r_s-r_{s-1}-1}, f_{s-1}], e_1] \subseteq < e_{r_s} >$$

hence, $[e_{r_s-r_{s-1}}, f_{s-1}] \subseteq < e_{r_s} >$.

Let suppose that embedding $[e_{r_s-r_t}, f_t] \subseteq < e_{r_s} >$ is true for any value greater than $t+1$. We prove it for t.

Consider the multiplication:

$$[e_{r_s-r_t}, f_t] = [[e_{r_s-r_t-1}, e_1], f_t] = [e_{r_s-r_t-1}, [e_1, f_t]]+ \\ +[[e_{r_s-r_t-1}, f_t], e_1]$$

As $[e_1, f_t] \in \mathcal{L}_{r_t+1}$, then in case $r_t + 1 = r_{t+1}$ we have $\mathcal{L}_{r_t+1} = \{e_{r_{t+1}}, f_{t+1} \vee f_{t+2} \vee \cdots \vee f_{s-1}\}$. Therefore the multiplication $[e_{r_s-r_t-1}, [e_1, f_t]]$ is contained in linear span $< e_{r_s} >$ by induction.

If $r_t + 1 \neq r_{t+1}$ the following equality $[e_1, f_t] =< e_{r_t+1} >$ is hold (because $\mathcal{L}_{r_t+1} =< e_{r_t+1} >$) and hence $[e_{r_s-r_t-1}, [e_1, f_t]] = 0$. Evidently, the second item also is contained in the linear span $< e_{r_s} >$.

Thus, $[e_{r_s-r_t}, f_t] \subseteq < e_{r_s} >, \ 1 \leq t \leq s-1$ is proved.

Let us prove that $\mathcal{L}_{r_s} \subseteq < e_{r_s} >$ supposing $r_s > s$. Consider the multiplication:

$$\mathcal{L}_{r_s} = [\mathcal{L}_{r_s-1}, \mathcal{L}_1] = [< e_{r_s-1} >, < e_1, f_1 \vee \cdots \vee f_{s-1} >]$$

From $[e_{r_s-r_t}, f_t] \subseteq < e_{r_s} >$, we have that $\mathcal{L}_{r_s} \subseteq < e_{r_s} >$, that is, we obtain the contradiction which completes the proof of theorem.

□

2.1. **Naturally graded 2-filiform Leibniz algebras.** In this section naturally graded 2-filiform Leibniz algebras will be classified.

The classification of the null-filiform Leibniz algebras is an easy task and one for naturally graded 1-filiform Leibniz algebras is similar to the case of Lie algebras. However, when p, increases the difficulties also increase exponentially in the study of Leibniz algebras with respect to Lie algebras.

From [5] we observe the existence of graded 2-filiform Lie algebras in arbitrary dimension. Let us demonstrate examples of graded 2-filiform Leibniz algebras of type I which obviously are not Lie algebras.

In this work, we use the following notation:

- $\{e_1, e_2, ..., e_{n-2}, e_{n-1}, e_n\}$ an adapted basis and
- $r_1, \ r_2$ the places of elements $e_{n-1}, \ e_n$.

Example 1. Let $\mathcal{L}^0_{n-2}$ be a graded nul-filiform Leibniz algebra of dimension $n-2$ and $\mathcal{L}^1_{n-1}$ be a graded filiform non Lie Leibniz algebra of dimension $n-1$ of type I. Then $\mathcal{L}^0_{n-2} \oplus \mathbf{C}^2$ and $\mathcal{L}^1_{n-1} \oplus \mathbf{C}$ are graded n-dimensional split 2-filiform Leibniz algebras of type I.

The following lemma establishes that a graded 2-filiform Leibniz algebra of type I with condition $r_1 = r_2 = 1$ is a split algebra from the above example.

Lemma 2.2. *Let $\mathcal{L}$ be a graded 2-filiform Leibniz algebra of type $\mu_{(I,1,1)}$. Then $\mathcal{L}$ is a split algebra from example 1.*

Proof:

Let algebra $\mathcal{L}$ has form $\mu_{(I,1,1)}$, then for an adapted basis $\{e_1, e_2, \dots, e_n\}$ the multiplications on the right side on e_1 are the following:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3 \\ [e_i, e_{n-1}] = \alpha_i e_{i+1}, & 1 \leq i \leq n-3 \\ [e_{n-1}, e_{n-1}] = \alpha_{n-1} e_2 & \\ [e_n, e_{n-1}] = \alpha_n e_2 & \\ [e_i, e_n] = \beta_i e_{i+1}, & 1 \leq i \leq n-3 \\ [e_{n-1}, e_n] = \beta_{n-1} e_2 & \\ [e_n, e_n] = \beta_n e_2 & \end{cases}$$

