This manuscript has been published in the Structural Safety, 2017, Vol. 67, pp.1-10.
doi.org/10.1016/j.strusafe.2017.03.005

A new unbiased metamodel method for efficient reliability analysis

Guofeng Xue[a,b], Hongzhe Dai[a,b,*], Hao Zhang[c]

[a]*School of Civil Engineering, Harbin Institute of Technology, Harbin 150090, China*
[b]*Key Lab of Structures Dynamic Behavior and Control (Harbin Institute of Technology), Ministry of Education, Harbin 150090, China*
[c]*School of Civil Engineering, the University of Sydney, NSW 2006, Australia*

Abstract

Metamodel method is widely used in structural reliability analysis. A main limitation of this method is that it is difficult or even impossible to quantify the model uncertainty caused by the metamodel approximation. This paper develops an improved metalmodel method which is unbiased and highly efficient. The new method formulates a probability of failure as a product of a metamodel-based probability of failure and a correction term, which accounts for the approximation error due to metamodel approximation. The correction term is constructed and estimated using the Markov chain simulation. An iterative scheme is further developed to adaptively improve the accuracy of the metamodel and the associated correction term. The accuracy and efficiency of the new metamodel method is illustrated and compared with the classical Kriging metamodel and high dimensional model representation methods using a number of numerical and structural examples.

Keywords: reliability, metamodelling, unbiased estimation, Markov chain simulation, adaptive refinement

*Corresponding author. Tel.: +86 451 86282075; E-mail address: hzdai@hit.edu.cn

This manuscript has been published in the Structural Safety, 2017, Vol. 67, pp.1-10.
doi.org/10.1016/j.strusafe.2017.03.005

1. Introduction

A common technique for evaluating structural reliabilities with complex limit state functions is to use the metamodel method. It uses a strategic design of experiments (DoE) to obtain an analytical approximation of the relationships between the input random variables and the limit state response of interest. Earlier application of this approach is the use of the response surface methods [1]. Construction of metalmodels is a challenging problem. Recent developments include but not limited to artificial neural networks [2–4], support vector machines [5–8], high dimensional model representation (HDMR) [9, 10], polynomial chaos expansion [11, 12] and Kriging [13, 14]. For the commonly used polynomial-based metamodel, the results may be sensitive to the selected interpolation polynomials and their parameters due to the rigid and non-adaptive structure of the polynomials [6]. For instance, although polynomial chaos can be used for local interpolation, the definitions of the design of numerical experiments and of the polynomial degrees are tricky [11]. The performance of artificial neural networks cannot be guaranteed due to the fitting problems as there is no efficient constructive method for choosing the structure and the learning parameters of artificial neural network [5]. In addition to these limitations, a general drawback of the metalmodel method is that it is difficult or even impossible to quantify the error caused by approximating the actual limit state function (LSF) by a metalmodel [15–17].

In order to overcome the aforementioned difficulties, this paper develops a new metamodel method which is unbiased and highly efficient. The basic

idea is to formulate an unknown probability of failure as the product of a metamodel-based failure probability and a correction term, which accounts for the approximation error due to metamodel approximation. Although this idea is mathematically straightforward and has been used in structural reliability analysis very recently [17, 18], the construction and the estimation of the correction term is a very challenging task in such methods. In this paper, the correction term is constructed by introducing an intermediate event, which is the union of the actual failure region and the metamodel-based failure region. The correction term is estimated efficiently using the Markov chain simulation. Furthermore, an adaptive refinement procedure is developed to simultaneously improve the metamodel and the corresponding correction term, to further improve the efficiency of the proposed method.

The paper is organized as follows: the Kriging metamodel is briefly introduced in Section 2, followed by the presentation of the proposed unbiased metamodel method in Section 3. The procedure of the proposed method is then summarized in Section 4. Three examples are then given to demonstrate the application and efficiency of the proposed method. Comparisons of the proposed method and the conventional metamodel methods, including Kriging metamodelling and high dimensional model representation, are made.

2. Kriging method

Among the available metamodel methods, herein we focus on the Kriging method, which has gained significant attention in the field of structural reliability theory in recent years [13, 14, 19]. It should be noted that the

proposed method of constructing and estimating the correction term is general and can be applied to any metalmodel method, and not restricted to the Kriging metamodel discussed here. This section briefly introduces the Kriging method for the completeness of introducing the proposed methodology. Details about Kriging method can be found elsewhere, e.g, [20, 21].

Kriging metamodel is an interpolation technique based on statistical theory, which consists of a parametric linear regression model and a nonparametric stochastic process [20]. It requires DoE to determine its stochastic parameters and then predictions of the response can be computed on any unknown sample. Given an initial DoE $\mathbf{X} = [\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(p)}]^{\mathrm{T}}$, with $\mathbf{x}^{(i)} \in \mathbb{R}^n$ $(i = 1, \ldots, p)$ the ith input, and $\mathbf{Y} = [g(\mathbf{x}^{(1)}), \ldots, g(\mathbf{x}^{(p)})]^{\mathrm{T}}$ with $g(\mathbf{x}^{(i)}) \in \mathbb{R}$ the corresponding response to $\mathbf{x}^{(i)}$. The approximate relationship between any sample $\mathbf{x}$ and the response $g(\mathbf{x})$ can be denoted as

$$g(\mathbf{x}) = F(\boldsymbol{\beta}, \mathbf{x}) + z(\mathbf{x}) = \mathbf{f}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\beta} + z(\mathbf{x}) \tag{1}$$

where $\mathbf{f}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\beta}$ is the regression model representing the trend of the model, which is defined by a set of basis functions $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})]^{\mathrm{T}}$ and the corresponding regression coefficients $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_m]^{\mathrm{T}}$. In the ordinary Kriging, $F(\boldsymbol{\beta}, \mathbf{x})$ is a scalar and always taken as $F(\boldsymbol{\beta}, \mathbf{x}) = \beta$. So the estimated $g(\mathbf{x})$ can be simplified as

$$g(\mathbf{x}) = \beta + z(\mathbf{x}). \tag{2}$$

Here $z(\mathbf{x})$ is a zero-mean stationary Gaussian process with autocovariance at points $\mathbf{x}$ and $\mathbf{w}$ defined as

$$\mathrm{cov}(z(\mathbf{x}), z(\mathbf{w})) = \sigma^2 R(\mathbf{x}, \mathbf{w}) \tag{3}$$

where cov = covariance, σ^2 is the process variance and $R(\mathbf{x}, \mathbf{w})$ is the autocorrelation function. The most widely used autocorrelation function is anisotropic Gaussian model and is adopted in this paper:

$$R(\mathbf{x}, \mathbf{w}) = \exp\left(-\sum_{i=1}^{n} \theta_i (x_i, w_i)^2\right) \tag{4}$$

where x_i and w_i are the ith component of the points $\mathbf{x}$ and $\mathbf{w}$ respectively, and θ_i is the correlation parameter in the ith dimension.

Define $\mathbf{R}$ as a $p \times p$ symmetric correlation matrix with $\mathbf{R}_{ij} = R(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$, $i, j = 1, \ldots, p$, and $\mathbf{F}$ as a $p \times 1$ unit vector, then β and σ^2 are estimated as

$$\hat{\beta} = \left(\mathbf{F}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{F}\right)^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{Y}, \tag{5}$$

$$\hat{\sigma}^2 = \frac{1}{p} \left(\mathbf{Y} - \mathbf{F}\boldsymbol{\beta}\right)^{\mathrm{T}} \mathbf{R}^{-1} \left(\mathbf{Y} - \mathbf{F}\boldsymbol{\beta}\right). \tag{6}$$

The correlation parameter θ can be obtained through the maximum likelihood estimation:

$$\boldsymbol{\theta} = \arg\min_{\boldsymbol{\theta}} (\det \mathbf{R})^{\frac{1}{p}} \hat{\sigma}^2. \tag{7}$$

Since there exists corresponding interpolation model for each θ, the best Kriging model can be obtained by optimizing θ.

