Integrality, complexity and colourings in polyhedral combinatorics

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Ph.D. thesis

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Notation

$[n]$ denotes the set $\{1, 2, \ldots n\}$ (for $n \in \mathbb{N}$).

$\mathbb{R}^n_+$ is the set of all nonnegative vectors.

$P + Q$ is the *Minkowski sum* of the two polyhedra $P, Q \subseteq \mathbb{R}^n$, that is, the set $\{x + y : \; x \in P, \; y \in P'\}$.

$P \times Q$ is the *direct product* or *Cartesian product* of the two polyhedra $P \subseteq \mathbb{R}^A$ and $Q \subseteq \mathbb{R}^B$, defined as $P \times Q := \{(x, y) \in \mathbb{R}^{A \cup B} : \; x \in P, y \in Q\}$ (we assume A and B are disjoint).

$\text{rows}(A)$ denotes the set of rows of the matrix A.

$\text{vert}(P)$ is the set of vertices of the polyhedron P.

$x(S)$ denotes the sum $\sum_{i \in S} x_i$ for a vector $x \in \mathbb{R}^n$ and a set $S \subseteq [n]$.

$\text{cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is the cone generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, that is, $\{\mathbf{z} \in \mathbb{R}^n : \; \mathbf{z} = \sum_{i=1}^m \lambda_i \mathbf{v}_i \; (\lambda_i \in \mathbb{R}_+)\}$.

$\text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ denotes the integer cone generated by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, that is, $\{\mathbf{z} \in \mathbb{Z}^n : \mathbf{z} = \sum_{i=1}^m \lambda_i \mathbf{v}_i \; (\lambda_i \in \mathbb{Z}_+)\}$.

Chapter 1

Introduction

1.1 Outline

Polyhedra are beautiful objects with broad applications beyond geometry. They are the main tool of combinatorial optimization as evidenced by the subtitle of Schrijver's combinatorial optimization "bible" [67]. They are also indispensable in applied mathematics as numerous practical problems can be formulated and solved with linear or integer programming.

In this thesis we explore different questions related to polyhedra and integrality. In Chapter 2 we examine several properties of polyhedra from a computational complexity point of view. We prove that testing whether a conic system is TDI – or, equivalently, testing whether a set of vectors forms a Hilbert-basis – is co-NP-complete. This answers a question raised by Papadimitriou and Yannakakis [56] in 1990. We prove also that deciding whether a system describes a generalized polymatroid can be done in polynomial time and the same is true for integer generalized polymatroids. We use a notion called total dual laminarity and prove that it is in contrast NP-hard. In addition, we prove that integer g-polymatroids form a maximal class for which it is true that every pairwise intersection is an integer polyhedron.

In Chapter 3 we state a polyhedral version of Sperner's lemma and deduce a variety of mostly known results from it. We show also that the corresponding complexity problem is PPAD-complete. A new application is a generalization of a theorem of Boros and Gurvich that perfect graphs are kernel-solvable.

In Chapter 4 we define a notion of idealness of set functions which generalizes ideal clutters to set functions instead of set systems, and also related notions like blocker, minors and minimally non-ideal set functions. We prove that many properties concerning these notions extend to set functions.

1.2 Preliminaries on integer polyhedra, TDI-ness and Hilbert bases

A polyhedron is *rational* if it can be described with a rational (or equivalently integral) linear system. A polyhedron is called *integer* if every face of it contains an integer vector. In the case of a polytope (that is, a bounded polyhedron), this is equivalent to the vertices being integer vectors. Edmonds and Giles [19] proved the following equivalent property for rational polyhedra.

Theorem 1.2.1 (Edmonds, Giles [19]). *A rational polyhedron P is integer if and only if for each integer c,* $\max cx : \ x \in P$ *is integer or infinite.*

Total dual integrality of a system of linear inequalities was introduced also by Edmonds and Giles in a different paper [20] and plays an important role in polyhedral combinatorics.

Definition 1.2.2. A linear system $Ax \leq b$ with rational A and b is called *totally dual integral* (or *TDI* for short) if for each integer vector c, the dual system $\{\min y^\top b : \ y \geq 0, \ y^\top A = c^\top\}$ has an integral optimal solution provided the optimum is finite.

They proved that it implies the integrality of the polyhedron:

Theorem 1.2.3 (Edmonds, Giles [20]). *If $Ax \leq b$ is TDI and b is integer, then $\{x : \ Ax \leq b\}$ is an integer polyhedron.*

In other words, for a TDI system, the duality theorem can be stated so that in both sides the vectors are required to be integers. By consequence, the TDI property is a common framework to prove min-max relations in combinatorial optimization.

Giles and Pulleyblank [38] introduced a related notion, namely that of Hilbert bases. A finite set of integer vectors is called a *Hilbert basis* if every integer vector in their cone can be written as a nonnegative integral combination of them. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_m\} \subset \mathbb{Z}^n$ is a Hilbert basis if $\text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) = \text{cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) \cap \mathbb{Z}^n$. Hilbert proved the following (in a different context).

Theorem 1.2.4 (Hilbert [40]). *For every rational polyhedral cone C there exists a Hilbert basis whose generated cone is C.*

In this case we say that it is a Hilbert basis of C.

If F is a face of a polyhedron defined by the system $Ax \leq b$, then a row a_i^T of A is called *active in F* if $a_i^\mathsf{T} x = b_i$ holds for every x in F. Giles and Pulleyblank established the following connection between Hilbert bases and total dual integrality.

Theorem 1.2.5 (Giles, Pulleyblank [38])**.** *For an integer matrix A, the system $Ax \leq b$ is TDI if and only if for every minimal face F of $\{x : \; Ax \leq b\}$ the active rows in F form a Hilbert basis.*

They used this characterization and the existence of a Hilbert basis to prove the following.

Theorem 1.2.6 (Giles, Pulleyblank [38])**.** *For every rational polyhedron P there exists a TDI system which describes P, with an integer constraint matrix.*

The following result on the uniqueness of a minimal Hilbert basis was first proved essentially by van der Corput [77, 76], then rediscovered by Jeroslow [43] and Schrijver [62].

Theorem 1.2.7. *Every pointed rational polyhedral cone has a unique (inclusionwise) minimal Hilbert basis.*

Using this, Schrijver proved the corresponding statement for TDI descriptions of polyhedra. Here a TDI description of a polyhedron is *minimal* if there is no subsystem of it which is TDI and describes the same polyhedron.

Theorem 1.2.8 (Schrijver [62])**.** *Every full-dimensional rational polyhedron P has a unique minimal TDI description $Ax \leq b$ for which A is integer and $P = \{x : \; Ax \leq b\}$. Furthermore, P is an integer polyhedron if and only if b is an integer vector.*

This unique TDI system is called the *Schrijver-system* of P.

Definition 1.2.9. The *tangent cone* of a polyhedron P at a point $x \in P$ is the set of directions in which one can move from x staying in P, that is, $\{v \in \mathbb{R}^n : \; \exists t > 0 : \; x + tv \in P\}$.

The *optimal cone* of a polyhedron P at a point $x \in P$ is the set of linear objective functions for which x is optimal in P.

1.3 Preliminaries on generalized polymatroids

The connection of matroids and linear programming was established by Edmonds who found an explicit inequality description for the independent set polytope of matroids, and showed that its dual linear program can be uncrossed [18]. Building on this, he proved the following polyhedral description of the convex hull of the common independent sets of two matroids [17].

Theorem 1.3.1 (Edmonds, [17])**.** *The common independent set polytope of two matroids M_1 and M_2 is described by*

$$\{x \in \mathbb{R}^n : \;\; x \geq 0,\; x(Z) \leq r_i(Z) \;\; \textit{for } i = 1,2 \textit{ and } Z \subseteq [n]\},$$

where r_1 and r_2 are the rank functions of M_1 and M_2 respectively. Furthermore, the above system is TDI.

TDI-ness implies that the result can be stated as a min-max theorem for the maximum weight of a common independent set of two matroids. Edmonds observed that his techniques and results immediately extended from independent set polytopes to the more general class of polymatroids.

Definition 1.3.2. A *polymatroid* is a polyhedron that can be described by a system

$$x \in \mathbb{R}^n : \;\; x \geq \mathbf{0}, \;\; x(S) \leq b(S) \;\; \forall S \subseteq [n],$$

where b is a nondecreasing submodular function, with $b(\emptyset) = 0$.

In other words, a packing LP with submodular upper bound, corresponding roughly to removing the subcardinality restriction from the rank function of matroids. The techniques of [17] also extend in a straightforward way when we replace one or both of the polymatroids by a contrapolymatroid – a covering LP with a supermodular lower bound.

The notion of generalized polymatroids (g-polymatroids for short) was introduced by Frank [23] to unify objects like polymatroids, contra-polymatroids, base-polyhedra, and submodular polyhedra. To define them, for arbitrary set-functions p, b with $p : 2^{[n]} \to \mathbb{R} \cup \{-\infty\}$ and $b : 2^{[n]} \to \mathbb{R} \cup \{+\infty\}$, let $Q(p, b)$ denote the packing-covering polyhedron

$$Q(p,b) := \{x \in \mathbb{R}^n : \; p(S) \leq x(S) \leq b(S) \;\; \forall S \subseteq [n]\}. \tag{1.1}$$

Note that infinities mean absent constraints. We treat $\pm\infty$ as "integers" for convenience.

Definition 1.3.3. The pair (p, b) is *paramodular* if p is supermodular, b is submodular, $p(\emptyset) = b(\emptyset) = 0$, and the "cross-inequality" $b(S) - p(T) \geq b(S \setminus T) - p(T \setminus S)$ holds for all $S, T \subseteq [n]$. A *g-polymatroid* is any polyhedron $Q(p, b)$ where (p, b) is paramodular; and also the empty set is considered a g-polymatroid.

Any g-polymatroid defined by a paramodular pair was shown by Frank [23] to be non-empty, and $\emptyset$ is included in the definition just for convenience.

Using the methods of Edmonds, Frank [23] showed that several properties of polymatroids extend to g-polymatroids.

Theorem 1.3.4 (Frank [23]). *Paramodular pairs have the following properties.*

(i) For a paramodular pair (p, b), the g-polymatroid $Q(p, b)$ is integral if and only if p and b are integral.

(ii) For two paramodular pairs (p_1, b_1) and (p_2, b_2), the linear system

$$\{x \in \mathbb{R}^n : \ p_i(S) \leq x(S) \leq b_i(S) \text{ for every } S \subseteq [n], i = 1, 2\}$$

describing the intersection of the two g-polymatroids is totally dual integral.

(iii) A g-polymatroid defined by a paramodular pair is never empty. Moreover the defining paramodular pair is unique, and for a g-polymatroid P it can be given as the minima and maxima

$$i(S) := \min_{x \in P} x(S) \qquad \text{and} \qquad a(S) := \max_{x \in P} x(S), \tag{1.2}$$

that is, $P = Q(i, a)$.

Part (ii) implies that the intersection is an integral polyhedron for integral p_i and b_i ($i = 1, 2$).

By part (iii), when (p, b) and (p', b') are paramodular and distinct, $Q(p, b)$ and $Q(p', b')$ are also distinct, or in other words, a non-empty g-polymatroid uniquely determines its defining paramodular pair. However, $Q(p, b)$ may be a g-polymatroid even if (p, b) is not paramodular. In fact, there are various relaxations of the notion of paramodularity that still define g-polymatroids, for example intersecting paramodularity. These kinds of weaker forms are important in several applications because they help recognizing polyhedra given in specific forms to be g-polymatroids.

Theorem 1.3.5 (Frank [23]). *The family of g-polymatroids is closed under the following operations:*

- *translation,*
- *reflection of all coordinates,*
- *projection along coordinate axes,*

- *intersection with a box,*
- *intersection with a plank*
- *taking a face,*
- *direct products,*
- *Minkowski sum,*
- aggregation *with respect to a surjective function* $\varphi : [n] \to [m]$*, which is defined as* $P_\varphi := \{(x(\varphi^{-1}(\{1\}), x(\varphi^{-1}(\{2\})), \dots, x(\varphi^{-1}(\{m\})) \in \mathbb{R}^m : \ x \in P\}$ *for a polyhedron* $P \subseteq \mathbb{R}^n$*.*

Moreover, if the operation involves integer numbers and we apply it to integer g-polymatroids, then the resulting g-polymatroid is also integer.

Here a *box* is a set $\{x \in \mathbb{R}^n : \ c_i \le x_i \le d_i \ \forall i \in [n]\}$, for some numbers c_i and d_i ($i \in [n]$), and a *plank* is a set $\{x \in \mathbb{R}^n : \ e \le x([n]) \le f\}$ for some numbers e and f.

Linear optimization over a bounded g-polymatroid is possible with a greedy algorithm [29]; conversely, a bounded polyhedron P is a g-polymatroid if and only if for every objective $\max\{cx : \ x \in P\}$, the following greedy algorithm is always correct: iteratively maximize the coordinates with positive c-coefficients in decreasing c-order, minimize those with negative c-coefficients similarly, and interleave the maximizations and minimizations arbitrarily [68].

A few characterizations of g-polymatroids are known. One uses base polyhedra, which generalize the convex hull of the bases of a matroid.

Definition 1.3.6. A *base polyhedron* is a set

$$\{x \in \mathbb{R}^n : \ x(S) \le b(S) \ \forall S \subset [n]; \ x([n]) = b([n])\}$$

where b is submodular with $b(\emptyset) = 0$ and $b([n])$ finite.

So each base polyhedron is a subset of the hyperplane $x([n]) = c$ for some constant c. Although base polyhedra are a subclass of g-polymatroids, there is also a useful bijection between the two classes, proved by Fujishige:

Theorem 1.3.7 (Fujishige [30])**.** *If B is a polyhedron in a hyperplane $x([n]) = c$ for some $c \in \mathbb{R}$, then it is a base polyhedron if and only if the projection $\{(x_1, \dots, x_{n-1}) : x \in B\}$ is a g-polymatroid.*

Tomizawa [75] proved another geometric characterization of g-polymatroids concerning the directions of the edges in the bounded case, or the tangent cones in the general case, see Theorem 2.2.3.

Many other general classes of polyhedra with somewhat esoteric definitions have been studied: for example lattice polyhedra [41], submodular flow polyhedra [19], bisubmodular polyhedra [67, §49.11d], and M-convex functions [52]. In some cases the definitions are chosen to be precisely as general as possible while allowing the proof techniques to go through, like Schrijver's framework for total dual integrality with cross-free families [67, §60.3c][63]. Relations among these complex classes are known: Schrijver [64] showed that P is a submodular flow polyhedron if and only if P is a lattice polyhedron for a distributive lattice; and Frank and Tardos [29] showed that P is a submodular flow polyhedron if and only if P is the projection along coordinate axes of the intersection of two g-polymatroids.

See also the surveys [27, 29] and the books [26, 31] as references.

1.4 Preliminaries on ideal and mni clutters

A set system $\mathcal{C}$ on a finite ground set S is called a *clutter* if no set of it contains another. We note that in the context of extremal combinatorics it is called a Sperner system or Sperner family. We will call the sets in a clutter its *edges*. The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is defined as the family of the (inclusionwise) minimal sets that intersect each set in $\mathcal{C}$, in other words the (inclusionwise) minimal transversals of $\mathcal{C}$ (a *transversal* of a set system is a set that intersects each set in the system). It is an easy observation that $b(b(\mathcal{C})) = \mathcal{C}$ for any clutter $\mathcal{C}$. (We regard $\emptyset$ and $\{\emptyset\}$ as clutters too, and they are blockers of each other.)

One of the most well-studied objects of polyhedral combinatorics is the *covering polyhedron* of a clutter:

$$P(\mathcal{C}) = \{x \in \mathbb{R}^S_+ : \ x(C) \geq 1 \text{ for every } C \in \mathcal{C}\}.$$

Of course we can define the covering polyhedron of an arbitrary set system, but it would be equal to the covering polyhedron of the minimal sets of the system, so it is enough to consider clutters. A clutter $\mathcal{C}$ is called *ideal* if the polyhedron $P(\mathcal{C})$ is integer, in which case it has 0-1 vertices.

It is easy to see that the $0-1$-elements in $P(\mathcal{C})$ are the characteristic vectors of the transversals of $\mathcal{C}$, and that $\mathcal{C}$ is ideal if and only if $P(\mathcal{C}) = \mathrm{conv}\{\chi_B : B \in b(\mathcal{C})\} + \mathbb{R}^S_+$. It is known that a clutter is ideal if and only if its blocker is.

We can define two types of minor operations of a clutter $\mathcal{C}$ corresponding to including or excluding an element $s \in S$ into the transversal:

- the *deletion minor* is the clutter $\mathcal{C}\backslash s$ on ground set $S - s$ with edge set $\{C : C \in \mathcal{C}, s \notin C\}$,
- the *contraction minor* is the clutter $\mathcal{C}/s$ on ground set $S - s$ whose edges are the inclusionwise minimal sets of $\{C \setminus s : C \in \mathcal{C}\}$.

A *minor* of $\mathcal{C}$ is a clutter obtained by these two operations (it is easy to see that the order of the operations does not matter).

The minor operations act nicely with the blocker operation: $b(\mathcal{C}/s) = b(\mathcal{C})\backslash s$ and $b(\mathcal{C}\backslash s) = b(\mathcal{C})/s$, and their covering polyhedra can be obtained from the covering polyhedron of $\mathcal{C}$:

$$P(\mathcal{C}/s) = \{x \in \mathbb{R}_+^{S-s} : (x,0) \in P(\mathcal{C})\} \cong P(\mathcal{C}) \cap \{x \in \mathbb{R}^S : x_s = 0\},$$
$$P(\mathcal{C}\backslash s) = \{x \in \mathbb{R}_+^{S-s} : \exists t : (x,t) \in P(\mathcal{C})\} = \mathrm{proj}_s(P(\mathcal{C})).$$

A clutter is *minimally non-ideal* (or *mni* for short) if it is not ideal but all its minors are ideal.

It follows from the above mentioned facts that a clutter is mni if and only if its blocker is. We note that an excluded minor characterization for mni clutters is not known (which would be a counterpart of the strong perfect graph theorem) but Lehman's fundamental theorem stated below says that mni clutters have special structure.

For an integer $t \geq 2$, the clutter $\mathcal{J}_t = \{\{1,2,\dots t\},\{0,1\},\{0,2\},\dots\{0,t\}\}$ on ground set $\{0,1,\dots t\}$ is called the finite degenerate projective plane. It is known that $\mathcal{J}_t$ is an mni clutter whose blocker is itself.

For a clutter $\mathcal{A}$ we denote its edge-element incidence matrix by $M_{\mathcal{A}}$.

Theorem 1.4.1 (Lehman [50]). *Let $\mathcal{A}$ be a minimally nonideal clutter nonisomorphic to $\mathcal{J}_t$ ($t \geq 2$) and let $\mathcal{B}$ be its blocker. Then $P(\mathcal{A})$ has a unique noninteger vertex, namely $\frac{1}{r}\mathbf{1}$, where r is the minimal size of an edge in $\mathcal{A}$, and $P(\mathcal{B})$ has a unique noninteger vertex, namely $\frac{1}{s}\mathbf{1}$, where s is the minimal size of an edge in $\mathcal{B}$. There are exactly n sets of size r in $\mathcal{A}$ and each element is contained in exactly r of them; and similarly for $\mathcal{B}$. Moreover if we denote the clutter of minimum size edges in $\mathcal{A}$ respectively $\mathcal{B}$ by $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, then the edges of $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ can be ordered in such a way that $M_{\overline{\mathcal{A}}}M_{\overline{\mathcal{B}}}^{\mathsf{T}} = M_{\overline{\mathcal{A}}}^{\mathsf{T}}M_{\overline{\mathcal{B}}} = J + dI$, where J is the $n \times n$ matrix of ones, and $d = rs - n$.*

Definition 1.4.2. The clutter $\overline{\mathcal{A}}$ defined above is called the *core* of the mni clutter $\mathcal{A}$.

For a survey of the topic see the book of Cornuéjols [15].

Chapter 2

Complexity of Hilbert bases, TDL-ness and g-polymatroids

A usual type of question in combinatorial optimization is to give the defining linear system of a certain class of combinatorially defined polytopes. Here, a combinatorially defined polytope means the convex hull of the characteristic vectors of some combinatorial objects. For example the polytope of network flows can be described by a small system (the flow conservation constraints for each vertex and the capacity constraints); while the defining system of the traveling salesman polytope is unknown and researchers keep finding new classes of valid inequalities for it. The tractability of a combinatorial problem is related to the size of the defining system: if the system is of polynomial size, then there is a polynomial time algorithm to find a weighted optimal solution, because a linear program can be solved in polynomial time, by the result of Khachiyan [44]. The defining linear system tells us a lot about the structure of the problem and helps to design algorithms. On the other hand, polyhedra are useful not only if a defining system is known, but for example for approximation algorithms, branch and bound methods and integer rounding algorithms.

In this chapter we consider mainly questions from a different point of view: can we decide whether a given linear system has a certain nice property? The analyzed properties – Hilbert bases, g-polymatroids, total dual laminarity – arise in proof techniques in polyhedral combinatorics. In Section 2.1 we prove that recognizing Hilbert bases is hard, which answers a longstanding question of Papadimitriou and Yannakakis; the result appeared in [54]. The results in the following sections of the present chapter are joint work with András Frank, Tamás Király and David Pritchard [28]. In Section 2.6 we give a polynomial-time algorithm to check whether a given linear program defines a generalized polymatroid, and whether it is integral if so. We prove also that in the full-dimensional case TDL-ness characterizes linear systems that define g-polymatroids (Corollary 2.6.9). Additionally, whereas it is known that the intersection of two integral generalized polymatroids is integral, we show in Section 2.3 that no larger class of polyhedra satisfies this property.

2.1 Testing Hilbert bases is hard

In 1990, Papadimitriou and Yannakakis [56] proved that it is co-NP-complete to decide whether a given rational polyhedron is integer. In their paper they raised the questions what the complexity of the recognition problems of TDI systems and Hilbert bases is. Both were open for a long time. First Cook, Lovász and Schrijver [14] showed that if the dimension is fixed then one can decide in polynomial time whether a system is TDI, which also implies that in fixed dimension Hilbert bases are also in P. Recently Ding, Feng and Zang [16] proved that the problem "Is $Ax \geq 1,\ x \geq 0$ TDI?" is co-NP-complete, even if A is the incidence matrix of a graph.

In this section we prove that recognizing Hilbert bases is also hard.

Theorem 2.1.1 ([54]). *The problem of deciding whether or not a set of integer vectors forms a Hilbert basis is co-NP-complete even if the set consists of 0–1 vectors having at most three ones.*

This is a strengthening of the result of Ding, Feng and Zang, since by the theorem of Giles and Pulleyblank 1.2.5, the problem is equivalent to deciding TDI-ness of a conic system $Ax \geq 0$.

Related questions were studied by Henk and Weismantel [39]: they proved that it is NP-complete to decide whether a given vector is in the minimal Hilbert basis of a given pointed cone, moreover in fixed dimension they gave an algorithm that enumerates the vectors of the minimal Hilbert basis of a given cone with polynomial delay. For a survey of the connection of Hilbert bases to combinatorial optimization see Sebő [69].

Let $\mathcal{H} = (V, E)$ be a hypergraph. We call an edge of $\mathcal{H}$ of size 2 a *2-edge* and one of size 3 a *3-edge*. Let us denote by $\mathrm{cone}(\mathcal{H})$ and $\mathrm{int.cone}(\mathcal{H})$ the cone and integer cone of the characteristic vectors of the edges of $\mathcal{H}$. Sometimes we will not distinguish between an edge and its characteristic vector. The binary vectors in Theorem 2.1.1 will consist of the characteristic vectors of 2- and 3-edges.

Proof of Theorem 2.1.1. For the sake of completeness we sketch the proof that the problem is in co-NP. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a set of integer vectors which is not a Hilbert basis and let F be the minimal face of $\text{cone}(S)$. It can be seen that $\text{int.cone}(S \cap F)$ is equal to the lattice generated by $S \cap F$. Thus if there exists an integer vector in F which can not be written as a nonnegative integer combination of vectors in S, then the lattice generated by $S \cap F$ is a proper subset of $F \cap \mathbb{Z}^n$, for which there is a certificate, see [66]. If there does not exist such a vector then it can be seen that there is an integer vector $\mathbf{z}$ in the zonotope of the vectors in S (that is in the set $\{\mathbf{v} : \ \mathbf{v} = \sum_{i=1}^m \lambda_i \mathbf{v}_i, \ 0 \le \lambda_i < 1\}$) for which $z \notin F$, and $\mathbf{z} - \mathbf{v}_i \notin \text{cone}(S)$ for all $\mathbf{v}_i \in S \setminus F$. In this case, $\mathbf{z}$ is a certificate.

To prove completeness we reduce the 3-SATISFIABILITY (3SAT) problem to the complement of this problem. Let $X = \{x_1, \dots, x_p\}$ be the set of variables and $\mathcal{C} = \{c_1, \dots, c_q\}$ be the set of clauses of an arbitrary 3SAT-instance.

Let the clause c_i be $c_i^1 \vee c_i^2 \vee c_i^3$, where $c_i^j \in X \cup \bar{X}$ ($j \in [3]$, $\bar{X}$ denotes the set of negated literals $\{\bar{x}_1, \dots, \bar{x}_p\}$).

We aim at constructing a hypergraph $\mathcal{H} = (V, E)$ (with $|V|$ and $|E|$ linear in p and q and maximal edge size three) such that $\mathcal{C}$ is satisfiable if and only if the characteristic vectors of the edges of $\mathcal{H}$ do not form a Hilbert basis.

Let the ground set V of the hypergraph $\mathcal{H}$ be

$$V = \{u_i, v_j, \bar{v}_j, w_k^l \ (i \in \{0, 1, \dots p + q\}, j \in [p], k \in [q], l \in [3])\},$$

where we say that the nodes v_j and $\bar{v}_j$ correspond to the literals x_j and $\bar{x}_j$, and nodes w_k^1, w_k^2, w_k^3 correspond to the three literals of clause c_k.

Let the edge-set of $\mathcal{H}$ be $E = E_1 \cup E_2 \cup E_3$ (in Figure 2.1, the black 2-edges are in E_1, the blue 2-edges in E_2 and the 3-edges in E_3), where

$$\begin{aligned} E_1 = \{u_0 u_{p+q}\} &\cup \{u_{i-1} v_i, u_{i-1} \bar{v}_i, u_i v_i, u_i \bar{v}_i \ (i \in [p])\} \\ &\cup \{u_{p+k-1} w_k^l, u_{p+k} w_k^l \ (k \in [q], l \in [3])\}, \end{aligned}$$

$$\begin{aligned} E_2 = \{v_j w_k^l &: \text{if } c_k^l = \bar{x}_j \ (j \in [p], k \in [q], l \in [3])\} \\ &\cup \{\bar{v}_j w_k^l : \text{if } c_k^l = x_j \ (j \in [p], k \in [q], l \in [3])\}, \end{aligned}$$

$$\begin{aligned} E_3 = \{u_p v_j w_k^l &: \text{if } c_k^l = \bar{x}_j \ (j \in [p], k \in [q], l \in [3])\} \\ &\cup \{u_p \bar{v}_j w_k^l : \text{if } c_k^l = x_j \ (j \in [p], k \in [q], l \in [3])\}. \end{aligned}$$

Notice that the 3-edges are exactly the 2-edges in E_2 together with u_p. We mention that it is necessary to add E_3 to the construction, since the characteristic vectors of the edges of a non-bipartite graph never form a Hilbert basis. The following claim will be useful:

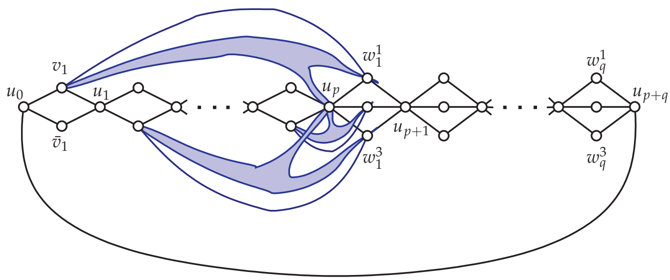

Figure 2.1: Part of hypergraph $\mathcal{H}$ where $c_1 = \bar{x}_1 \vee x_p \vee x_2$

Claim 2.1.2. *$\mathcal{H} - u_p$ is a bipartite graph.*

Proof. The following is a good bipartition of the vertices: $\{u_i \in V : \ i < p\} \cup \{w_k^l \ (k \in [q], l \in [3])\}$ and $\{u_i \in V : \ i > p\} \cup \{v_j, \bar{v}_j \ (j \in [p])\}$. $\diamond$

We call a cycle a *choice-cycle* if its edges are in E_1 and has length $2(p+q)+1$ (these are the only odd cycles in E_1). Such a cycle uses exactly one of $\{v_j, \bar{v}_j\}$ for each $j \in [p]$ and exactly one of $\{w_k^1, w_k^2, w_k^3\}$ for each $k \in [q]$. A cycle is *induced* if its node set does not induce other edges from E.