Using Leibniz identity it is not difficult to obtain the following restrictions:

$$\begin{cases} \alpha_i = \alpha, & 1 \leq i \leq n-3 \\ \beta_i = \beta, & 1 \leq i \leq n-3 \\ \alpha_{n-1} = \alpha_n = 0 & \\ \beta_{n-1} = \beta_n = 0 & \end{cases}$$

Let us rewrite the multiplications of basis elements taking into account the above restrictions:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3 \\ [e_i, e_{n-1}] = \alpha e_{i+1}, & 1 \leq i \leq n-3 \\ [e_i, e_n] = \beta e_{i+1}, & 1 \leq i \leq n-3 \end{cases}$$

If $\alpha \neq 0$ we take the change of basis: $e'_i = e_i,\ 1 \leq i \leq n-1,\ e'_n = \alpha e_n - \beta e_{n-1}$, we can suppose that the coefficient β is equal to zero, that is, we have the multiplications:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3 \\ [e_i, e_{n-1}] = \alpha e_{i+1}, & 1 \leq i \leq n-3 \end{cases}$$

If $\alpha = 0$, then taking $e'_i = e_i,\ 1 \leq i \leq n-2,\ e'_{n-1} = e_n,\ e'_n = e_{n-1}$, we can also suppose that coefficient $\beta = 0$. In this case it is easy to see that $\mathcal{L} = \mathcal{L}^0_{n-2} \oplus \mathbb{C}^2$.

If $\alpha \neq 0$, the change of basis $e'_{n-1} = \dfrac{1}{\alpha} e_{n-1}$ (and $e'_i = e_i,\ i \neq n-1$) allows us to suppose $\alpha = 1$ and so $\mathcal{L} = \mathcal{L}^1_{n-1} \oplus \mathbb{C}$.

□

For graded non split 2-filiform Leibniz algebra of type I with condition $r_2 = 2$ the following theorem is hold.

The next results were supported by *Mathematica* package.

Proposition 2.3. *Let $\mathcal{L}$ be an 4-dimensional graded 2-filiform non split Leibniz algebra of type $\mu_{(I,1,2)}$. Then $\mathcal{L}$ is isomorphic to the following algebra:*

$$\begin{cases} [e_1, e_1] = e_2 \\ [e_1, e_3] = e_4 \end{cases}$$

Proof:

We have that the natural gradation is:

$$< e_1, e_3 > \oplus < e_2, e_4 >$$

and the multiplication is:

$$\begin{cases} [e_1, e_1] = e_2 \\ [e_1, e_3] = \alpha_1 e_2 + \beta_1 e_4 \\ [e_3, e_3] = \alpha_2 e_2 + \beta_2 e_4 \end{cases}$$

with $\beta_1 \neq 0$ or $\beta_2 \neq 0$.

If we make the following change of basis $\beta_2' e_4' = \alpha_2 e_2 + \beta_2 e_4$ it is possible to suppose $\alpha_2 = 0$ and

$$\begin{cases} [e_1, e_1] = e_2 \\ [e_1, e_3] = \alpha_1 e_2 + \beta_1 e_4 \\ [e_3, e_3] = \beta_2 e_4 \end{cases}$$

with $\beta_1 \neq 0$ or $\beta_2 \neq 0$.

According to the characteristic sequence we have that $rank(R_{e_1 + Ae_3}) \leq 1$, it implies that $\beta_2 = 0$ and $\beta_1 \neq 0$. An elementary change of basis permits to prove this result.

□

Proposition 2.4. *Let $\mathcal{L}$ be a 5-dimensional naturally graded 2-filiform Leibniz algebra of type $\mu_{(I,1,2)}$. Then, $\mathcal{L}$ is isomorphic to the one of the following pairwise non isomorphic algebras:*

$$\mu^1 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2 \\ [e_1, e_4] = e_2 + e_5, & \\ [e_2, e_4] = e_3, & \end{cases} \qquad \mu^2 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2 \\ [e_1, e_4] = e_5. & \end{cases}$$

$$\mu^3 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2 \\ [e_1, e_4] = ie_2 + e_5, & \\ [e_2, e_4] = ie_3, & i^2 = -1 \\ [e_5, e_4] = e_3. & \end{cases} \qquad \mu^4 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2 \\ [e_1, e_4] = e_5. & \\ [e_5, e_4] = e_3. & \end{cases}$$

