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Discrete methods in geometric measure theory

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Notation

λ	1-dimensional Lebesgue measure.
λ^n	n-dimensional Lebesgue measure.
diam	Diameter of a set.
$B(x,r)$	Closed ball with center x and radius r.
$S(x,r)$	The sphere with center x and radius r.
$\mathcal{H}^s(A)$	s-dimensional Hausdorff measure of A.
$\mathcal{H}^s_\delta(A)$	$= \inf\left\{\sum \operatorname{diam}(A_i)^s \,:\, A \subset \bigcup A_i;\, \operatorname{diam}(A_i) \le \delta\right\}$.
$\mathcal{H}^s_\infty(A)$	$= \inf\left\{\sum \operatorname{diam}(A_i)^s \,:\, A \subset \bigcup A_i\right\}$.
dim	Hausdorff dimension.
$\overline{\dim}_{\mathrm{M}}$	Upper Minkowski dimension.
$\underline{\dim}_{\mathrm{M}}$	Lower Minkowski dimension.
$[x]$	The greatest integer not more than x.
$\lceil x \rceil$	The smallest integer not less than x.
$\angle(\mathbf{u},\mathbf{v})$	The angle enclosed by the vectors $\mathbf{u}$ and $\mathbf{v}$.

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Chapter 1

Introduction

The task of guaranteeing given patterns in a sufficiently large set has been a central problem in various areas of mathematics for a long time. Perhaps the most famous example is the celebrated theorem of Szemerédi claiming that any sequence of positive integers with positive upper density contains arbitrarily long arithmetic progressions. Such problems are often studied in the field of geometric measure theory, as well. The following problem was proposed by Tamás Keleti. We say that a set $A \subset \mathbb{R}^n$ contains the angle α if there exist distinct points $P, Q, R \in A$ such that $\angle PQR = \alpha$. How large (in terms of Hausdorff dimension) can a compact set $A \subset \mathbb{R}^n$ be if it does not contain some given angle α? Or equivalently, how large dimension guarantees that our set must contain α? These questions will be investigated in Chapter 2.

We also study an approximate version of this problem, where we only want our set to contain angles close to α rather than contain the exact angle α. This version turns out to be completely different from the original one, which is best illustrated by the case $\alpha = \pi/2$. If the dimension of our set is greater than 1, then it must contain angles arbitrarily close to $\pi/2$. However, if we want to make sure that it contains the exact angle $\pi/2$, then we need to assume that its dimension is greater than $n/2$.

Another interesting phenomenon is that different angles show different behaviour. In the approximate version the angles $\pi/3$, $\pi/2$ and $2\pi/3$ play special roles, while in the original version $\pi/2$ seems to behave differently than other angles.

One of our goals will be to construct large dimensional sets not containing some angle α. Our strategy will be that we first construct large discrete sets, which then can be *blown up* to large dimensional self-similar sets. Such a discrete set that stands behind one of our constructions will be the following. Erdős and Füredi used the probabilistic method to prove that for any $\delta > 0$ there is a constant $c_\delta > 1$ such that there exist c_δ^n points in $\mathbb{R}^n$ with the property that the angle determined by any three points is less than $\pi/3 + \delta$.

This result is related to the next topic studied by this thesis, as well.

We say that a finite set of points is an acute set if any angle determined by three points of the set is acute. In Chapter 3 we examine the maximal cardinality $\alpha(n)$ of an n-dimensional acute set. The above result of Erdős and Füredi tells us that $\alpha(n)$ is exponentially large. (Before their random construction it was conjectured by Danzer and Grünbaum that $\alpha(n) < Cn$ for some constant C.) The exact value of $\alpha(n)$ is known only for $n \leq 3$. For each $n \geq 4$ we improve on the best known lower bound for $\alpha(n)$. We present different approaches. On one hand, we give a probabilistic proof that $\alpha(n) > c \cdot 1.2^n$. (This improves the random construction given by Erdős and Füredi.) On the other hand, we give an *almost* exponential constructive example which outdoes the random construction in low dimension ($n \leq 250$). Both approaches use the small dimensional examples that we found partly by hand ($n = 4, 5$), partly by computer ($6 \leq n \leq 10$).

We also investigate the following variant of this problem: what is the maximal size $\kappa(n)$ of an n-dimensional cubic acute set (that is, an acute set contained in the vertex set of an n-dimensional hypercube). We give an *almost* exponential constructive lower bound, and we improve on the best known upper bound.

Finally, in Chapter 4 we show that the Koch curve is tube-null, that is, it can be covered by strips of arbitrarily small total width. In fact, we prove the following stronger result: the Koch curve can be decomposed into three sets such that each can be projected to a line in such a way that the image has Hausdorff dimension less than 1. The proof contains geometric, combinatorial, algebraic and probabilistic arguments.

Chapter 2 is based on [19] and [20]. The latter is a joint paper with Keleti, Kiss, Maga, Máthé, Mattila and Strenner. For the sake of completeness some constructions due to Máthé are also included in this thesis (Section 2.5). Chapter 3 and 4 are based on [21] and [22], respectively.

Chapter 2

How large dimension guarantees a given angle?

An easy consequence of Lebesgue's density theorem claims that for any Lebesgue measurable set $A \subset \mathbb{R}^n$ with positive Lebesgue measure it holds that a similar copy of any finite configuration of points can be found in A.

What can be said about infinite configurations? Erdős asked whether there is a sequence $x_n \to 0$ such that a similar copy of this sequence can be found in every measurable set $A \subset \mathbb{R}$ with $\lambda(A) > 0$. This question is usually referred to as Erdős similarity problem and still unsolved.

And what about finite configurations in null sets? The following problem was also posed by Erdős. How large (in terms of Hausdorff dimension) can a set $A \subset \mathbb{R}^2$ be if there is no equilateral triangle with all three vertices in A? Falconer answered this question by showing that there exists a compact set A on the plane with Hausdorff dimension 2 such that A does not contain three points that form an equilateral triangle. In fact, it was shown in [15, 26, 27] that for any three points in $\mathbb{R}$ or in $\mathbb{R}^2$ there exists a compact set (in $\mathbb{R}$ or in $\mathbb{R}^2$) of full Hausdorff dimension, which does not contain a similar copy of the three points. It is open whether the analogous result holds in higher dimension.

It would be interesting to find *patterns*, which can be found in every full dimensional set. In this chapter we investigate such a pattern. We say that a set $A \subset \mathbb{R}^n$ contains the angle α if there exist distinct points $P, Q, R \in A$ such that $\angle PQR = \alpha$. Keleti posed the following question: how large can a set $A \subset \mathbb{R}^n$ be if it does not contain α? If there is no restriction on A, then for any given $\alpha \in [0, \pi]$ one can use transfinite recursion to construct a full dimensional set not containing α, see Section 2.6. The problem is more interesting, though, if we restrict ourselves to, for example, compact sets. What is the smallest s for which $\dim(A) > s$ implies that A must contain α provided that $A \subset \mathbb{R}^n$ is compact? (Or equivalently, what is the maximal Hausdorff dimension s of a compact set $A \subset \mathbb{R}^n$ with the property that A does not contain the angle α?) This minimal (maximal) value of s will be denoted by $C(n, \alpha)$. It is not hard to show that $C(n, \alpha) \le n-1$ for arbitrary α, in other words, if the Hausdorff dimension of a compact set $A \subset \mathbb{R}^n$ is greater than $n-1$, then A contains every angle $\alpha \in [0, \pi]$. For $\alpha = \pi/2$ we even prove that compact sets in $\mathbb{R}^n$ with Hausdorff dimension greater than $\lceil n/2 \rceil$ must contain $\pi/2$, that is $C(n, \pi/2) \le \lceil n/2 \rceil$, see Section 2.4.

As far as lower bounds are concerned, the line segment shows that $C(n, \alpha) \ge 1$ for any $\alpha \in (0, \pi)$. Our first goal is to improve on this obvious lower bound by constructing a compact set of Hausdorff dimension greater than 1 which does not contain some angle $\alpha \in (0, \pi)$. In Section 2.1 for any $\delta > 0$ we present a self-similar set $K \subset \mathbb{R}^n$ of dimension $c_\delta n$ such that all angles contained by A are from the δ-neighbourhood of the set $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$. It implies that $C(n, \alpha) \ge c(\alpha)n$ given that $\alpha \in (0, \pi)$ and $\alpha \ne \pi/3, \pi/2, 2\pi/3$.

What about the exceptional angles $\pi/3$, $\pi/2$, $2\pi/3$? In Section 2.2 we present a more involved construction of a self-similar set in $\mathbb{R}^n$ with dimension $c\sqrt[3]{n}/\log n$ that contains neither $\pi/3$, nor $2\pi/3$. The constructed sets also avoid a small neighbourhood of $\pi/3$ and $2\pi/3$. To be more precise, for any $\delta > 0$ we prove the existence of a set (in some Euclidean space of sufficiently large dimension) which has dimension $c\delta^{-1}/\log(\delta^{-1})$ and which contains no angle in the δ-neighbourhood of $\pi/3$ and $2\pi/3$. This latter result is essentially sharp: if the dimension of A is at least $C\delta^{-1}\log(\delta^{-1})$ for some $\delta > 0$, then A must contain an angle in the δ-neighbourhood of $\pi/3$ as well in the δ-neighbourhood of $2\pi/3$ (see Section 2.3). (Throughout this chapter c and C denote absolute constants but different appearances may denote different values.)

Both above constructions (the one for general angles and the one for $\pi/3$, $2\pi/3$) are so called *homothetic self-similar sets* (see Section 2.1 for details) and have the property that they avoid not only the given angle α but also a small neighbourhood of α. As Theorem 2.37 will show, one cannot avoid $\pi/2$ with such homothetic self-similar sets given that the dimension of the set is greater than 1. Moreover, it is shown in Section 2.3 that if the dimension of any set A is greater than 1, then A must contain angles arbitrarily close to $\pi/2$. In other words, it is impossible to construct sets of dimension greater than 1 that avoid a neighbourhood of $\pi/2$.

We outline another type of constructions in Section 2.5. These constructions use number theoretic methods and they are due to András Máthé. He proves, for example, that there exist compact sets in $\mathbb{R}^n$ with Hausdorff dimension $n/2$ such that they do not contain the angle $\pi/2$. Consequently, $C(n, \pi/2) \ge n/2$. As we have mentioned before,

$C(n,\pi/2) \le \lceil n/2 \rceil$. In particular, if n is even, we have $C(n,\pi/2) = n/2$.

We emphasize the difference between the tasks of finding an angle precisely and finding it approximately. For example, we can find angles arbitrarily close to $\pi/2$ given that the dimension of our set is greater than 1, while if we want to find the exact angle $\pi/2$ in our set, we need to know that its dimension is greater than $n/2$.

Sections 2.3, 2.4, 2.6 and parts of Section 2.1 are from [20] and thus joint work with Keleti, Kiss, Maga, Máthé, Mattila and Strenner. Section 2.2 and parts of Section 2.1 are from [19]. Section 2.5 contains results due to Máthé.

2.1 Avoiding general angles

In this section we construct sets with the property that any angle contained by the set is close to one of the following angles: $0, \pi/3, \pi/2, 2\pi/3, \pi$.

First we define homothetic self-similar sets and prove some simple facts about them. Let us take points $P_1, \dots, P_m$ in some Euclidean space $\mathbb{R}^n$. We denote the convex hull of these points by K_0. For every $i = 1, \dots, m$ we take a homothety φ_i with center P_i and scale factor $0 < q_i < 1$. Let K be the unique non-empty compact set satisfying $K = \bigcup_i \varphi_i(K)$. One can get this homothetic self-similar set K by setting

$$K_r \stackrel{\text{def}}{=} \bigcup_{i=1}^{m} \varphi_i\left(K_{r-1}\right) = \bigcup_{i_1,\dots,i_r} \varphi_{i_1} \circ \dots \circ \varphi_{i_r}(K_0),$$

then $K = \bigcap_{r=1}^{\infty} K_r$. We will use the following notations:

$$d_{\min} = \min\{|P_i - P_j| : i \ne j\}; d_{\max} = \max\{|P_i - P_j| : i \ne j\}; q_{\max} = \max\{q_1, \dots, q_m\}.$$

Set $\eta \stackrel{\text{def}}{=} q_{\max} d_{\max}/d_{\min}$. We will assume that $\eta < 1/2$ which clearly implies that the sets $\varphi_i(K_0)$ $(i = 1, \dots, m)$ are pairwise disjoint. Therefore the well-known Moran equation for the dimension s of the self-similar K holds:

$$q_1^s + \dots + q_m^s = 1,$$

which yields that in the special case $q_1 = \dots = q_m = q$ the dimension is

$$s = \frac{\log m}{\log\left(1/q\right)}.$$

For these sets most of the dimension notions (like Hausdorff or Minkowski dimension) coincide, so for the sake of simplicity in this and the next section we simply say dimension.

The next proposition says that the set of directions in K is *close* to the set of directions in $\{P_1, \dots, P_m\}$.

Proposition 2.1. *Suppose that $\eta = q_{\max} d_{\max}/d_{\min} < 1/2$. Then for any two distinct points $A, B \in K$ there exist $i \neq j$ such that the angle between the vectors $A - B$ and $P_i - P_j$ is less than $\pi\eta$.*

Proof. There exist unique sequences $i_1, i_2, \dots$ and $j_1, j_2, \dots$ such that

$$A \in \varphi_{i_1} \circ \cdots \circ \varphi_{i_r}(K) \text{ and } B \in \varphi_{j_1} \circ \cdots \circ \varphi_{j_r}(K)$$

for any positive integer r. Let r be the smallest index with $i_r \neq j_r$. Now let ψ be the homothety defined as $\varphi_{i_1} \circ \cdots \circ \varphi_{i_{r-1}} = \varphi_{j_1} \circ \cdots \circ \varphi_{j_{r-1}}$. Clearly $A' \stackrel{\text{def}}{=} \psi^{-1}(A) \in \varphi_{i_r}(K)$ and $B' \stackrel{\text{def}}{=} \psi^{-1}(B) \in \varphi_{j_r}(K)$. It also follows that $A' - B'$ and $A - B$ are parallel (one is a positive scalar multiple of the other).

So we can assume that A and B are in different level 1 parts of K, that is, there exist indices $i \neq j$ such that $A \in \varphi_i(K)$ and $B \in \varphi_j(K)$. Thus $|A - P_i|, |B - P_j| \leq q_{\max} d_{\max}$. Let us now translate the segment $P_i P_j$ by the vector $A - P_i$ so that P_i goes to A, and P_j goes to some point Q. Then the angle in question is equal to $\angle BAQ$. We have $|B - Q| \leq |A - P_i| + |B - P_j| \leq 2 q_{\max} d_{\max}$. On the other hand, $|A - Q| = |P_i - P_j| \geq d_{\min}$. Since $\eta < 1/2$, it follows that $|B - Q| < |A - Q|$. Under this condition the angle $\angle BAQ$ is clearly at most

$$\arcsin\left(\frac{|B - Q|}{|A - Q|}\right) \leq \arcsin(2\eta) \leq \pi\eta.$$

□

Corollary 2.2. *Suppose that $\eta < 1/2$. Then for any three distinct points A, B, C of K there exist indices i_1, i_2, i_3, i_4 such that*

$$|\angle ABC - \angle(P_{i_1} - P_{i_2}, P_{i_3} - P_{i_4})| < 2\pi\eta.$$

Proof. Let $A, B, C, D \in K$ with $A \neq B$ and $C \neq D$. We apply the above proposition for the vectors $A - B$ and $C - D$. It follows that there exist indices i_1, i_2, i_3, i_4 such that

$$|\angle(A - B, C - D) - \angle(P_{i_1} - P_{i_2}, P_{i_3} - P_{i_4})| < 2\pi\eta.$$

Setting $B = D$ completes the proof. □

In [20] this self-similar construction was used in the special case when the points P_i are the vertices of a regular simplex in $\mathbb{R}^n$. Then $m = n+1$; $d_{\min} = d_{\max}$ and the possible values of $\angle(P_{i_1} - P_{i_2}, P_{i_3} - P_{i_4})$ are 0, $\pi/3$, $\pi/2$, $2\pi/3$ and π. So setting $q_1 = \cdots = q_m = q < 1/2$ yields that K has dimension $\log(n+1)/\log(1/q)$ and all the angles contained by K are in the $2\pi q$-neighbourhood of the set $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$. So for any angle α not in this set there is a constant $c(\alpha)$ such that in $\mathbb{R}^n$ a set K of dimension $c(\alpha)\log(n+1)$ can be given with the property that K does not contain α as an angle.

The following simple observation enables us to do better than that, namely, we show the existence of a set of dimension $c(\alpha)n$ having the same property. For the above construction to work, it suffices to know that the distances $|P_i - P_j|$ are approximately the same (equal with some small error δ). And there are a lot of points in a Euclidean space with this property: in 1983 Erdős and Füredi proved [12] that for any $\delta > 0$ there exist at least $(1 + c\delta^2)^n$ points in $\mathbb{R}^n$ such that the distance of any two is between 1 and $1 + \delta$. This is also a special case of the well-known lemma of Johnson and Lindenstrauss which was first published in 1984 (see Lemma 2.7 in the next section).

Now we prove the simple fact that if we have four points with each pair having approximately the same distance, then the angles enclosed by the segments are close to either $\pi/3$ or $\pi/2$.

Lemma 2.3. *Suppose that the distance of any two of some given points is between* 1 *and* $1+\delta$ *for some* $\delta > 0$*. Then the angle between two arbitrary nonzero vectors with endpoints from the given set is in the* $C\delta$*-neighbourhood of the set* $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$.

Proof. We assume that $0 < \delta < 0.1$. If the lemma holds under this assumption, then it must also hold for arbitrary $\delta > 0$ (possibly with some larger C).

Take the endpoints of the two vectors. The set of these endpoints consists of either two, three or four points.

In the first case the two vectors coincide or they are the negative of each other. So the enclosed angle is 0 or π.

In the second case the two vectors share exactly one common endpoint which we denote by A. Let the two other endpoints be B_1, B_2 and let $\alpha = \angle B_1AB_2$. (So the angle enclosed by the vectors is α or $\pi - \alpha$.) By the cosine law we have

$$\cos\alpha = \frac{|A - B_1|^2 + |A - B_2|^2 - |B_1 - B_2|^2}{2|A - B_1||A - B_2|}.$$

Using this and the inequalities $(1+\delta)^2 < 1 + 3\delta$ and $1 - 3\delta < 1/(1 + 3\delta)$ we obtain that

$$\frac{1}{2} - 3\delta < \frac{(1 - 3\delta)^2}{2} \leq \frac{2 - (1 + 3\delta)}{2(1 + 3\delta)} \leq \cos\alpha \leq \frac{2(1 + 3\delta) - 1}{2} = \frac{1}{2} + 3\delta.$$

Since arccos is a Lipschitz function on the interval $[0.2, 0.8]$, it follows that $|\alpha - \pi/3| < C\delta$. Therefore, in this case the enclosed angle is in the $C\delta$-neighbourhood of $\pi/3$ or $2\pi/3$.

