LEIBNIZ ALGEBRAS ASSOCIATED WITH REPRESENTATIONS OF FILIFORM LIE ALGEBRAS

SH.A. AYUPOV, L.M. CAMACHO, A.KH. KHUDOYBERDIYEV, B.A. OMIROV

ABSTRACT. In this paper we investigate Leibniz algebras whose quotient Lie algebra is a naturally graded filiform Lie algebra $n_{n,1}$. We introduce a Fock module for the algebra $n_{n,1}$ and provide classification of Leibniz algebras L whose corresponding Lie algebra L/I is the algebra $n_{n,1}$ with condition that the ideal I is a Fock $n_{n,1}$-module, where I is the ideal generated by squares of elements from L.

We also consider Leibniz algebras with corresponding Lie algebra $n_{n,1}$ and such that the action $I \times n_{n,1} \to I$ gives rise to a minimal faithful representation of $n_{n,1}$. The classification up to isomorphism of such Leibniz algebras is given for the case of $n = 4$.

AMS Subject Classifications (2010): 17A32, 17B30, 17B10.
Key words: Filiform algebra, Heisenberg algebra, Fock representation, minimal faithful representation.

1. INTRODUCTION

Leibniz algebras are generalizations of Lie algebras and they have been firstly introduced by Loday in [16] as a non-antisymmetric version of Lie algebras.

However this kind of algebras was previously introduced and studied under the notion of D-algebras by D. Bloh [5]. Since the 1993 when Loday's work was published, many researchers have been attracted to Leibniz algebras, with remarkable activity during the last decade. Namely, the investigations have been mainly focused on low dimensional, nilpotent, solvable and other special classes of algebras (see [1, 3, 6, 8, 9, 11, 13, 18]).

Recall that the variety of Leibniz algebras is defined by the fundamental identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

In fact, each non-Lie Leibniz algebra L contains a non-trivial ideal (further denoted by I), which is the subspace spanned by squares of elements of the algebra L . Moreover, it is readily to see that this ideal belongs to right annihilator of L, that is $[L, I] = 0$. Note also that the ideal I is the minimal ideal with the property that the quotient algebra L/I is a Lie algebra.

One of the approaches to investigation of Leibniz algebras is a description of such algebras whose quotient algebra with respect to the ideal I is a given Lie algebra. In particular in [17] the description has been obtained for finite-dimensional complex Leibniz algebras whose quotient algebra is isomorphic to the simple Lie algebra sl_2. In [4] D. Barnes showed that any finite-dimensional complex Leibniz algebra can be decomposed into a semidirect sum of the solvable radical and a semi-simple Lie algebra (the analogue of Levi's theorem). Hence, we conclude that if the quotient algebra is isomorphic to a semi-simple Lie algebra, then knowing a module over this semi-simple Lie algebra, one can easily obtain the description of Leibniz algebras with this properties.

Therefore, it is important to study the case when the quotient Lie algebra is solvable, or moreover is nilpotent. Since the Heisenberg and filiform Lie algebras are well-known, it is natural to consider a Leibniz algebra whose quotient Lie algebra is the Heisenberg algebra H_n or the filiform Lie algebra $n_{n,1}$. On the other hand, we recall that Heisenberg and filiform Lie algebras play an important role in mathematical physics and geometry, in particular in Quantum Mechanics (see for instance [10, 12, 14, 15]). Indeed, the Heisenberg Uncertainty Principle implies the non-compatibility of position and momentum observables acting on fermions. In [7] some Leibniz algebras with the quotient algebra being Heisenberg algebra are described. In particular, a classification theorem was obtained for Leibniz algebras whose corresponding Lie algebra is H_n and that the H_n-module I is isomorphic to its Fock module.

The third named author was partially supported by IMU/CDC-program, and he also supported by Instituto de Matématicas de la Universidad de Sevilla. The authors were supported by Ministerio de Economía y Competitividad (Spain), grant MTM2013-43687-P (European FEDER support included).

In order to achieve the goal of our study we organize the paper as follows. The first two sections are devoted to introduction and preliminaries. In section 3 we introduce the Fock module for the filiform Lie algebra $n_{n,1}$ and give the classification of Leibniz algebras with corresponding Lie algebra $n_{n,1}$ under the condition that the $n_{n,1}$-module I is Fock module. In this section we also consider a generalization of this class of algebras by considering the direct sum of filiform Lie algebras as the corresponding Lie algebras, and provide a classification theorem. Finally, in Section 4 we deal with the category of Leibniz algebras with $n_{n,1}$ as corresponding Lie algebra and such that the action $I \times n_{n,1} \to I$ gives rise to a minimal faithful representation of $n_{n,1}$. A complete description of this category of algebras is given when dimension is equal to 4.

2. Preliminaries

In this section we give necessary definitions and preliminary results.

Definition 1. *An algebra $(L, [-,-])$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x, y, z \in L$, the so-called Leibniz identity*

$$\big[[x,y],z\big] = \big[[x,z],y\big] + \big[x,[y,z]\big]$$

holds.

For a Leibniz algebra L consider the following lower central series:

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1] \qquad k \geq 1.$$

Since the notions of right nilpotency and nilpotency coincide, we can define nilpotency as follows:

Definition 2. *A Leibniz algebra L is called nilpotent if there exists $s \in \mathbb{N}$ such that $L^s = 0$.*

Definition 3. *A Leibniz algebra L is said to be filiform if $\dim L^i = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.*

Now let us define a natural gradation for a filiform Leibniz algebra.

Definition 4. *Given a filiform Leibniz algebra L, put $L_i = L^i/L^{i+1}$, $1 \leq i \leq n-1$, and $Gr(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $Gr(L)$. If $Gr(L)$ and L are isomorphic, then we say that the algebra L is naturally graded.*

From [19] it is well known that there are two types of naturally graded filiform Lie algebras. In fact, the second type will appear only in the case when the dimension of the algebra is even.

Theorem 1 ([19])**.** *Any complex naturally graded filiform Lie algebra is isomorphic to one of the following non isomorphic algebras:*

$$n_{n,1}: \quad \{[x_i, x_1] = -[x_1, x_i] = x_{i+1}, \quad 2 \leq i \leq n-1.$$

$$Q_{2n}: \quad \begin{cases} [x_i, x_1] = -[x_1, x_i] = x_{i+1}, & 2 \leq i \leq 2n-2, \\ [x_i, x_{2n+1-i}] = -[x_{2n+1-i}, x_i] = (-1)^i x_{2n}, & 2 \leq i \leq n. \end{cases}$$

Let L be a Leibniz algebra. The ideal I generated by the squares of elements of the algebra L, that is by the set $\{[x,x] : x \in L\}$, plays an important role in the theory since it determines the (possible) non-Lie character of L. From the Leibniz identity, this ideal satisfies

$$[L, I] = 0.$$

Clearly, quotient algebra L/I is a Lie algebra, called the *corresponding Lie algebra* of L. The map $I \times L/I \to I$, $(i, \overline{x}) \mapsto [i, x]$ endows I with a structure of L/I-module (see [2, 17]).

Denote by $Q(L) = L/I \oplus I$, then the operation $(-,-)$ defines the Leibniz algebra structure on $Q(L)$, where

$$(\overline{x}, \overline{y}) = \overline{[x,y]}, \quad (\overline{x}, i) = [x,i], \quad (i, \overline{x}) = 0, \quad (i,j) = 0, \qquad x, y \in L,\ i, j \in I.$$

Therefore, given a Lie algebra G and a $G-$module M, we can construct a Leibniz algebra (G, M) by the above construction.

The main problem which occurs in this connections is a description of Leibniz algebras L, such that the corresponding Leibniz algebra $Q(L)$ is isomorphic to an a priory given algebra (G, M).

In the present paper we restrict our attention on the case where the Lie algebra G is the naturally graded filiform Lie algebra $n_{n,1}$ and the $G-$module M is the Fock module or a minimal faithful module.

2.1. **Fock module over the algebra $n_{n,1}$.** First we recall the notion of Fock module over the Heisenberg algebra H_1, which was introduced in [7]. It is known that if we denote by $\overline{x}$ the operator associated to position and by $\frac{\overline{\partial}}{\partial x}$ the one associated to momentum (acting for instance on the space V of differentiable functions on a single variable), then $[\overline{x}, \frac{\overline{\partial}}{\partial x}] = \overline{1}_V$. Thus we can identify the subalgebra generated by $\overline{1}, \overline{x}$ and $\frac{\overline{\partial}}{\partial x}$ with the three-dimensional Heisenberg algebra H_1 whose multiplication table in the basis $\{\overline{1}, \overline{x}, \frac{\overline{\partial}}{\partial x}\}$ has as unique non-zero product $[\overline{x}, \frac{\overline{\partial}}{\partial x}] = \overline{1}$.

For a given Heisenberg algebra H_1 this representation gives rise to the so-called *Fock module* over H_1, the linear space $\mathbb{F}[x]$ of polinomials on x ($\mathbb{F}$ denotes the algebraically closed field with zero characteristic) with the action induced by

$$\begin{array}{lcl} (p(x), \overline{1}) & \mapsto & p(x) \\ (p(x), \overline{x}) & \mapsto & xp(x) \\ (p(x), \frac{\overline{\partial}}{\partial x}) & \mapsto & \frac{\partial}{\partial x}(p(x)) \end{array} \tag{1}$$

for any $p(x) \in \mathbb{F}[x]$.

Now for any filiform Lie algebra $n_{n,1}$ the define Fock module. Algebra $n_{n,1}$ is characterized by the existence of a basis $\{x_1, x_2, \ldots, x_n\}$ (see Theorem 1) and we denote

$$\frac{\overline{\delta}}{\delta x} = x_1, \quad \overline{x^{n-i}} = (n-i)!x_i, \quad 2 \leq i \leq n. \tag{2}$$

Then the action on $n_{n,1}$ is the linear space $\mathbb{F}[x]$, defined by

$$\begin{array}{lcl} (p(x), \overline{1}) & \mapsto & p(x) \\ (p(x), \overline{x^i}) & \mapsto & x^i p(x) \\ (p(x), \frac{\overline{\delta}}{\delta x}) & \mapsto & \frac{\delta(p(x))}{\delta x} \end{array} \tag{3}$$

In Section 3 we are interested in studying the class of Leibniz algebras L satisfying that its corresponding Lie algebra is a filiform Lie algebra $n_{n,1}$ and the $n_{n,1}$-module I is isomorphic to its Fock module.