Proof:

Analogously as in above we can assume that

$$\mathcal{L}_1 =< e_1, e_4 >, \ \mathcal{L}_2 =< e_2, e_5 >, \ \mathcal{L}_3 =< e_3 >, \ \mathcal{L}_4 =< 0 >$$

Put $[e_5, e_4] = \gamma e_3$. If $\gamma = 0$, then we obtain algebra $L(\alpha, 0)$, otherwise not restricted of generality we obtain algebra $L(\alpha, 1)$. Since dimension of left annihilator of the algebra $L(\alpha, 0)$ is equal to 2 ($e_4, e_5 \in L(\mathcal{L})$) and dimension of left annihilator of the algebra $L(\alpha, 1)$ is equal to 1 ($e_4 \in L(\mathcal{L})$) there are not isomorphic.

From the above argumentation we have following algebras

$$L(\alpha,1):\begin{cases}[e_i,e_1]=e_{i+1}, & 1\le i\le 2\\ [e_1,e_4]=\alpha e_2+f_2\\ [e_2,e_4]=\alpha e_3\\ [f_2,e_4]=e_3\end{cases}\qquad L(\alpha,0):\begin{cases}[e_i,e_1]=e_{i+1}, & 1\le i\le 2\\ [e_1,e_4]=\alpha e_2+f_2\\ [e_2,e_4]=\alpha e_3\end{cases}$$

If we considered algebra

$$L(\alpha,0):\begin{cases}[e_i,e_1]=e_{i+1} & 1\le i\le 2\\ [e_1,e_4]=\alpha e_2+f_2\\ [e_2,e_4]=\alpha e_3\end{cases}$$

We have $\alpha=1$ or 0. And we obtain the two first algebras of proposition. By standard way it is not difficult to check that these algebras are not isomorphic.

Consider algebra $L(\alpha,1)$, we make the general change of basis

$$e_1'=a_1e_1+a_2e_4,\quad e_4'=b_1e_1+b_2e_4$$

where $a_1b_2-a_2b_1\neq 0$.

In other hand $[e_4',e_1']=0$ and we have

$$b_1a_1+b_1a_2\alpha=0$$

$$b_1a_2=0$$

it implies that $b_1=0$. Finally we obtain

$$\alpha'=\frac{b_2[a_1\alpha+a_2(\alpha^2+1)]}{[(a_1+a_2\alpha)^2+a_2^2]}$$

Comparing the coefficients at the basic element we obtain restriction

$$b_2^2=\frac{[(a_1+a_2\alpha)^2+a_2^2]^2}{a_1^2}$$

It is not difficult to check that the nullity of the following expression is invariant because:

$$1+\alpha'^2=\frac{(1+\alpha^2)((a_1+a_2\alpha)^2+a_2^2)}{a_1^2}=$$

Case 1. $\alpha^2+1\neq 0$ then putting $a_2=-\frac{a_1\alpha}{1+\alpha^2}$ implies $\alpha'=0$. Thus, in this case we obtain μ_4.

Case 2. $\alpha^2+1=0$ (i.e $\alpha=\pm i$) then we have that $b_2=\pm\frac{(a_1+a_2\alpha^2)+a_2^2}{a_1}$ and $\alpha'=\pm\alpha$ we obtain $\alpha'=i$. Thus, in this case we obtain μ_3. □

Theorem 2.5. *Let $\mathcal{L}$ be an n-dimensional ($n\ge 6$) graded 2-filiform non split Leibniz algebra of type $\mu_{(I,1,2)}$. Then $\mathcal{L}$ is isomorphic to the one of the following pairwise non isomorphic algebras:*

$$\begin{cases}[e_i,e_1]=e_{i+1}, & 1\le i\le n-3\\ [e_1,e_{n-1}]=e_2+e_n\\ [e_i,e_{n-1}]=e_{i+1}, & 2\le i\le n-3\end{cases}\qquad \begin{cases}[e_i,e_1]=e_{i+1}, & 1\le i\le n-3\\ [e_1,e_{n-1}]=e_n\end{cases}$$

Proof:

According to the theorem conditions we have the following multiplications in an adapted basis $\{e_1, e_2, \dots, e_n\}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_1, e_{n-1}] = \alpha_1 e_2 + \gamma_1 e_n \\ [e_i, e_{n-1}] = \alpha_i e_{i+1}, & 2 \le i \le n-3 \\ [e_{n-1}, e_{n-1}] = \alpha_{n-1} e_2 + \gamma_{n-1} e_n \\ [e_n, e_{n-1}] = \alpha_n e_3 \\ [e_i, e_n] = \beta_i e_{i+2}, & 1 \le i \le n-4 \\ [e_{n-1}, e_n] = \beta_{n-1} e_3 \\ [e_n, e_n] = \beta_n e_4 \end{cases}$$

where either $\gamma_1 \neq 0$ or $\gamma_{n-1} \neq 0$.