Claim 2.1.3. *$\mathcal{C}$ is satisfiable if and only if there exists an induced choice-cycle in $\mathcal{H}$.*

Proof. Suppose that $\tau : X \mapsto \{\text{TRUE}, \text{FALSE}\}$ is a satisfying truth assignment for $\mathcal{C}$. Then the nodes u_i $(i \in \{0, 1, \dots p+q\})$, and the nodes in $\{v_j, \bar{v}_j : \ j \in [p]\}$ corresponding to the true literals, and for each $k \in [q]$ one node from $\{w_k^1, w_k^2, w_k^3\}$ which corresponds to a true literal induce a choice-cycle.

On the other hand, if Q is an induced choice-cycle then the assignment

$$\tau(x_j) := \begin{cases} \text{TRUE if } v_j \in V(Q) \\ \text{FALSE if } \bar{v}_j \in V(Q) \end{cases}$$

satisfies $\mathcal{C}$. $\diamond$

Using Claim 2.1.3 we can show that the satisfiability of $\mathcal{C}$ implies that $\{\chi_e : e \in E\}$ is not a Hilbert basis: for an induced choice-cycle Q the incidence vector of its vertex-set, $\chi_{V(Q)}$ is in cone$(\mathcal{H})$ but is not in int.cone$(\mathcal{H})$. This is because every nonnegative integer linear combination which gives $\chi_{V(Q)}$ can only use the edges of Q (Q being an induced cycle, any other edge would contribute with a positive coefficient to some $v \notin V(Q)$), and the characteristic vectors of these edges are linearly independent so there is a unique linear combination of edges of Q that gives $\chi_V(Q)$ and that is the all-1/2 vector.

It remains to prove that if $\mathcal{C}$ is not satisfiable then the incidence vectors of E form a Hilbert basis. Let $0 \neq \mathbf{z} \in \mathbb{Z}^V \cap \text{cone}(\mathcal{H})$. Since $\mathbf{z} \in \text{cone}(\mathcal{H})$, using Carathéodory's theorem, $\mathbf{z} = \sum_{e \in E} \lambda_e \chi_e$ ($\lambda_e \geq 0\ \forall e \in E$), where $\{\chi_e : \ \lambda_e > 0\}$ are linearly independent. We have to show that there exist $\lambda'_e \in \mathbb{Z}_+$ ($e \in E$) for which $\mathbf{z} = \sum_{e \in E} \lambda'_e \chi_e$. It suffices to show that $\sum_{e \in E} \{\lambda_e\} \chi_e$ can be obtained as a nonnegative integer combination of edges ($\{\ .\ \}$ denotes the fractional part), so we can assume that $\lambda_e < 1$ ($\forall e \in E$).

Let us call an edge $e \in E$ positive if $\lambda_e > 0$ (these are exactly the edges with non-integer coefficient) and let us denote the set of positive edges by E^+. For an edge e, let $t(e)$ denote e itself if it is a 2-edge and $e \setminus \{u_p\}$ if it is a 3-edge, and let $G = (V, E')$ be the multigraph with $E' = \{t(e) : \ e \in E^+\}$.

Claim 2.1.4. *G is a cycle (and isolated nodes).*

Proof. A node $v \in V \setminus \{u_p\}$ cannot be a leaf of G because then $\mathbf{z}_v$ would be non-integer.

If Q is a cycle in G then adding the vectors $\{\chi_e : \ \ e \in E^+, t(e) \in Q\}$ with coefficients +1 and -1 alternately regarding $t(e)$ going round Q, starting at u_p if it lies on Q, we get $k\chi_{\{u_p\}}$ where $k \neq 0$ because of the linear independence of the positive edges.

From this and the linear independence of the positive edges it follows that there cannot be two different cycles in G.

From the above observations and Claim 2.1.2 it follows that either G is a cycle or an even cycle and a path from u_p to a node v on the cycle with no other common nodes. But the latter cannot happen either because then the coefficients on the cycle could only be alternately λ and $1 - \lambda$ for some $0 < \lambda < 1$, so $\mathbf{z}_v$ would be non-integer. $\diamond$

Let us denote this cycle by Q. $|V(Q)|$ is greater than 2 because if $|V(Q)| = 2$, then in E^+ vertex u_p would have degree one and hence $\mathbf{z}_{u_p}$ would be non-integer. So by Claim 2.1.4 the hypergraph of the positive edges looks like in Figure 2.2. The cycle Q can be odd or even, and u_p can be on the cycle or not, but if it is not on Q then Q is even because of Claim 2.1.2.

Let us denote the edges of Q by $h_1, h_2, \ldots h_{|E(Q)|}$, beginning from u_p if it lies on Q. We colour an edge $e \in E^+$ red or green if $t(e)$ has an odd respectively even index, and we colour a 2-edge vw red or green if the 3-edge $u_p vw$ is already

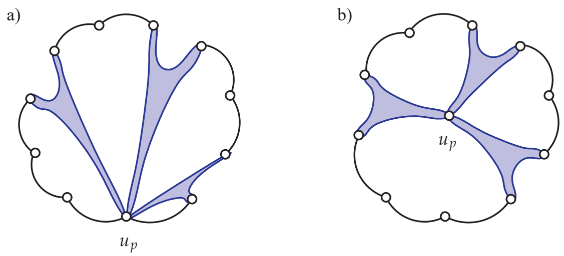

Figure 2.2: Structure of hypergraph (V, E^+) if a) $u_p \in V(Q)$ and b) $u_p \notin V(Q)$

red respectively green. So we coloured every positive edge and $t(e)$ for every positive 3-edge e.

It follows from Claim 2.1.4 that there is a $0 < \lambda < 1$ for which $\lambda_e = \lambda$ if $e \in E_+$ is red and $\lambda_e = 1 - \lambda$ if $e \in E_+$ is green. Thus $\mathbf{z} = \chi_{V(Q)} + c\chi_{\{u_p\}}$ where $c \in \mathbb{Z}_+$.

Suppose there are r red and g green 3-edges.

If $|Q|$ is even then (no matter whether u_p is on Q or not) $c = r\lambda + g(1-\lambda) \leq \max(r, g)$. Let us assume that $r \leq g$ (the other case is similar). Then $\mathbf{z}$ can be obtained as the sum of characteristic vectors of only green edges: we can take c arbitrary green 3-edges and the $|Q|/2 - c$ green 2-edges disjoint from them (except in u_p).

Thus we can suppose that $|Q|$ is odd. In this case u_p is on Q and the two 2-edges in E^+ incident to it have coefficient λ so $c = 2\lambda - 1 + r\lambda + g(1-\lambda) = (r+1)\lambda + (g-1)\lambda \leq \max(r+1, g-1)$.

All vectors of the form $\chi_{V(Q)} + c'\chi_{\{u_p\}}$ (where $c' \in \{1, 2, \ldots, r+1\}$) can be obtained as the sum of $(|Q|+1)/2$ red edges which are disjoint except in u_p. On the other hand, all vectors of the form $\chi_{V(Q)} + c''\chi_{\{u_p\}}$ (where $c'' \in \{0, 1, \ldots, g-1\}$) can be obtained as the sum of $(|Q|-1)/2$ green edges which are disjoint except in u_p. Thus we may assume that $\mathbf{z}$ is not among these from which follows that $\mathbf{z} = \chi_{V(Q)}$ and $g = 0$.

If Q is a choice cycle then because of Claim 2.1.3 $V(Q)$ induces a 3-edge Δ. It follows from the construction of $\mathcal{H}$ that Δ divides Q into three odd length paths so $\mathbf{z}$ can be obtained by adding the characteristic vectors of Δ and every second edge on these paths.

If Q is not a choice cycle then there is an edge vw on Q for which $u_pvw \in E$. We claim that there is one for which the two edge-disjoint paths on Q from u_p

to v and w are odd. If the two paths are even then each path either contains the edge u_0u_{p+q} or contains another edge $v'w'$ with $u_pv'w' \in E$. So in one of the two directions the first edge from u_p with this property will have odd paths from u_p to its endnodes. Adding the characteristic vectors of this 3-edge and every second edge on the two odd length paths yields $\mathbf{z}$ and the proof is complete. □

Remark. In some papers Hilbert bases are defined slightly differently: the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ form a Hilbert basis (in general sense) if

$$\text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) = \text{cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) \cap \Lambda,$$

where Λ is the lattice generated by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. Our proof applies to this definition as well, in fact it can be seen that the lattice generated by the edges in our construction is $\mathbb{Z}^n$: for an arbitrary 3-edge $u_pv_jw_k^l$, $\chi_{u_p} = \chi_{u_pv_jw_k^l} - \chi_{v_jw_k^l}$, and to get χ_v for an arbitrary $v \in V$, we can take an even-length path from u_p to v in E_1 and add it to χ_{u_p} with alternately -1 and 1 coefficients.

We note that in the case when the vectors contain at most two ones, then the problem becomes tractable.

Proposition 2.1.5. *Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{Z}^n$ be 0-1 vectors that contain two ones, and let G be the graph on vertex set $[n]$ of which $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are the characteristic vectors of the edges. Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ form a Hilbert basis if and only if G is bipartite.*

Proof. Suppose first that G is bipartite. Take $\mathbf{x} \in \text{cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) \cap \mathbb{Z}^n$. Thus there exist coefficients $\lambda_i \geq 0$ for which $\mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{v}_i$. This inequality system is described by the incidence matrix of the bipartite graph G, which is well known to be totally unimodular. Therefore the system has an integral solution, that is, $\mathbf{x} \in \text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$.

Now suppose that G is not bipartite. Let $\mathbf{x}$ be the characteristic vector χ_C of a shortest odd cycle C. Then $\mathbf{x} \in \text{cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) \cap \mathbb{Z}^n$ (since it is half the sum of the characteristic vectors of the edges of C) but $\mathbf{x} \notin \text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$. □

Proposition 2.1.6. *If the 0-1 vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{Z}^n$ contain at least two ones, then it can be checked in polynomial time whether they form a Hilbert basis.*

Proof. We can assume that $\mathbf{v}_i \neq \mathbf{0}$ for each $i \in [m]$. Let Z be the set $\{i \in [n] : \mathbf{v}_j = \chi_i \text{ for some } j \in [m]\}$. Let G be the graph on vertex set $[n]$ with edge set $\{ij : i, j \in [n], \mathbf{v}_k = \chi_{\{i,j\}} \text{ for some } k \in [m]\}$. We claim that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ is a Hilbert basis if and only if $G[[n] \setminus Z]$ is bipartite. If $G[[n] \setminus Z]$ is not bipartite, then as in Proposition 2.1.5, the vectors do not form a Hilbert basis.

If $G[[n] \setminus Z]$ is bibartite, then take $\mathbf{x} \in \text{cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) \cap \mathbb{Z}^n$. There exist coefficients $\lambda_i \geq 0$ for which $\mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{v}_i$. Choose them so that their support contains a minimal number of odd cycles. We claim that then the support is bipartite (and singletons). If not, then let C be an odd cycle in the support. Since $G[[n] \setminus Z]$ is bibartite, C contains a vertex i in Z. If we decrease and increase the coefficients corresponding to the edges of C alternatingly by the same amount ϵ, beginning from i, and increase that of χ_i, by 2ϵ, then by choosing ϵ appropriately, we can achieve that the coefficient of one of the edges becomes zero. This is a contradiction, so we proved that the support is bipartite and singletons.

If we restrict the linear inequality system to this support, then the matrix will be totally unimodular, hence the system has an integral solution too. This proves that $\mathbf{x} \in \text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$.

Clearly it can be decided in polynomial time whether $G[[n] \setminus Z]$ is bibartite.

□

2.2 Total dual laminarity and generalized polymatroids

One property of generalized polymatroids used widely in the literature without a name is what we will call "total dual laminarity". Consider a packing-covering system, where every constraint is of the form $x(S) \geq \beta$ or $x(S) \leq \beta$, that is, a polyhedron $Q(p, b)$ for some set functions p, b, as defined in 1.1. In LP duality each such constraint gives rise to a dual variable corresponding to S. Let y^ℓ and y^u be the dual variable vector corresponding to the lower respectively upper bound constraints. If in the primal problem we want to maximize cx over $Q(p, b)$, then the dual is:

$$\{\min y^u b - y^\ell p : \; y^u, y^\ell \geq \mathbf{0}, \; (y^u - y^\ell)\chi = c\}, \tag{2.1}$$

where χ denotes the matrix whose rows are the characteristic vectors χ_S of the subsets S of $[n]$. As a technicality, when $b(S) = +\infty$ (or likewise $p(S) = -\infty$) for some S, the dual variable y^u_S does not really exist, but the notation (2.1) still accurately represents the dual provided that y^u_S is fixed at 0 and the constant $y^u_S b(S)$ term in the objective is ignored – all duals we deal with will have finite objective value, so $y^u_S = 0$ is without loss of generality.

The *support* of a dual solution is the set system consisting of all sets for whom at least one dual variable is nonzero. A set system is *laminar* if for any two members S_i, S_j, either $S_i \subseteq S_j$, or $S_j \subseteq S_i$, or $S_i \cap S_j = \emptyset$. A dual solution is *laminar* if its support is laminar.

Definition 2.2.1. The pair (p, b) is *totally dual laminar* (TDL) if for every primal objective with finite optimal value, some optimal dual solution to (2.1) is laminar.

We note that it would be enough to require a laminar optimal solution for integer objective functions, because then for rational ones we can multiply by the least common denominator; and it can be seen that for a fixed set of constraints, the set of objective functions for which there is an optimal dual solution whose support is contained in the set, is a closed set, which implies that the property would hold for arbitrary objective functions.

As a general application of Edmonds' methods, two key steps in [23] were proving that every paramodular pair is TDL, and the following lemma which is proved by an uncrossing argument.

Lemma 2.2.2 (Implicitly in Frank [23])**.** *The intersection of two TDL systems is totally dual integral.*

One notable application of g-polymatroids is in network design. Frank [24] addresses two flavours of network design problems by using g-polymatroids – undirected pair-requirements and directed uniform requirements. He gives min-max relations and algorithms for edge connectivity augmentation, even subject to degree bounds. In these applications, it is important that g-polymatroids can be defined by skew-submodular or intersecting-submodular functions. Total dual laminarity is the typical property used to show that such functions define g-polymatroids: it is therefore natural that we try to properly understand this property.

Frank [25] and Pritchard [58] showed independently that if (p, b) is totally dual laminar, then the polyhedron $Q(p, b)$ is a g-polymatroid, see Theorem 2.4.1. If in addition p and b are integral, then $Q(p, b)$ is an integral g-polymatroid. This characterizes g-polymatroids as the set of all polyhedra that have at least one TDL formulation. As a negative result, we show in Section 2.4.2 that testing if a given system is TDL is NP-hard.

The main question we are led to consider is: what exactly is necessary and sufficient to define a g-polymatroid? Also, does there exist a polynomial algorithm that, given a linear system, decides if the polyhedron described by it is a(n integral) g-polymatroid? We will answer these questions in Section 2.6.

One might ask if it is true for every g-polymatroid P that *every* (p, b) for which $Q(p, b) = P$ holds is TDL? This is, however, false, as Example 2.6.11 shows. But it is a consequence of our Theorem 2.6.3 that it holds in the special case when P is full-dimensional. In Section 2.6 we also show that there is a polynomial-time algorithm which, for a given system of linear inequalities, determines whether the polyhedron it describes is a g-polymatroid (Theorem 2.6.1). Despite that testing for TDL is NP-hard, the proof uses Theorem 2.4.1, uncrossing methods, and a decomposition theorem for non-full-dimensional g-polymatroids. The method also gives a polynomial-time algorithm to tell whether a g-polymatroid is integral, see Theorem 2.6.15. In contrast, testing an arbitrary polyhedron for integrality [56] or TDI-ness is co-NP-complete [16], the latter even for cones, see Section 2.1.

In Section 2.7 we give a relaxation of paramodularity, called truncation-paramodularity, that guarantees TDL-ness, and can be verified in polynomial time if the finite values of the functions are given as an input. This relaxation enables us to give a short proof of a slight generalization of Schrijver's supermodular colouring theorem.

The following figure summarizes our results for an integer valued pair (p, b) whose finite values are given explicitly as an input.

$$\begin{array}{rl}
 & \begin{array}{ll} (p,b) \text{ paramodular} & \in \mathsf{P} \\ \quad\Downarrow & \\ (p,b) \text{ truncation-paramodular} & \in \mathsf{P} \\ \quad\Downarrow & \end{array} \\
\begin{array}{c} \text{equivalent if } Q(p,b) \\ \text{is full-dimensional} \end{array} & \left\{ \begin{array}{ll} (p,b) \text{ TDL} & \mathsf{NP}\text{-hard} \\ \quad\Downarrow & \\ Q(p,b) \text{ integer g-polymatroid} & \in \mathsf{P} \\ \quad\Downarrow & \\ Q(p,b) \text{ g-polymatroid} & \in \mathsf{P} \end{array} \right.
\end{array}$$

In the proof of Theorem 2.3.1, we will exploit another known characterization, implicitly by Tomizawa [75]. For the sake of completeness, we include a proof, which follows the line of the proof in [31, Thm. 17.1]. Here e_i denotes the ith unit basis vector.

Theorem 2.2.3 (Tomizawa [75])**.** *A polyhedron $Q \subseteq \mathbb{R}^n$ is a g-polymatroid if and only if its tangent cone at each point in Q is generated by some vectors of the form $\pm e_i$ ($i \in [n]$) and $e_i - e_j$ ($i, j \in [n]$).*

Also, a polyhedron $B \subseteq \mathbb{R}^n$ is a base polyhedron if and only if its tangent cone at each point in B is generated by some vectors of the form $e_i - e_j$ ($i, j \in [n]$).

We will use the following lemma.

Lemma 2.2.4. *Let $G = (V, A)$ be a directed graph and $\mathcal{I}(G)$ the set of closed sets of G (that is, the subsets of V with out-degree 0). Then*

$$\mathrm{cone}\{e_i - e_j : (i, j) \in A\} = \{x \in \mathbb{R}^V : x(S) \leq 0\ \forall S \in \mathcal{I}(G);\ x(V) = 0\}.$$

Proof. $\subseteq$: All vectors $e_i - e_j$ for $(i, j) \in A$ clearly satisfy the inequalities, since $(e_i - e_j)(S)$ would be positive only if $i \in S$ and $j \notin S$, but if $S \in \mathcal{I}(G)$, then there is no such arc.

$\supseteq$: Let RS denote the set on the right side and take a vector w in RS. Let us take a set F of arcs in G which forms a spanning forest in the underlying graph of the digraph obtained by contracting the strong components of G. As a first step we assign nonnegative coefficients λ_{ij} to the arcs (i, j) in F so that $w' := w - \sum_{ij \in T} \lambda_{ij}(e_i - e_j)$ is still in RS and that $w'(C) = 0$ for every strong component C. This can be done by taking an arc incident to a leaf in the unassigned subtree of F.

If C is a strong component of G, then it is easy to see that the cone of the vectors $e_i - e_j$ for $i, j \in C$, $(i, j) \in A$ is the entire subspace $\{x \in \mathbb{R}^V : x_i = 0\ \forall i \notin C,\ x(V) = 0\}$. Thus the vector w' is in the cone of arcs going inside a strong component, so we are done. $\square$

Proof of Theorem 2.2.3. By Theorem 1.3.7, it is enough to prove the statement on base polyhedra.

For the "only if" part, suppose that B is a base polyhedron described by the submodular function b and take a vector v in B. Let $\mathcal{D}_v$ denote the family of sets that are tight at v, that is, the sets $S \subseteq [n]$ for which $v(S) = b(S)$. Thus the tangent cone T_v at v is

$$\{x \in \mathbb{R}^n : x(S) \leq 0\ \forall S \in \mathcal{D}_v;\ x([n]) = 0\}.$$

Note that submodularity implies that $\mathcal{D}_v$ is a ring family. We claim that $\mathcal{D}_v$ is the set of closed sets of the digraph G_v on $[n]$ with arc set $A_v = \{(i, j) : \nexists S \in \mathcal{D}_v$ for which $i \in S$ and $j \notin S\}$. Clearly every set in $\mathcal{D}_v$ is a closed sets of G_v. Suppose now that S is a closed set of G_v. It means that for every $i \in S$ and $j \notin S$, there is a set $D_{i,j} \in \mathcal{D}_v$ for which $i \in D_{i,j}$ and $j \notin D_{i,j}$. Thus S can be described as $\cup_{i \in S} \cap_{j \notin S} D_{i,j}$, which, since $\mathcal{D}_v$ is a ring family, is in $\mathcal{D}_v$.

Therefore, by Lemma 2.2.4, we have $T_v = \mathrm{cone}\{e_i - e_j : (i, j) \in A_v\}$, so we are done.

For the "if" part suppose that for every vector v in a polyhedron B, the tangent cone T_v at $v \in B$ is generated by the vectors $e_i - e_j$, $(i, j) \in A_v$ for some $A_v \subseteq [n] \times [n]$, that is, $T_v = \mathrm{cone}\{e_i - e_j : (i, j) \in A_v\}$. Let G_v be the directed graph with vertex set $[n]$ and arc set A_v for $v \in B$. Denote the set of closed sets of G_v by $\mathcal{I}(G_v)$. By Lemma 2.2.4 we have

$$T_v = \{x \in \mathbb{R}^n : \ x(S) \leq 0 \ \forall S \in \mathcal{I}(G_v); \ x([n]) = 0\}. \tag{2.2}$$

This implies that B is described by a system

$$\{x \in \mathbb{R}^n : \ x(S) \leq f(S) \ \forall S \in \mathcal{F}; \ x([n]) = f([n])\}$$

for a set function $f : \mathcal{F} \to \mathbb{R}$, where $\mathcal{F} = \cup\{\mathcal{I}(G_v) : \ v \in B\}$. We take f to be minimal, that is, $f(S) = a(S)$, in which case $v(S) = f(S)$ if $v \in B$ and $S \in \mathcal{I}(G_v)$.

Our goal is to prove that $\mathcal{F}$ is a ring family and f is submodular. Take two sets S and T in $\mathcal{F}$ with $S \setminus T \neq \emptyset$ and $T \setminus S \neq \emptyset$. Since $\max_{x \in B}(\chi_S + \chi_T)x \leq f(S) + f(T) < +\infty$, there is a vector v in B where $\chi_S + \chi_T$ attains its maximum over B. This means that $\chi_S + \chi_T$ is in the optimal cone at v, which is, by equation 2.2, $\text{cone}(\mathcal{I}(G_v) \cup \{\mathbf{1}, -\mathbf{1}\})$. Since $\mathcal{I}(G_v)$ is a ring family, a standard uncrossing argument implies that $S \cap T$ and $S \cup T$ are both in $\mathcal{I}(G_v)$, and thus also in $\mathcal{F}$, so we proved that $\mathcal{F}$ is a ring family. Using the minimality of f, we have

$$f(S) + f(T) \geq v(S) + v(T) = v(S \cup T) + v(S \cap T) = f(S \cup T) + f(S \cap T),$$

so f is submodular. Hence b is a base polyhedron. □

2.3 Intersection integrality

Edmonds' polymatroid intersection theorem was shown in [23] to extend to integral g-polymatroids as well. In this section we prove the following converse statement: if the intersection of a polyhedron P with each integral g-polymatroid is integral, then P is an integral g-polymatroid. By combining this with the g-polymatroid intersection theorem, one obtains that a polyhedron P is an integral g-polymatroid if and only if its intersection with every integral g-polymatroid is integral. In other words, the family of integral g-polymatroids is maximal subject to integral pairwise intersections. We rely on Tomizawa's Theorem 2.2.3 in the proof.

Theorem 2.3.1 ([28])**.** *If P is a polyhedron whose intersection with each integral g-polymatroid is integral, then P is an integral g-polymatroid.*

Proof. Suppose that the nonempty polyhedron P is not an integer g-polymatroid. We want to give an integral g-polymatroid Q for which $P \cap Q$ is not integral. We can assume that P is an integer polyhedron since if not, then $Q_1 = \mathbb{R}^n$ will do.

Assume that P is bounded and integer. Then Theorem 2.2.3 implies that there is an edge of P whose direction v is not in $E := \{\chi_i : i \in [n]\} \cup \{-\chi_i : i \in [n]\} \cup \{\chi_i - \chi_j : i, j \in [n]\}$. Let z be an integer point on this edge. The cube $z + [-1,1]^n$ is a g-polymatroid, thus we can assume that its intersection with P is integer. This implies that v can be chosen $\{0,1,-1\}^n$ and $z + v$ is in P. Since $v \notin E$, there are two coordinates of v which are the same, both 1 or -1, we can assume that $v_1 = v_2 = 1$. The g-polymatroid Q_2 defined by the paramodular pair

$$p(S) := \begin{cases} z_1 + z_2 + 1 & \text{if } S = \{1,2\}, \\ -\infty & \text{otherwise,} \end{cases}$$

$$b(S) := \begin{cases} z_1 + z_2 + 1 & \text{if } S = \{1,2\}, \\ \infty & \text{otherwise,} \end{cases}$$

is the affine hyperplane $z + \{x \in \mathbb{R}^n : x_1 + x_2 = 1\}$ which intersects the edge $z + tv$ in a noninteger vector $z + \frac{1}{2}v$. Thus Q_2 intersects P in a noninteger polyhedron, too.

Assume now that P is an unbounded integer polyhedron. By Theorem 2.2.3, there is a vector z such that the tangent cone of P at z is not generated by vectors in the set E. Since P is integral, we can choose z to be an integral vector. Let C be the cube $z + [-1,1]^n$. Then $P \cap C$ is a bounded polyhedron which is – again by Theorem 2.2.3 – not a g-polymatroid, since the tangent cone at z did not change. Thus we can use the bounded case, which implies that there is a polymatroid Q_3 for which $P \cap C \cap Q_3$ is non-integer. Since the intersection of an integral g-polymatroid with an integral box is again an integral g-polymatroid [29], $C \cap Q_3$ is an integral g-polymatroid which intersects P in a non-integer polyhedron. □

The pseudo-recursive characterization in Theorem 2.3.1 can be refined to ones less dependent on external definitions:

Corollary 2.3.2. *A polyhedron $P \subseteq \mathbb{R}^n$ is an integral g-polymatroid if and only if it has integral intersection with each polyhedron Q of the following form: Q has some fixed integral coordinates $\{c_i\}_{i \in F}$, optionally two distinct coordinates $j, k \notin F$ with fixed integral sum c, and the remaining coordinates free, that is,*

$$\begin{aligned} &Q = \{x \in \mathbb{R}^n : x_i = c_i, \forall i \in F;\ x_j + x_k = c\} \\ &\qquad \textit{or} \\ &Q = \{x \in \mathbb{R}^n : x_i = c_i, \forall i \in F\}. \end{aligned} \tag{2.3}$$

Proof. To prove the easy $\Rightarrow$ direction, it is enough to verify that each such Q is an integral g-polymatroid. This follows from Theorem 1.3.5: Q is a direct product of copies of $\mathbb{R}$, integer singleton sets, and possibly the plank $x_j + x_k = c$.

So now we focus on the $\Leftarrow$ direction: given a polyhedron P which is not an integral g-polymatroid, find an integral g-polymatroid Q of the desired form such that $P \cap Q$ is non-integral. According to the proof of Theorem 2.3.1, there is an integer g-polymatroid Q – either $\mathbb{R}^n$, or an integer box, or the intersection of an integer box with the integer plank $\{x : \ x_j + x_k = c\}$ – so that $P \cap Q$ has a non-integer vertex z. In the third case, direct computation shows that Q is either an $(n-2)$-dimensional box with two fixed integer coordinates, or the direct product of an $(n-2)$-dimensional box with a line segment of the form $\{x : \ x_i + x_j = c, \ell \leq x_i \leq u\}$.