This algebra will be called *filiform Fock type* Leibniz algebra and denoted by $FR(n_{n,1})$, hence we will consider the filiform Lie algebra together with its Fock representation.

2.2. **Minimal faithful representation on the algebra $n_{n,1}$.** It is known that the minimal faithful representations of $n_{n,1}$ have dimension n. More precisely, if $\{x_1, x_2, \ldots, x_n\}$ is a basis of $n_{n,1}$, then as a minimal faithful representations we take linear transformations with the matrices $\sum_{i=1}^{n-2} E_{i,i+1}$, $E_{1,n}, E_{2,n}, \ldots, E_{n-1,n}$, on the linear space $V = \{e_1, e_2, \ldots, e_n\}$, where $E_{i,j}$ is a matrix with (i,j) -th entry equal to 1 and others are zero.

In other words, these linear transformations have the form

$$\begin{pmatrix} 0 & a_1 & 0 & \ldots & 0 & a_2 \\ 0 & 0 & a_1 & \ldots & 0 & a_3 \\ 0 & 0 & 0 & \ldots & 0 & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 & a_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & a_n \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

The faithful representation (isomorphism) $\varphi : n_{n,1} \to End(V)^-$ is defined as follows

$$\varphi(x_1) = \sum_{i=1}^{n-2} E_{i,i+1}, \quad \varphi(x_i) = E_{n+1-i,n} \quad 2 \leq i \leq n.$$

i.e.,

$$\varphi([x,y])(e) = [\varphi(x), \varphi(y)](e) = \varphi(y)\big(\varphi(x)(e)\big) - \varphi(x)\big(\varphi(y)(e)\big),$$

where $x, y \in n_{n,1}$, $\varphi(x) \in End(V)$, $e \in V$.

Now, we construct a module $V \times H_{2m+1} \to V$, such that

$$(e, x) = \varphi(x)e.$$

Then we obtain

$$\begin{cases} (e_i, x_1) = e_{i-1}, & 2 \le i \le n-1, \\ (e_n, x_j) = e_{n+1-j}, & 2 \le j \le n, \end{cases}$$

the remaining products in the action being zero.

In Section 4 we deal with the category of Leibniz algebras with $n_{n,1}$ as corresponding algebra and such that the action $I \times n_{n,1} \to I$ gives rise to a minimal faithful representation of $n_{n,1}$.

3. Classification of Filiform Fock type Leibniz algebras

3.1. **Classification of $FR(n_{n,1})$.** Consider a naturally graded filiform Lie algebra $n_{n,1}$ with its Fock module $\mathbb{F}[x]$ under the action (3). Since $\mathbb{F}[x]$ is infinite-dimensional we obtain a family of infinite-dimensional Leibniz algebras.

Theorem 2. *The Leibniz algebra $FR(n_{n,1})$ admits a basis*

$$\{\overline{1}, \overline{x}, \overline{x^2}, \dots, \overline{x^{n-2}}, \frac{\overline{\delta}}{\delta x}, \ x^t \mid t \in \mathbb{N} \cup \{0\}\}$$

such that the multiplication table in this basis has the form:

$$\begin{array}{ll} [\overline{x^i}, \frac{\overline{\delta}}{\delta x}] = i\overline{x^{i-1}}, & 1 \le i \le n-2, \\ [\frac{\overline{\delta}}{\delta x}, \overline{x^i}] = -i\overline{x^{i-1}}, & 1 \le i \le n-2, \\ [x^t, \overline{x^i}] = x^{t+i}, & 1 \le i \le n-2, \\ [x^t, \overline{1}] = x^t, & [x^t, \frac{\overline{\delta}}{\delta x}] = tx^{t-1}, \end{array}$$

where the omitted products are equal to zero.

Proof. Taking into account the action (3) we conclude that

$$\{\overline{1}, \overline{x}_i, \frac{\overline{\delta}}{\delta x_i}, \ x_1^{t_1} x_2^{t_2} \dots x_k^{t_k} \mid t_i \in \mathbb{N} \cup \{0\}, \ 1 \le i \le k\}$$

is a basis of $FR(n_{n,1})$ and

$$[x^t, \overline{1}] = x^t, \quad [x^t, \frac{\overline{\delta}}{\delta x}] = tx^{t-1}, \quad [x^t, \overline{x^i}] = x^{t+i}, \quad 1 \le i \le n-2.$$

Let us denote

$$[\frac{\overline{\delta}}{\delta x}, \overline{1}] = q(x), \quad [\overline{1}, \overline{1}] = r(x), \quad [\overline{x^i}, \overline{1}] = p_i(x), \quad 1 \le i \le n-2,$$

Taking the following change of basis:

$$\frac{\overline{\delta}}{\delta x}' = \frac{\overline{\delta}}{\delta x} - q(x), \quad \overline{1}' = \overline{1} - r(x), \quad \overline{x^i}' = \overline{x^i} - p_i(x), \quad 1 \le i \le n-2,$$

we derive

$$[\overline{x^i}, \overline{1}] = 0, \quad [\frac{\overline{\delta}}{\delta x}, \overline{1}] = 0, \quad [\overline{1}, \overline{1}] = 0, \quad 1 \le i \le n-2.$$

We denote

$$\begin{array}{lll} [\frac{\overline{\delta}}{\delta x}, \frac{\overline{\delta}}{\delta x}] = a(x), & [\overline{x^i}, \overline{x^j}] = b_{i,j}(x), & 1 \le i, j \le n-2, \\ [\overline{1}, \frac{\overline{\delta}}{\delta x}] = c(x), & [\overline{x^i}, \frac{\overline{\delta}}{\delta x}] = i\overline{x^{i-1}} + d_i(x), & 1 \le i \le n-2, \\ [\overline{1}, \overline{x^i}] = g_i(x), & [\frac{\overline{\delta}}{\delta x}, \overline{x^i}] = -i\overline{x^{i-1}} + h_i(x), & 1 \le i \le n-2. \end{array}$$

Consider the Leibniz identity

$$[[\frac{\overline{\delta}}{\delta x}, \frac{\overline{\delta}}{\delta x}], \overline{1}] = [\frac{\overline{\delta}}{\delta x}, [\frac{\overline{\delta}}{\delta x}, \overline{1}]] + [[\frac{\overline{\delta}}{\delta x}, \overline{1}], \frac{\overline{\delta}}{\delta x}] = 0,$$

On the other hand,

$$[[\frac{\overline{\delta}}{\delta x}, \frac{\overline{\delta}}{\delta x}], \overline{1}] = [a(x), \overline{1}] = a(x),$$

which implies $a(x) = 0$.

Similarly, from the Leibniz identities

$$\begin{array}{lllllllllll}
b_{i,j}(x) & = & [b_{i,j}(x),\overline{1}] & = & [[\overline{x^i},\overline{x^j}],\overline{1}] & = & [\overline{x^i},[\overline{x^j},\overline{1}]] & + & [[\overline{x^i},\overline{1}],\overline{x^j}] & = & 0,\\
c(x) & = & [c(x),\overline{1}] & = & [[\overline{1},\frac{\overline{\delta}}{\delta x_i}],\overline{1}] & = & [\overline{1},[\frac{\overline{\delta}}{\delta x_i},\overline{1}]] & + & [[\overline{1},\overline{1}],\frac{\overline{\delta}}{\delta x_i}] & = & 0,\\
d_i(x) & = & [i\overline{x^{i-1}}+d_i(x),\overline{1}] & = & [[\overline{x^i},\frac{\overline{\delta}}{\delta x}],\overline{1}] & = & [\overline{x^i},[\frac{\overline{\delta}}{\delta x},\overline{1}]] & + & [[\overline{x^i},\overline{1}],\frac{\overline{\delta}}{\delta x}] & = & 0,\\
g_i(x) & = & [g_i(x),\overline{1}] & = & [[\overline{1},\overline{x^i}],\overline{1}] & = & [\overline{1},[\overline{x^i},\overline{1}]] & + & [[\overline{1},\overline{1}],\overline{x^i}] & = & 0,\\
h_i(x) & = & [-i\overline{x^{i-1}}+h_i(x),\overline{1}] & = & [[\frac{\overline{\delta}}{\delta x},\overline{x^i}],\overline{1}] & = & [\frac{\overline{\delta}}{\delta x},[\overline{x^i},\overline{1}]] & + & [[\frac{\overline{\delta}}{\delta x},\overline{1}],\overline{x^i}] & = & 0,
\end{array}$$

we obtain

$$\begin{array}{c}
c(x)=0, \quad b_{i,j}=0, \qquad 1\le i,j\le n-2,\\
d_i(x)=0, \quad g_i(x)=0, \quad h_i(x)=0, \quad 1\le i\le n-2.
\end{array}$$

□

3.2. **Classification of generalized Filiform Fock type Leibniz algebras.** In this subsection we are focused in classifying of the class of (infinite-dimensional) Leibniz algebras L such that their corresponding Lie algebras are finite direct sums of filiform Lie algebras $n_{n_1,1}\oplus n_{n_2,1}\oplus\cdots\oplus n_{n_s,1}$ and that their actions on I are induced by Fock representations.

Since each algebra $n_{n_i,1}$ has a standard basis $\{x_{i,1},x_{i,2},\dots,x_{i,n}\}$ we put

$$\frac{\overline{\delta}}{\delta x_i}=x_{i,1}, \qquad \overline{x_i^{n-j}}=(n_i-j)!x_{i,j}, \quad 2\le j\le n_i. \tag{4}$$

For the algebra $n_{n_1,1}\oplus n_{n_2,1}\oplus\cdots\oplus n_{n_s,1}$ the *Fock module* on $n_{n_1,1}\oplus n_{n_2,1}\oplus\cdots\oplus n_{n_s,1}$ is the linear space $\mathbb{F}[x_1,x_2,\dots,x_s]$ with the action induced by

$$\begin{array}{lll}
(p(x_1,x_2,\dots,x_s),\overline{1_i}) & \mapsto & p(x_1,x_2,\dots,x_s), \quad 1\le i\le s,\\
(p(x_1,x_2,\dots,x_s),\overline{x_i^j}) & \mapsto & x_i^j p(x_1,x_2,\dots,x_s),\\
(p(x_1,x_2,\dots,x_s),\frac{\overline{\delta}}{\delta x_i}) & \mapsto & \frac{\delta(p(x_1,x_2,\dots,x_s))}{\delta x_i},
\end{array}$$

for any $p(x_1,x_2,\dots,x_s)\in\mathbb{F}[x_1,x_2,\dots,x_s]$.