Finally, in the third case we have four distinct points A_1, A_2, B_1, B_2. Using coordinates, one can easily obtain the following formula for the inner product of the vectors $A_1 - A_2$

and $B_1 - B_2$:

$$\langle A_1 - A_2, B_1 - B_2 \rangle = \left(|A_1 - B_2|^2 + |A_2 - B_1|^2 - |A_1 - B_1|^2 - |A_2 - B_2|^2\right)/2,$$

which yields that for the angle β enclosed by $A_1 - A_2$ and $B_1 - B_2$ it holds that

$$\cos\beta = \frac{|A_1 - B_2|^2 + |A_2 - B_1|^2 - |A_1 - B_1|^2 - |A_2 - B_2|^2}{2|A_1 - A_2||B_1 - B_2|}.$$

Using that each distance is between 1 and $1 + \delta$ we obtain that

$$|\cos\beta| \leq \frac{2(1+\delta)^2 - 2}{2} = 2\delta + \delta^2 \leq 3\delta.$$

It follows that $|\beta - \pi/2| < C\delta$. □

In the next theorem we put together the above results to obtain large dimensional sets with all angles close to the special angles $0, \pi/3, \pi/2, 2\pi/3, \pi$.

Theorem 2.4. *There is a $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ there exists a self-similar set in $\mathbb{R}^n$ of dimension at least*

$$c_\delta n = c\delta^2 \log^{-1}(1/\delta) \cdot n$$

such that the angle determined by any three points of the set is in the δ-neighbourhood of the set $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$.

Proof. Take some real number $0 < \delta \leq 1/3$. As we mentioned before Lemma 2.3, there exist $m \geq (1 + c\delta^2)^n$ points $P_1, \dots, P_m \in \mathbb{R}^n$ such that the distance of any two of them is between 1 and $1 + \delta$. Take the homotheties with center P_i and ratio $q_i = q = \delta$, and consider the corresponding self-similar set K. On one hand, the dimension of K is

$$\frac{\log m}{\log(1/q)} \geq \frac{n \log(1 + c\delta^2)}{\log(1/\delta)} \geq c\frac{\delta^2}{\log(1/\delta)}n.$$

On the other hand, Lemma 2.3 and Corollary 2.2 imply that any angle in our self-similar set is in the $C\delta$-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$. Changing δ to δ/C completes the proof. □

2.2 Avoiding angles $\pi/3$ and $2\pi/3$

Our goal in this section is to construct large dimensional sets avoiding the angles $\pi/3$ and $2\pi/3$. Again, we will use the self-similar construction described at the beginning of the previous section. The idea is to find (many) points P_i such that any angle determined by them is in a small neighbourhood of $\pi/3$ but avoids an even smaller neighbourhood of $\pi/3$.

We were inspired by the following r-colouring of the complete graph on 2^r vertices. Let $C_1, \ldots, C_r$ denote the colours and let us associate to each vertex a 0-1 sequence of length r. Consider the edge between the vertices corresponding to the sequences $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$. We colour this edge with C_k where k denotes the first index where the sequences differ, that is, $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}, i_k \neq j_k$. Let us denote this coloured graph by $\mathcal{G}_r = \mathcal{G}_r(C_1, \ldots, C_r)$. This is a folklore graph colouring showing that the multicolour Ramsey number $R_r(3)$ is greater than 2^r.

One can obtain $\mathcal{G}_r$ recursively as well. Consider the colouring $\mathcal{G}_{r-1}(C_2, \ldots, C_r)$, and take two copies of this coloured graph. Let the edges going between the two copies be all coloured with C_1. It is easy to see that this way we get $\mathcal{G}_r(C_1, \ldots, C_r)$. This colouring clearly has the property that there is no monochromatic triangle in the graph. Moreover, every triangle has two sides with the same colour and a third side with a different colour of higher index.

The idea is to realize $\mathcal{G}_r$ geometrically in the following manner: the vertices of the graph will be represented by points of a Euclidean space and edges with the same colour will correspond to equal distances. In the sequel we will show that $\mathcal{G}_r$ can be represented in the above sense. First we prove a simple geometric fact.

Proposition 2.5. *Let m be a non-negative integer and R, l be positive real numbers with $R \le l/\sqrt{2}$. Take a $(2m+2)$-dimensional sphere $\mathcal{S}$ with radius*

$$R' \stackrel{def}{=} \sqrt{\frac{1}{4}l^2 + \frac{1}{2}R^2} \le \sqrt{\frac{1}{4}l^2 + \frac{1}{2}\left(\frac{l}{\sqrt{2}}\right)^2} = \frac{l}{\sqrt{2}}.$$

Then there exist two m-dimensional spheres $\mathcal{X}, \mathcal{Y} \subset \mathcal{S}$ with radius R such that $|X - Y| = l$ for any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$.

Proof. We may assume that $\mathcal{S} = \{P \in \mathbb{R}^{2m+3} : |P| = R'\}$. Set $t \stackrel{\text{def}}{=} \sqrt{l^2 - 2R^2}/2$ and take the spheres

$$\mathcal{X} \stackrel{\text{def}}{=} \left\{X = (x_1, \ldots, x_{m+1}, -t, 0, \ldots, 0) \in \mathbb{R}^{2m+3} : x_1^2 + \cdots + x_{m+1}^2 = R^2\right\},$$
$$\mathcal{Y} \stackrel{\text{def}}{=} \left\{Y = (0, \ldots, 0, t, y_1, \ldots, y_{m+1}) \in \mathbb{R}^{2m+3} : y_1^2 + \cdots + y_{m+1}^2 = R^2\right\}.$$

For any $X \in \mathcal{X}$ we have $|X| = \sqrt{R^2 + t^2} = R'$ and thus $\mathcal{X} \subset \mathcal{S}$. Similarly, $\mathcal{Y} \subset \mathcal{S}$. On the other hand, for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ it clearly holds that $|X - Y| = \sqrt{R^2 + (2t)^2 + R^2} = l$. $\square$

Lemma 2.6. *Let $l_1 \geq l_2 \geq \ldots \geq l_r > 0$ be a decreasing sequence of positive reals. By $\mathcal{I}_r$ we denote the set of 0-1 sequences of length r. Then 2^r points $P_{i_1,\ldots,i_r}$, $(i_1,\ldots,i_r) \in \mathcal{I}_r$ can be given in some Euclidean space in such a way that for two distinct 0-1 sequences $(i_1,\ldots,i_r) \neq (j_1,\ldots,j_r)$ the distance of $P_{i_1,\ldots,i_r}$ and $P_{j_1,\ldots,j_r}$ is equal to l_k where k denotes the first index where the sequences differ, that is, $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}, i_k \neq j_k$.*

Proof. For the sake of simplicity, we say that the points $P_{i_1,\ldots,i_r}$, $(i_1,\ldots,i_r) \in \mathcal{I}_r$ have configuration $\mathcal{P}_r(l_1,\ldots,l_r)$ if the distances between the points are as in the claim of the lemma.

We will prove by induction that there exist points with configuration $\mathcal{P}_r(l_1,\ldots,l_r)$ on a (2^r-2)-dimensional sphere with radius at most $l_1/\sqrt{2}$. This is clearly true for $r=1$. Suppose that it holds for $r-1$. The induction hypothesis applied for the distances $l_2 \geq \ldots \geq l_r$ yields that there exist points with configuration $\mathcal{P}_{r-1}(l_2,\ldots,l_r)$ on a $(2^{r-1}-2)$-dimensional sphere with radius $R \leq l_2/\sqrt{2}$.

Since $R \leq l_2/\sqrt{2} \leq l_1/\sqrt{2}$, Proposition 2.5 implies that there is a (2^r-2)-dimensional sphere $\mathcal{S}$ with some radius $R' \leq l_1/\sqrt{2}$ such that it contains two $(2^{r-1}-2)$-dimensional spheres with common radius R such that no matter how we take one point from each sphere their distance is l_1.

We can take a copy of the configuration $\mathcal{P}_{r-1}(l_2,\ldots,l_r)$ on each of these two spheres. The union of them clearly have configuration $\mathcal{P}_r(l_1,\ldots,l_r)$. $\square$

Using the above lemma we now construct a large set of points with the property that any angle determined by them is in a small neighbourhood of $\pi/3$ but avoids an even smaller neighbourhood of $\pi/3$. We will need the previously mentioned Johnson-Lindenstrauss lemma.

Lemma 2.7 (Johnson-Lindenstrauss lemma, [25])**.** *Suppose that m points $P_1,\ldots,P_m$ are given in some Euclidean space $\mathbb{R}^d$. For any $\delta > 0$ one can find points $P'_1,\ldots,P'_m$ in the $\lceil C\log m/\delta^2\rceil$-dimensional Euclidean space in such a way that*

$$|P_i - P_j| \leq |P'_i - P'_j| \leq (1+\delta)|P_i - P_j| \quad (1 \leq i,j \leq m).$$

Theorem 2.8. *There exist absolute constants $c, C > 0$ such that for any positive integer r and positive real $\varepsilon < 1$, 2^r points can be given in the $\lceil Cr^3/\varepsilon^2\rceil$-dimensional Euclidean space with the property that for any angle α determined by three given points the following holds:*

$$c\frac{\varepsilon}{r} < \left|\alpha - \frac{\pi}{3}\right| < \varepsilon.$$

Moreover, for any four distinct points A, B, C, D of these points we have

$$\left|\angle(A-B, C-D) - \frac{\pi}{2}\right| < \varepsilon.$$

Proof. Let $\lambda > 1$ be a real number. We use Lemma 2.6 with $l_i = \lambda^{r-i}$ $(i = 1, \dots, r)$. The lemma gives us 2^r points which have configuration $\mathcal{P}_r(\lambda^{r-1}, \dots, \lambda, 1)$. Let us denote the set of these points by S, and take three distinct points in S. By construction, the triangle determined by these points has two sides with the same length λ^s and a third side with a smaller length λ^t for some integers $0 \le t < s \le r-1$. Let this third side be A_1A_2 and let B denote the remaining vertex. (That is, $|A_1 - A_2| = \lambda^t < \lambda^s = |A_1 - B| = |A_2 - B|$.)

Now we apply the Johnson-Lindenstrauss lemma for the points in S with some $0 < \delta < 1$; by S' we will denote the set of the points obtained. We consider the points $A_1', A_2', B' \in S'$ corresponding to the points A_1, A_2, B. Using the fact that $(1+\delta)^2 < 1+3\delta$ we get that

$$\lambda^{2t} \le |A_1' - A_2'|^2 < (1+3\delta)\lambda^{2t}; \quad \lambda^{2s} \le |A_i' - B'|^2 < (1+3\delta)\lambda^{2s} \quad (i = 1, 2).$$

By the cosine law we have

$$\cos\left(\angle A_1'A_2'B'\right) = \frac{|A_1' - A_2'|^2 + |A_2' - B'|^2 - |A_1' - B'|^2}{2|A_1' - A_2'||A_2' - B'|} <$$
$$\frac{(1+3\delta)\left(\lambda^{2s} + \lambda^{2t}\right) - \lambda^{2s}}{2\lambda^s\lambda^t} = \frac{1}{2\lambda^{s-t}} + 3\delta\frac{\lambda^{2s} + \lambda^{2t}}{2\lambda^s\lambda^t} \le \frac{1}{2\lambda} + 3\delta\frac{\lambda^r + 1}{2}.$$

Set $\lambda = 1 + \frac{c\varepsilon}{r}$ and $\delta = \frac{c\varepsilon}{36r}$ with a sufficiently small constant c. Then

$$\lambda^r = \left(1 + \frac{c\varepsilon}{r}\right)^r < \exp(c\varepsilon) < 1 + 2c\varepsilon < 2.$$

Thus

$$\cos\left(\angle A_1'A_2'B'\right) < \frac{1}{2\lambda} + 3\delta\frac{\lambda^r + 1}{2} < \left(\frac{1}{2} - \frac{\lambda - 1}{2\lambda}\right) + \frac{9}{2}\delta < \frac{1}{2} - \frac{c\varepsilon}{4r} + \frac{c\varepsilon}{8r} = \frac{1}{2} - \frac{c\varepsilon}{8r}.$$

Since cos is a Lipschitz function with Lipschitz constant 1, it follows that $\angle A_1'A_2'B' > \pi/3 + c\varepsilon/8r$. The same holds for the angle $\angle A_2'A_1'B'$. Therefore for the third angle in the triangle we get $\angle A_1'B'A_2' < \pi/3 - c\varepsilon/4r$.

On the other hand, the distance of any two points in S' is at least 1 and at most $(1+\delta)\lambda^{r-1} < \lambda^r < 1+2c\varepsilon$. Now let us take four distinct points A, B, C, D in S'. As we have seen in the proof of Lemma 2.3, $|\angle(ABC) - \pi/3| < \varepsilon$ and $|\angle(A - B, C - D) - \pi/2| < \varepsilon$ provided that c is sufficiently small.

Finally, by the Johnson-Lindenstrauss lemma the set S' is contained in a Euclidean space of dimension at most $\lceil C\log(2^r)/\delta^2\rceil = \lceil Cr^3/\varepsilon^2\rceil$. $\square$

This discrete set of points can be blown up (using the self-similar construction described in Section 2.1) to a large dimensional set that does not contain the angles $\pi/3$ and $2\pi/3$.

Theorem 2.9. *There exist absolute constants $c, C > 0$ such that for any $0 < \delta < \varepsilon < 1$ with $\varepsilon/\delta > C$ there exists a self-similar set of dimension*

$$s \geq \frac{c\varepsilon/\delta}{\log(1/\delta)}$$

in a Euclidean space of dimension

$$n \leq \frac{C\varepsilon}{\delta^3}$$

such that any angle determined by three points of the set is inside the ε-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$ but outside the δ-neighbourhood of $\{\pi/3, 2\pi/3\}$.

Proof. Set $r = [c\varepsilon/\delta]$. The previous theorem claims that for

$$n = \lceil Cr^3/\varepsilon^2 \rceil \leq C\varepsilon/\delta^3; \quad m = 2^r,$$

there exist m points $P_1, \ldots, P_m \in \mathbb{R}^n$ such that for any three distinct points P_i, P_j, P_k

$$2\delta < \left| \angle P_i P_j P_k - \frac{\pi}{3} \right| < \frac{\varepsilon}{2},$$

and for any four different points P_i, P_j, P_k, P_l

$$\left| \angle(P_i - P_j, P_k - P_l) - \frac{\pi}{2} \right| < \frac{\varepsilon}{2}.$$

Now we take the self-similar set of Section 2.1 with $q_i = q = c\delta$. The set obtained has dimension

$$\frac{\log m}{\log(1/(c\delta))} \geq \frac{cr}{\log(1/\delta)} \geq \frac{c\varepsilon/\delta}{\log(1/\delta)}.$$

Moreover, Corollary 2.2 implies that all the angles occurring in this set are inside the ε-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$ but outside the δ-neighbourhood of $\{\pi/3, 2\pi/3\}$. $\square$

By fixing a small ε and setting $\delta = c/\sqrt[3]{n}$ in the above theorem, we obtain the following corollaries.

Corollary 2.10. *A self-similar set $K \subset \mathbb{R}^n$ can be given such that the dimension of K is at least*

$$s \geq \frac{c\sqrt[3]{n}}{\log n},$$

and K does not contain the angle $\pi/3$ and $2\pi/3$ (moreover, K does not contain any angle in the $c/\sqrt[3]{n}$-neighbourhood of $\pi/3$ and $2\pi/3$).

So there exists a compact set in $\mathbb{R}^n$ of dimension at least $\frac{c\sqrt[3]{n}}{\log n}$ that avoids a small neighbourhood of the angles $\pi/3$ and $2\pi/3$. Probably, this result is quite far from being sharp. However, as we will see in the next section, the following corollary is surprisingly sharp.

Corollary 2.11. *For any $0 < \delta < 1$ there exists a self-similar set K of dimension at least $c\delta^{-1}/\log(\delta^{-1})$ in some Euclidean space such that K does not contain any angle in $(\pi/3 - \delta, \pi/3 + \delta) \cup (2\pi/3 - \delta, 2\pi/3 + \delta)$.*

Finally, we mention that the constructions of this and the previous section have the additional property that the constructed self-similar sets K avoid α even in the sense that for any $A, B, C, D \in K$ with $A \neq B$ and $C \neq D$ we have $\angle(A - B, C - D) \neq \alpha$ (see the proof of Corollary 2.2).

2.3 Finding angles close to a given angle

We start this section by proving that a set that does not contain angles near to $\pi/2$ must be very small, it cannot have Hausdorff dimension bigger than 1. This makes $\pi/2$ very special since, as we have seen, the analogous statement would be false for any other angle $\alpha \in (0, \pi)$.

Theorem 2.12. *Any analytic (compact) set A in $\mathbb{R}^n$ $(n \geq 2)$ with Hausdorff dimension greater than 1 contains angles arbitrarily close to the right angle.*

Proof. It is a well-known fact that any analytic set A with positive $\mathcal{H}^s$ measure contains a compact s-set (see e.g. [16, 2.10.47-48]). Consequently, we can assume that $0 < \mathcal{H}^s(A) < \infty$ for some $s > 1$. Then for $\mathcal{H}^s$ almost all $x \in A$ it holds that for almost all $(n-1)$-dimensional hyperplane through x intersect A in a set of dimension $s-1$ (see Theorem 2.30 of the next section). Let us fix a point x with this property and let $y \neq x$ be an arbitrary point in A. Since the set of hyperplanes forming an angle at least $\pi/2 - \delta$ with the vector $y - x$ has positive measure for any $\delta > 0$, while the set of exceptional hyperplanes has measure zero, the theorem follows. $\square$

Now we prove the same result for upper Minkowski dimension instead of Hausdorff dimension. It is well-known that the upper Minkowski dimension is always greater or equal than the Hausdorff dimension. Hence the following theorem is stronger than the previous one.

Theorem 2.13. *Any set A in $\mathbb{R}^n$ $(n \geq 2)$ with upper Minkowski dimension greater than 1 contains angles arbitrarily close to the right angle.*

The upper Minkowski dimension can be defined in many different ways, we will use the following definition (see [31, Section 5.3] for details).

Definition 2.14. By $B(x,r)$ we denote the closed ball with center $x \in \mathbb{R}^n$ and radius r. For a non-empty bounded set $A \subset \mathbb{R}^n$ let $P(A,\varepsilon)$ denote the greatest integer k for which there exist disjoint balls $B(x_i,\varepsilon)$ with $x_i \in A$, $i = 1,\dots,k$. The *upper Minkowski dimension* of A is defined as

$$\overline{\dim}_{\mathrm{M}}(A) \stackrel{\text{def}}{=} \sup\{s : \limsup_{\varepsilon \to 0+} P(A,\varepsilon)\varepsilon^s = \infty\}.$$

Note that we get an equivalent definition if we consider the lim sup for ε's only in the form $\varepsilon = 2^{-k}$, $k \in \mathbb{N}$.

The next lemma is mainly technical. It roughly says that in a set of large upper Minkowski dimension one can find many points such that the distance of each pair is more or less the same.

Lemma 2.15. *Suppose that $\overline{\dim}_{\mathrm{M}}(A) > t$ for a set $A \subset \mathbb{R}^n$ and a positive real t. Then for infinitely many positive integers k it holds that for any integer $0 < l < k$ there are more than $2^{(k-l)t}$ points in A with the property that the distance of any two of them is between 2^{-k+1} and 2^{-l+2}.*

Proof. Let

$$r_k = P(A, 2^{-k})2^{-kt}.$$

Due to the previous definition $\limsup_{k\to\infty} r_k = \infty$. It follows that there are infinitely many values of k such that $r_k > r_l$ for all $l < k$. Let us fix such a k and let $0 < l < k$ be arbitrary.

By the definition of r_k, there are $r_k 2^{kt}$ disjoint balls with radii 2^{-k} and centers in A. Let $\mathcal{S}$ denote the set of the centers of these balls. Clearly the distance of any two of them is at least 2^{-k+1}.