Next, let Q' be the minimal face of Q containing z, and let Q'' be the affine hull of Q'. Now z is a vertex of $P \cap Q'$ since $Q' \subseteq Q$. Also, z is a vertex of $P \cap Q''$ since Q' and Q'' are identical in a neighbourhood of z (by our choice of Q').

We claim Q'' is the desired integral g-polymatroid. This is accomplished by the straightforward verification that no matter which of the three cases we are in, and no matter which face of Q is Q', we can describe Q'' in the desired form. This completes the proof. □

2.4 Properties of total dual laminarity

2.4.1 All TDL systems define generalized polymatroids

The following result was found independently by Frank [25] and Pritchard [58], see also [28]. Here we show that in the rational case it follows easily from Theorem 2.3.1 and Lemma 2.2.2.

Theorem 2.4.1 (Frank [25], Pritchard [58])**.** *If (p, b) is totally dual laminar, then the polyhedron $Q(p, b)$ is a g-polymatroid. If in addition p and b are integral, then $Q(p, b)$ is an integral g-polymatroid.*

Proof. We only proove the result in the case when (p, b) is rational.

Suppose first that (p, b) is integral and TDL. Our goal is to prove that the intersection of $Q(p, b)$ with an integer g-polymatroid Q' is integer and call Theorem 2.3.1 by which in this case $Q(p, b)$ is also a g-polymatroid. Let (p', b') be an integer paramodular pair which defines Q'. By Lemma 2.2.2, the system

$$\begin{aligned}\{x \in \mathbb{R}^n: \quad p(S) &\leq x(S) \leq b(S) \quad \forall S \subseteq [n],\\ p'(S) &\leq x(S) \leq b'(S) \quad \forall S \subseteq [n]\}\end{aligned}$$

defining $Q(p,b) \cap Q'$ is TDI, therefore $Q(p,b) \cap Q'$ is an integer polyhedron. Thus by Theorem 2.3.1 $Q(p,b)$ is an integer g-polymatroid.

If (p,b) is rational, then we can multiply it by the least common denominator and use the same argument. □

The following theorem of Frank is an easy consequence of Theorem 2.4.1.

Corollary 2.4.2 (Frank [23])**.** *If (p,q) is an intersecting paramodular pair, then $Q(p,q)$ is a g-polymatroid.*

Proof. By Theorem 2.4.1 we only need to prove that an intersecting paramodular pair is TDL. This can be proved with a standard uncrossing argument. □

2.4.2 Hardness of total dual laminarity

Theorem 2.4.3 ([28])**.** *Deciding whether a given system is TDL is* NP*-hard.*

Proof. We reduce the 3-dimensional perfect matching problem to it, which is well known to be NP-complete. Let $\mathcal{H} = (V_1, V_2, V_3; \mathcal{E})$ be an instance of the 3-dimensional perfect matching problem, that is, a 3-uniform hypergraph on vertex set $V_1 \cup V_2 \cup V_3$ (where V_1, V_2 and V_3 are disjoint and equal in size) and edge set $\mathcal{E} \subseteq V_1 \times V_2 \times V_3$, where the goal is to find a matching $M \subseteq \mathcal{E}$ which covers all vertices. For convenience we assume that the edges cover V_3. We construct the following linear system consisting only of homogeneous equalities.

$$\begin{aligned}\{x \in \mathbb{R}^{V_1 \cup V_2 \cup V_3}: \quad x(e) &= 0 \quad \forall e \in \mathcal{E},\\ x(v) &= 0 \quad \forall v \in V_1 \cup V_2\}.\end{aligned}$$

The dual system is

$$\begin{aligned}\{y \in \mathbb{R}^{\mathcal{E} \cup V_1 \cup V_2}: \quad \sum_{e: v \in e} y_e &= c_v \quad \forall v \in V_3,\\ y_v + \sum_{e: v \in e} y_e &= c_v \quad \forall v \in V_1 \cup V_2\}.\end{aligned}$$

We claim that this system is TDL if and only if $\mathcal{H}$ has a perfect matching. Since V_3 is covered, a dual solution always exists, and all are optimal, thus the system is TDL if and only if for every objective function c there is a dual solution $y \in \mathbb{R}^{\mathcal{E} \cup V_1 \cup V_2}$ for which $\mathrm{supp}(y)$ is laminar.

Suppose that the system is TDL, and take such a y for $c = \mathbf{1}$. Now every vertex in V_3 has to be covered with an edge e with positive dual variable y_e, and these have to be disjoint. In other words, $\mathrm{supp}(y)$ has to contain a perfect matching.

For the other direction, suppose that M is a perfect matching in $\mathcal{H}$, and let c be an objective function. Let us define y by

$$\begin{aligned} y_e &:= \begin{cases} c_{v_3} & \text{if } e = \{v_1, v_2, v_3\} \in M, \\ 0 & \text{if } e \notin M, \end{cases} \\ y_v &:= c_v - y_e \quad \text{if } v \in e \in M, v \in V_1 \cup V_2. \end{aligned}$$

The support of y is laminar, so we are done. □

2.5 Decomposition of generalized polymatroids

Recall that in n dimensions, a base polyhedron is contained within a hyperplane and thus has dimension at most $(n-1)$. If this holds with equality, we call the base polyhedron *max-dimensional*. We note that a base-polyhedron is max-dimensional if and only if $b(X) + b([n] \setminus X) > b([n])$ for every $\emptyset \neq X \subset [n]$.

Theorem 2.5.1. *Every g-polymatroid is the direct product of at most one full-dimensional g-polymatroid and some (possibly zero) max-dimensional base-polyhedra.*

Every non-max-dimensional base polyhedron is the direct product of some max-dimensional base polyhedra.

Proof. The two statements are equivalent by Fujishige's Theorem 1.3.7, we prove the one concerning g-polymatroids.

Let (p, b) be a paramodular pair which defines the g-polymatroid Q. First let us prove that the affine hull of Q is of the form $\{x \in \mathbb{R}^n : x(A_i) = a_i \ \forall i \in [t]\}$ for some subpartition $\mathcal{A} = \{A_1, A_2, \dots A_t\}$ of $[n]$ and some numbers $a_i \in \mathbb{R}$. We know that the affine hull is the intersection of the implicit equalities (from the system). An equality $x(S) = b(S)$ is implicit if and only if $p(S) = b(S)$ if and only if the equality $x(S) = p(S)$ is implicit. Let us call such a set *fixed-sum*.

If S and T are fixed-sum, then so are $S \cap T$ and $S \cup T$:

$$\begin{aligned} &b(S \cap T) + b(S \cup T) \leq b(S) + b(T) = p(S) + p(T) \leq \\ &\qquad \leq p(S \cap T) + p(S \cup T) \leq b(S \cap T) + b(S \cup T). \end{aligned}$$

Also, if S and T are fixed-sum, then $S \setminus T$ and $T \setminus S$ are also fixed-sum:

$$b(S \setminus T) - p(T \setminus S) \leq b(S) - p(T) = p(S) - b(T) \leq \\ \leq p(S \setminus T) - b(T \setminus S) \leq b(S \setminus T) - p(T \setminus S).$$

It follows that the inclusion-minimal fixed-sum sets form a subpartition, and that every other fixed-sum set is a disjoint union of them. So they form the desired subpartition $\mathcal{A}$.

The empty set is trivially fixed-sum, and if no other set is fixed-sum, then Q is full-dimensional and we are done. If the only fixed-sum sets are the empty set and $[n]$, then Q is a max-dimensional base polyhedron and we are done again. Otherwise, take a fixed-sum set A other than $[n]$ and $\emptyset$. We claim that $Q = Q_1 \times Q_2$, where Q_1 is a base polyhedron on A and Q_2 is a g-polymatroid on $[n] \setminus A$, then we are done by induction. For this, it is enough to prove that Q is the direct product of a polyhedron in $\mathbb{R}^A$ and one in $\mathbb{R}^{[n]\setminus A}$, since we know that by fixing some coordinates in Q, we get a g-polymatroid.

Take $x = (x_A, x_{[n]\setminus A})$, $y = (y_A, y_{[n]\setminus A}) \in Q$. We need to prove that the vector $(x_A, y_{[n]\setminus A})$ is also in Q. For a set $S \subseteq [n]$, by the cross-inequality and that $x(A) = y(A) = p(A) = b(A)$, we have

$$(x_A, y_{[n]\setminus A})(S) = x(S \cap A) + y(S \setminus A) = y(A) - x(A \setminus S) + y(S \setminus A) \\ \leq p(A) - p(A \setminus S) + b(S \setminus A) \leq b(S),$$

and similarly

$$(x_A, y_{[n]\setminus A})(S) = x(S \cap A) + y(S \setminus A) = y(A) - x(A \setminus S) + y(S \setminus A) \\ \geq b(A) - b(A \setminus S) + p(S \setminus A) \geq p(S),$$

so $(x_A, y_{[n]\setminus A}) \in Q$, which completes the proof. □

2.6 Recognizing generalized polymatroids

In this section we give a polynomial-time algorithm that decides whether a given LP of the form (1.1) describes a g-polymatroid. Here the inequalities where $b(S) = +\infty$ or $p(S) = -\infty$ are not part of the input.

Theorem 2.6.1 ([28]). *There is a polynomial-time algorithm that, on input (A, b), determines whether the polyhedron $\{x : \ Ax \leq b\}$ is a g-polymatroid.*

First we deal with the case when the polyhedron is full-dimensional, and also characterize the linear systems that define g-polymatroids; afterwards, we show how to reduce the general case to the full-dimensional one with the help of Theorem 2.5.1.

We can always make the following assumption:

Assumption 2.6.2. *The input polyhedron is* minimally described *in the sense that deleting any inequality would yield a strictly larger polyhedron.*

This is without loss of generality because we can convert an arbitrary description to a minimal one in polynomial time using linear programming.

2.6.1 The full-dimensional case

For full-dimensional polyhedra, the minimal description is known to be unique up to scaling inequalities by a positive scalar; also, every inequality in the minimal description defines a facet. Moreover, a g-polymatroid's facet-defining inequalities are obviously of the form $x(S) \geq \beta$ or $x(S) \leq \beta$ for some S and β. So by scaling we assume all input inequalities are represented by the following families $\mathcal{B}$ and $\mathcal{P}$.

Let $\mathcal{B}$ be the family of all S where $x(S) \leq b(S)$ is part of the input (that is, $b(S) \neq +\infty$). Similarly let $\mathcal{P}$ be the family of all S where $x(S) \geq p(S)$ is part of the input.

Our proof method will use the functions $i(S)$ and $a(S)$ described by (1.2), where $P = Q(p,b)$ is the input polyhedron. Note that for any particular set S, $i(S)$ and $a(S)$ can be computed in polynomial time. Moreover, $i(S) = p(S)$ holds for all $S \in \mathcal{P}$ and similarly for $\mathcal{B}$, by the minimality of the description. The core of our approach is the following new theorem:

Theorem 2.6.3 ([28])**.** *Suppose that for a pair (p,b), the polyhedron $Q(p,b)$ is full-dimensional. Then $Q(p,b)$ is a g-polymatroid if and only if*

(i) for every $S,T \in \mathcal{B}$, $a(S \cup T) + a(S \cap T) \leq b(S) + b(T)$ holds,

(ii) for every $S,T \in \mathcal{P}$, $i(S \cup T) + i(S \cap T) \geq p(S) + p(T)$ holds, and

(iii) for every $S \in \mathcal{B}$ and $T \in \mathcal{P}$, $a(S \setminus T) - i(T \setminus S) \leq b(S) - p(T)$ holds.

The theorem yields our polynomial-time algorithm (the full-dimensional special case of Theorem 2.6.1): simply iterate through every pair of sets in the input, and check these conditions.

Proof. The "only if" direction is the easy one. If $Q(p,b)$ is a g-polymatroid, then (i,a) is paramodular and $Q(p,b) = Q(i,a)$. Since $a(X) \le b(X)$ for all sets X, and a is submodular, we have $a(S \cup T) + a(S \cap T) \le a(S) + a(T) \le b(S) + b(T)$. The other cases are similar.

To prove the "if" part, we will show that (p,b) is TDL, that is, for every objective function $c \in \mathbb{R}^V$ for which a dual optimal solution exists, there is a laminar one. Using Theorem 2.4.1 it follows that $Q(p,b)$ is a g-polymatroid.

Let $M_{\mathcal{B}}$ and $M_{\mathcal{P}}$ be the matrices whose rows are indexed by $\mathcal{B}$ and $\mathcal{P}$ respectively, and where the rows are the characteristic vectors of their indices. Let M denote the matrix $\binom{M_{\mathcal{B}}}{-M_{\mathcal{P}}}$.

For every set $S \subseteq [n]$ with $a(S)$ finite, let $(\beta^S, \pi^S) \in \mathbb{R}_+^{\mathcal{B} \cup \mathcal{P}}$ be an optimal dual for the objective function χ_S, that is,

$$\chi_S = (\beta^S, \pi^S) M \tag{2.4}$$
$$a(S) = (\beta^S, \pi^S)(b, -p). \tag{2.5}$$

Likewise when $i(S)$ is finite, let $(\beta^{-S}, \pi^{-S}) \in \mathbb{R}_+^{\mathcal{B} \cup \mathcal{P}}$ be an optimal dual for the objective function $-\chi_S$, that is,

$$-\chi_S = (\beta^{-S}, \pi^{-S}) M \tag{2.6}$$
$$-i(S) = (\beta^{-S}, \pi^{-S})(b, -p). \tag{2.7}$$

For certain sets S and T we define a vector in $\mathbb{R}^{\mathcal{B} \cup \mathcal{P}}$, which will be used for modifying the dual. Let e_S denote the vector with 1 in the S component and 0 elsewhere – it lies in $\mathbb{R}^{\mathcal{B}}$ or $\mathbb{R}^{\mathcal{P}}$ depending on context. Say that two sets S and T *conflict* if all of $S \cap T, S \setminus T, T \setminus S$ are nonempty; note that a set system is laminar if and only if it has no conflicting pair of sets. Then,

- if $S, T \in \mathcal{B}$ conflict and $a(S)$ and $a(T)$ are bounded, define $u(S,T)$ to be

$$u(S,T) = -(e_S, \mathbf{0}) - (e_T, \mathbf{0}) + (\beta^{S \cup T}, \pi^{S \cup T}) + (\beta^{S \cap T}, \pi^{S \cap T}); \tag{2.8}$$

- if $S, T \in \mathcal{P}$ conflict and $i(S)$ and $i(T)$ are bounded, define $v(S,T)$ to be

$$v(S,T) = -(\mathbf{0}, e_S) - (\mathbf{0}, e_T) + (\beta^{-S \cup T}, \pi^{-S \cup T}) + (\beta^{-S \cap T}, \pi^{-S \cap T}); \tag{2.9}$$

- if $S \in \mathcal{B}$ and $T \in \mathcal{P}$ conflict and $a(S)$ and $i(T)$ are bounded, define $w(S,T)$ to be

$$w(S,T) = -(e_S, \mathbf{0}) - (\mathbf{0}, e_T) + (\beta^{S \setminus T}, \pi^{S \setminus T}) + (\beta^{-T \setminus S}, \pi^{-T \setminus S}). \tag{2.10}$$

Claim 2.6.4. *For the vectors defined above, the following properties hold:*

(a) *The vectors $u(S,T)$, $v(S,T)$, $w(S,T)$ are always nonzero.*

(b) $u(S,T)M = v(S,T)M = w(S,T)M = \mathbf{0}$.

(c) *$u(S,T)$, $v(S,T)$ and $w(S,T)$ are weakly improving directions for the objective function $(b,-p)$.*

Proof. *(a)* If $u(S,T)$ were $\mathbf{0}$, then $\mathrm{supp}((\beta^{S\cup T},\pi^{S\cup T})+(\beta^{S\cap T},\pi^{S\cap T})) = \{S,T\}$. But, using the fact that S and T conflict, it is easy to see that no dual can meet condition (2.4) in the definition of $(\beta^{S\cap T},\pi^{S\cap T})$ and also have support that is a subset of $\{S,T\}$. The arguments for $v(S,T)$ and $w(S,T)$ are similar.

(b) $u(S,T)M = -\chi_S - \chi_T + \chi_{S\cup T} + \chi_{S\cap T} = \mathbf{0}$, and similarly for the other cases.

(c) $u(S,T)(b,-p) = -b(S) - b(T) + a(S\cup T) + a(S\cap T) \leq 0$, this was condition (i). The other cases follow likewise from conditions (ii) and (iii). ◇

Let C be the cone generated by these vectors:

$$C := \mathrm{cone}(\{u(S,T) : S,T\in\mathcal{B} \text{ conflict}\} \cup \{v(S,T) : S,T\in\mathcal{P} \text{ conflict}\} \\ \cup \{w(S,T) : S\in\mathcal{B}, T\in\mathcal{P} \text{ conflict}\}).$$

Claim 2.6.5. *The cone C is pointed, that is, it does not contain any line.*

Proof. For some number N, let z be the vector whose value in the coordinate indexed by each set S is $N+(n-|S|)^2$. We claim that for N sufficiently large, z has positive scalar product with all the generators of C, which will complete the proof. To see this, one part is to observe that $\mathbf{1}\cdot(\beta^X,\pi^X)\geq 1$ for any nonempty X, with equality if and only if $(\beta^X,\pi^X) = (e_X,\mathbf{0})$; and similarly for $-X$. It follows that $\mathbf{1}\cdot u(S,T)$ is nonnegative, with equality only when $(\beta^{S\cup T},\pi^{S\cup T}) = (e_{S\cup T},\mathbf{0})$ and $(\beta^{S\cap T},\pi^{S\cap T}) = (e_{S\cap T},\mathbf{0})$. Furthermore in this case, $z\cdot u(S,T) = (n-|S\cup T|)^2 + (n-|S\cap T|)^2 - (n-|S|)^2 - (n-|T|)^2 > 0$, since S and T conflict. The proof for the other generators of C is similar. ◇

Claim 2.6.6. *If for a dual solution y the affine cone $y+C$ intersects the dual polyhedron only in y, then $\mathrm{supp}(y)$ is laminar.*

Proof. Write $y = (y^u, y^\ell)$. Suppose in contradiction of the claim that there are two conflicting sets $S,T\in\mathcal{B}$, for which y^u_S and y^u_T are positive; the other cases are similar. Then for sufficiently small $\epsilon > 0$, $y' := y + \epsilon u(S,T)$ lies in $y+C$ and has $y'\geq 0$. Moreover, y' is dual feasible because of part *(b)* of Claim 2.6.4, and $y'\neq y$ because of part *(a)*. This contradicts the assumption of the claim. ◇

Due to the above claim it is enough to give an optimal dual solution y for which the intersection of $y + C$ and the dual polyhedron is $\{y\}$. The existence of such a vector follows from the next two claims.

Claim 2.6.7. *If P is a bounded polytope and C is a pointed cone, then there exists a vector $y \in P$ such that $(y + C) \cap P = \{y\}$.*

Proof. Since C is pointed, there is a vector c with which every vector in C has positive scalar product. Let y be maximal in P for the objective c. Then $(y + C) \cap P = \{y\}$. $\diamond$

Claim 2.6.8. *If a linear program with no all-zero rows defines a full-dimensional polyhedron, then the optimal face of the dual is bounded.*

Proof. Write $Ax \leq b$ for the linear program. Suppose for contradiction that the optimal dual face contains a ray. This implies that there is a dual combination $y \geq 0$ of primal inequalities, $y \neq 0$, such that $yA = 0$ and (by optimality) $yb = 0$. Consequently the negative of some constraint can be obtained as a nonnegative combination of other constraints, so this constraint always holds with equality, contradicting full-dimensionality (using that the constraint is not all-zero). $\diamond$

The claims combine as follows: since $Q(p, b)$ is full-dimensional, Claim 2.6.8 implies the optimal face of its dual is bounded. Apply Claim 2.6.7 to the optimal face, obtaining an optimal y such that the only optimal point of $y + C$ is y. Further, by part *(c)* of Claim 2.6.4, any feasible point of $y + C$ is optimal, so y is the only *feasible* point of $y + C$. So Claim 2.6.6 applies and the proof of Theorem 2.6.3 is complete. $\square$

The proof of Theorem 2.6.3 implies the following.

Corollary 2.6.9. *If $Q(p, b)$ is a full-dimensional g-polymatroid, then (p, b) is TDL.*

Theorem 2.6.3 also implies a test for max-dimensional base polyhedra, which will be useful later.

Corollary 2.6.10. *Let $P = \{x : \ Ax \leq b\} \cap \{x : \ x([n]) = c\}$ be of dimension $n - 1$. Then we can test in polynomial time whether P is a base polyhedron.*

Proof. We can convert the linear system to the form $\{x : \ x(S) \leq b(S) \ \forall S \subset [n]; \ x([n]) = c\}$, since if an irredundant inequality is not of the form $\alpha x(S) + \beta x([n] \setminus S) \leq \gamma$, then P is not a base polyhedron.

By Fujishige's theorem 1.3.7 we know that P is a base polyhedron if and only if by projecting away some variable x_n, we get a g-polymatroid in $n - 1$

dimensions. The pair of functions p', b' that define the projection as $Q(p', b') \subseteq \mathbb{R}^{n-1}$ can be obtained easily by

$$p'(S) = c - b([n] \setminus S), \text{ and}$$
$$b'(S) = b(S).$$

We can test whether $Q(p', b')$ is an $(n-1)$-dimensional g-polymatroid by Theorem 2.6.3. $\square$

2.6.2 The general case

The proof method of Theorem 2.6.3 does not work directly in the non-full-dimensional case, because the system is not necessarily TDL, as the following example shows.

Example 2.6.11. Consider the LP with 6 constraints $\{(x_1, x_2, x_3) : x_i + x_j \geq 1,\ x_i + x_j \leq 1\ (i, j \in [3], i \neq j)\}$. It defines a g-polymatroid (the single point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$), but it is not totally dual laminar.

We use the decomposition from Theorem 2.5.1 to get around this obstacle.

Proof of Theorem 2.6.1. It is useful to first check whether the affine hull has the correct form.

Claim 2.6.12. *The affine hull of any g-polymatroid is of the form $\{x : x(A_i) = c_i\ \forall i \in [t]\}$ for some subpartition $\mathcal{A} = \{A_i : i \in [t]\}$ of $[n]$.*

Proof. This follows from Theorem 2.5.1, by observing that the affine hull of a full-dimensional g-polymatroid is all of its ambient space, and the analogue for max-dimensional base polyhedra. $\diamond$

Our algorithm begins by checking whether the polyhedron's affine hull has the form in Claim 2.6.12. Notice that an inequality $a_i x \leq b_i$ is an implicit equality if the minimum of $a_i x$ is b_i, and in this way we can compute a system $A^= x = b^=$ of linear equalities defining the affine hull.

Claim 2.6.13. *We can check in polynomial time whether a given affine subspace $L = \{x : A^= x = b^=\}$ is of the form $\{x : x(A_i) = c_i\ \forall i \in [t]\}$ for some subpartition $\mathcal{A} = \{A_i : i \in [t]\}$ of $[n]$, and find $\mathcal{A}, c$ if so.*

Proof. We may assume that L has this form, and concentrate on the problem of finding $\mathcal{A}, c$. This is because we can run such an algorithm on any L, and then merely check that the output of the algorithm (if it does not crash) satisfies

$\{x : A^= x = b^=\} = L$, which is a matter of seeing if each equality defining one system is implied by the other system, which can be done using a subroutine to compute matrix ranks.

We start identifying parts of the subpartition. For $I \subseteq [n]$ let L_I be the projection of L on to the variables $\{x_i\}_{i\in I}$. We can check in polynomial time whether L_I has full dimension $|I|$, by testing whether there is any vector y such that $yA^=$ is zero on all coordinates of $[n] \setminus I$, and nonzero on at least one coordinate of I.

Observe that $\dim(L_{A_i}) < |A_i|$, and moreover that $\dim(L_I) < |I|$ if and only if I contains some A_i. To begin with, if $\dim(L) = n$ then $L = \mathbb{R}^n$ and the algorithm returns "yes," with $\mathcal{A} = c = \emptyset$. Otherwise, initialize $I = [n]$, then for each element $j \in I$ in turn, delete j from I unless it would cause the new I to satisfy $\dim(L_I) = |I|$. We may set A_1 equal to this final I. Similarly, if $\dim(L_{[n]\setminus A_1}) = n - |A_1|$ then we are done, otherwise we let A_2 be an inclusion-minimal subset of $[n] \setminus A_1$ with $\dim(L_{A_2}) < |A_2|$. Iterating this gives $\mathcal{A}$, then computing c is easy. $\diamond$

Now that we have the subpartition we want to check whether Q is a direct product of some polyhedra on the sets in $\mathcal{A}$ and on $[n] \setminus \cup\mathcal{A}$. Using the following lemma we can compute the linear systems describing these polyhedra if they exist. We denote the ith row of a matrix M by m_i and of a vector v by v_i.

Lemma 2.6.14. *If a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a direct product of two polyhedra $P = P_1 \times P_2$ where $P_1 \subseteq \mathbb{R}^I$ and $P_2 \subseteq \mathbb{R}^{[n]\setminus I}$, then P_1 is described by the system $\{x \in \mathbb{R}^I : A'x \leq b'\}$ and P_2 by the system $\{x \in \mathbb{R}^{[n]\setminus I} : A''x \leq b''\}$, where A' and A'' are the submatrices of A restricted to I and $[n] \setminus I$ respectively and the right hand sides are $b'_i := \max_{x\in P} a'_i x$ and $b''_i := \max_{x\in P} a''_i x$.*

Proof. Let x_I and $x_{[n]\setminus I}$ denote the restrictions of x to I and $[n] \setminus I$ respectively. Let $P' := \{x : A'x_I \leq b',\ A''x_{[n]\setminus I} \leq b''\}$. It is clear that $P \subseteq P'$ since P' consists of inequalities that are valid for P. For the other direction, it is enough to show each $a_i \leq b_i$ is valid for P'. Let x^1 and x^2 maximize a'_i and a''_i respectively in P, then the vector $(x^1_I, x^2_{[n]\setminus I}) \in P$ maximizes both a'_i and a''_i (it is in P because P is a direct product). Thus

$$b'_i + b''_i = a'_i(x^1_I, x^2_{[n]\setminus I}) + a''_i(x^1_I, x^2_{[n]\setminus I}) = a_i(x^1_I, x^2_{[n]\setminus I}) \leq b_i.$$

which shows $a_i x \leq b_i$ is implied by the two inequalities $a'_i x_I \leq b'_i$ and$a''_i x_{[n]\setminus I} \leq b''_i$ that define P'. $\diamond$

With these tools, our algorithm goes as follows. First, check whether the affine hull could be the affine hull of a g-polymatroid, using Claim 2.6.13, and compute the subpartition $\mathcal{A}$. Next we check whether Q is the direct product of some polyhedra on the sets A_i and on $[n] \setminus \cup\mathcal{A}$: using Lemma 2.6.14 we compute the possible linear descriptions of the factors Q_i and then check whether their direct product is Q. We then use Theorem 2.6.3 (respectively Corollary 2.6.10) to check whether Q_i is a g-polymatroid (respectively base polyhedron). □

2.6.3 Recognizing integral generalized polymatroids

We can also decide whether a given linear system of the form (1.1) describes an integer g-polymatroid. Again, there is a difference between the full-dimensional case and the non-full-dimensional case. Suppose p and b are integral. If $Q(p, b)$ is a full-dimensional g-polymatroid, then it is an integral one, since by Corollary 2.6.9, the system is TDL, thus TDI. But $Q(p, b)$ may be a non-integral g-polymatroid when it is non-full-dimensional, see the example at the start of Section 2.6.2.

Nonetheless, we now describe an algorithm to determine whether an arbitrary polyhedron is an integral g-polymatroid. Assume without loss of generality that the system is given by a minimal description, and as in the proof of Theorem 2.6.1 we may assume the description is $Q(p, b)$. Note that p and b must be integral in order for $Q(p, b)$ to be integral. In the full-dimensional case we are done by the above remark. In the case that $Q(p, b)$ is a max-dimensional base polyhedron with $x([n]) = c$, it is additionally necessary that c is integral, but also sufficient by considering the correspondence between base polyhedra and g-polymatroids. Finally, in the general case, observe that the direct product of several g-polymatroids is integral if and only if each individual one is integral, and so it is necessary and sufficient that (p, b) and all c_i are integral.

