ON THE DESCRIPTION OF LEIBNIZ SUPERALGEBRAS OF NILINDEX $n + m$

L.M. CAMACHO, J. R. GÓMEZ, A.KH. KHUDOYBERDIYEV AND B.A. OMIROV

ABSTRACT. In this work we investigate the complex Leibniz superalgebras with characteristic sequence $(n_1, \ldots, n_k|m)$ and nilindex $n + m$, where $n = n_1 + \cdots + n_k$, n and m $(m \neq 0)$ are dimensions of even and odd parts, respectively. Such superalgebras with condition $n_1 \geq n - 1$ were classified in [4]–[5]. Here we prove that in the case $n_1 \leq n - 2$ the Leibniz superalgebras have nilindex less than $n + m$. Thus, we get the classification of Leibniz superalgebras with characteristic sequence $(n_1, \ldots, n_k|m)$ and nilindex $n + m$.

Mathematics Subject Classification 2000: 17A32, 17B30, 17B70, 17A70.

Key Words and Phrases: Lie superalgebras, Leibniz superalgebras, nilindex, characteristic sequence, naturally gradation.

1. INTRODUCTION

The paper is devoted to the study of the nilpotent Leibniz superalgebras. The notion of Leibniz superalgebras appears comparatively recently [1], [8]. Leibniz superalgebras are generalizations of Leibniz algebras [9] and, on the other hand, they naturally generalize well-known Lie superalgebras. The elementary properties of Leibniz superalgebras were obtained in [1].

In the work [7] the Lie superalgebras with maximal nilindex were classified. The distinguishing property of such superalgebras is that they are two-generated and its nilindex equal to $n + m$ (where n and m are dimensions of even and odd parts, respectively). In fact, there exists unique Lie superalgebra of the maximal nilindex and its characteristic sequence is equal to $(1, 1|m)$. This superalgebra is filiform Lie superalgebra (the characteristic sequence equal to $(n - 1, 1|m)$) and we mention about paper [3], where some crucial properties of filiform Lie superalgebras are given.

In case of Leibniz superalgebras the property of maximal nilindex is equivalent to the property of single-generated superalgebras and they are described in [1]. However, the description of Leibniz superalgebras of nilindex $n + m$ is an open problem and it needs to solve many technical tasks. Therefore, they can be studied by applying restrictions on their characteristic sequences. In the present paper we consider Leibniz superalgebras with characteristic sequence $(n_1, \ldots, n_k|m)$ and nilindex $n + m$. Since such superalgebras in the case $n_1 \geq n - 1$ have been already classified in [4]-[5] we need to study the case $n_1 \leq n - 2$.

In fact, in the previous cases (cases where $n_1 \geq n - 1$) due to work [2] we have used some information on the structure of even part of the superalgebra and it played the crucial role in that classifications. In the rest case (case where $n_1 \leq n - 2$) the structure of even part is unknown, but we used the properties of natural

gradation and the naturally graded basis (so-called adapted basis) of even part of the superalgebra.

All over the work we consider spaces over the field of complex numbers. By asterisks $(*)$ we denote the appropriate coefficients at the basic elements of superalgebra.

2. Preliminaries

Recall the notion of Leibniz superalgebras.

Definition 2.1. *A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:*
  1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta (mod\ 2)}$,
  2. $[x, [y, z]] = [[x, y], z] - (-1)^{\alpha\beta}[[x, z], y] -$ Leibniz superidentity,
for all $x \in L$, $y \in L_\alpha$, $z \in L_\beta$ and $\alpha, \beta \in \mathbb{Z}_2$.

Evidently, even part of the Leibniz superalgebra is a Leibniz algebra.

The vector spaces L_0 and L_1 are said to be even and odd parts of the superalgebra L, respectively.

Note that if in Leibniz superalgebra L the identity

$$[x, y] = -(-1)^{\alpha\beta}[y, x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity can be transformed into the Jacobi superidentity. Thus, Leibniz superalgebras are a generalization of Lie superalgebras and Leibniz algebras.

The set of all Leibniz superalgebras with the dimensions of the even and odd parts, respectively equal to n and m, we denote by $Leib_{n,m}$.

For a given Leibniz superalgebra L we define the descending central sequence as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

Definition 2.2. *A Leibniz superalgebra L is called nilpotent, if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The minimal number s with this property is called nilindex of the superalgebra L.*

The following theorem describes nilpotent Leibniz superalgebras with maximal nilindex.

Theorem 2.1. [1] *Let L be a Leibniz superalgebra of $Leib_{n,m}$ with nilindex equal to $n + m + 1$. Then L is isomorphic to one of the following non-isomorphic superalgebras:*

$$[e_i, e_1] = e_{i+1},\ 1 \leq i \leq n-1,\ m = 0; \quad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n+m-1, \\ [e_i, e_2] = 2e_{i+2}, & 1 \leq i \leq n+m-2, \end{cases}$$

(omitted products are equal to zero).

Remark 2.1. From the assertion of Theorem 2.1 we have that in case of non-trivial odd part L_1 of the superalgebra L there are two possibility for n and m, namely, $m = n$ if $n + m$ is even and $m = n + 1$ if $n + m$ is odd. Moreover, it is clear that the Leibniz superalgebra has the maximal nilindex if and only if it is single-generated.

Let $L = L_0 \oplus L_1$ be a nilpotent Leibniz superalgebra. For an arbitrary element $x \in L_0$, the operator of right multiplication R_x defined as $R_x(y) = [y, x]$ is

a nilpotent endomorphism of the space L_i, where $i \in \{0, 1\}$. Taking into account the property of complex field we can consider the Jordan form for R_x. For operator R_x denote by $C_i(x)$ ($i \in \{0, 1\}$) the descending sequence of its Jordan blocks dimensions. Consider the lexicographical order on the set $C_i(L_0)$.