Using Leibniz identity it is not difficult to obtain the following restrictions:

$$\begin{cases} \alpha_i = \alpha, & 1 \le i \le n-3 \\ \beta_i \gamma_1 = 0, & 1 \le i \le n-4 \\ \beta_i \gamma_{n-1} = 0, & 1 \le i \le n-4 \\ \gamma_1 \beta_{n-1} + \alpha_{n-1} = 0 \\ \alpha_{n-1} = \alpha_n = 0 \\ \beta_i \gamma_j = 0, & i \in \{n-1, n\},\ j \in \{1, n-1\} \end{cases}$$

Since either $\gamma_1 \neq 0$ or $\gamma_{n-1} \neq 0$, we have that $\beta_i = 0$ for $1 \le i \le n-4$ and $\beta_{n-1} = \beta_n = 0$. Thus, the multiplications have the following form:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_1, e_{n-1}] = \alpha e_2 + \gamma_1 e_n \\ [e_i, e_{n-1}] = \alpha e_{i+1}, & 2 \le i \le n-3 \\ [e_{n-1}, e_{n-1}] = \gamma_{n-1} e_n \end{cases}$$

It is possible to suppose that

$$R_{e_1 + A e_{n-1}} = \begin{pmatrix} & & & 0 & 0 & 0 \\ & (1+A\alpha) I_{n-3} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ A\gamma_1 & 0 & \cdots & 0 & 0 & A\gamma_{n-1} \end{pmatrix}$$

where I_{n-3} is the unit matrix of size $n-3$ and $1 + A\alpha \neq 0$.

As $rang(R_{e_1 + A e_{n-1}}) \le n-3$ (otherwise the characteristic sequence for element $e_1 + A e_{n-1}$ would be greater than $(n-p, 1, \dots, 1)$), then $(1 + A\alpha)^{n-3} A \gamma_{n-1} = 0$, hence $\gamma_{n-1} = 0$ and $\gamma_1 \neq 0$. By an elementary change of basis, it is possible to suppose that $\gamma_1 = 1$.

By a general change of basis the expression for the new generators is

$$e_1' = \sum_{i=1}^{n-1} A_i e_i, \qquad e_{n-1}' = \sum_{i=1}^{n-1} B_i e_i$$

obtaining $\alpha' = \dfrac{B_{n-1}\alpha}{A_1 + A_{n-1}\alpha}$.

It is easy to see that if $\alpha \neq 0$ we have the first algebra of the theorem and if $\alpha = 0$ the second algebra.

□

Consider now graded 2-filiform Leibniz algebras of type II.

Let $\mathcal{L}$ be a graded n-dimensional p-filiform Leibniz algebra. Then there exists a basis $\{e_1, e_2, \ldots, e_{n-p}, f_1, f_2, \ldots, f_p\}$ of $\mathcal{L}$ such that multiplications on the right side on element e_1 will have the form:

$$\begin{cases} [e_1, e_1] = 0 & \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-p-1 \\ [f_j, e_1] = 0, & 1 \leq j \leq p \end{cases}$$

From these multiplications we have:

$$< e_1 >\subseteq \mathcal{L}_1, \qquad < e_{i+1} >\subseteq \mathcal{L}_i, \quad 2 \leq i \leq n-2$$

But again we do not know about the position of elements $\{e_2, f_2, f_3, \ldots, f_p\}$ in natural gradation.

Let denote by $r_1, r_2, \ldots, r_p$ $(r_1 \leq r_2 \leq \cdots \leq r_p)$ the places of elements e_2, f_2, f_3, $\ldots$, f_p correspondingly, that is, $e_2 \in \mathcal{L}_{r_1}$, $f_i \in \mathcal{L}_{r_i}$, $2 \leq i \leq p$.

Let $\mathcal{L}$ be a graded 2-filiform Leibniz algebra. Since $r_1 = 1$, further we shall denote r_2 by r.