We denote

$$\begin{array}{l}
[\overline{x_i^j},\overline{1_k}]=a_{i,k}^j(x_1,x_2,\dots,x_s), \quad 1\le i,k\le s, \quad 1\le j\le n_i-2,\\
[\frac{\overline{\delta}}{\delta x_i},\overline{1_j}]=b_{i,j}(x_1,x_2,\dots,x_s), \quad 1\le i,j\le s,\\
[\overline{1_i},\overline{1_j}]=c_{i,j}(x_1,x_2,\dots,x_s), \quad 1\le i,j\le s.
\end{array}$$

Taking the change of basis

$$\begin{array}{l}
\overline{x_i^j}'=\overline{x_i^j}-a_{i,i}^k(x_1,x_2,\dots,x_s), \qquad 1\le i\le s,\ 1\le j\le n_i-2,\\
\frac{\overline{\delta}}{\delta x_i}'=\frac{\overline{\delta}}{\delta x_i}-b_{i,i}(x_1,x_2,\dots,x_s),\\
\overline{1_i}'=\overline{1_i}-c_{i,i}(x_1,x_2,\dots,x_s),
\end{array}$$

we derive

$$[\overline{x_i^j},\overline{1_i}]=0, \quad [\frac{\overline{\delta}}{\delta x_i},\overline{1_i}]=0, \quad [\overline{1_i},\overline{1_i}]=0, \quad 1\le i\le s,\ 1\le j\le n_i-2.$$

Let us introduce notations:

$$
\begin{aligned}
&[\overline{x_i^j},\overline{x_k^t}] = d_{i,k}^{j,t}(x_1,x_2,\dots,x_s), && 1\le i,k\le s,\quad 1\le j\le n_i-2,\quad 1\le t\le n_k-2,\\
&[\tfrac{\overline{\delta}}{\delta x_i},\tfrac{\overline{\delta}}{\delta x_j}] = e_{i,j}(x_1,x_2,\dots,x_s), && 1\le i,j\le s,\\
&[\overline{x_i^j},\tfrac{\overline{\delta}}{\delta x_i}] = i\overline{x_i^{j-1}} + f_{i,i}^j(x_1,x_2,\dots,x_s), && 1\le i\le s,\quad 1\le j\le n_i-2,\\
&[\overline{x_i^j},\tfrac{\overline{\delta}}{\delta x_k}] = f_{i,k}^j(x_1,x_2,\dots,x_s), && 1\le i,k\le s,\quad 1\le j\le n_i-2,\quad i\ne k,\\
&[\tfrac{\overline{\delta}}{\delta x_i},\overline{x_i^j}] = -i\overline{x_i^{j-1}} + g_{i,i}^j(x_1,x_2,\dots,x_s), && 1\le i\le s,\quad 1\le j\le n_i-2,\\
&[\tfrac{\overline{\delta}}{\delta x_k},\overline{x_i^j}] = g_{k,i}^j(x_1,x_2,\dots,x_s), && 1\le i\le s,\quad 1\le j\le n_i-2,\quad i\ne k,\\
&[\overline{1_k},\overline{x_i^j}] = h_{k,i}^j(x_1,x_2,\dots,x_s), && 1\le i,k\le s,\quad 1\le j\le n_i-2,\\
&[\overline{1_i},\tfrac{\overline{\delta}}{\delta x_j}] = q_{i,j}(x_1,x_2,\dots,x_s), && 1\le i,k\le s.
\end{aligned}
$$

Consider the Leibniz identity

$$[[\overline{x_i^j},\overline{1_k}],\overline{1_i}] = [\overline{x_i^j},[\overline{1_k},\overline{1_i}]] + [[\overline{x_i^j},\overline{1_i}],\overline{1_k}] = 0.$$

On the other hand

$$[[\overline{x_i^j},\overline{1_k}],\overline{1_i}] = [a_{i,k}^j(x_1,x_2,\dots,x_s),\overline{1_i}] = a_{i,k}^j(x_1,x_2,\dots,x_s)$$

which implies

$$a_{i,k}^j(x_1,x_2,\dots,x_s) = 0,\quad 1\le i,k\le s,\ 1\le j\le n_i-2.$$

Similarly from the Leibniz identities

$$
\begin{aligned}
b_{i,j}(x_1,x_2,\dots,x_s) &= [[\tfrac{\overline{\delta}}{\delta x_i},\overline{1_j}],\overline{1_i}] = [\tfrac{\overline{\delta}}{\delta x_i},[\overline{1_j},\overline{1_i}]] + [[\tfrac{\overline{\delta}}{\delta x_i},\overline{1_i}],\overline{1_j}] = 0,\\
c_{i,j}(x_1,x_2,\dots,x_s) &= [[\overline{1_i},\overline{1_j}],\overline{1_i}] = [\overline{1_i},[\overline{1_j},\overline{1_i}]] + [[\overline{1_i},\overline{1_i}],\overline{1_j}] = 0,
\end{aligned}
$$

we obtain

$$b_{i,j}(x_1,x_2,\dots,x_s) = 0,\quad c_{i,j}(x_1,x_2,\dots,x_s) = 0,\quad 1\le i,j\le s.$$

In a similar way, from the Leibniz identities

$$
\begin{aligned}
&d_{i,k}^{j,t}(x_1,x_2,\dots,x_s) = [[\overline{x_i^j},\overline{x_k^t}],\overline{1_i}] = [\overline{x_i^j},[\overline{x_k^t},\overline{1_i}]] + [[\overline{x_i^j},\overline{1_i}],\overline{x_k^t}] = 0,\\
&e_{i,j}(x_1,x_2,\dots,x_s) = [[\tfrac{\overline{\delta}}{\delta x_i},\tfrac{\overline{\delta}}{\delta x_j}],\overline{1_i}] = [\tfrac{\overline{\delta}}{\delta x_i},[\tfrac{\overline{\delta}}{\delta x_j},\overline{1_i}]] + [[\tfrac{\overline{\delta}}{\delta x_i},\overline{1_i}],\tfrac{\overline{\delta}}{\delta x_j}] = 0,\\
&f_{i,i}^j(x_1,x_2,\dots,x_s) = [i\overline{x_i^{j-1}} + f_{i,i}^j(x_1,x_2,\dots,x_s),\overline{1_i}] = [[\overline{x_i^j},\tfrac{\overline{\delta}}{\delta x_i}],\overline{1_i}] = [\overline{x_i^j},[\tfrac{\overline{\delta}}{\delta x_i},\overline{1_i}]] + [[\overline{x_i^j},\overline{1_i}],\tfrac{\overline{\delta}}{\delta x_i}] = 0,\\
&f_{i,k}^j(x_1,x_2,\dots,x_s) = [[\overline{x_i^j},\tfrac{\overline{\delta}}{\delta x_k}],\overline{1_i}] = [\overline{x_i^j},[\tfrac{\overline{\delta}}{\delta x_k},\overline{1_i}]] + [[\overline{x_i^j},\overline{1_i}],\tfrac{\overline{\delta}}{\delta x_k}] = 0,\\
&g_{i,i}^j(x_1,x_2,\dots,x_s) = [-i\overline{x_i^{j-1}} + g_{i,i}^j(x_1,x_2,\dots,x_s),\overline{1_i}] = [[\tfrac{\overline{\delta}}{\delta x_i},\overline{x_i^j}],\overline{1_i}] = [\tfrac{\overline{\delta}}{\delta x_i},[\overline{x_i^j},\overline{1_i}]] + [[\tfrac{\overline{\delta}}{\delta x_i},\overline{1_i}],\overline{x_i^j}] = 0,\\
&g_{k,i}^j(x_1,x_2,\dots,x_s) = [g_{k,i}^j(x_1,x_2,\dots,x_s),\overline{1_i}] = [[\tfrac{\overline{\delta}}{\delta x_k},\overline{x_i^j}],\overline{1_k}] = [\tfrac{\overline{\delta}}{\delta x_k},[\overline{x_i^j},\overline{1_k}]] + [[\tfrac{\overline{\delta}}{\delta x_k},\overline{1_k}],\overline{x_i^j}] = 0,\\
&h_{k,i}^j(x_1,x_2,\dots,x_s) = [h_{k,i}^j(x_1,x_2,\dots,x_s),\overline{1_k}] = [[\overline{1_k},\overline{x_i^j}],\overline{1_k}] = [\overline{1_k},[\overline{x_i^j},\overline{1_k}]] + [[\overline{1_k},\overline{1_k}],\overline{x_i^j}] = 0,\\
&q_{i,j}(x_1,x_2,\dots,x_s) = [q_{i,j}(x_1,x_2,\dots,x_s),\overline{1_i}] = [[\overline{1_i},\tfrac{\overline{\delta}}{\delta x_j}],\overline{1_i}] = [\overline{1_i},[\tfrac{\overline{\delta}}{\delta x_j},\overline{1_i}]] + [[\overline{1_i},\overline{1_i}],\tfrac{\overline{\delta}}{\delta x_j}] = 0,
\end{aligned}
$$

we derive

$$
\begin{aligned}
&d_{i,k}^{j,t}(x_1,x_2,\dots,x_s) && 1\le i,k\le s,\quad 1\le j\le n_i-2,\quad 1\le t\le n_k-2,\\
&e_{i,j}(x_1,x_2,\dots,x_s) = 0, && 1\le i,j\le s,\\
&f_{i,k}^j(x_1,x_2,\dots,x_s) = 0, && 1\le i,k\le s,\quad 1\le j\le n_i-2,\\
&g_{k,i}^j(x_1,x_2,\dots,x_s) = 0, && 1\le i\le s,\quad 1\le j\le n_i-2,\\
&h_{k,i}^j(x_1,x_2,\dots,x_s) = 0, && 1\le i,k\le s,\quad 1\le j\le n_i-2,\\
&q_{i,j}(x_1,x_2,\dots,x_s) = 0, && 1\le i,k\le s.
\end{aligned}
$$

