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Characterization of Bimodal Extension of the Generalized Gamma Distribution

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Abstract

Cankaya et al. (2015) [1] introduced a bimodal extension of the generalized gamma distribution and studied certain properties and applicability of this distribution. This is a continuous distribution whose probability density function is defined via two branches. These types of distributions are very interesting but not easy to characterize. In this short note we try to present a characterization of this distribution which we believe, it may possibly be the only one for this rather complicated distribution.

Mathematics Subject Classification: 60E05

Keywords: Bimodal distribution; Generalized gamma; Truncated moments; characterizations

1 Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers.

Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. This work deals with a characterization of a six-parameter distribution called "Bimodal Extension of the Generalized Gamma (BEGG) Distribution" based on two truncated moments.

Cankaya et al. (2015) [1] proposed (BEGG) distribution whose probability density function (pdf) is given by

$$f\left(x\right) = f\left(\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon\right) = \begin{cases} C_1(-x)^{\delta_1} \exp\{-C_2(-x)^{\alpha\beta}\}, & x < 0 \\ C_3 x^{\delta_0} \exp\{-C_4 x^{\alpha\beta}\}, & x \geq 0 \end{cases}, \quad (1)$$

where $\alpha, \beta, \delta_0, \delta_1, \eta, \varepsilon$ are all positive parameters and $C_1 = \dfrac{\alpha\beta}{2\eta^{\frac{\delta_1+1}{\alpha}}(1+\varepsilon)^{\delta_1}\Gamma\left(\frac{\delta_1+1}{\alpha\beta}\right)}$, $C_2 = \dfrac{1}{\eta^\beta(1+\varepsilon)^{\alpha\beta}}$, $C_3 = \dfrac{\alpha\beta}{2\eta^{\frac{\delta_0+1}{\alpha}}(1-\varepsilon)^{\delta_0}\Gamma\left(\frac{\delta_0+1}{\alpha\beta}\right)}$ and $C_4 = \dfrac{1}{\eta^\beta(1-\varepsilon)^{\alpha\beta}}$ are normalizing constants.

The characterization presented here requires that the cumulative distribution function to be twice continuously differentiable. Thus, pdf (1) should be differentiable. Clearly the derivative of $f\left(x\right)$ exists for $x < 0$ and $x > 0$ and the left and the right derivatives of $f\left(x\right)$ exist at $x = 0$ and are equal to 0, if $\delta_1 > 1$ and $\delta_0 > 1$, so, under these conditions, $f\left(x\right)$ is differentiable on $\mathbb{R}$.

Our characterization employs a theorem of Glänzel (1987), [2] see Theorem 1 below . The result, however, holds also when the interval H is not closed. Furthermore, it does not require that the cdf have a closed form as it is the case with BEGG distribution.

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty, \ e = \infty$ might as well be allowed) . Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right) \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right) \mid X \geq x\right] \xi\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1\left(H\right)$, $\xi \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ , particularly

$$F(x) = \int_a^x C \left| \frac{\xi'(u)}{\xi(u)\, q_1(u) - q_2(u)} \right| \exp(-s(u))\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi'\, q_1}{\xi\, q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel 1990), [3] in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_{1n}, q_{2n} and ξ_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1$, $q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if ξ_n converges to ξ, where

$$\xi(x) = \frac{E\left[q_2(X) \mid X \geq x\right]}{E\left[q_1(X) \mid X \geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1, q_2 and ξ, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glänzel and Hamedani (2001) [4].

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, ξ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take $q_1(x) \equiv 1$, which reduces the condition of Theorem 1 to $\mathbf{E}\left[q_2(X) \mid X \geq x\right] = \xi(x)$, $x \in H$. We, however, believe that employing three functions q_1, q_2 and ξ will enhance the domain of applicability of Theorem 1.

2 Characterizations based on two truncated moments

This section deals with the characterization of the EBGG distribution based on the ratio of two truncated moments.

Proposition 1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let

$$q_1\left(x\right) = \begin{cases} \frac{(\delta_1+1)}{C_1} \exp\left\{-(-x)^{\delta_1+1}+C_2(-x)^{\alpha\beta}\right\}, & x < 0 \\ \frac{(\delta_0+1)}{C_3} \exp\left\{-x^{\delta_0+1}+C_4 x^{\alpha\beta}\right\}, & x \geq 0 \end{cases},$$

and

$$q_2\left(x\right) = \begin{cases} \frac{2(\delta_1+1)}{C_1} \exp\left\{-2(-x)^{\delta_1+1}+C_2(-x)^{\alpha\beta}\right\}, & x < 0 \\ \frac{2(\delta_0+1)}{C_3} \exp\left\{-2x^{\delta_0+1}+C_4 x^{\alpha\beta}\right\}, & x \geq 0 \end{cases}.$$

Then, the random variable X has pdf (1), for $\delta_1 > 1$, $\delta_0 > 1$ and $\alpha\beta > 1$, if and only if the function ξ defined in Theorem 1 is of the form

$$\xi\left(x\right) = \begin{cases} \frac{2-\exp\left\{-2(-x)^{\delta_1+1}\right\}}{2-\exp\left\{-(-x)^{\delta_1+1}\right\}}, & x < 0 \\ \exp\left\{-x^{\delta_0+1}\right\}, & x \geq 0 \end{cases}.$$

Proof.   As mentioned in the Introduction, the conditions $\delta_1 > 1$, $\delta_0 > 1$ are required to assure the differentiability of the pdf (1). We need to show that $q_1, q_2 \in C^1\left(\mathbb{R}\right)$ and $\xi \in C^2\left(\mathbb{R}\right)$. Clearly $q_1\left(x\right)$ is differentiable for $x < 0$ and $x > 0$. The left and the right derivatives of $q_1\left(x\right)$ exist at $x = 0$ and will be equal to 0 if $\alpha\beta > 1$. Similarly, $q_2\left(x\right)$ is differentiable on $\mathbb{R}$. Observe that $\xi \in C^2\left(\mathbb{R}\right)$ if $\delta_1 > 1$, $\delta_0 > 1$.