Similarly, we can find a maximal system of disjoint balls $B(x_i, 2^{-l})$ with $x_i \in A$, $i = 1,\dots,r_l 2^{lt}$. Consider the balls $B(x_i, 2^{-l+1})$ of doubled radii. These doubled balls are covering the whole A (otherwise the original system would not be maximal). By the pigeonhole principle, one of these doubled balls contains at least

$$\frac{r_k 2^{kt}}{r_l 2^{lt}} = \frac{r_k}{r_l} 2^{(k-l)t} > 2^{(k-l)t}$$

points of $\mathcal{S}$. These points clearly have the desired property. □

Now we are ready to prove the theorem.

Proof of Theorem 2.13. We can assume that $\operatorname{diam}(A) > 2$. Fix a t such that $\overline{\dim}_{\mathrm{M}}(A) > t > 1$. Lemma 2.15 tells us that there are arbitrarily large integers k such that for any

$l < k$ one can have more than $2^{(k-l)t}$ points in A such that each distance is between 2^{-k+1} and 2^{-l+2}. Let $\mathcal{S}$ be a set of such points and pick an arbitrary point $O \in \mathcal{S}$. Since $\operatorname{diam}(A) > 2$, there exists a point $P \in A$ with $OP \geq 1$. Now we project the points of $\mathcal{S}$ to the line OP. There must be two distinct points $Q_1, Q_2 \in \mathcal{S}$ such that the distance of their projection is at most

$$\frac{2^{-l+2}}{2^{(k-l)t}} = 2^{-l+2-(k-l)t},$$

It follows that

$$\cos\angle(Q_2 - Q_1, P - O) \leq \frac{2^{-l+2-(k-l)t}}{2^{-k+1}} = 2^{-(k-l)(t-1)+1}.$$

Since $Q_1O \leq 2^{-l+2}$ and $OP \geq 1$, the angle of the lines OP and Q_1P is at most $C_1 2^{-l}$ with some constant C_1. Combining the previous results we get that

$$|\angle PQ_1Q_2 - \pi/2| \leq C_1 2^{-l} + C_2 2^{-(k-l)(t-1)}$$

with some constants C_1, C_2. The right hand side can be arbitrarily small since $t - 1$ is positive and both l and $k - l$ can be chosen to be large. □

Now we try to find angles close to $\pi/3$. We will do that by finding three points forming an *almost regular* triangle provided that the dimension of the set is sufficiently large.

We will need a simple result from Ramsey theory. Let $R_r(3)$ denote the least positive integer k for which it holds that no matter how we colour the edges of a complete graph on k vertices with r colours it contains a monochromatic triangle. The next inequality can be obtained easily:

$$R_r(3) \leq r \cdot R_{r-1}(3) - (r-2).$$

(A more general form of the above inequality can be found in e.g. [17, p. 90, Eq. 2].) It readily implies the following upper bound for $R_r(3)$.

Lemma 2.16. *For any positive integer $r \geq 2$*

$$R_r(3) \leq 3r!,$$

that is, any complete graph on at least $3r!$ vertices edge-coloured by r colours contains a monochromatic triangle.

Using this lemma we can prove the following theorem.

Theorem 2.17. *There exists an absolute constant C such that whenever $\overline{\dim}_{\mathrm{M}}(A) > C\delta^{-1}\log(\delta^{-1})$ for some set $A \subset \mathbb{R}^n$ and $\delta > 0$ the following holds: A contains three points that form a δ-almost regular triangle, that is, the ratio of the longest and shortest side is at most $1 + \delta$.*

As an immediate consequence, we can find angles close to $\pi/3$.

Corollary 2.18. *Suppose that* $\overline{\dim}_{\mathrm{M}}(A) > C\delta^{-1}\log(\delta^{-1})$ *for some set* $A \subset \mathbb{R}^n$ *and* $\delta > 0$. *Then* A *contains angles from the interval* $(\pi/3-\delta, \pi/3]$ *and also from* $[\pi/3, \pi/3+\delta)$.

Remark 2.19. The above theorem and even the corollary is essentially sharp: in the previous section we constructed a set with Hausdorff dimension $c\delta^{-1}/\log(\delta^{-1})$ and without any angles from the interval $(\pi/3-\delta, \pi/3+\delta)$.

Proof of Theorem 2.17. Let $t = C\delta^{-1}\log(\delta^{-1})$ and apply Lemma 2.15 for $l = k-1$. We obtain at least 2^t points in A such that each distance is in the interval $[2^{-k+1}, 2^{-k+3}]$. Let $a = 2^{-k+1}$ and divide $[a, 4a]$ into $N = \lceil \frac{3}{\delta} \rceil$ disjoint intervals of length at most δa. Regard the points of A as the vertices of a graph. Colour the edges of this graph with N colours according to which interval contains the distance of the corresponding points.

Easy computation shows that $2^t > 3N!$ (with a suitable choice of C). Therefore the above graph contains a monochromatic triangle by Lemma 2.16. It easily follows that the three corresponding points form a δ-almost regular triangle in $\mathbb{R}^n$. □

Remark 2.20. The same proof yields the following: for any positive integer d and positive real δ there is a number $K(d, \delta)$ such that whenever $\overline{\dim}_{\mathrm{M}}(A) > K(d, \delta)$ for some set A, one can find d points in A with the property that the ratio of the largest and the smallest distance among these points is at most $1+\delta$. (One needs to use the fact that the Ramsey number $R_r(d)$ is finite.)

In order to derive similar results for $2\pi/3$ instead of $\pi/3$ we show that if large Hausdorff dimension implies the existence of an angle near α, then it also implies the existence of an angle near $\pi - \alpha$.

Proposition 2.21. *Suppose that* $s = s(\alpha, \delta, n)$ *is a positive real number such that any analytic set* $A \subset \mathbb{R}^n$ *with* $\mathcal{H}^s(A) > 0$ *contains an angle from the interval* $(\alpha-\delta, \alpha+\delta)$. *Then any analytic set* $B \subset \mathbb{R}^n$ *with* $\mathcal{H}^s(B) > 0$ *contains an angle from the interval* $(\pi-\alpha-\delta', \pi-\alpha+\delta')$ *for any* $\delta' > \delta$.

Proof. Again, we can assume that $0 < \mathcal{H}^s(B) < \infty$. It is well-known that for $\mathcal{H}^s$ almost all $x \in B$ the set $B \cap B(x, r)$ has positive $\mathcal{H}^s$ measure for any $r > 0$ [31, Theorem 6.2]. If we omit the exceptional points from B, this will be true for every point of the obtained set. Assume that B had this property in the first place. Then, by the assumptions of the proposition, any ball around any point of B contains an angle from the δ-neighbourhood of α.

We define the points $P_m, Q_m, R_m \in B$ recursively in the following way. Fix a small ε. First take P_0, Q_0, R_0 such that the angle $\angle P_0Q_0R_0$ falls into the interval $(\alpha-\delta, \alpha+$

δ). If the points P_m, Q_m, R_m are given, then choose points $P_{m+1}, Q_{m+1}, R_{m+1}$ from the $\varepsilon \cdot \min(Q_m P_m, Q_m R_m)$-neighbourhood of P_m such that $\angle P_{m+1} Q_{m+1} R_{m+1} \in (\alpha - \delta, \alpha + \delta)$.

We can find two indices $k > l$ such that the angle enclosed by the vectors $\overrightarrow{Q_l P_l}$ and $\overrightarrow{Q_k P_k}$ is less than ε. It is clear that if we choose ε sufficiently small, then $\angle(Q_l, Q_k, R_k) \in (\pi - \alpha - \delta', \pi - \alpha + \delta')$. □

Remark 2.22. Proposition 2.21 holds for $\delta' = \delta$ as well. Surprisingly, it even holds for some $\delta' < \delta$. The reason behind is the following. If every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $(\alpha - \delta, \alpha + \delta)$, then there necessarily exists a closed subinterval $[\alpha - \gamma, \alpha + \gamma]$ ($\gamma < \delta$) such that every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $[\alpha - \gamma, \alpha + \gamma]$. We prove this statement at the end of this section (Theorem 2.25).

Theorem 2.23. *There exists an absolute constant C such that any analytic set $A \subset \mathbb{R}^n$ with $\dim(A) > C\delta^{-1} \log(\delta^{-1})$ contains an angle from the δ-neighbourhood of $2\pi/3$.*

Proof. The claim readily follows from Corollary 2.18, Proposition 2.21 and the fact that the upper Minkowski dimension is greater or equal than the Hausdorff dimension. □

To find angles arbitrarily close to 0 and π, it suffices to have infinitely many points.

Proposition 2.24. *Any $A \subset \mathbb{R}^n$ of infinite cardinality contains angles arbitrarily close to 0 and angles arbitrarily close to π.*

Sketch of the proof. We claim that given N points in $\mathbb{R}^n$ they must contain an angle less than $\delta_1 = \frac{C}{\sqrt[n-1]{N}}$ and an angle greater than $\pi - \delta_2$ with $\delta_2 = \frac{C}{\sqrt[n-1]{\log N}}$. The former follows easily from the pigeonhole principle. The latter is a result of Erdős and Füredi [12, Theorem 4.3]. □

In this section we have seen results saying that large dimensional sets contain angles close to a given angle $\alpha \in \{0, \pi/3, \pi/2, 2\pi/3, \pi\}$. Note that in these results the dimension of the Euclidean space (n) did not play any role. To sum up the results we introduce the following function $\widetilde{C}$ depending on an angle $\alpha \in [0, \pi]$ and a small positive δ.

$$\begin{aligned}\widetilde{C}(\alpha, \delta) \stackrel{\text{def}}{=} \sup\{\dim(A) : A \subset \mathbb{R}^n \text{ for } \textbf{some } n; A \text{ is analytic};\\ A \text{ does not contain any angle from } (\alpha - \delta, \alpha + \delta)\}.\end{aligned}$$

Remark 2.22 implies that $\widetilde{C}$ satisfies the symmetry property

$$\widetilde{C}(\alpha, \delta) = \widetilde{C}(\pi - \alpha, \delta).$$

In Section 2.1 for any positive ε we constructed sets of arbitrarily large dimension such that all the angles fall into the ε-neighbourhood of the special angles 0, $\pi/3$, $\pi/2$, $2\pi/3$, π (Theorem 2.4). So for any angle α other than the special angles $\widetilde{C}(\alpha,\delta)=\infty$ if δ is smaller than the distance of α from the special angles. Therefore this construction and the results of this section give essentially all the values of $\widetilde{C}(\alpha,\delta)$, see the table below.

Table 2.1: Smallest dimensions that guarantee angle in the δ-neighbourhood of α

α	$\widetilde{C}(\alpha,\delta)$	
$0,\pi$	$=0$	
$\pi/2$	$=1$	
$\pi/3, 2\pi/3$	$\approx 1/\delta$	apart from a multiplicative error $C\cdot\log(1/\delta)$
other angles	$=\infty$	provided that δ is sufficiently small

Finally, we prove the following theorem, which was claimed in Remark 2.22.

Theorem 2.25. *Suppose that $s=s(\alpha,\delta,n)$ is a positive real number such that every analytic set $A\subset\mathbb{R}^n$ with $\mathcal{H}^s(A)>0$ contains an angle from the interval $(\alpha-\delta,\alpha+\delta)$. Then there exists a closed subinterval $[\alpha-\gamma,\alpha+\gamma]$ ($\gamma<\delta$) such that every analytic set $A\subset\mathbb{R}^n$ with $\mathcal{H}^s(A)>0$ contains an angle from the interval $[\alpha-\gamma,\alpha+\gamma]$.*

To prove this theorem, we need two lemmas. For $r\in(0,\infty]$ let

$$\mathcal{H}^s_r(A)=\inf\left\{\sum_{i=1}^{\infty}\operatorname{diam}(U_i)^s \,:\, \operatorname{diam}(U_i)\le r,\ A\subset\cup_{i=1}^{\infty}U_i\right\},$$

thus $\mathcal{H}^s(A)=\lim_{r\to0+}\mathcal{H}^s_r(A)$.

Lemma 2.26. *Let A_i be a sequence of compact sets converging in the Hausdorff metric to a set A. Then the following two statements hold.*

(i) $\mathcal{H}^s_\infty(A)\ge\limsup_{i\to\infty}\mathcal{H}^s_\infty(A_i)$.

(ii) Suppose that for every $i=1,2,\dots$ the set A_i does not contain an angle from $[\alpha-\delta+\varepsilon_i,\ \alpha+\delta-\varepsilon_i]$, where $\varepsilon_i\to0+$. Then A does not contain an angle from $(\alpha-\delta,\ \alpha+\delta)$.

Proof. The first statement is well-known and easy. To prove the second, notice that for any three points x,y,z of A there exist three points in A_i arbitrarily close to x,y,z, for sufficiently large i. □

The next lemma follows easily from [16, Theorem 2.10.17 (3)]. For the sake of completeness, we give a short direct proof.

Lemma 2.27. *Let $A \subset \mathbb{R}^n$ be a compact set satisfying $\mathcal{H}^s(A) > 0$. Then there exists a ball B such that $\mathcal{H}^s_\infty(A \cap B) \geq c \cdot diam(B)^s$, where $c > 0$ depends only on s.*

Proof. We may suppose without loss of generality that $\mathcal{H}^s(A) < \infty$. (Otherwise we choose a compact subset of A with positive and finite $\mathcal{H}^s$ measure. If the theorem holds for a subset of A, then it clearly holds for A as well.)

Choose $r > 0$ so that $\mathcal{H}^s_r(A) > \mathcal{H}^s(A)/2$. Cover A by sets U_i of diameter at most $r/2$ such that $\sum_i \operatorname{diam}(U_i)^s \leq 2\mathcal{H}^s(A)$. Cover each U_i by a ball B_i of radius at most the diameter of U_i. Then the balls B_i cover A, have diameter at most r, and $\sum_i \operatorname{diam}(B_i)^s \leq 2^{1+s}\mathcal{H}^s(A)$.

We claim that one of these balls B_i satisfies the conditions of the Lemma for $c = 2^{-2-s}$. Otherwise we have

$$\mathcal{H}^s_\infty(A \cap B_i) < 2^{-2-s} \operatorname{diam}(B_i)^s$$

for every i. Since the sets $A \cap B_i$ have diameter at most r, clearly $\mathcal{H}^s_r(A \cap B_i) = \mathcal{H}^s_\infty(A \cap B_i)$. Therefore

$$\mathcal{H}^s_r(A) \leq \sum_i \mathcal{H}^s_r(A \cap B_i) < \sum_i 2^{-2-s} \operatorname{diam}(B_i)^s \leq 2^{-2-s}2^{1+s}\mathcal{H}^s(A) = \mathcal{H}^s(A)/2,$$

which contradicts the choice of r. □

Proof of Theorem 2.25. Suppose on the contrary that there exist analytic sets $K_i \subset \mathbb{R}^n$ with $\mathcal{H}^s(K_i) > 0$ such that K_i does not contain an angle from $[\alpha - \delta + 1/i,\ \alpha + \delta - 1/i]$. We can clearly assume that the sets K_i are compact. Choose a ball B_i for each compact set K_i according to Lemma 2.27. Let B be a ball of diameter 1. Let K_i' be the image of $K_i \cap B_i$ under a similarity transformation which maps B_i to the ball B. Thus $\mathcal{H}^s_\infty(K_i') \geq c$. Let K denote the limit of a convergent subsequence of the sets K_i. We can apply Lemma 2.26 to this subsequence and obtain $\mathcal{H}^s_\infty(K) \geq c$, implying $\mathcal{H}^s(K) > 0$. Also, K does not contain any angle from the interval $(\alpha - \delta,\ \alpha + \delta)$, which is a contradiction. □

2.4 Finding a given angle

In this section we give upper bounds for $C(n, \alpha)$ which is defined as follows.

Definition 2.28. If $n \geq 2$ is an integer and $\alpha \in [0, \pi]$, then let

$$C(n, \alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ compact such that } \operatorname{dim}(A) = s \text{ and } A \text{ does not contain the angle } \alpha\}.$$

As we already mentioned, any analytic set A with positive $\mathcal{H}^s$ measure contains a compact s-set. Consequently, whenever we want to prove that every compact set of dimension greater than s contains the angle α, then instead of compactness it is enough to assume that the set is analytic (or Borel) and on the other hand, we can always suppose that the given compact or analytic set is a compact t-set for some $t > s$. Thus $C(n, \alpha)$ can be also expressed as

$$C(n,\alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ analytic such that } \dim(A) = s \text{ and } A \text{ does not contain the angle } \alpha\},$$

or

$$C(n,\alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ compact such that } 0 < \mathcal{H}^s(A) < \infty \text{ and } A \text{ does not contain the angle } \alpha\}.$$

However, as we prove it in Section 2.6, some assumption about the set is necessary, otherwise the above function would be n for any α. In fact, for any given n and α we construct by transfinite recursion a set in $\mathbb{R}^n$ with positive Lebesgue outer measure that does not contain the angle α.

The following theorem, which is the first statement of [31, Theorem 10.11], plays essential role in some of our proofs.

Notation 2.29. The set of k-dimensional subspaces of $\mathbb{R}^n$ will be denoted by $G(n, k)$ and the natural probability measure on it by $\gamma_{n,k}$ (see e.g. [31] for more details).

Theorem 2.30. *If $m < s < n$ and A is an $\mathcal{H}^s$ measurable subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$, then*

$$\dim\big(A \cap (W + x)\big) = s - m$$

for $\mathcal{H}^s \times \gamma_{n,n-m}$ almost all $(x, W) \in A \times G(n, n-m)$.

In two dimensions it says that for $\mathcal{H}^s$ almost all $x \in A$, almost all lines through x intersect A in a set of dimension $s - 1$. One would expect that this theorem also holds for half-lines instead of lines. Indeed, Marstrand proved it in [28, Lemma 17]. Although the lemma only says that it holds for lines, he actually proves it for half-lines. Therefore the following theorem is also true.

Theorem 2.31. *Let $1 < s < 2$ and let $A \subset \mathbb{R}^2$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$. For any $x \in \mathbb{R}^2$ and $\vartheta \in [0, 2\pi)$ let $L_{x,\vartheta} = \{x + te^{i\vartheta} : t \geq 0\}$. Then*

$$\dim\big(A \cap L_{x,\vartheta}\big) = s - 1$$

for $\mathcal{H}^s \times \lambda$ almost all $(x, \vartheta) \in A \times [0, 2\pi)$.

In this section we give estimates to $C(n,\alpha)$. For $n = 2$ we get the following exact result.

Theorem 2.32. *For any $\alpha \in [0,\pi]$ we have $C(2,\alpha) = 1$.*

Proof. A line has dimension 1 and it contains only the angles 0 and π. A circle also has dimension 1, but does not contain the angles 0 and π. Therefore $C(2,\alpha) \geq 1$ for all $\alpha \in [0,\pi]$.

For the other direction let $\alpha \in [0,\pi]$ and $s > 1$ fixed. We have to prove that any compact s-set contains the angle α. By Theorem 2.31, there exists $x \in K$ such that $\dim(K \cap L_{x,\vartheta}) = s-1$ for almost all $\vartheta \in [0,2\pi)$, where $L_{x,\vartheta} = \{x + te^{i\vartheta} : t \geq 0\}$. Hence we can take $\vartheta_1, \vartheta_2 \in [0,2\pi)$ such that $|\vartheta_1 - \vartheta_2| = \alpha$, and $\dim(K \cap L_{x,\vartheta_i}) = s-1$ for $i = 1,2$. If $x_i \in L_{x,\vartheta_i} \setminus \{x\}$, then the angle between the vectors $x_1 - x$ and $x_2 - x$ is α, so indeed, K contains the angle α. $\square$

An analogous theorem holds for higher dimensions.