Therefore, we have proved

Theorem 3. *The above Leibniz algebra denoted by $FR(n_{n_1,1} \oplus n_{n_2,1} \oplus \cdots \oplus n_{n_s,1})$ admits a basis*

$$\{\overline{1}_i, \overline{x}_i^j, \frac{\overline{\delta}}{\delta x_i},\ x_1^{t_1}x_2^{t_2}\dots x_k^{t_k} \mid t_i \in \mathbb{N}\cup\{0\},\ 1\le i\le s, 1\le j\le n_i\}$$

in such that the multiplication table in this basis has the form:

$$\begin{aligned}
&[\overline{x_i}, \tfrac{\overline{\delta}}{\delta x_i}] = \overline{1}, \qquad [\tfrac{\overline{\delta}}{\delta x_i}, \overline{x_i}] = -\overline{1}, \qquad 1\le i\le k,\\
&[x_1^{t_1}x_2^{t_2}\dots x_k^{t_k}, \overline{1}] = x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\\
&[x_1^{t_1}x_2^{t_2}\dots x_k^{t_k}, \overline{x}_i^j] = x_1^{t_1}\dots x_{i-1}^{t_{i-1}}x_i^{t_i+j}x_{i+1}^{t_{i+1}}\dots x_k^{t_k},\\
&[x_1^{t_1}x_2^{t_2}\dots x_k^{t_k}, \tfrac{\overline{\delta}}{\delta x_i}] = t_i x_1^{t_1}\dots x_{i-1}^{t_{i-1}}x_i^{t_i-1}x_{i+1}^{t_{i+1}}\dots x_k^{t_k}.
\end{aligned}$$

where the omitted products are equal to zero.

4. Leibniz algebras associated with minimal faithful representation of $n_{n,1}$

In this section we are going to study the Leibniz algebras L such that $L/I \cong n_{n,1}$ and the $n_{n,1}$-module I is the minimal faithful representation. In this case we have that $\dim L = 2n$ and $\{x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_n\}$ is a basis of L. We also have

$$\begin{cases} [e_i, x_1] = e_{i-1}, & 2\le i\le n-1,\\ [e_n, x_j] = e_{n+1-j}, & 2\le j\le n. \end{cases} \tag{5}$$

Further we should define the multiplications $[x_i, x_j]$ for $1\le i,j\le n$. We put

$$[x_i,x_j] = \begin{cases} x_{i+1} + \sum\limits_{k=1}^{n}\alpha_{i,1}^k e_k, & j=1,\ 2\le i\le n-1,\\ -x_{j+1} + \sum\limits_{k=1}^{n}\alpha_{1,j}^k e_k, & i=1,\ 2\le j\le n-1,\\ \sum\limits_{k=1}^{n}\alpha_{i,j}^k e_k, & otherwise. \end{cases} \tag{6}$$

In the following Lemma we define the multiplications in the case of $i=1$ or $j=1$ and $j=n$.

Lemma 1. *There exists a basis $\{x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_n\}$ of L such that*

$$\begin{cases} [x_1,x_1] = \alpha_1 e_{n-1} + \alpha_2 e_n, & [x_1,x_j] = -x_{j+1}, & 2\le j\le n-1,\\ [x_2,x_1] = x_3 + \alpha_3 e_n, & [x_i,x_1] = x_{i+1} - \alpha_2 e_{n+2-i}, & 3\le i\le n-1,\\ [x_1,x_n] = \alpha_4 e_1 + \alpha_2 e_2, & [x_n,x_1] = -\alpha_4 e_1 - 2\alpha_2 e_2, & \\ [x_2,x_n] = \alpha_5 e_1 + \alpha_3 e_2, & [x_i,x_n] = 0, & 3\le i\le n. \end{cases} \tag{7}$$

Proof. In the multiplication (6) taking the transformation of basis

$$\begin{aligned}
x_1' &= x_1 - \sum_{k=1}^{n-2}\alpha_{1,1}^k e_{k+1} - (\alpha_{2,1}^{n-1} + \alpha_{1,2}^{n-1})e_n, \qquad x_2' = x_2 - \sum_{k=1}^{n-2}(\alpha_{2,1}^k + \alpha_{1,2}^k)e_{k+1},\\
x_j' &= x_j - \sum_{k=1}^{n}\alpha_{1,j-1}^k e_k + (\alpha_{2,1}^{n-1} + \alpha_{1,2}^{n-1})e_{n+2-j}, \quad 3\le j\le n,
\end{aligned}$$

we obtain $[x_1,x_1] = \alpha_{1,1}^{n-1}e_{n-1} + \alpha_{1,1}^n e_n$, $[x_2,x_1] = x_3 + \alpha_{2,1}^n e_n$, $[x_1,x_j] = -x_{j+1}$, $2\le j\le n-1$.

Using the Leibniz identity we derive

$$\begin{aligned}
[x_3,x_1] &= -[[x_1,x_2],x_1] = -[x_1,[x_2,x_1]] - [[x_1,x_1],x_2] =\\
&= [-x_1, x_3 + \alpha_{2,1}^{n-1}e_{n-1} + \alpha_{2,1}^n e_n] - [\alpha_{1,1}^{n-1}e_{n-1} + \alpha_{1,1}^n e_n, x_2] = x_4 - \alpha_{1,1}^n e_{n-1}.
\end{aligned}$$

From the Leibniz identity, $[[x_1,x_i],x_1] = [x_1,[x_i,x_1]] + [[x_1,x_1],x_i]$ recurrently we obtain

$$\begin{cases} [x_i,x_1] = x_{i+1} - \alpha_{1,1}^n e_{n+2-i}, & 3\le i\le n-1,\\ [x_n,x_1] = -\alpha_{1,1}^n e_2 - \sum\limits_{k=1}^{n}\alpha_{1,n}^k e_k. \end{cases}$$

On the other hand, from

$$0=[x_1,[x_n,x_1]]=[[x_1,x_n],x_1]-[[x_1,x_1],x_n]=\sum_{k=2}^{n-1}\alpha_{1,n}^k e_{k-1}-\alpha_{1,1}^n e_1,$$

we get

$$\alpha_{1,n}^2=\alpha_{1,1}^n,\qquad \alpha_{1,n}^k=0,\quad 3\le k\le n-1.$$

Now, we consider the Leibniz identity

$$\begin{aligned}0&=[x_1,[x_n,x_j]]=[[x_1,x_n],x_j]-[[x_1,x_j],x_n]=\\&=[\alpha_{1,n}^1e_1+\alpha_{1,1}^n e_2+\alpha_{1,n}^n e_n,x_j]+[x_{j+1},x_n]=\alpha_{1,n}^n e_{n+1-j}+[x_{j+1},x_n].\end{aligned}$$

Hence, we have

$$[x_{j+1},x_n]=-\alpha_{1,n}^n e_{n+1-j},\quad 2\le j\le n-1.$$

On the other hand, from the equalities

$$\begin{aligned}0&=[x_2,[x_1,x_n]]=[[x_2,x_1],x_n]-[[x_2,x_n],x_1]=\\&=[x_3+\alpha_{2,1}^n e_n,x_n]-[\sum_{k=1}^{n-1}\alpha_{2,n}^k e_k,x_1]=-\alpha_{1,n}^n e_{n-1}+\alpha_{2,1}^n e_1-\sum_{k=1}^{n-2}\alpha_{2,n}^{k+1}e_k,\end{aligned}$$

we obtain

$$\alpha_{2,1}^n=\alpha_{2,n}^2,\quad \alpha_{1,n}^n=0,\quad \alpha_{2,n}^k=0,\quad 3\le k\le n-1.$$

□

Put

$$Q_{0,1}=1,\qquad Q_{0,k}=\tfrac{1}{2},\quad k\ge 2,\qquad Q_{1,k}=\tfrac{k+1}{2},$$
$$Q_{m,k}=\frac{k(k+1)\dots(k+m-2)(k+2m-1)}{2(m!)},\quad m\ge 2.$$

It is not difficult to check that

$$Q_{m,k}=Q_{m,k-1}+Q_{m-1,k}. \tag{8}$$

Now we will define the products $[x_i,x_j]$ for $i+j\le n+1$.

Lemma 2. *We have*

$$\begin{cases}
[x_2,x_2] &= \sum\limits_{k=1}^{n-2}\beta_k e_k,\\
[x_{i+1},x_i] &= \sum\limits_{k=1}^{n-1}\gamma_{i,k}e_k, & 2\le i\le \lfloor\frac{n}{2}\rfloor,\\
[x_i,x_{i+j}] &= \sum\limits_{s=0}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^s Q_{s,j+2-2s}\sum\limits_{k=1}^{n-2-j+2s}\gamma_{i+s-1,j+1-2s+k}e_k, & 0\le j\le n-5,\quad 3\le i\le\lfloor\frac{n+1-j}{2}\rfloor,\\
[x_2,x_j] &= -(j-2)\alpha_3 e_{n+2-j}+\sum\limits_{k=1}^{n-j}\beta_{j-2+k}e_k+\\
&\quad+\sum\limits_{s=2}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^{s+1}Q_{s-1,j+2-2s}\sum\limits_{k=1}^{n-2-j+2s}\gamma_{s,j+1-2s+k}e_k, & 3\le j\le n-1,
\end{cases} \tag{9}$$

where $\lfloor a\rfloor$ is the integer part of a.

