

# LEIBNIZ ALGEBRAS WHOSE SEMISIMPLE PART IS RELATED TO $sl_2$ .

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ABSTRACT. In this paper we identify the structure of complex finite-dimensional Leibniz algebras with associated Lie algebras  $sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$ , where  $R$  is a solvable radical. The classifications of such Leibniz algebras in the cases  $\dim R = 2, 3$  and  $\dim I \neq 3$  have been obtained. Moreover, we classify Leibniz algebras with  $L/I \cong sl_2^1 \oplus sl_2^2$  and some conditions on ideal  $I = id \langle [x, x] \mid x \in L \rangle$ .

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**Key Words and Phrases:** Leibniz algebra, simple algebra  $sl_2$ , direct sum of algebras, right module, irreducible module.

## 1. INTRODUCTION.

The notion of Leibniz algebra was first introduced by Loday in [7], [8] as a non-antisymmetric generalization of Lie algebra. During the last 20 years the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras. A lot of papers have been devoted to the description of finite-dimensional nilpotent Leibniz algebras [1], [2] so far. However, just a few works are related to the semisimple part of Leibniz algebras [5], [4], [10].

We know that an arbitrary Lie algebra can be decomposed into a semidirect sum of the solvable radical and its semisimple subalgebra (Levi's Theorem [6]). According to the Cartan-Killing theory, a semisimple Lie algebra can be represented as a direct sum of simple ideals, which are completely classified [6].

In a recent study, Barnes has proved an analogue of Levi's Theorem for the case of Leibniz algebras [4]. Namely, a Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra.

The inherent properties of non-Lie Leibniz algebras imply that the subspace spanned by squares of elements of the algebra is a non-trivial ideal (denoted by  $I$ ). Moreover, the ideal  $I$  is abelian and hence, it is contained in the solvable radical. Thanks to result of Barnes in order to describe Leibniz algebras it is enough to investigate the relationship between products of a semisimple Lie algebra and the radical (see [5], [9] and [10]).

The present work devoted to describing the structure of Leibniz algebras with the associated Lie algebras isomorphic to  $sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$  with  $I$  an irreducible right  $sl_2^k$ -module for some  $k$ . Since the description of such Leibniz algebras is very complicated, we have focused on Leibniz algebras with semisimple part  $sl_2^1 \oplus sl_2^2$  under some conditions on the ideal  $I$ .

In order to achieve our goal we organize the paper as follows. In Section 2, we give some necessary notions and preliminary results about Leibniz algebras with associated Lie algebra  $sl_2 \dot{+} R$ . Section 3 is devoted to the study of the structure of Leibniz algebras, whose semisimple part is a direct sum of copies of  $sl_2$  algebras and with some conditions on the ideal  $I$ . In Section 4, we classify Leibniz algebras whose semisimple part is a direct sum  $sl_2^1 \oplus sl_2^2$  and  $I$  is decomposed into a direct sum of two irreducible modules  $I_{1,1}, I_{1,2}$  over  $sl_2^1$  such that  $\dim I_{1,1} = \dim I_{1,2}$ . The description of the structure of Leibniz algebras with associated Lie algebra  $sl_n$  and  $\dim I = 1, 2$  is obtained in Section 5.

Throughout the work, vector spaces and algebras are finite-dimensional over the field of complex numbers. Moreover, in the table of multiplication of an algebra the omitted products are assumed to be zero. We shall use the following symbols:  $+$ ,  $\oplus$  and  $\dot{+}$  for notations of the direct sum of the vector spaces, the direct and semidirect sums of algebras, respectively.

## 2. PRELIMINARIES

In this section we give some necessary definitions and preliminary results.

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**Definition 2.1.** [7] *An algebra  $(L, [\cdot, \cdot])$  over a field  $F$  is called a Leibniz algebra if for any  $x, y, z \in L$  the so-called Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

*holds.*

Let  $L$  be a Leibniz algebra and let  $I = \text{ideal} \langle [x, x] \mid x \in L \rangle$  be an ideal of  $L$  generated by all squares. The natural epimorphism  $\varphi : L \rightarrow L/I$  determines the associated Lie algebra  $L/I$  of the Leibniz algebra  $L$ . It is clear that ideal  $I$  is the minimal ideal with respect to the property that the quotient algebra by this ideal is a Lie algebra.

It is noted that in [4] the ideal  $I$  coincides with the space spanned by squares of elements of an algebra.

According to [6] there exists a unique (up to isomorphism) 3-dimensional simple Lie algebra with the following table of multiplication:

$$sl_2 : \quad [e, h] = -[h, e] = 2e, \quad [h, f] = -[f, h] = 2f, \quad [e, f] = -[f, e] = h,$$

The basis  $\{e, f, h\}$  is called the *canonical basis*.

In [9], Leibniz algebras whose quotient Lie algebra is isomorphic to  $sl_2$  are described. Let us present a Leibniz algebra  $L$  with the table of multiplication in a basis  $\{e, f, h, x_0^1, \dots, x_{t_1}^1, x_0^2, \dots, x_{t_2}^2, \dots, x_0^p, \dots, x_{t_p}^p\}$  and the quotient algebra  $L/I$  is  $sl_2$ :

$$\begin{aligned} [e, h] &= -[h, e] = 2e, & [h, f] &= -[f, h] = 2f, & [e, f] &= -[f, e] = h, \\ [x_k^j, h] &= (t_j - 2k)x_k^j, & & & & & 0 \leq k \leq t_j, \\ [x_k^j, f] &= x_{k+1}^j, & & & & & 0 \leq k \leq t_j - 1, \\ [x_k^j, e] &= -k(t_j + 1 - k)x_{k-1}^j, & & & & & 1 \leq k \leq t_j. \end{aligned}$$

where  $L = sl_2 + I_1 + I_2 + \dots + I_p$  and  $I_j = \langle x_1^j, \dots, x_{t_j}^j \rangle$ ,  $1 \leq j \leq p$ .

The last three types of products of the above table of multiplication are characterized as an irreducible  $sl_2$ -module with the canonical basis of  $sl_2$  [6].

Now we give the notion of semisimplicity for Leibniz algebras.