Note that we change the system during the algorithm, so we may ask whether this is a sufficient condition in any system of the form (1.1). The answer is positive:

Theorem 2.6.15 ([28]). *Suppose that $Q(p, b)$ is a g-polymatroid, and that it is minimally described. Then $Q(p, b)$ is an integer g-polymatroid if and only if p and b are integral and on every fixed-sum set, the sum is integer.*

Proof. The conditions are clearly necessary, because of minimality of the description. For sufficiency suppose that p and b are integral and on every fixed-sum set the sum is integer. It is enough to show that when the full dimensional g-polymatroid respectively max dimensional base polyhedra according to Theorem 2.5.1 have integral describing systems, then by the above remark, they are integer polyhedra and so is $Q(p,b)$. This is implied by the following claim, together with the fact that $b_i' + b_i'' = b_i$.

Claim 2.6.16. *Let Q be a polyhedron for which $Q = Q_1 \times Q_2$ where $Q_1 \subseteq \mathbb{R}^I$ and $Q_2 \subseteq \mathbb{R}^{[n]\setminus I}$. Suppose that $a_i x \leq b_i$ is an inequality in a system of Q which is not redundant and let $a_i' x \leq b_i'$ and $a_i'' x \leq b_i''$ be the inequalities for Q_1 resp. Q_2 according to Lemma 2.6.14. Then one of them is an implicit equality.*

Proof. Let $\dim(Q) = d$. Because $a_i x \leq b_i$ is not redundant, the face $F := \{x \in Q : a_i x = b_i\}$ has dimension at least $d-1$. Let F' and F'' be the faces of Q given by $F' = \{x \in Q : \; a_i' x_I = b_i'\}$ and $F'' = \{x \in Q : \; a_i'' x_{[n]\setminus I} = b_i''\}$. Then $F \subseteq F' \cap F''$. Suppose that $a_i' x \leq b_i'$ and $a_i'' x \leq b_i''$ are not implicit equalities. Then there exists a vector $x^1 \in Q^1$ such that $a_i' x^1 < b_i'$. Let x^2 be a vector in F'' (which is nonempty since F is nonempty). Then $x^3 := (x^1, x^2)$ is in $F'' \setminus F'$. Similarly there exists a vector $x^4 \in F' \setminus F''$. But there can not be two different faces of Q which strictly contain its $(d-1)$-dimensional face F. $\diamond$

This completes the proof of Theorem 2.6.15. $\square$

2.6.4 Oracle model

Since we came up with a polynomial-time algorithm to recognize g-polymatroids when they are presented explicitly, it is also interesting to consider whether the same could be accomplished when the input polyhedron is given in an implicit form. Say that a *linear optimization oracle* for a polyhedron P takes an objective function c as input, and returns a point in P which maximizes cx. The following argument shows that we cannot recognize g-polymatroids with polynomial (in the dimension) number of queries. Consider the *permutahedron*

$$\Pi := \{x \in \mathbb{R}^n : \; x([n]) = \tbinom{n+1}{2}; \; x(S) \geq \tbinom{|S|+1}{2} \; \forall S \subset [n]\}$$

which is a max-dimensional base polytope, whose vertices are the permutations of $[n]$. One may show that, when $n \equiv 2 \pmod 4$, if we delete any one constraint for some S with $|S| = n/2$, the modified polyhedron Π_S is no longer a g-polymatroid. Furthermore, it can be shown that if a query can distinguish Π from Π_S, then that query cannot distinguish Π from $\Pi_{S'}$, where S' is any other $(n/2)$-subset of $[n]$. Therefore, no deterministic algorithm can recognize g-polymatroids with fewer than $\binom{n}{n/2} = \Omega(2^n/\sqrt{n})$ queries.

2.7 Truncation-paramodularity

We introduce a notion which is a relaxation of paramodularity and still implies TDL-ness. It also generalizes some known notions like intersecting paramodularity and near paramodularity ([26]).

Definition 2.7.1. The *upper truncation* of a set function $p : 2^{[n]} \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$p^{\wedge}(S) = \max\{\sum_{Z \in \mathcal{F}} p(Z) : \mathcal{F} \text{ is a partition of } S\},$$

where the trivial partition $\{S\}$ is also allowed. Similarly, the *lower truncation* of a set function $b : 2^{[n]} \to \mathbb{R} \cup \{+\infty\}$ is

$$b^{\vee}(S) = \min\{\sum_{Z \in \mathcal{F}} b(Z) : \mathcal{F} \text{ is a partition of } S\}.$$

We call a set S *separable from below* with respect to b if there is a non-trivial partition $\{S_i : i \in [t]\}$ of S for which $\sum b(S_i) \leq b(S)$. Similarly, S is *separable from above* with respect to p if there is a non-trivial partition $\{S_i : i \in [t]\}$ of S for which $\sum p(S_i) \geq p(S)$.

Note that if S is not separable from below with respect to b, then $b^{\vee}(S) = b(S)$, and if S is not separable from above with respect to p, then $p^{\wedge}(S) = p(S)$.

Definition 2.7.2. The pair (p, b) is *truncation-paramodular* if it satisfies the following:

(i) $b^{\vee}$ satisfies the submodular inequality $b^{\vee}(S \cap T) + b^{\vee}(S \cup T) \leq b^{\vee}(S) + b^{\vee}(T)$ for conflicting sets S, T which are not separable from below with respect to b,

(ii) $p^{\wedge}$ satisfies the supermodular inequality $p^{\wedge}(S \cap T) + p^{\wedge}(S \cup T) \geq p^{\wedge}(S) + p^{\wedge}(T)$ for conflicting sets S, T which are not separable from above with respect to p,

(iii) the cross-inequality $b^{\vee}(S) - p^{\wedge}(T) \geq b^{\vee}(S \setminus T) - p^{\wedge}(T \setminus S)$ holds for every conflicting S and T where S is not separable from below with respect to b and T is not separable from above with respect to p.

Theorem 2.7.3 ([28])**.** *If the pair (p, b) is truncation-paramodular, then it is TDL.*

Proof. We have to show that there is a laminar optimal dual solution for any integral objective function c. We can assume that there is an integral optimal dual solution, since if y is an arbitrary rational optimal dual solution, and N is the lowest common denominator of y, then for the objective function Nc, Ny is an integral optimal dual solution and the set of possible support systems did not change.

Let us order the subsets of $[n]$ in such a way that if $X \subset Y$ then X comes first, that is, we take a linear extension of the poset $(2^{[n]}, \subseteq)$. Let $y = (y^l, y^u)$ be the integral optimal dual solution for which y^l is lexicographically maximal in the above order, and with respect to this, y^u is lexicographically maximal.

We claim that no set in $\mathrm{supp}(y^l)$ is separable from above with respect to p and no set in $\mathrm{supp}(y^u)$ is separable from below with respect to b. Suppose indirectly that for a partition $\{X_i : i \in [t]\}$ of $X \in \mathrm{supp}(y^l)$, $\sum p(X_i) \geq p(X)$ holds. Then by decreasing y^l on X by one and increasing it on each X_i by one, we get an integral optimal dual solution for which the first part is lexicographically larger than y^l, a contradiction. The other part is similar.

Now we claim that $\mathrm{supp}(y^l) \cup \mathrm{supp}(y^u)$ is laminar. Suppose first that there are conflicting sets X, Y in $\mathrm{supp}(y^l)$. Since X and Y are not separable from above with respect to p, inequality $p^\wedge(X \cap Y) + p^\wedge(X \cup Y) \geq p(X) + p(Y)$ holds, with partitions $\mathcal{F}^\cap$ and $\mathcal{F}^\cup$ giving the upper truncation values. Thus if we decrease y^l on X and Y by 1 and increase it on the elements of $\mathcal{F}^\cap$ and $\mathcal{F}^\cup$ by 1, we get again an integral optimal dual solution for which the first part is lexicographically larger than y^l, a contradiction. We can prove similarly that $\mathrm{supp}(y^u)$ is laminar. Now suppose that for $X \in \mathrm{supp}(y^u)$ and $Y \in \mathrm{supp}(y^l)$, X and Y are conflicting. Since X is not separable from below with respect to b and Y is not separable from above with respect to p, inequality $b(X) - p(Y) \geq b^\vee(X \setminus Y) - p^\wedge(Y \setminus X)$ holds, with partitions $\mathcal{F}^1$ and $\mathcal{F}^2$ giving the upper truncation values. Thus if we decrease y^l on X and Y by 1 and increase it on the elements of $\mathcal{F}^1$ and $\mathcal{F}^2$ by 1, we get again an integral optimal dual solution for which the first part is lexicographically larger than y^l, a contradiction. This proves TDL-ness. □

2.7.1 An application: the supermodular colouring theorem

Given a set function f, consider the problem of k-colouring the ground set so that each set S gets at least $f(S)$ different colours. When f is supermodular, or the maximum of two supermodular functions, this "supermodular colouring" problem can be attacked with g-polymatroids, as shown by Schrijver [65].

Theorem 2.7.4 (Schrijver [65]). *Let k be a positive integer and p_1 and p_2 be supermodular functions on ground set $[n]$, for which $p_i(S) \leq \min\{k, |S|\}$ for every set $S \subseteq [n]$.*

Then $[n]$ can be coloured with k colours so that every set $S \subseteq [n]$ contains at least $\max\{p_1(S), p_2(S)\}$ *colours. Moreover there is such a colouring where each colour is used $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ times.*

The proof was simplified by Tardos [74] and Schrijver [67]. Bernáth and Király proved that the statement stays true for skew-supermodular functions as well (as described in [6]). We give a further extension, in order to show an example of using total dual laminarity. Our proof is a descendant of Schrijver's proof [67, §49.11c] and relies on Theorems 2.7.3 and 2.4.1.

Theorem 2.7.5 ([28])**.** *Let k be a positive integer and let f_1 and f_2 be nonnegative integer-valued set functions on ground set $[n]$, which satisfy the following properties:*

(i) $\max\{f_1(S), f_2(S)\} \leq \min\{k, |S|\}$ *for each $S \subseteq [n]$,*

(ii) for every conflicting $S, T \subset [n]$, there exist $U \subseteq S \cup T$ and $I \subseteq S \cap T$ such that $f(U) + f(I) \geq f(S) + f(T)$.

Then $[n]$ can be coloured with k colours so that every set $S \subseteq [n]$ contains at least $\max\{f_1(S), f_2(S)\}$ *colours. Moreover there is such a colouring where each colour is used $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ times.*

Proof. We can assume without loss of generality that f_1 and f_2 have value 1 on every singleton. We use induction on k; the claim is evident for $k = 1$. For the inductive step, we want to define the k-th colour class C so that $f_i'(S) := \max\{f_i(S), \max_{X \subseteq C} f_i(S \cup X) - 1\}$ $(i = 1, 2)$ fulfill the criteria on ground set $[n] \setminus C$ with $k-1$ colours. Equivalently, C has to satisfy $p_i(S) \leq |C \cap S| \leq b_i(S)$ $(i = 1, 2)$ for every set $S \subseteq [n]$, where

$$\begin{aligned} p_i(S) &:= \begin{cases} 1 & \text{if } S \text{ is minimal such that } f_i(S) = k, \\ -\infty & \text{otherwise}, \end{cases} \\ b_i(S) &:= |S| - f_i(S) + 1. \end{aligned}$$

In other words, $\chi_C \in Q(p_1, b_1) \cap Q(p_2, b_2)$. In addition, we also require that $\lfloor n/k \rfloor \leq |C| \leq \lceil n/k \rceil$.

We claim that (p_i, b_i) is a truncation-paramodular pair for $i = 1, 2$. First, p_i clearly satisfies (ii) of Definition 2.7.2, since the minimal sets on which f_i has value k are disjoint.

Let S and T be conflicting and not separable from below with respect to b_i. There exist $U \subseteq S \cup T$ and $I \subseteq S \cap T$ such that $f_i(U) + f_i(I) \geq f_i(S) + f_i(T)$.

Using that b_i is 1 on each singleton, we have

$$b_i^\vee(S\cup T) \le b_i(U) + |(S\cup T)\setminus U| = |S\cup T| - f_i(U) + 1 \text{ and}$$
$$b_i^\vee(S\cap T) \le b_i(I) + |(S\cap T)\setminus I| = |S\cap T| - f_i(I) + 1,$$

hence

$$\begin{aligned} b_i(S) + b_i(T) &= |S| + |T| - f_i(S) - f_i(T) + 2 \\ &\ge |S\cup T| + |S\cap T| - f_i(U) - f_i(I) + 2 \ge b_i^\vee(S\cup T) + b_i^\vee(S\cap T). \end{aligned}$$

Finally we show that (iii) of Definition 2.7.2 is trivially satisfied because there are no conflicting sets S and T with that property. Let S be a minimal set such that $f_i(S) = k$, and let T be a conflicting set; we claim that T is separable from below with respect to b_i. Indeed, we know that there are sets $U \subseteq S\cup T$ and $I \subseteq S\cap T$ such that $f_i(U) + f_i(I) \ge f_i(S) + f_i(T)$. We have $f_i(U) \le k = f_i(S)$, hence $f_i(I) \ge f_i(T)$. This gives $b_i(T) \ge b_i(I) + |T\setminus I|$, so the partition $\{I, \{v : v \in T\setminus I\}\}$ shows that T is separable from below with respect to b_i.

Since (p_i, b_i) is truncation-paramodular and integer, it is TDL by Theorem 2.7.3. Therefore by Theorem 2.4.1, $Q(p_i, b_i)$ is an integer g-polymatroid ($i = 1, 2$). Thus the polyhedron $Q(p_1, b_1) \cap Q(p_2, b_2) \cap \{x : \lfloor n/k \rfloor \le \mathbf{1}x \le \lceil n/k \rceil\}$ is integral. It is also non-empty, because the vector $\frac{1}{k}\mathbf{1}$ is an element. We can choose an arbitrary set C whose characteristic vector is in the polyhedron, and get the remaining $k-1$ colour classes by induction. □

Remark. If f is a skew-supermodular function, then we can construct a function f' by $f'(S) = 0$ if $f(S) \le 0$ or there is a set $T \subsetneq S$ such that $f(T) \ge f(S)$, and $f'(S) = f(S)$ otherwise. The set function f' satisfies the properties of Theorem 2.7.5, and a feasible colouring for f' is also feasible for f. Thus Theorem 2.7.5 is a generalization of the skew-supermodular colouring theorem in [6].

2.7.2 Checking truncation-paramodularity in polynomial time

In contrast to the hardness of checking total dual laminarity, truncation-paramodularity of a pair (p, b) can be checked in polynomial time if the input consists of the finite values of the two functions.

Theorem 2.7.6 ([28]). *Let $p : 2^{[n]} \to \mathbb{Z} \cup \{-\infty\}$ and $b : 2^{[n]} \to \mathbb{Z} \cup \{+\infty\}$ be set functions, given by an explicit enumeration of their finite values. We can decide in polynomial time if (p, b) is a truncation-paramodular pair.*

Proof. Let $\mathcal{B}$ and $\mathcal{P}$ be the families of all sets where b respectively p is finite. We first show an algorithm that decides if $b^\vee$ satisfies the submodular inequality for conflicting sets which are not separable from below with respect to b (non-separable for short), and at the same time identifies all non-separable sets in $\mathcal{B}$.

We enumerate all sets in $\mathcal{B}$ and all conflicting pairs $S, T \in \mathcal{B}$ in one series $A_1, A_2, \ldots, A_k$ in an order of increasing size, where the size of a pair is the size of the union. We consider the sets in this order. Suppose that for a given index t we have already identified all non-separable sets with index smaller than t, and we have established that the submodular inequality for $b^\vee$ holds for all conflicting non-separable pairs of index smaller than t.

Suppose first that A_t is a set $S \in \mathcal{B}$.

Claim 2.7.7. *For any $T \subsetneq S$, $b^\vee(T) = \max\{x(T) : x(Z) \le b(Z)\ \forall Z \subseteq T\}$.*

Proof. Let $\gamma = \max\{x(T) : x(Z) \le b(Z)\ \forall Z \subseteq T\}$. At this point of the algorithm we know that the set function $b^\vee|_{\{Z: Z\subseteq T\}}$ is submodular on conflicting non-separable pairs. Therefore the LP $\max\{x(T) : x(Z) \le b(Z)\ \forall Z \subseteq T\}$ has a laminar dual optimal solution y, which satisfies $yb = \gamma$ and $y\chi = \chi_T$. By laminarity, the inclusionwise maximal elements of $\operatorname{supp}(y)$ form a partition $\mathcal{F}$ of T.

We claim that $\sum_{Z\in\mathcal{F}} b(Z) = \gamma$. Indeed, let $\epsilon = \min\{y_Z : Z \in \mathcal{F}\}$. If $\sum_{Z\in\mathcal{F}} b(Z) > \gamma$, then we can construct a dual solution y' of objective value smaller than γ by

$$y'_Z = \begin{cases} \frac{y_Z - \epsilon}{1-\epsilon} & \text{if } Z \in \mathcal{F}, \\ \frac{y_Z}{1-\epsilon} & \text{if } Z \notin \mathcal{F}. \end{cases}$$

This would contradict the optimality of y, thus $b^\vee(T) = \gamma$. $\diamond$

Due to the claim we can test in polynomial time whether S is non-separable: we can compute $b^\vee(S \setminus T)$ for every $T \in \mathcal{B}$ which is a subset of S. Then S is non-separable if and only if $b^\vee(S \setminus T) + b(T) > b(S)$ for any such T.

Suppose now that A_t is a conflicting pair $S, T \in \mathcal{B}$. We have already checked if both are non-separable; let us assume that they are. A proof similar to the proof of the above claim shows that we can compute $b^\vee(U)$ for any $U \subsetneq S \cup T$. Thus we can compute $b^\vee(S \cap T)$, and we can also determine $b^\vee(S \cup T)$ by computing $b^\vee((S \cup T) \setminus U)$ for every $U \in \mathcal{B}$ which is a subset of $S \cup T$. Therefore we can decide whether $b(S) + b(T) \ge b^\vee(S \cap T) + b^\vee(S \cup T)$ holds. This concludes the description of the first algorithm.

An analogous algorithm can be used to decide if $p^\wedge$ satisfies the supermodular inequality for conflicting sets which are not separable from above with respect to p, and to identify all non-separable sets in $\mathcal{P}$.

It remains to check whether the cross-inequality for $p^\wedge$ and $b^\vee$ holds for conflicting non-separable pairs. Since we have already identified non-separable sets, and we can compute $p^\wedge$ and $b^\vee$ on any set by linear programming, this can be done in polynomial time. $\square$

Chapter 3

Polyhedral Sperner's Lemma and applications

In this chapter we will present some results which are related to the multidimensional generalization of Sperner's well-known lemma about colourings of triangulations. Our versions deal with colourings of the vertices or facets of polytopes and polyhedra. Then we will show how to apply one of these versions to combinatorially defined polyhedra in order to get short proofs of several purely combinatorial results. Most of these results are known in the literature, but the results in Section 3.3.2 and 3.3.3 and the method are joint work with Király [46, 47]. After the applications we will address possible converses [53] and also the complexity of corresponding computational problems; this is also joint work with Király [49].

3.1 About polarity

First we prove a generalization of the well-known fact that the polar of a polytope is combinatorially polar to it (see for example [78]). We show that we do not have to restrict ourselves to polytopes which have the origin in the interior: the property holds for pointed polyhedra which contain the origin, provided that we extend the definition of faces and the face lattice to unbounded pointed polyhedra in the appropriate way. We will use this to prove versions of Sperner's Lemma for unbounded polyhedra.

Let us first introduce the extended notion of vertices.

Definition 3.1.1. The *ends* of a pointed polyhedron P are its vertices and its extreme directions (an *extreme direction* of a polyhedron is an extreme ray of its characteristic cone).

We also extend the notions of faces, face lattice, and combinatorial equivalence slightly to pointed polyhedra, which we now describe. In addition to usual faces we consider "faces at infinity" in the following way. For an infinite direction d (that is, an element of the characteristic cone) and an objective function c, the *objective value* of d is $+\infty$, $-\infty$ or "finite" depending on whether cd is positive, negative or 0. A subset Φ of the characteristic cone of P is a *face at infinity* if there is an objective function c for which the elements in Φ have finite objective value, and all other elements of the characteristic cone have objective value $-\infty$. This implies that Φ is a cone. The face-dimension of a face at infinity is the dimension of it as a set in $\mathbb{R}^n$ minus 1. The set of inequalities that are incident to a face Φ at infinity can be described as those whose hyperplanes contain a certain infinite direction (one in the relative interior of Φ). The set of all infinite directions is also a face at infinity because of the objective function $\mathbf{0}$; it is a facet if and only if the dimension of the characteristic cone of P equals the dimension of P.

For a face or face at infinity of P, its *end-set* is the set of ends it contains. In the face lattice of P we include also the faces at infinity and extend the lattice operations according to the containment of the end-sets, or equivalently, according to the containment of the faces themselves together with the infinite directions they contain.

Definition 3.1.2. Two polyhedra are called *combinatorially equivalent* if their face lattices are isomorphic. Two polyhedra are called *combinatorially polar* if their face lattices are opposite to each other.

Definition 3.1.3. The *polar* of a polyhedron P is the polyhedron

$$P^{\Delta} := \{c \in \mathbb{R}^n : cx \leq 1 \text{ for all } x \in P\}.$$

We note that the polar of a polyhedron equals the polar of the convex hull of the polyhedron and the origin.

Claim 3.1.4. *The polar can be described the following way.*

(i) If $P = \operatorname{conv}(V) + \operatorname{cone}(D)$, *then*

$$P^{\Delta} = \{c \in \mathbb{R}^n : cv \leq 1 \ \forall v \in V,\ cd \leq 0 \ \forall d \in D\}.$$

(ii) If $\mathbf{0} \in P$ *and* $P = \{x \in \mathbb{R}^n : Ax \leq \mathbf{1},\ Bx \leq \mathbf{0}\}$, *then*

$$P^{\Delta} = \operatorname{conv}(\mathbf{0}, \operatorname{rows}(A)) + \operatorname{cone}(\operatorname{rows}(B)).$$

Proof. (i) $\subseteq$: It is obvious that $cv \leq 1$ for each $c \in P^\Delta$ and $v \in V$. If $x \in P$ and $d \in D$, then $x + td \in P$ for each nonnegative t, thus $c(x + td) \leq 1$, which means that cd has to be nonnegative.

$\supseteq$: If $x \in P$, then there are coefficients $\lambda_v \geq 0$ ($v \in V$) and $\mu_d \geq 0$ ($d \in D$) for which $\sum_{v \in V} \lambda_v = 1$ and $x = \sum_{v \in V} \lambda_v v + \sum_{d \in D} \mu_d d$, thus if c is in the set in the claim, then $cx = c(\sum_{v \in V} \lambda_v v + \sum_{d \in D} \mu_d d) \leq \sum_{v \in V} \lambda_v = 1$.

(ii) $\supseteq$: if $c = \sum_i \lambda_i A_i + \sum_i \mu_i B_i$, where $\lambda_i, \mu_i \geq 0$ and $\sum_i \lambda_i \leq 1$, then $cx = \sum_i \lambda_i A_i x + \sum_i \mu_i B_i x \leq \sum_i \lambda_i \leq 1$ for every $x \in P$, so c is in P^Δ.

$\subseteq$: Suppose that $c \notin \mathrm{conv}(\mathbf{0}, \mathrm{rows}(A)) + \mathrm{cone}(\mathrm{rows}(B))$, that is $\nexists \lambda_i, \mu_i \geq 0$ for which $\lambda_i, \mu_i \geq 0$ and $\sum_i \lambda_i \leq 1$ and $c = \sum_i \lambda_i A_i + \sum_i \mu_i B_i$. By Farkas' Lemma, we get that $\exists x \in \mathbb{R}^n, \xi \in \mathbb{R}$ for which $\xi \leq 0$, $A_i x + \xi \leq 0$ for every row A_i, $B_i x \leq 0$ for every row B_i, and $cx + \xi > 0$. If $\xi < 0$ then this implies that $x' := \frac{1}{-\xi} x \in P$ and $cx' > 1$, thus $c \notin P^\Delta$. If $\xi = 0$ then x is an infinite direction of P, but for $x' \in P$ and large enough t, $c(x' + tx) > 1$ although $x' + tx \in P$, which implies that $c \notin P^\Delta$. $\square$

Proposition 3.1.5. *If P is a full-dimensional pointed polyhedron and $\mathbf{0} \in P$, then P^Δ is combinatorially polar to P.*

Proof. Suppose that $P = \mathrm{conv}(V) + \mathrm{cone}(D)$, where $V = \{v_1, v_2, \dots v_k\}$ and $D = \{d_1, d_2, \dots d_l\}$ are the sets of finite and infinite ends. By (i) of Claim 3.1.4, $P^\Delta = \{c \in \mathbb{R}^n : cv \leq 1 \ \forall v \in V,\ cd \leq 0 \ \forall d \in D\}$. We claim that a subset S of $V \cup D$ forms a set of ends of a face if and only if the corresponding inequalities of P^Δ are the ones containing a face of P^Δ.

We can assume that $S = \{v_1, v_2, \dots v_r\} \cup \{d_1, d_2, \dots d_s\}$ (where r and s might be 0). S is the set of ends of a face of P if and only if there is an objective function $c \in \mathbb{R}^n$ and a number α for which

$$\begin{aligned} cv_i &= \alpha \qquad && \forall i \leq r, \\ cv_i &< \alpha && \forall i > r, \\ cd_i &= 0 && \forall i \leq s, \\ cd_i &< 0 && \forall i > s. \end{aligned}$$

Since $\mathbf{0} \in P$, $\alpha \geq 0$. If $\alpha > 0$, then c and α can be scaled so that $\alpha = 1$, in which case the above mean that c is in P^Δ and the tight inequalities are those corresponding to S.

If $\alpha = 0$, then the above conditions mean that c as an infinite direction is contained by the hyperplanes of the inequalities corresponding to S. This proves our claim.

This gives a bijection between the faces of P and the faces of P^{Δ}, which clearly reverses containment, so we are done. □

Proposition 3.1.6. *For every pointed polyhedron P there is a polytope that is combinatorially equivalent to it.*

Proof. We can assume that P is full-dimensional, otherwise we restrict ourselves to the affine hull of P. Let us translate P to contain the origin in its interior and then take its polar P'. By the full-dimensionality of P, P' is a polytope, and it is combinatorially polar to P; moreover, P' is full-dimensional since P is pointed. If we do the same a second time, we get a polytope P'' which is combinatorially equivalent to P. □

We note that the above claims are true for arbitrary polyhedra as well, provided we extend the above definitions even more in the appropriate way. The face lattice of a polyhedron P will be the same as the face lattice of the pointed polyhedron $P \cap L^{\perp}$, where L is the lineality space of P.

3.2 Polyhedral versions of Sperner's Lemma

Let us recall the multidimensional Sperner Lemma first. A *triangulation* of an n-dimensional simplex Σ is a simplicial complex of which the underlying space is Σ. We colour the vertices of the triangulation with $n+1$ colours in such a way that the vertices of Σ get different colours and if a vertex of the triangulation lies on a face of Σ, then its colour also appears on one of the vertices of that face. We call a simplex of the triangulation *multicoloured* if it has a vertex of every colour.

Theorem 3.2.1 (Sperner's Lemma, [72])**.** *For any triangulation of a simplex Σ and any colouring of the form above, there is a multicoloured simplex in the triangulation.*

For a colouring of the vertices of a polytope P, a facet of P is *multicoloured* if it contains vertices of every colour. For a colouring of the facets of P, a vertex of P is *multicoloured* if it lies on facets of every colour.

The following theorem is a variant of the multidimensional Sperner Lemma. It is basically Sperner's Lemma applied to (a subdivision of) the Schlegel diagram of a polytope (the Schlegel diagram of a polytope is a polyhedral complex obtained by projecting the complex of the faces to a facet from a point near the facet outside the polytope; see [78] for a reference). We present a simple direct proof.

Theorem 3.2.2. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P with n colours such that F_0 is multicoloured. Then there is another multicoloured facet.*

Proof. Let us divide the non-simplex facets of P into simplices. We show that there is a multicoloured simplex. Let $[n]$ be the set of all colours.