Definition 2.3. *A sequence*

$$C(L) = \left(\max_{x \in L_0 \setminus L_0^2} C_0(x) \ \middle| \ \max_{\widetilde{x} \in L_0 \setminus L_0^2} C_1(\widetilde{x}) \right)$$

is said to be the characteristic sequence of the Leibniz superalgebra L.

Similarly to [6] (corollary 3.0.1) it can be proved that the characteristic sequence is invariant under isomorphism.

Since Leibniz superalgebras from $Leib_{n,m}$ with characteristic sequence equal to $(n_1, \ldots, n_k | m)$ and nilindex $n + m$ are already classified for the case $n_1 \geq n - 1$, henceforth we shall reduce our investigation to the case of $n_1 \leq n - 2$.

From the Definition 2.3 we conclude that a Leibniz algebra L_0 has characteristic sequence $(n_1, \ldots, n_k)$. Let $s \in \mathbb{N}$ be a nilindex of the Leibniz algebra L_0. Since $n_1 \leq n - 2$ then we have $s \leq n - 1$ and Leibniz algebra L_0 has at least two generators (the elements which belong to the set $L_0 \setminus L_0^2$).

For the completeness of the statement below we present the classifications of the papers [4], [5] and [7].

$Leib_{1,m}$:

$$\{ \ [y_i, x_1] = y_{i+1}, \quad 1 \leq i \leq m - 1,$$

$Leib_{n,1}$:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n - 1, \\ [y_1, y_1] = \alpha x_n, & \alpha = \{0, \ 1\}, \end{cases}$$

$Leib_{2,2}$:

$$\begin{cases} [y_1, x_1] = y_2, \\ [x_1, y_1] = \dfrac{1}{2} y_2, \\ [x_2, y_1] = y_2, \\ [y_1, x_2] = 2y_2, \\ [y_1, y_1] = x_2, \end{cases} \qquad \begin{cases} [y_1, x_1] = y_2, \\ [x_2, y_1] = y_2, \\ [y_1, x_2] = 2y_2, \\ [y_1, y_1] = x_2, \end{cases}$$

$Leib_{2,m}$:

$$\begin{cases} [x_1, x_1] = x_2, & m \geq 3 \\ [y_i, x_1] = y_{i+1}, & 1 \leq i \leq m - 1, \\ [x_1, y_i] = -y_{i+1}, & 1 \leq i \leq m - 1, \\ [y_i, y_{m+1-i}] = (-1)^{i+1} x_2, & 1 \leq i \leq m - 1. \end{cases} \qquad \begin{cases} [y_i, x_1] = -[x_1, y_i] = y_{i+1}, & 1 \leq i \leq m - 1, \\ [y_{m+1}, y_i] = (-1)^{i+1} x_2, & 1 \leq i \leq \frac{m+1}{2}, \end{cases}$$

In order to present the classification of Leibniz superalgebras with characteristic sequence $(n - 1, 1 | m)$, $n \geq 3$ and nilindex $n + m$ we need to introduce the following families of superalgebras:

$Leib_{n,n-1}$:

$L(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta):$

$$
\begin{cases}
[x_1, x_1] = x_3, & \\
[x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\
[y_j, x_1] = y_{j+1}, & 1 \le j \le n-2, \\
[x_1, y_1] = \frac{1}{2} y_2, & \\
[x_i, y_1] = \frac{1}{2} y_i, & 2 \le i \le n-1, \\
[y_1, y_1] = x_1, & \\
[y_j, y_1] = x_{j+1}, & 2 \le j \le n-1, \\
[x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \ldots + \alpha_{n-1} x_{n-1} + \theta x_n, & \\
[x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \ldots + \alpha_{n+2-j} x_n, & 2 \le j \le n-2, \\
[y_1, x_2] = \alpha_4 y_3 + \alpha_5 y_4 + \ldots + \alpha_{n-1} y_{n-2} + \theta y_{n-1}, & \\
[y_j, x_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \ldots + \alpha_{n+1-j} y_{n-1}, & 2 \le j \le n-3.
\end{cases}
$$

$G(\beta_4, \beta_5, \ldots, \beta_n, \gamma):$

$$
\begin{cases}
[x_1, x_1] = x_3, & \\
[x_i, x_1] = x_{i+1}, & 3 \le i \le n-1, \\
[y_j, x_1] = y_{j+1}, & 1 \le j \le n-2, \\
[x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \ldots + \beta_n x_n, & \\
[x_2, x_2] = \gamma x_n, & \\
[x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \ldots + \beta_{n+2-j} x_n, & 3 \le j \le n-2, \\
[y_1, y_1] = x_1, & \\
[y_j, y_1] = x_{j+1}, & 2 \le j \le n-1, \\
[x_1, y_1] = \frac{1}{2} y_2, & \\
[x_i, y_1] = \frac{1}{2} y_i, & 3 \le i \le n-1, \\
[y_j, x_2] = \beta_4 y_{j+2} + \beta_5 y_{j+3} + \ldots + \beta_{n+1-j} y_{n-1}, & 1 \le j \le n-3.
\end{cases}
$$