Proof. Taking into account the notation (6), from the Leibniz identity

$$\begin{aligned}0&=[x_i,[x_j,x_k]]=[[x_i,x_j],x_k]-[[x_i,x_k],x_j]=\left[\sum_{t=1}^{n}\alpha_{i,j}^t e_t,x_k\right]+\left[\sum_{t=1}^{n}\alpha_{i,k}^t e_t,x_i\right]=\\&=\alpha_{i,j}^n e_{n+2-k}-\alpha_{i,k}^n e_{n+2-j},\quad 2\le i,j,k(j\ne k)\le n,\end{aligned}$$

we get

$$\alpha_{i,j}^n=0,\quad 2\le i,j\le n.$$

From the Leibniz identities for the triples of elements $[x_1,[x_i,x_j]],[x_i,[x_1,x_j]]$, we derive the following relations

$$(10)\qquad \begin{cases} [x_{i+1},x_j]=[x_{j+1},x_i], & 2\le i,j\le n-1,\\ [x_2,x_{j+1}]+[x_3,x_j]=-\alpha_3 e_{n+1-j}+[[x_2,x_j],x_1], & 2\le j\le n-1,\\ [x_i,x_{j+1}]+[x_{i+1},x_j]=[[x_i,x_j],x_1], & 3\le i\le n-1,\ 2\le j\le n-1,\\ [x_n,x_{j+1}]=[[x_n,x_j],x_1] & 2\le j\le n-1. \end{cases}$$

From the first equality in (10) it is easy to see that it is sufficient to define the multiplications $[x_i,x_j]$ for $j\ge i-1$. Put,

$$[x_2,x_2]=\sum_{k=1}^{n-1}\beta_k e_k,\quad [x_{i+1},x_i]=\sum_{k=1}^{n-1}\gamma_{i,k}e_k,\quad 2\le i\le \lfloor\frac{n}{2}\rfloor.$$

Applying, if necessary, the change of basis $x_2'=x_2-\beta_{n-1}e_n$ we can suppose that $\beta_{n-1}=0$ and we will express other product by means of the structure constants β_i and $\gamma_{i,j}$.

First we will proof the third equation from (9) by induction on j. From the relations (10) we get $[x_i,x_i]=[x_{i+1},x_{i-1}]$ and $[x_i,x_i]+[x_{i+1},x_{i-1}]=[[x_i,x_{i-1}],x_1]$ which imply the assertion of Lemma for $j=0$, i.e.,

$$[x_i,x_i]=\frac{1}{2}[[x_i,x_{i-1}],x_1]=\frac{1}{2}\sum_{k=1}^{n-2}\gamma_{i-1,k+1}e_k,\quad 3\le i\le\lfloor\frac{n+1}{2}\rfloor.$$

Then from the relations (1) we obtain $[x_i,x_{i+1}]+[x_{i+1},x_i]=[[x_i,x_i],x_1]$. Using the assertion of Lemma for $j=0$, we get

$$[x_i,x_{i+1}]=\frac{1}{2}[[[x_i,x_{i-1}],x_1],x_1]-[x_{i+1},x_i]=\frac{1}{2}\sum_{k=1}^{n-3}\gamma_{i-1,k+2}e_k-\sum_{k=1}^{n-1}\gamma_{i,k}e_k,\quad 3\le i\le\lfloor\frac{n}{2}\rfloor.$$

Hence the assertion of the Lemma is true for $j=1$.

Let us suppose that the assertion of the Lemma is true for indices less or equal to j and we will prove it for $j+1$.

From the relations (10) we obtain $[x_i,x_{i+j+1}]+[x_{i+1},x_{i+j}]=[[x_i,x_{i+j}],x_1]$. Using the assumption of the induction we get

$$\begin{aligned}[x_i,x_{i+j+1}]&=[[x_i,x_{i+j}],x_1]-[x_{i+1},x_{i+j}]=\\ &=\sum_{s=0}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^sQ_{s,j+2-2s}\sum_{k=1}^{n-3-j+2s}\gamma_{i+s-1,j+2-2s+k}e_k-\sum_{s=0}^{\lfloor\frac{j}{2}\rfloor}(-1)^sQ_{s,j+1-2s}\sum_{k=1}^{n-1-j+2s}\gamma_{i+s,j-2s+k}e_k=\\ &=Q_{0,j+2}\sum_{k=1}^{n-3-j}\gamma_{i-1,j+2+k}e_k+\sum_{s=1}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^sQ_{s,j+2-2s}\sum_{k=1}^{n-3-j+2s}\gamma_{i+s-1,j+2-2s+k}e_k-\\ &-\sum_{s=1}^{\lfloor\frac{j}{2}\rfloor+1}(-1)^{s-1}Q_{s-1,j+3-2s}\sum_{k=1}^{n-3-j+2s}\gamma_{i+s-1,j+2-2s+k}e_k.\end{aligned}$$

If j is odd then $\lfloor\frac{j+1}{2}\rfloor=\lfloor\frac{j}{2}\rfloor+1=\lfloor\frac{j+2}{2}\rfloor$ and using the equality (8) we get

$$\begin{aligned}[x_i,x_{i+j+1}]&=Q_{0,j+2}\sum_{k=1}^{n-3-j}\gamma_{i-1,j+2+k}e_k+\sum_{s=1}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^sQ_{s,j+3-2s}\sum_{k=1}^{n-3-j+2s}\gamma_{i+s-1,j+2-2s+k}e_k=\\ &=\sum_{s=0}^{\lfloor\frac{j+2}{2}\rfloor}(-1)^sQ_{s,j+3-2s}\sum_{k=1}^{n-3-j+2s}\gamma_{i+s-1,j+2-2s+k}e_k.\end{aligned}$$

If j is even then $\lfloor\frac{j+1}{2}\rfloor=\lfloor\frac{j}{2}\rfloor$ and we get

$$\begin{aligned}[x_i,x_{i+j+1}]&=Q_{0,j+2}\sum_{k=1}^{n-3-j}\gamma_{i-1,j+2+k}e_k+\sum_{s=1}^{\lfloor\frac{j}{2}\rfloor}(-1)^s(Q_{s,j+3-2s})\sum_{k=1}^{n-3-j+2s}\gamma_{i+s-1,j+2-2s+k}e_k+\\ &+(-1)^{\lfloor\frac{j+2}{2}\rfloor}\sum_{k=1}^{n-1}\gamma_{i+\lfloor\frac{j}{2}\rfloor,k}e_k=\sum_{s=0}^{\lfloor\frac{j+2}{2}\rfloor}(-1)^sQ_{s,j+3-2s}\sum_{k=1}^{n-3-j+2s}\gamma_{i+s-1,j+2-2s+k}e_k.\end{aligned}$$

The products $[x_2, x_j]$ are also obtained by induction on j, using the equality (10) and the multiplication $[x_3, x_{j-1}]$. □

In the following lemma we define the products $[x_i, x_j]$ for $i + j \geq n + 2$.

Lemma 3. *We have*

$$
(11)\quad \begin{cases}
[x_i, x_{n+2-i}] &= (-1)^i\alpha_5 e_1 + (-1)^i(n-5)\alpha_3 e_2 + (-1)^{i+1}\beta_{n-2}e_1 + \\
&+ \sum\limits_{s=2}^{i-2}(-1)^{s+i}\sum\limits_{t=1}^{s} Q_{s-t,n+1-2s}\sum\limits_{k=1}^{2s-2}\gamma_{s,n+1-2s+k}e_k + \\
&+ \sum\limits_{s=i-1}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+i}\sum\limits_{t=1}^{i-2} Q_{s-t,n+1-2s}\sum\limits_{k=1}^{2s-2}\gamma_{s,n+1-2s+k}e_k, \\
[x_i, x_{n+3-i}] &= (-1)^{i+1}(i-3)(n-5)\alpha_3 e_1 + \\
&+ \sum\limits_{s=2}^{i-3}(-1)^{s+i+1}\sum\limits_{t=1}^{s}(i-2-t)Q_{s-t,n+1-2s}\sum\limits_{k=1}^{2s-3}\gamma_{s,n+2-2s+k}e_k + \\
&+ \sum\limits_{s=i-2}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+i+1}\sum\limits_{t=1}^{i-3}(i-2-t)Q_{s-t,n+1-2s}\sum\limits_{k=1}^{2s-3}\gamma_{s,n+2-2s+k}e_k, \\
[x_i, x_{n+p-i}] &= \sum\limits_{s=\lfloor\frac{p}{2}\rfloor+1}^{i-p}(-1)^{s+i+p}\sum\limits_{t=1}^{s}\binom{i-2-t}{p-2}Q_{s-t,n+1-2s}\sum\limits_{k=1}^{2s-p}\gamma_{s,n+p-1-2s+k}e_k + \\
&+ \sum\limits_{s=\max\{i-p+1;\lfloor\frac{p}{2}\rfloor+1\}}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+i+p}\sum\limits_{t=1}^{i-p}\binom{i-2-t}{p-2}Q_{s-t,n+1-2s}\sum\limits_{k=1}^{2s-p}\gamma_{s,n+p-1-2s+k}e_k,
\end{cases}
$$

where $4 \leq p \leq n-1$, $p+1 \leq i \leq \lfloor\frac{n+p+1}{2}\rfloor$.

Proof. First we will find the products $[x_i, x_{n+2-i}]$.

According to Lemma 1 we have $[x_2, x_n] = \alpha_5 e_1 + \alpha_3 e_2$ and using Lemma 2 from relations (10) we obtain

$$
\begin{aligned}
[x_3, x_{n-1}] &= -\alpha_3 e_2 - [x_2, x_n] + [[x_2, x_{n-1}], x_1] = -\alpha_5 e_1 - (n-5)\alpha_3 e_2 + \\
&+ \sum_{s=2}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+1}Q_{s-1,n+1-2s}\sum_{k=1}^{2s-2}\gamma_{s,n+1-2s+k}e_k.
\end{aligned}
$$

Similarly, from the equality $[x_{i-1}, x_{n+3-i}] + [x_i, x_{n+2-i}] = [[x_{i-1}, x_{n+2-i}], x_1]$, for $3 \leq i \leq k+1$, by induction we obtain

$$
\begin{aligned}
[x_i, x_{n+2-i}] &= (-1)^i\alpha_5 e_1 + (-1)^i(n-5)\alpha_3 + (-1)^{i+1}\beta_{n-2}e_1 + \\
&+ \sum_{s=2}^{i-3}(-1)^{s+i}\sum_{t=1}^{s} Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2}\gamma_{s,n+1-2s+k}e_k + \sum_{s=i-2}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+i}\sum_{t=1}^{i-3} Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2}\gamma_{s,n+1-2s+k}e_k + \\
&+ \sum_{s=0}^{\lfloor\frac{n+4-2i}{2}\rfloor}(-1)^s Q_{s,n+5-2i-2s}\sum_{k=1}^{2s+2i-6}\gamma_{i+s-2,n+5-2i-2s+k}e_k.
\end{aligned}
$$

Taking $s' = s + i - 2$ in the last sum, we obtain

$$
\begin{aligned}
&\sum_{s=0}^{\lfloor\frac{n+4-2i}{2}\rfloor}(-1)^s Q_{s,n+5-2i-2s}\sum_{k=1}^{2s+2i-6}\gamma_{i+s-2,n+5-2i-2s+k}e_k = \\
&= \sum_{s=i-2}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+i}Q_{s-i+2,n+1-2s}\sum_{k=1}^{2s-2}\gamma_{s,n+1-2s+k}e_k.
\end{aligned}
$$

Placing this equality to above one we obtain the first equality of the lemma.