**Definition 2.2.** [5] *A Leibniz algebra is called semisimple if its maximal solvable ideal is equal to  $I$ .*

Since in the Lie algebras case the ideal  $I$  is equal to zero, this definition also agrees with the definition of semisimple Lie algebra.

Although Levi's Theorem is proved for the left Leibniz algebras [4], it is also true for right Leibniz algebras (here we consider the right Leibniz algebras).

**Theorem 2.3.** [4] (*Levi's Theorem*). *Let  $L$  be a finite dimensional Leibniz algebra over a field of characteristic zero and  $R$  be its solvable radical. Then there exists a semisimple subalgebra  $S$  of  $L$ , such that  $L = S \dot{+} R$ .*

An algebra  $L$  is called *simple* if its only ideals are  $\{0\}, I, L$  and  $L^2 \neq I$ . From the proof of Theorem 2.3, it is not difficult to see that  $S$  is a semisimple Lie algebra. Therefore, we have that a simple Leibniz algebra is a semidirect sum of simple Lie algebra  $S$  and the irreducible right module  $I$  over  $S$ , i.e.  $L = S \dot{+} I$ . Hence, we get the description of the simple Leibniz algebras in terms of simple Lie algebras and ideal  $I$ .

**Definition 2.4.** [6] *A non-zero module  $M$  over a Lie algebra whose only submodules are the module itself and zero module is called irreducible module. A non-zero module  $M$  which admits decomposition into a direct sum of irreducible modules is said to be completely reducible.*

Further, we shall use the following Weyl's semisimplicity theorem.

**Theorem 2.5.** [6] *Let  $G$  be a semisimple Lie algebra over a field of characteristic zero. Then every finite dimensional module over  $G$  is completely reducible.*

Now we present results on classification of Leibniz algebras with the conditions  $L/I \cong sl_2 \oplus R$ ,  $\dim R = 2, 3$ , where  $I$  is an irreducible right module over  $sl_2$  ( $\dim I \neq 3$ ).

**Theorem 2.6.** [3] *Let  $L$  be a Leibniz algebra whose quotient  $L/I \cong sl_2 \oplus R$ , where  $R$  is a two-dimensional solvable ideal and  $I$  is a right irreducible module over  $sl_2$  ( $\dim I \neq 3$ ). Then there exists a*

basis  $\{e, h, f, x_0, x_1, \dots, x_m, y_1, y_2\}$  of the algebra  $L$  such that the table of multiplication in  $L$  has the following form:

$$\left\{ \begin{array}{ll} [e, h] = -[h, e] = 2e, & [h, f] = -[f, h] = 2f, \quad [e, f] = -[f, e] = h, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & [x_k, y_2] = ax_k, \quad 0 \leq k \leq m, \quad a \in \mathbb{C}, \\ [x_k, h] = (m - 2k)x_k & 0 \leq k \leq m, \\ [x_k, f] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m. \end{array} \right.$$

The following theorem extends Theorem 2.6 for  $\dim R = 3$ .

**Theorem 2.7.** [10] *Let  $L$  be a Leibniz algebra whose quotient  $L/I \cong sl_2 \oplus R$ , where  $R$  is a three-dimensional solvable ideal and  $I$  is an irreducible right module over  $sl_2$  ( $\dim I \neq 3$ ). Then there exists a basis  $\{e, h, f, x_0, x_1, \dots, x_m, y_1, y_2, y_3\}$  of the algebra  $L$  such that the table of multiplication in  $L$  has one of the following two forms:*

$$L_1(\alpha, a) : \left\{ \begin{array}{ll} [e, h] = -[h, e] = 2e, & [h, f] = -[f, h] = 2f, \quad [e, f] = -[f, e] = h, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & [y_3, y_2] = -[y_2, y_3] = \alpha y_3, \\ [x_k, h] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m. \end{array} \right.$$

$$L_2(a) : \left\{ \begin{array}{ll} [e, h] = -[h, e] = 2e, & [h, f] = -[f, h] = 2f, \quad [e, f] = -[f, e] = h, \\ [y_1, y_2] = -[y_2, y_1] = y_1 + y_3, & [y_3, y_2] = -[y_2, y_3] = y_3, \\ [x_k, h] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m. \end{array} \right.$$

For a semisimple Lie algebra  $S$  we consider a semisimple Leibniz algebra  $L$  such that  $L = (sl_2 \oplus S) \dot{+} I$ . We put  $I_1 = [I, sl_2]$ .

Let  $I_1$  be a reducible over  $sl_2$ . Then by Theorem 2.5 we have a decomposition:

$$I_1 = I_{1,1} \oplus I_{1,2} \oplus \dots \oplus I_{1,p},$$

where  $I_{1,j}$  are irreducible modules over  $sl_2$  for every  $j$ ,  $1 \leq j \leq p$ .

**Theorem 2.8.** [5] *Let  $\dim I_{1,j_1} = \dim I_{1,j_2} = \dots = \dim I_{1,j_s} = t + 1$ ,  $1 \leq s \leq p$ . Then there exist  $t + 1$  submodules  $I_{2,1}, I_{2,2}, \dots, I_{2,t+1}$  of dimension  $s$  of the module  $I_2 = [I, S]$  such that*

$$I_{2,1} + I_{2,2} + \dots + I_{2,t+1} = I_1 \cap I_2.$$

### 3. ON LEIBNIZ ALGEBRAS WHOSE ASSOCIATED LIE ALGEBRA IS ISOMORPHIC TO

$$sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R.$$

In this section, we will consider a Leibniz algebra satisfying the following conditions:

- (i) the quotient algebra  $L/I$  is isomorphic to the direct sum  $sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$ , where  $R$  is  $n$ -dimensional solvable Lie algebra;
- (ii) the ideal  $I$  is a right irreducible  $sl_2^k$ -module for some  $k \in \{1, \dots, s\}$ .

By reordering direct sums and changing indexes in the condition (ii) we can assume  $k = 1$ .