$$\mathbf{Leib_{n,n}}:$$

$M(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta, \tau):$

$$
\begin{cases}
[x_1, x_1] = x_3, & \\
[x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\
[y_j, x_1] = y_{j+1}, & 1 \le j \le n-1, \\
[x_1, y_1] = \frac{1}{2} y_2, & \\
[x_i, y_1] = \frac{1}{2} y_i, & 2 \le i \le n, \\
[y_1, y_1] = x_1, & \\
[y_j, y_1] = x_{j+1}, & 2 \le j \le n-1, \\
[x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \ldots + \alpha_{n-1} x_{n-1} + \theta x_n, & \\
[x_2, x_2] = \gamma_4 x_4, & \\
[x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \ldots + \alpha_{n+2-j} x_n, & 3 \le j \le n-2, \\
[y_1, x_2] = \alpha_4 y_3 + \alpha_5 y_4 + \ldots + \alpha_{n-1} y_{n-2} + \theta y_{n-1} + \tau y_n, & \\
[y_2, x_2] = \alpha_4 y_4 + \alpha_5 y_4 + \ldots + \alpha_{n-1} y_{n-1} + \theta y_n, & \\
[y_j, x_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \ldots + \alpha_{n+2-j} y_n, & 3 \le j \le n-2.
\end{cases}
$$

$H(\beta_4, \beta_5, \ldots, \beta_n, \delta, \gamma):$

$$
\begin{cases}
[x_1, x_1] = x_3, & \\
[x_i, x_1] = x_{i+1}, & 3 \leq i \leq n-1, \\
[y_j, x_1] = y_{j+1}, & 1 \leq j \leq n-2, \\
[x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \ldots + \beta_n x_n, & \\
[x_2, x_2] = \gamma x_n, & \\
[x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \ldots + \beta_{n+2-j} x_n, & 3 \leq j \leq n-2, \\
[y_1, y_1] = x_1, & \\
[y_j, y_1] = x_{j+1}, & 2 \leq j \leq n-1, \\
[x_1, y_1] = \frac{1}{2} y_2, & \\
[x_i, y_1] = \frac{1}{2} y_i, & 3 \leq i \leq n-1, \\
[y_1, x_2] = \beta_4 y_3 + \beta_5 y_4 + \ldots + \beta_n y_{n-1} + \delta y_n, & \\
[y_j, x_2] = \beta_4 y_{j+2} + \beta_5 y_{j+3} + \ldots + \beta_{n+2-j} y_n, & 2 \leq j \leq n-2.
\end{cases}
$$

Let us introduce also the following operators which act on k-dimensional vectors:

$$V_{j,k}^m(\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, \overset{j-1}{0}, 1, S_{m,j}^{j+1}\alpha_{j+1}, S_{m,j}^{j+2}\alpha_{j+2}, \ldots S_{m,j}^{k-1}\alpha_{k-1}, S_{m,j}^k\alpha_k),$$

$$V_{k+1,k}^m(\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, 0),$$

$$W_{s,k}^m(0, 0, \ldots, \overset{j-1}{0}, \overset{j}{1}, S_{m,j}^{j+1}\alpha_{j+1}, S_{m,j}^{j+2}\alpha_{j+2}, \ldots, S_{m,j}^k\alpha_k, \gamma) =$$

$$= (0, 0, \ldots, \overset{j}{1}, 0, \ldots, \overset{s+j}{1}, S_{m,s}^{s+1}\alpha_{s+j+1}, S_{m,s}^{s+2}\alpha_{s+j+2}, \ldots, S_{m,s}^{k-j}\alpha_k, S_{m,s}^{k+6-2j}\gamma),$$

$$W_{k+1-j,k}^m(0, 0, \ldots, \overset{j-1}{0}, \overset{j}{1}, S_{m,j}^{j+1}\alpha_{j+1}, S_{m,j}^{j+2}\alpha_{j+2}, \ldots, S_{m,j}^k\alpha_k, \gamma) =$$

$$= (0, 0, \ldots, \overset{j}{1}, 0, \ldots, 1),$$

$$W_{k+2-j,k}^m(0, 0, \ldots, \overset{j-1}{0}, \overset{j}{1}, S_{m,j}^{j+1}\alpha_{j+1}, S_{m,j}^{j+2}\alpha_{j+2}, \ldots, S_{m,j}^k\alpha_k, \gamma) =$$

$$= (0, 0, \ldots, \overset{j}{1}, 0, \ldots, 0),$$

where $k \in N$, $1 \leq j \leq k$, $1 \leq s \leq k-j$, $S_{m,t} = \cos\dfrac{2\pi m}{t} + i\sin\dfrac{2\pi m}{t}$ ($m = 0, 1, \ldots, t-1$).

Below we present the complete list of pairwise non-isomorphic Leibniz superalgebras with characteristic sequence equal to $(n-1, 1|m)$ and nilindex $n+m$:

$L\left(V_{j,n-3}(\alpha_4, \alpha_5, \ldots, \alpha_n), S_{m,j}^{n-3}\theta\right),$ $\quad 1 \leq j \leq n-3,$

$L(0, 0, \ldots, 0, 1),$ $L(0, 0, \ldots, 0),$ $G(0, 0, \ldots, 0, 1),$ $G(0, 0, \ldots, 0),$

$G\left(W_{s,n-2}(V_{j,n-3}(\beta_4, \beta_5, \ldots, \beta_n), \gamma)\right),$ $\quad 1 \leq j \leq n-3,\ 1 \leq s \leq n-j,$

$M\left(V_{j,n-2}(\alpha_4, \alpha_5, \ldots, \alpha_n), S_{m,j}^{n-3}\theta\right),$ $\quad 1 \leq j \leq n-2,$

$M(0, 0, \ldots, 0, 1),$ $M(0, 0, \ldots, 0),$ $H(0, 0, \ldots, 0, 1),$ $H(0, 0, \ldots, 0)$

$H\left(W_{s,n-1}(V_{j,n-2}(\beta_4, \beta_5, \ldots, \beta_n), \gamma)\right),$ $\quad 1 \leq j \leq n-2,\ 1 \leq s \leq n+1-j,$

where omitted products are equal to zero.