For the 2-filiform Leibniz algebras of type II the following lemma is hold.

Lemma 2.6. *Let $\mathcal{L}$ be an n-dimensional 2-filiform Leibniz algebra. Then the following conditions are hold:*

a) $\mathcal{L}$ has nilindex $n-1$;

b) or $dim(\mathcal{L}^i) = n-1-i$, $2 \leq i \leq n-2$
or $dim(\mathcal{L}^i) = \begin{cases} n-i, & 2 \leq i \leq r \\ n-1-i\ , & r+1 \leq i \leq n-2 \end{cases}$ *for some r, $2 \leq r \leq n-2$*

Proof:

a) Let $x \in \mathcal{L} - [\mathcal{L}, \mathcal{L}]$ such that $C(x) = (n-2, 1, 1)$. Hence, $R_x^{n-2} = 0$ and $R_x^{n-3} \neq 0$ and, consequently, there exists element $y \in \mathcal{L}$, such that $R_x^{n-3}(y) \neq 0$. Therefore $\mathcal{L}^{n-2} \neq 0$ and $\mathcal{L}^{n-1} = 0$ (when $\mathcal{L}^{n-1} \neq 0$, then by [[1],lemma 1, lemma 4] the algebra $\mathcal{L}$ would be either nul-filiform or filiform).

b) Let $e_1 \in \mathcal{L} - [\mathcal{L}, \mathcal{L}]$ be a maximal characteristic vector of ll, where $\mathcal{L}$ is of type I. Then for $r = 1$, that is, $dim(\mathcal{L}/\mathcal{L}^2) = 3$ we have that $dim(\mathcal{L}^i) = n-1-i$, $2 \leq i \leq n-2$. For $r = 2$, that is, $dim(\mathcal{L}/\mathcal{L}^2) = 2$ we get:

$$dim(\mathcal{L}^i) = \begin{cases} n-2, & i = 2 \\ n-1-i, & 3 \leq i \leq n-2 \end{cases}$$

Let algebra $\mathcal{L}$ has the type II. For $r_2 = 1$ we obtain that $dim(\mathcal{L}/\mathcal{L}^2) = 3$, that is, $dim(\mathcal{L}^i) = n-1-i$, $2 \leq i \leq n-2$. For $r_2 \in \{2, 3, \ldots, n-2\}$ we get: $dim(\mathcal{L}/\mathcal{L}^2) = 2$, that is, $dim(\mathcal{L}^i) = \begin{cases} n-i, & 2 \leq i \leq r \\ n-1-i, & r+1 \leq i \leq n-2 \end{cases}$ □

Lemma 2.7. *Let $\mathcal{L}$ be a complex n-dimensional ($n \geq 5$) graded 2-filiform Leibniz algebra of type II and $r > 2$. Then $\mathcal{L}$ is a Lie algebra.*

Proof:
Let (1) be the family of laws of $\mathcal{L}$:

$$(1) \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-2 \\ [e_1, e_i] = \alpha_{1,i} e_{i+1}, & 2 \leq i \leq n-2, \ i \neq r \\ [e_1, e_r] = \alpha_{1,r} e_{r+1} + \gamma_1 e_n & \\ [e_1, e_n] = \alpha_{1,n} e_{r+2} & \\ [e_i, e_j] = \alpha_{i,j} e_{i+j-1}, & 2 \leq i, j \leq n-2, \ i+j \leq n, \ i+j \neq r+2 \\ [e_i, e_{r+2-i}] = \alpha_{i,r+2-i} e_{r+1} + \gamma_i e_n, & 2 \leq i \leq r \\ [e_n, e_i] = \alpha_{n,i} e_{i+r}, & 2 \leq i \leq n-r-1 \\ [e_i, e_n] = \alpha_{i,n} e_{i+r}, & 2 \leq i \leq n-r-1 \\ [e_n, e_n] = \alpha_{n,n} e_{2r+1}, & r \leq \frac{n-2}{2} \end{cases}$$

where omitted products are zero and $(\gamma_1, \gamma_2, \ldots, \gamma_r) \neq (0, 0, \ldots, 0)$.
Using Leibniz identity we get the following restrictions:

$$\begin{cases} \alpha_{1,i} = \alpha, & 2 \leq i \leq n-2 \\ \gamma_1 = 0 & \\ \alpha_1(\alpha_1 + 1) = 0 & \\ \alpha_{1,n} = 0, & r \leq n-4 \\ \alpha_{n,n} = 0, & r \leq \dfrac{n-3}{2} \end{cases}$$