We denote by  $\langle e_i, f_i, h_i \rangle$ ,  $1 \leq i \leq s$  the basis elements of  $L$  which are preimage of standard basis of  $sl_2^i$  in the homomorphism  $L \rightarrow L/I$  and we put

$$I = \langle x_0, \dots, x_m \rangle, \quad R = \langle y_1, \dots, y_n \rangle.$$

Then due to [6] we have

$$\begin{aligned} [e_1, h_1] &= -[h_1, e_1] = 2e_1, & [h_1, f_1] &= -[f_1, h_1] = 2f_1, & [e_1, f_1] &= -[f_1, e_1] = h_1, \\ [x_i, h_1] &= (m-2i)x_i, & & 0 \leq i \leq m, \\ [x_i, f_1] &= x_{i+1}, & & 0 \leq i \leq m-1, \\ [x_i, e_1] &= -i(m+1-i)x_{i-1}, & & 1 \leq i \leq m. \end{aligned}$$

The correctness of the next lemma follows from Lemma 3.3 in [5].

**Lemma 3.1.** *Let  $L$  be a Leibniz algebra satisfying the conditions (i)-(ii). Then  $[I, sl_2^j] = 0$  for any  $j \in \{2, \dots, s\}$ .*

The following lemma trivially follows from Theorem 2.3.

**Lemma 3.2.** *Let  $L$  be a Leibniz algebra satisfying the conditions (i)-(ii). Then  $[sl_2^t, sl_2^t] = sl_2^t$  for  $1 \leq t \leq s$  and  $[sl_2^i, sl_2^j] = 0$  for  $1 \leq i, j \leq s$ ,  $i \neq j$ .*

The next lemma establish that the solvable ideal  $R$  is contained in two sided annihilator of each  $sl_2^i$ ,  $2 \leq i \leq s$ .

**Lemma 3.3.** *Let  $L$  be a Leibniz algebra satisfying the conditions (i)-(ii). Then*

$$[R, sl_2^i] = [sl_2^i, R] = 0, \quad 2 \leq i \leq s.$$

*Proof.* Applying Leibniz identity for the following triples

$$(y_s, e_1, a), (y_s, f_1, a), (a, y_s, e_1), (a, y_s, f_1)$$

lead  $[y_s, a] = 0$ ,  $[a, y_s] = 0$ ,  $1 \leq s \leq n$  for an arbitrary element  $a \in sl_2^i$ ,  $2 \leq i \leq s$ . □

Summarizing the results of Lemmas 3.1-3.3, we obtain the following theorem.

**Theorem 3.4.** *Let  $L$  be a finite-dimensional Leibniz algebra satisfying conditions:*

(i)  $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$ , where  $R$  is an  $n$ -dimensional solvable Lie algebra;

(ii) the ideal  $I$  is a right irreducible module over  $sl_2^1$ .

Then,  $L \cong ((sl_2^1 \oplus R) \dot{+} I) \oplus sl_2^2 \oplus \dots \oplus sl_2^s$ .

As a result of Theorems 2.6–2.7 and 3.4, we have the following corollaries.

**Corollary 3.5.** *Let  $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$  with  $\dim R = 2$  and  $\dim I \neq 3$ . Then  $L$  is isomorphic to the following algebra:*

$$\left\{ \begin{array}{ll} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \leq j \leq s, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & \\ [x_k, h_1] = (m-2k)x_k, & 0 \leq k \leq m, \\ [x_k, f_1] = x_{k+1}, & 0 \leq k \leq m-1, \\ [x_k, e_1] = -k(m+1-k)x_{k-1}, & 1 \leq k \leq m, \\ [x_k, y_2] = ax_k, & 0 \leq k \leq m, \quad a \in \mathbb{C}. \end{array} \right.$$

**Corollary 3.6.** *Let  $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$ , with  $\dim R = 3$  and  $\dim I \neq 3$ . Then  $L$  is isomorphic to the following non-isomorphic algebras:*

$$L_1(\alpha, a) : \begin{cases} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \leq j \leq s, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & [y_3, y_2] = -[y_2, y_3] = \alpha y_3, \\ [x_k, h_1] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f_1] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e_1] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m, \end{cases}$$

$$L_2(a) : \begin{cases} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \leq j \leq s, \\ [y_1, y_2] = -[y_2, y_1] = y_1 + y_3, & [y_3, y_2] = -[y_2, y_3] = y_3, \\ [x_k, h_1] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f_1] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e_1] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m. \end{cases}$$

#### 4. ON LEIBNIZ ALGEBRAS WITH SEMISIMPLE PART $sl_2^1 \oplus sl_2^2$ .

Let the quotient Lie algebra  $L/I$  for a Leibniz algebra  $L$  be isomorphic to a direct sum of two copies of the  $sl_2$ , i.e.,  $L/I \cong sl_2 \oplus sl_2$ . In this section we shall investigate the case when the ideal  $I$  is reducible over each copy of  $sl_2$ . In order to distinguish copies of  $sl_2$  we shall denote them by  $sl_2^1$  and  $sl_2^2$ . One can assume that  $I$  is reducible over  $sl_2^1$ . Due to Theorem 2.5 we have the following decomposition:

$$I = I_{1,1} \oplus I_{1,2} \oplus \dots \oplus I_{1,s+1},$$

where  $I_{1,j}$ ,  $1 \leq j \leq s + 1$  are irreducible  $sl_2^1$ -modules.

We shall focus our study on the case when  $\dim I_{1,1} = \dim I_{1,2} = \dots = \dim I_{1,s+1} = m + 1$ .

Let us introduce notations as follows:

$$I_{1,j} = \langle x_0^j, x_1^j, \dots, x_m^j \rangle, \quad 1 \leq j \leq s + 1.$$

In the proof of Theorem 3.7 of the paper [5] it was proved that

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} a_{i,j}^k x_i^k, \quad [x_i^j, f_2] = \sum_{k=1}^{s+1} b_{i,j}^k x_i^k, \quad [x_i^j, h_2] = \sum_{k=1}^{s+1} c_{i,j}^k x_i^k,$$

where  $0 \leq i \leq m$ ,  $1 \leq j \leq s + 1$ .