Definition 2.4. *For a given Leibniz algebra A of the nilindex s we put $gr(A)_i = A^i/A^{i+1}$, $1 \leq i \leq s-1$ and $gr(A) = gr(A)_1 \oplus gr(A)_2 \oplus \cdots \oplus gr(A)_{s-1}$. Then $[gr(A)_i, gr(A)_j] \subseteq gr(A)_{i+j}$ and we obtain the graded algebra $gr(A)$. The gradation constructed in this way is called the natural gradation and if Leibniz algebra A is isomorphic to $gr(A)$ we say that the algebra A is naturally graded Leibniz algebra.*

Further we shall consider the basis of even part L_0 of the superalgebra L which corresponds with the natural gradation, i.e. $\{x_1, \ldots, x_{t_1}\} = L_0 \backslash L_0^2$, $\{x_{t_1+1}, \ldots, x_{t_2}\} = L_0^2 \setminus L_0^3, \ldots, \{x_{t_{s-2}+1}, \ldots, x_n\} = L_0^{s-1}$.

Since the second part of the characteristic sequence of a Leibniz superalgebra L is equal to m then there exists a nilpotent endomorphism R_x $(x \in L_0 \setminus L_0^2)$ of the space L_1 such that its Jordan form consists of one Jordan block. Therefore, we can assume the existence of an adapted basis $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ such that

$$[y_j, x_1] = y_{j+1}, \quad 1 \leq j \leq m-1. \tag{1}$$

3. The main result

Let L be a Leibniz superalgebra with characteristic sequence $(n_1, \ldots, n_k|m)$, $n_1 \leq n-2$ and let $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ be the adapted basis of L. In this section we shall prove that nilindex of such superalgebra is less than $n+m$. According to the Theorem 2.1 we have a description of single-generated Leibniz superalgebras, which have nilindex $n+m+1$. If the number of generators is greater than two, then evidently superalgebra has nilindex less than $n+m$. Therefore, we should be consider case of two generators.

Note that, the case where both generators lie in even part is not possible (since $m \neq 0$). The equality (1) implies that basic elements $y_2, y_3, \ldots, y_m$ can not to be generators. Therefore, the first generator belongs to L_0 and the second one lies in L_1. Moreover, without loss of generality we can suppose that y_1 is a generator. Let us find the generator of Leibniz superalgebra L which lies in L_0.

Lemma 3.1. *Let $L = L_0 \oplus L_1$ be a two generated Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n_1, \ldots, n_k|m)$. Then x_1 and y_1 can be chosen as generators of the L.*

Proof. As mentioned above y_1 can be chosen as first generator of L. If $x_1 \in L \setminus L^2$, then the assertion of the lemma is evident. If $x_1 \in L^2$, then there exists some i_0 $(2 \leq i_0 \leq t_1)$ such that $x_{i_0} \in L \setminus L^2$. Put $x_1' = Ax_1 + x_{i_0}$, then x_1' is a generator of the superalgebra L (since $x_1' \in L \setminus L^2$). Moreover, making transformation of the basis of L_1 as follows

$$y_1' = y_1, \quad y_j' = [y_{j-1}', x_1'], \quad 2 \leq j \leq m$$

and taking sufficiently big value of the parameter A we preserve the equality (1). Thus, in the basis $\{x_1', x_2, \ldots, x_n, y_1', y_2', \ldots, y_m'\}$ of the L the elements x_1' and y_1' are generators. $\square$

Due to Lemma 3.1 further we shall suppose that $\{x_1, y_1\}$ are generators of the Leibniz superalgebra L.

Let us introduce the notations:

$$[x_i, y_1] = \sum_{j=2}^m \alpha_{i,j} y_j, \ 1 \leq i \leq n, \quad [y_i, y_1] = \sum_{j=2}^n \beta_{i,j} x_j, \ 1 \leq i \leq m. \tag{2}$$

Since x_1 and y_1 are generators of Leibniz superalgebra L, we have

$$L = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\},$$

$$L^2 = \{x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_m\}.$$

If we consider the next power of L, then from the multiplication (1), obviously, we get $\{y_3, \ldots, y_m\} \subseteq L^3$. However, do not have information about the position of the element y_2.

Theorem 3.1. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n_1, \ldots, n_k | m)$ and let $y_2 \notin L^3$. Then L has a nilindex less than $n + m$.*

Proof. Let us assume the contrary, i.e. nilindex of the superalgebra L is equal to $n + m$. Then the condition $y_2 \notin L^3$ deduce $\{x_2, x_3, \ldots, x_n\} \subseteq L^3$. Therefore,

$$L^3 = \{x_2, x_3, \ldots, x_n, y_3, \ldots, y_m\}.$$

Let $s \in \mathbb{N}$ be a number such that $x_2 \in L^s \setminus L^{s+1}$, that is

$$L^s = \{x_2, x_3, \ldots, x_n, y_s, \ldots, y_m\}, \ s \geq 3,$$

$$L^{s+1} = \{x_3, x_4, \ldots, x_n, y_s, \ldots, y_m\}.$$

It means that x_2 can be obtained only from product $[y_{s-1}, y_1]$ and thereby $\beta_{s-1,2} \neq 0$.