It is necessary to consider separately the cases $r = n-3$, $r = n-2$ and $r = \frac{n-2}{2}$ (n even).
Case 1. $\alpha = 0$. Then (1) will have the following form:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-2 \\ [e_i, e_j] = \alpha_{i,j} e_{i+j-1}, & 2 \leq i, j \leq n-2, \ i+j \leq n, \ i+j \neq r+2 \\ [e_i, e_{r+2-i}] = \alpha_{i,r+2-i} e_{r+1} + \gamma_i e_n, & 2 \leq i \leq r \\ [e_n, e_i] = \alpha_{n,i} e_{i+r}, & 2 \leq i \leq n-r-1 \\ [e_i, e_n] = \alpha_{i,n} e_{i+r}, & 2 \leq i \leq n-r-1 \end{cases}$$

Using Leibniz identity for elements $\{e_i, e_{r+1-i}, e_1\}$ for $2 \leq i \leq r$ and $\{e_i, e_1, e_{r+1-i}\}$ for $2 \leq i \leq r$, that is,

$$[e_i, [e_{r+1-i}, e_1]] = [[e_i, e_{r+1-i}], e_1] - [[e_i, e_1], e_{r+1-i}]$$

$$[e_i, [e_1, e_{r+1-i}]] = [[e_i, e_1], e_{r+1-i}] - [[e_i, e_{r+1-i}], e_1]$$

we obtain that $\gamma_i = 0$ for $2 \leq i \leq r$. Hence $e_n \notin \mathcal{L}^2$ and $r = 1$ we have the contradiction to the condition of the lemma.

Case 2. $\alpha = -1$. Then the multiplications (1) will have the form:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-2 \\ [e_1, e_i] = -e_{i+1}, & 2 \leq i \leq n-2 \\ [e_i, e_j] = \alpha_{i,j} e_{i+j-1}, & 2 \leq i, j \leq n-2, \quad i+j \leq n \quad i+j \neq r+2 \\ [e_i, e_{r+2-i}] = \alpha_{i,r+2-i} e_{r+1} + \gamma_i e_n, & 2 \leq i \leq r \\ [e_n, e_i] = \alpha_{n,i} e_{i+r}, & 2 \leq i \leq n-r-1 \\ [e_i, e_n] = \alpha_{i,n} e_{i+r}, & 2 \leq i \leq n-r-1 \end{cases}$$

where $(\gamma_2, \dots, \gamma_r) \neq (0, \dots, 0)$.

Using Leibniz identity it is not difficult to get that

$$\begin{array}{ll} \alpha_{n,i} = \alpha_n, & 2 \leq i \leq n-r-1 \\ \alpha_{i,n} = \overline{\alpha_n}, & 2 \leq i \leq n-r-1 \\ \alpha_n = -\overline{\alpha_n} & \end{array}$$

From equality $[e_1, [e_i, e_i]] = 0$ for $2 \leq i \leq \frac{n-3}{2}$, we have $\alpha_{i,i} = 0$ for $2 \leq i \leq \frac{n-1}{2}$. When $i = \frac{n}{2}$ (when n is even), we consider the following equalities:

$$[e_{\frac{n}{2}}[e_{\frac{n}{2}-1}, e_1]] = [[e_{\frac{n}{2}}, e_{\frac{n}{2}-1}], e_1] - [[e_{\frac{n}{2}}, e_1], e_{\frac{n}{2}-1}] \quad (2)$$

$$[e_{\frac{n}{2}-1}[e_{\frac{n}{2}-1}, e_1]] = [[e_{\frac{n}{2}-1}, e_{\frac{n}{2}-1}], e_1] - [[e_{\frac{n}{2}-1}, e_1], e_{\frac{n}{2}-1}] \quad (3)$$

$$[e_1, [e_{\frac{n}{2}-1}, e_{\frac{n}{2}}]] = [[e_1, e_{\frac{n}{2}-1}], e_{\frac{n}{2}}] - [[e_1, e_{\frac{n}{2}}], e_{\frac{n}{2}-1}] \quad (4)$$

From equalities (2) up to (4) we obtain the restrictions:

$$(5) \begin{cases} \alpha_{\frac{n}{2},\frac{n}{2}} = \alpha_{\frac{n}{2},\frac{n}{2}-1} - \alpha_{\frac{n}{2}+1,\frac{n}{2}-1} \\ \alpha_{\frac{n}{2},\frac{n}{2}-1} = -\alpha_{\frac{n}{2}-1,\frac{n}{2}} \\ \alpha_{\frac{n}{2},\frac{n}{2}} = \alpha_{\frac{n}{2}+1,\frac{n}{2}-1} + \alpha_{\frac{n}{2}-1,\frac{n}{2}} \end{cases}$$

From (5) we have that $\alpha_{\frac{n}{2},\frac{n}{2}} = 0$.