Without loss of generality, one can assume that the products  $[I_{1,j}, sl_2^1]$ ,  $1 \leq j \leq s + 1$  are expressed as follows:

$$\begin{aligned} [x_i^j, e_1] &= -i(m + 1 - i)x_{i-1}^j, & 1 \leq i \leq m, \\ [x_i^j, f_1] &= x_{i+1}^j, & 0 \leq i \leq m - 1, \\ [x_i^j, h_1] &= (m - 2i)x_i^j, & 0 \leq i \leq m. \end{aligned}$$

**Proposition 4.1.** *Let  $L/I$  be isomorphic to  $sl_2^1 \oplus sl_2^2$ , where  $I = I_{1,1} \oplus I_{1,2} \oplus \dots \oplus I_{1,s+1}$  with  $\dim I_{1,j} = m + 1$  and  $I_{1,j}$  are irreducible  $sl_2^1$ -modules for  $1 \leq j \leq s + 1$ . Then*

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} a_j^k x_i^k, \quad [x_i^j, f_2] = \sum_{k=1}^{s+1} b_j^k x_i^k, \quad [x_i^j, h_2] = \sum_{k=1}^{s+1} c_j^k x_i^k,$$

where  $0 \leq i \leq m$ ,  $1 \leq j \leq s + 1$ .

*Proof.* Applying Leibniz identity for the following triples of elements:

$$(x_1^j, e_1, e_2), \quad 1 \leq j \leq s + 1$$

we derive the restrictions:

$$a_{1,j}^k = a_{0,j}^k, \quad 1 \leq k \leq s + 1.$$

Consequently, we obtain

$$[x_0^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_0^k, \quad [x_1^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_1^k, \quad 1 \leq j \leq s+1.$$

By induction, we shall prove the equality

$$(4.1) \quad [x_i^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_i^k, \quad 0 \leq i \leq m.$$

Using the assumption of correctness of Equality 4.1 for  $i$  in the following chain of equalities:

$$\begin{aligned} 0 &= [x_{i+1}^j, [e_1, e_2]] = [[x_{i+1}^j, e_1], e_2] - [[x_{i+1}^j, e_2], e_1] = -[(i+1)(m-i)x_{i+1}^j, e_2] - \\ &- \sum_{k=1}^{s+1} a_{i+1,j}^k [x_{i+1}^k, e_1] = -(i+1)(m-i) \sum_{k=1}^{s+1} a_{0,j}^k x_i^k + \sum_{k=1}^{s+1} a_{i+1,j}^k (i+1)(m-i)x_i^k, \end{aligned}$$

we conclude that  $a_{i+1,j}^k = a_{0,j}^k$  for  $1 \leq k \leq s+1$ , that is,  $[x_{i+1}^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_{i+1}^k$  and Equality 4.1 is proved.

Putting  $a_j^k = a_{0,j}^k$ , we have  $[x_i^j, e_2] = \sum_{k=1}^{s+1} a_j^k x_i^k$ ,  $1 \leq j \leq s+1$ ,  $0 \leq i \leq m$ .

Applying Leibniz identity for the triples of elements:

$$(x_1^j, e_1, f_2), \quad 1 \leq j \leq s+1,$$

we get

$$b_{1,j}^k = b_{0,j}^k, \quad 1 \leq k \leq s+1.$$

Therefore, we obtain

$$[x_0^j, f_2] = \sum_{k=1}^{s+1} b_{0,j}^k x_0^k, \quad [x_1^j, f_2] = \sum_{k=1}^{s+1} b_{0,j}^k x_1^k, \quad 1 \leq j \leq s+1.$$

Applying induction and the following chain of equalities

$$\begin{aligned} 0 &= [x_{i+1}^j, [e_1, f_2]] = [[x_{i+1}^j, e_1], f_2] - [[x_{i+1}^j, f_2], e_1] = -[(i+1)(m-i)x_{i+1}^j, f_2] - \\ &- \sum_{k=1}^{s+1} b_{i+1,j}^k [x_{i+1}^k, e_1] = -(i+1)(m-i) \sum_{k=1}^{s+1} b_{0,j}^k x_i^k + \sum_{k=1}^{s+1} b_{i+1,j}^k (i+1)(m-i)x_i^k, \end{aligned}$$

we derive the equality

$$[x_i^j, f_2] = \sum_{k=1}^{s+1} b_{0,j}^k x_i^k, \quad 0 \leq i \leq m, \quad 1 \leq j \leq s+1.$$

Setting  $b_j^k = b_{0,j}^k$ , we obtain  $[x_i^j, f_2] = \sum_{k=1}^{s+1} b_j^k x_i^k$ ,  $0 \leq i \leq m$ ,  $1 \leq j \leq s+1$ .

Analogously, one can prove the equality  $[x_i^j, h_2] = \sum_{k=1}^{s+1} c_j^k x_i^k$  with  $1 \leq j \leq s+1$ .  $\square$

Now we shall describe Leibniz algebras such that  $L/I \cong sl_2^1 \oplus sl_2^2$  and  $I = I_{1,1} \oplus I_{1,2}$ , where  $I_{1,1}, I_{1,2}$  are irreducible  $sl_2^1$ -modules. Without loss of generality we can suppose

$$\begin{aligned} [x_i^j, h_1] &= (m-2i)x_i^j, & 0 \leq i \leq m, \\ [x_i^j, f_1] &= x_{i+1}^j, & 0 \leq i \leq m-1, \\ [x_i^j, e_1] &= -i(m+1-i)x_{i-1}^j, & 1 \leq i \leq m. \end{aligned}$$

for  $j = 1, 2$ .

According to Proposition 4.1, one can assume that

$$\begin{aligned} [x_i^1, e_2] &= a_1 x_i^1 + a_2 x_i^2, & [x_i^2, e_2] &= a_3 x_i^1 + a_4 x_i^2, \\ [x_i^1, f_2] &= b_1 x_i^1 + b_2 x_i^2, & [x_i^2, f_2] &= b_3 x_i^1 + b_4 x_i^2, \\ [x_i^1, h_2] &= c_1 x_i^1 + c_2 x_i^2, & [x_i^2, h_2] &= c_3 x_i^1 + c_4 x_i^2, \end{aligned}$$

where  $0 \leq i \leq m$ .