Now we will deduce the products $[x_i, x_{n+3-i}]$.

Using $[x_3, x_n] = 0$ from the relation $[x_3, x_n] + [x_4, x_{n-1}] = [[x_3, x_{n-1}], x_1]$ we get

$$
[x_4, x_{n-1}] = -(n-5)\alpha_3 e_1 + \sum_{s=2}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+1}Q_{s-1,n+1-2s}\sum_{k=1}^{2s-3}\gamma_{s,n+2-2s+k}e_k
$$

Using the products $[x_i, x_{n+2-i}]$ from the equality $[x_{i-1}, x_{n+4-i}] + [x_i, x_{n+3-i}] = [[x_{i-1}, x_{n+3-i}], x_1]$, for $4 \leq i \leq k+2$, by induction on i similarly to the previous case we obtain

$$[x_i, x_{n+3-i}] = (-1)^{i+1}(i-3)(n-5)\alpha_3 e_1 + \sum_{s=2}^{i-3}(-1)^{s+i+1}\sum_{t=1}^{s}(i-2-t)Q_{s-t,n+1-2s}\sum_{k=1}^{2s-3}\gamma_{s,n+2-2s+k}e_k +$$
$$+ \sum_{s=i-2}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+i+1}\sum_{t=1}^{i-3}(i-2-t)Q_{s-t,n+1-2s}\sum_{k=1}^{2s-3}\gamma_{s,n+2-2s+k}e_k.$$

The last equality from (11) follows by the induction on p and i (first by p, then by i.) □

Now we define some restrictions to the structure constants β_i and $\gamma_{i,j}$.
From the equality (10) we obtain

$$[x_{\lfloor\frac{n}{2}\rfloor+l}, x_{\lfloor\frac{n}{2}\rfloor+l}] = \frac{1}{2}[[x_{\lfloor\frac{n}{2}\rfloor+l}, x_{\lfloor\frac{n}{2}\rfloor+l-1}], x_1], \quad 1 \leq l \leq \lfloor\frac{n}{2}\rfloor.$$

Let n be even. Then in the case of $l = 1$ we get

$$[x_{\lfloor\frac{n}{2}\rfloor+1}, x_{\lfloor\frac{n}{2}\rfloor+1}] = \frac{1}{2}[[x_{\lfloor\frac{n}{2}\rfloor+1}, x_{\lfloor\frac{n}{2}\rfloor}], x_1] = \frac{1}{2}\sum_{k=1}^{n-2}\gamma_{\lfloor\frac{n}{2}\rfloor,k+1}e_k$$

On the other hand, from Lemma 3 we obtain

$$[x_{\lfloor\frac{n}{2}\rfloor+1}, x_{\lfloor\frac{n}{2}\rfloor+1}] = (-1)^{\lfloor\frac{n}{2}\rfloor+1}\alpha_5 e_1 + (-1)^{\lfloor\frac{n}{2}\rfloor+1}(n-5)\alpha_3 e_2 + (-1)^{\lfloor\frac{n}{2}\rfloor}\beta_{n-2}e_1 +$$
$$+ \sum_{s=2}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^{s+\lfloor\frac{n}{2}\rfloor+1}\sum_{t=1}^{s}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2}\gamma_{s,n+1-2s+k}e_k - \sum_{t=1}^{\lfloor\frac{n}{2}\rfloor-1}Q_{\lfloor\frac{n}{2}\rfloor-t,1}\sum_{k=1}^{n-2}\gamma_{\lfloor\frac{n}{2}\rfloor,k+1}e_k.$$

Comparing the coefficients at the basis elements we obtain the following restrictions:

$$(12)\quad \begin{cases} \alpha_5 - \beta_{n-2} + \sum\limits_{s=2}^{\lfloor\frac{n}{2}\rfloor}(-1)^s\gamma_{s,n+2-2s}\Big(\sum\limits_{t=1}^{s}Q_{s-t,n+1-2s}\Big) - (-1)^{\lfloor\frac{n}{2}\rfloor}\frac{1}{2}\gamma_{\lfloor\frac{n}{2}\rfloor,2} = 0, \\ (n-5)\alpha_3 + \sum\limits_{s=2}^{\lfloor\frac{n}{2}\rfloor}(-1)^s\gamma_{s,n+3-2s}\Big(\sum\limits_{t=1}^{s}Q_{s-t,n+1-2s}\Big) - (-1)^{\lfloor\frac{n}{2}\rfloor}\frac{1}{2}\gamma_{\lfloor\frac{n}{2}\rfloor,3} = 0, \\ \sum\limits_{s=\lfloor\frac{k+3}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}(-1)^s\gamma_{s,n+1-2s+k}\Big(\sum\limits_{t=1}^{s}Q_{s-t,n+1-2s}\Big) - (-1)^{\lfloor\frac{n}{2}\rfloor}\frac{1}{2}\gamma_{\lfloor\frac{n}{2}\rfloor,k+1} = 0, \qquad 3 \leq k \leq n-2. \end{cases}$$

If $l \geq 2$, then we have

$$[x_{\lfloor\frac{n}{2}\rfloor+l}, x_{\lfloor\frac{n}{2}\rfloor+l}] = \tfrac{1}{2}[[x_{\lfloor\frac{n}{2}\rfloor+l}, x_{\lfloor\frac{n}{2}\rfloor+l-1}], x_1] =$$
$$= \tfrac{1}{2}\sum_{s=l+1}^{\lfloor\frac{n}{2}\rfloor-l+1}(-1)^{s+\lfloor\frac{n}{2}\rfloor+l-1}\sum_{t=1}^{s}\binom{\lfloor\frac{n}{2}\rfloor+l-2-t}{2l-3}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l}\gamma_{s,n+2l-1-2s+k}e_k +$$
$$+\tfrac{1}{2}\sum_{s=\max\{\lfloor\frac{n}{2}\rfloor-l+2;l+1\}}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+\lfloor\frac{n}{2}\rfloor+l-1}\sum_{t=1}^{\lfloor\frac{n}{2}\rfloor-l+1}\binom{\lfloor\frac{n}{2}\rfloor+l-2-t}{2l-3}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l}\gamma_{s,n+2l-1-2s+k}e_k.$$

On the other hand, in the equality (11) for $p = 2l$ and $i = \lfloor\frac{n}{2}\rfloor + l$ we deduce

$$[x_{\lfloor\frac{n}{2}\rfloor+l}, x_{\lfloor\frac{n}{2}\rfloor+l}] = \sum_{s=l+1}^{\lfloor\frac{n}{2}\rfloor-l}(-1)^{s+\lfloor\frac{n}{2}\rfloor+l}\sum_{t=1}^{s}\binom{\lfloor\frac{n}{2}\rfloor+l-2-t}{2l-2}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l}\gamma_{s,n+2l-1-2s+k}e_k +$$
$$+\sum_{s=\max\{\lfloor\frac{n}{2}\rfloor-l+1;l+1\}}^{\lfloor\frac{n}{2}\rfloor}(-1)^{s+\lfloor\frac{n}{2}\rfloor+l}\sum_{t=1}^{\lfloor\frac{n}{2}\rfloor-l}\binom{\lfloor\frac{n}{2}\rfloor+l-2-t}{2l-2}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l}\gamma_{s,n+2l-1-2s+k}e_k.$$

Comparing the coefficients at the basis elements we get

$$(13)\quad \begin{cases} \sum\limits_{s=\lfloor\frac{k+1}{2}\rfloor+l}^{\lfloor\frac{n}{2}\rfloor}(-1)^s\gamma_{s,n+2l-1-2s+k}\sum\limits_{t=1}^{\min\{s,\lfloor\frac{n}{2}\rfloor-l\}}\left[\binom{\lfloor\frac{n}{2}\rfloor+l-2-t}{2l-2} + \frac{1}{2}\binom{\lfloor\frac{n}{2}\rfloor+l-2-t}{2l-3}\right]Q_{s-t,n+1-2s} + \\ \sum\limits_{s=\max\{\lfloor\frac{n}{2}\rfloor-l+1;\lfloor\frac{k+1}{2}\rfloor+l\}}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{2}(-1)^s\gamma_{s,n+2l-1-2s+k}Q_{s-\lfloor\frac{n}{2}\rfloor+l-1,n+1-2s} = 0, \end{cases}$$

where $2 \le l \le \lfloor \frac{n}{2} \rfloor - 1,\ 1 \le k \le n-2l$.