Then at an unknown point $\mathbf{x}^{(0)}$, the Best Linear Unbiased Predictor (BLUP) of the response $\tilde{g}(\mathbf{x}^{(0)})$ and Kriging variance $\sigma^2_{\tilde{g}}(\mathbf{x}^{(0)})$ are computed as

$$\tilde{g}(\mathbf{x}^{(0)}) = \mathbf{q}^{\mathrm{T}}(\mathbf{x}^{(0)})\boldsymbol{\beta} + \mathbf{r}(\mathbf{x}^{(0)})^{\mathrm{T}} \mathbf{R}^{-1} (\mathbf{Y} - \mathbf{F}\boldsymbol{\beta}), \tag{8}$$

$$\sigma^2_{\tilde{g}}(\mathbf{x}^{(0)}) = \hat{\sigma}^2 \left(1 + u(\mathbf{x}^{(0)})^{\mathrm{T}} (\mathbf{F}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{F})^{-1} u(\mathbf{x}^{(0)}) - \mathbf{r}(\mathbf{x}^{(0)})^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{r}(\mathbf{x}^{(0)})\right) \tag{9}$$

where $\mathbf{r}(\mathbf{x}^{(0)}) = [R(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}), \ldots, R(\mathbf{x}^{(0)}, \mathbf{x}^{(p)})]^{\mathrm{T}}$ and $u(\mathbf{x}^{(0)}) = \mathbf{F}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{r}(\mathbf{x}^{(0)}) - 1$.

3. An unbiased metamodel method

Although some metamodels like Kriging can provide a measure of the local uncertainty of the prediction of new samples, i.e., Kriging variance, the overall error resulting from replacing the actual LSF with the metamodel cannot be quantified. This model uncertainty is the epistemic uncertainty of the metamodel. It cannot be quantified by the metamodel itself. As a consequence, the direct use of Kriging metamodel will inevitably result in a biased estimator of the probability of failure. Having identified this issue, we propose a correction term to quantify the bias of the metamodel-based failure probability, and formulate the unknown probability of failure as a product of the metamodel-based failure probability and a correction term. In this manner, the bias of the metamodel-based failure probability can be accounted for and an unbiased estimator of the failure probability is obtained.

Let $\tilde{g}(\mathbf{x})$ be a Kriging metamodel for the real LSF $g(\mathbf{x})$, and $\tilde{F} = \{\mathbf{x} \mid \tilde{g}(\mathbf{x}) \leq 0\}$ be the metamodel-based failure region for the real failure region $F = \{\mathbf{x} \mid g(\mathbf{x}) \leq 0\}$. The correction term, denoted by K, is defined as

$$K = \frac{P(F)}{P(\tilde{F})} \tag{10}$$

where $P(F)$ and $P(\tilde{F})$ is the failure probability and the metamodel-based failure probability, respectively. Then $P(F)$ can be written as

$$P(F) = K \cdot P(\tilde{F}). \tag{11}$$

Eq. (10) shows that the correction term K quantifies the error resulting from substituting $g(\mathbf{x})$ with $\tilde{g}(\mathbf{x})$, thus it can be used to consider the bias of the metamodel-based failure probability $P(\tilde{F})$ even a poor metamodel $\tilde{g}(\mathbf{x})$ is

employed. By multiplying $P(\tilde{F})$ with K, an unbiased estimator of $P(F)$ is achieved as shown in Eq. (11).

Clearly, the key issue of the method is the computation of the correction term K. Since one cannot guarantee that the metamodel-based failure region $\tilde{F}$ covers the real failure region F, sampling methods such as importance sampling or Markov chain simulation, cannot be used to estimate K. We introduce an intermediate event $F \cup \tilde{F}$ and reformulate the correction term in Eq. (10) as

$$K = \frac{P(F \cup \tilde{F})}{P(\tilde{F})} \frac{P(F)}{P(F \cup \tilde{F})} = \frac{1}{K_1} K_2 \tag{12}$$

where $K_1 = P(\tilde{F})/P(F \cup \tilde{F})$, and $K_2 = P(F)/P(F \cup \tilde{F})$. Therefore, K can be estimated provided that K_1 and K_2 are obtained.

3.1. Estimation of the correction term

By introducing an importance sampling density $h_{F \cup \tilde{F}}(\mathbf{x})$, $P(\tilde{F})$ is formulated as

$$P(\tilde{F}) = \int_{\mathbb{R}^n} \mathbb{I}_{\tilde{F}}(\mathbf{x}) f_X(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} \mathbb{I}_{\tilde{F}}(\mathbf{x}) \frac{f_X(\mathbf{x})}{h_{F \cup \tilde{F}}(\mathbf{x})} h_{F \cup \tilde{F}}(\mathbf{x}) d\mathbf{x} \tag{13}$$

where $\mathbb{I}_{\tilde{F}}(\mathbf{x})$ is the indicator function of $\tilde{F}$: $\mathbb{I}_{\tilde{F}}(\mathbf{x}) = 1$ if $\mathbf{x} \in \tilde{F}$ and $\mathbb{I}_{\tilde{F}}(\mathbf{x}) = 0$ otherwise. $h_{F \cup \tilde{F}}(\mathbf{x})$ denotes the conditional distribution of X given that it lies in the region of $F \cup \tilde{F}$:

$$h_{F \cup \tilde{F}}(\mathbf{x}) = \frac{\mathbb{I}_{F \cup \tilde{F}}(\mathbf{x}) f_X(\mathbf{x})}{\int_{\mathbb{R}^n} \mathbb{I}_{F \cup \tilde{F}}(\mathbf{x}) f_X(\mathbf{x}) d\mathbf{x}} = \frac{\mathbb{I}_{F \cup \tilde{F}}(\mathbf{x}) f_X(\mathbf{x})}{P(F \cup \tilde{F})}. \tag{14}$$

Utilizing Eq. (14) and Eq. (13), K_1 is then formulated as

$$K_1 = \frac{P(\tilde{F})}{P(F \cup \tilde{F})} = \int_{\mathbb{R}^n} \frac{\mathbb{I}_{\tilde{F}}(\mathbf{x})}{\mathbb{I}_{F \cup \tilde{F}}(\mathbf{x})} h_{F \cup \tilde{F}}(\mathbf{x}) d\mathbf{x}. \tag{15}$$

Similarly, K_2 can be also formulated as

$$K_2 = \frac{P(F)}{P(F \cup \tilde{F})} = \int_{\mathbb{R}^n} \frac{\mathbb{I}_F(\mathbf{x})}{\mathbb{I}_{F \cup \tilde{F}}(\mathbf{x})} h_{F \cup \tilde{F}}(\mathbf{x}) d\mathbf{x}. \tag{16}$$

Since the conditions that $\tilde{F} \subseteq F \cup \tilde{F}$ and $F \subseteq F \cup \tilde{F}$ always hold, we have $0 \leq K_1 \leq 1$ and $0 \leq K_2 \leq 1$, and it is thus possible to estimate K_1 and K_2 by sampling the regions of $F \cup \tilde{F}$.

Note that $P(F \cup \tilde{F})$ in the denominator of $h_{F \cup \tilde{F}}(\mathbf{x})$ is unknown, thus the conventional Monte Carlo sampling procedure cannot be used to sample $h_{F \cup \tilde{F}}(\mathbf{x})$. The Markov chain simulation is used to generate samples from $h_{F \cup \tilde{F}}(\mathbf{x})$ since only the ratio of $h_{F \cup \tilde{F}}(\mathbf{x})$ between consecutive states are required, and the unknown constant $P(F \cup \tilde{F})$ is not needed in the Markov chain simulations. In particular, when $F \cup \tilde{F}$ has multiple sub-regions, especially when those sub-regions are disconnected, multiple Markov chain with different initial states are used to generate samples from such regions in the present study.