Similarly, we assume that k is a number for which $x_3 \in L^{s+k} \setminus L^{s+k+1}$. Then for the powers of superalgebra L we have the following

$$L^{s+k} = \{x_3, x_4, \ldots, x_n, y_{s+k-1}, \ldots, y_m\}, k \geq 1,$$

$$L^{s+k+1} = \{x_4, \ldots, x_n, y_{s+k-1}, \ldots, y_m\}.$$

Let us suppose $k = 1$, then

$$L^{s+2} = \{x_4, \ldots, x_n, y_s, \ldots, y_m\}.$$

Since $x_3 \notin L^{s+2}$ and the vector space L^{s+1} is generated by multiplying the space L^s to elements x_1 and y_1 on the right side (because of Leibniz superidentity), it follows that element x_3 is obtained by the product $[x_2, x_1]$, i.e. $[x_2, x_1] = ax_3 + \sum_{i \geq 4}(*)x_i$, $a \neq 0$. Making the change of basic element as $x_3' = ax_3 + \sum_{i \geq 4}(*)x_i$, we can conclude that $[x_2, x_1] = x_3$.

Let us define the products $[y_{s-j}, y_j], 1 \leq j \leq s - 1$.

Applying the Leibniz superidentity and induction on j we prove

$$[y_{s-j}, y_j] = (-1)^{j+1}\beta_{s-1,2}x_2 + \sum_{i \geq 3}(*)x_i. \tag{3}$$

Indeed, for $j = 1$ it is true by notation (2). Let us suppose that equality (3) holds for $j = t$. Then for $j = t + 1$ we have

$$[y_{s-t-1}, y_{t+1}] = [y_{s-t-1}, [y_t, x_1]] = [[y_{s-t-1}, y_t], x_1] - [[y_{s-t-1}, x_1], y_t] =$$

$$= -[y_{s-t}, y_t] + [\sum_{i \geq 2}(*)x_i, x_1] = -(-1)^{t+1}\beta_{s-1,2}x_2 + \sum_{i \geq 3}(*)x_i +$$

$$+[\sum_{i \geq 2}(*)x_i, x_1] = (-1)^{t+2}\beta_{s-1,2}x_2 + \sum_{i \geq 3}(*)x_i.$$

It should be noted that in the decompositions of $[y_s, y_1]$ and $[y_1, y_s]$ the coefficients at the basic elements x_2 and x_3 are equal to zero.

Let us define the products $[y_{s+1-j}, y_j], 2 \leq j \leq s$. In fact, if $j = 2$, then

$$[y_{s-1}, y_2] = [y_{s-1}, [y_1, x_1]] = [[y_{s-1}, y_1], x_1] - [y_s, y_1] =$$

$$= [\sum_{i=2}^n \beta_{s-1,i} x_i, x_1] - \sum_{i=4}^n \beta_{s,i} x_i = \beta_{s-1,2} x_3 + \sum_{i \geq 4}(*) x_i,$$

Inductively applying the above arguments for $j \geq 3$ and using equality (3) we obtain

$$[y_{s+1-j}, y_j] = (-1)^j (j-1) \beta_{s-1,2} x_3 + \sum_{i \geq 4}(*) x_i, \quad 2 \leq j \leq s. \tag{4}$$

In particular, $[y_1, y_s] = (-1)^s (s-1) \beta_{s-1,2} x_3 + \sum_{i \geq 4}(*) x_i$. On the other hand (as mentioned above), in the decompositions of $[y_1, y_s]$ the coefficient at the basic element x_3 is equal to zero. Therefore, $\beta_{s-1,2} = 0$, which contradicts to the condition $\beta_{s-1,2} \neq 0$. Thus, our assumption $k = 1$ is not possible.

Hence, $k \geq 2$ and we have

$$L^{s+2} = \{x_3, \ldots, x_n, y_{s+1}, \ldots, y_m\}.$$

Since $y_s \notin L^{s+2}$, it follows that

$$\alpha_{2,s} \neq 0, \quad \alpha_{2,j} = 0 \quad \text{for} \quad j < s,$$

$$\alpha_{i,j} = 0, \quad \text{for any} \quad i \geq 3, \quad j < s+1.$$

Consider the product

$$[[y_{s-1}, y_1], y_1] = \frac{1}{2}[y_{s-1}, [y_1, y_1]] = \frac{1}{2}[y_{s-1}, \sum_{i=2}^n \beta_{1,i} x_i].$$

The element y_{s-1} belongs to L^{s-1} and elements $x_2, x_3, \ldots, x_n$ lie in L^3. Hence $\frac{1}{2}[y_{s-1}, \sum_{i=2}^n \beta_{1,i} x_i] \in L^{s+2}$. Since $L^{s+2} = \{x_3, \ldots, x_n, y_{s+1}, \ldots, y_m\}$, we obtain that $[[y_{s-1}, y_1], y_1] = \sum_{j \geq s+1}(*) y_j$.

On the other hand,

$$[[y_{s-1}, y_1], y_1] = [\sum_{i=2}^n \beta_{s-1,i} x_i, y_1] = \sum_{i=2}^n \beta_{s-1,i}[x_i, y_1] =$$

$$= \sum_{i=2}^n \beta_{s-1,i} \sum_{j \geq s} \alpha_{i,j} y_j = \beta_{s-1,2} \alpha_{2,s} y_s + \sum_{j \geq s+1}(*) y_j.$$

Comparing the coefficients at the basic elements we obtain $\beta_{s-1,2} \alpha_{2,s} = 0$, which contradicts the conditions $\beta_{s-1,2} \neq 0$ and $\alpha_{2,s} \neq 0$.

Thus, we get a contradiction with assumption that the superalgebra L has nilindex equal to $n + m$ and therefore the assertion of the theorem holds. $\qquad\square$

The investigation of the case $y_2 \in L^3$ shows that it depends on the structure of the Leibniz algebra L_0. So, we present some remarks on naturally graded nilpotent Leibniz algebras.