From the following chains of the equalities obtained applying Leibniz identity

$$\begin{aligned}
2(a_1x_0^1 + a_2x_0^2) &= 2[x_0^1, e_2] = [x_0^1, [e_2, h_2]] = (a_2c_3 - c_2a_3)x_0^1 + (a_1c_2 + a_2c_4 - c_1a_2 - c_2a_4)x_0^2, \\
2(a_3x_0^1 + a_4x_0^2) &= 2[x_0^2, e_2] = [x_0^2, [e_2, h_2]] = (a_3c_1 + a_4c_3 - c_3a_1 - c_4a_3)x_0^1 + (a_3c_2 - a_2c_3)x_0^2, \\
-2(b_1x_0^1 + b_2x_0^2) &= -2[x_0^1, f_2] = [x_0^1, [f_2, h_2]] = (b_2c_3 - c_2b_3)x_0^1 + (b_1c_2 + b_2c_4 - c_1b_2 - c_2b_4)x_0^2, \\
-2(b_3x_0^1 + b_4x_0^2) &= 2[x_0^2, e_2] = [x_0^2, [f_2, h_2]] = (b_3c_1 + b_4c_3 - c_3b_1 - c_4b_3)x_0^1 + (b_3c_2 - b_2c_3)x_0^2, \\
-c_1x_1^1 - c_2x_1^2 &= -[x_1^1, h_2] = [x_1^1, [f_2, e_2]] = (a_3b_2 - a_2b_3)x_1^1 + 2(a_2b_1 - a_1b_2)x_1^2, \\
-c_3x_1^1 - c_4x_1^2 &= -[x_1^2, h_2] = [x_1^2, [f_2, e_2]] = 2(a_1b_3 - a_3b_1)x_1^1 + (a_2b_3 - a_3b_2)x_1^2.
\end{aligned}$$

we derive:

$$(4.2) \quad \left\{ \begin{array}{l} 2a_1 = a_2c_3 - a_3c_2, \\ 2a_2 = a_1c_2 + a_2c_4 - c_1a_2 - c_2a_4, \\ 2a_3 = a_3c_1 + a_4c_3 - c_3a_1 - c_4a_3, \\ 2a_4 = a_3c_2 - a_2c_3, \\ -2b_1 = b_2c_3 - c_2b_3, \\ -2b_2 = b_1c_2 + b_2c_4 - c_1b_2 - c_2b_4, \\ -2b_3 = b_3c_1 + b_4c_3 - c_3b_1 - c_4b_3, \\ -2b_4 = b_3c_2 - b_2c_3, \\ c_1 = a_2b_3 - a_3b_2, \\ c_2 = 2(a_1b_2 - a_2b_1), \\ c_3 = 2(a_3b_1 - a_1b_3), \\ c_4 = a_3b_2 - a_2b_3. \end{array} \right.$$

It is easy to see that  $a_4 = -a_1$ ,  $b_4 = -b_1$  and  $c_4 = -c_1$ .

Thus, we obtain the following products:

$$(4.3) \quad \begin{aligned} [x_i^1, e_2] &= a_1x_i^1 + a_2x_i^2, & [x_i^1, f_2] &= b_1x_i^1 + b_2x_i^2, \\ [x_i^2, e_2] &= a_3x_i^1 - a_1x_i^2, & [x_i^2, f_2] &= b_3x_i^1 - b_1x_i^2, \\ [x_i^1, h_2] &= (a_2b_3 - a_3b_2)x_i^1 + 2(a_1b_2 - a_2b_1)x_i^2, \\ [x_i^2, h_2] &= 2(a_3b_1 - a_1b_3)x_i^1 - (a_2b_3 - a_3b_2)x_i^2, \end{aligned}$$

where the structure constants  $a_1, a_2, a_3$  and  $b_1, b_2, b_3$  satisfy the relations (4.2).

We present the classification of Leibniz algebras satisfying the following conditions:

- (a)  $L/I \cong sl_2^1 \oplus sl_2^2$ ;
- (b)  $I = I_{1,1} \oplus I_{1,2}$  such that  $I_{1,1}, I_{1,2}$  are irreducible  $sl_2^1$ -modules and  $\dim I_{1,1} = \dim I_{1,2}$ ;
- (c)  $I = I_{2,1} \oplus I_{2,2} \oplus \dots \oplus I_{2,m+1}$  such that  $I_{2,k}$  are irreducible  $sl_2^2$ -modules with  $1 \leq k \leq m+1$ .

**Theorem 4.2.** *An arbitrary Leibniz algebra satisfying the conditions (a)-(c) is isomorphic to the following algebra:*

$$\left\{ \begin{array}{l} [e_i, h_i] = -[h_i, e_i] = 2e_i, \\ [e_i, f_i] = -[f_i, e_i] = h_i, \\ [h_i, f_i] = -[f_i, h_i] = 2f_i, \\ [x_k^i, h_1] = (m-2k)x_k^i, & 0 \leq k \leq m, \\ [x_k^i, f_1] = x_{k+1}^i, & 0 \leq k \leq m-1, \\ [x_k^i, e_1] = -k(m+1-k)x_{k-1}^i, & 1 \leq k \leq m, \\ [x_j^1, e_2] = [x_j^2, h_2] = x_j^2, \\ [x_j^1, h_2] = [x_j^2, f_2] = -x_j^1, \end{array} \right.$$

with  $1 \leq i \leq 2$  and  $0 \leq j \leq m$ .

*Proof.* We set  $\dim I_{1,1} = \dim I_{1,2} = m+1$ . Then, according to Theorem 2.8, we obtain  $\dim I_{2,k} = 2$  for  $1 \leq k \leq m+1$ .