Suppose that the total number of Markov chain samples is N_K, and N_C Markov chains are generated, so that N_K/N_C samples are simulated for each chain. Let $\{\mathbf{x}^{(i,j)}, i = 1, 2, \ldots, N_C, j = 1, 2, \ldots, N_K/N_C\}$ be the Markov chain samples drawn from $h_{F \cup \tilde{F}}(\mathbf{x})$, then the estimators of K_1 and K_2 are given by

$$\hat{K}_1 = \frac{1}{N_K} \sum_{i=1}^{N_C} \sum_{j=1}^{N_K/N_C} \frac{\mathbb{I}_{\tilde{F}}(\mathbf{x}^{(ij)})}{\mathbb{I}_{F \cup \tilde{F}}(\mathbf{x}^{(ij)})} \tag{17}$$

and

$$\hat{K}_2 = \frac{1}{N_K} \sum_{i=1}^{N_C} \sum_{j=1}^{N_K/N_C} \frac{\mathbb{I}_{F}(\mathbf{x}^{(ij)})}{\mathbb{I}_{F \cup \tilde{F}}(\mathbf{x}^{(ij)})} \tag{18}$$

respectively.

Theoretically, the Markov chain samples are asymptotically distributed according to $h_{F\cup\tilde{F}}(\mathbf{x})$, thus the estimators $\hat{K}_1$ and $\hat{K}_2$ in Eqs.(17) and (18) are asymptotically unbiased [22]. Although the Markov chain samples are correlated, the estimators of K_1 and K_2 still have the usual convergence properties of estimators according to the *Strong Law of Large Numbers* [22]. Therefore, $\hat{K}_1$ and $\hat{K}_2$ converge almost surely to K_1 and K_2.

Once the estimators of K_1 and K_2 are obtained by Eqs.(17) and (18), the estimator of the correction term K can be computed by substituting Eqs.(17) and (18) into Eq. (12) as

$$K \approx K(\hat{K}_1, \hat{K}_2) = \frac{\hat{K}_2}{\hat{K}_1}. \tag{19}$$

However, the estimator of K given by Eq. (19) is unbiased only if the total number of Markov chain samples N_K is infinite. In practice, the estimator of K is biased. The bias of $\hat{K}_2/\hat{K}_1$, defined as the difference between the expectation of the estimator and the true value of K, is given by:

$$\text{Bias}(\hat{K}) = \frac{K_2}{N_K K_1^3}\text{Var}(\hat{K}_1). \tag{20}$$

The derivation of Eq. (20) is found in Appendix A. In order to construct an unbiased estimator of K, the bias in Eq. (20) should be subtracted from the biased estimator in Eq. (19). Since $\text{Bias}(\hat{K})$ involves the variance of $\hat{K}_1$, the following devotes to the computation of the variance of $\hat{K}_1$.

As shown in [23], the variance of the estimator $\hat{K}_1$ is given by

$$\begin{aligned} &\text{Var}(\hat{K}_1) = E(\hat{K}_1 - K_1)^2 \\ &= \frac{1}{N_K}\left[R_1(0) + 2\sum_{l=1}^{N_K/N_C-1}\left(1 - \frac{lN_C}{N_K}\right)R_1(l)\right] \end{aligned} \tag{21}$$

where $R_1(l)$ is the covariance between $I_{\tilde{F}}(\mathbf{x}^{(i,m)})/I_{F\cup\tilde{F}}(\mathbf{x}^{(i,m)})$ and $I_{\tilde{F}}(\mathbf{x}^{(i,m+l)})/I_{F\cup\tilde{F}}(\mathbf{x}^{(i,m+l)})$ for $l = 0, 1, \cdots, N_K - 1$. $R_1(l)$ can be estimated using the Markov chain samples $\{\mathbf{x}^{(i,j)}, i = 1, 2, \ldots, N_C, j = 1, 2, \ldots, N_K/N_C\}$ by

$$\begin{aligned} & R_1(l) \approx \hat{R}_1(l) \\ & = \frac{1}{N_K - lN_C} \sum_{i=1}^{N_C} \sum_{m=1}^{N_K/N_C - l} \left[\frac{\mathbb{I}_{\tilde{F}}(\mathbf{x}^{(i,m)})}{\mathbb{I}_{F\cup\tilde{F}}(\mathbf{x}^{(i,m)})} \frac{\mathbb{I}_{\tilde{F}}(\mathbf{x}^{(i,m+l)})}{\mathbb{I}_{F\cup\tilde{F}}(\mathbf{x}^{(i,m+l)})} \right] - \hat{K}_1^2. \end{aligned} \tag{22}$$

$R_1(l)$ depends on the correlation between the samples. It is positive in general and equal to zero when the samples are independent. Thus the correlation of the Markov chain samples has to be considered when computing the variance of $\hat{K}_1$, as is shown in Eq. (21). However, for large values of l, the estimated $R_1(l)$ will be too noisy. Hence, a truncated summation, which starts from $l = 0$ until the sum of covariance estimates for two successive lags $\hat{R}_1(T_1 + 1) + \hat{R}_1(T_1 + 2)$ is negative, is used when computing the summations of $R_1(l)$ in Eq. (21) [24]. Consequently, $\mathrm{Var}(\hat{K}_1)$ is approximated as

$$\mathrm{Var}(\hat{K}_1) \approx \frac{1}{N_K} \left[R_1(0) + 2 \sum_{l=1}^{T_1} \left(1 - \frac{lN_C}{N_K} \right) R_1(l) \right] \tag{23}$$

where T_1 is the first odd positive integer for which $\hat{R}_1(T_1 + 1) + \hat{R}_1(T_1 + 2)$ is negative.

Similarly, $\mathrm{Var}(\hat{K}_2)$ is approximated as

$$\mathrm{Var}(\hat{K}_2) \approx \frac{1}{N_K} \left[R_2(0) + 2 \sum_{l=1}^{T_2} \left(1 - \frac{lN_C}{N_K} \right) R_2(l) \right] \tag{24}$$

where

$$\begin{aligned} & R_2(l) \approx \hat{R}_2(l) \\ & = \frac{1}{N_K - lN_C} \sum_{i=1}^{N_C} \sum_{m=1}^{N_K/N_C - l} \left[\frac{\mathbb{I}_{F}(\mathbf{x}^{(i,m)})}{\mathbb{I}_{F\cup\tilde{F}}(\mathbf{x}^{(i,m)})} \frac{\mathbb{I}_{F}(\mathbf{x}^{(i,m+l)})}{\mathbb{I}_{F\cup\tilde{F}}(\mathbf{x}^{(i,m+l)})} \right] - \hat{K}_2^2 \end{aligned} \tag{25}$$

and T_2 is the first odd positive integer for which $\hat{R}_2(T_1+1)+\hat{R}_2(T_1+2)$ is negative.

Combining Eqs.(19) and (20), an unbiased estimator of K is constructed as

$$\hat{K} \approx \frac{\hat{K}_2}{\hat{K}_1} - \frac{\hat{K}_2}{\hat{K}_1^3}\mathrm{Var}(\hat{K}_1). \tag{26}$$

The estimator of K given in Eq. (26) is asymptotically unbiased and consistent, as proved in Appendix A.