Define a graph whose nodes are the simplices in the division and there is an edge between two simplices if and only if they share an $(n-2)$-dimensional facet whose vertices use each colour in $[n-1]$ exactly once. It is easy to see that in this graph the multicoloured simplices are of degree one, while the simplices which use one colour in $[n-1]$ twice, and the others once, are of degree two. The other simplices are of degree zero, so the graph is the disjoint union of paths and cycles (and isolated vertices). The assumption implies that F_0 is a node of degree one, so there has to be another node of degree one which gives a multicoloured simplex. □

A *simple vertex* of an n-dimensional polyhedron is a vertex that lies on exactly n facets. By polarity, the following theorem is also true.

Theorem 3.2.3. *Let P be an n-dimensional polytope, with a simple vertex v_0. Suppose we have a colouring of the facets of P with n colours such that v_0 is multicoloured. Then there is another multicoloured vertex.*

Now we generalize the above results to unbounded pointed polyhedra, this is the version that we will apply later.

Theorem 3.2.4. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If the ends (that is the vertices and extreme directions) of P are coloured with n colours such that the extreme directions receive different colours, then there is a multicoloured facet.*

Proof. Let us take a polytope P' that is combinatorially equivalent to P, which exists by Proposition 3.1.6. Let F be the facet of P' which corresponds to the infinite facet of P. The assumption on P implies that F is a multicoloured simplex facet. We can apply Theorem 3.2.2. □

Theorem 3.2.5. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If the facets of the polyhedron are coloured with n colours such that facets containing the i-th extreme direction do not get colour i, then there is a multicoloured vertex.*

Proof. Let us take again a polytope P' which is combinatorially equivalent to P, and let F be the simplex facet of P' which corresponds to the infinite facet of P. Let us attach a (sufficiently flat) simplex to P' on facet F, and colour the new facets so that the facet opposite (in the simplex) to the vertex corresponding to the i-th extreme direction gets colour i. Applying Theorem 3.2.3 we get that there is another multicoloured vertex of P' (besides the new vertex of the simplex) and from the assumption it follows that this cannot correspond to an extreme direction, so it corresponds to a vertex of P. □

Remark. We will use the above result for polyhedra given by a linear system, where we assign colors to the inequalities. This is not the same as colouring the facets, since here, non-facet-defining inequalities are also coloured, and a facet can get more colours. But this is not a problem, since we can ignore the non-facet-defining and multiple inequalities and then use Theorem 3.2.5. The asserted multicoloured vertex is a multicoloured vertex in the original system too.

We will apply Theorem 3.2.5 to polyhedra of the form $P = Q - \mathbb{R}^n_+$ where Q is a bounded polytope. We will need the following lemma. If $a \in \mathbb{R}^n_+$ and $J \subseteq [n]$, then we denote by a^J the vector whose j-th coordinate is

$$a^J(j) := \begin{cases} a(j) & \text{if } j \in J \\ 0 & \text{if } j \notin J. \end{cases}$$

Lemma 3.2.6. *If $P = Q - \mathbb{R}^n_+$ where $Q = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ is a bounded polytope and A and b are nonnegative, then P is described by inequalities of the form $a_i^J x \leq b_i$, where a_i is the i-th row of A, b_i is the i-th coordinate of b, and $\emptyset \neq J \subseteq \operatorname{supp}(a_i)$. The extreme directions of P are $-e_j$ ($j = 1, \ldots n$).*

Proof. It is clear that the vectors in Q satisfy the inequalities. If $p \in P$, then there is a vector in Q such that $p \leq q$, so p also satisfies the inequalities since their coefficients are nonnegative.

For the other direction, let x be a vector that satisfies all inequalities. Let y be a maximal vector with the same property for which $y \geq x$. Suppose that $y_j < 0$; then $j \notin J$ whenever y satisfies the inequality $a_i^J y \leq b_i$ with equality, since otherwise $a_i^{J \setminus \{j\}} y > b_i$. This means that y_j could be increased without violating the conditions, which contradicts the maximality of y. We can conclude that $y \geq 0$, so $y \in Q$, which means that $x \in P$. □

3.3 Using the polyhedral Sperner Lemma

3.3.1 Kernel-solvability of perfect graphs

In a directed graph $D = (V, A)$, a stable set $S \subseteq V$ is said to be a *kernel* if from every node of $V \setminus S$ there is an arc to S. Kernels have several applications in combinatorics and game theory, and there has been extensive work on the characterization of digraphs that have kernels. See [9] for a survey on the topic.

One approach to characterize the existence of kernels has been to identify undirected graphs for which every "nice" orientation has a kernel. This led to the introduction of the notion of kernel-solvability by Berge and Duchet [5].

Definition 3.3.1. Let $G = (V, E)$ be an undirected graph. A *superorientation* of G is a directed graph $\vec{G}$ obtained by replacing each edge uv of G by an arc uv or an arc vu or both. A *one-way cycle* in a superorientation is a directed cycle consisting of arcs that are not present reversed in the digraph. We define a *source node* of an induced subdigraph $\vec{G}[U]$ as a node in U from which there are arcs to all of its neighbours in $G[U]$. A superorientation is *clique-acyclic* if no clique contains a one-way cycle (equivalently, if every clique contains a source node). A graph G is *kernel solvable* if every clique-acyclic superorientation of G has a kernel.

Berge and Duchet [5] conjectured that a graph is kernel solvable if and only if it is perfect. The kernel-solvability of perfect graphs was proved by Boros and Gurvich [8].

Theorem 3.3.2 (Boros, Gurvich [8])**.** *Every perfect graph is kernel solvable.*

Their proof uses elaborate game-theoretic machinery, and is based on Scarf's Lemma [61], a result originating in game theory. Later Aharoni and Holzman [2] gave a short proof using Scarf's Lemma directly; a concise version of the proof can be found in section 65.7b of Schrijver's book [67]. We present a similarly short proof that relies on the more familiar Sperner's Lemma instead.

In the proof we use the following result, which follows from Fulkerson's theorem [32] on the description of the clique-polytope of a perfect graph and Lovász' weak perfect graph theorem [51].

Theorem 3.3.3 (Fulkerson [32] + Lovász [51])**.** *If G is a perfect graph, then its stable set polytope* $\mathrm{STAB}(G)$ *is described by the clique-inequalities:*

$$\mathrm{STAB}(G) = \{x \in \mathbb{R}^n_+ : x(C) \leq 1 \text{ for every maximal clique } C\}.$$

Proof of Theorem 3.3.2. Let $G = (V, E)$ be a perfect graph, with $V = [n]$, and let $\vec{G}$ be a clique-acyclic superorientation of G. Let $\mathcal{C}$ denote the set of all (not necessarily maximal) cliques of G. We consider the polyhedron $P := \mathrm{STAB}(G) - \mathbb{R}^n_+$. Theorem 3.3.3 and Lemma 3.2.6 imply that

$$P = \{x \in \mathbb{R}^n : x(C) \leq 1 \text{ for every } C \in \mathcal{C}\},$$

and the extreme directions of P are $-e_j$ ($j = 1, \ldots n$).

Let the colour of a facet $\{x \in P : x(C) = 1\}$ be a source node j of clique C. Clearly the extreme direction $-e_j$ does not belong to a facet of colour j, so by applying Theorem 3.2.5 we get that there exists a multicoloured vertex x^* of P. By the definition of P, $x^* = \chi_S$ for a maximal stable set S.

Since x^* is multicoloured, for each node j of V, there is a clique C such that the facet $\{x \in P : x(C) = 1\}$ contains x^* and has colour j. This means that $|C \cap S| = 1$ and j is a source node of C. Thus from each node $j \notin S$ there is an arc to S, so S is a kernel. □

Note that it follows easily from the Strong Perfect Graph Theorem [13] that non-perfect graphs are not kernel solvable. One needs the observations that odd holes and odd antiholes are not kernel solvable, and that induced subgraphs of kernel solvable graphs are kernel solvable. Thus we can state the conjecture of Berge and Duchet as a theorem.

Theorem 3.3.4 (Boros, Gurvich [8] + Chudnovsky, Robertson, Seymour, Thomas [13])**.** *A graph is perfect if and only if it is kernel solvable.*

We mention that of the "if" direction no proof is known that does not rely on the Strong Perfect Graph Theorem.

In the following sections we present new results on kernels in superorientations of non-perfect graphs. Since these graphs are not kernel solvable, we have to make additional restrictions on the superorientation. In Section 3.3.2 we show a result where these restrictions depend on the facets of $\mathrm{STAB}(G)$ (the convex hull of the characteristic vectors of the stable sets of G). The result is specialized to h-perfect graphs in Section 3.3.3, where it is shown that every clique-acyclic and odd-hole-acyclic superorientation of an h-perfect graph has a kernel. The reverse implication is not true here, but the result can be slightly strengthened (Theorem 3.3.8), and we conjecture that this stronger property characterizes h-perfect graphs.

3.3.2 Generalization based on the facets of STAB(G)

In this section we extend Theorem 3.3.2 to arbitrary undirected graphs, provided some conditions hold that depend on the facets of STAB(G). Let $\vec{G}$ be a superorientation of a graph $G = (V, E)$, and let U be a set of vertices. We say that $\vec{G}$ is *one-way-cycle-free* (or *owc-free*) *in* U if there is no one-way cycle in $\vec{G}[U]$.

Theorem 3.3.5. *If* STAB$(G) = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$, *where A and b are nonnegative, and $\vec{G}$ is a superorientation of G which is owc-free in* supp(a) *for every row a of A, then there is a kernel in $\vec{G}$.*

Proof. Let $P = \text{STAB}(G) - \mathbb{R}^n_+$. By Lemma 3.2.6, P is described by the inequalities of the form $a_i^J x \leq b_i$, where a_i is the i-th row of A, b_i is the i-th coordinate of b, and $\emptyset \neq J \subseteq \text{supp}(a_i)$.

Let the colour of a facet of the form $P \cap \{x : a_i^J x = b_i\}$ be a source node of the subdigraph of $\vec{G}$ induced by J. Such a source node exists because $\vec{G}$ is owc-free in supp(a_i). In order to apply Theorem 3.2.5, we have to show that a facet containing the j-th extreme direction does not have colour j. This is true because in this case $j \notin J$. Thus Theorem 3.2.5 implies that P has a multi-colured vertex $x^* = \chi_S$ for a maximal stable set S.

For every node j, there is a facet F of colour j containing x^*. Let F be $P \cap \{x : a_i^J x = b_i\}$. Then $S \cap J$ is a maximal stable set in $G[J]$ because $a_i^J x^* = b_i$ and $\text{supp}(a_i^J) = J$. This and $j \in J$ imply that either $j \in S$ or j has a neighbour in $S \cap J$, in which case j has an out-neighbour in $S \cap J$, since j is a source node of J. We proved that S is a kernel of $\vec{G}$. $\square$

3.3.3 Kernels in h-perfect graphs

Sbihi and Uhri [60] introduced the class of *h-perfect graphs* as the graphs for which the stable set polytope is described by the following set of inequalities:

$$x_v \geq 0 \qquad \text{for every } v \in V, \tag{3.1}$$
$$x(C) \leq 1 \qquad \text{for every maximal clique } C, \tag{3.2}$$
$$x(Z) \leq \frac{|Z|-1}{2} \qquad \text{for every odd hole } Z. \tag{3.3}$$

In addition to perfect graphs, it is known that the class of h-perfect graphs includes

- all graphs containing no odd-K_4-subdivision (see [37]),

- all near-bipartite graphs containing no odd wheel and no prime antiweb except for cliques and odd holes (this is implicitly in [71]),
- line graphs of graphs that contain no odd subdivision of $C_5 + e$ (see [11]).

To apply Theorem 3.3.5 to h-perfect graphs, let us call a superorientation of a graph *odd-hole-acyclic* if no oriented odd hole is a one-way cycle. Our result is as follows.

Theorem 3.3.6. *If $\vec{G}$ is a superorientation of an h-perfect graph and is clique-acyclic and odd-hole-acyclic, then it has a kernel.*

Proof. Directly follows from Theorem 3.3.5. □

A notion related to kernel-solvability is kernel-perfectness: a digraph is called *kernel-perfect* if all of its induced subdigraphs have kernels. Theorem 3.3.6 has the following consequence.

Corollary 3.3.7. *It is in co-NP to decide whether a given superorientation of an h-perfect graph is kernel-perfect.*

Proof. If a superorientation of a clique has a one-way Hamiltonian cycle, then it has no kernel; and a one-way odd hole has also no kernel. Thus by Theorem 3.3.6, a superorientation of an h-perfect graph is kernel-perfect if and only if it is clique- and odd-hole-acyclic (since an induced subgraph of an h-perfect graph is also h-perfect). If not, it can be witnessed by a one-way cycle which forms an odd hole or is in a clique. □

Obviously a superorientation of a perfect graph is always odd-hole-acyclic, thus Theorem 3.3.6 is an extension of Theorem 3.3.2.

We give here a less elegant but stronger theorem for which we conjecture that the converse also holds (see Section 3.4.1).

Let G be an h-perfect graph, and let $\vec{G}$ be a clique-acyclic superorientation of G. Some odd holes of G may become one-way cycles; let us denote these by $Z_1, \dots, Z_k$. Let us select nodes $v_1, \dots, v_k$ such that $v_i \in Z_i$ for $i = 1, \dots, k$ (the selected nodes need not be distinct). We call this a *superorientation with special nodes*. An *almost-kernel* for a superorientation with special nodes is a stable set S with the following property:

> If a node $v \notin S$ has no outgoing arc into S, then $v = v_i$ for some i and $|Z_i \cap S| = (|Z_i| - 1)/2$.

Theorem 3.3.8. *If G is an h-perfect graph, then every clique-acyclic superorientation with special nodes has an almost-kernel.*

Proof. The proof is almost the same as the proof of Theorem 3.3.5. Let $P = \mathrm{STAB}(G) - \mathbb{R}^n_+$. Here again we colour a facet of the form $P \cap \{x : a_i^J x = b_i\}$ with a source node of the subdigraph of $\vec{G}$ induced by J, if there is such a node. If not, then a_i^J is the characteristic vector of a one-way odd hole Z_l; in this case, let its colour be the selected v_l.

Theorem 3.2.5 implies that P has a multicoloured vertex $x^* = \chi_S$ for a maximal stable set S, so for every node j, there is a facet $F = P \cap \{x : a_i^J x = b_i\}$ of colour j containing x^*. If j is a source node of the subdigraph of $\vec{G}$ induced by J, then we obtain (by the same argument as in the proof of Theorem 3.3.5) that either $j \in S$ or there is an arc from j to S. Otherwise a_i^J is the characteristic vector of a one-way odd hole Z_l, $j = v_l$, and $|Z_l \cap S| = (|Z_l| - 1)/2$ since $a_i^J x^* = b_i$. Thus S is an almost-kernel. □

Note that this theorem is stronger than Theorem 3.3.6 since every almost-kernel in a clique-acyclic and odd-hole-acyclic orientation is a kernel.

3.3.4 Scarf's Lemma

In one of his fundamental papers on game theory [61], Scarf proved that a balanced n-person game with non-transferable utilities (NTU) always has a non-empty core. The proof is based on a theorem on the existence of a dominating vertex in certain polyhedra, which became known as "Scarf's Lemma". The proof he gave is based on a finite (not necessarily polynomial) algorithm.

The interest in the lemma has been renewed in combinatorics when Aharoni and Holzman used it to give a short and elegant proof of the kernel-solvability of perfect graphs.

The relation of Scarf's Lemma and Sperner's Lemma has already been mentioned in Scarf's original paper [61], and later it has been studied by other authors (see for example [59]). We show an even stronger link between the two theorems: essentially, Scarf's Lemma for a polyhedron P corresponds to the polyhedral Sperner's Lemma 3.2.5 for the polyhedron $P - \mathbb{R}^n_+$. This gives a new proof of Scarf's Lemma.

In Scarf's Lemma we consider a bounded polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ where A is an $m \times n$ nonnegative matrix (with non-zero columns) and $b \in \mathbb{R}^m$ is a positive vector. In addition, for every row $i \in [m]$ of A, a total order $<_i$ of the columns (or a subset of them) is given. We denote the domain of $<_i$ by

$\mathrm{Dom}(<_i)$. If $j \in \mathrm{Dom}(<_i)$ and $K \subseteq \mathrm{Dom}(<_i)$, we use the notation $j \leq_i K$ as an abbreviation for "$j \leq_i k$ for every $k \in K$".

The central notion in Scarf's lemma is that of a dominating vertex.

Definition 3.3.9. A vertex x^* of P *dominates* column j if there is a row i where $a_i x^* = b_i$ and $j \leq_i \mathrm{supp}(x^*) \cap \mathrm{Dom}(<_i)$ (this implies that $j \in \mathrm{Dom}(<_i)$).

Theorem 3.3.10 (Scarf's Lemma, [61])**.** *Let P be as above and let $<_i$ be a total order on $[n]$ ($i \in [m]$). Then P has a nonzero vertex that dominates every column.*

We state another version, which will be more convenient to prove. A vertex x^* of P is *maximal* if by increasing any coordinate of x^* we leave P (or formally, $(\{x^*\} + \mathbb{R}^n_+) \cap P = \{x^*\}$).

Theorem 3.3.11 (Scarf's Lemma, alternate version)**.** *Let P be as above and let $<_i$ be a total order on $\mathrm{supp}(a_i)$ ($i \in [m]$), where a_i is the i-th row of A. Then P has a maximal vertex that dominates every column.*

Note that in Theorem 3.3.10 we cannot guarantee the maximality of the dominating vertex. Consider the following two-dimensional example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; 1 <_1 2, \; 1 <_2 2.$$

Here the only vertex that dominates every column is $(0,1)$, which is not maximal since $(1,1)$ is also a vertex.

On the other hand, Theorem 3.3.10 follows fairly easily from Theorem 3.3.11 by changing the 0 coefficients in the matrix A to some small positive values such that the facet-defining inequalities remain the same and the vertex sets of the facets remain also the same except for possible fission.

We now show that Scarf's Lemma (Theorem 3.3.11) follows from Theorem 3.2.5.

Proof. Let $P = \{x \in \mathbb{R}^n : Ax \leq b, \; x \geq 0\}$ be the polyhedron as in Scarf's Lemma, and consider the polyhedron $Q := P - \mathbb{R}^n_+$. Because P is bounded, Q has n extreme directions: $-e_j$ ($j \in [n]$). Since A and b are nonnegative, the vertices of Q are the maximal vertices of P. By Lemma 3.2.6, the inequalities which define Q are of the form $a_i^J x \leq b_i$, where $a_i^J := a_i \chi_J$ for an index set J, and we can assume that $J = \mathrm{supp}(a_i^J)$.

Let us colour a face which is defined by inequality $a_i^J x \leq b_i$ with the index $j \in J$ which is the smallest in the ordering $<_i$. If a facet contains the extreme direction $-e_l$ for some l, then the l-th component of its defining inequality is zero, so the colour of the facet is different from l. So we can apply Theorem 3.2.5, and get that there is a vertex x^* of Q (thus a maximal vertex of P) which is multicoloured. We have to show that x^* satisfies the criteria of Scarf's Lemma. If j is an arbitrary index, then there is a j-coloured facet $a_i^J x = b_i$ containing x^*, which means that $j \leq_i \mathrm{supp}(a_i^J) = J$. Since x^* is also a vertex of P, it is nonnegative, so $a_i x^* \geq a_i^J x^* = b_i$, but we know that $a_i x^* \leq b_i$, thus the facet $a_i x = b_i$ of P contains x^*. On the other hand this implies also that $\mathrm{supp}(x^*) \cap \mathrm{supp}(a_i) \subset J$ which with $j \leq_i J$ means that $j \leq_i \mathrm{supp}(x^*) \cap \mathrm{supp}(a_i) = \mathrm{supp}(x^*) \cap \mathrm{Dom}(<_i)$. Thus x^* dominates column j. □

3.3.5 Fractional core of NTU games and stable matchings of hypergraphs

The role of Scarf's Lemma in game theory can be described in several different ways. Here we use a combinatorial approach that does not require the definition of all the basic terms of game theory. We prove the result using the polyhedral version of Sperner's lemma instead of Scarf's Lemma.

A possible definition of a *finitely generated non-transferable utility (NTU) game* is as follows. There are *m players*, and a finite multiset of *basic coalitions* $S_j \subseteq [m]$ ($j \in [n]$). We may interpret a coalition as a possible action performed by a set of players, thus several different coalitions may be formed by the same set of players. Each player i has a total order $<_i$ of the basic coalitions that he participates in; $S_j <_i S_k$ means that the player i prefers coalition S_k to coalition S_j. We can assume that every player is in at least one coalition.

A set $\mathcal{S}$ of basic coalitions is said to be in the *core* of the game if they are disjoint and for each basic coalition S' not in $\mathcal{S}$ there is a player $i \in S'$ and a basic coalition $S \in \mathcal{S}$ such that $S' <_i S$. In other words, an element of the core is a subpartition formed of basic coalitions, such that every basic coalition S' not in the subpartition has a player who is in a block of the subpartition that she prefers to S'.

A related concept is the *fractional core* of the game: a vector $x \in \mathbb{R}_+^n$ is in the fractional core if for each player i,

$$\sum_{j: i \in S_j} x(j) \leq 1,$$

and for each $j \in [n]$ there is a player i in S_j such that

$$\sum_{k:i\in S_k} x(k) = 1$$

and $S_j \leqslant_i S_k$ whenever $i \in S_k$ and $x(k) > 0$.

To motivate this definition, we can imagine that the action performed by each basic coalition can have an intensity (between 0 and 1), and the condition is that the sum of the intensities of the actions that a given player participates in is at most 1. Such a vector of intensities is in the fractional core if there is no basic coalition where every member wants to increase its intensity. It is an easy observation that integer-valued elements in the fractional core are exactly the elements of the core.

Let us call a vector $x \in \mathbb{R}^n_+$ *admissible* if

$$\sum_{j:i\in S_j} x(j) \leq 1$$

for every player i. We prove the following version of Scarf's Theorem.

Theorem 3.3.12 (Scarf [61]). *The fractional core of a finitely generated NTU-game is always non-empty. If the polyhedron of admissible vectors is integer, then the core is also non-empty.*

Proof. The admissible vectors form the polytope $P = \{x \in \mathbb{R}^n_+ : \quad \sum_{j:i\in S_j} x(j) \leq 1 \ \forall i \in [m]\}$. Let Q be the polyhedron $P - \mathbb{R}^n_+$. By Lemma 3.2.6, Q is described by the inequalities $\sum_{j\in J} x_j \leq 1$, where $i \in [m]$ and $J \subseteq \{k \in [n] : \ i \in S_k\}$. Let us colour this facet by the smallest index in J in the order $<_i$. Q has as extreme directions $-e_i$ ($i \in [n]$), and the colouring fulfils the criterion in Theorem 3.2.5 since the coefficient of the colour of an inequality is nonzero. Thus by Theorem 3.2.5 there exists a multicoloured vertex x^*. That means, for each colour (or basic coalition) $j \in [n]$, there is a player $i \in [m]$ and a set $J \subseteq \{k \in [n] : \ i \in S_k\}$ such that x^* satisfies the inequality corresponding to the pair $\{i, J\}$ with equality, that is, $\sum_{l\in J} x^*_l = 1$ and the smallest index in J in the order $<_i$ is j. This implies that $\sum_{k:i\in S_k} x^*_k = 1$ and that every l for which $i \in S_l$ and $x^*_l > 0$ has to be in J. This means that x^* is in the fractional core of the game.

If P is integer then the vertex x^* guaranteed by Theorem 3.2.5 is integer, which implies that it is in the core. □

Biró and Fleiner [7] noticed that the above game-theoretic result is the same as a result of Aharoni and Fleiner [1] about stable matchings in hypergraphs, which we now explain. They define a *hypergraphic preference system* as a pair

$(\mathcal{H}, \mathcal{O})$, where $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph and $\mathcal{O}$ is a family of linear orders $\{\leq_v : v \in V\}$, $\leq_v$ being an order on the set $\delta(v)$ of edges in $\mathcal{H}$ containing vertex v. A *matching* in a hypergraph is a set of disjoint edges. A *stable matching* of a hypergraphic preference system $(H, \mathcal{O})$ is a matching M such that for every $e \in \mathcal{E}$ there is a vertex $v \in e$ an edge $m \in M$ for which $m \geq_v e$.

A *fractional matching* in a hypergraph is a nonnegative vector $x \in \mathbb{R}^{\mathcal{E}}$ for which $\sum_{e \in \delta(v)} x_e \leq 1$ for each $v \in V$; their polytope is called the *fractional matching polytope*. A fractional matching is called *stable* if every edge $e \in \mathcal{E}$ contains a vertex v for which $\sum_{f \in \delta(v): f \geq_v e} x_f = 1$. Aharoni and Fleiner proved the following.

Theorem 3.3.13 (Aharoni, Fleiner [1]). *Every hypergraphic preference system has a fractional stable matching.*

Proof. Theorem 3.3.12 directly implies this, with the following "dictionary": the players and the coalitions of the game correspond to the vertices and edges of the hypergraph, thus a set of coalitions is in the core if the corresponding set of edges form a stable matching, and a vector in the fractional core of the game corresponds to a stable fractional matching. □

It also follows that there is a stable matching if the fractional matching polytope of the hypergraph is integer. A result of Lovász tells us when this is the case.

Definition 3.3.14. A *partial hypergraph* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph $\mathcal{H}' = (V, \mathcal{E}')$ for which $\mathcal{E}' \subseteq \mathcal{E}$. The *chromatic index* of a hypergraph is the least number of colours by which the edges can be coloured so that edges with the same colour are disjoint. A hypergraph $\mathcal{H}$ is called *normal* if for every partial hypergraph $\mathcal{H}'$ of $\mathcal{H}$, the chromatic index of $\mathcal{H}'$ equals the minimum degree of $\mathcal{H}'$.

Theorem 3.3.15 (Lovász [51]). *The fractional matching polytope of a hypergraph $\mathcal{H}$ is integer if and only if $\mathcal{H}$ is normal.*

This implies the following.

Corollary 3.3.16. *If $\mathcal{H}$ is a normal hypergraph, then every hypergraphic preference system $(\mathcal{H}, \mathcal{O})$ has a stable matching.*

In the context of NTU games this translates to the following result, which was first proved by Boros, Gurvich and Vasin [10].

Corollary 3.3.17 (Boros, Gurvich, Vasin [10]). *If the hypergraph defined by the basic coalitions is normal, then the core of the game is non-empty.*

3.3.6 Stable half-matchings

The traditional interpretation of stable matchings in a graph is the so-called stable roommates problem, where we want to assign pairs of students to college rooms so that there are no two students who prefer each other to their assigned roommates. Formally, let $G = (V, E)$ be an undirected graph, possibly with parallel edges, but no loops and no isolated vertices. For every $v \in V$ we are given a total order $<_v$ of the edges incident to v, where $uv <_v wv$ means that v prefers w to u. The set of these total orders is denoted by $\mathcal{O}$, and the pair $(G, \mathcal{O})$ is called a *graphic preference system*. For two edges e and f with a common endnode v, the notation $e \leqslant_v f$ is used if $e <_v f$ or $e = f$.

Definition 3.3.18. A *stable matching* of the graphic preference system $(G, \mathcal{O})$ is a matching M of G with the property that every edge $e \in E$ has an endnode v that is covered by a matching edge $vw \in M$ for which $e \leqslant_v vw$.

A *stable half-matching* is a vector $x : E \to \{0, \frac{1}{2}, 1\}$, for which

- $\sum_{v:uv \in E} x(uv) \leq 1$ for every $u \in V$,
- every edge $e \in E$ has an endnode v where $\sum_{f \geqslant_v e} x(f) = 1$.

In their celebrated paper [33], Gale and Shapley proved that every bipartite preference system has a stable matching, and they provided an efficient algorithm that finds one. However, if we consider arbitrary graphic preference systems, it is easy to see that not all of them have a stable matching, for example if G is a triangle and the nodes prefer the edge going to the next node in cyclic order to the edge going to the previous node. Nevertheless, Irving [42] gave a polynomial algorithm that decides if there is a stable matching, and, relying on this, Tan [73] observed the following.