Let  $\{x_0^1, x_1^1, \dots, x_m^1\}$ ,  $\{x_0^2, x_1^2, \dots, x_m^2\}$  and  $\{y_{i+1}^1, y_{i+1}^2\}$  be bases of  $I_{1,1}$ ,  $I_{1,2}$  and  $I_{2,i+1}$ ,  $0 \leq i \leq m$ , respectively. We set

$$(4.4) \quad y_{i+1}^1 = \alpha_i^1 x_i^1 + \alpha_i^2 x_i^2, \quad y_{i+1}^2 = \beta_i^1 x_i^1 + \beta_i^2 x_i^2, \quad 0 \leq i \leq m.$$

Taking into account the products (4.3) for  $0 \leq i \leq m$  we consider the equalities

$$0 = [y_{i+1}^2, f_2] = [\beta_i^1 x_i^1 + \beta_i^2 x_i^2, f_2] = \beta_i^1 (b_1 x_i^1 + b_2 x_i^2) + \beta_i^2 (b_3 x_i^1 - b_1 x_i^2) = (\beta_i^1 b_1 + \beta_i^2 b_3) x_i^1 + (\beta_i^1 b_2 - \beta_i^2 b_1) x_i^2.$$

Therefore,

$$(4.5) \quad \begin{cases} \beta_i^1 b_1 + \beta_i^2 b_3 = 0, \\ \beta_i^1 b_2 - \beta_i^2 b_1 = 0, \end{cases}$$

with  $0 \leq i \leq m$ .

If  $b_1^2 + b_2 b_3 \neq 0$ , then the system of equations (4.5) has only trivial solution, which is a contradiction. Hence,  $b_1^2 + b_2 b_3 = 0$ .

Similarly, from

$$0 = [y_{i+1}^1, e_2] = (\alpha_i^1 a_1 + \alpha_i^2 a_3) x_i^1 + (\alpha_i^1 a_2 - \alpha_i^2 a_1) x_i^2$$

we derive  $a_1^2 + a_2 a_3 = 0$ .

Thus, we have  $a_1 = \pm i \sqrt{a_2 a_3}$  and  $b_1 = \pm i \sqrt{b_2 b_3}$ .

Let us summarize the obtained products:

$$(4.6) \quad \begin{aligned} [x_i^1, e_2] &= a_1 x_i^1 + a_2 x_i^2, & [x_i^1, f_2] &= b_1 x_i^1 + b_2 x_i^2, \\ [x_i^2, e_2] &= a_3 x_i^1 - a_1 x_i^2, & [x_i^2, f_2] &= b_3 x_i^1 - b_1 x_i^2, \\ [x_i^1, h_2] &= (a_2 b_3 - a_3 b_2) x_i^1 + 2(a_1 b_2 - a_2 b_1) x_i^2, \\ [x_i^2, h_2] &= 2(a_3 b_1 - a_1 b_3) x_i^1 - (a_2 b_3 - a_3 b_2) x_i^2, \end{aligned}$$

with  $0 \leq i \leq m$  and the relations  $a_1^2 + a_2 a_3 = b_1^2 + b_2 b_3 = 0$ .

Taking the following basis transformation:

$$x_i^{1'} = A x_i^1 + B x_i^2, \quad x_i^{2'} = (A a_1 + B a_3) x_i^1 + (A a_2 - B a_1) x_i^2, \quad 0 \leq i \leq m$$

we can assume that the products (5.6) have the following form:

$$\begin{aligned} [x_i^1, e_2] &= x_i^2, & [x_i^2, e_2] &= 0, \\ [x_i^1, f_2] &= b_1 x_i^1 + b_1^2 x_i^2, & [x_i^2, f_2] &= -x_i^1 - b_1 x_i^2, \\ [x_i^1, h_2] &= -x_i^1 - 2b_1 x_i^2, & [x_i^2, h_2] &= x_i^2. \end{aligned}$$

Applying the change of basis as follows

$$x_i^{1'} = x_i^1 + b_1 x_i^2, \quad x_i^{2'} = x_i^2, \quad 0 \leq i \leq m,$$

we complete the proof of the theorem.  $\square$

The following theorem establishes that condition (c) can be omitted because if conditions (a)-(b) are true, then condition (c) is always executable.

**Theorem 4.3.** *Let  $L$  be a Leibniz algebra satisfying the conditions (a)-(b). Then either  $L$  satisfies the condition (c) or  $L \cong (sl_2^1 + I) \oplus sl_2^2$ .*

*Proof.* Let  $L$  be a Leibniz algebra satisfying conditions (a) and (b), but not (c). In order to prove the assertion of theorem we have to establish that all modules  $I_{2,i}$ ,  $1 \leq i \leq m+1$  are reducible over  $sl_2^2$ . Indeed, according to Theorem 2.5 we conclude that  $I_{2,i}$  are completely reducible modules over  $sl_2^2$ . In denotation of (4.4) we have  $I_{2,i} = \langle y_i^1 \rangle \oplus \langle y_i^2 \rangle$ , where  $\langle y_i^1 \rangle$ ,  $\langle y_i^2 \rangle$  are one-dimensional trivial  $sl_2^2$ -modules, that is,

$$[y_i^1, e_2] = [y_i^2, e_2] = [y_i^1, f_2] = [y_i^2, f_2] = [y_i^1, h_2] = [y_i^2, h_2] = 0.$$

We shall prove by contrary method, that is, we shall assume that not all modules  $I_{2,i}$  are reducible over  $sl_2^2$ . Then we can assume that there exist some  $s$ ,  $1 \leq s \leq m+1$  and  $t$ ,  $1 \leq t \leq m+1$ ,  $t \neq s$  such that  $I_{2,s}$  is irreducible, but  $I_{2,t}$  is reducible  $sl_2^2$ -modules. By renumerating of indexes, without loss of generality, we can suppose  $s = 2$  and  $t = 1$ .