When Eq. (26) is used to estimate the correction term K, the variance of the estimator K is given by

$$\mathrm{Var}(\hat{K}) \approx \frac{\mathrm{Var}(\hat{K}_2)}{\hat{K}_1^2} + \frac{\hat{K}_2^2}{\hat{K}_1^4}\mathrm{Var}(\hat{K}_1). \tag{27}$$

The proof of Eq. (27) can be found in Appendix A. The coefficient of variation of $\hat{K}$ is then estimated as

$$\mathrm{Cov}(\hat{K}) \approx \frac{\sqrt{\mathrm{Var}(\hat{K})}}{\hat{K}}, \tag{28}$$

in which Cov = coefficient of variation. $\mathrm{Cov}(\hat{K})$ measures the accuracy of the estimator $\hat{K}$. Besides, it is used as the convergence criterion for the adaptive refinement of the metamodel, which will be introduced in Section 3.2.

3.2. Adaptive refinement of the metamodel

Although the proposed method is independent of the adopted metamodel, a metamodel of high accuracy is still preferred since it affects the efficiency of computing the correction term K. Because of this, an adaptive strategy is developed to simultaneously refine the metamodel and update the correction term. The general idea is to enrich the DoE with additional 'useful' samples

until a more strict criterion is satisfied. A straightforward way is to add N_K Markov chain samples used in the computation of the correction term to the initial DoE. However, the target distribution of the Markov chain is $h_{F\cup\tilde{F}}(\mathbf{x})$. These N_K samples tend to concentrate in the regions of $F\cup\tilde{F}$. Therefore, the real failure region F may not be adequately explored if only these samples are added to the DoE. It is proposed to further enrich the DoE by adding another N_M samples that distributed as $h_{\tilde{F}}(\mathbf{x}) = I_{\tilde{F}}(\mathbf{x})f_X(\mathbf{x})/P(\tilde{F})$. The N_M samples are selected as the K-means clusters' center of a large sample population (say 10^4 samples) generated by Markov chain simulation from the metamodel-based failure region $\tilde{F}$. As the centroids of the clusters identified by the K-means algorithm, these N_M samples are more likely to dispersely populate the region of $\tilde{F}$. Although $\tilde{F}$ deviates from F at the initial stage, it is expected that $\tilde{F}$ will approach to F as the refinement continues and the clusted N_M centroids can enhance the exploration of F. As a consequence, by adding N_K Markov chain samples and N_M centroids of the clusters to the DoE, the failure region F can be better approximated and thus an improved metamodel can be obtained.

Based on the refined metamodel $\tilde{g}(\mathbf{x})$ and corresponding metamodel-based failure region $\tilde{F}$ obtained above, one can generate another N_K Markov chain samples that distributed according to the new event $F\cup\tilde{F}$ to estimate the new correction term. These new N_K Markov chain samples, together with new N_M centroids of the clusters of a large sample population from the refine failure region $\tilde{F}$, are used to further enrich the current DoE, and to refine the current metamodel $\tilde{g}(\mathbf{x})$. The above adaptive refinement is repeated until either of the following criterion is satisfied: the coefficient of variation

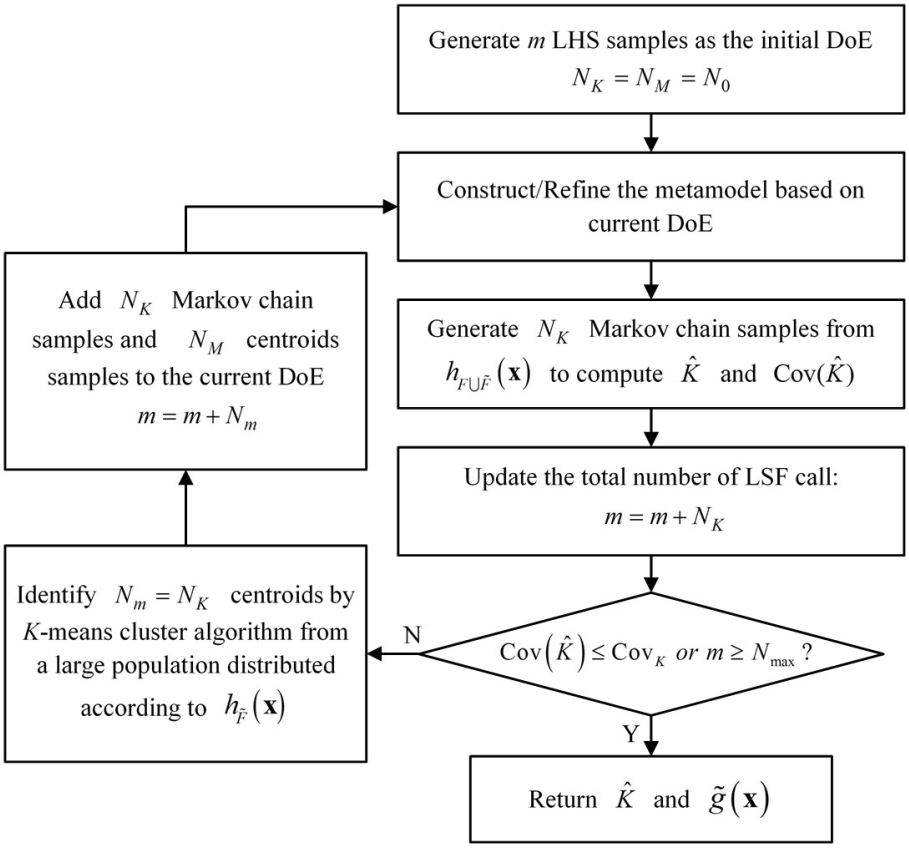


Figure 1: Flowchart of the adaptively refinement scheme.

of K reaches the target value, i.e., $\mathrm{Cov}(\hat{K}) \leqslant \mathrm{Cov}_K$, or the total number of LSF calls exceeds the threshold, i.e., $N_{\mathrm{call}} > N_{\max}$. The algorithm for adaptively refining the metamodel is summarized in Figure 1

According to the authors' experience, a satisfied mix proportion of N_K and N_M can be adopted as $N_K/N_M = 1$, and their initial values are both selected as N_0. Thus, the total number of samples used to adaptively refine

the metamodel is $2iN_0$, where i represents the number of the iterations.

4. Procedure of the unbiased metamodel method

The proposed methodology can be summarized as follows.

Step 1 Construction of the initial metamodel: Generate an initial Kriging metamodel as described in Section 2 based on m initial DoE samples.

Step 2 Adaptive refinement of the metamodel and correction term: Generate N_K Markov chain samples to estimate the correction term for the Kriging metamodel obtained in Step 1. Enrich the initial DoE by adding these N_K samples and another N_M centroids of the clusters of a large sample population from the failure region $\tilde{F}$, and then adaptively refine the metamodel and its correction term as described in Section 3.2, until the convergence criterion is satisfied.

Step 3 Estimation of the failure probability: Compute the failure probability based on the refined metamodel and its correction term that obtained in Step 2 according to Eq. (11). Since the expression of the refined metamodel has been extracted in Step 2, the metamodal-based failure probability $P(\tilde{F})$ can be readily estimated by the direct Monte Carlo simulation. The coefficient of variation of the failure probability is approximated by

$$\mathrm{Cov}(\hat{P}_F) \approx \sqrt{\mathrm{Cov}^2(\hat{P}_{\tilde{F}}) + \mathrm{Cov}^2(\hat{K})}. \tag{29}$$

The derivation of Eq. (29) can be found in Appendix B.

In the proposed method, Steps 1 and 2 requires multiple evaluations of the LSF. Since m DoE samples are used to construct the initial Kriging metamodel in Step 1, and $2iN_0$ samples are used to adaptively refine the metamodel in Step 2, the total number of function evaluations of LSF is $m + 2iN_0$. For most reliability analysis of structures of practical interest, majority of the computational cost is expended on the multiple evaluations of LSF. The CPU time needed for constructing the Kriging metamodel is insignificant in comparison with that of performing multiple limit state analyses. Therefore, the total number of function calls of the LSF to achieve a given accuracy is used in this work as the relevant measure of the computational cost.