Theorem 2.33. *If $n \geq 2$ and $\alpha \in [0,\pi]$, then $C(n,\alpha) \leq n-1$.*

Proof. We have already seen the case $n = 2$, so we may assume that $n \geq 3$. It is enough to show that if $s > n-1$ and K is a compact s-set, then K contains the angle α. By Theorem 2.30, there exists $x \in K$ such that there exists a $W \in G(n,2)$ with $\dim(B) = s-n+2 > 1$ for $B \stackrel{\text{def}}{=} A \cap (W + a)$. The set B lies in a two-dimensional plane, so we can think of B as a subset of $\mathbb{R}^2$. Applying Theorem 2.32 completes the proof. $\square$

Now we are able to give the exact value of $C(n,0)$ and $C(n,\pi)$.

Theorem 2.34. *$C(n,0) = C(n,\pi) = n-1$ for all $n \geq 2$.*

Proof. One of the inequalities was proven in the previous theorem, while the other one is shown by the $(n-1)$-dimensional sphere. $\square$

We prove a better upper bound for $C(n,\pi/2)$.

Theorem 2.35. *If n is even, then $C(n,\pi/2) \leq n/2$. If n is odd, then $C(n,\pi/2) \leq (n+1)/2$.*

Proof. First suppose that n is even. Let $s > n/2$ and let K be a compact s-set. From Theorem 2.30 we know that there exists a point $x \in K$ such that

$$\dim\big(K \cap (x + W)\big) = s - n/2 > 0 \tag{2.1}$$

for $\gamma_{n,n/2}$ almost all $W \in G(n,n/2)$. There exists a $W \in G(n,n/2)$ such that (2.1) holds both for W and $W^\perp$. As $(x+W) \cap (x+W^\perp) = \{x\}$, by choosing a $y \in K \cap (x+W)$ and

$z \in K \cap (x + W^{\perp})$ such that $x \neq y$ and $x \neq z$, we find a right angle at x in the triangle xyz.

Now suppose that n is odd, $s > (n+1)/2$ and K is a compact s-set. With a similar argument we can conclude that $\exists x \in K$ and $W \in G(n, (n+1)/2)$ such that $\dim\big(K \cap (x + W)\big) = s - (n+1)/2 > 0$ and $\dim\big(K \cap (x + W^{\perp})\big) = s - (n-1)/2 > 1$. If $y \in K \cap (x + W) \setminus \{x\}$ and $z \in K \cap (x + W^{\perp}) \setminus \{x\}$, then there is again a right angle at x in the triangle xyz. $\square$

Remark 2.36. By the following result of András Máthé the above estimate is sharp if n is even: for any n there exists a compact set of Hausdorff dimension $n/2$ in $\mathbb{R}^n$ that does not contain $\pi/2$. Therefore if n is even, we have $C(n, \pi/2) = n/2$. We outline this construction in the next section.

Finally, we prove that if we have a homothetic self-similar set K with $\dim(K) > 1$ and the strong separation condition is satisfied, then K must contain the vertices of a rectangle, in particular, it contains the angle $\pi/2$. It means that it is impossible to avoid $\pi/2$ with constructions like the ones presented in Section 2.1 and 2.2.

Theorem 2.37. *Let $K \subset \mathbb{R}^n$ be a homothetic self-similar set, that is, K is compact and there exist homotheties $\varphi_1, \dots, \varphi_m$ with ratios less than 1 such that $K = \varphi_1(K) \cup \varphi_2(K) \cup \dots \cup \varphi_m(K)$. Suppose that the sets $\varphi_i(K)$ are pairwise disjoint (that is, the strong separation condition is satisfied). Then K contains four points that form a non-degenerate rectangle given that* $\dim(K) > 1$.

Proof. We begin the proof by defining the following map:

$$D: \ K \times K \setminus \{(x,x) : x \in K\} \to S^{n-1}; \quad (x,y) \mapsto \frac{x-y}{|x-y|}.$$

We denote the range of D by Range(D). The set Range(D) can be considered as the set of directions in K. First we are going to prove that if K is such a self-similar set, then Range(D) is closed.

As we have seen in the proof of Proposition 2.1, for any $x, y \in K$, $x \neq y$ there exist $x' \in \varphi_i(K)$ and $y' \in \varphi_j(K)$ for some $i \neq j$ such that $x = \psi(x')$ and $y = \psi(y')$ where ψ is the composition of finitely many φ_i's. The important thing for us is that $x - y$ is parallel to $x' - y'$. If $d(\cdot,\cdot)$ denotes the Euclidean distance, then

$$\min_{0 \le i < j \le k} d(\varphi_i(K), \varphi_j(K)) = c > 0,$$

so Range(D) actually equals to the image of D restricted to the set $K \times K \setminus \{(x,y) : d(x,y) < c\}$. As this is a compact set, the continuous image is also compact. That is what we wanted to prove.

Next we show that for any $v \in S^{n-1}$ there exist $x, y \in K$, $x \neq y$ such that the vectors v and $D(x, y)$ are perpendicular. If this was not true, the compactness of $\mathrm{Range}(D)$ would imply that the orthogonal projection p to a line parallel to v would be a one-to-one map on K with p^{-1} being a Lipschitz map on $p(K)$. This would imply $\dim(K) \leq 1$, which is a contradiction.

For simplifying our notation, let $f \stackrel{\text{def}}{=} \varphi_1$, $g \stackrel{\text{def}}{=} \varphi_2$. The homotheties $f \circ g$ and $g \circ f$ have the same ratio. Denote their fixed points by P and Q, respectively. Since $P \neq Q$, there are $x, y \in K$, $x \neq y$ such that $x - y$ is perpendicular to $P - Q$. It is easy to check that the points $f(g(x))$, $f(g(y))$, $g(f(y))$ and $g(f(x))$ form a non-degenerate rectangle. $\square$

2.5 Number theoretic constructions

Although the constructions of this section are due to András Máthé, we include them in this thesis for the sake of completeness.

The starting point is Falconer's famous *distance set problem*. Instead of regarding the angles contained by our set A, we now consider the set of distances occurring in A, that is

$$D(A) \stackrel{\text{def}}{=} \{|x - y| \, : \, x, y \in A\} .$$

Now it does not make much sense to ask whether a particular distance is in $D(A)$ or not. Instead, we are interested in the size of $D(A)$. The next theorem was proved by Falconer.

Theorem 2.38 (Falconer, [13])**.** *If $A \subset \mathbb{R}^n$ is an analytic set with $\dim(A) > n/2 + 1/2$, then the distance set $D(A)$ has positive Lebesgue measure.*

Certain improvements have been done by Bourgain [5], Mattila [30] and Wolff [35]. Recently it was proved by Erdoğan that $n/2 + 1/2$ can be replaced with $n/2 + 1/3$ in the above theorem given that $n \geq 3$ [11]. It is generally believed that it can be replaced even with $n/2$. As we will see, one cannot do better than that.

One can use the above theorem to say something about angles, as well. The following simple observation is due to Máthé. Let $A \subset \mathbb{R}^n$ be analytic with $\dim(A) > n/2 + 3/2$. Let us take an arbitrary point $x \in A$ and project the set A from x onto $S(x, 1)$, the unit sphere centered at x; we denote the image of the projection by $A_x \subset S(x, 1)$. It is easy to see that $\dim(A_x) \geq \dim(A) - 1 > n/2 + 1/2$. Thus Falconer's theorem yields that the distance set of A_x has positive Lebesgue measure. However, if $y, z \in S(x, 1)$, then the angle $\angle yxz$ depends only on the distance of y and z. It follows that the set of angles contained by A has positive Lebesgue measure.

Let us now turn our attention to constructions. First we show how to construct large dimensional sets with distance set of measure zero. The following construction is due to

Falconer [13, Theorem 2.4]. Let N_i be a sufficiently fast growing sequence and let $\kappa > 1$. By E_κ we denote the set of those $x \in [0,1]$ for which $\forall i\ \exists m_i \in \mathbb{Z}$ such that

$$\left| x - \frac{m_i}{N_i} \right| \leq \frac{1}{N_i^\kappa}.$$

Theorem 2.39. *E_κ is a compact set with* $\dim(E_\kappa) = 1/\kappa$.

For a proof see [14, Theorem 8.15]. The key property of this set is that the Minkowski sum $E_\kappa + E_\kappa + \ldots + E_\kappa$ also has Hausdorff dimension $1/\kappa$.

Now let us consider the set

$$A_\kappa = E_\kappa \times E_\kappa \times \cdots \times E_\kappa \subset \mathbb{R}^n.$$

It can be shown that $\dim(A_\kappa) = n/\kappa$. It is not hard to prove that $\lambda(D(A_\kappa)) = 0$ given that $\kappa > 2$. If $\kappa \to 2+$, then we get compact sets in $\mathbb{R}^n$ with Hausdorff dimension arbitrarily close to $n/2$ such that their distance sets are null sets. (With a little more effort, one can construct sets with the same property and of dimension precisely $n/2$.)

As Máthé proved, the set of angles contained by A_κ has Lebesgue measure zero provided that $\kappa > 6$. It immediately follows that for almost all $\alpha \in [0, \pi]$ we have $C(n, \alpha) \geq n/6$.

Moreover, using similar number theoretic techniques, for any given angle $\alpha \in (0, \pi)$ Máthé constructed sets in $\mathbb{R}^n$ of Hausdorff dimension cn that avoid α. Even though the constructed sets contain angles arbitrarily close to α, they succeed to avoid α. (Recall that the constructions presented in Section 2.1 and 2.2 had the property that they avoided not only a certain angle α but also a whole neighbourhood of α.) Here we outline the construction only for the simplest case $\alpha = \pi/2$.

Theorem 2.40 (Máthé, [29])**.** *There exists a compact set $K \subset \mathbb{R}^n$ such that* $\dim(K) = n/2$ *and K does not contain the angle $\pi/2$.*

It follows from Theorem 2.35 that this result is sharp given that n is even.

Sketch of the proof. Let us take the points

$$P_0 = (0, 0, \ldots, 0);\ P_1 = (1, 0, \ldots, 0);\ P_2 = (0, 1, \ldots, 0)$$

in the n-dimensional Euclidean space; $P_0P_1P_2$ is clearly a right-angled triangle. As a first step, we make sure that our set contains no right-angled triangle lying close to $P_0P_1P_2$.

Let B_i be the closed ball with center P_i and radius $1/100$; $i = 0, 1, 2$. Fix some positive integer N and consider the following point lattice in the n-dimensional Euclidean space:

$$\left\{ \left(\frac{m_1}{N}, \frac{m_2}{N}, \ldots, \frac{m_n}{N} \right) \ :\ m_i \in \mathbb{Z} \right\}.$$

The $1/N^{2+\varepsilon}$-neighbourhood of this lattice will be denoted by L. We take the following sets:

$$K_0 = B_0 \cap L;\ K_1 = B_1 \cap L;\ K_2 = B_2 \cap \left(L + \left(\frac{1}{2N^2}, 0, 0, \dots, 0\right)\right).$$

If we take a point Q_i from each K_i, then $\angle Q_1Q_0Q_2 \neq \pi/2$, which can be seen easily by computing the scalar product $\langle Q_1 - Q_0, Q_2 - Q_0\rangle$. Now let us consider the set

$$K_0 \cup K_1 \cup K_2 \cup (\mathbb{R}^n \setminus (B_0 \cup B_1 \cup B_2)). \tag{2.2}$$

As we have just seen, this set has the property that it contains no right-angled triangles lying close to $P_0P_1P_2$.

Note that in the above argument we can replace $P_0P_1P_2$ by any right-angled triangle with all three vertices having rational coordinates. Let us take the set (2.2) for each of these countably many triangles (we might use different N's for different triangles). The intersection of these sets clearly contains no right angle at all. It can be also shown that if we choose the N's carefully, then the intersection will have Hausdorff dimension $n/(2+\varepsilon)$. (It is not hard to modify this proof in such a way that the Hausdorff dimension of the constructed set is precisely $n/2$.) □

Finally, to sum up the results of this and the previous section, we gathered the best known bounds for $C(n, \alpha)$ in the following table.

Table 2.2: Best known bounds for $C(n, \alpha)$

α	lower bound	upper bound
$0, \pi$	$n-1$	$n-1$
$\alpha \in (0, \pi);\ \alpha \neq \pi/2$	cn	$n-1$
$\pi/2$	$n/2$	$\lceil n/2 \rceil$

2.6 A construction using transfinite recursion

Now we show that if we allowed arbitrary sets in Definition 2.28, then $C(n, \alpha)$ would be n.

Theorem 2.41. *Let $n \geq 2$. For any $\alpha \in [0, \pi]$ there exists $H \subset \mathbb{R}^n$ such that H does not contain the angle α, and H has positive Lebesgue outer measure. In particular,* $\dim(H) =$ *n.*

Proof. Take a well-ordering $\{B_\beta : \beta < \mathfrak{c}\}$ of the Borel null-sets of $\mathbb{R}^n$ (with respect to the n-dimensional Lebesgue measure). We will construct a sequence of points $\{x_\beta : \beta < \mathfrak{c}\}$ of $\mathbb{R}^n$ using transfinite recursion, and define H as $\{x_\beta : \beta < \mathfrak{c}\}$.

We introduce the following notation. If $y, z \in \mathbb{R}^n$ and $y \neq z$, then

$$C_{yz} \overset{\text{def}}{=} \{x \in \mathbb{R}^n \setminus \{y,z\} : \text{ the angle between } x-y \text{ and } z-y \text{ is } \alpha\} \cup \{y,z\};$$
$$D_{yz} \overset{\text{def}}{=} \{x \in \mathbb{R}^n \setminus \{y,z\} : \text{ the angle between } y-x \text{ and } z-x \text{ is } \alpha\} \cup \{y,z\}.$$

C_{yz} is a cone with vertex y, while D_{yz} has the property that a 2-dimensional plane containing y and z intersects it in the union of two circular arcs going between y and z. When $\alpha = 0$ or π, both sets are degenerate: C_{yz} becomes a half-line, D_{yz} becomes a segment or the union of two half-lines.

First we show that if v is a vector such that the angle between v and $z-y$ is not α or $\pi - \alpha$, then any line l parallel to v intersects C_{yz} in at most two points. Let $x \in l$ be arbitrary. Then $l = \{x + tv : t \in \mathbb{R}\}$. Suppose that $t_0 \in \mathbb{R}$ such that $x + t_0 v \in C_{yz}$. Then

$$\cos^2 \alpha = \frac{\langle (x+t_0 v) - y, z - y\rangle^2}{|(x+t_0v) - y|^2 |z-y|^2} = \frac{p_1(t_0)}{p_2(t_0)},$$

where $p_1(t)$ and $p_2(t)$ are polynomials of degree 2, with leading coefficients $\langle v, z-y\rangle^2$ and $|v|^2|z-y|^2$, respectively. The number t_0 is a root of

$$p(t) \overset{\text{def}}{=} p_2(t)\cos^2\alpha - p_1(t),$$

which has degree 2, as the coefficient of t^2 is

$$|v|^2|z-y|^2\cos^2\alpha - \langle v, z-y\rangle^2 \neq 0.$$

Hence $p(t)$ has at most two roots which means that l intersects C_{yz} in at most two points.

Similarly, we prove that if D_{yz} is non-degenerate, then any line l intersects it in at most four points. Let $l = \{x + tv : t \in \mathbb{R}\}$ again, and suppose that $x + t_0 v \in D_{yz}$ for some $t_0 \in \mathbb{R}$. Then

$$\cos^2\alpha = \frac{\langle y - (x+t_0v), z - (x+t_0v)\rangle^2}{|y-(x+t_0v)|^2|z-(x+t_0v)|^2} = \frac{p_1(t_0)}{p_2(t_0)},$$

where $p_1(t)$ and $p_2(t)$ now denote polynomials of degree 4. Again, t_0 is a root of the polynomial $p_2(t)\cos^2\alpha - p_1(t)$ which has degree exactly 4 as the leading coefficient of both p_1 and p_2 are $|v|^4$, and $\cos^2\alpha \neq 1$. As it has at most four roots, we are done. When D_{yz} is degenerate, any line that does not go through both y and z intersects D_{yz} in at most one point.

Now we move on to the construction. Suppose that $\beta < \mathfrak{c}$ and we have already defined x_γ for all $\gamma < \beta$. Let $H_\beta = \{x_\gamma : \gamma < \beta\}$.

We want the point x_β to satisfy the following properties:

(i) $x_\beta \notin C_{yz}$ for any $y, z \in H_\beta$ with $y \neq z$;

(ii) $x_\beta \notin D_{yz}$ for any $y, z \in H_\beta$ with $y \neq z$;

(iii) $x_\beta \notin B_\beta$.

If we prove that it is possible to define x_β this way, then we are done, because (i) and (ii) guarantee that the resulting set H will not contain the angle α, while (iii) ensures that H will not be a null set as each null set is contained by a Borel null set.

First we show that there is a direction $v \in S^{n-1}$ such that each line parallel to v intersects the set

$$A_\beta \stackrel{\text{def}}{=} \bigcup_{y,z \in H_\beta, y \neq z} (C_{yz} \cup D_{yz})$$

in less than $\mathfrak{c}$ points. We say that v is *good* for C_{yz} (or D_{yz}) if each line parallel to v intersects C_{yz} (or D_{yz}) in less than $\mathfrak{c}$ points. We have already shown that for each D_{yz} there are at most two $v \in S^{n-1}$ which are not good for D_{yz}. Therefore there are less than $\mathfrak{c}$ directions that are not good for some D_{yz}.

For a fixed y and z the set of directions which are not good for C_{yz} is

$$\{v \in S^{n-1} : \langle v, z-y \rangle = \pm |z-y| \cos\alpha\} = S^{n-1} \cap (\Sigma_{y-z} \cup \Sigma_{z-y}),$$

where Σ_w denotes the hyperplane $\{x \in \mathbb{R}^n : \langle x, w \rangle = |w| \cos\alpha\}$ for $w \in \mathbb{R}^n \setminus \{0\}$. First, suppose that $\alpha \neq \pi/2$, whence $0 \notin \Sigma_w$. Take arbitrary v_1 and v_2 with $v_2 \neq \pm v_1$, and denote the two-dimensional plane $\{sv_1 + tv_2 : s, t \in \mathbb{R}\}$ by F. The set $C \stackrel{\text{def}}{=} S^{n-1} \cap F$ is an ordinary circle. It is clear that the set $S^{n-1} \cap \Sigma_w \cap C = S^{n-1} \cap (\Sigma_w \cap F)$ has at most two elements for all $w \in \mathbb{R}^n \setminus \{0\}$, because $\Sigma_w \cap F$ is an at most one-dimensional affine subspace of $\mathbb{R}^n$ as $0 \notin \Sigma_w$. From this we can conclude that there are less than $\mathfrak{c}$ points on C which are not good for some C_{yz}, hence there is a point on C which is good for every C_{yz} and D_{yz}.

This method does not work if $\alpha = \pi/2$. In this case take a subset V of S^{n-1} such that $\mathrm{card}(V) = \mathfrak{c}$ and no n distinct elements of V are linearly dependent. For example, the set $U = \{(1, t, \ldots, t^{n-1}) : t \in [0,1]\}$ does not contain n distinct points which are linearly dependent (their determinant is a Vandermonde determinant), so we may get a good V by normalizing each $u \in U$ to $u/|u|$. As Σ_w goes through the origin in this case, it can contain at most $n-1$ points of V. It follows that the union of the hyperplanes Σ_{y-z} and Σ_{z-y} can cover only less than $\mathfrak{c}$ points of V $(y, z \in H_\beta, y \neq z)$. Hence there exists a $v \in V$ which is good for every C_{yz} and D_{yz} in this case, too.