Thus, we prove that $[e_i, e_i] = 0$ for $1 \leq i \leq n$.

From the following chain of equalities:

$$\begin{aligned} [e_i, e_j] &= [e_i, [e_{j-1}, e_1]] = [[e_i, e_{j-1}], e_1] - [[e_i, e_1], e_{j-1}] = \\ &= -[e_1, [e_i, e_{j-1}]] + [[e_1, e_i], e_{j-1}] = \\ &= -([[e_1, e_i], e_{j-1}] - [[e_1, e_{j-1}], e_i]) + [[e_1, e_i], e_{j-1}] = [[e_1, e_{j-1}], e_i] = \\ &= -[e_j, e_i] \end{aligned}$$

we obtain that $[e_i, e_j] = -[e_j, e_i]$ for $1 \leq i < j \leq n$, that is, is a Lie algebra.

The cases $r = n-3$, $r = n-2$ and $r = \frac{n-2}{2}$ (when n is even) are proved analogously. □

Next, we will see some examples of graded filiform Leibniz algebras of type II.

Example 2. Let $\mathcal{L}$ be a graded filiform Leibniz algebra of type II. Then $\mathcal{L} \oplus \mathbb{C}$ is graded 2-filiform Leibniz algebra of type II.

And now, we prove some lemmas for a graded non split and non Lie 2-filiform Leibniz algebra of type II.

Lemma 2.8. *There exits no a graded non split and non Lie 2-filiform Leibniz algebra of type II and $r = 1$.*

Proof:
Let $\mathcal{L}$ be a Leibniz algebra which satisfies the condition of the lemma. Then the table of multiplication is

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-2 \\ [e_1, e_i] = \alpha_{1,i} e_{i+1}, & 2 \le i \le n-2 \\ [e_1, e_n] = \alpha_{1,n} e_3 & \\ [e_i, e_j] = \alpha_{i,j} e_{i+j-1}, & 2 \le i, j \le n-2, \quad i+j \le n \\ [e_n, e_i] = \alpha_{n,i} e_{i+1}, & 2 \le i \le n-2 \\ [e_i, e_n] = \alpha_{i,n} e_{i+1}, & 2 \le i \le n-2 \\ [e_n, e_n] = \alpha_{n,n} e_3 & \end{cases}$$

From Leibniz identity we have the following restrictions:

$$\begin{array}{ll} \alpha_{1,i} = \alpha, & 2 \le i \le n-2 \\ \alpha_1(\alpha_1 + 1) = 0 & \\ \alpha_{1,n} = \alpha_{n,n} = 0 & \end{array}$$

Case 1. $\alpha = 0$. Using Leibniz identity we get

$$\begin{array}{ll} \alpha_{i,j} = \alpha_j, & 2 \le i \le n-2 \\ \alpha_j = 0, & 3 \le j \le n-2 \\ \alpha_{i,n} = \alpha_n & 2 \le i \le n-2 \\ \alpha_{n,i} = 0 & 2 \le i \le n-2 \end{array}$$

and taking a change of basis: $e_2' = e_2 - \alpha_2 e_1$, $e_n' = e_n - \alpha_n e_1$, $e_i' = e_i$ for $i \neq 2, n$, we have that $\alpha_2 = \alpha_n = 0$, that is, $\mathcal{L}$ is split.
Case 2. $\alpha = -1$. Analogous to lemma 2.7, we get a Lie algebra.

□

Lemma 2.9. *There exits no a graded non split and non Lie 2-filiform Leibniz algebra of type II and $r = 2$.*

Proof:
Let $\mathcal{L}$ be a Leibniz algebra satisfying the conditions of the lemma. Then, there exists an adapted basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathcal{L}$ such that the multiplications will be the following:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-2 \\ [e_1, e_2] = \alpha_{1,2} e_3 + \gamma_1 e_n & \\ [e_1, e_i] = \alpha_{1,i} e_{i+1}, & 3 \le i \le n-2 \\ [e_1, e_n] = \alpha_{1,n} e_4 & \\ [e_2, e_2] = \alpha_{2,2} e_3 + \gamma_2 e_n & \\ [e_i, e_j] = \alpha_{i,j} e_{i+j-1}, & 2 \le i, j \le n-2, \ i+j \le n, \ (i,j) \neq (2,2) \\ [e_n, e_i] = \alpha_{n,i} e_{i+2}, & 2 \le i \le n-3 \\ [e_i, e_n] = \alpha_{i,n} e_{i+2}, & 2 \le i \le n-3 \\ [e_n, e_n] = \alpha_{n,n} e_5, & \end{cases}$$

where either $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$.