5. Examples

Three examples from literature were selected to demonstrate the proposed method. The performance of the proposed method is examined through comparison with importance sampling (IS) and directional sampling (DS) in [25], and two commonly used metamodel methods including the Kriging and high dimensional model representation (HDMR) method [10]. The accuracy of the methods are assessed by comparing with the 'exact' probability of failure given by standard Monte Carlo simulation. To obtain a variance-type error estimate, each method was repeated 20 times to obtain a sample of 20 results. On the basis of the 20 calculations, the sample coefficient of variation (Cov) for the estimated failure probability by each method was calculated and compared. In all the metalmodel methods, the initial DoE were generated by using Latin Hypercube sampling.

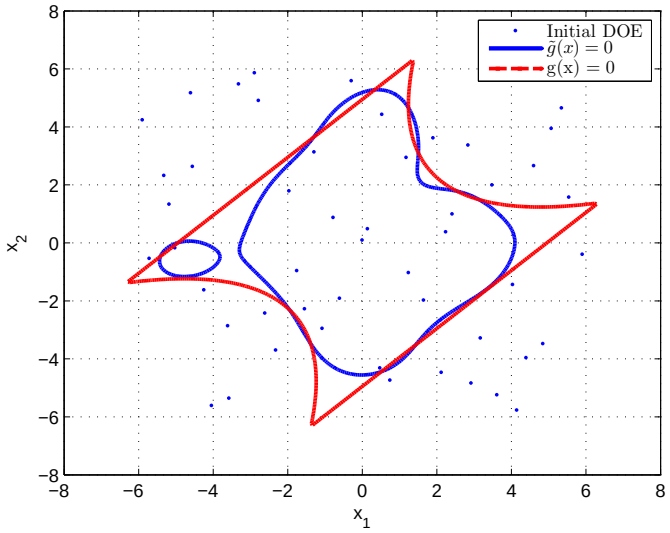
5.1. *Example 1: a series system with four branches*

The first example is a series system with four branches which has been studied in [15, 19, 26]. The failure probability is controlled by two linear and two nonlinear limit states defined as follows:

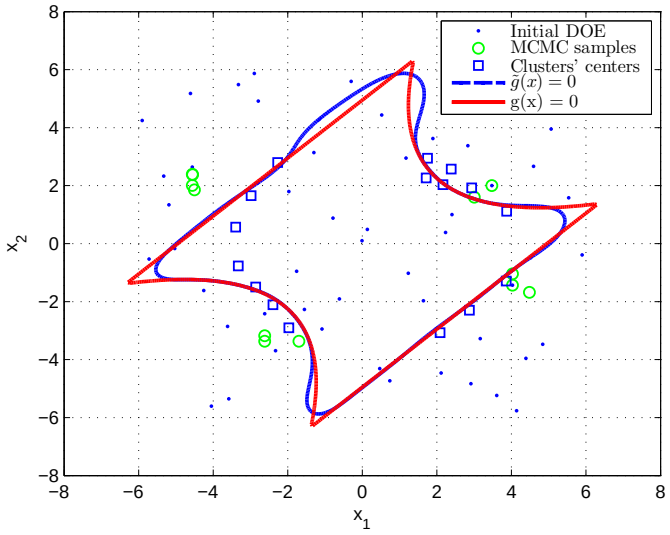
$$g\left(x_1, x_2\right) = \min \begin{cases} 3 + 0.1(x_1 - x_2)^2 - \frac{(x_1+x_2)}{\sqrt{2}} \\ 3 + 0.1(x_1 - x_2)^2 + \frac{(x_1+x_2)}{\sqrt{2}} \\ (x_1 - x_2) + \frac{7}{\sqrt{2}} \\ (x_2 - x_1) + \frac{7}{\sqrt{2}} \end{cases} \tag{30}$$

where x_1 and x_2 are independent standard normal variables. The different branches have comparable contribution to the system failure probability. As the limit state is explicit and relatively simple, using the metamodel method for this example offers no particular advantage. The purpose of the example is to graphically demonstrate the refined Kriging metamodel, together with the Markov chain samples in the computation of correction term and the K-means clusters' centers.

Figure (2a) and (2b) plot the actual and the approximated limit state surface, 50 initial DoE samples, 66 additional samples that are used to adaptively refine the metamodel, including the Markov chain samples and centroids of the clusters identified by the K-means algorithm. It can be seen that, with the initial DoE samples, the direct Kriging metamodel leads to noticeable error in the LSF approximation. By enriching the DoE with some 'useful' samples, the accuracy of the adaptively refined Kriging metamodel has been significantly improved, especially in the region of most likely failure. This illustrates the effectiveness of the adaptive refinement strategy developed in this study. From Figure (2b), it is also clear that the Markov chain samples



(a) The Kriging metamodel constructed with the initial DoE.



(b) The refined Kriging metamodel.

Figure 2: Example 1: Adaptive refinement of the Kriging metamodel.

concentrate in the region of higher probability density in the failure region, thus the real failure region cannot be sufficiently approximated if just adding the Markov chain samples into the initial DoE. The supplement of K-means clusters center from the metamodel-based failure region $\tilde{F}$ to DoE helps to refine the Kriging metamodel.

Table 1: Reliability results of Example 1.

Method	N_{call}	$\hat{P}_F$	Cov	$\varepsilon_{\hat{P}_F}(\%)$
MCS	10^6	2.233×10^{-3}	0.021	-
DS [25]	9192	2.6×10^{-3}	-	16.44
IS [25]	4750	2.2×10^{-3}	-	1.48
Kriging	116	2.889×10^{-3}	0.4128	29.4
	400	2.782×10^{-3}	0.4686	24.59
	600	2.450×10^{-3}	0.1597	9.72
Proposed method	116	2.22×10^{-3}	0.047	0.58

Table 1 compares the failure probabilities and the number of function calls of LSF of different methods. The 'exact' probability of failure was found to be 2.233×10^{-3} using 10^6 Monte Carlo simulations. It can be seen that the proposed method agrees reasonably well with the exact solution at the expense of a significantly smaller N_{call} than the simulation methods. Specifically, with the same number of N_{call} (i.e., $N_{call} = 116$), the relative error of the proposed method and the direct Kriging method is 0.58% and 29.4%, respectively. This illustrates that the correction term can eliminate the approximation error induced by direct Kriging metamodel and achieve

an accurate estimate of the failure probability even a poor metamodel is used. With the increasing number of DoE samples (e.g., $N = 600$), the accuracy of the direct Kriging metamodel method is improved, and comparable accuracy can be achieved as the proposed method. This suggests that the proposed method can be particularly advantageous when the number of N_{call} is relatively small.

5.2. Example 2: a nonlinear oscillator

A non-linear undamped single degree of freedom system shown in Figure 3 is considered next. The problem involves six random variables and is a classic illustration in the literatures [19, 26, 27]. The statistics of the basic random variables are given in Table 2. The limit state is defined by

$$g(\mathbf{x}) = 3r - |z_{\max}| = 3r - \left| \frac{2F_1}{m\omega_0^2} \sin\left(\frac{\omega_0^2 t_1}{2}\right) \right| \tag{31}$$

where z_{max} represents the maximum displacement response of the system, $\omega_0 = \sqrt{(c_1 + c_2)/m}$, and r is the displacement at which one of the springs yields.