Let $A = \{z_1, z_2, \ldots, z_n\}$ be an n-dimensional nilpotent Leibniz algebra of nilindex p $(p < n)$. Note that algebra A is not single-generated.

Consider the case where $gr(A)$ is a non-Lie Leibniz algebra.

Lemma 3.2. *Let $gr(A)$ be a naturally graded non-Lie Leibniz algebra. Then $dim(gr(A)_1) + dim(gr(A)_2) \geq 4$.*

Proof. The construction of $gr(A)$ implies that every subspace $gr(A)_i$ for $1 \leq i \leq p - 1$ is not empty. Obviously, $dim(gr(A)_1) \geq 2$ (otherwise $p = n + 1$). If the dimension of subspace $gr(A)_1$ is greater than two, then the statement of the lemma is true. If $dim(gr(A)_1) = 2$ and $dim(gr(A)_2) = 2$, then the assertion of the lemma is evident.

Let us suppose that $dim(gr(A)_1) = 2$ and $dim(gr(A)_2) = 1$. Then taking into account the condition $p < n$ we conclude that there exists some t $(t > 2)$ such that $dim(gr(A)_t) \geq 2$ (otherwise the nilindex equals n).

Let t_0 $(t_0 > 2)$ be the smallest number with condition $dim(gr(A)_{t_0}) \geq 2$. Then

$$gr(A)_1 = \{\overline{z}_1, \overline{z}_2\},\ gr(A)_2 = \{\overline{z}_3\},\ \ldots,\ gr(A)_{t_0-1} = \{\overline{z}_{t_0}\},\ gr(A)_{t_0} = \{\overline{z}_{t_0+1}, \overline{z}_{t_0+2}\}.$$

Using the argumentation similar to the one from [10] we obtain that

$$\begin{cases} [\overline{z}_1, \overline{z}_1] = \alpha_1 \overline{z}_3, \\ [\overline{z}_2, \overline{z}_1] = \alpha_2 \overline{z}_3, \\ [\overline{z}_1, \overline{z}_2] = \alpha_3 \overline{z}_3, \\ [\overline{z}_2, \overline{z}_2] = \alpha_4 \overline{z}_3, \\ [\overline{z}_i, \overline{z}_1] = \overline{z}_{i+1}, \quad 3 \leq i \leq t_0. \end{cases}$$

Since $gr(A)$ is non-Lie Leibniz algebra then there exists an element of $gr(A)_1$ the square of which is non zero. It is not difficult to see that $\overline{z}_3, \ldots, \overline{z}_{t_0+1}$ belong to the right annihilator $\Re(gr(A))$, which is defined as

$$\Re(gr(A)) = \{\overline{z} \in gr(A)|\ [\overline{y}, \overline{z}] = 0 \text{ for any } \overline{y} \in gr(A)\}.$$

Moreover, one can assume $[\overline{z}_{t_0}, \overline{z}_2] = \overline{z}_{t_0+2}$.

On the other hand,

$$[\overline{z}_{t_0}, \overline{z}_2] = [[\overline{z}_{t_0-1}, \overline{z}_1], \overline{z}_2] = [\overline{z}_{t_0-1}, [\overline{z}_1, \overline{z}_2]] + [[\overline{z}_{t_0-1}, \overline{z}_2], \overline{z}_1] =$$
$$= [\overline{z}_{t_0-1}, \alpha_3 \overline{z}_3] + [\beta \overline{z}_{t_0}, \overline{z}_1] = \beta \overline{z}_{t_0+1}.$$

The obtained equality $\overline{z}_{t_0+2} = \beta_{t_0-1,2} \overline{z}_{t_0+1}$ derives a contradiction, which leads to the assertion of the lemma. $\qquad \square$

From the Lemma 3.2 the corollary follows.

Corollary 3.1. *Let A be a Leibniz algebra satisfying the condition of the Lemma 3.2. Then $dim(A^3) \leq n - 4$.*

The result on nilindex of the superalgebra under condition $dim(L_0^3) \leq n - 4$ is established in the following proposition.

Proposition 3.1. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence $(n_1, \ldots, n_k|m)$ and $dim(L_0^3) \leq n - 4$. Then L has a nilindex less than $n + m$.*

Proof. Let us assume the contrary, i.e. the nilindex of the superalgebra L is equal to $n + m$. According to the Theorem 3.1 we need to consider the case where y_2 belongs to L^3, which leads to $x_2 \notin L^3$. Thus, we have

$$L^3 = \{x_3, x_4, \ldots, x_n, y_2, y_3, \ldots, y_m\}.$$

From the condition $dim(L_0^3) \leq n - 4$ it follows that there exist at least two basic elements, that belong to $L_0^2 \setminus L_0^3$. Without loss of generality, one can assume $x_3, x_4 \in L_0^2 \setminus L_0^3$.

Let s be a natural number such that $x_3 \in L^{s+1} \setminus L^{s+2}$,

$$L^{s+1} = \{x_3, x_4, \ldots, x_n, y_s, y_{s+1}, \ldots, y_m\}, \ s \geq 2,$$

$$L^{s+2} = \{x_4, \ldots, x_n, y_s, y_{s+1}, \ldots, y_m\}.$$

The condition $x_4 \notin L_0^3$ implies that x_4 can not be obtained by the products $[x_i, x_1]$, with $3 \leq i \leq n$. Therefore, it is generated by products $[y_j, y_1], s \leq j \leq m$. Hence, $L^{s+3} = \{x_4, \ldots, x_n, y_{s+1}, \ldots, y_m\}$ and $y_s \in L^{s+2} \setminus L^{s+3}$, which implies $\alpha_{3,s} \neq 0$.