Now, suppose the random variable X has pdf (1), then for $x < 0$,

$$\left(1 - F\left(x\right)\right) E\left[q_1\left(x\right) \mid X \geq x\right] = \int_x^0 \left(\delta_1 + 1\right)\left(-u\right)^{\delta_1} \exp\left\{-\left(-u\right)^{\delta_1+1}\right\} du +$$

$$\int_0^\infty \left(\delta_0 + 1\right) u^{\delta_0} \exp\left\{-u^{\delta_0+1} \quad du\right.$$

$$= 2 - \exp\left\{-\left(-x\right)^{\delta_1+1}\right\},$$

and

$$\left(1 - F\left(x\right)\right) E\left[q_2\left(x\right) \mid X \geq x\right] = 2 - \exp\left\{-2\left(-x\right)^{\delta_1+1}\right\}.$$

Further,

$$\xi\left(x\right)q_1\left(x\right)-q_2\left(x\right)=q_1\left(x\right)\left\{\frac{\left(2-\exp\left\{-\left(-x\right)^{\delta_1+1}\right\}\right)^2-2}{2-\exp\left\{-\left(-x\right)^{\delta_1+1}\right\}}\right\}<0.$$

For $x\geq 0$,

$$\left(1-F\left(x\right)\right)E\left[q_1\left(x\right)\mid X\geq x\right]=\int_x^\infty\left(\delta_0+1\right)\left(-u\right)^{\delta_0}\exp\left\{-u^{\delta_0+1}\right\}\,du$$
$$=\exp\left\{-x^{\delta_0+1}\right\}$$

and

$$\left(1-F\left(x\right)\right)E\left[q_2\left(x\right)\mid X\geq x\right]=\exp\left\{-2x^{\delta_0+1}\right\}\quad.$$

Further

$$\xi\left(x\right)q_1\left(x\right)-q_2\left(x\right)=-q_1\left(x\right)\exp\left\{-x^{\delta_0+1}\right\}<0.$$

Conversely, if ξ is of the above form, then ,after some computations and arrangement of terms, we arrive at

$$s'\left(x\right)=\frac{\xi'\left(x\right)q_1\left(x\right)}{\xi\left(x\right)q_1\left(x\right)-q_2\left(x\right)}=\begin{cases}\frac{\left(\delta_1+1\right)\left(-x\right)^{\delta_1}\exp\left\{-\left(-x\right)^{\delta_1+1}\right\}}{2-\exp\left\{-\left(-x\right)^{\delta_1+1}\right\}}, & x<0\\ \left(\delta_0+1\right)x^{\delta_0}, & x\geq 0\end{cases},$$

and consequently

$$s\left(x\right)=\begin{cases}-\log\left[2-\exp\left\{-\left(-x\right)^{\delta_1+1}\right\}\right], & x<0\\ x^{\delta_0+1}, & x\geq 0\end{cases}.$$

Now, according to Theorem 1, X has density (1).

Corollary 1. Let $X:\mathbb{R}$ be a continuous random variable and let $q_1\left(x\right)$ be as in Proposition 1. The random variable X has pdf (1), for $\delta_1>1$, $\delta_0>1$ and $\alpha\beta>1$, if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)\,q_1(x)}{\xi(x)\,q_1(x) - q_2(x)} = \begin{cases} \frac{(\delta_1+1)(-x)^{\delta_1}\exp\left\{-(-x)^{\delta_1+1}\right\}}{2-\exp\left\{-(-x)^{\delta_1+1}\right\}}, & x < 0 \\ (\delta_0+1)\,x^{\delta_0}, & x \geq 0 \end{cases}.$$

Corollary 2. The general solution of the differential equation in Corollary 1 is

$$\xi(x) = \begin{cases} \left[2-\exp\left\{-(-x)^{\delta_1+1}\right\}\right]^{-1}\left[\begin{array}{c} -\int(\delta_1+1)(-x)^{\delta_1}\exp\left\{-(-x)^{\delta_1+1}\right\} \times \\ (q_1(x))^{-1}\,q_2(x)\,dx + D_1 \end{array}\right], & x < 0 \\ \exp\left\{x^{\delta_0+1}\right\}\left[-\int(\delta_0+1)x^{\delta_0}(q_1(x))^{-1}q_2(x)dx+D_2\right], & x \geq 0 \end{cases},$$

where D_1 and D_2 are constants. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 1 with $D_1 = D_2 = 0$. Clearly, there are other triplets (q_1, q_2, ξ) which satisfy conditions of Theorem 1.

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