Theorem 3.3.19 (Tan [73])**.** *Every graphic preference system has a stable half-matching.*

Proof. Let $(G, \mathcal{O})$ be a graphic preference system and V the vertex set of G. We consider the fractional matching polytope $P = \{x \in \mathbb{R}^E_+ : \; x(\delta(v)) \leq 1 \; \forall v \in V\}$, where $\delta(v)$ denotes the set of edges incident to v. By a result of Balinski [3], the polyhedron P is half-integral.

By Lemma 3.2.6, the down hull $Q := P - \mathbb{R}^E_+$ of P is $\{x \in \mathbb{R}^E : x(Y) \leq 1 \;\text{ for every star } Y \subseteq E\}$. Let us colour the facets of Q in the following way: if a facet corresponds to a star Y which has v as center vertex, then its colour shall be the smallest edge in Y according to $<_v$.

Theorem 3.2.5 asserts that there is a multicoloured vertex x^* of Q, which is half-integer by Balinski's Theorem. Multicolouredness means that for every edge e there is a star Y with center vertex v whose smallest edge according to $<_v$ is e and for which $x^*(Y) = 1$. Thus x^* is a stable half-matching. □

We remark that stable half-matchings that are maximal vertices of P have an interesting property that seems to be peculiar to this problem: all of them are non-integer on the same set of edges. More precisely, the graph has a given set of disjoint odd cycles so that every stable half-matching that is a vertex of P has value $\frac{1}{2}$ on exactly the edges of these cycles. This immediately gives the following corollary.

Theorem 3.3.20 (Tan [73])**.** *Let x^* be a stable half-matching that is a vertex of P. Then x^* is integer if and only if the preference system has a stable matching.*

3.3.7 A matroidal generalization of kernels

Tamás Fleiner defined in [21] a notion of matroid-kernels which we describe in the following. An *ordered matroid* $\mathcal{M} = \{S, \mathcal{I}, <\}$ is a matroid on ground set S with independent sets $\mathcal{I}$ together with a linear order $<$ of its ground set. In an ordered matroid, a set $X \subseteq S$ is said to *dominate* an element e if either $e \in X$ or there is an independent set $Y \subseteq X$ for which $Y \cup \{e\} \notin \mathcal{I}$ (that is, Y spans e) and $e < y$ for each $y \in Y$.

Matroid-kernels concern two ordered matroids on the same ground set – let $\mathcal{M}_1 = \{S, \mathcal{I}_1, <_1\}$ and $\mathcal{M}_2 = \{S, \mathcal{I}_2, <_2\}$ be ordered matroids. A set $K \subset S$ is called an *$\mathcal{M}_1\mathcal{M}_2$-kernel* if it is a common independent set of the two matroids (that is, $K \in \mathcal{I}_1 \cap \mathcal{I}_2$) and every element $e \in S$ is dominated by K in (at least) one of the two matroids.

How does this specialize to stable matchings in bipartite graphs? Let $G = (V_1, V_2; E)$ be a bipartite graph and $\mathcal{O}$ a preference system. Let us define $\mathcal{M}_1$ and $\mathcal{M}_2$ as the ordered matroids on ground set E (the edge set of the graph), which are the partition matroids of the stars on V_1 and V_2, respectively (that is, a set of edges is in $\mathcal{I}_i$ if their endpoints in V_i are distinct), with linear orders $<_i$ which extend all orders $<_v \in \mathcal{O}$ where $v \in V_i$ ($i = 1, 2$). This way a set F of edges form a matching if and only if it is a common independent set, and it is a stable matching if and only if it is an $\mathcal{M}_1\mathcal{M}_2$-kernel.

In our proof of the theorem of Fleiner we use the classic result of Edmonds (Theorem 1.3.1) that the convex hull of the common independent sets of two matroids is described by the rank inequalities.

Theorem 3.3.21 (Fleiner [21])**.** *For every two ordered matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, there exists an $\mathcal{M}_1\mathcal{M}_2$-kernel.*

Proof. Let P be the polytope of the common independent sets of $\mathcal{M}_1$ and $\mathcal{M}_2$, which by Edmonds' Theorem 1.3.1 is described by the system

$$P = \{x \in \mathbb{R}^S : \ x \geq 0, \ x(Z) \leq r_i(Z) \ \text{ for } i = 1,2 \text{ and } Z \subseteq S\}.$$

Thus, by Lemma 3.2.6 the down hull $Q = P - \mathbb{R}^S_+$ of P is $\{x \in \mathbb{R}^S : \ x(Z) \leq r_i(Z) \ \text{ for } i = 1,2 \text{ and } Z \subseteq S\}$.

For a facet that is defined by the inequality $x(Z) \leq r_i(Z)$, let us colour it with the element e in Z which is the smallest according to the linear order $<_i$. Since e has positive coefficient in $x(Z) \leq r_i(Z)$, the conditions of Theorem 3.2.5 hold. Thus by Theorem 3.2.5 there exists a multicoloured vertex x^* of Q. This means that x^* is the characteristic vector of a(n inclusionwise maximal) common independent set K of the two matroids, and for each element $e \in S$, there is a set Z that has colour e and $x^*(Z) = r_i(Z)$ for $i = 1$ or 2. This implies that $K \cap Z$ spans Z and by the definition of the colouring, e is the smallest among Z in $<_i$, so Z dominates e in $\mathcal{M}_i$, and we are done. □

3.3.8 Orientation of clutters

In Section 1.4 we defined ideal clutters as Sperner systems of which the covering polyhedron is integral. Using our method we get the following new result for clutters. Partial results about a possible converse statement will be described in Section 3.4.3.

Theorem 3.3.22. *Let $\mathcal{A}$ be an ideal clutter on ground set $[n]$ and let $\mathcal{B}$ be its blocker. Then there are no functions $p : \mathcal{A} \to S$ and $q : \mathcal{B} \to S$ such that*

(i) $p(A) \in A \ \forall A \in \mathcal{A}$,

(ii) $q(B) \in B \ \forall B \in \mathcal{B}$ *and*

(iii) if $p(A) = q(B)$ *for some* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$, *then* $|A \cap B| > 1$.

Proof. Suppose that there are functions p and q with the above properties. Let us examine the following colouring of the facets of $P(\mathcal{A})$: we colour a facet corresponding to a set $A \in \mathcal{A}$ (namely the facet defined by $x(A) \geq 1$) with colour $p(A)$, and a facet corresponding the i-th nonnegativity constraint gets the i-th colour. This colouring satisfies the condition in Theorem 3.2.5 since the extreme directions of $P(\mathcal{A})$ are the unit vectors and if a facet has colour i then the i-th coordinate of its normal vector is nonzero, thus the i-th unit vector is not an extreme direction of the facet. Thus we can apply Theorem 3.2.5 which asserts the existence of a vertex of $P(\mathcal{A})$ which is incident to every colour. Since $\mathcal{A}$ is ideal, we know that the vertex is the characteristic vector of a set in the blocker, say $B \in \mathcal{B}$. It follows that for every $i \in B$ there exists a set A_i for which $|A_i \cap B| = 1$ (that is, the facet corresponding to A_i is incident to χ_B) and $p(A_i) = i$ (that is, A_i has colour i). However for $i = q(B)$ there can not be such a set, which is a contradiction. □

3.3.9 Stable flows

As another generalization of stable marriages, Fleiner [22] introduced a notion of stable flows, just as network flows generalize bipartite matchings.

In an instance of the stable flow problem we have a network on digraph $D = (V, A)$ with $s, t \in V$ and capacities $c \in \mathbb{R}_+^A$, and additionally linear orders $\leq_v$ for each node v on the arc set incident to v. The network along with the set of these preference orders is called a *network with preferences*. We note that we will only compare outgoing arcs or incoming arcs, so the information that we really need is a linear order on the set of outgoing arcs $\delta^{out}(v)$ and one on the set of incoming arcs $\delta^{in}(v)$.

Let f be a flow of network (D, s, t, c). A *rooted cycle* is a directed cycle in which one node is designated as the root. It can be regarded as a path which ends at its starting node. A path or rooted cycle $P = (v_1, a_1, v_2, a_2, \ldots, a_{k-1}, v_k)$ is said to *block* f if the following hold:

(i) $v_i \neq s, t$ if $i \in \{2, 3, \ldots, k-1\}$,

(ii) each arc a_i is unsaturated in f,

(iii) $v_1 = s$ or $v_1 = t$ or there is an arc $a' = v_1 u$ for which $f(a') > 0$ and $a' <_{v_1} a_1$,

(iv) $v_k = s$ or $v_k = t$ or there is an arc $a'' = w v_k$ for which $f(a'') > 0$ and $a'' <_{v_1} a_{k-1}$.

A flow is called *stable* if there is no path or cycle blocking it.

The problem can be motivated by a network trading model: the nodes are traders that can buy and sell amounts of a certain product along the arcs of the digraph, and have preferences with whom they would like to trade (an arc that is bigger in the linear order is more preferred). A blocking path represents a possible chain of transactions with which the starting and ending trader would be happier than with some transaction they make in f. In this interpretation, nodes s and t represent the producers and the consumers.

Fleiner proved the following result, for which we give a new proof using the polyhedral Sperner's Lemma 3.2.5.

Theorem 3.3.23 (Fleiner [22])**.** *In every network with preferences there exists a stable flow. If the capacity function is integral, then there is an integral stable flow.*

Proof. Let $(D = (V, A), s, t, c)$ and $\{<_v : v \in V\}$ be an instance of the problem. Let $\mathcal{P}$ denote the set of paths and rooted cycles in D, and $\mathcal{P}_a$ the set of paths and rooted cycles that contain arc a. Furthermore let $\mathcal{P}_v^{\text{out}}$ and $\mathcal{P}_v^{\text{in}}$ denote the set of paths and rooted cycles that start and end at vertex v, respectively. We consider the polyhedron Π in $\mathbb{R}^{\mathcal{P} \cup A}$ described by the following inequalities:

$$y_a \geq 0 \qquad \forall a \in A, \tag{3.4}$$

$$x(\mathcal{P}') - y_a \leq 0 \qquad \forall \mathcal{P}' \subseteq \mathcal{P}_a \ \forall a \in A, \tag{3.5}$$

$$x(\mathcal{P}') \leq c_a \qquad \forall \mathcal{P}' \subseteq \mathcal{P}_a \ \forall a \in A, \tag{3.6}$$

$$x(\mathcal{P}') - y(\delta') \leq c(\delta^{in}(v) \setminus \delta') \qquad \forall \emptyset \neq \mathcal{P}' \subseteq \mathcal{P}_v^{\text{out}},\ \delta' \subseteq \delta^{in}(v),\ v \in V \setminus \{s, t\}, \tag{3.7}$$

$$x(\mathcal{P}') - y(\delta') \leq c(\delta^{out}(v) \setminus \delta') \qquad \forall \emptyset \neq \mathcal{P}' \subseteq \mathcal{P}_v^{\text{in}},\ \delta' \subseteq \delta^{out}(v),\ v \in V \setminus \{s, t\}. \tag{3.8}$$

First let us check the set of extreme directions of Π. Clearly $(-\chi_P, \mathbf{0})$ for $P \in \mathcal{P}$ and $(\mathbf{0}, \chi_a)$ for $a \in A$ are infinite directions. Because x_P is bounded from above and y_a from below, there is no infinite direction which is not in the cone of the above. So the number of extreme directions equals the dimension.

Now let us assign colours (that is, coordinates of $\mathbb{R}^{\mathcal{P} \cup A}$) to each inequality:

- to an inequality of type (3.4) or type (3.5) we assign y_a,
- to an inequality of type (3.6) we assign x_P for a longest possible path $P \in \mathcal{P}'$,
- to an inequality of type (3.7) we assign x_P for a path $P \in \mathcal{P}'$ in which the outgoing edge from v is smallest possible in the order $<_v$ from $\mathcal{P}'$, and among these, we choose P to be one of the longest paths,
- to an inequality of type (3.8) we assign x_P for a path $P \in \mathcal{P}'$ in which the incoming edge to v is smallest possible in the order $<_v$ from $\mathcal{P}'$, and among these, we choose P to be one of the longest paths.

Since the assigned colour of each inequality is a coordinate with nonzero coefficient, the colouring fulfils the criteria of Theorem 3.2.5. Thus there exists a multicoloured vertex (x^*, y^*) of Π.

Claim 3.3.24. $x^* \geq \mathbf{0}$, $y^* \leq c$.

Proof. Suppose that x^*_P is negative for some $P \in \mathcal{P}$. Then by increasing x^*_P to zero we get a vector that is still in Π, because every inequality where x_P has positive coefficient is also present with the coefficient changed to zero (except for the inequalities where $\mathcal{P}' = \{P\}$, but changing the coefficient of x_p in those to zero is satisfied too since y_a and c_a are nonnegative). On the other hand we know that $-x_P$ is an infinite direction, so (x^*, y^*) could not have been a vertex. Thus x^*_P is nonnegative for every $P \in \mathcal{P}$. Similarly we get that $y_a \leq c_a$ for $a \in A$. $\diamond$

Claim 3.3.25. $\sum_{P \in \mathcal{P}_a} x^*_P = y_a$ *for every arc* $a \in A$.

Proof. Since (x^*, y^*) is multicoloured, there is a tight inequality which has colour y_a. If this inequality is of type (3.4), then using Claim 3.3.24, $0 \leq \sum_{P \in \mathcal{P}_a} x^*_P \leq y_a = 0$, thus equality holds. If the tight inequality is of type (3.5) for some $\mathcal{P}' \subseteq \mathcal{P}_a$, then by Claim 3.3.24, $\sum_{P \in \mathcal{P}_a} x^*_P \geq \sum_{P \in \mathcal{P}'} x^*_P = y_a \geq \sum_{P \in \mathcal{P}_a} x^*_P$, thus equality holds. $\diamond$

Claim 3.3.26. $x^*_{P^*} = 0$ *for every path or rooted cycle* P^* *which has at least 2 edges.*

Proof. Suppose that $x^*_{P^*} > 0$ for a path or rooted cycle $P^* = (v_1, a_1, \ldots, v_k)$, where $k \geq 3$. Let Q be the one-edge path (v_1, a_1, v_2) and R be the path $(v_2, a_2, \ldots v_k)$.

Consider the inequality that (x^*, y^*) satisfies with equality and has colour x_Q. It can not be of type (3.6), since because of P^*, we would not have chosen Q as colour. It also can not be of type (3.7) for the same reason. Thus it is of type (3.8). This means that there exists some $\mathcal{P}' \subseteq \mathcal{P}^{\text{in}}_{v_2}$ and $\delta' \subseteq \delta^{out}(v_2)$ for which $x^*(\mathcal{P}') - y^*(\delta') = c(\delta^{out}(v) \setminus \delta')$. Using Claim 3.3.24 this holds also for the whole sets $\mathcal{P}^{\text{in}}_{v_2}$ and $\delta^{out}(v_2)$, that is, $x^*(\mathcal{P}^{\text{in}}_{v_2}) = y^*(\delta^{out}(v_2))$.

Adding the inequalities of type (3.5) for the arcs in $\delta^{out}(v_2)$, and using $x^*_{P^*} > 0$, we get

$$y^*(\delta^{out}(v_2)) = x^*(\mathcal{P}^{\text{in}}_{v_2}) < x^*(\bigcup_{a \in \delta^{in}(v_2)} \mathcal{P}_a) \leq y^*(\delta^{in}(v_2)).$$

By the same argument for the subpath R, we get

$$y^*(\delta^{in}(v_2)) = x^*(\mathcal{P}^{\text{out}}_{v_2}) < y^*(\delta^{out}(v_2)),$$

which is a contradiction. $\diamond$

We obtained that x^* is positive only on the edges, and by Claim 3.3.25, $x^*_a = y^*_a$ for every arc a. By inequalities (3.7) and (3.8), we have

$$x^*(\delta^{out}(v)) \leq y^*(\delta^{in}(v)) = x^*(\delta^{in}(v)) \leq y^*(\delta^{out}(v)) = x^*(\delta^{out}(v)),$$

so y^*_a $(a \in A)$ is a flow, and by inequality (3.6), it satisfies the capacity constraints also. We are done by proving our last claim.

Claim 3.3.27. *y^* is a stable flow.*

Proof. Let $P^* = (v_1, a_1, \dots, v_k)$ be an arbitrary path or rooted cycle. Since (x^*, y^*) is a multicoloured vertex, there is a tight inequality of colour P^*. If it is of the form (3.6), then the arc a is saturated, so P^* does not block y^*.

If it is of the form (3.7) for some vertex v, $\mathcal{P}' \subseteq \mathcal{P}_v^{\text{out}}$ and $\delta' \subseteq \delta^{in}(v)$, then whenever $a' \in \delta^{out}(v_1)$ and $a' <_{v_1} a_1$, then $a' \notin \mathcal{P}'$, thus $x_{a'} = 0$, because if $x^*_{a'}$ would be positive, then adding $x_{a'}$ to this tight inequality would not hold for (x^*, y^*), although it is also an inequality of the system. This also implies that P^* does not block the flow y^*.

The case when the tight inequality of colour P^* is of type 3.8 is analogous. ◇

We proved the existence of a stable flow. Now suppose that the capacity c is integral. We know that (x^*, y^*) is a vertex of

$$\Pi \cap \{(x, y) \in \mathbb{R}^{\mathcal{P} \cup A} :\ x_P = 0\ \forall P \in \mathcal{P} \text{ with } \geq 2 \text{ arcs}, x_a = y_a\ \forall a \in A\},$$

which is basically a flow polyhedron (with duplicated coordinates and extra coordinates set to 0), hence it is an integer polyhedron. Thus (x^*, y^*) is integer. □

3.4 Attempts at converse statements

We mentioned in Section 3.3.1 that the Strong Perfect Graph Theorem implies the converse of the theorem of Boros and Gurvich 3.3.2. So the question arises, whether the converse of the other applications of Theorem 3.2.5 is also true, or even the converse of the theorem itself. In this section we give some counterexamples and a few special cases in which the converse holds.

3.4.1 A conjecture on the characterization of h-perfect graphs

First we show that the converse of Theorem 3.3.6 does not hold, and a counterexample is given here. The graph on Figure 3.1 is not h-perfect (this follows from the results of Barahona and Mahjoub [4]), but it can be seen by case analysis that every clique- and odd-hole-acyclic superorientation of it has a kernel.

However, we conjecture that the converse of Theorem 3.3.8 holds:

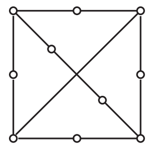

Figure 3.1: A non-h-perfect graph whose clique- and odd-hole-acyclic superorientations all have kernels

Conjecture 3.4.1. *A graph G is h-perfect if and only if every clique-acyclic superorientation with special nodes has an almost-kernel.*

3.4.2 A possible converse of Sperner's and Scarf's Lemma

It would be tempting to formulate a more general conjecture, which is a kind of converse to the polyhedral Sperner's Lemma.

Question 3.4.2. *Let P be a d-dimensional polytope, and let x^1 and x^2 be two distinct vertices of P, where x^1 is simple. Is it true that the facets of P can be coloured by d colours so that x^1 and x^2 are precisely the vertices that are incident to facets of all colours?*

We now show that the answer to this question is 'No'. Note that it is true in 3 dimensions: the skeleton of P contains 3 vertex-disjoint paths between x^1 and x^2; these paths partition the set of facets into 3 classes, and the colouring given by these 3 colour classes satisfies the conditions. However, it turns out to be false in 4 dimensions, as the following polytope shows:

Facets:	Vertices:	
$-x_1 - x_3 + x_4 \leq 1$	$(0,0,0,1)$,	$(0,0,0,-1)$,
$x_1 + x_2 + x_4 \leq 1$	$(-2,2,2,1)$,	$(2,2,2,-3)$,
$x_2 - x_3 + x_4 \leq 1$	$(1,2,0,-2)$,	$(2/3,4/3,-2/3,-1)$,
$-x_1 - x_2 - x_3 + x_4 \leq 1$	$(0,2,0,-1)$,	$(-1,1,0,0)$,
$x_1 - x_2 - x_3 - x_4 \leq 1$	$(2,-3,2,2)$,	$(2/3,-1/3,-2/3,2/3)$,
$-x_1 - x_3 - x_4 \leq 1$	$(-1,0,2,2)$,	$(-1/3,-2/3,-1/3,-1/3)$,
$-x_1 - x_4 \leq 1$	$(0,0,-1,0)$,	$(-1,0,0,0)$.
$-x_1 - x_2 + x_3 - x_4 \leq 1$		

Let the two designated vertices be $x^1 := (0,0,0,1)$ and $x^2 := (0,0,0,-1)$ (both of them are simple). The first four facets are incident to the vertex x^1, while the last four facets are incident to the vertex x^2. It can be shown by case analysis that no matter how we colour the first four facets by four different colours and the last four facets by the same four colours, there will be another vertex incident to facets of all four colours. The counterexample was found with the help of the software package polymake [36].

The following is a similar question concerning the converse of Scarf's Lemma.

Question 3.4.3. *Let A be a nonnegative $m \times n$ matrix and let $b \in \mathbb{R}^m$ be a positive vector so that the polyhedron $P = \{x : \; Ax \leq b, \; x \geq 0\}$ is bounded. Let x^* be a maximal vertex of P. Is it true that for each row a_i of A we can give a total order on supp(a_i), so that x^* is the only maximal vertex of P that dominates every column?*

Now we show that the above counterexample can be transformed into a counterexample for Question 3.4.3 using the technique in Section 3.2. Let P be the polytope defined above. First, we cut off the vertex x^1 with a hyperplane to obtain a simplex facet F_0. Then we take the polar of this polytope and affinely transform it into a polytope P' so that the image of F_0 is the origin and the facets containing it are $\{x \in P' : x_i = 0\}$ $(i = 1, \ldots, 4)$. If we now take the polar from the origin, we obtain a polyhedron whose extreme directions are $-e_i$ $(i = 1, \ldots, 4)$; we can translate this to a polyhedron P'' whose vertices are all in $\mathbb{R}^4_+$. Let x^* be the image of x^2; we know that this is a maximal vertex of $P''' := P'' \cap \mathbb{R}^4_+$. We claim that P''' and x^* give a counterexample for Question 3.4.3. Suppose we have total orders $<_i$ on the supports of the rows such that x^* dominates every column. These can be transformed into a colouring on the facets of $P'' = P''' - \mathbb{R}^4_+$ as in the proof of Theorem 3.3.11, such that x^* is multicoloured. Furthermore, such a colouring of the facets of P'' defines a colouring of the facets of P where x^1 and x^2 are multicoloured. Since P is a counterexample for Question 3.4.2, there is a third multicoloured vertex x^3. The polyhedron P''' has a corresponding maximal vertex, and this vertex dominates every column by the construction.

It may be interesting to know special classes of polyhedra where the answer to Question 3.4.3 is affirmative. We have no counterexamples for the following conjecture.

Conjecture 3.4.4. *Let A be an $m \times n$ matrix with $0-1$ coefficients and let $b \in \mathbb{R}^m$ be a positive vector so that the polyhedron $P = \{x : \; Ax \leq b, \; x \geq 0\}$ is bounded. Suppose that P has a non-integer maximal vertex. Then for each row a_i of A we can give a total order on supp(a_i) so that for every $0-1$ vertex x' of P there is a column that it does not dominate.*

We note that this is more general than Conjecture 3.4.1. To see this, consider a non-h-perfect graph G. The polyhedron P defined by inequalities (3.1) $-$ (3.3) has a non-integral vertex, hence it has a non-integral maximal vertex. Let $<_i$ $(i = 1, \ldots, m)$ denote the total orders given by Conjecture 3.4.4. These total orders define a clique-acyclic superorientation with special vertices:

- For each maximal clique, we orient the edges of the clique according to the total ordering of the clique. (An edge may appear in two cliques and its endpoints may be in different order in the two total orders; in this case, we orient the edge in both directions.) This defines the superorientation.
- If an odd hole is a one-way cycle in this superorientation, we define its special node to be the smallest node in its total order.

Let S be an arbitrary stable set of G. The characteristic vector of S is a $0-1$ vertex of the polyhedron P. By the properties of the partial orders, there is a node $v \in V$ with the following properties:

- If there is a maximal clique K_i with $|K_i \cap S| = 1$ and $v \in K_i$, then there is a node $u \in K_i \cap S$ with $u <_i v$.
- If there is an odd hole Z_i with $|Z_i \cap S| = (|Z_i| - 1)/2$ and $v \in Z_i$, then there is a node $u \in Z_i \cap S$ with $u <_i v$.

The first property means that $v \notin S$ and the out-neighbours of v in the superorientation are not in S, so v is not dominated by S. The second property implies that if v is the special node of an odd hole Z (that is, it is the smallest node in the total order) then $|Z \cap S| < (|Z| - 1)/2$. Therefore the existence of v proves that S is not an almost-kernel.

3.4.3 A conjecture on clutters

Király conjectures that the converse of Theorem 3.3.22 also holds, and we prove some partial results on it, see [53].

Conjecture 3.4.5. *Let $\mathcal{A}$ be a clutter on ground set $[n]$ and let $\mathcal{B}$ be its blocker. Then $\mathcal{A}$ and $\mathcal{B}$ are nonideal if and only if there exist functions $p : \mathcal{A} \to S$ and $q : \mathcal{B} \to S$ such that*

(i) $p(A) \in A$ for every $A \in \mathcal{A}$,

(ii) $q(B) \in B$ for every $B \in \mathcal{B}$, and

(iii) if $p(A) = q(B)$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|A \cap B| > 1$.

We now prove that it is enough to prove Conjecture 3.4.5 for minimally non-ideal clutters.

Claim 3.4.6. *If Conjecture 3.4.5 holds for $\mathcal{A}/s$ or $\mathcal{A}\backslash s$ then it holds for $\mathcal{A}$ as well.*

Proof. First suppose that the conjecture is true for $\mathcal{A}/s$, and let $p : \mathcal{A}/s \to S$ and $q : \mathcal{B}\backslash s \to S$ be functions satisfying the conditions (in which case $\mathcal{A}/s \neq \{\emptyset\}$).

For a set $A \in \mathcal{A}$ let $p'(A) := p(A')$ for an arbitrary set $A' \in \mathcal{A}/s$ for which $A' \subseteq A$, and for a set $B \in \mathcal{B}$ let

$$q'(B) := \begin{cases} s & \text{if } s \in B \\ q(B) & \text{otherwise.} \end{cases}$$

So $p'(A) \neq s$ for any $A \in \mathcal{A}$. The conditions $p'(A) \in A$ and $q'(B) \in B$ hold. If $p'(A) = q'(B)$, then this element is not s, so $q'(B) = q(B)$, and for some $A' \subseteq A$, $p'(A) = p(A')$. Thus $|A' \cap B| > 1$, so $|A \cap B| > 1$ and the second condition also holds.

The other case follows from the first because of symmetry. □

In the following we show on one hand that the conjecture is true if we restrict it to the cores and on the other hand that it is true if the core of the clutter (or of its blocker) is cyclic.

Theorem 3.4.7. *Let $\mathcal{A}$ be a minimally nonideal clutter on ground set S and let $\mathcal{B}$ be its blocker. Then there exist functions $p : \overline{\mathcal{A}} \to S$ and $q : \overline{\mathcal{B}} \to S$ such that*

(i) $p(A) \in A$ for every $A \in \overline{\mathcal{A}}$,

(ii) $q(B) \in B$ for every $B \in \overline{\mathcal{B}}$, and

(iii) if $p(A) = q(B)$ for some $A \in \overline{\mathcal{A}}$ and $B \in \overline{\mathcal{B}}$, then $|A \cap B| > 1$.

Proof. Due to Lehman's Theorem 1.4.1 the sets in $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ can be indexed as $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots B_n$ such that $|A_i \cap B_j| > 1$ if and only if $i = j$. So we want to choose $p(A_i) = q(B_i) \in A_i \cap B_i$ so they are all different. To this end we construct the bipartite graph $G = (S, T; E)$ where $T = \{t_1, t_2, \ldots t_n\}$ and $s_i t_j \in E \Leftrightarrow s_i \in A_j \cap B_j$. Lehman's Theorem implies that G is $d+1$-regular: on the side of S because $\overline{A}\overline{B}^{\mathsf{T}} = I + dJ$, and on the side of T because $\overline{B}^{\mathsf{T}}\overline{A} = I + dJ$. So there is a perfect matching in G which gives functions p, q with the desired properties. □

Definition 3.4.8. A clutter is *cyclic* if it is isomorphic to a clutter of all the sets containing r consecutive elements in cyclic order for some r.