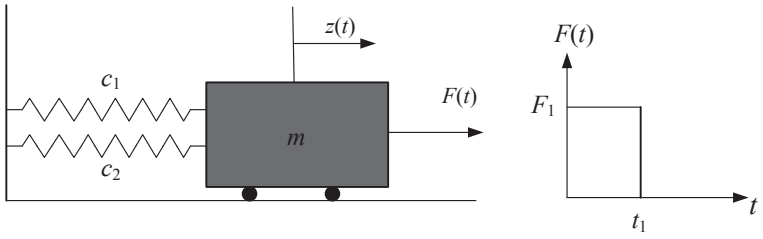


Figure 3: Example 2: a nonlinear oscillator.

In the proposed method, the initial DoE size is adopted as 100, and the number of the samples added to the DoE in each round of iteration is 6.

Table 2: Random variables of the non-linear oscillator.

Variable	Distribution	Mean	Standard deviation
m	Normal	1	0.05
c_1	Normal	1	0.1
c_2	Normal	0.1	0.01
r	Normal	0.5	0.05
F_1	Normal	1	0.2
t_1	Normal	1	0.2

Since a moderate number of random variables is involved in this example, the commonly used 2nd-order HDMR metamodel method is also used to compare with the proposed method.

Table 3 compares the failure probabilities and the number of function calls of LSF of different methods. The probability of failure is found to be 2.834×10^{-2} using 7×10^4 direct Monte Carlo simulation. Similar observations are made as in Example 1, the proposed method achieved the highest accuracy among all methods with the smallest number of function calls of LSF. The proposed method converges to the 'exact' solution with only 233 samples, while 2nd-order HDMR metamodel requires at least 577 samples to achieve the comparable accuracy. With a relative large number of DoE samples, e.g., $N_{call} = 600$, the relative error obtained from the Kriging metamodel method is still noticeably larger than that of the new method with 233 samples. The $\mathrm{Cov}(\hat{P}_F)$ for these two cases are 0.17 and 0.072, respectively. This demonstrates the advantage of the proposed method over the direct

Table 3: Reliability results of Example 2.

Method	N_{call}	$\hat{P}_F$	Cov	$\varepsilon_{\hat{P}_F}(\%)$
MCS	7×10^4	2.834×10^{-2}	0.022	-
DS	1281	3.5×10^{-2}	-	23.5
IS	6144	2.7×10^{-2}	-	4.73
Kriging	233	5.077×10^{-2}	0.26	79.15
	400	3.763×10^{-2}	0.28	32.78
	600	3.086×10^{-2}	0.17	8.89
2nd-order HDMR	577	2.936×10^{-2}	-	3.60
Proposed method	233	2.844×10^{-2}	0.072	0.35

Kriging metamodel method.

5.3. Example 3: a roof structure

Consider a roof structure, which is subjected to a uniformly distributed vertical load q, as shown in Figure 4. The example is adopted from [28–30]. The top cords and the compression bars are concrete, and the bottom cords and the tension bars are steel. In structural analysis, the uniformly distributed load q was transformed into three nodal loads, each is $P = ql/4$. The serviceability limit state of the structure with respect to its maximum vertical displacement was considered. The limit state function is given by

$$g = u_a - \frac{ql^2}{2}\left(\frac{3.81}{A_c E_c} + \frac{1.13}{A_s E_s}\right) \tag{32}$$

in which u_a is the allowable displacement and is set to be 0.03 m, E and A denote the Modulus of elasticity and cross-sectional area, and the sub-

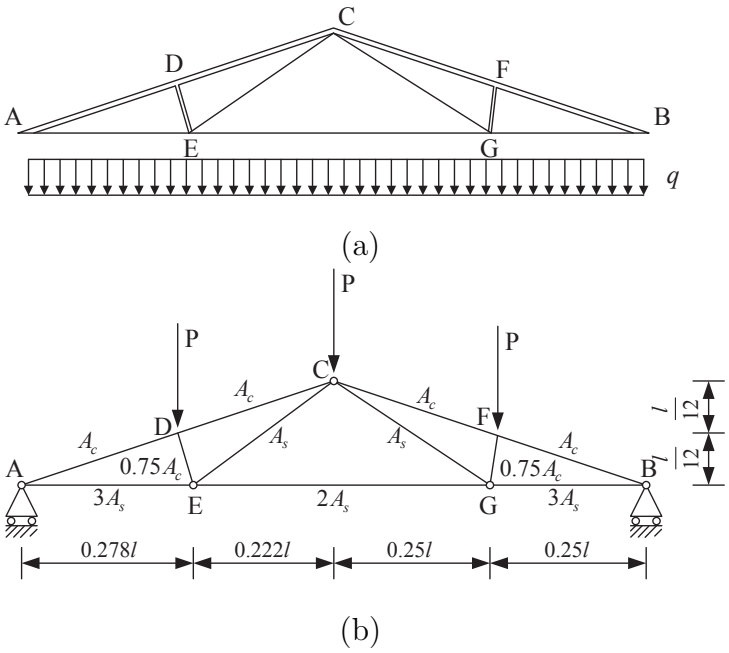


Figure 4: Example 3: a roof structure.

scripts s and c indicate the material steel and concrete, respectively. Table 4 summarizes the statistical information of the random variables. All random variables are assumed independent normals.

Table 5 lists the reliability results of different methods.The probability of failure is found to be 9.37×10^{-3} after 5×10^7 direct Monte Carlo simulations. Again, it can be observed that the proposed method gives a reasonable result at a small computational effort. With a relative small size of DoE (i.e., $N_{call} < 600$), the Kriging metamodel method is less accurate than the proposed method, illustrating the importance of the correction term. With 220 samples, $\mathrm{Cov}(\hat{P}_F)$ of the proposed method is 0.035, while $\mathrm{Cov}(\hat{P}_F)$ of the Kriging metamodel method is still 0.36 even 600 DoE samples are used. Noted that the above sample coefficient of variation is computed from 20

runs of the proposed method, and is very close to the theoretical value of 0.032 which is calculated from Eq. (29). This shows that the coefficient of variation obtained from Eq. (29) agrees well with the empirical result. It should be noted that, with 800 samples, the relative error of the Kriging method is even larger than that of with 600 samples, illustrating that the accuracy of the direct metamodel method cannot be guaranteed even a large number of function calls of LSF is used.

Table 4: Random variables of the roof truss.

Variable	Distribution	Mean	Cov
q(N/m)	Normal	20000	0.07
l(m)	Normal	12	0.01
A_S(m^2)	Normal	9.82×10^{-4}	0.06
A_C(m^2)	Normal	400×10^{-4}	0.12
E_S(N/m^2)	Normal	1×10^{11}	0.06
E_C(N/m^2)	Normal	2×10^{10}	0.06

6. Conclusion

A new unbiased metamodel method has been developed for efficient reliability assessment. The method formulates the probability of failure as a product of a metamodel-based failure probability and a correction term. The correction term is used to quantify and further eliminate the error resulting from approximating the real limit state function with the metamodel. Due to the introduction of correction term, the new method can obtain an

Table 5: Reliability results of Example 3.

Method	N_{call}	$\hat{P}_F$	Cov	$\varepsilon_{\hat{P}_F}(\%)$
MCS	5×10^7	9.373×10^{-3}	0.001	-
Subset Simulation	54×10^3	9.647×10^{-3}	-	2.92
IS	2000	9.361×10^{-3}	-	0.13
Kriging	220	2.534×10^{-2}	0.59	170.35
	400	1.231×10^{-2}	0.44	31.35
	600	7.265×10^{-3}	0.36	22.49
	800	6.732×10^{-3}	0.27	28.81
2nd-order HDMR	577	9.3×10^{-3}	-	0.78
Proposed method	220	9.555×10^{-3}	0.035	1.94

asymptotically unbiased estimate of the probability of failure even when a poor metamodel is used. The developed iterative procedure can efficiently improve the accuracy of the metalmodel and the associated correction term. The proposed methodology is general and applicable to any metamodel methods.