Let n be odd. Then in the case of $l=2$ we get $[x_{\lfloor \frac{n}{2} \rfloor+2}, x_{\lfloor \frac{n}{2} \rfloor+2}] = \frac{1}{2}[[x_{\lfloor \frac{n}{2} \rfloor+2}, x_{\lfloor \frac{n}{2} \rfloor+1}], x_1]$ and using the first equality of (11) we obtain

$$[x_{\lfloor \frac{n}{2} \rfloor+2}, x_{\lfloor \frac{n}{2} \rfloor+2}] = \frac{1}{2}(-1)^{\lfloor \frac{n}{2} \rfloor}(n-5)\alpha_3 e_1 + \frac{1}{2}\sum_{s=2}^{\lfloor \frac{n}{2} \rfloor}(-1)^{s+\lfloor \frac{n}{2} \rfloor}\sum_{t=1}^{s} Q_{s-t,n+1-2s}\sum_{k=1}^{2s-3}\gamma_{s,n+2-2s+k}e_k.$$

On the other hand, from the second equality of (11) for $i = \lfloor \frac{n}{2} \rfloor + 2$ we get

$$\begin{aligned}[x_{\lfloor \frac{n}{2} \rfloor+2}, x_{\lfloor \frac{n}{2} \rfloor+2}] &= (-1)^{\lfloor \frac{n}{2} \rfloor+1}(\lfloor \tfrac{n}{2} \rfloor - 1)(n-5)\alpha_3 e_1 + \\ &+ \sum_{s=2}^{\lfloor \frac{n}{2} \rfloor-1}(-1)^{s+\lfloor \frac{n}{2} \rfloor+1}\sum_{t=1}^{s}(\lfloor \tfrac{n}{2} \rfloor - t)Q_{s-t,n+1-2s}\sum_{k=1}^{2s-3}\gamma_{s,n+2-2s+k}e_k - \\ &- \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor-1}(\lfloor \tfrac{n}{2} \rfloor - t)Q_{\lfloor \frac{n}{2} \rfloor-t,2}\sum_{k=1}^{n-4}\gamma_{\lfloor \frac{n}{2} \rfloor,k+3}e_k.\end{aligned}$$

Comparing the coefficients at the basis elements we derive

(14)
$$\begin{cases} (\lfloor \frac{n}{2} \rfloor - \frac{1}{2})(n-5)\alpha_3 + \sum\limits_{s=2}^{\lfloor \frac{n}{2} \rfloor}(-1)^s\gamma_{s,n+3-2s}\sum\limits_{t=1}^{s}(\lfloor \frac{n}{2} \rfloor - t + \frac{1}{2})Q_{s-t,n+1-2s} = 0, \\ \sum\limits_{s=\lfloor \frac{k+4}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor}(-1)^s\gamma_{s,n+2-2s+k}\sum\limits_{t=1}^{s}(\lfloor \frac{n}{2} \rfloor - t + \frac{1}{2})Q_{s-t,n+1-2s} = 0, \qquad 2 \le k \le n-4. \end{cases}$$

If $l \ge 3$, then we have

$$\begin{aligned}&[x_{\lfloor \frac{n}{2} \rfloor+l}, x_{\lfloor \frac{n}{2} \rfloor+l}] = \tfrac{1}{2}[[x_{\lfloor \frac{n}{2} \rfloor+l}, x_{\lfloor \frac{n}{2} \rfloor+l-1}], x_1] = \\ &= \tfrac{1}{2}\sum_{s=l}^{\lfloor \frac{n}{2} \rfloor-l+2}(-1)^{s+\lfloor \frac{n}{2} \rfloor+l}\sum_{t=1}^{s}\binom{\lfloor \frac{n}{2} \rfloor+l-2-t}{2l-4}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l+1}\gamma_{s,n+2l-2-2s+k}e_k + \\ &+ \tfrac{1}{2}\sum_{s=\max\{\lfloor \frac{n}{2} \rfloor-l+3;l\}}^{\lfloor \frac{n}{2} \rfloor}(-1)^{s+\lfloor \frac{n}{2} \rfloor+l}\sum_{t=1}^{\lfloor \frac{n}{2} \rfloor-l+2}\binom{\lfloor \frac{n}{2} \rfloor+l-2-t}{2l-4}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l+1}\gamma_{s,n+2l-2-2s+k}e_k.\end{aligned}$$

On the other hand, in the equality (11) for $p = 2l-1$ and $i = \lfloor \frac{n}{2} \rfloor + l$ we have

$$\begin{aligned}[x_{\lfloor \frac{n}{2} \rfloor+l}, x_{\lfloor \frac{n}{2} \rfloor+l}] &= \sum_{s=l}^{\lfloor \frac{n}{2} \rfloor-l+1}(-1)^{s+\lfloor \frac{n}{2} \rfloor+l-1}\sum_{t=1}^{s}\binom{\lfloor \frac{n}{2} \rfloor+l-2-t}{2l-3}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l+1}\gamma_{s,n+2l-2-2s+k}e_k + \\ &+ \sum_{s=\max\{\lfloor \frac{n}{2} \rfloor-l+2;l\}}^{\lfloor \frac{n}{2} \rfloor}(-1)^{s+\lfloor \frac{n}{2} \rfloor+l-1}\sum_{t=1}^{\lfloor \frac{n}{2} \rfloor-l+1}\binom{\lfloor \frac{n}{2} \rfloor+l-2-t}{2l-3}Q_{s-t,n+1-2s}\sum_{k=1}^{2s-2l+1}\gamma_{s,n+2l-2-2s+k}e_k.\end{aligned}$$

Comparing the coefficients at the basis elements we get

(15)
$$\begin{cases} \sum\limits_{s=\lfloor \frac{k}{2} \rfloor+l}^{\lfloor \frac{n}{2} \rfloor}(-1)^s\gamma_{s,n+2l-2-2s+k}\sum\limits_{t=1}^{\min\{s,\lfloor \frac{n}{2} \rfloor-l+1\}}\left[\binom{\lfloor \frac{n}{2} \rfloor+l-2-t}{2l-3} + \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor+l-2-t}{2l-4}\right]Q_{s-t,n+1-2s} + \\ \sum\limits_{s=\max\{\lfloor \frac{n}{2} \rfloor-l+2;\lfloor \frac{k}{2} \rfloor+l\}}^{\lfloor \frac{n}{2} \rfloor}\tfrac{1}{2}(-1)^s\gamma_{s,n+2l-2-2s+k}Q_{s-\lfloor \frac{n}{2} \rfloor+l-2,n+1-2s} = 0, \end{cases}$$

where $3 \le l \le \lfloor \frac{n}{2} \rfloor,\ 1 \le k \le n-2l$.

Therefore, we obtain following main Theorem of this section.

Theorem 4. *Let L be a Leibniz algebra such that $L/I \cong n_{n,1}$ and I is the L/I-module with the minimal faithful representation. Then L admits a basis $\{x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_n\}$ such that the multiplications table for this basis has the form* (5), (7), (9), (11) *with the restrictions* (12), (13), (14), (15).

Now we are in position to give a classification of such algebras up to isomorphism for the case $n = 4$. In this case we get the following family of algebras denoted by $\mu(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \gamma_1, \gamma_2)$:

$$\begin{cases} [e_2, x_1] = e_1, & [x_3, x_1] = x_4 - \alpha_2 e_3, \\ [e_3, x_1] = e_2, & [x_4, x_1] = -\alpha_4 e_1 - 2\alpha_2 e_2, \\ [e_4, x_2] = e_3, & [x_2, x_2] = \beta_1 e_1 + \beta_2 e_2, \\ [e_4, x_3] = e_2, & [x_3, x_2] = \gamma_1 e_1 + \gamma_2 e_2 - 2\alpha_3 e_3, \\ [e_4, x_4] = e_1, & [x_4, x_3] = -\alpha_3 e_1, \\ [x_1, x_2] = -x_3, & [x_3, x_3] = \frac{1}{2}\gamma_2 e_1 - \alpha_3 e_2, \\ [x_1, x_3] = -x_4, & [x_4, x_2] = \frac{1}{2}\gamma_2 e_1 - \alpha_3 e_2 \\ [x_1, x_1] = \alpha_1 e_3 + \alpha_2 e_4, & [x_2, x_3] = (\beta_2 - \gamma_1) e_1 - \gamma_2 e_2 + \alpha_3 e_3, \\ [x_2, x_1] = x_3 + \alpha_3 e_4, & [x_2, x_4] = -\frac{3}{2}\gamma_2 e_1 - \alpha_3 e_2, \\ [x_1, x_4] = \alpha_4 e_1 + \alpha_2 e_2. \end{cases}$$

Theorem 5. *Let L be an 8-dimensional Leibniz algebra such that $L/I \cong n_{4,1}$ and I is the L/I-module with the minimal faithful representation. Then L is isomorphic to the one of the following pairwise non isomorphic algebras:*

$\mu(\alpha_1, \alpha_2, 1, 1, \beta_1, \beta_2, 0, 1)$	$\mu(\alpha_1, 1, 1, 0, \beta_1, \beta_2, 0, 1)$	$\mu(1, 0, 1, 0, \beta_1, \beta_2, 0, 1)$	$\mu(\alpha_1, 1, 1, 1, \beta_1, \beta_2, 0, 0)$
$\mu(1, 0, 1, 1, \beta_1, \beta_2, 0, 0)$	$\mu(0, 0, 1, 1, 1, \beta_2, 0, 0)$	$\mu(0, 0, 1, 1, 0, 1, 0, 0)$	$\mu(0, 0, 1, 1, 0, 0, 0, 0)$
$\mu(1, 1, 1, 0, \beta_1, \beta_2, 0, 0)$	$\mu(0, 1, 1, 0, 1, \beta_2, 0, 0)$	$\mu(0, 1, 1, 0, 0, 1, 0, 0)$	$\mu(0, 1, 1, 0, 0, 0, 0, 0)$
$\mu(1, 0, 1, 0, 1, \beta_2, 0, 0)$	$\mu(1, 0, 1, 0, 0, 1, 0, 0)$	$\mu(0, 0, 1, 0, 1, \beta_2, 0, 0)$	$\mu(0, 0, 1, 0, 0, 1, 0, 0)$
$\mu(0, 0, 1, 0, 0, 0, 0, 0)$	$\mu(\alpha_1, 1, 0, 1, 0, \beta_2, \gamma_1, 1)$	$\mu(\alpha_1, 0, 0, 1, 0, 1, \gamma_1, 1)$	$\mu(\alpha_1, 0, 0, 1, 0, 0, 1, 1)$
$\mu(\alpha_1, 0, 0, 1, 0, 0, 0, 1)$	$\mu(\alpha_1, 1, 0, 0, 0, 1, \gamma_1, 1)$	$\mu(\alpha_1, 1, 0, 0, 0, 0, 1, 1)$	$\mu(\alpha_1, 1, 0, 0, 0, 0, 0, 1)$
$\mu(1, 0, 0, 0, 0, 1, \gamma_1, 1)$	$\mu(1, 0, 0, 0, 0, 0, 1, 1)$	$\mu(1, 0, 0, 0, 0, 0, 0, 1)$	$\mu(0, 0, 0, 0, 0, 1, \gamma_1, 1)$
$\mu(0, 0, 0, 0, 0, 0, 1, 1)$	$\mu(0, 0, 0, 0, 0, 0, 0, 1)$	$\mu(1, 1, 0, 1, \beta_1, \beta_2, \gamma_1, 0)$	$\mu(0, 1, 0, 1, 1, \beta_2, \gamma_1, 0)$
$\mu(0, 1, 0, 1, 0, 1, \gamma_1, 0)$	$\mu(0, 1, 0, 1, 0, 0, 1, 0)$	$\mu(0, 1, 0, 1, 0, 0, 0, 0)$	$\mu(1, 0, 0, 1, 1, \beta_2, \gamma_1, 0)$
$\mu(1, 0, 0, 1, 0, 1, \gamma_1, 0)$	$\mu(1, 0, 0, 1, 0, 0, 1, 0)$	$\mu(1, 0, 0, 1, 0, 0, 0, 0)$	$\mu(0, 0, 0, 1, 1, 1, \gamma_1, 0)$
$\mu(0, 0, 0, 1, 1, 0, 1, 0)$	$\mu(0, 0, 0, 1, 1, 0, 0, 0)$	$\mu(0, 0, 0, 1, 0, 1, 1, 0)$	$\mu(0, 0, 0, 1, 0, 1, 0, 0)$
$\mu(0, 0, 0, 1, 0, 0, 1, 0)$	$\mu(0, 0, 0, 1, 0, 0, 0, 0)$	$\mu(1, 1, 0, 0, 1, \beta_2, \gamma_1, 0)$	$\mu(1, 1, 0, 0, 0, 1, \gamma_1, 0)$
$\mu(1, 1, 0, 0, 0, 0, 1, 0)$	$\mu(1, 1, 0, 0, 0, 0, 0, 0)$	$\mu(1, 0, 0, 0, 1, 1, \gamma_1, 0)$	$\mu(1, 0, 0, 0, 1, 0, 1, 0)$
$\mu(1, 0, 0, 0, 1, 0, 0, 0)$	$\mu(1, 0, 0, 0, 0, 1, 1, 0)$	$\mu(1, 0, 0, 0, 0, 1, 0, 0)$	$\mu(1, 0, 0, 0, 0, 0, 1, 0)$
$\mu(1, 0, 0, 0, 0, 0, 0, 0)$	$\mu(0, 1, 0, 0, 1, 1, \gamma_1, 0)$	$\mu(0, 1, 0, 0, 1, 0, 1, 0)$	$\mu(0, 1, 0, 0, 1, 0, 0, 0)$
$\mu(0, 1, 0, 0, 0, 1, \gamma_1, 0)$	$\mu(0, 1, 0, 0, 0, 0, 1, 0)$	$\mu(0, 1, 0, 0, 0, 0, 0, 0)$	$\mu(0, 0, 0, 0, 1, 1, \gamma_1, 0)$
$\mu(0, 0, 0, 0, 0, 1, \gamma_1, 0)$	$\mu(0, 0, 0, 0, 1, 0, 1, 0)$	$\mu(0, 0, 0, 0, 1, 0, 0, 0)$	$\mu(0, 0, 0, 0, 0, 0, 1, 0)$
$\mu(0, 0, 0, 0, 0, 0, 0, 0)$			