Consider the chain of equalities

$$[[x_3, y_1], y_1] = [\sum_{j=s}^{m} \alpha_{3,j} y_j, y_1] = \sum_{j=s}^{m} \alpha_{3,j}[y_j, y_1] = \alpha_{3,s}\beta_{s,4}x_4 + \sum_{i \geq 5}(*)x_i.$$

On the other hand,

$$[[x_3, y_1], y_1] = \frac{1}{2}[x_3, [y_1, y_1]] = [x_3, \sum_{i=2}^{n} \beta_{1,i}x_i] = \sum_{i=2}^{n} \beta_{1,i}[x_3, x_i] = \sum_{i \geq 5}(*)x_i.$$

Comparing the coefficients at the corresponding basic elements in these equations we get $\alpha_{3,s}\beta_{s,4} = 0$, which implies $\beta_{s,4} = 0$. Thus, we conclude that $x_4 \in L^{s+4}$ and

$$L^{s+4} = \{x_4, \ldots, x_n, y_{s+2}, \ldots, y_m\}.$$

Let k $(4 \leq k \leq m - s + 2)$ be a natural number such that $x_4 \in L^{s+k} \setminus L^{s+k+1}$. Then for the powers of descending lower sequences we have

$$L^{s+k-2} = \{x_4, \ldots, x_n, y_{s+k-4}, \ldots, y_m\},$$

$$L^{s+k-1} = \{x_4, \ldots, x_n, y_{s+k-3}, \ldots, y_m\},$$

$$L^{s+k} = \{x_4, \ldots, x_n, y_{s+k-2}, \ldots, y_m\},$$

$$L^{s+k+1} = \{x_5, \ldots, x_n, y_{s+k-2}, \ldots, y_m\}.$$

It is easy to see that in the decomposition $[y_{s+k-3}, y_1] = \sum_{i=4}^{n} \beta_{s+k-3,i}x_i$ we have $\beta_{s+k-3,4} \neq 0$.

Consider the equalities

$$[y_{s+k-4}, y_2] = [y_{s+k-4}, [y_1, x_1]] = [[y_{s+k-4}, y_1], x_1] - [[y_{s+k-4}, x_1], y_1] =$$

$$= [\sum_{i=3}^{n} \beta_{s+k-3,i}x_i, x_1] - [y_{s+k-3}, y_1] = -\beta_{s+k-3,4}x_4 + \sum_{i \geq 5}(*)x_i.$$

Since $y_{s+k-4} \in L^{s+k-2}, y_2 \in L^3$ and $\beta_{s+k-3,4} \neq 0$, then the element x_4 should lie in L^{s+k+1}, but it contradicts to $L^{s+k+1} = \{x_5, \ldots, x_n, y_{s+k-2}, \ldots, y_m\}$. Thus, the superalgebra L has a nilindex less than $n + m$. $\qquad\square$

From Proposition 3.1 we conclude that Leibniz superalgebra $L = L_0 \oplus L_1$ with the characteristic sequence $(n_1, \ldots, n_k | m)$ and nilindex $n + m$ can appear only if $dim(L_0^3) \geq n - 3$. Taking into account the condition $n_1 \leq n - 2$ and properties of naturally graded subspaces $gr(L_0)_1$ and $gr(L_0)_2$ we get $dim(L_0^3) = n - 3$.

Let $dim(L_0^3) = n - 3$. Then

$$gr(L_0)_1 = \{\overline{x}_1, \overline{x}_2\}, \ gr(L_0)_2 = \{\overline{x}_3\}.$$

Then, by Corollary 3.1 the naturally graded Leibniz algebra $gr(L_0)$ is a Lie algebra, i.e. the following multiplication rules are true

$$\begin{cases} [\overline{x}_1, \overline{x}_1] = 0, \\ [\overline{x}_2, \overline{x}_1] = \overline{x}_3, \\ [\overline{x}_1, \overline{x}_2] = -\overline{x}_3, \\ [\overline{x}_2, \overline{x}_2] = 0. \end{cases}$$

Using these products for the corresponding products in the Leibniz algebra L_0 with the basis $\{x_1, x_2, \ldots, x_n\}$ we have

$$\begin{cases} [x_1, x_1] = \gamma_{1,4} x_4 + \gamma_{1,5} x_5 + \cdots + \gamma_{1,n} x_n, \\ [x_2, x_1] = x_3, \\ [x_1, x_2] = -x_3 + \gamma_{2,4} x_4 + \gamma_{2,5} x_5 + \cdots + \gamma_{2,n} x_n, \\ [x_2, x_2] = \gamma_{3,4} x_4 + \gamma_{3,5} x_5 + \cdots + \gamma_{3,n} x_n. \end{cases} \tag{5}$$

Proposition 3.2. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence $(n_1, \ldots, n_k | m)$ and $dim(L_0^3) = n - 3$. Then L has a nilindex less than $n + m$.*

Proof. Let us suppose the contrary, i.e. the nilindex of the superalgebra L equals $n + m$. Then by Theorem 3.1 we can assume $x_2 \notin L^3$. Hence,

$$L^2 = \{x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_m\},$$

$$L^3 = \{x_3, x_4, \ldots, x_n, y_2, y_3, \ldots, y_m\}.$$

If $y_2 \in L^4$, then it should be generated from the products $[x_i, y_1], 3 \leq i \leq n$, but elements $x_i, (3 \leq i \leq n)$ are in L_0^2. Therefore, they are generated by linear combinations of products of elements from L_0. The equalities

$$[[x_i, x_j], y_1] = [x_i, [x_j, y_1]] + [[x_i, y_1], x_j] = [x_i, \sum_{t=2}^{m} \alpha_{j,t} y_t] + [\sum_{t=2}^{m} \alpha_{i,t} y_t, x_i] = \sum_{t \geq 3} (*) y_t$$

show that the element y_2 can not be obtained by the products $[x_i, y_1], 3 \leq i \leq n$, i.e. $y_2 \notin L^4$. Thus, we have

$$L^4 = \{x_3, x_4, \ldots, x_n, y_3, \ldots, y_m\}.$$

The simple analysis of descending lower sequences L^3 and L^4 derives $\alpha_{2,2} \neq 0$.