Theorem 3.4.9. *If the core of clutter $\mathcal{A}$ is cyclic, then Conjecture 3.4.5 is true for $\mathcal{A}$.*

Proof. For easier notation let S be the set $[n]$ and let the sets in $\overline{\mathcal{A}}$ consist of r consecutive elements in cyclic order. For $A \in \overline{\mathcal{A}}$ let $p(A)$ be the last element of A in cyclic order. For $A \in \mathcal{A} \setminus \overline{\mathcal{A}}$ take a modulo r congruence class which has more than one elements in A and let $p(A)$ be the smallest element among these.

For $B \in \mathcal{B}$ let $q(B)$ be the largest element such that the preceding $r-1$ elements intersect B. There is such an element because otherwise every rth element would be in B but $rs = n + d > n$.

Now suppose that $p(A) = q(B) = i$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $A \in \overline{\mathcal{A}}$ then by the definition of p and q, A and B meet at an element among the $r-1$ elements preceding i. If $A \in \mathcal{A} \setminus \overline{\mathcal{A}}$ then A and B meet in an element from the congruence class of i, larger than i since this set is contained in B (by the definition of the blocker and of $q(B)$) and is met by A. So p and q fulfill the criteria. □

An example

Let us examine the clutter $\mathcal{O}_{K_5}$ whose ground set is the edge set of the graph K_5 and which consists of the odd cycles. The blocker $b(\mathcal{O}_{K_5})$ consists of the K_4 subgraphs and the subgraphs formed by a triangle together with the edge disjoint from it. These clutters are minimally nonideal as was shown by Seymour [57].

Figure 3.4.3 shows functions on the cores $\overline{\mathcal{O}_{K_5}}$ and $\overline{b(\mathcal{O}_{K_5})}$ whose existence is guaranteed by Claim 3.4.7: it satisfies the conditions for the cores (the graph-edges selected by p_{bad} and q_{bad} are drawn in thick; on the other clutter-edges p_{bad} and q_{bad} act with a rotation symmetry).

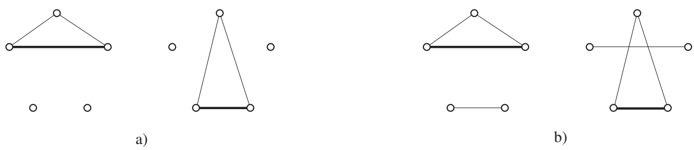

Figure 3.2: a) Function p_{bad} on $\overline{\mathcal{O}_{K_5}}$, b) function q_{bad} on $\overline{b(\mathcal{O}_{K_5})}$

However these functions can not be extended to the whole clutter, as shown by the 5-cycle of the "outer" edges. This shows that good functions on the cores may not be extendable to the entire clutter. On the other hand, in the present case there are functions p_{good} and q_{good} which satisfy all the requirements of Conjecture 3.4.5; Figure 3.4.3 shows such a function (again the clutter-edges not shown have their selected edges symmetrically).

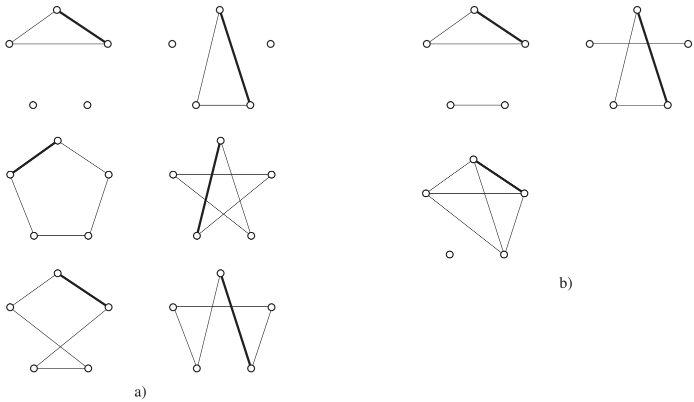

Figure 3.3: a) Function p_{good} on $\mathcal{O}_{K_5}$, b) function q_{good} on $b(\mathcal{O}_{K_5})$

3.5 PPAD-completeness

We will prove that computational versions of Theorems 3.2.2 and 3.2.5 are PPAD-complete. These are joint results with Tamás Király [49].

The following is the computational version of Theorem 3.2.2 that we are interested in.

Polytopal Sperner:

Input: vectors $v^i \in \mathbb{Q}^n$ $(i = 1, \ldots, m)$ whose convex hull is a full-dimensional polytope P; a colouring of the vertices by n colours; a multicoloured simplex facet F_0 of P.

Output: n affine independent vectors $v^{i_1}, \ldots, v^{i_n}$ with different colours which lie on a facet of P different from F_0.

Another computational problem we consider corresponds to Theorem 3.2.5.

Extreme direction Sperner:

Input: matrix $A \in \mathbb{Q}^{m \times n}$ and vector $b \in \mathbb{Q}^m$ such that $P = \{x : Ax \leq b\}$ is a pointed polyhedron whose characteristic cone is generated by n linearly independent vectors; a colouring of the facets by n colours such that facets containing the i-th extreme direction do not get colour i.

Output: a multicoloured vertex of P.

The complexity class PPAD is defined as the set of problems which are Karp-reducible to its prototypical problem, the following END OF THE LINE.

END OF THE LINE:

Input: a directed graph on $\{0, 1\}^n$ given implicitly by an algorithm polynomial in n. It is required that in the graph, every vertex has at most one out-neighbour and at most one in-neighbour, and $\mathbf{0}$ has no in-neighbour, but it has an out-neighbour. The input of the algorithm is a vertex, that is, an n-bit binary string, and its output is the out-neighbour and the in-neighbour of the vertex.

Output: any vertex in $\{0, 1\}^n \setminus \{\mathbf{0}\}$ that has degree 1 (where the degree is the in-degree plus the out-degree).

Finally, a problem in PPAD is called *PPAD-complete* if every other problem in PPAD is Karp-reducible to it. The class PPAD was introduced by Papadimitriou [55], who proved among other results that a computational version of 3D Sperner's Lemma is PPAD-complete. Later Chen and Deng [12] proved that the 2 dimensional problem is also PPAD-complete. The input of these computational versions is a polynomial algorithm that computes a legal colouring, while the number of vertices to be coloured is exponential in the input size. This is conceptually different from POLYTOPAL SPERNER, where the input explicitly contains the vertices and the colouring. In POLYTOPAL SPERNER the difficulty lies not in the large number of vertices but in that the structure is encoded as a polytope. We note that in fixed dimension POLYTOPAL SPERNER is solvable in polynomial time since then the number of facets is polynomial in the number of vertices.

Our proof of PPAD-hardness is essentially the same as the proof by Kintali [45] of PPAD-hardness of the computational problem SCARF which is related to Scarf's Lemma (Theorem 3.3.10). Let us prove first that EXTREME DIRECTION SPERNER is reducible to POLYTOPAL SPERNER.

Proposition 3.5.1. EXTREME DIRECTION SPERNER *is Karp-reducible to* POLYTOPAL SPERNER.

Proof. Suppose that matrix A and vector b are an instance of EXTREME DIRECTION SPERNER and let $P = \{x : Ax \leq b\}$. We can translate P so that it contains the origin in its interior. In this case its polar P^Δ is a polytope whose vertices can be obtained easily from A and b. The colouring of P defines a colouring of the vertices of P^Δ except for the origin which corresponds to the infinite facet of P. Let us cut off the origin with a hyperplane H, such a hyperplane can be computed in polynomial time. This way, since the origin is a simple vertex of P, we introduce exactly n new vertices and a simplex facet. The i-th new vertex lies on the facets that correspond to all but the i-th extreme direction of P; let the colour of it be i. We obtained a colouring of $P^\Delta \cap H^+$ (where H^+ is the halfspace bounded by H not containing the origin) which satisfies the criteria, so it is an instance of POLYTOPAL SPERNER. A multicoloured facet of $P^\Delta \cap H^+$ corresponds to a multicoloured vertex of P. □

Now let us prove that POLYTOPAL SPERNER is in PPAD, which by Proposition 3.5.1 proves also that EXTREME DIRECTION SPERNER is in PPAD.

Proposition 3.5.2. POLYTOPAL SPERNER *is in PPAD.*

Proof. We reduce it to the problem END OF THE LINE. We can compute in polynomial time a perturbation of the vertices in the input such that every facet becomes a simplex, and every facet (as a vertex set) is a subset of an original facet. Assume that $[n]$ is the set of colours. We define a digraph whose nodes are the facets that contain all colours in $[n-1]$ (formally, we may associate a node to each n-tuple of vertices, all other nodes being isolated). Each $(n-2)$-dimensional face with all colours in $[n-1]$ is in exactly two facets. We can say that one of them is on the left side of the face and the other is on the right side, with respect to a fixed orientation: we compute the sign of the two determinants of the vectors going from a fixed inner point of P to the $n-1$ vertices of the $(n-2)$-dimensional face (in a fixed order) and the n-th vertex of the two facets; the facet whose determinant is positive is on the left side, the other is on the right. For each such $(n-2)$-dimensional face, we introduce an arc from the node corresponding to the facet on the left side to the node corresponding to the facet on the right side.

The obtained digraph has in-degree and out-degree at most 1 in every node, and the neighbours of a node can be computed in polynomial time. A node has degree 1 if and only if the corresponding facet is multicoloured. We may assume w.l.o.g. that the node corresponding to F_0 is a source, so the solution of END OF THE LINE for this digraph corresponds to finding a multicoloured facet different from F_0. □

It suffices to prove completeness for EXTREME DIRECTION SPERNER.

Theorem 3.5.3. EXTREME DIRECTION SPERNER *is PPAD-complete.*

Proof. The proof is analogous to the proof of PPAD-completeness of SCARF by Kintali [45], who proves that the problem 3-STRONG KERNEL defined below is PPAD-complete, and reduces it to SCARF. Recall that a digraph $D = (V, E)$ is *clique-acyclic* if for every directed cycle either the reverse of one of the arcs is also in E, or there are two nodes of the cycle that are not connected by an arc in E. A *strong fractional kernel* of D is a vector $x : V \to \mathbb{R}_+$ such that $x(K) \leq 1$ for every clique K, and for each node v there is at least one clique K in the out-neighbourhood of v (including v) such that $x(K) = 1$.

3-STRONG KERNEL:

Input: A clique-acyclic digraph $D = (V, E)$ with maximum clique size at most 3.

Output: A strong fractional kernel of D.

To reduce 3-STRONG KERNEL to EXTREME DIRECTION SPERNER, let $n = |V|$, and let us consider the polyhedron

$$P = \{x \in \mathbb{R}^n : x(K) \leq 1 \text{ for every clique } K \text{ of } D\}.$$

Since every clique has size at most 3, the number of cliques is polynomial in n. The extreme directions of P are $-e_j$ ($j \in [n]$). As a set of colours, we use the nodes of V. Let the colour of the facet $x(K) = 1$ be a source node of K. This colouring satisfies the criterion in Theorem 3.2.5, so we have a valid input for EXTREME DIRECTION SPERNER. Let x^* be a mulitcoloured vertex. For each node $v \in V$, there is a clique K such that the facet $x(K) = 1$ contains x^* and has colour v, hence v is a source of K, that is, K is in the out-neighbourhood of v. This means that x^* is a strong fractional kernel. □

Theorem 3.5.4. POLYTOPAL SPERNER *is PPAD-complete.*

Proof. It follows directly from Proposition 3.5.2, Proposition 3.5.1 and Theorem 3.5.3. □

Chapter 4

Ideal set functions

The aim of the present chapter is to propose an extension of the notions of the blocking relation and idealness of clutters to set functions. We show that several properties of ideal clutters are maintained: idealness is preserved for taking minors and blockers. We also show that new types of minimally non-ideal structures emerge. This is joint work with Tamás Király, most of the results appeared in [48].

There have been several attempts at generalizing the notion of ideal clutters and Lehman's theorem ([34, 70, 35]), but their aim was to incorporate packing polyhedra, and in particular the stable set polytopes of graphs, to the framework. Our approach is different: for set functions on a ground set of size n, we define a polyhedron in dimension $n+1$, and show that the notion of minors, idealness, and blocking carry over naturally to this framework.

We also define a new natural operation, which we call twisting of the set function, that preserves idealness. This gives rise to new types of minimally non-ideal structures, namely the set functions we get by twisting the functions corresponding to clutters.

We provide some examples beyond clutters in Section 4.4. We show that both the rank function and the co-rank function of a matroid are ideal. Concerning 0-1-2 set functions where $f(\emptyset) = 0$ and $f(v) = 1$ for every $v \in V$, we show that these are ideal if and only if the maximal sets with value 1 are the stable sets of a nearly bipartite graph.

4.1 Gradual set functions

Let V be a finite ground set, and let $f : 2^V \to \mathbb{Z}$ be an integer-valued set function that satisfies $f(X) \leq f(X+v) \leq f(X)+1$ for every $X \subsetneq V$ and $v \in V \setminus X$. We call such a set function *gradual*. Let the *blocker* $b(f) : 2^V \to \mathbb{Z}$ of f be the (gradual) set function defined by

$$b(f)(X) := -f(V \setminus X)$$

for any set $X \subseteq V$. Obviously, $b(b(f)) = f$.

We define the following two minor operations on gradual functions. By *deleting* a node v we mean the function denoted by $f \backslash v$ on ground set $V - v$ for which

$$f \backslash v(X) := f(X) \quad \text{for every } X \subseteq V - v.$$

Similarly by *contracting* v we mean the function f/v on ground set $V - v$ for which

$$f/v(X) := f(X + v) \quad \text{for every } X \subseteq V - v.$$

A function f' is a *minor* of f if it can be obtained from f using deletions and contractions. It is easy to see that the order of the operations does not affect the minor we get, thus we will denote a minor by $f \backslash A/B$ if it is obtained by deleting the nodes in A and contracting the nodes in B.

Proposition 4.1.1. *If f is gradual, then its minors are also gradual.*

Proof. Straightforward. □

Proposition 4.1.2. *For any gradual function f, $b(f \backslash v) = b(f)/v$ and $b(f/v) = b(f) \backslash v$.*

Proof. For $X \subseteq V - v$ we have

$$\begin{aligned} b(f \backslash v)(X) &= -f \backslash v((V - v) \setminus X) = -f((V - v) \setminus X) = \\ &= b(f)(X + v) = b(f)/v(X), \end{aligned}$$

and

$$b(f/v)(X) = -f/v((V - v) \setminus X) = -f(V \setminus X) = b(f)(X) = b(f) \backslash v(X).$$

□

We call two set functions f_1 and f_2 *translations* of each other if for every $X \subseteq V$, $f_2(X) = f_1(X) + c$ for a constant c; we will use the notation $f_1 \cong f_2$. It can be seen that the notions we discuss in this chapter are preserved under translation.

4.1.1 Polyhedra of gradual functions

Let us assign the following three $(n+1)$-dimensional polyhedra to a gradual set function f:

$$\begin{aligned}
P(f) &:= \{(y,\beta) \in \mathbb{R}^{n+1}:\ 0 \le y \le 1,\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\},\\
Q(f) &:= \{(y,\beta) \in \mathbb{R}^{n+1}:\ y \ge 0,\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\},\\
R(f) &:= \{(y,\beta) \in \mathbb{R}^{n+1}:\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\}.
\end{aligned}$$

In the following we prove some basic properties of these polyhedra.

Proposition 4.1.3. *If f is a gradual set function, then $Q(f) = P(f) + \mathbb{R}^n_+$.*

Proof. The $Q(f) \supseteq P(f) + \mathbb{R}^n_+$ inclusion is easy, since the describing matrix of $Q(f)$ has nonnegative coefficients in the first n variables.

For the $Q(f) \subseteq P(f) + \mathbb{R}^n_+$ inclusion, let $(y,\beta) \in Q(f)$. We want to show that there is a $(y',\beta) \in P(f)$ for which $y' \le y$. Let $y'_i := \min(y_i, 1)$. Then $y' \le y$ and $0 \le y' \le 1$ hold, so it remains to show that $y'(X) - \beta \ge f(X)$ for each $X \subseteq V$. We have $y'(X) = |X \cap \{i : y_i > 1\}| + y(X \cap \{i : y_i \le 1\}) \ge |X \cap \{i : y_i > 1\}| + f(X \cap \{i : y_i \le 1\}) + \beta \ge f(X) + \beta$, since f is gradual. □

Let C be the cone generated by $\{e_i :\ i \in [n]\} \cup \{-e_i - e_{n+1} :\ i \in [n]\}$. We call a set X *tight* with respect to f and a vector (y,β) if $y(X) - \beta = f(X)$.

Proposition 4.1.4. *Every vertex (y^*,β^*) of $R(f)$ satisfies $0 \le y^* \le 1$, and the characteristic cone of $R(f)$ is C, hence $R(f) = P(f) + C$.*

Proof. First let us show that the characteristic cone is C. It is easy to see that all the vectors e_i and $-e_i - e_{n+1}$ are in the characteristic cone of $R(f)$. If a vector (z,γ) is in the characteristic cone of $R(f)$, then for every $X \subseteq V$, $z(X) - \gamma \ge 0$ holds. For $X := \{i : z_i < 0\}$ we have $(z,\gamma) = \sum_{i\in X} -z_i(-e_i - e_{n+1}) + (z',\gamma')$, where $z' \ge 0$ and $\gamma' \le 0$, and it is easy to see that $(z',\gamma') \in C$.

Now let (y^*,β^*) be a vertex, and suppose that $y^*_v < 0$. Then every tight set X contains v, because otherwise the inequality for $X + v$ would be violated since $f(X+v) \ge f(X)$. Now, if every tight set X contains v, then $(y^*,\beta^*) + \varepsilon(\chi_v, 1)$ is in $R(f)$ for some positive ε. This contradicts the fact that (y^*,β^*) is a vertex and $(-\chi_v, -1)$ is an extreme direction.

Now suppose that $y^*_v > 1$ for a vertex (y^*,β^*). Then no tight set contains v, since otherwise the inequality for $X - v$ would be violated: $y^*(X - v) - \beta < y^*(X) - 1 - \beta = f(X) - 1 \le f(X - v)$, a contradiction. This implies that for some positive ε, the vector $(y^*,\beta^*) - \varepsilon(\chi_v, 0)$ is in $R(f)$, which contradicts the fact that (y^*,β^*) is a vertex and e_v is an extreme direction. □

Corollary 4.1.5. *If f is a gradual set function, then $R(f) = Q(f) + C$.*

Proof. It follows from Propositions 4.1.3 and 4.1.4 and that $\mathbb{R}^n_+ \subset C$. $\square$

Corollary 4.1.6. *For a gradual function f,* $\mathrm{vert}(P(f)) \supseteq \mathrm{vert}(Q(f)) \supseteq \mathrm{vert}(R(f))$.

Proposition 4.1.7. *For any gradual function f, the following hold:*

$$P(f\backslash v) = \{(y,\beta) \in \mathbb{R}^{n-1+1} : (y,1,\beta) \in P(f)\}, \text{ and}$$

$$P(f/v) = \{(y,\beta) \in \mathbb{R}^{n-1+1} : (y,0,\beta) \in P(f)\},$$

that is, both $P(f\backslash v)$ and $P(f/v)$ are facets of $P(f)$.

Proof. It is easy to see that for a vector $(y,1,\beta) \in P(f)$, (y,β) satisfies the inequalities of $P(f\backslash v)$, since they are present in the system of $P(f)$ also.

If $(y,\beta) \in P(f\backslash v)$ and $X \subseteq V - v$, then on one hand we have $(y,1)(X) - \beta = y(X) - \beta \geq f\backslash v(X) = f(X)$, and on the other hand $(y,1)(X+v) - \beta = y(X) + 1 - \beta \geq f\backslash v(X) + 1 = f(X) + 1 \geq f(X+v)$, since f is gradual. So $(y,1,\beta) \in P(f)$.

It is easy to see that for a vector $(y,0,\beta) \in P(f)$, (y,β) satisfies the inequalities of $P(f/v)$, since $y(X) - \beta = (y,0)(X+v) - \beta \geq f(X+v) = f/v(X)$.

If $(y,\beta) \in P(f/v)$ and $X \subseteq V - v$, then on one hand we have $(y,0)(X) - \beta = y(X) - \beta \geq f/v(X) = f(X+v) \geq f(X)$, since f is gradual, and on the other hand $(y,0)(X+v) - \beta = y(X) - \beta \geq f/v(X) = f(X+v)$, thus $(y,0,\beta) \in P(f)$. $\square$

4.2 Ideal gradual set functions

Definition 4.2.1. The gradual set function f is called *ideal* if the polyhedron $P(f)$ is integral.

For a gradual set function f, let us define the following finite set of vectors in $\mathbb{R}^{n+1}$:

$$S(f) := \{(\chi_X, f(X)) : X \subseteq V\}.$$

We denote the set $S(f) - \mathrm{cone}\{(\mathbf{0},-1)\}$ by $S^{\downarrow}(f)$.

We note that the idealness of f is equivalent to $P(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\}$.

Theorem 4.2.2. *For a gradual set function, the following are equivalent:*

(i) *f is ideal, that is,* $P(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\}$

(ii) *$b(f)$ is ideal, that is, $P(b(f)) = \mathrm{conv}\{S^{\downarrow}(f)\}$*

(iii) *$R(f)$ is an integer polyhedron, that is, $R(f) = \mathrm{conv}\{S(b(f))\} + C$*

(iv) *$R(b(f))$ is an integer polyhedron, that is, $R(b(f)) = \mathrm{conv}\{S(f)\} + C$*

(v) *$Q(f)$ is an integer polyhedron*

(vi) *$Q(b(f))$ is an integer polyhedron*

In the proof we will use an operation B on polyhedra in $\mathbb{R}^{n+1}$, which is similar to taking the blocker of a polyhedron, it differs only in the last coordinate. For a polyhedron $P \subseteq \mathbb{R}^{n+1}$, let us define $B(P)$ as follows:

$$B(P) := \{(y, \beta) \in \mathbb{R}^{n+1} : x^{\mathsf{T}} y \geq \alpha + \beta \text{ for every } (x, \alpha) \in P\}.$$

Note that $B(P)$ is indeed a polyhedron, since using standard polyhedral techniques one can prove that if $P = \mathrm{conv}\{S\} + \mathrm{cone}\{T\}$ for finite vector sets S and T in $\mathbb{R}^{n+1}$, then

$$B(P) = \{(y, \beta) \in \mathbb{R}^{n+1} : s_{[n]}^{\mathsf{T}} y \geq s_{n+1} + \beta \ \forall s \in S \text{ and } t_{[n]}^{\mathsf{T}} y \geq t_{n+1} \ \forall t \in T\}. \quad (4.1)$$

Suppose that the polyhedron $P \subset \mathbb{R}^{n+1}$ has the following properties:

(a) $\exists \bar{\alpha} : (\mathbf{0}, \bar{\alpha}) \in P$

(b) P is bounded from above in the last coordinate

(c) $(\mathbf{0}, -1)$ is in the characteristic cone of P

Proposition 4.2.3. *If a polyhedron P satisfies properties (a)-(c) then so does $B(P)$.*

Proof. To see property (a), we can observe that if $P = \mathrm{conv}\{S\} + \mathrm{cone}\{T\}$, then from (4.1) we get that for $\bar{\beta} = \min(-s_{n+1} : s \in S)$, $(\mathbf{0}, \bar{\beta}) \in P$. For property (b) we can take an $\bar{\alpha}$ such that $(\mathbf{0}, \bar{\alpha}) \in P$ which implies that $\beta \leq \mathbf{0}^{\mathsf{T}} y - \bar{\alpha} = -\bar{\alpha}$. For property (c) we need that $x^{\mathsf{T}} \mathbf{0} \geq -1$ which is obvious, and that $B(P)$ is nonempty which follows from (a). □

Lemma 4.2.4. *If P satisfies properties (a)-(c) then $B(B(P)) = P$.*

Proof. For every $(x, \alpha) \in P$ and $(y, \beta) \in B(P)$ we have $x^{\mathsf{T}} y \geq \alpha + \beta$ which shows that $P \subseteq B(B(P))$.

Suppose that there is a vector $(x^*, \alpha^*) \in B(B(P))$ which is not in P. Then there is a vector (z, γ) and a number ξ such that ${x^*}^{\mathsf{T}} z + \alpha^* \gamma < \xi$, but for every $(x, \alpha) \in P$, $x^{\mathsf{T}} z + \alpha \gamma \geq \xi$. From (c) it follows that $\gamma \leq 0$.

Case 1: $\gamma = 0$. We show that there is an $\varepsilon > 0$ such that ${x^*}^\mathsf{T} z + \alpha^*(-\varepsilon) < x^\mathsf{T} z + \alpha(-\varepsilon)$ for each $(x, \alpha) \in P$. Because of (b) we know that there is an $a \in \mathbb{R}$ such that $\alpha \leq a$ for every $(x, \alpha) \in P$. We can assume that $a > \alpha^*$. If $\varepsilon < \frac{\xi - {x^*}^\mathsf{T} z}{a - \alpha^*}$, then for every $(x, \alpha) \in P$, $\varepsilon(\alpha - \alpha^*) \leq \varepsilon(a - \alpha^*) < \xi - {x^*}^\mathsf{T} z \leq x^\mathsf{T} z - {x^*}^\mathsf{T} z$. Since $x^\mathsf{T} z + \alpha(-\varepsilon)$ attains its minimum on P, we have an instance of Case 2.

Case 2: $\gamma < 0$. We can assume that $\gamma = -1$, since we can scale the inequalities with a positive multiplier. So we have ${x^*}^\mathsf{T} z - \alpha^* < \xi$, and for each $(x, \alpha) \in P$, $x^\mathsf{T} z - \alpha \geq \xi$. That means the vector $(z, \xi) \in B(P)$ but for this vector (x^*, α^*) does not fulfil the required inequality to be in the blocker of $B(P)$, which contradicts $(x^*, \alpha^*) \in B(B(P))$. □

Notice that for a gradual function f, the polyhedron $P(f)$ satisfies properties (a)-(c).

Proposition 4.2.5. *$B(P(f)) = \operatorname{conv}\{S(f)\} + C$ and $B(R(f)) = \operatorname{conv}\{S^\downarrow(f)\}$*

Proof. First we prove that $B(\operatorname{conv}\{S(f)\} + C) = P(f)$, by Lemma 4.2.4 this implies the first equation. Using (4.1), we have

$$\begin{aligned} B(\operatorname{conv}\{S(f)\} + C) = \{\ (y, \beta) \in \mathbb{R}^{n+1} :\ & y(X) \geq f(X) + \beta \ \ \forall X \subseteq V, \\ & y_i \geq 0 \ \ \forall i \in [n], \\ & -y_i \geq -1 \ \ \forall i \in [n]\ \}, \end{aligned}$$

which is equal to $P(f)$.

Now let us prove that $B(\operatorname{conv}\{S^\downarrow(f)\}) = R(f)$, which implies the second equation. Using (4.1), we have

$$B(\operatorname{conv}\{S^\downarrow(f)\}) = \{\ (y, \beta) \in \mathbb{R}^{n+1} :\ y(X) \geq f(X) + \beta\ \forall X \subseteq V\},$$

which is $R(f)$. □

Proof of Theorem 4.2.2. Using Propositions 4.1.4 and 4.2.5 and Lemma 4.2.4 we have

$$\begin{aligned} & P(f) = \operatorname{conv}\{S^\downarrow(b(f))\} \overset{+C}{\Longrightarrow} R(f) = \operatorname{conv}\{S^\downarrow(b(f))\} + C \overset{B(.)}{\Longrightarrow} \\ & \overset{B(.)}{\Longrightarrow} \operatorname{conv}\{S^\downarrow(f)\} = P(b(f)) \overset{+C}{\Longrightarrow} \operatorname{conv}\{S^\downarrow(f)\} + C = R(b(f)), \end{aligned}$$

which shows the equivalence of (i)-(iv). Corollary 4.1.6 implies that if $P(f)$ is integral then so is $Q(f)$, and if $Q(f)$ is integral then so is $R(f)$, which together with the above equivalences imply the equivalence of (v) (and also (vi)) and the other statements. □

Proposition 4.2.6. *If f is ideal, then any minor of it is also ideal.*

Proof. It follows from Proposition 4.1.7. □

Definition 4.2.7. A gradual set function f is called *minimally nonideal* (*mni*) if it is not ideal but every minor is ideal.