Take such a v. The only thing we need to prove in order to finish the proof of the theorem is that $A_\beta \cup B_\beta \neq \mathbb{R}^n$. Taking a Cartesian coordinate system with one axis in the direction of v, and applying Fubini's Theorem for the characteristic function of B_β gives that $\mathcal{H}^1(l_x \cap B_\beta) = 0$ for almost all $x \in \{v\}^\perp$, where l_x denotes the line $\{x + tv : t \in \mathbb{R}\}$. We also have $\operatorname{card}(l_x \cap A_\beta) < \mathfrak{c}$ for all $x \in \{v\}^\perp$, therefore it remains to show that the complement of a null set of $\mathbb{R}$ has cardinality $\mathfrak{c}$. But this is clear, as the complement of a null set contains a compact set with positive measure, which is the union of a non-empty perfect set and a countable set. $\square$

Chapter 3

Acute sets in Euclidean spaces

Around 1950 Erdős conjectured that given more than 2^d points in $\mathbb{R}^d$ there must be three of them determining an obtuse angle. The vertices of the d-dimensional cube show that 2^d points exist such that the angle determined by any three of them is at most $\pi/2$.

In 1962 Danzer and Grünbaum proved this conjecture [10] (their proof can also be found in [2]). They posed the following question in the same paper: what is the maximal number of points in $\mathbb{R}^d$ such that all angles determined are acute (in other words, this time we want to exclude right angles as well as obtuse angles). A set of such points will be called an *acute set* or *acute d-set* in the sequel.

The exclusion of right angles seemed to decrease the maximal number of points dramatically: they could only give $2d-1$ points, and they conjectured that this is the best possible. However, this was only proved for $d = 2, 3$. (For the non-trivial case $d = 3$, see Croft [8], Schütte [33], Grünbaum [18].)

Then in 1983 Erdős and Füredi disproved the conjecture of Danzer and Grünbaum. They used the probabilistic method to show the existence of an acute d-set of cardinality exponential in d. Their idea was to choose random points from the vertex set of the d-dimensional unit cube, that is $\{0,1\}^d$. Actually they even proved the following result: for any fixed $\delta > 0$ there exist exponentially many points in $\mathbb{R}^d$ with the property that the angle determined by any three points is less than $\pi/3 + \delta$. We used this result in the previous chapter to construct large dimensional sets such that each angle contained by the sets is close to one of the angles 0, $\pi/3$, $\pi/2$, $2\pi/3$, π.

We denote the maximal size of acute sets in $\mathbb{R}^d$ and in $\{0,1\}^d$ by $\alpha(d)$ and $\kappa(d)$, respectively; clearly $\alpha(d) \geq \kappa(d)$. Our goal in this chapter is to give good bounds for $\alpha(d)$ and $\kappa(d)$. The random construction of Erdős and Füredi implied the following lower bound for $\kappa(d)$ (thus for $\alpha(d)$ as well)

$$\kappa(d) > \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d > 0.5 \cdot 1.154^d. \tag{3.1}$$

The best known lower bound both for $\alpha(d)$ and for $\kappa(d)$ (for large values of d) is due to Ackerman and Ben-Zwi from 2009 [1]. They improved (3.1) with a multiplicative factor $\sqrt{d}$:

$$\alpha(d) \geq \kappa(d) > c\sqrt{d}\left(\frac{2}{\sqrt{3}}\right)^d. \tag{3.2}$$

In Section 3.1 we modify the random construction of Erdős and Füredi to get

$$\alpha(d) > c\left(\sqrt[10]{\frac{144}{23}}\right)^d > c \cdot 1.2^d. \tag{3.3}$$

A different approach can be found in Section 3.2 where we recursively construct acute sets. These constructions outdo (3.3) up to dimension 250. In Theorem 3.18 we will show that this constructive lower bound is *almost* exponential in the following sense: given any positive integer r, for infinitely many values of d we have an acute d-set of cardinality at least

$$\exp(d/\underbrace{\log\log\cdots\log}_{r}(d)).$$

See Table 3.2 in Section 3.4 for the best known lower bounds of $\alpha(d)$ ($d \leq 84$). These bounds are new results except for $d \leq 3$.

Both the probabilistic and the constructive approach use small dimensional acute sets as building blocks. So it is crucial for us to construct small dimensional acute sets of large

Table 3.1: Results for $\alpha(d)$ ($d \leq 10$)

dim(d)	D,G[10]	*Bevan*[3]	*Our result*
2	$= 3$		
3	$= 5$		
4	≥ 7		≥ 8
5	≥ 9		≥ 12
6	≥ 11		≥ 16
7	≥ 13	≥ 14	≥ 20
8	≥ 15	≥ 16	≥ 23
9	≥ 17	≥ 19	≥ 27
10	≥ 19	≥ 23	≥ 31

cardinality. In Section 3.3 we present an acute set of 8 points in $\mathbb{R}^4$ and an acute set of 12 points in $\mathbb{R}^5$ (disproving the conjecture of Danzer and Grünbaum for $d \geq 4$ already). We used computer to find acute sets in dimension $6 \leq d \leq 10$, for details see Section 3.3. Table 3.1 shows our results compared to the construction of Danzer and Grünbaum $(2d-1)$ and the examples found by Bevan using computer.

As far as $\kappa(d)$ is concerned, in large dimension (3.2) is still the best known lower bound. Bevan used computer to determine the exact values of $\kappa(d)$ for $d \leq 9$ [3]. He also gave a recursive construction improving upon the random constructions in low dimension. The constructive approach of Section 3.2 yields a lower bound not only for $\alpha(d)$ but also for $\kappa(d)$, which further improves the bounds of Bevan in low dimension. Table 3.3 in Section 3.4 shows the best known lower bounds for $\kappa(d)$ ($d \leq 82$). These bounds are new results except for $d \leq 12$ and $d = 27$.

The following notion plays an important role in both approaches.

Definition 3.1. A triple A, B, C of three points in $\mathbb{R}^d$ will be called *bad* if for each integer $1 \leq i \leq d$ the i-th coordinate of B equals the i-th coordinate of A or C.

We denote by $\kappa_n(d)$ the maximal size of a set $S \subset \{0, 1, \ldots, n-1\}^d$ that contains no *bad triples*. It is easy to see that $\kappa_2(d) = \kappa(d)$ but our main motivation to investigate $\kappa_n(d)$ is that we can use sets without bad triples to construct acute sets recursively (see Lemma 3.2). We give an upper bound for $\kappa_n(d)$ (Theorem 3.8) and two different lower bounds (Theorem 3.3 and 3.12). In the special case $n = 2$ the upper bound yields that

$$\kappa(d) \leq 3(\sqrt{2})^{d-1},$$

which improves the bound $\sqrt{2}\left(\sqrt{3}\right)^d$ given by Erdős and Füredi in [12]. Note that for $\alpha(d)$ the best known upper bound is $2^d - 1$.

Although we can make no contribution to it, we mention that there is an affine variant of this problem. A finite set $\mathcal{H}$ in $\mathbb{R}^d$ is called *strictly antipodal* if for any two distinct points $P, Q \in \mathcal{H}$ there exist two parallel hyperplanes, one through P and the other through Q, such that all other points of $\mathcal{H}$ lie strictly between them. Let $\alpha'(d)$ denote the maximal cardinality of a d-dimensional strictly antipodal set. An acute set is strictly antipodal, thus $\alpha'(d) \geq \alpha(d)$. For $\alpha'(d)$ Talata gave the following constructive lower bound [34]:

$$\alpha'(d) \geq \sqrt[4]{5}^{\,d}/4 > 0.25 \cdot 1.495^d.$$

A weaker result (also due to Talata) can be found in [4, Lemma 9.11.2].

This chapter is based on [21].

3.1 The probabilistic approach

As we have mentioned, in 1983 Erdős and Füredi proved the existence of an acute d-set of exponential cardinality [12]. Since then their proof has become a well-known example to demonstrate the probabilistic method. In this section we use similar arguments to prove a better lower bound for $\alpha(d)$.

We shall study the following problem: what is the maximal cardinality $\kappa_n(d)$ of a set $S \subset \{0,1,\ldots,n-1\}^d$ that contains no bad triples? (Recall Definition 3.1.)

In the case $n = 2$, given three distinct points $A, B, C \in \{0,1\}^d$, $\angle ABC = \pi/2$ holds if and only if A, B, C is a bad triple, otherwise $\angle ABC < \pi/2$. So a set $S \subset \{0,1\}^d$ contains no bad triples if and only if S is an acute set, thus $\kappa_2(d) = \kappa(d)$.

If $n > 2$, then a triple being bad still implies that the angle determined by the triple is $\pi/2$ but we can get right angles from good triples as well, moreover, we can even get obtuse angles. So for $n > 2$ the above problem is not directly related to acute sets. However, the following simple lemma shows how one can use sets without bad triples to construct acute sets recursively.

Lemma 3.2. *Suppose that $\mathcal{H} = \{h_0, h_1, \ldots, h_{n-1}\} \subset \mathbb{R}^m$ is an acute m-set of cardinality n. If $S \subset \{0,1,\ldots,n-1\}^d$ contains no bad triples, then the set*

$$\mathcal{H}^S \stackrel{\text{def}}{=} \{(h_{i_1}, h_{i_2}, \ldots, h_{i_d}) : (i_1, i_2, \ldots, i_d) \in S\} \subset \underbrace{\mathcal{H} \times \mathcal{H} \times \ldots \times \mathcal{H}}_{d} \subset \mathbb{R}^{md}$$

is an acute (md)-set. Consequently,

$$\alpha(md) \geq \kappa_{\alpha(m)}(d) \quad \text{and} \quad \kappa(md) \geq \kappa_{\kappa(m)}(d). \tag{3.4}$$

Proof. Take three distinct points of S:

$$i = (i_1, i_2, \ldots, i_d); \quad j = (j_1, j_2, \ldots, j_d); \quad k = (k_1, k_2, \ldots, k_d),$$

and the corresponding points in $\mathcal{H}^S$:

$$h_i = (h_{i_1}, h_{i_2}, \ldots, h_{i_d}); \quad h_j = (h_{j_1}, h_{j_2}, \ldots, h_{j_d}); \quad h_k = (h_{k_1}, h_{k_2}, \ldots, h_{k_d}).$$

We show that $\angle h_i h_j h_k$ is acute by proving that the scalar product

$$\langle h_i - h_j, h_k - h_j \rangle = \sum_{r=1}^{d} \langle h_{i_r} - h_{j_r}, h_{k_r} - h_{j_r} \rangle$$

is positive. Since $\mathcal{H}$ is an acute set, the summands on the right-hand side are positive with the exception of those where j_r equals i_r or k_r, in which case the r-th summand is 0. This cannot happen for each r though, else i, j, k would be a bad triple in S.

To prove (3.4) we set $|\mathcal{H}| = n = \alpha(m)$ and $|S| = \kappa_n(d) = \kappa_{\alpha(m)}(d)$. Then $\alpha(md) \geq |\mathcal{H}^S| = |S| = \kappa_{\alpha(m)}(d)$. A similar argument works for $\kappa(md)$. (Note that if $\mathcal{H} \subset \{0,1\}^m$, then $\mathcal{H}^S \subset \{0,1\}^{md}$.) □

In view of the above lemma, it would be useful to construct large sets without bad triples. One possibility is using the probabilistic method. The next theorem is a generalization of the original random construction of Erdős and Füredi.

Theorem 3.3.

$$\kappa_n(d) > \frac{1}{2}\left(\frac{n^2}{2n-1}\right)^{\frac{d}{2}} > \frac{1}{2}\left(\frac{n}{2}\right)^{\frac{d}{2}} = \left(\frac{1}{2}\right)^{\frac{d+2}{2}} n^{\frac{d}{2}}.$$

Proof. For a positive integer m, we take $2m$ (independent and uniformly distributed) random points in $\{0,1,\ldots,n-1\}^d$: $A_1, A_2, \ldots, A_{2m}$. What is the probability that the triple A_1, A_2, A_3 is bad? For a fixed i, the probability that the i-th coordinate of A_2 is equal to the i-th coordinate of A_1 or A_3 is clearly $(2n-1)/n^2$. These events are independent so the probability that this holds for every i (that is to say A_1, A_2, A_3 is a bad triple) is

$$p = \left(\frac{2n-1}{n^2}\right)^d.$$

We get the same probability for all triples, thus the expected value of the number of bad triples is

$$3\binom{2m}{3}p = \frac{2m(2m-1)(2m-2)}{2}p < 4m^3 p \leq m, \text{ where we set } m = \left\lfloor \frac{1}{2\sqrt{p}} \right\rfloor.$$

Consequently, the $2m$ random points determine less than m bad triples with positive probability. Now we take out one point from each bad triple. Then the remaining at least $m+1$ points obviously contain no bad triples. So we have proved that there exist

$$m+1 > \frac{1}{2\sqrt{p}} = \frac{1}{2}\left(\sqrt{\frac{n^2}{2n-1}}\right)^d$$

points in $\{0,1\}^d$ without a bad triple. (Note that the original $2m$ random points might contain duplicated points. However, a triple of the form A, A, B is always bad, thus the final (at least) $m+1$ points contain no duplicated points.) □

Combining Lemma 3.2 and Theorem 3.3 we readily get the following.

Corollary 3.4. *Suppose that we have an m-dimensional acute set of size n. Then for any positive integer t*

$$\alpha(mt) > \frac{1}{2}\left(\sqrt{\frac{n^2}{2n-1}}\right)^t,$$

which yields the following lower bound in general dimension:

$$\alpha(d) \geq \alpha\left(m\left\lfloor\frac{d}{m}\right\rfloor\right) > \frac{1}{2}\left(\sqrt[2m]{\frac{n^2}{2n-1}}\right)^{m\left\lfloor\frac{d}{m}\right\rfloor} \geq c\left(\sqrt[2m]{\frac{n^2}{2n-1}}\right)^{d}.$$

Using this corollary with $m = 5$ and $n = 12$ (see Example 3.26 for a 5-dimensional acute set with 12 points) we obtain the following.

Theorem 3.5.

$$\alpha(d) > c\left(\sqrt[10]{\frac{144}{23}}\right)^{d} > c \cdot 1.2^d,$$

that is, there exist at least $c \cdot 1.2^d$ points in $\mathbb{R}^d$ such that any angle determined by three of these points is acute. (If d is divisible by 5, then c can be chosen to be $1/2$, for general d we need to use a somewhat smaller c.)

Remark 3.6. We remark that one can improve the above result with a factor $\sqrt{d}$ by using the method suggested by Ackerman and Ben-Zwi in [1].

Remark 3.7. We could have applied Corollary 3.4 with any specific acute set. The larger the value $\sqrt[2m]{n^2/(2n-1)}$ is, the better the lower bound we obtain. For $m = 1, 2, 3$ the largest values of n are known.

$$\left.\begin{matrix} m=1 \\ n=2 \end{matrix}\right\} \sqrt[2]{\frac{4}{3}} \approx 1.154 \qquad \left.\begin{matrix} m=2 \\ n=3 \end{matrix}\right\} \sqrt[4]{\frac{9}{5}} \approx 1.158 \qquad \left.\begin{matrix} m=3 \\ n=5 \end{matrix}\right\} \sqrt[6]{\frac{25}{9}} \approx 1.185$$

We will construct small dimensional acute sets in Section 3.3 (see Table 3.1 for the results). For $m = 4, 5, 6$ these constructions yield the following values for $\sqrt[2m]{n^2/(2n-1)}$.

$$\left.\begin{matrix} m=4 \\ n=8 \end{matrix}\right\} \sqrt[8]{\frac{64}{15}} \approx 1.198 \qquad \left.\begin{matrix} m=5 \\ n=12 \end{matrix}\right\} \sqrt[10]{\frac{144}{23}} \approx \mathbf{1.201} \qquad \left.\begin{matrix} m=6 \\ n=16 \end{matrix}\right\} \sqrt[12]{\frac{256}{31}} \approx 1.192$$

However, we do not know whether these acute sets are optimal or not. If we found an acute set of 9 points in $\mathbb{R}^4$, 13 points in $\mathbb{R}^5$ or 18 points in $\mathbb{R}^6$, we could immediately improve Theorem 3.5.

3.2 The constructive approach

3.2.1 On the maximal cardinality of sets without bad triples

Lemma 3.2 of the previous section shows how sets without bad triples (recall Definition 3.1) can be used to construct acute sets. In this subsection we investigate the maximal cardinality $\kappa_n(d)$ of a set in $\{0,1,\dots,n-1\}^d$ containing no bad triples. We have already seen a probabilistic lower bound for $\kappa_n(d)$ (Theorem 3.3). Now we first give an upper bound. As we will see it, this upper bound is essentially sharp if n is large enough (compared to d).

Theorem 3.8. *For even d*

$$\kappa_n(d) \leq 2n^{d/2},$$

and for odd d

$$\kappa_n(d) \leq n^{(d+1)/2} + n^{(d-1)/2}.$$

Proof. Suppose that $S \subset \{0,1,\dots,n-1\}^d$ contains no bad triples. Let $0 < r < d$ be an integer, and consider the following two projections:

$$\pi_1\left((x_1,\dots,x_d)\right) = (x_1,\dots,x_r)\ ;\ \pi_2\left((x_1,\dots,x_d)\right) = (x_{r+1},\dots,x_d).$$

Now we take the set

$$S_0 \stackrel{\text{def}}{=} \left\{x \in S : \exists y \in (S \setminus \{x\})\ \ \pi_1(x) = \pi_1(y)\right\}.$$

By definition π_1 is injective on $S \setminus S_0$, thus $|S \setminus S_0| \leq n^r$. We claim that π_2 is injective on S_0, so $|S_0| \leq n^{d-r}$. Otherwise there would exist $x, y \in S_0$ such that $\pi_2(x) = \pi_2(y)$. Since $y \in S_0$, there exists $z \in S$ such that $\pi_1(y) = \pi_1(z)$. It follows that the triple x, y, z is bad, contradiction.

Consequently, $|S| \leq n^r + n^{d-r}$. Setting $r = \left\lfloor \frac{d}{2} \right\rfloor$ we get the desired upper bound. □

Setting $n = 2$ and using that $\kappa_2(d) = \kappa(d)$ the next corollary readily follows.

Corollary 3.9. *For even d*

$$\kappa(d) \leq 2^{(d+2)/2} = 2\left(\sqrt{2}\right)^d,$$

and for odd d

$$\kappa(d) \leq 2^{(d+1)/2} + 2^{(d-1)/2} = \frac{3}{\sqrt{2}}\left(\sqrt{2}\right)^d.$$

This corollary improves the upper bound $\sqrt{2}(\sqrt{3})^d$ given by Erdős and Füredi in [12]. (We note though that they proved not only that a subset of $\{0,1\}^d$ of size larger than $\sqrt{2}(\sqrt{3})^d$ must contain three points determining a right angle but they also showed that such a set cannot be strictly antipodal which is a stronger assertion.)

If n is a prime power greater than d, then the following constructive method gives better lower bound than the random construction of the previous section. We will need matrices over finite fields with the property that every square submatrix of theirs is invertible. In coding theory the so-called *Cauchy matrices* are used for that purpose.