The efficiency and accuracy of the proposed methodology was demonstrated through three examples, including a series system with multiple design points, and two structural problems involving moderate dimensions. For all examples, it was observed that the proposed method is more accurate and efficient than the conventional Kriging metamodel and high dimensional model representation method. The proposed unbiased metamodel method can be a useful tool for structural reliability analysis, particularly for prob-

lems with complex implicit limit state functions.

Acknowledgments

This research was supported, in part, by Grants from the National Natural Science Foundation of China (Project 11572106), and the Australian Research Council Discovery Project DP150104873. These supports are gratefully acknowledged.

Appendix A. Estimation of correction term

In this appendix, we will derive the bias that induced by using Eq. (19) to estimate the correction term K. Based on this bias, we will further construct an asymptotically unbiased estimator of K. The variance of the modified estimator will also be derived.

In Section 3.1, it has been pointed out that the estimator of K in Eq. (19) is biased. Although $\hat{K}_2$ and $\hat{K}_1$ are asymptotically unbiased estimators, their ratio is biased. In order to estimate this bias, we take the 2nd-order Taylor's

expansion on the estimator K in Eq. (19) as

$$\begin{aligned} K(\hat{K}_1, \hat{K}_2) \approx\ & K(\hat{K}_1, \hat{K}_2)\Big|_{K_1,K_2} + \frac{\partial K(\hat{K}_1, \hat{K}_2)}{\partial \hat{K}_1}\Big|_{K_1,K_2} (\hat{K}_1 - K_1) \\ & + \frac{\partial K(\hat{K}_1, \hat{K}_2)}{\partial \hat{K}_2}\Big|_{K_1,K_2} (\hat{K}_2 - K_2) \\ & + \frac{1}{2}\frac{\partial^2 K(\hat{K}_1, \hat{K}_2)}{\partial \hat{K}_1^2}\Big|_{K_1,K_2} (\hat{K}_1 - K_1)^2 \\ & + \frac{1}{2}\frac{\partial^2 K(\hat{K}_1, \hat{K}_2)}{\partial \hat{K}_2^2}\Big|_{K_1,K_2} (\hat{K}_2 - K_2)^2 \\ & + \frac{\partial^2 K(\hat{K}_1, \hat{K}_2)}{\partial \hat{K}_1 \partial \hat{K}_2}\Big|_{K_1,K_2} (\hat{K}_1 - K_1)(\hat{K}_2 - K_2). \end{aligned} \tag{A.1}$$

By taking expectation on both sides of Eq. (A.1), we have

$$E[K(\hat{K}_1, \hat{K}_2)] \approx \frac{K_2}{K_1} + \frac{K_2}{K_1^3}\mathrm{Var}(\hat{K}_1). \tag{A.2}$$

Noted that $E[(\hat{K}_1 - K_1)(\hat{K}_2 - K_2)]$ can be considered as zero in the above derivation since the value of the 2nd-order term $(\hat{K}_1 - K_1)(\hat{K}_2 - K_2)$ is significantly less than the other terms. In Eq. (A.2), the term of $(K_2/K_1^3)\mathrm{Var}(\hat{K}_1)$ is the bias that induced by using Eq. (19) to estimate the correction term K. By substituting Eq. (21) to the formula of the bias, we have

$$\mathrm{Bias}(\hat{K}) = \frac{K_2}{N_K K_1^3}\left[R_1(0) + 2\sum_{l=1}^{N_K/N_C - 1}\left(1 - \frac{lN_C}{N_K}\right)R_1(l)\right]. \tag{A.3}$$

From Eq. (A.3), it can be seen that the bias tends to zero as the number of Markov chain samples N_K approaches to infinity. Only in such a case, the estimator of K given in Eq. (19) is asymptotically unbiased. However, in practice, one can never generate infinite Markov chain samples and thus the

estimator of K is biased. In order to eliminate this bias, a new estimator of K is constructed by subtracting the above computed bias from the estimator given in Eq. (19) as

$$\begin{aligned} & K_m(\hat{K}_1, \hat{K}_2) \\ &= K(\hat{K}_1, \hat{K}_2) - \frac{K_2}{K_1^3}\mathrm{Var}(\hat{K}_1) \\ &\approx K(\hat{K}_1, \hat{K}_2) - \frac{\hat{K}_2}{\hat{K}_1^3}\mathrm{Var}(\hat{K}_1). \end{aligned} \tag{A.4}$$

Taking expectation on both sides of Eq. (A.4), we have

$$E\left[K_m(\hat{K}_1, \hat{K}_2)\right] = E[K(\hat{K}_1, \hat{K}_2)] - \frac{K_2}{K_1^3}\mathrm{Var}(\hat{K}_1^3) \approx \frac{K_2}{K_1}. \tag{A.5}$$

Eq. (A.5) illustrates that the constructed estimator K_m in Eq. (A.4) is an asymptotically unbiased estimation of the correction term K.

The variance of the new estimator K_m is formulated as

$$\begin{aligned} & \mathrm{Var}[K_m(\hat{K}_1, \hat{K}_2)] \\ &= E\left[\left\{K_m(\hat{K}_1, \hat{K}_2) - E[K_m(\hat{K}_1, \hat{K}_2)]\right\}^2\right] \\ &= E\left[\left\{K\left(\hat{K}_1, \hat{K}_2\right) - E[K(\hat{K}_1, \hat{K}_2)]\right\}^2\right] \\ &\approx E\left[\left(\frac{\hat{K}_2}{\hat{K}_1} - \frac{K_2}{K1}\right)^2\right] \approx \frac{\mathrm{Var}(\hat{K}_2^2)}{K_1^2} + \frac{K_2^2}{K_1^4}\mathrm{Var}(\hat{K}_1). \end{aligned} \tag{A.6}$$

By substituting Eq. (23) and Eq. (24) into Eq. (A.6), the variance of K_m

becomes

$$\begin{aligned}
&\mathrm{Var}[K_m(\hat{K}_1,\hat{K}_2)] \\
&= \frac{1}{N_K K_1^2}\left[R_2(0) + 2\sum_{l=1}^{N_K/N_C-1}\left(1-\frac{lN_C}{N_K}\right)R_2(l)\right] \\
&+ \frac{K_2^2}{N_K K_1^4}\left[R_1(0) + 2\sum_{l=1}^{N_K/N_C-1}\left(1-\frac{lN_C}{N_K}\right)R_1(l)\right].
\end{aligned} \tag{A.7}$$

It can be concluded from Eq. (A.7) that the variance of the new estimator K_m tends to zero as N_K approaches to infinity, illustrating that the constructed estimator of K is consistent.