4.3 Twisting

Definition 4.3.1. Let f be a gradual set function on ground set V, and let U be a subset of V. The *twisting of f at U* is the set function f^U on ground set V defined by

$$f^U(X) := f(X\Delta U) + |X \cap U|.$$

The *complement* of f is the set function $\bar{f} := f^V$.

Proposition 4.3.2. *Every twisting of a gradual set function is a gradual set function.*

Proof. Straightforward. □

Proposition 4.3.3. *For a set $U \subseteq V$ and an element $v \in V$ the following hold.*

(i)

$$(f\backslash v)^{U-v} \cong \begin{cases} f^U/v & \text{if } v \in U, \\ f^U\backslash v & \text{if } v \notin U, \end{cases}$$

(ii)

$$(f/v)^{U-v} \cong \begin{cases} f^U\backslash v & \text{if } v \in U, \\ f^U/v & \text{if } v \notin U. \end{cases}$$

Proof. Suppose first that $v \in U$ and take a set $X \subseteq V - v$. Then

$$\begin{aligned}(f\backslash v)^{U-v}(X) &= f\backslash v(X\Delta(U-v)) + |X \cap (U-v)| = \\ &= f((X+v)\Delta U) + |(X+v) \cap U| - 1 = \\ &= f^U(X+v) - 1 = f^U/v(X) - 1, \quad \text{and}\end{aligned}$$

$$\begin{aligned}(f/v)^{U-v}(X) &= f/v(X\Delta(U-v)) + |X \cap (U-v)| = \\ &= f(X\Delta U) + |X \cap U| = f^U(X) = f^U\backslash v(X).\end{aligned}$$

The other cases are similar. □

Proposition 4.3.4. *Every twisting of an ideal set function is also ideal.*

Proof. Let f be an ideal set function on V, and U be a subset of V. Consider the following $(|V|+1) \times (|V|+1)$ matrix:

$$M_U := \begin{pmatrix} -1 & & & & & & & & \\ & -1 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & -1 & & & & & \\ & & & & 1 & & & & \\ & & 0 & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} -1 \\ -1 \\ \ddots \\ -1 \end{matrix}}\right\} U \\ \vphantom{\begin{matrix} 1 \\ 1 \\ \ddots \\ 1 \\ 1 \end{matrix}} \end{array}$$

It is easy to check that $M_U^{-1} = M_U$, so M_U is unimodular. We claim that

$$R(f) = M_U R(f^U) + (\chi_U, |U|).$$

Indeed, if we denote the describing matrix of $R(f)$ by A (that is, the matrix whose rows are the vectors $(\chi_X, -1)^\mathsf{T}$), then by $(\chi_X, -1)^\mathsf{T} M_U^{-1} = (\chi_{X\Delta U}, -1)^\mathsf{T}$, we have

$$\begin{aligned} &M_U R(f^U) + (\chi_U, |U|) = \{M_U(y,\beta) : \ A(y,\beta) \geq f^U\} + (\chi_U, |U|) = \\ &= \{(z,\gamma) : \ AM_U^{-1}(z,\gamma) \geq f^U + AM_U^{-1}(\chi_U, |U|)\} = \\ &= \{(z,\gamma) : \ (\chi_{X\Delta U}, -1)^\mathsf{T}(z,\gamma) \geq f^U(X) + (\chi_{X\Delta U}, -1)^\mathsf{T}(\chi_U, |U|) \quad \forall X \subseteq V\} = \\ &= \{(z,\gamma) : \ z(X\Delta U) - \gamma \geq f(X\Delta U) + |X \cap U| + |U \setminus X| - |U| \quad \forall X \subseteq V\} = \\ &= \{(z,\gamma) : \ z(Y) - \gamma \geq f(Y) \quad \forall Y \subseteq V\} = R(f). \end{aligned}$$

Hence we also have $R(f^U) = M_U^{-1}(R(f) - (\chi_U, |U|)) = M_U R(f) + (\chi_U, 0)$. Therefore $R(f)$ is integer if and only if $R(f^U)$ is integer. □

Corollary 4.3.5. *Every twisting of an mni set function is also mni.*

Proof. This follows from Propositions 4.3.3 and 4.3.4. □

4.4 Examples

4.4.1 Clutters

Let $\mathcal{C}$ be a clutter on ground set V. Let $\mathcal{C}^\uparrow$ denote the uphull of $\mathcal{C}$, that is, $\{X \subseteq V : \ \exists C \in \mathcal{C} : \ C \subseteq X\}$. We associate the following gradual set function to

$\mathcal{C}$:

$$f_{\mathcal{C}}(X) = \begin{cases} 1 & \text{if } X \in \mathcal{C}^{\uparrow}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that this works well with the minor operations: for any $v \in V$, $f_{\mathcal{C}\setminus v} = f_{\mathcal{C}} \setminus v$ and $f_{\mathcal{C}/v} = f_{\mathcal{C}}/v$. Likewise, one can check that the blocker $b(f_{\mathcal{C}})$ is a translation of the set function corresponding to the blocker of $\mathcal{C}$, namely $f_{b(\mathcal{C})}$ (they differ by 1).

Proposition 4.4.1. *A clutter $\mathcal{C}$ is ideal if and only if $f_{\mathcal{C}}$ is ideal.*

Proof. It is easy to see that

$$\begin{aligned} Q(f_{\mathcal{C}}) &= \{(y, \beta) \in \mathbb{R}^{n+1} : \ y \geq 0, \ \ y(X) - \beta \geq f_{\mathcal{C}}(X) \ \ \forall X \subseteq V\} = \\ &= \{(y, \beta) \in \mathbb{R}^{n+1} : \ y \geq 0, \ \ -\beta \geq 0, \ y(X) - \beta \geq 1 \ \ \forall X \in \mathcal{C}\}. \end{aligned}$$

So the face of $Q(f_{\mathcal{C}})$ in the $\beta = 0$ hyperplane is the covering polyhedron of $\mathcal{C}$, thus if $f_{\mathcal{C}}$ is ideal then $\mathcal{C}$ is ideal too.

To see the other direction, note that in the above description all inequalities but $-\beta \geq 0$ are incident to the vector $(\mathbf{0}, -1)$. Therefore $Q(f_{\mathcal{C}})$ is the intersection of a cone pointed at $(\mathbf{0}, -1)$ and the halfspace $\{(y, \beta) : \ \beta \leq 0\}$. It follows that if $\mathcal{C}$ is ideal then $f_{\mathcal{C}}$ is also ideal. $\square$

Corollary 4.4.2. *A clutter $\mathcal{C}$ is mni if and only if $f_{\mathcal{C}}$ is mni.*

We note that Lehman's Theorem 1.4.1 has the consequence that if $\mathcal{C}$ is mni, then the polyhedron $Q(f_{\mathcal{C}})$ has a unique fractional vertex and it is simple (here a vertex is simple if it lies on $n+1$ facets).

4.4.2 Matroid rank functions

For a matroid $\mathcal{M}$, its rank function r and also its corank function cr are gradual functions.

Proposition 4.4.3. *Both the rank function r and the corank function cr of a matroid are ideal gradual functions.*

Proof. First we prove that cr is ideal. It is enough to show that the polyhedron

$$R(cr) = \{(x, \alpha) \in \mathbb{R}^{n+1} : \ x(Y) - \alpha \geq cr(Y) \ \forall Y \subseteq V\}$$

is integer. In fact it can be proved using standard uncrossing techniques that this system is TDI, since cr is supermodular.

The blocker of r is $b(r)(Y) = -r(V - Y) = cr(Y) - r(V)$, which is a translation of cr, thus $b(r)$ is ideal, hence r is also ideal by Theorem 4.2.2. $\square$

4.4.3 Nearly bipartite graphs

Let $G = (V, E)$ be a graph and f_G the following gradual set function on V:

$$f_G(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ 1 & \text{if } X \text{ is a stable set of } G \\ 2 & \text{otherwise.} \end{cases}$$

The graph G is called *nearly bipartite* if for every node v, the graph $G[V - N(v)]$ is bipartite, where $N(v)$ denotes the (closed) neighbourhood of v.

Proposition 4.4.4. *Let f be a gradual function with values in $\{0, 1, 2\}$ such that $f(\emptyset) = 0$ and $f(v) = 1$ ($\forall v \in V$). Then f is ideal if and only if $f = f_G$ for a nearly bipartite graph G.*

Proof. If there is a set $Z \subseteq V$ which is of size at least 3 and which is a minimal set with $f(Z) = 2$, then it can be seen that $f \backslash (V - Z)$ is nonideal (in fact it is mni, see the function θ_n Section 4.4.4).

If there is no such Z, then $f = f_G$ for a graph G.

If G is not nearly bipartite, then let v be a node for which $G[V - N(v)]$ is not bipartite. In this case $f/v\backslash(N(v))$ is a translation of the gradual function of the clutter formed by the edges of the graph $G[V - N(v)]$. Since this graph is not bipartite, the clutter is nonideal.

Suppose now that G is nearly bipartite. It can be seen that the point $(\mathbf{1}, 0)$ lies on each facet of $P(f_G)$ except for the nonnegativity facets (this is true for any graph, by graduality). Thus it is enough to show that for any node v, $P(f_G) \cap \{x : x_v = 0\}$ is an integer polyhedron, that is, f_G/v is ideal. This follows from the fact that $G[V - N(v)]$ is bipartite. □

4.4.4 A class of mni gradual set functions

We saw already that the gradual functions associated with mni clutters are minimally non-ideal.

Proposition 4.4.5. *The following two types of gradual functions are mni, if $n \geq 3$:*

$$\theta_n(X) = \begin{cases} 0 & \textit{if } X = \emptyset, \\ 2 & \textit{if } X = V, \\ 1 & \textit{otherwise.} \end{cases}$$

$$\overline{\theta}_n(X) = \begin{cases} 0 & \textit{if } X = \emptyset, \\ n-2 & \textit{if } X = V, \\ |X|-1 & \textit{otherwise.} \end{cases}$$

They are translations of each others complements; and their blockers, $b(\theta_n)$ and $b(\overline{\theta}_n)$ are translations of θ_n and $\overline{\theta}_n$, respectively.

We note that the above set functions can be obtained by twisting the degenerate projective plane.

4.4.5 An mni set function with non-simple fractional vertex

We mentioned in Section 4.4.1 that if our mni set function $f_{\mathcal{C}}$ is defined by an mni clutter $\mathcal{C}$, then $Q(f_{\mathcal{C}})$ has a unique fractional vertex and it is simple. This raises the question whether this is true for arbitrary mni set functions as well. The following mni set function f on ground set $[5]$ is an example where the unique fractional vertex of $Q(f)$ is not simple. The properties were checked using the software polymake [36].

$$f(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ 1 & \text{if } |X| = 1 \text{ or } X \in \{\{1,2\},\{2,3\},\{3,4\}\{4,5\}\}, \\ 2 & \text{if } |X| = 3 \text{ or } X \in \{\{1,3\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,5\}\} \\ & \quad \text{or } X \in \{\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\}\}, \\ 3 & \text{if } X \in \{\{1,2,3,4\},\{2,3,4,5\},\{1,2,3,4,5\}\}. \end{cases}$$

The question of uniqueness of the fractional vertex remains open.

4.5 Convex and concave gradual extensions

Let g be a function on a box B in $\mathbb{R}^n$. We call g *gradual* if for every $x, z \in B$ for which $x \leq z$, $g(x) \leq g(z) \leq g(x) + \|z - x\|_1$ holds. In this section we consider gradual extensions of a gradual set function f to the unit cube $\{x \in \mathbb{R}^n : \mathbf{0} \leq x \leq \mathbf{1}\}$, and prove that f is ideal if and only if it has a unique convex gradual extension to the unit cube.

Proposition 4.5.1. *The maximal convex extension of a gradual set function f to the unit cube $\{x \in \mathbb{R}^n : \mathbf{0} \leq x \leq \mathbf{1}\}$ is*

$$\hat{f}(z) := \min\{\sum_{Y \subseteq V} \lambda_Y f(Y) : \lambda_Y \geq 0 \ \forall Y \subseteq V, \sum_{Y \subseteq V} \lambda_Y = 1, \sum_{Y \subseteq V} \lambda_Y \chi_Y = z\},$$

which is moreover gradual. The minimal convex gradual extension of f to the unit cube is

$$\tilde{f}(z) := \max\{f(Y) + z(Y) - |Y| : \ Y \subseteq V\}.$$

Proof. It is easy to check that $\hat{f}$ is a convex function, and that it is indeed an extension of f.

Let us first check that $\hat{f}$ is gradual. It is enough to prove the condition for x and $z \in [0,1]^n$ for which $z = x + \alpha e_i$ for some $i \in V$ and positive α. To prove that $\hat{f}(x) \leq \hat{f}(z)$, let λ^z attain the minimum in the definition of $\hat{f}(z)$. Let us modify λ^z as follows, to get λ': we decrease some positive λ^z_Y for sets Y that contain i, in total with α, and increase the corresponding λ^z_{Y-i} with the same amount. Then $\lambda' \geq \mathbf{0}$, $\sum_{Y \subseteq V} \lambda'_Y = 1$ and $\sum_{Y \subseteq V} \lambda'_Y \chi_Y = x$, and since $f(Y) \geq f(Y - i)$, we have

$$\hat{f}(z) = \sum_{Y \subseteq V} \lambda^z_Y f(Y) \geq \sum_{Y \subseteq V} \lambda'_Y f(Y) \geq \hat{f}(x).$$

To prove that $\hat{f}(z) \leq \hat{f}(x) + \alpha$, let λ^x attain the minimum in the definition of $\hat{f}(x)$. Now we modify λ^x as follows, to get λ'': we decrease some positive λ^x_Y for sets Y not containing i, in total with α and increase λ^x_{Y+i} with the same amount. Then $\lambda'' \geq \mathbf{0}$, $\sum_{Y \subseteq V} \lambda''_Y = 1$ and $\sum_{Y \subseteq V} \lambda''_Y \chi_Y = z$, and since $f(Y) \geq f(y + i) - 1$, we have that

$$\hat{f}(x) = \sum_{Y \subseteq V} \lambda^x_Y f(Y) \geq \sum_{Y \subseteq V} \lambda''_Y f(Y) - \alpha \geq \hat{f}(z) - \alpha.$$

To prove that $\hat{f}$ is the maximal convex extension of f, let g be an arbitrary convex extension of f. Then for the coefficient vector $\lambda \in \mathbb{R}^{2^V}_+$ for which $\sum_{Y \subseteq V} \lambda_Y = 1$, $\hat{f}(z) = \sum_{Y \subseteq V} \lambda_Y f(Y)$ and $\sum_{Y \subseteq V} \lambda_Y \chi_Y = z$, we have $g(z) \leq \sum \lambda_Y g(Y) = \sum \lambda_Y f(Y) \leq \hat{f}$.

Now let us turn to $\tilde{f}$. It is an extension of f, because

$$\begin{aligned}\tilde{f}(\chi_Z) &= \max\{f(Y) + |Z \cap Y| - |Y| : \ Y \subseteq V\} \\ &= \max\{f(Y) - |Y \setminus Z| : \ Y \subseteq V\} = f(Z),\end{aligned}$$

where we use that $f(Z) \geq f(Y) - |Y \setminus Z|$, since f is gradual.

The function $\tilde{f}$ is convex, because it is the maximum of finitely many linear functions.

To prove that $\tilde{f}$ is gradual, it is again enough to consider $x, z \in [0,1]^n$ for which $z = x + \alpha e_i$ for some $i \in V$ and $\alpha > 0$. If Y^x attains the maximum in the definition of $\tilde{f}(x)$, then

$$\tilde{f}(x) = f(Y^x) + x(Y^x) - |Y^x| \leq f(Y^x) + z(Y^x) - |Y^x| \leq \tilde{f}(z).$$

On the other hand, if Y^z attains the maximum in the definition of $\tilde{f}(z)$, then

$$\tilde{f}(z) = f(Y^z) + z(Y^z) - |Y^z| \leq f(Y^z) + x(Y^z) - |Y^z| + \alpha \leq \tilde{f}(x) + \alpha,$$

using that $z(Y^z)$ either equals to $x(Y^z)$ (if $i \notin Y^z$), or is $x(Y^z) + \alpha$ (if $i \in Y^z$), so in either case $z(Y^z) \leq x(Y^z) + \alpha$.

It remains to prove that $\tilde{f}$ is the minimal convex gradual extension of f. Let g be a convex gradual extension of f. We have to show that for any vector $z \in [0,1]^n$ and set $Y \subseteq V$, $g(z) \geq f(Y) + z(Y) - |Y|$ holds. Let z' be the following vector:

$$z'_i := \begin{cases} z_i & \text{if } i \in Y, \\ 0 & \text{if } i \notin Y. \end{cases}$$

Obviously, $z' \leq z$ and $z' \leq \chi_Y$, thus, since g is a gradual function, we have $g(z') \leq g(z)$ and $g(\chi_Y) \leq g(z') + \|\chi_Y - z'\|_1$, thus

$$g(z) \geq g(z') \geq g(\chi_Y) - \|\chi_Y - z'\|_1 = f(Y) - \sum_{i \in Y}(1 - z_i) = f(Y) - |Y| + z(Y),$$

so we are done. □

Proposition 4.5.2. *A gradual set function f is ideal if and only if it has a unique convex gradual extension to the unit cube.*

Proof. By Proposition 4.5.1 it is enough to prove that f is ideal if and only if $\hat{f} = \tilde{f}$. By definition, f is ideal if and only if there is no $(x, \alpha) \in P(f) \setminus \text{conv}\{S^{\downarrow}(b(f))\}$. Substituting $z = 1 - x$ and $\gamma = -\alpha$, this is equivalent to that there is no $(z, \gamma) \in \mathbb{R}^{n+1}$ for which $\mathbf{0} \leq z \leq \mathbf{1}$, and which satisfies $|Y| - z(Y) + \gamma \geq f(Y)$ for each set $Y \subseteq V$, and which is not in $\text{conv}\{(\chi_{V \setminus Y}, -\beta) : \; b(f)(Y) \geq \beta\}$.

The convex hull can be written as $\text{conv}\{(\chi_Y, \beta) : \; b(f)(V \setminus Y) \geq -\beta\} = \text{conv}\{(\chi_Y, \beta) : \; f(Y) \leq \beta\}$. We claim that $(z, \gamma) \in \text{conv}\{(\chi_Y, \beta) : \; f(Y) \leq \beta\}$ if and only if $\gamma \geq \hat{f}(z)$. For the "only if" direction, if $(z, \gamma) \in \text{conv}\{(\chi_Y, \beta) : f(Y) \leq \beta\}$, then there exist nonnegative coefficients λ_Y for $Y \subseteq V$ which sum up to 1, and for which $\sum_{Y \subseteq V} \lambda_Y \chi_Y = z$, and numbers $\beta_Y \geq f(Y)$ (for $Y \subseteq V$) for which $\sum_{Y \subseteq V} \lambda_Y \beta_Y = \gamma$. This implies

$$\gamma = \sum_{Y \subseteq V} \lambda_Y \beta_Y \geq \sum_{Y \subseteq V} \lambda_Y f(Y) \geq \hat{f}(z).$$

For the "if" direction, if $\gamma \geq \hat{f}(z)$ then there exist nonnegative coefficients λ_Y for $Y \subseteq V$ which sum up to 1 and for which $\sum_{Y \subseteq V} \lambda_Y \chi_Y = z$ and $\sum_{Y \subseteq V} \lambda_Y f(Y) \leq \gamma$.

Then it is easy to see that we can choose numbers $\beta_Y \geq f(Y)$ (for $Y \subseteq V$) for which $\sum_{Y \subseteq V} \lambda_Y \beta_Y = \gamma$, which implies that (z, γ) is in the convex hull.

Using this, we have that f is ideal if and only if there is no $(z, \gamma) \in \mathbb{R}^{n+1}$ for which $\gamma \geq \tilde{f}(z)$ and $\gamma < \hat{f}(z)$, which means that $\tilde{f} \geq \hat{f}$, which is equivalent to $\tilde{f} = \hat{f}$, since Proposition 4.5.1 implies that $\tilde{f} \leq \hat{f}$ always holds. □

Theorem 4.5.3. *For a gradual set function f, the following are equivalent.*

(i) f is ideal,

(ii) f has a unique convex gradual extension to the unit cube,

(iii) f has a unique concave gradual extension to the unit cube,

(iv) $\mathrm{conv}(S(f))$ *can be described as*

$$\begin{aligned} \{(x,\alpha) \in \mathbb{R}^{n+1} : \mathbf{0} &\leq x \leq \mathbf{1}, \\ x(Y) - \alpha &\leq u(Y) \quad \forall Y \subseteq V, \\ x(Y) - \alpha &\geq l(Y) \quad \forall Y \subseteq V \,\} \end{aligned}$$

for some set functions u and l,

(v) $\mathrm{conv}(S(f))$ *can be described as*

$$\begin{aligned} \{(x,\alpha) \in \mathbb{R}^{n+1} : \mathbf{0} &\leq x \leq \mathbf{1}, \\ x(Y) - \alpha &\leq |Y| - f(Y) \quad \forall Y \subseteq V, \\ x(Y) - \alpha &\geq b(f)(Y) \quad \forall Y \subseteq V \,\}. \end{aligned}$$

Proof. Proposition 4.5.2 states that (i) and (ii) are equivalent.

By Theorem 4.2.2, f is ideal if and only if its blocker $b(f)$ is ideal. The graph of $b(f)$ is obtained by a point reflection through the point $(\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2}, 0)$, thus f has a unique convex gradual extension if and only if $b(f)$ has a unique concave gradual extension, so we have (ii) $\Leftrightarrow$ (iii).

By Propositions 4.5.1 and 4.5.2, if f is ideal, then $\hat{f} = \tilde{f}$, thus the facets of the convex hull for which the last coordinate of their normal vector is negative, are of the form $\alpha \geq f(Y) + x(Y) - |Y|$ for some $Y \subseteq V$. By the same argument for the blocker, the facets with positive last coordinate are of the form $-\alpha \geq b(f)(Y) + (1-x)(Y) - |Y|$ for some $Y \subseteq V$. The facets with last coordinate 0 are among $\mathbf{0} \leq x \leq \mathbf{1}$. Thus ideality implies (v).

(iv) trivially follows from (v).

Now suppose that the set functions u and l describe $\mathrm{conv}(S(f))$ as in (iv). We want to prove that f has a unique convex gradual extension, namely (ii). For $x \in [0,1]^n$ let $g(x)$ be $\max\{x(Y) - u(Y) : Y \subseteq V\}$, that is, the "lower envelope" of $\mathrm{conv}(S(f))$. We claim that g is the only convex gradual extension of f. It is certainly convex and gradual, since it is the maximum of (finitely many) linear gradual functions, and we know that it is the maximal convex extension of f. Let g' be a convex gradual extension of f, and take $x \in [0,1]^n$. Let Y be the set for which $g(x) = x(Y) - u(Y)$, and let x' be the vector for which

$$x'_i := \begin{cases} x_i & \text{if } i \in Y \\ 0 & \text{if } i \notin Y \end{cases}.$$

Since g' is gradual, we have

$$\begin{aligned} g'(x) \geq g'(x') \geq g'(\chi_Y) - \|\chi_Y - x'\|_1 = f(Y) - (1-x)(Y) = \\ = f(Y) - |Y| + x(Y) = f(Y) - |Y| + g(x) + u(Y) \geq g(x), \end{aligned}$$

where the last inequality is valid since $(\chi_Y, f(Y)) \in \mathrm{conv}(S(f))$. We proved that $g' \geq g$, thus g is the minimal convex gradual extension of f, so we are done. □

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Summary

In this thesis we explore questions related to polyhedra and integrality. The three main chapters are loosely connected and contain results of different nature.

In Chapter 2 we examine several properties of polyhedra from a computational complexity point of view. We prove that testing whether a conic system is TDI – or, equivalently, testing whether a set of vectors forms a Hilbert-basis – is co-NP-complete, even for binary rows (respectively vectors) with at most three 1s. This answers a question raised by Papadimitriou and Yannakakis in 1990. We prove also that it can be decided in polinomial time whether a system describes a generalized polymatroid and the same is true for integer generalized polymatroids. We use a notion called total dual laminarity and prove that it is in contrast NP-hard. In addition, we prove that integer g-polymatroids form a maximal class for which it is true that every pairwise intersection is an integer polyhedron.

We begin Chapter 3 by stating a polyhedral version of Sperner's lemma and then we deduce a variety of mostly known results from it, such as the theorem of Boros and Gurvich that perfect graphs are kernel-solvable and a new generalization of it, a game theoretic result of Scarf, a result by Fleiner on a matroidal generalization of kernels, and another one on stable flows. Motivated by the fact that the converse of the Boros–Gurvich Theorem is also true by the Strong Perfect Graph Theorem, we investigate possible converse statements of the polyhedral Sperner's lemma and some of its applications – we give some counterexamples and a few special cases when the converse remains true. We show also that the complexity problem corresponding to the polyhedral Sperner's Lemma is PPAD-complete.

In Chapter 4 we define a notion of idealness of set functions which generalizes ideal clutters to set functions instead of set systems. We generalize several related notions like the blocking relation, minors, minimally non-idealness and prove that many of their properties stay true. We prove several equivalent formulations of idealness and minimally non-idealness, one of which is new to this general framework: a set function is ideal if and only if it has a unique convex (or concave) gradual extension to the unit cube. By an example we show that Lehman's Theorem in its full generality does not extend to this setting.

Összefoglaló

A disszertációban poliéderekkel és egészértékűséggel kapcsolatos kérdéseket derítünk fel. A három fő fejezet lazán kapcsolódik egymáshoz, és különböző jellegű eredményeket tartalmaz.

A második fejezetben poliéderek néhány tulajdonságát vizsgáljuk bonyolultságelméleti szempontból. Bebizonyítjuk, hogy co-NP-teljes eldönteni, hogy egy homogén lineáris rendszer TDI-e, vagy ekvivalens módon, hogy vektorok egy halmaza Hilbert-bázist alkot-e, abban az esetben is, ha a sorok illetve vektorok binárisak, legfeljebb három egyessel. Ez megválaszolja Papadimitriou és Yannakakis 1990-ben felvetett kérdését. Emellett polinomiális algoritmust adunk annak eldöntésére, hogy egy adott rendszer g-polimatroidot (illetve egész g-poliamtroidot) határoz-e meg. A teljesen duálisan laminaritás fogalmát használjuk, amiről viszont belátjuk, hogy eldöntése NP-nehéz. Ezen felül belátjuk, hogy az egész g-polimatroidok halmaza maximális olyan, amiben minden páronkénti metszet egész poliéder.

A harmadik fejezetet azzal kezdjük, hogy megfogalmazzuk a Sperner lemma egy poliéderes változatát, amiből ezután számos, főként ismert tételt vezetünk le egyszerűen, például Boros és Gurvich tételét, miszerint a perfekt gráfok kernel-megoldhatók, amit általánosítunk is, Scarf egy játékelméleti eredményét, Fleiner egy eredményét a kernelek matroidos általánosításáról, és egy másik eredményét stabil folyamokról. Azáltal ösztönözve, hogy a Boros-Gurvich tétel fordított iránya is igaz az erős perfekt gráf tétel miatt, megvizsgáljuk a poliéderes Sperner lemma illetve néhány alkalmazásának lehetséges megfordítását – adunk néhány ellenpéldát, és mutatunk néhány speciális esetet, ahol igaz a fordított irány. Azt is belátjuk, hogy a poliéderes Sperner lemmának megfelelő bonyolultságelméleti feladat PPAD-teljes.

A negyedik fejezetben kiterjesztjük az idealitás fogalmát halmazrendszerekről halmazfüggvényekre. Néhány további kapcsolódó fogalmat is általánosítunk, mint a blokker, a minorok, a minimálisan nemideális halmazrendszerek, és megmutatjuk, hogy számos tulajdonságuk szintén kiterjed. Belátjuk néhány ekvivalens jellemzését az idealitásnak és az mni-ségnek, amik egyike újdonság ebben a kontextusben: egy halmazfüggvény pontosan akkor ideális, ha egyértelmű a konvex (vagy konkáv) kiterjesztése az egységkockára. Egy példával megmutatjuk, hogy Lehman tétele teljes általánosságában nem terjeszthető ki.