Definition 3.10. Let $\mathbb{F}_q$ denote the finite field of order q. A $k \times l$ matrix A over $\mathbb{F}_q$ is called a *Cauchy matrix* if it can be written in the form

$$A_{i,j} \stackrel{\text{def}}{=} (x_i - y_j)^{-1} \quad (i = 1, \dots, k; j = 1, \dots, l), \tag{3.5}$$

where $x_1, \dots, x_k, y_1, \dots, y_l \in \mathbb{F}_q$ and $x_i \neq y_j$ for any pair of indices i, j.

In the case $k = l = r$, the determinant of a Cauchy matrix A is given by

$$det(A) = \frac{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}{\prod_{1\leq i,j\leq r}(x_i - y_j)}.$$

This well-known fact can be easily proved by induction. It follows that A is invertible provided that the elements $x_1, \dots, x_r, y_1, \dots, y_r$ are pairwise distinct.

Lemma 3.11. *Let q be a prime power and k, l be positive integers. Suppose that $q \geq k+l$. Then there exists a $k \times l$ matrix over $\mathbb{F}_q$ any square submatrix of which is invertible.*

Proof. Let $x_1, \dots, x_k, y_1, \dots, y_l$ be pairwise distinct elements of $\mathbb{F}_q$, and take the $k \times l$ Cauchy matrix A as in (3.5). Clearly, every submatrix of A is also a Cauchy matrix thus the determinant of every square submatrix of A is invertible. □

Now let $k + l = d \geq 2$ and n be a prime power greater than or equal to d. Due to the lemma, there exists a $k \times l$ matrix A over the field $\mathbb{F}_n$ such that each square submatrix of A is invertible. Let us think of $\{0, 1, \dots, n-1\}^d$ as the d-dimensional vector space $\mathbb{F}_n^d$. We define an $\mathbb{F}_n$-linear subspace of $\mathbb{F}_n^d$: take all points $(x, Ax) \in \mathbb{F}_n^d$ as x runs through $\mathbb{F}_n^l$ (thus $Ax \in \mathbb{F}_n^k$). This is an l-dimensional subspace consisting n^l points. We claim that each of its points has at least $k+1$ nonzero coordinates. We prove this by contradiction. Assume that there is a point (x, Ax) which has at most k nonzero coordinates. Let the number of nonzero coordinates of x be r. It follows that the number of nonzero coordinates of Ax is at most $k - r$, in other words, Ax has at least r zero coordinates. Consequently, A has an $r \times r$ submatrix which takes a vector with nonzero elements to the null vector. This contradicts the assumption that every square submatrix is invertible.

Setting $k = \left\lfloor \frac{d}{2} \right\rfloor$ and $l = \left\lceil \frac{d}{2} \right\rceil$ we get a subspace of dimension $\left\lceil \frac{d}{2} \right\rceil$, every point of which has at least $\left\lfloor \frac{d}{2} \right\rfloor + 1 > \frac{d}{2}$ nonzero coordinates. We claim that this subspace does not contain bad triples. Indeed, taking distinct points $x_1, x_2, x_3 \in \mathbb{R}^l$, the points $(x_1 - x_2, A(x_1 - x_2))$ and $(x_3 - x_2, A(x_3 - x_2))$ are elements of the subspace, thus both have more than $\frac{d}{2}$ nonzero coordinates which means that there is a coordinate where both of them take nonzero value. We have proved the following theorem.

Theorem 3.12. *If $d \geq 2$ is an integer and $n \geq d$ is a prime power, then*

$$\kappa_n(d) \geq n^{\left\lceil \frac{d}{2} \right\rceil}.$$

If n is not a prime power, then there exists no finite field of order n. We can still consider matrices over the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. If we could find a $\lfloor \frac{d}{2} \rfloor \times \lceil \frac{d}{2} \rceil$ matrix with all of its square submatrices invertible, it would imply the existence of a set without bad triples and of cardinality $n^{\lceil \frac{d}{2} \rceil}$. For example, in the case $d = 3$ the matrix $(1 \ \ 1)$ over $\mathbb{Z}_n$ is clearly good for any n so the next theorem follows.

Theorem 3.13. *For arbitrary positive integer n it holds that $\kappa_n(3) \geq n^2$.*

Proof. We can prove this directly by taking all points in the form $(i, j, i+j)$ where i, j run through $\mathbb{Z}_n$ (addition is meant modulo n). Clearly, there are no bad triples among these n^2 points. □

Finally we show that the upper bound given in Theorem 3.8 is sharp apart from a constant factor provided that $n \geq d^3$.

Theorem 3.14. *Suppose that $n \geq d^3$ for some positive integers $n, d \geq 2$. If n is sufficiently large, then $\kappa_n(d) > n^{\lceil \frac{d}{2} \rceil}/64$.*

Proof. Let k be the unique positive integer for which $k^3 \leq n < (k+1)^3$. Obviously $k \geq d$. If k is large enough, then there is a prime number q between the consecutive cubes $(k-1)^3$ and k^3 [24, 7]. Since $q \geq d$, by Theorem 3.12 we can find a set $S \subset \{0, 1, \ldots, q-1\}^d \subset \{0, 1, \ldots, n-1\}^d$ such that S contains no bad triples and

$$|S| \geq q^{\lceil \frac{d}{2} \rceil} > (k-1)^{3\lceil \frac{d}{2} \rceil} > \left(\frac{k-1}{k+1}\right)^{3\lceil \frac{d}{2} \rceil} n^{\lceil \frac{d}{2} \rceil} \geq \left(\frac{k-1}{k+1}\right)^{3\lceil \frac{k}{2} \rceil} n^{\lceil \frac{d}{2} \rceil} \geq \frac{1}{64} n^{\lceil \frac{d}{2} \rceil},$$

where the last inequality holds because the expression $((k-1)/(k+1))^{\lceil \frac{k}{2} \rceil}$ takes its minimum value at $k = 3$. (For $k = 2$ and $k = 3$ it equals $1/3$ and $1/4$, respectively, and it is monotone increasing for even values of k as well as for odd values of k, which follows easily from the well-known fact that $((k-1)/(k+1))^k$ is monotone increasing.) □

Remark 3.15. The claim that there is a prime number between any two consecutive cubes $(k-1)^3$ and k^3 has been only verified if k is large enough. It is widely conjectured though that the claim holds for any $k > 1$. If this was true, we could omit the condition that n should be sufficiently large in the theorem.

3.2.2 Constructive lower bounds for $\alpha(d)$ and $\kappa(d)$

Random constructions of acute sets (as the original one of Erdős and Füredi or the one given in Section 3.1) give exponential lower bound for $\alpha(d)$. However, these only prove existence without telling us exactly how to find such large acute sets. Also, one can give better (constructive) lower bound if the dimension is small.

The first (non-linear) constructive lower bound is due to Bevan[3]:

$$\alpha(d) \geq \kappa(d) > \exp\left(cd^{\mu}\right), \text{ where } \mu = \frac{\log 2}{\log 3} = 0.631... \tag{3.6}$$

For small d this is a better bound than the probabilistic ones.

Our goal in this section is to obtain even better constructive bounds. The key will be the next theorem which follows readily from Lemma 3.2, Theorem 3.12 and Theorem 3.13 setting $d = 2s - 1$. (In fact, the special case $s = 2$ was already proved by Bevan, see [3, Theorem 4.2]. He obtained (3.6) by the repeated application of this special case.)

Theorem 3.16. *Let $s \geq 2$ be an integer, and suppose that $n \geq 2s - 1$ is a prime power. (In the case $s = 2$ the theorem holds for arbitrary positive integer n.) If $\mathcal{H} \subset \mathbb{R}^m$ is an acute m-set of cardinality n, then we can choose n^s points of the set*

$$\underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{2s-1} \subset \mathbb{R}^{(2s-1)m}$$

that form an acute set.

Remark 3.17. If $\mathcal{H}$ is cubic (that is, $\mathcal{H} \subset \{0,1\}^m$), then the obtained acute set is also cubic (that is, it is in $\{0,1\}^{(2s-1)m}$).

Now we start with an acute set $\mathcal{H}$ of prime power cardinality and we apply the previous theorem with the largest possible s. Then we do the same for the obtained larger acute set (the cardinality of which is also a prime power). How large acute sets do we get if we keep doing this? For the sake of simplicity, let us start with the $d_0 = 4$ dimensional acute set of size $n_0 = 8$ that we will construct in Section 3.3. Let us denote the dimension and the size of the acute set we obtain in the k-th step by d_k and n_k, respectively. Clearly n_k is a power of 2, thus at step $(k+1)$ we can apply Theorem 3.16 with $s_k = n_k/2$. Setting $u_k = \log_2 n_k$ we get the following:

$$\begin{aligned} d_{k+1} = d_k(2s_k - 1) < d_k n_k; \quad n_{k+1} = n_k^{s_k} = n_k^{n_k/2}; \\ u_{k+1} = u_k(n_k/2) = u_k 2^{u_k - 1} \geq 2 \cdot 2^{u_k - 1} = 2^{u_k}. \end{aligned}$$

It follows that $d_{k+1}/u_{k+1} \leq 2d_k/u_k$ so

$$d_k \leq \frac{d_0}{u_0} u_k 2^k = \frac{4}{3} 2^k u_k.$$

It yields that in dimension d_k we get an acute set of size

$$n_k = 2^{u_k} \geq 2^{(3/4)2^{-k} d_k}.$$

Due to the factor 2^{-k} in the exponent, n_k is not exponential in d_k. However, the inequality $u_{k+1} \geq 2^{u_k}$ implies that u_k grows extremely fast (and so does n_k and d_k) which means that n_k is *almost* exponential. For instance, we can easily obtain that for any positive integer r there exists k_0 such that for $k \geq k_0$ it holds that

$$n_k > \exp(d_k/\underbrace{\log\log\cdots\log}_{r}(d_k)).$$

We have given a constructive proof of the following theorem.

Theorem 3.18. *For any positive integer r we have infinitely many values of d such that*

$$\alpha(d) > \exp(d/\underbrace{\log\log\cdots\log}_{r} d).$$

We can also get a constructive lower bound for $\kappa(d)$. We do the same iterated process but this time we start with an acute set in $\{0,1\}^{d_0}$. (For instance, we can set $d_0 = 3$ and $n_0 = 4$.) Then the acute set obtained in step k will be in $\{0,1\}^{d_k}$. This way we get an almost exponential lower bound for $\kappa(d)$ as well.

However, Theorem 3.16 gives acute sets only in certain dimensions. In the remainder of this section we consider the problems investigated so far in a slightly more general setting to get large acute sets in any dimension. (The proofs of these more general results are essentially the same as the original ones. Thus we could have considered this general setting in the first place, but for the sake of better understanding we opted not to.)

Let $n_1, n_2, \ldots, n_d \geq 2$ be positive integers and consider the $n_1 \times \cdots \times n_d$ lattice, that is the set $\{0, 1, \ldots, n_1 - 1\} \times \cdots \times \{0, 1, \ldots, n_d - 1\}$. What is the maximal cardinality of a subset S of the $n_1 \times \cdots \times n_d$ lattice containing no bad triples?

We claim that if $n \geq \max\{n_1, \ldots, n_d\}$ and the set $S_0 \subset \{0, 1, \ldots, n-1\}^d$ contains no bad triples, then we can get a set in the $n_1 \times \cdots \times n_d$ lattice without bad triples and of cardinality at least

$$\frac{n_1}{n} \cdots \frac{n_d}{n} |S_0|.$$

Indeed, starting with the $n \times \ldots \times n$ lattice, we replace the n's one-by-one with the n_i's; in each step we keep those n_i *sections* that contain the biggest part of S_0. Combining this argument with Theorem 3.12 and 3.13 we get the following for the odd case $d = 2s - 1$.

Theorem 3.19. *Let $s \geq 2$, and suppose that $n \geq 2s-1$ is a prime power (in the case $s = 2$ the theorem holds for arbitrary positive integer n). For positive integers $n_1, \ldots, n_{2s-1} \leq n$ in the $n_1 \times n_2 \times \cdots \times n_{2s-1}$ lattice at least $\lceil n_1 n_2 \cdots n_{2s-1}/n^{s-1} \rceil$ points can be chosen without any bad triple.*

Also, one can get a more general version of Lemma 3.2 with the same proof.

Lemma 3.20. *Suppose that the set* $\mathcal{H}_t = \{h_0^t, h_1^t, \ldots, h_{n_t-1}^t\} \subset \mathbb{R}^{m_t}$ *is acute for each* $1 \le t \le d$. *If* $S \subset \{0, 1, \ldots, n_1 - 1\} \times \cdots \times \{0, 1, \ldots, n_d - 1\}$ *contains no bad triples, then the set*

$$\left\{\left(h_{i_1}^1, h_{i_2}^2, \ldots, h_{i_d}^d\right) : (i_1, i_2, \ldots, i_d) \in S\right\} \subset \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_d \subset \mathbb{R}^{m_1+\cdots+m_d}$$

is an $(m_1 + \cdots + m_d)$*-dimensional acute set.*

Putting these results together we obtain a more general form of Theorem 3.16.

Theorem 3.21. *Let* $s \ge 2$, *and suppose that* $n \ge 2s-1$ *is a prime power (in the case* $s = 2$ *the theorem holds for arbitrary positive integer* n*). Assume that for each* $t = 1, \ldots, 2s - 1$ *we have an acute set of* $n_t \le n$ *points in* $\mathbb{R}^{m_t}$. *Then in* $\mathbb{R}^{m_1+\cdots+m_{2s-1}}$ *there exists an acute set of cardinality at least*

$$\left\lceil n_1 n_2 \cdots n_{2s-1} / n^{s-1} \right\rceil .$$

The obtained acute set is cubic provided that all acute sets used are cubic.

Remark 3.22. We also note that in the case $s = 3$ the theorem can be applied for $n = 4$ as well. Consider the 4-element field $\mathbb{F}_4 = \{0, 1, a, b\}$. Then the 2×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \end{pmatrix}$$

has no singular square submatrix which implies that Theorem 3.12 holds for $d = 5; n = 4$, thus Theorem 3.19 and Theorem 3.21 hold for $s = 3; n = 4$.

Now we can use the small dimensional acute sets of Section 3.3 as building blocks to build higher dimensional acute sets by Theorem 3.21. Table 3.2 in Section 3.4 shows the lower bounds we get this way for $d \le 84$. (We could keep doing that for larger values of d and up to dimension 250 we would get better bound than the probabilistic one given in Section 3.1.) These bounds are all new results except for $d \le 3$.

We can do the same for $\kappa(d)$, see Table 3.3 in Section 3.4 for $d \le 82$. This method outdoes the random construction up to dimension 200. (We need small dimensional cubic acute sets as building blocks. We use the ones found by Bevan who used computer to determine the exact values of $\kappa(d)$ for $d \le 9$. He also used a recursive construction to obtain bounds for larger d's. His method is similar but less effective: our results are better for $d \ge 13; d \ne 27$. In dimension $d = 63$ we get a cubic acute set of size 65536. This is almost ten times bigger than the one Bevan obtained which contains 6561 points.)

Tables 3.4 and 3.5 in Section 3.5 compare the probabilistic and constructive lower bounds for $\alpha(d)$ and $\kappa(d)$.

Finally we prove the simple fact that $\alpha(d)$ is strictly monotone increasing. We will need this fact in Table 3.2.

Lemma 3.23. *$\alpha(d+1) > \alpha(d)$ holds for any positive integer d.*

Proof. Assume that we have an acute set $\mathcal{H} = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$. Let P be the convex hull of $\mathcal{H}$ and y be any point in $P \setminus \mathcal{H}$. We claim that $\angle yx_ix_j < \pi/2$ for any $i \neq j$. Let $H_{i,j}$ be the hyperplane that is perpendicular to the segment x_ix_j and goes through x_i. Let $S_{i,j}$ be the open half-space bounded by $H_{i,j}$ that contains x_j. For a point $z \in \mathbb{R}^d$ the angle $\angle zx_ix_j$ is acute if and only if $z \in S_{i,j}$. It follows that $\mathcal{H} \setminus \{x_i\} \subset S_{i,j}$ while x_i lies on the boundary of $S_{i,j}$. Thus $y \in P \setminus \{x_i\} \subset S_{i,j}$ which implies that $\angle yx_ix_j < \pi/2$.

Now let us consider the usual embedding of $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ and let $\mathbf{v}$ denote the unit vector $(0, \ldots, 0, 1)$. Consider the point $y^t = y + t\mathbf{v}$ for sufficiently large t. It is easy to see that $\angle y^tx_ix_j < \pi/2$ still holds, but now even the angles $\angle x_iy^tx_j$ are acute. It follows that $\mathcal{H} \cup \{y^t\} \subset \mathbb{R}^{d+1}$ is an acute set. □

Remark 3.24. For $\kappa(d)$ it is only known that $\kappa(d+2) > \kappa(d)$ [3, Theorem 4.1]. In Table 3.3 we will refer to this result as *almost strict monotonicity.*

3.3 Small dimensional acute sets

In this section we construct acute sets in dimension $m = 4, 5$ and use computer to find such sets for $6 \leq m \leq 10$. These small dimensional examples are important because the random construction of Section 3.1 and the recursive construction of Section 3.2 use them to find higher dimensional acute sets of large cardinality.

Danzer and Grünbaum presented an acute set of $2m - 1$ points in $\mathbb{R}^m$ [10]. It is also known that for $m = 2, 3$ this is the best possible [8, 33, 18]. Bevan used computer to find small dimensional acute sets by generating random points on the unit sphere. For $m \geq 7$ he found more than $2m - 1$ points [3].

Our approach starts similarly as the construction of Danzer and Grünbaum. We consider the following $2m - 2$ points in $\mathbb{R}^m$:

$$P_i^{\pm 1} = (0, \ldots, 0, \underbrace{\pm 1}_{i\text{-th}}, 0, \ldots, 0) \qquad (i = 1, 2, \ldots, m-1).$$

What angles do these points determine? Clearly, $\angle P_i^{-1}P_j^{\pm 1}P_i^{+1} = \pi/2$ for $i \neq j$ and all other angles are acute. We can get rid of the right angles by slightly perturbing the points in the following manner:

$$\tilde{P}_i^{\pm 1} = (0, \ldots, 0, \underbrace{\pm 1}_{i\text{-th}}, 0, \ldots, 0, \varepsilon_i) \qquad (i = 1, 2, \ldots, m-1),$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}$ are pairwise distinct real numbers.

Our goal is to complement the points $\tilde{P}_i^{\pm 1}$ with some additional points such that they still form an acute set. In fact, we will complement the points $P_i^{\pm 1}$ such that all *new* angles are acute. (Then changing points $P_i^{\pm 1}$ to $\tilde{P}_i^{\pm 1}$ we get an acute set provided that the ε_i's are small enough.)

Under what condition can a point $\mathbf{x} = (x_1, \dots, x_m)$ be added in the above sense? Simple computation shows that the exact condition is

$$\|\mathbf{x}\| > 1 \text{ and } |x_i| + |x_j| < 1 \text{ for } 1 \le i, j \le m-1; i \ne j. \tag{3.7}$$

For example, the point $A = (0, \dots, 0, a)$ can be added for $a > 1$. This way we get an acute set of size $2m-1$. Basically, this was the construction of Danzer and Grünbaum. We know that this is the best possible for $m = 2, 3$. However, we can do better if $m \ge 4$.