As in the above two lemmas we obtain:

$$\begin{cases} \alpha_{1,i} = \alpha, & 2 \le i \le n-2 \\ \alpha(1+\alpha) = 0 \\ \alpha_{1,n} = \alpha_{n,n} = 0 \end{cases}$$

Let us consider two possible cases for parameter α.

Case 1. $\alpha = 0$. Then, the multiplications in $\mathcal{L}$ have the form:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-2 \\ [e_1, e_2] = \gamma_1 e_n \\ [e_2, e_2] = \alpha_{2,2} e_3 + \gamma_2 e_n \\ [e_i, e_j] = \alpha_{i,j} e_{i+j-1}, & 2 \le i, j \le n-2, \ i+j \le n, \ (i,j) \ne (2,2) \\ [e_n, e_i] = \alpha_{n,i} e_{i+2}, & 2 \le i \le n-3 \\ [e_i, e_n] = \alpha_{i,n} e_{i+2}, & 2 \le i \le n-3 \end{cases}$$

where either $\gamma_1 \ne 0$ or $\gamma_2 \ne 0$.

Using Leibniz identity leads us to the following restrictions:

$$\begin{array}{ll} \alpha_{i,j} = \alpha_j, & 2 \le i, j \le n-2 \\ \alpha_j = 0, & 3 \le j \le n-2 \\ \alpha_{i,n} = \alpha_n, & 2 \le i \le n-3 \\ \alpha_{n,i} = 0, & 2 \le i \le n-3 \\ \alpha_n \gamma_2 = 0 \\ \alpha_n \gamma_1 = 0 \end{array}$$

Either $\gamma_1 \ne 0$ or $\gamma_2 \ne 0$ (otherwise algebra $\mathcal{L}$ is split), then $\alpha_n = 0$.

The change of basis given by $e_2' = e_2 - \alpha_2 e_1$, $e_i' = e_i$, $i \ne 2$, allows to suppose $\alpha_2 = 0$.

Consider the operator of right multiplication $R_{e_1 + Ae_{n-1}}$, where $0 \ne A \in \mathbb{C}$ such that $e_1 + Ae_{n-1} \ne 0$. $\gamma_2 = 0$ and hence $\gamma_1 \ne 0$ may be proved in much the same way as the proof of theorem 2.5. Without loss of generality we can assume that $\gamma = 1$.

Thus, we have the following multiplications in algebra $\mathcal{L}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-2 \\ [e_1, e_2] = e_n \end{cases}$$

Taking the change of basis in form:

$$\begin{array}{l} e_1' = e_1 + e_2, \quad e_2' = e_3 + e_n, \\ e_i' = e_{i+1}, \quad 3 \le i \le n-2, \\ e_{n-1}' = e_1, \quad e_n' = e_n \end{array}$$

we obtain the algebra of type I.

Case 2. $\alpha = -1$. As in above cases, we get a Lie algebra.

□

From lemmas 2.7, 2.8 and 2.9 we can conclude that there exist no graded non split and non Lie 2-filiform Leibniz algebras of type II.

Thus, according to theorem 2.5 we have the classification of non split and non Lie 2-filiform Leibniz algebras. Summing the classification of non split graded 2-filiform Lie algebras [5] and the result of theorem 2.5 we have completed the classification of graded non split 2-filiform Leibniz algebras.

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[L.M. Camacho – J.R. Gómez] Dpto. Matemática Aplicada I. Universidad de Sevilla. Avda. Reina Mercedes, s/n. 41012 Sevilla. (Spain)

E-mail address: lcamacho@us.es --- jrgomez@us.es

[A.J. González] Dpto. de Matemáticas. Universidad de Extremadura. Avda. de la Universidad, s/n. Badajoz. (Spain)

E-mail address: agonzale@unex.es

[B.A. Omirov] Institute of Mathematics of Academy of Uzbekistan. 29, F.Khodjaev str. 700125, Tashkent, Uzbekistan

E-mail address: omirovb@mail.ru