From the products in the proof of Theorem 4.2 we have

$$[x_1^1, e_2] = [x_1^2, h_2] = x_1^2, \quad [x_1^1, h_2] = [x_1^2, f_2] = -x_1^1, \quad [x_1^2, e_2] = [x_1^1, f_2] = 0.$$

Consider the chain of equalities

$$\begin{aligned} 0 &= [y_1^1, [e_2, f_1]] = [[y_1^1, e_2], f_1] - [[y_1^1, f_1], e_2] = -[[y_1^1, f_1], e_2] = \\ &= -[[\alpha_0^1 x_0^1 + \alpha_0^2 x_0^2, f_1], e_2] = -[\alpha_0^1 x_1^1 + \alpha_0^2 x_1^2, e_2] = -\alpha_0^1 x_1^2, \end{aligned}$$



$$0 = [y_1^1, [f_2, f_1]] = [[y_1^1, f_2], f_1] - [[y_1^1, f_1], f_2] = -[[y_1^1, f_1], f_2] = \\ -[[\alpha_0^1 x_0^1 + \alpha_0^2 x_0^2, f_1], e_2] = -[\alpha_0^1 x_1^1 + \alpha_0^2 x_1^2, f_2] = \alpha_0^2 x_1^1.$$

Therefore,  $\alpha_0^1 = \alpha_0^2 = 0$ , which means  $y_1^1 = 0$ . Thus, we get a contradiction.  $\square$

### 5. SOME REMARKS ON LEIBNIZ ALGEBRAS WITH SEMISIMPLE PART $sl_n$ .

In this section we present the structure of Leibniz algebras with associated Lie algebra  $sl_n$  and with dimension of ideal  $I$  equal to 1 and 2.

**Proposition 5.1.** *Let  $L/I$  be isomorphic to  $G$  where  $G = \langle e_1, e_2, \dots, e_m \rangle$  is a semisimple Lie algebra and  $\dim I = 1$ . Then  $L = G \oplus I$ .*

*Proof.* We put  $I = \langle x \rangle$  and  $[x, e_i] = \alpha_i x$ . For the semisimple Lie algebra  $G$  we have  $[G, G] = G$ , that is, for any  $e_i$ ,  $1 \leq i \leq m$  there exist  $e_p$  and  $e_q$  such that  $e_i = \sum_{p,q} \beta_{p,q} [e_p, e_q]$ .

The proof of proposition completes the following chain of equalities:

$$[x, e_i] = [x, \sum_{p,q} \beta_{p,q} [e_p, e_q]] = \sum_{p,q} \beta_{p,q} [x, [e_p, e_q]] = \sum_{p,q} \beta_{p,q} ([x, e_p], e_q) - [x, e_q], e_p) = \\ \sum_{p,q} \beta_{p,q} (\alpha_p [x, e_q] - \alpha_q [x, e_p]) = \sum_{p,q} \beta_{p,q} (\alpha_p \alpha_q x - \alpha_q \alpha_p x) = 0.$$

$\square$

Let  $G$  be a simple Lie algebra with a basis  $\{e_1, e_2, \dots, e_m\}$  which satisfies the condition that for any  $e_i$  there exist  $p, q$  such that  $e_i = [e_p, e_q]$ .

We consider the case of  $L/I \cong G$  and  $\dim I = 2$ . We set  $\{x, y\}$  the basis elements of  $I$  and

$$[x, e_i] = \alpha_i x + \beta_i y, \quad [y, e_i] = \gamma_i x + \delta_i y, \quad 1 \leq i \leq n.$$

Consider

$$[x, e_i] = [x, [e_p, e_q]] = [[x, e_p], e_q] - [[x, e_q], e_p] = (\beta_p \gamma_q - \beta_q \gamma_p)x + (\alpha_p \beta_q + \beta_p \delta_q - \alpha_q \beta_p - \beta_q \delta_p)y.$$

On the other hand,  $[x, e_i] = \alpha_i x + \beta_i y$ .

Comparing the coefficients we derive

$$(5.1) \quad \begin{cases} \alpha_i = \beta_p \gamma_q - \beta_q \gamma_p, \\ \beta_i = \alpha_p \beta_q + \beta_p \delta_q - \alpha_q \beta_p - \beta_q \delta_p, \end{cases}$$

where  $1 \leq i \leq n$ .

Similarly, if we consider  $[y, e_i]$ , we deduce

$$(5.2) \quad \begin{cases} \gamma_i = \gamma_p \alpha_q + \delta_p \gamma_q - \gamma_q \alpha_p - \delta_q \gamma_p, \\ \delta_i = \gamma_p \beta_q - \gamma_q \beta_p, \end{cases}$$

where  $1 \leq i \leq n$ .

From systems (5.1) and (5.2) we obtain

$$(5.3) \quad [x, e_i] = \alpha_i x + \beta_i y, \quad [y, e_i] = \gamma_i x - \alpha_i y, \quad 1 \leq i \leq n.$$

Let  $L/I$  be isomorphic to  $sl_n$ . From [6] we have the standard basis of  $sl_n = \langle h_k, e_{i,j} \rangle$  with  $1 \leq k \leq n-1$ ,  $1 \leq i, j \leq n$  and  $i \neq j$ . We recall the table of multiplication of  $sl_n$

$$\begin{array}{lll} [e_{ij}, e_{jk}] = e_{ik}, & & 1 \leq i, j, k \leq n, i \neq j, \quad j \neq k, \quad k \neq i, \\ [h_k, e_{in}] = e_{in}, & [h_k, e_{ni}] = -e_{ni}, & 1 \leq i, k \leq n-1, \quad k \neq i, \\ [h_i, e_{ij}] = e_{ij}, & [h_j, e_{ij}] = -e_{ij}, & [e_{ij}, e_{ji}] = h_i - h_j, \quad 1 \leq i, j \leq n-1, \\ [h_i, e_{in}] = 2e_{in}, & [h_i, e_{ni}] = -2e_{ni}, & [e_{in}, e_{ni}] = h_i, \quad 1 \leq i \leq n-1. \end{array}$$

**Theorem 5.2.** *Let  $L/I$  be isomorphic to  $sl_n$  ( $n \geq 3$ ), where  $\dim I = 2$ . Then  $L = sl_n \oplus I$ .*

*Proof.* From (5.3) we have

$$\begin{aligned} [x, h_i] &= \alpha_i x + \beta_i y, \quad 1 \leq i \leq n-1, & [y, h_i] &= \gamma_i x - \alpha_i y, \quad 1 \leq i \leq n-1, \\ [x, e_{ij}] &= \alpha_{ij} x + \beta_{ij} y, & [y, e_{ij}] &= \gamma_{ij} x - \alpha_{ij} y, \quad 1 \leq i, j \leq n, \quad i \neq j. \end{aligned}$$