Suppose that we have two points $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$ both satisfying (3.7) (that is, they can be separately added). Both points can be added (at the same time) if and only if

$$|x_i + y_i| < 1 + \langle \mathbf{x}, \mathbf{y} \rangle \text{ and } |x_i - y_i| < \min\left(\|\mathbf{x}\|^2, \|\mathbf{y}\|^2\right) - \langle \mathbf{x}, \mathbf{y} \rangle \text{ for } 1 \le i \le m-1. \tag{3.8}$$

We can find two such points in the following simple form: $A_1 = (a_1, a_1, \dots, a_1, a_2)$ and $A_2 = (-a_1, -a_1, \dots, -a_1, a_2)$. Then points A_1 and A_2 can be added if and only if

$$\frac{1}{m-1} < a_1 < \frac{1}{2} \text{ and } a_2^2 > \left|1 - (m-1)a_1^2\right|. \tag{3.9}$$

Such a_1 and a_2 clearly exist if $m \ge 4$.

Example 3.25. For sufficiently small and pairwise distinct ε_i's the 8 points below form an acute set in $\mathbb{R}^4$.

$$\begin{array}{cccccc}
(& 1 & 0 & 0 & \varepsilon_1 &) \\
(& -1 & 0 & 0 & \varepsilon_1 &) \\
(& 0 & 1 & 0 & \varepsilon_2 &) \\
(& 0 & -1 & 0 & \varepsilon_2 &) \\
(& 0 & 0 & 1 & \varepsilon_3 &) \\
(& 0 & 0 & -1 & \varepsilon_3 &) \\
(& 0.4 & 0.4 & 0.4 & 1 &) \\
(& -0.4 & -0.4 & -0.4 & 1 &)
\end{array}$$

For $m = 5$, we can even add four points of the following form:

$$A_1 = (a_1, a_1, a_1, a_1, a_2); A_2 = (-a_1, -a_1, -a_1, -a_1, a_2);$$
$$B_1 = (b_1, b_1, -b_1, -b_1, -b_2); B_2 = (-b_1, -b_1, b_1, b_1, -b_2).$$

We have seen that $1/4 < a_1, b_1 < 1/2$ must hold so we set $a_1 = 1/4 + \delta$ and $b_1 = 1/2 - \delta$. Then we set $a_2 = \sqrt{3}/2$ and $b_2 = 2\sqrt{\delta}$ so that $\|A_i\|$ and $\|B_i\|$ are slightly bigger than 1.

Example 3.26. Let us fix a positive real number $\delta < 1/48$ and consider the points below.

$$\begin{array}{rccccccl}
A_1 = (& 1/4+\delta & 1/4+\delta & 1/4+\delta & 1/4+\delta & \sqrt{3}/2 &) \\
A_2 = (& -1/4-\delta & -1/4-\delta & -1/4-\delta & -1/4-\delta & \sqrt{3}/2 &) \\
B_1 = (& 1/2-\delta & 1/2-\delta & -1/2+\delta & -1/2+\delta & -2\sqrt{\delta} &) \\
B_2 = (& -1/2+\delta & -1/2+\delta & 1/2-\delta & 1/2-\delta & -2\sqrt{\delta} &)
\end{array}$$

Then the set $\{\tilde{P}_i^{\pm 1} : i = 1,2,3,4\} \cup \{A_1, A_2, B_1, B_2\}$ is an acute set of 12 points in $\mathbb{R}^5$ assuming that ε_i's are sufficiently small and pairwise distinct.
(This specific example is important because the random method presented in Section 3.1 gives the best result starting from this example.)

Proof. We need to prove that A_1, A_2, B_1, B_2 can be added to $P_i^{\pm 1}$'s in such a way that all *new* angles are acute. First we prove that any pair of these 4 points can be added. Since each of them satisfies (3.7), we only have to check that each pair satisfies (3.8). For the pair A_1, A_2 it is done since they satisfy (3.9). It goes similarly for the pair B_1, B_2. For the pairs A_i, B_j (3.8) yields the condition $3/4 < 1 - \sqrt{3\delta} \Leftrightarrow \delta < 1/48$.

Now we have checked all new angles except those that are determined by three new points. The squares of the distances between the 4 new points are:

$$d(A_1,A_2)^2 = 1+8\delta+16\delta^2; d(B_1,B_2)^2 = 4-16\delta+16\delta^2; d(A_i,B_j)^2 = 2+2\sqrt{3\delta}+2\delta+8\delta^2.$$

Now for any triangle in $\{A_1, A_2, B_1, B_2\}$ the square of the length of each side is less than the sum of the squares of the two other side lengths which means that the triangle is acute-angled. □

For $m \geq 6$ we used computer to find additional points. We generated random points on the sphere with radius $1+\delta$ and we added the point whenever it was possible. Table 3.1 shows the cardinality of acute sets we found this way compared to previous results. Below the reader can find examples for $m = 6, 7, 8$. For $m \geq 11$, the recursive construction presented in Section 3.2 gives better result than the computer search (see Table 3.2 in Section 3.4 for the best known lower bounds of $\alpha(d)$ for $d \leq 84$).

The following 16 points form an acute 6-set.

$$\begin{array}{cccccccc}
(& \dots & 0 & \pm 999 & 0 & \dots & \varepsilon_i &) \\
(& -88 & 2 & -244 & -35 & 124 & -957 &) \\
(& 1 & -448 & -458 & -482 & 485 & 349 &) \\
(& -537 & 364 & -358 & -227 & -426 & 466 &) \\
(& -386 & 473 & 494 & -420 & 455 & -18 &) \\
(& 455 & 467 & -47 & 490 & 296 & 494 &) \\
(& 435 & 411 & -431 & -533 & -39 & -413 &)
\end{array}$$

The following 20 points form an acute 7-set.

$$\begin{array}{ccccccccc}
(& \dots & 0 & \pm 999 & 0 & \dots & 0 & \varepsilon_i &) \\
(& -398 & -425 & -271 & 548 & 316 & -191 & -389 &) \\
(& -29 & 174 & -320 & 278 & 322 & 250 & 789 &) \\
(& -413 & -261 & -498 & -295 & -263 & -288 & 524 &) \\
(& 453 & -273 & -380 & -241 & -493 & 438 & -288 &) \\
(& -224 & 473 & -260 & -410 & 73 & 319 & -619 &) \\
(& -398 & 28 & 348 & 475 & -511 & 479 & 60 &) \\
(& -117 & -420 & 377 & -422 & 548 & 386 & 199 &) \\
(& 506 & -444 & 490 & 292 & -233 & -409 & -20 &)
\end{array}$$

The following 23 points form an acute 8-set.

$$\begin{array}{cccccccccc}
(& 0 & \dots & 0 & \pm 999 & 0 & \dots & 0 & \varepsilon_i &) \\
(& -403 & 160 & 381 & 120 & -438 & 470 & 435 & -226 &) \\
(& -3 & 470 & -158 & -424 & -375 & 423 & 233 & 447 &) \\
(& -456 & 349 & 387 & -135 & -32 & -538 & -438 & 145 &) \\
(& 166 & -170 & -16 & 286 & -35 & -314 & 188 & 853 &) \\
(& 239 & -281 & 451 & -297 & -521 & 255 & -454 & 173 &) \\
(& 271 & 273 & 438 & -543 & 204 & 446 & 148 & -321 &) \\
(& 384 & 149 & -408 & 476 & -499 & 116 & -195 & -370 &) \\
(& -239 & -414 & -499 & -151 & -230 & -273 & 99 & -603 &) \\
(& 563 & 410 & 93 & 219 & -399 & -415 & 354 & 26 &)
\end{array}$$

(The ε_i's denote *small* and pairwise distinct real numbers.)

3.4 Best known bounds in low dimension

The following tables show the best known lower bounds for $\alpha(d)$ and $\kappa(d)$. Beside the dimension and the bound itself, we stated the value of s, n and the product $n_1 \cdots n_{2s-1}/n^{s-1}$ with which Theorem 3.21 is applied. From the n_i's the reader can easily obtain the m_i's. *Str. mon.* and *a. str. mon.* stand for *strict monotonicity* (cf. Lemma 3.23) and *almost strict monotonicity* (cf. Remark 3.24).

For example, in dimension 39 in Table 3.2 we see that $s = 5$ and $n = 9$. (Note that n is indeed a prime power and $n \geq 2s - 1$ holds.) The expression $8^6 \cdot 9^3/9^4$ means that we need to apply Theorem 3.21 with $n_1 = n_2 = \ldots = n_6 = 8$ and $n_7 = n_8 = n_9 = 9$. (Note that they are all indeed at most n.) Then for each i we take the smallest dimension m_i in which we have an acute set containing at least n_i points. In our case the corresponding dimensions are $m_1 = m_2 = \ldots = m_6 = 4$ and $m_7 = m_8 = m_9 = 5$. Consequently, the total dimension is $6 \cdot 4 + 3 \cdot 5 = 39$. We obtain that in $\mathbb{R}^{39}$ there exists an acute set of cardinality at least $\lceil 8^6 \cdot 9^3/9^4 \rceil = 29128$.

Recall that in the case $s = 2$ we can take arbitrary n (it does not need to be a prime power). Also, according to Remark 3.22, in the case $s = 3$ we can have $n = 4$ (even though $n \geq 2s - 1$ does not hold). See dimension 13 and 15 in Table 3.3.

Table 3.2: Best known lower bound for $\alpha(d)$ $(1 \le d \le 84)$

dim	l. b.	s	n		dim	l. b.	s	n	
1	2				43	85184	5	11	$8^2 \cdot 11^7/11^4$
2	3				44	120439	5	13	$8^1 \cdot 12^8/13^4$
3	5				45	180659	5	13	$12^9/13^4$
4	8			construction	46	195714	5	13	$12^8 \cdot 13^1/13^4$
5	12			construction	47	212023	5	13	$12^7 \cdot 13^2/13^4$
6	16			computer	48	229692	5	13	$12^6 \cdot 13^3/13^4$
7	20			computer	49	262144	6	11	$8^6 \cdot 11^5/11^5$
8	23			computer	50	360448	6	11	$8^5 \cdot 11^6/11^5$
9	27			computer	51	495616	6	11	$8^4 \cdot 11^7/11^5$
10	31			computer	52	681472	6	11	$8^3 \cdot 11^8/11^5$
11	40	2	8	$5^1 \cdot 8^2/8^1$	53	937024	6	11	$8^2 \cdot 11^9/11^5$
12	64	2	8	$8^3/8^1$	54	1334092	6	13	$8^1 \cdot 12^{10}/13^5$
13	65			str. mon.	55	2001138	6	13	$12^{11}/13^5$
14	96	2	12	$8^1 \cdot 12^2/12^1$	56	2167900	6	13	$12^{10} \cdot 13^1/13^5$
15	144	2	12	$12^3/12^1$	57	2348558	6	13	$12^9 \cdot 13^2/13^5$
16	145			str. mon.	58	2544271	6	13	$12^8 \cdot 13^3/13^5$
17	192	2	16	$12^1 \cdot 16^2/16^1$	59	2756293	6	13	$12^7 \cdot 13^4/13^5$
18	256	2	16	$16^3/16^1$	60	2985984	6	16	$12^6 \cdot 16^5/16^5$
19	320	3	8	$5^1 \cdot 8^4/8^2$	61	4378558	7	13	$8^4 \cdot 12^9/13^6$
20	512	3	8	$8^5/8^2$	62	6567837	7	13	$8^3 \cdot 12^{10}/13^6$
21	513			str. mon.	63	9851755	7	13	$8^2 \cdot 12^{11}/13^6$
22	514			str. mon.	64	14777632	7	13	$8^1 \cdot 12^{12}/13^6$
23	704	3	11	$8^2 \cdot 11^3/11^2$	65	22166447	7	13	$12^{13}/13^6$
24	982	3	13	$8^1 \cdot 12^4/13^2$	66	24013651	7	13	$12^{12} \cdot 13^1/13^6$
25	1473	3	13	$12^5/13^2$	67	26014789	7	13	$12^{11} \cdot 13^2/13^6$
26	1600	4	8	$5^2 \cdot 8^5/8^3$	68	28182688	7	13	$12^{10} \cdot 13^3/13^6$
27	2560	4	8	$5^1 \cdot 8^6/8^3$	69	30531245	7	13	$12^9 \cdot 13^4/13^6$
28	4096	4	8	$8^7/8^3$	70	33075516	7	13	$12^8 \cdot 13^5/13^6$
29	4097			str. mon.	71	35831808	7	16	$12^7 \cdot 16^6/16^6$
30	4098			str. mon.	72	47775744	7	16	$12^6 \cdot 16^7/16^6$
31	4099			str. mon.	73	63700992	7	16	$12^5 \cdot 16^8/16^6$
32	5632	4	11	$8^3 \cdot 11^4/11^3$	74	84934656	7	16	$12^4 \cdot 16^9/16^6$
33	7744	4	11	$8^2 \cdot 11^5/11^3$	75	113246208	7	16	$12^3 \cdot 16^{10}/16^6$
34	10873	4	13	$8^1 \cdot 12^6/13^3$	76	150994944	7	16	$12^2 \cdot 16^{11}/16^6$
35	16310	4	13	$12^7/13^3$	77	201326592	7	16	$12^1 \cdot 16^{12}/16^6$
36	20457	5	9	$8^9/9^4$	78	268435456	7	16	$16^{13}/16^6$
37	23015	5	9	$8^8 \cdot 9^1/9^4$	79	268435457			str. mon.
38	25891	5	9	$8^7 \cdot 9^2/9^4$	80	268435458			str. mon.
39	29128	5	9	$8^6 \cdot 9^3/9^4$	81	322486272	8	16	$12^9 \cdot 16^6/16^7$
40	36864	4	16	$12^2 \cdot 16^5/16^3$	82	429981696	8	16	$12^8 \cdot 16^7/16^7$
41	49152	4	16	$12^1 \cdot 16^6/16^3$	83	573308928	8	16	$12^7 \cdot 16^8/16^7$
42	65536	4	16	$16^7/16^3$	84	764411904	8	16	$12^6 \cdot 16^9/16^7$

Table 3.3: Best known lower bound for $\kappa(d)$ $(1 \leq d \leq 82)$

dim	l. b.	s	n		dim	l. b.	s	n	
1	2				42	4096	4	8	$8^7/8^3$
2	2				43	4096			
3	4				44	4097			a. str. mon.
4	5			Bevan	45	4097			
5	6			Bevan	46	4608	4	9	$8^3 \cdot 9^4/9^3$
6	8			Bevan	47	5184	4	9	$8^2 \cdot 9^5/9^3$
7	9			Bevan	48	5832	4	9	$8^1 \cdot 9^6/9^3$
8	10			Bevan	49	6561	4	9	$9^7/9^3$
9	16	2	4	$4^3/4^1$	50	7991	5	9	$5^2 \cdot 8^7/9^4$
10	16				51	10229	5	9	$4^1 \cdot 8^8/9^4$
11	20	2	5	$4^1 \cdot 5^2/5^1$	52	12786	5	9	$5^1 \cdot 8^8/9^4$
12	25	2	5	$5^3/5^1$	53	15343	5	9	$6^1 \cdot 8^8/9^4$
13	32	3	4	$2^1 \cdot 4^4/4^2$	54	20457	5	9	$8^9/9^4$
14	32				55	23015	5	9	$8^8 \cdot 9^1/9^4$
15	64	3	4	$4^5/4^2$	56	25891	5	9	$8^7 \cdot 9^2/9^4$
16	64				57	29128	5	9	$8^6 \cdot 9^3/9^4$
17	65			a. str. mon.	58	32768	5	9	$8^5 \cdot 9^4/9^4$
18	80	3	5	$4^2 \cdot 5^3/5^2$	59	36864	5	9	$8^4 \cdot 9^5/9^4$
19	100	3	5	$4^1 \cdot 5^4/5^2$	60	41472	5	9	$8^3 \cdot 9^6/9^4$
20	125	3	5	$5^5/5^2$	61	46656	5	9	$8^2 \cdot 9^7/9^4$
21	125				62	52488	5	9	$8^1 \cdot 9^8/9^4$
22	126			a. str. mon.	63	65536	4	16	$16^7/16^3$
23	126				64	65536			
24	133	3	7	$5^1 \cdot 6^4/7^2$	65	65537			a. str. mon.
25	160	3	8	$4^1 \cdot 5^1 \cdot 8^3/8^2$	66	65537			
26	200	3	8	$5^2 \cdot 8^3/8^2$	67	65538			a. str. mon.
27	256	2	16	$16^3/16^1$	68	67505	6	11	$8^9 \cdot 9^2/11^5$
28	320	3	8	$5^1 \cdot 8^4/8^2$	69	75943	6	11	$8^8 \cdot 9^3/11^5$
29	384	3	8	$6^1 \cdot 8^4/8^2$	70	85436	6	11	$8^7 \cdot 9^4/11^5$
30	512	3	8	$8^5/8^2$	71	102400	5	16	$5^2 \cdot 16^7/16^4$
31	512				72	131072	5	16	$4^1 \cdot 8^1 \cdot 16^7/16^4$
32	513			a. str. mon.	73	163840	5	16	$5^1 \cdot 8^1 \cdot 16^7/16^4$
33	576	3	9	$8^2 \cdot 9^3/9^2$	74	196608	5	16	$6^1 \cdot 8^1 \cdot 16^7/16^4$
34	681	4	7	$5^1 \cdot 6^6/7^3$	75	262144	5	16	$4^1 \cdot 16^8/16^4$
35	817	4	7	$6^7/7^3$	76	327680	5	16	$5^1 \cdot 16^8/16^4$
36	1024	4	8	$4^2 \cdot 8^5/8^3$	77	393216	5	16	$6^1 \cdot 16^8/16^4$
37	1280	4	8	$4^1 \cdot 5^1 \cdot 8^5/8^3$	78	524288	5	16	$8^1 \cdot 16^8/16^4$
38	1600	4	8	$5^2 \cdot 8^5/8^3$	79	589824	5	16	$9^1 \cdot 16^8/16^4$
39	2048	4	8	$4^1 \cdot 8^6/8^3$	80	655360	5	16	$10^1 \cdot 16^8/16^4$
40	2560	4	8	$5^1 \cdot 8^6/8^3$	81	1048576	5	16	$16^9/16^4$
41	3072	4	8	$6^1 \cdot 8^6/8^3$	82	1048576			

3.5 Comparing the two approaches

Finally, we compare the lower bounds given by the probabilistic and the constructive approach. For a small m we take an m-dimensional acute set of prime power cardinality n, then we apply Theorem 3.16 with the largest possible s to get an acute set of size n^s in dimension $d = (2s-1)n$. Then we compare this to the probabilistic bound $\alpha(d) > (1/2)(144/23)^{d/10}$ (in fact, we obtained this result only for d divisible by 5; for general d it only holds with a somewhat smaller constant factor). For the sake of simplicity we consider the base-10 logarithm of the bounds. (See Table 3.2 for values of n used here.)

Table 3.4: Comparing constructive and probabilistic lower bound of $\alpha(d)$

m	n	s	dimension $d = (2s-1)m$	constructive l.b. $s \lg n$	probabilistic l.b. $\lg \frac{1}{2} + \frac{d}{10} \lg \frac{144}{23}$
4	8	4	28	**3.61**	1.92
5	11	6	55	**6.24**	4.08
6	16	8	90	**9.63**	6.86
7	19	10	133	**12.78**	10.29
8	23	12	184	**16.34**	14.35
9	27	14	243	**20.03**	19.05
10	31	16	310	23.86	**24.39**
11	37	19	407	29.79	**32.12**
12	64	32	756	57.79	**59.92**

We can do the same for $\kappa(d)$. We apply Theorem 3.16 for small dimensional acute sets in $\{0,1\}^d$ with the largest possible s and compare what we get to the bound $\kappa(d) > (1/2)(4/3)^{d/2}$ given by Erdős and Füredi. (See Table 3.3 for values of n used here.)