Let s be a natural number such that $x_3 \in L^{s+1} \setminus L^{s+2}$, i.e.

$$L^s = \{x_3, x_4, \ldots, x_n, y_{s-1}, y_s, \ldots, y_m\}, s \geq 3,$$

$$L^{s+1} = \{x_3, x_4, \ldots, x_n, y_s, y_{s+1}, \ldots, y_m\},$$

$$L^{s+2} = \{x_4, \ldots, x_n, y_s, y_{s+1}, \ldots, y_m\} \ \text{and} \ \beta_{s-1,3} \neq 0.$$

If $s = 3$, then $\beta_{2,3} \neq 0$ and we consider the product

$$[[x_2, y_1], y_1] = [\sum_{j=2}^{m} \alpha_{2,j} y_j, y_1] = \sum_{j=2}^{m} \alpha_{2,j}[y_j, y_1] = \alpha_{2,2}\beta_{2,3}x_3 + \sum_{i \geq 4}(*)x_4.$$

On the other hand,

$$[[x_2, y_1], y_1] = \frac{1}{2}[x_2, [y_1, y_1]] = \frac{1}{2}[x_2, \sum_{i=2}^{n} \beta_{1,i}x_i] = \sum_{i \geq 4}(*)x_i.$$

Comparing the coefficients at the corresponding basic elements we get equality $\alpha_{2,2}\beta_{2,3} = 0$, i.e. we have a contradiction with supposition $s = 3$.

If $s \geq 4$, then consider the chain of equalities

$$[y_{s-2}, y_2] = [y_{s-2}, [y_1, x_1]] = [[y_{s-2}, y_1], x_1] - [[y_{s-2}, x_1], y_1] =$$

$$= [\sum_{i=3}^{n} \beta_{s-2,i}x_i, x_1] - [y_{s-1}, y_1] = -\beta_{s-1,3}x_3 + \sum_{i \geq 4}(*)x_i.$$

Since $y_{s-2} \in L^{s-1}$ and $y_2 \in L^3$ then $x_3 \in L^{s+2} = \{x_4, \ldots, x_n, y_{s-1}, \ldots, y_m\}$, which is a contradiction with the assumption that the nilindex of L is equal to $n + m$. □

Summarizing the Theorem 3.1 and Propositions 3.1, 3.2 we have the following result

Theorem 3.2. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n_1, \ldots, n_k|m)$. Then the nilindex of the Leibniz superalgebra L is less than $n + m$.*

REFERENCES

[1] Albeverio S., Ayupov Sh.A., Omirov B.A. *On nilpotent and simple Leibniz algebras.* Comm. in Algebra, 33(1), (2005), p. 159-172.
[2] Ayupov Sh.A., Omirov B.A. *On some classes of nilpotent Leibniz algebras*, Sib. Math. J., vol. 42(1), 2001, p.18-29.
[3] Bordemann M., Gómez J.R., Khakimdjanov Yu., Navarro R.M. *Some deformations of nilpotent Lie superalgebras*, J. Geom. and Phys., 57, (2007), p. 1391-1403
[4] Ayupov Sh. A., Khudoyberdiyev A. Kh., Omirov B. A. *The classification of filiform Leibniz superalgebras of nilindex n+m*, Acta Math.Sinica (english series), 25(2), 2009, p. 171-190.
[5] Camacho L.M., Gómez J.R., Omirov B.A, Khudoyberdiyev A.Kh *On complex Leibniz superalgebras of nilindex n+m*, submitted to J. Geom. and Phys., arXiv:math/0812.2156.
[6] Gilg M. *Super-algèbres de Lie nilpotentes:* PhD thesis. University of Haute Alsace, 2000. – 126 p.
[7] Gómez J.R., Khakimdjanov Yu., Navarro R.M. *Some problems concerning to nilpotent Lie superalgebras*, J. Geom. and Phys., 51(4), (2004), p. 473-486.
[8] Livernet M. *Rational homotopy of Leibniz algebras*, Manuscripta Mat., vol. 96, 1998, p. 295-315.
[9] Loday J.-L. *Une version non commutative des algébres de Lie: les algébres de Leibniz*, Ens. Math., 39, (1993), p. 269-293.
[10] Vergne M. *Cohomologie des de Lie nilpotentes.* Application de Lie nilpotentes. Bull. Soc. Math. France. (1970), vol. 98, p. 81-116.

[L.M. Camacho – J.R. Gómez] Dpto. Matemática Aplicada I. Universidad de Sevilla. Avda. Reina Mercedes, s/n. 41012 Sevilla. (Spain)
E-mail address: lcamacho@us.es --- jrgomez@us.es

[A.Kh. Khudoyberdiyev – B.A. Omirov] Institute of Mathematics and Information Technologies of Academy of Uzbekistan, 29, F.Hodjaev srt., 100125, Tashkent (Uzbekistan)

E-mail address: `khabror@mail.ru --- omirovb@mail.ru`