Appendix B. Coefficient of variation of $\hat{P}_F$

This section shows the derivation of the coefficient of variation of the failure probability given in Eq. (29). The variance of $\hat{P}_F$ can be formulated as

$$\mathrm{Var}(\hat{P}_F) = E[\hat{P}_F^2] - E[\hat{P}_F]^2. \tag{B.1}$$

We rewrite $\hat{P}_F$ as $\hat{P}_F = \hat{P}_{\tilde{F}} \cdot \hat{K}$, and noticing that $\hat{P}_{\tilde{F}}$ and $\hat{K}$ are independent, Eq. (B.1) then becomes

$$\begin{aligned}
\mathrm{Var}(\hat{P}_F) &= E[\hat{P}_{\tilde{F}}^2]E[\hat{K}^2] - E[\hat{P}_{\tilde{F}}]^2 E[\hat{K}]^2 \\
&= \left(E[\hat{P}_{\tilde{F}}]^2 + \mathrm{Var}(\hat{P}_{\tilde{F}})\right)\left(E[\hat{K}]^2 + \mathrm{Var}(\hat{K})\right) - E[\hat{P}_{\tilde{F}}]^2 E[\hat{K}]^2.
\end{aligned} \tag{B.2}$$

Since the estimators of both $\hat{P}_{\tilde{F}}$ and $\hat{K}$ are unbiased, Eq. (B.2) can be further simplified as

$$\begin{aligned}
\mathrm{Var}(\hat{P}_F) &= \left(\hat{P}_{\tilde{F}}^2 + \mathrm{Var}(\hat{P}_{\tilde{F}})\right)\left(\hat{K}^2 + \mathrm{Var}(\hat{K})\right) - \hat{P}_{\tilde{F}}^2\hat{K}^2 \\
&= \mathrm{Var}(\hat{P}_{\tilde{F}}^2)\mathrm{Var}(\hat{K}) + \hat{P}_{\tilde{F}}^2\mathrm{Var}(\hat{K}) + \hat{K}^2\mathrm{Var}(\hat{P}_{\tilde{F}}).
\end{aligned} \tag{B.3}$$

Thus, the coefficient of variation of $\hat{P}_F$ can be formulated as

$$\begin{aligned} \mathrm{Cov}(\hat{P}_F) &= \frac{\sqrt{\mathrm{Var}(\hat{P}_F)}}{\hat{P}_F} = \frac{\sqrt{\mathrm{Var}(\hat{P}_F)}}{\hat{P}_F \hat{K}} \\ &= \sqrt{\mathrm{Cov}^2(\hat{P}_{\tilde{F}}) + \mathrm{Cov}^2(\hat{K}) + \mathrm{Cov}^2(\hat{P}_{\tilde{F}})\mathrm{Cov}^2(\hat{K})}. \end{aligned} \tag{B.4}$$

Since the product of $\mathrm{Cov}^2(\hat{P}_{\tilde{F}})$ and $\mathrm{Cov}^2(\hat{K})$ in Eq. (B.4) is significantly less than the first two terms, the coefficient of variation of $\hat{P}_F$ is simplified as

$$\mathrm{Cov}(\hat{P}_F) \approx \sqrt{\mathrm{Cov}^2(\hat{P}_{\tilde{F}}) + \mathrm{Cov}^2(\hat{K})}. \tag{B.5}$$

References

[1] C. Bucher, T. Most, A comparison of approximate response functions in structural reliability analysis, Probabilistic Engineering Mechanics 23 (2-3) (2008) 154–163.

[2] H. Dai, H. Zhang, W. Wang, A multiwavelet neural network-based response surface method for structural reliability analysis, Computer-Aided Civil and Infrastructure Engineering 30 (2) (2015) 151–162.

[3] H. Dai, G. Xue, W. Wang, An adaptive wavelet frame neural network method for efficient reliability analysis, Computer-Aided Civil and Infrastructure Engineering 29 (10) (2014) 801–814.

[4] J. E. Hurtado, D. A. Alvarez, Neural-network-based reliability analysis: a comparative study, Computer Methods in Applied Mechanics and Engineering 191 (1-2) (2001) 113–132.

[5] H. Dai, B. Zhang, W. Wang, A multiwavelet support vector regression method for efficient reliability assessment, Reliability Engineering & System Safety 136 (2015) 132–139.

[6] J. E. Hurtado, An examination of methods for approximating implicit limit state functions from the viewpoint of statistical learning theory, Structural Safety 26 (3) (2004) 271–293.

[7] J. E. Hurtado, D. A. Alvarez, Classification approach for reliability analysis with stochastic finite-element modeling, Journal of Structural Engineering 129 (8) (2003) 1141–1149.

[8] H. Dai, H. Zhang, W. Wang, A support vector density-based importance sampling for reliability assessment, Reliability Engineering & System Safety 106 (2012) 86–93.

[9] R. Chowdhury, B. N. Rao, Assessment of high dimensional model representation techniques for reliability analysis, Probabilistic Engineering Mechanics 24 (1) (2009) 100–115.

[10] R. Chowdhury, B. N. Rao, A. M. Prasad, High-dimensional model representation for structural reliability analysis, Communications in Numerical Methods in Engineering 25 (4) (2009) 301–337.

[11] G. Blatman, B. Sudret, An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis, Probabilistic Engineering Mechanics 25 (2) (2010) 183–197.

[12] J. Li, D. Xiu, Evaluation of failure probability via surrogate models, Journal of Computational Physics 229 (23) (2010) 8966–8980.

[13] I. Kaymaz, Application of kriging method to structural reliability problems, Structural Safety 27 (2) (2005) 133–151.

[14] B. J. Bichon, M. S. Eldred, L. P. Swiler, S. Mahadevan, J. M. McFarland, Efficient global reliability analysis for nonlinear implicit performance functions, AIAA Journal 46 (10) (2008) 2459–2468.

[15] F. Cadini, F. Santos, E. Zio, An improved adaptive kriging-based importance technique for sampling multiple failure regions of low probability, Reliability Engineering & System Safety 131 (2014) 109–117.

[16] Z. Lv, Z. Lu, P. Wang, A new learning function for kriging and its applications to solve reliability problems in engineering, Computers & Mathematics with Applications 70 (5) (2015) 1182–1197.

[17] V. Dubourg, B. Sudret, F. Deheeger, Metamodel-based importance sampling for structural reliability analysis, Probabilistic Engineering Mechanics 33 (2013) 47–57.

[18] I. Depina, T. M. H. Le, G. Fenton, G. Eiksund, Reliability analysis with metamodel line sampling, Structural Safety 60 (2016) 1–15.

[19] B. Echard, N. Gayton, M. Lemaire, AK-MCS: An active learning reliability method combining kriging and monte carlo simulation, Structural Safety 33 (2) (2011) 145–154.

[20] D. R. Jones, M. Schonlau, W. J. Welch, Efficient global optimization of expensive black-box functions, Journal of Global Optimization 13 (4) (1998) 455–492.

[21] G. Matheron, The intrinsic random functions and their applications, Advances in Applied Probability 5 (3) (1973) 439–468.

[22] C. P. Robert, G. Casella, Monte Carlo statistical methods, Springer, 2004.

[23] S. K. Au, J. L. Beck, Estimation of small failure probabilities in high dimensions by subset simulation, Probabilistic Engineering Mechanics 16 (4) (2001) 263–277.

[24] A. Gelman, J. B. Carlin, H. Stern, D. B. Rubin, Bayesian Data Analysis, Vol. Third Edition, CRC Press, London, 2014.

[25] L. Schueremans, D. Van Gemert, Benefit of splines and neural networks in simulation based structural reliability analysis, Structural Safety 27 (3) (2005) 246–261.

[26] X. Huang, J. Chen, H. Zhu, Assessing small failure probabilities by AK-SS: An active learning method combining kriging and subset simulation, Structural Safety 59 (2016) 86–95.

[27] H. Dai, H. Zhang, K. J. R. Rasmussen, W. Wang, Wavelet density-based adaptive importance sampling method, Structural Safety 52 (2015) 161–169.

[28] S. Song, Z. Lu, H. Qiao, Subset simulation for structural reliability sensitivity analysis, Reliability Engineering & System Safety 94 (2) (2009) 658–665.

[29] H. Zhao, Z. Yue, Y. Liu, Z. Gao, Y. Zhang, An efficient reliability method combining adaptive importance sampling and kriging metamodel, Applied Mathematical Modelling 39 (7) (2015) 1853–1866.

[30] V. Dubourg, B. Sudret, Meta-model-based importance sampling for reliability sensitivity analysis, Structural Safety 49 (2014) 27–36.