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 \in \mathbb{C}$.

Proof. Let L be an 8-dimensional Leibniz algebra given by $\mu(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \gamma_1, \gamma_2)$. We make the following change of basis:

$$x_1' = \sum_{k=1}^{4} P_k x_k + \sum_{k=1}^{4} Q_k e_k,$$
$$x_2' = \sum_{k=1}^{4} M_k x_k + \sum_{k=1}^{4} N_k e_k,$$
$$e_4' = \sum_{k=1}^{4} R_k x_k + \sum_{k=1}^{4} T_k e_k,$$

while the other elements of the new basis (i.e. e_1', e_2', e_3', x_3' and x_4') are obtained as products of the above elements.

The table of multiplication in this new basis implies the following restrictions on the coefficients:

$$P_2 = M_1 = R_k = 0, \qquad 1 \le k \le 4,$$

$$T_3 = -\frac{T_4 P_3}{P_1}, \qquad T_2 = -\frac{T_4 P_4}{P_1}, \qquad N_4 = \alpha_3 M_3,$$

$$Q_4 = \frac{\alpha_2 P_1 M_3}{M_2},$$

$$Q_3 = -\frac{-\alpha_3 P_3^2 M_2 - \alpha_1 P_1^2 M_3 + \alpha_2 P_1 P_3 M_3}{P_1 M_2},$$

$$Q_2 = -\frac{-2\alpha_2 T_1 P_1^2 M_2 + \gamma_2 T_4 P_3^2 M_2 - 2\alpha_3 T_4 P_3 P_4 M_2 + 2\alpha_2 T_4 P_1 P_4 M_3 - 2\alpha_1 T_4 P_1^2 M_4}{2T_4 P_1 M_2},$$

$$N_3 = -\frac{\alpha_3 P_4 M_2^2 - \alpha_3 P_3 M_2 M_3 + \alpha_2 P_1 M_3^2 - \alpha_2 P_1 M_2 M_4}{P_1 M_2},$$

$$N_2 = -\frac{-\alpha_3 T_1 P_1 M_2^2 + \beta_2 T_4 P_3 M_2^2 - \gamma_2 T_4 P_4 M_2^2 + \gamma_2 T_4 P_3 M_2 M_3 - \alpha_3 T_4 P_3 M_2 M_4 + \alpha_2 T_4 P_1 M_3 M_4}{T_4 P_1 M_2},$$

$T_4 P_1 M_2 \neq 0$.

Calculating new parameters we obtain:

$$\alpha_1' = \frac{\alpha_1 P_1^2}{T_4 M_2}, \qquad \alpha_2' = \frac{\alpha_2 P_1^2}{T_4},$$
$$\alpha_3' = \frac{\alpha_3 P_1 M_2}{T_4}, \qquad \alpha_4' = \frac{\alpha_4 P_1}{T_4},$$
$$\beta_1' = \frac{2\beta_1 M_2^2 + \gamma_2 M_3^2 - 2\gamma_2 M_2 M_4}{2T_4 P_1^2 M_2},$$
$$\gamma_1' = \frac{\gamma_1 M_2^2 - \alpha_3 M_3^2 + 2\alpha_3 M_2 M_4}{T_4 P_1 M_2},$$
$$\beta_2' = \frac{\beta_2 M_2}{T_4 P_1}, \qquad \gamma_2' = \frac{\gamma_2 M_2}{T_4}.$$

Considering all the possible cases we obtain the families of algebras listed in the theorem. □

REFERENCES

[1] Abdykassymova, S., Dzhumaldil'daev, A.: Leibniz algebras in characteristic p. C. R. Acad. Sci. Paris Sér. I Math. 332(12), (2001), 1047–1052.

[2] Albeverio, S., Ayupov, Sh.A. and Omirov, B.A.: On nilpotent and simple Leibniz algebras. Comm. Algebra. (2005), 159–172.

[3] Ayupov, Sh.A. and Omirov, B.A.: On some classes of nilpotent Leibniz algebras. Siberian Math. Journal 42(1), (2001), 18–29.

[4] Barnes, D.W.: On Levi's theorem for Leibniz algebras, Bull. Aust. Math. Soc., 86(2), (2012), 184-185.

[5] Bloh, A.: On a generalization of the concept of Lie algebra. Dokl. Akad. Nauk SSSR 165, (1965), 471–473.

[6] Cabezas, J.M., Camacho, L.M. and Rodriguez, I.M.: On filiform and 2-filiform Leibniz algebras of maximum length. J. Lie Theory 18(2), (2008), 335–350.

[7] Calderón, A.J., Camacho, L.M., Omirov, B.A.: Leibniz algebras of Heisenberg type. arXiv:1411.3861v1, 14 pp.

[8] Camacho, L.M., Gómez, J.R., González, A.J. and Omirov, B.A.: The classification of naturally graded p-filiform Leibniz algebras. Comm. Algebra 39, no. 1, (2011), 153–168.

[9] Cañete, E.M., Khudoyberdiyev, A.Kh. : The classification of 4-dimensional Leibniz algebras, Linear Algebra and its Applications 439(1), 2013, 273–288.

[10] de Jeu, M., Svensson, C. and Silvestrov, S.: Algebraic curves for commuting elements in the q-deformed Heisenberg algebra. J. Algebra 321, no. 4, (2009), 1239–1255.

[11] Fialowski, A., Khudoyberdiyev, A.Kh. and Omirov, B.A.: A Characterization of Nilpotent Leibniz Algebras. Algebr. Represent. Theory 16, no. 5, (2013), 1489–1505.

[12] Geloun, B., Hounkonnou, J. and Mahouton, N.: q-graded Heisenberg algebras and deformed supersymmetries. J. Math. Phys. 51 (2010), no. 2, 023502, 14 pp.

[13] Gómez-Vidal, S., Omirov, B.A., Khudoyberdiyev, A.Kh.: Some remarks on semisimple Leibniz algebras, Journal of Algebra. 410, 2014, 526–540.

[14] Hengyun, Y. and Naihong, H. Heisenberg Lie color algebras. Comm. Algebra 39(5), (2011), 1782–1795.

[15] Konstantina, C.: Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras. J. Algebra 320, no. 7, (2008), 2871–2890.

[16] Loday, J.L.: Une version non commutative des algébres de Lie: les algébres de Leibniz. L'Ens. Math. 39, (1993), 269–293.

[17] Omirov, B.A., Rakhimov, I.S. and Turdibaev, R.M.: On description of Leibniz algebras corresponding to sl_2. Algbr. Represent. Theor. 16 (2013), 1507–1519.
[18] Rakhimov, I.S. and Hassan, M.A. On low-dimensional filiform Leibniz algebras and their invariants. Bull. Malays. Math. Sci. Soc. (2) 34, no. 3, (2011), 475–485.
[19] Vergne, M. Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes. Bull. Soc. Math. France, 98:81–116, 1970.

[SH.A. AYUPOV — A.KH. KHUDOYBERDIYEV — B.A. OMIROV] INSTITUTE OF MATHEMATICS. NATIONAL UNIVERSITY OF UZBEKISTAN, DORMON YOLI STR. 29, 100125, TASHKENT (UZBEKISTAN)
E-mail address: sh_ayupov@mail.ru --- khabror@mail.ru --- omirovb@mail.ru

[L.M. CAMACHO] DPTO. MATEMÁTICA APLICADA I. UNIVERSIDAD DE SEVILLA. AVDA. REINA MERCEDES, S/N. 41012 SEVILLA. (SPAIN)
E-mail address: lcamacho@us.es