Applying Leibniz identity for the following triples of elements

$$(x, h_i, e_{ij}), \quad (y, h_i, e_{ij}), \quad (x, h_i, e_{in}), \quad (y, h_i, e_{in}), \quad (x, h_i, e_{ni}), \quad (y, h_i, e_{ni}).$$

we deduce

$$(5.4) \quad \begin{cases} \alpha_{ij} + \gamma_i \beta_{ij} - \beta_i \gamma_{ij} = 0, \\ 2\beta_i \alpha_{ij} + (1 - 2\alpha_i) \beta_{ij} = 0, \\ -2\gamma_i \alpha_{ij} + (1 + 2\alpha_i) \gamma_{ij} = 0, \end{cases}$$

$$(5.5) \quad \begin{cases} 2\alpha_{in} + \gamma_i \beta_{in} - \beta_i \gamma_{in} = 0, \\ \beta_i \alpha_{in} + (1 - \alpha_i) \beta_{in} = 0, \\ -\gamma_i \alpha_{in} + (1 + \alpha_i) \gamma_{in} = 0, \end{cases}$$

$$(5.6) \quad \begin{cases} -2\alpha_{ni} + \gamma_i \beta_{ni} - \beta_i \gamma_{ni} = 0, \\ \beta_i \alpha_{ni} - (\alpha_i + 1) \beta_{ni} = 0, \\ -\gamma_i \alpha_{ni} + (\alpha_i - 1) \gamma_{ni} = 0. \end{cases}$$

Determinants of systems (5.4) - (5.6) have the following values:

$$\begin{aligned} \text{Det} \begin{pmatrix} 1 & \gamma_i & -\beta_i \\ 2\beta_i & 1 - 2\alpha_i & 0 \\ -2\gamma_i & 0 & 1 + 2\alpha_i \end{pmatrix} &= 1 - 4(\alpha_i^2 + \beta_i \gamma_i), \\ \text{Det} \begin{pmatrix} 2 & \gamma_i & -\beta_i \\ \beta_i & 1 - \alpha_i & 0 \\ -\gamma_i & 0 & 1 + \alpha_i \end{pmatrix} &= 2 - 2(\alpha_i^2 + \beta_i \gamma_i), \\ \text{Det} \begin{pmatrix} -2 & \gamma_i & -\beta_i \\ \beta_i & -1 - \alpha_i & 0 \\ -\gamma_i & 0 & \alpha_i - 1 \end{pmatrix} &= -2 + 2(\alpha_i^2 + \beta_i \gamma_i). \end{aligned}$$

It is easy to see that these determinants do not equal to zero simultaneously. Therefore, we conclude that either

$$\alpha_{in} = \beta_{in} = \gamma_{in} = \alpha_{ni} = \beta_{ni} = \gamma_{ni} = 0, \quad 1 \leq i \leq n-1$$

or

$$\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0, \quad 1 \leq i, j \leq n-1.$$

**Case 1.** Let  $\alpha_{in} = \beta_{in} = \gamma_{in} = \alpha_{ni} = \beta_{ni} = \gamma_{ni} = 0$  be where  $1 \leq i \leq n-1$ . Then the non-zero products characterized  $[I, sl_n]$  are the following

$$\begin{aligned} [x, h_i] &= \alpha_i x + \beta_i y, & [y, h_i] &= \gamma_i x - \alpha_i y, \quad 1 \leq i \leq n-1, \\ [x, e_{ij}] &= \alpha_{ij} x + \beta_{ij} y, & [y, e_{ij}] &= \gamma_{ij} x - \alpha_{ij} y, \quad 1 \leq i, j \leq n-1, \quad i \neq j. \end{aligned}$$

From the equalities

$$\begin{aligned} [x, h_i] &= [x, [e_{in}, e_{ni}]] = [[x, e_{in}], e_{ni}] - [[x, e_{ni}], e_{in}] = 0, \\ [y, h_i] &= [y, [e_{in}, e_{ni}]] = [[y, e_{in}], e_{ni}] - [[y, e_{ni}], e_{in}] = 0, \\ [x, e_{ij}] &= [x, [e_{in}, e_{nj}]] = [[x, e_{in}], e_{nj}] - [[x, e_{nj}], e_{ni}] = 0, \\ [y, e_{ij}] &= [y, [e_{in}, e_{nj}]] = [[y, e_{in}], e_{nj}] - [[y, e_{nj}], e_{ni}] = 0, \end{aligned}$$

we obtain that  $[I, sl_n] = 0$ .

**Case 2.** Let  $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$  be where  $1 \leq i, j \leq n-1$ . Then we have

$$\begin{aligned} [x, h_i] &= \alpha_i x + \beta_i y, & [y, h_i] &= \gamma_i x - \alpha_i y, & 1 \leq i \leq n-1, \\ [x, e_{in}] &= \alpha_{in} x + \beta_{in} y, & [y, e_{in}] &= \gamma_{in} x - \alpha_{in} y, & 1 \leq i \leq n-1, \\ [x, e_{ni}] &= \alpha_{ni} x + \beta_{ni} y, & [y, e_{ni}] &= \gamma_{ni} x - \alpha_{ni} y, & 1 \leq i \leq n-1, \\ [x, e_{ij}] &= [y, e_{ij}] = 0, & & & 1 \leq i, j \leq n-1, \quad i \neq j. \end{aligned}$$

Using Leibniz identity for the following triples

$$\begin{array}{cccc} (x, e_{ij}, e_{ji}), & (y, e_{ij}, e_{ji}), & (x, h_j, e_{in}), & (y, h_j, e_{in}), \\ (x, h_j, e_{ni}), & (y, h_j, e_{ni}), & (x, e_{in}, e_{ni}), & (y, e_{in}, e_{ni}), \end{array}$$

we obtain

$$[x, h_i] = [y, h_i] = [x, e_{in}] = [y, e_{in}] = [x, e_{ni}] = [y, e_{ni}] = 0, \quad 1 \leq i \leq n-1.$$

□

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