Table 3.5: Comparing constructive and probabilistic lower bound of $\kappa(d)$

m	n	s	dimension $d = (2s-1)m$	constructive l.b. $s \lg n$	probabilistic l.b. $\lg \frac{1}{2} + \frac{d}{2} \lg \frac{4}{3}$
4	5	3	20	**2.09**	0.94
6	8	4	42	**3.61**	2.32
9	16	8	135	**9.63**	8.13
11	19	10	209	**12.78**	12.75
12	25	13	300	18.17	**18.43**
13	32	16	403	24.08	**24.87**
15	64	32	945	57.79	**58.73**

Chapter 4

The Koch curve is tube-null

In this chapter we show that the Koch curve is tube-null, that is, it can be covered by strips of arbitrarily small total width. In fact, we prove the following stronger result: the Koch curve can be decomposed into three sets such that each can be projected to a line in such a way that the image has Hausdorff dimension less than 1. The proof contains geometric, combinatorial, algebraic and probabilistic arguments. This chapter is based on [22].

4.1 Tube-nullity

In $\mathbb{R}^n$ an infinite tube is the closed r-neighbourhood of l for some positive real r and some straight line l. The *tube-measure* of a set $E \subset \mathbb{R}^n$ is defined as

$$\mu(E) = \inf \left\{ \sum_i \gamma_{n-1} r_i^{n-1} : \bigcup_i T_i \supset E \right\},$$

where T_i is a tube with cross-sectional radius r_i, and γ_{n-1} denotes the volume of the unit ball of $\mathbb{R}^{n-1}$. The set E is called *tube-null* if $\mu(E) = 0$.

Csörnyei and Wisewell showed that the only μ-measurable sets are the tube-null sets and their complements [9]. Tube-null sets come up in Fourier analysis: Carbery, Soria and Vargas proved that every tube-null set is a "set of divergence" for the localisation problem [6]. From this point of view, it could be useful to see non-trivial examples for tube-null sets. In many cases, it is hard to tell whether a set is tube-null or not (even for *simple* sets). The following question was posed by, among others, Marianna Csörnyei: is the Koch snowflake curve tube-null?

In this chapter we answer this question affirmatively. In the plane tubes are infinite strips and tube-nullity simply means the existence of a covering with strips of arbitrarily small total width. Actually, we will prove more than that. For some $s < 1$ we will show that K can be covered by strips such that the sum of the s-powers of the widths is arbitrarily small, and we will get such coverings by using strips in only three directions. This will give a decomposition of the Koch curve into three sets, each of which can be projected to a line in such a way that the image has Hausdorff dimension less than 1.

Theorem 4.1. *The Koch curve K is tube-null, that is, it can be covered by strips of arbitrarily small total width.*

Moreover, there exists a decomposition $K = K_0 \cup K_1 \cup K_2$ and projections π_0, π_1, π_2 such that the Hausdorff dimension of $\pi_i(K_i)$ is less than 1 for $i = 0, 1, 2$.

We mention that in a conference talk T.C. O'Neil proved that a certain variant of the Koch curve (which uses only right angles) is tube-null [32]. He also asked whether this holds for the Koch curve.

4.2 Covering the Koch curve with strips

Let $A_0A_1A_2$ be an equilateral triangle with side length $2/\sqrt{3}$ so that each height of the triangle is 1. This is our level 0 triangle. Let e_i be the line that is parallel to $A_{i+1}A_{i+2}$ and goes through A_i (indices are cyclic). The strip bounded by the lines $A_{i+1}A_{i+2}$ and e_i is the level 0 strip in direction i. For some positive integer n we decompose this strip into 3^n strips with equal width 3^{-n}. These strips will be called the level n strips in direction i. The boundary lines of these strips (in all three directions) determine a triangle grid. The triangles in this grid are called level n triangles.

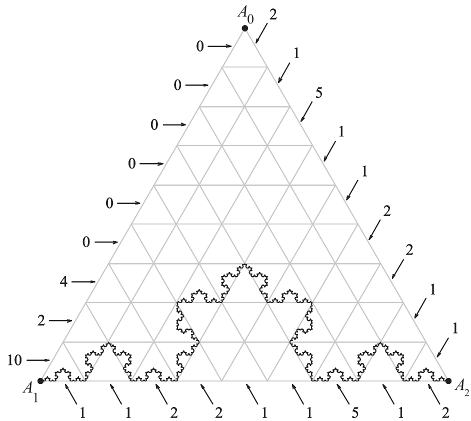
Let us consider the Koch curve K connecting A_1 with A_2 and contained in the triangle $A_0A_1A_2$. It is a self-similar set: it is the union of 4^n pieces, each similar to K. Each of these level n pieces is contained in one of the level n triangle of the grid and connects two vertices of that triangle.

Our goal is to find a collection of level n strips such that they cover K and they have a small total width. For a level n strip we define its *covering number* as the number of level n pieces covered by the strip (see Figure 4.1). The idea is to use strips with large covering number. The next lemma shows that each piece is covered by at least one strip with a large covering number.

Lemma 4.2. *For each level n piece (at least) one of the three level n strips through this piece contains at least $2^{n/3}$ level n pieces.*

Proof. For an arbitrary level n piece take all three level n strips covering this piece. It is sufficient to prove that the product of the covering numbers of these strips is at least

Figure 4.1: The covering numbers corresponding to level 2 strips



2^n. We prove this by induction on n. It clearly holds for $n = 0$. For arbitrary $n \geq 1$, a level n piece can be viewed as a level $(n-1)$ piece in one of the four level 1 pieces. Due to the reflection symmetry of K the level n strip in direction 0 covers at least twice as many level n pieces in the whole curve as it covers in any of the level 1 pieces. For the other two directions, we simply use the fact that the strips cover at least as many pieces in K as in a level 1 piece. It follows that the product is at least the double of the product corresponding to the same piece when it is considered as a level $(n-1)$ piece of a level 1 piece, which completes the proof. □

Now take all level n strips that contain at least $2^{n/3}$ pieces. The lemma yields that these strips cover K. Our goal is to prove that the number of such strips is *very small* (compared to 3^n). Since the width of a level n strip is 3^{-n}, this would imply that the total width is also *very small*.

For a given strip we distinguish two different ways it can cover a piece. A piece connects two points lying on the border lines of the strip. If these endpoints lie on the same border line, then we say that it is a *border piece*. If, on the other hand, its endpoints are on different border lines, then it is a *crossing piece*. Note that a piece can have different types when covered by different strips. In fact, for each level n piece out of the three level n strips covering the piece, two cover it as a crossing piece and one covers it as a border piece.

To every strip we associate a two dimensional vector called the *covering vector*, the first and second coordinate of which denotes the number of border and the number of crossing pieces in the strip, respectively. Clearly, the covering number of a strip is simply the sum of the coordinates of the covering vector. First we show that the covering vector of a strip determines the covering vectors corresponding to the three offspring strips. (By *offspring strips* of a level n strip we mean the three level $n+1$ strips contained in the strip.)

Proposition 4.3. *A covering vector (v_1, v_2) yields the following three vectors on the next level:*

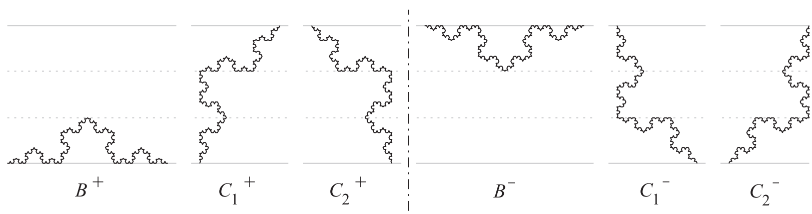
$$(2v_1, 2v_1 + v_2); \quad (0, v_2); \quad (v_2, v_2).$$

In other words, to get a next level covering vector we simply right-multiply with one of the three 2×2 matrices below:

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proof. Take an arbitrary strip and the pieces covered by the strip. Theoretically, there are 6 possible types of these pieces (two types of border pieces labelled with B^+ and B^- in Figure 4.2 and four types of crossing pieces labelled with C_1^+, C_2^+, C_1^- and C_2^-.) However, the truth is that each strip has an *orientation* and depending on this orientation either all the pieces (covered by the strip) are of types B^+, C_1^+, C_2^+ or all of them are of types B^-, C_1^-, C_2^-. This can be proved by induction on the level of the strip: using Figure 4.2 the reader can easily check that the middle offspring strip always changes orientation while the other two offspring strips have the same orientation as the original strip. Now the statement of the proposition is immediate. ☐

Figure 4.2: The different types of pieces covered by a strip



Now we fix a direction (0, 1 or 2) and take the level 0 strip in this direction. The covering vector $\mathbf{v}$ associated to this strip is either $(1,0)$ or $(0,1)$ depending on the direction. The covering vectors of level n strips in the fixed direction can be obtained in the following way. We take the product of n matrices, each matrix being A, B or C and right-multiply $\mathbf{v}$ with this product matrix. If we do this for all possible 3^n products, then we get the covering vectors of all 3^n level n strips in the fixed direction.

So we need to compute such matrix products. It is not that complicated due to the following relations between A,B and C:

$$BA = B; \quad BB = B; \quad BC = C; \quad CC = C. \tag{4.1}$$

So there are a lot of cancellations in such a product: a matrix B cancels all the subsequent A's and B's until a C comes which cancels B. (For example, $BAABAC = BC = C$.) Also, if there are more than one successive C's, then we can write only one C instead. After all possible cancellations have been done we get a product of the following form:

$$(C)A^{k_1}CA^{k_2}C \cdots CA^{k_r}(B \text{ or } C).$$

By induction, we get that

$$A^k = \begin{pmatrix} 2^k & 2^{k+1}-2 \\ 0 & 1 \end{pmatrix}, \text{ so } CA^k = \begin{pmatrix} 0 & 0 \\ 2^k & 2^{k+1}-1 \end{pmatrix}.$$

Now it is easy to see that the sum of the elements in the product matrix is at most

$$L \cdot 2^{(k_1+1)+(k_2+1)+\cdots+(k_r+1)} \leq 2^{c_0+reduced\_length},$$

where L, c_0 are absolute constants and *reduced_length* denotes the length of the product after the cancellations.

The covering number of a strip is the sum of the elements in the covering vector which is bounded above by the sum of the elements in the corresponding product matrix that has been shown to be at most $2^{c_0+reduced\_length}$. So we have proved that

$$covering\_number \leq 2^{c_0+reduced\_length}. \tag{4.2}$$

Now we forget for a moment that A, B, C denote matrices. We just take a random sequence of letters A, B, C, choosing every letter independently and with uniform distribution. We do all the cancellations implied by the relations in (4.1). The reduced length of the sequence is defined as the number of letters that *survive* cancellation. The next lemma claims that the reduced length of a random sequence of length n is less than $n/3 - c_0$ with high probability.

Lemma 4.4. *There exists a constant $a < 1$ such that*

$$P(\text{the reduced length of a random sequence of length } n \text{ is at least } n/3 - c_0) < a^n.$$

Before proving this lemma, we first show how it can be used to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\mathcal{S}_i^n$ be the set of level n strips in direction i with covering number at least $2^{n/3}$, and put $\mathcal{S}^n = \mathcal{S}_0^n \cup \mathcal{S}_1^n \cup \mathcal{S}_2^n$. On one hand, Lemma 4.2 yields that $\mathcal{S}^n$ is a covering of K. On the other hand, (4.2) and Lemma 4.4 entail that a random level n strip is in $\mathcal{S}^n$ with probability less than a^n for some constant $a < 1$. Thus $|\mathcal{S}^n| < 3a^n 3^n$. It follows that K is tube-null for $\mathcal{S}^n$ has total width at most $3a^n$.

To obtain the decomposition claimed in the theorem we define the set K_i^n as the set of those points in K which are covered by at least one strip in $\mathcal{S}_i^n$. Since $\mathcal{S}^n$ is a covering of K, $K = K_0^n \cup K_1^n \cup K_2^n$. Set

$$K_i := \{x : x \in K_i^n \text{ for infinitely many values of } n\}.$$

Clearly, $K = K_0 \cup K_1 \cup K_2$. By definition, K_i is covered by $\mathcal{S}_i^m \cup \mathcal{S}_i^{m+1} \cup \ldots$ for any positive integer m. Let π_i be the projection in direction i. Then $\pi_i(K_i)$ is covered by $\pi_i(\cup \mathcal{S}_i^m) \cup \pi_i(\cup \mathcal{S}_i^{m+1}) \cup \ldots$ where $\pi_i(\cup \mathcal{S}_i^n)$ is the union of at most $(3a)^n$ segments of length 3^{-n}. It easily follows that $\pi_i(K_i)$ has Hausdorff dimension at most $s = \log_3(3a) < 1$. $\square$

Proof of Lemma 4.4. First we give a heuristic proof. A typical sequence contains about $n/3$ of each letter. About half of the A's survive (depending on whether the first preceding non-A letter is B or C), basically no B's survive and about one third of the C's survive (depending on whether the next letter is A or not). Thus the reduced length of a typical sequence is about $n/3(1/2 + 0 + 1/3) = 5n/18$. In the sequel we make these heuristics precise.

First we compute the expected value of the reduced length of a random sequence of length n. Consider the letter in position k. We will determine the probability that this letter survives cancellation. Clearly, the sum of these probabilities is the expected value in question. However, for these probabilities to be well defined we need to agree on which letter is cancelled in case of two successive B's or C's. When we have two successive B's, let the first B survive and the second one be cancelled. On the other hand, for two successive C's let the first be cancelled and the second survive. (In other words, B's have a *forward-mouth* and they *eat* A's and other B's, while C's have a *backward-mouth* eating B's and other C's.) Now it is a well-defined question whether a letter survives or not. Let the random sequence be $M_1 M_2 \cdots M_n$.

Case $M_k = A$: with probability $1/3^{k-1}$ it holds that for each $i \leq k-1$ $M_i = A$ when M_k survives. If it is not so, then there is an index $i < k$ for which $M_i \neq A$ but $M_{i+1} = M_{i+2} = \cdots = M_k = A$. If $M_i = B$, then M_i cancels all the subsequent A's so it cancels M_k. If $M_i = C$, then M_k survives. The probability of this is clearly $(1-1/3^{k-1})/2$. Consequently:

$$P(M_k \text{ survives}|M_k = A) = \frac{1}{2} + \frac{1}{2 \cdot 3^{k-1}}.$$

Case $M_k = B$: it survives only if M_i equals A or B for each $i \geq k+1$ (and even in this case it might be cancelled due to a preceding B):

$$P(M_k \text{ survives}|M_k = B) \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-k}.$$

Case $M_k = C$: if $M_{k+1} = A$, then M_k survives; if $M_{k+1} = C$, then M_k is cancelled. If $M_{k+1} = B$, then M_k survives if and only if M_{k+1} survives which holds if and only if M_i equals A or B for each $i \geq k+2$. Thus

$$P(M_k \text{ survives}|M_k = C) = \frac{1}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^{n-k-1} \quad (1 \leq k \leq n-1),$$
$$P(M_n \text{ survives}|M_n = C) = 1.$$

It follows that

$$P(M_k \text{ survives}) \leq \frac{5}{18} + \frac{1}{2 \cdot 3^k} + \frac{1}{9}\left(\frac{2}{3}\right)^{n-k} + \frac{1}{9}\left(\frac{2}{3}\right)^{n-k-1} \quad (1 \leq k \leq n-1).$$

When we add up these terms, the sum of the geometric progressions will be bounded so there exists an absolute constant c_1 such that

$$E_n := E(\text{reduced length of a random sequence of length } n) \leq \frac{5}{18}n + c_1.$$

Let $0 < \varepsilon < 1/36$ and let us fix n_0 in such a way that $E_{n_0} < (1/3 - 2\varepsilon)n_0$. Now let $n = kn_0$ for some positive integer k. We take a random sequence of length n and split it up into subsequences of length n_0. Let X_j be the random variable defined as the reduced length of the j-th subsequence ($j = 1, 2, \ldots, k$), and let X be the reduced length of the whole sequence. Clearly, $X \leq X_1 + \cdots + X_k$. The X_j's are independent random variables with $E(X_j) = E_{n_0}$ and $X_j \in (0, n_0]$. We know that under these conditions the sum $X_1 + \cdots + X_k$ is highly concentrated around its expectation which is $kE_{n_0} < (1/3 - 2\varepsilon)n$. For example, we can use Hoeffding's inequality [23] (since $X_1, \ldots, X_k$ are independent and bounded). For sufficiently large k it holds that $c_0 < \varepsilon n$, thus

$$P\left(X \geq \frac{n}{3} - c_0\right) \leq P\left(X > \left(\frac{1}{3} - \varepsilon\right)n\right) \leq P\left(\sum_{j=1}^{k}(X_j - E_{n_0}) > \varepsilon n_0 k\right) <$$
$$\exp\left(-\frac{2\varepsilon^2 n_0^2 k^2}{kn_0^2}\right) = a^n$$

for some constant $a < 1$. This already proves the lemma for n's that are sufficiently large multiples of n_0. However, with a larger $a < 1$ the lemma clearly holds for arbitrary positive integer n. $\square$

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Abstract

The thesis addresses problems from the field of geometric measure theory. It turns out that discrete methods can be used efficiently to solve these problems. Let us summarize the main results of the thesis.

In Chapter 2 we investigate the following question proposed by Tamás Keleti. How large (in terms of Hausdorff dimension) can a compact set $A \subset \mathbb{R}^n$ be if it does not contain some given angle α, that is, it does not contain distinct points $P, Q, R \in A$ with $\angle PQR = \alpha$? Or equivalently, how large dimension guarantees that our set must contain α?

We also study an approximate version of this problem, where we only want our set to contain angles close to α rather than contain the exact angle α. This version turns out to be completely different from the original one, which is best illustrated by the case $\alpha = \pi/2$. If the dimension of our set is greater than 1, then it must contain angles arbitrarily close to $\pi/2$. However, if we want to make sure that it contains the exact angle $\pi/2$, then we need to assume that its dimension is greater than $n/2$.

Another interesting phenomenon is that different angles show different behaviour. In the approximate version the angles $\pi/3$, $\pi/2$ and $2\pi/3$ play special roles, while in the original version $\pi/2$ seems to behave differently than other angles.

The investigation of the above problems led us to the study of the so-called acute sets. A finite set $\mathcal{H}$ in $\mathbb{R}^n$ is called an acute set if any angle determined by three points of $\mathcal{H}$ is acute. Chapter 3 of the thesis studies the maximal cardinality $\alpha(n)$ of an n-dimensional acute set. The exact value of $\alpha(n)$ is known only for $n \leq 3$. For each $n \geq 4$ we improve on the best known lower bound for $\alpha(n)$. We present different approaches. On one hand, we give a probabilistic proof that $\alpha(n) > c \cdot 1.2^n$. (This improves a random construction given by Erdős and Füredi.) On the other hand, we give an *almost* exponential constructive example which outdoes the random construction in low dimension ($n \leq 250$). Both approaches use the small dimensional examples that we found partly by hand ($n = 4, 5$), partly by computer ($6 \leq n \leq 10$).

Finally, in Chapter 4 we show that the Koch curve is tube-null, that is, it can be covered by strips of arbitrarily small total width. In fact, we prove the following stronger result: the Koch curve can be decomposed into three sets such that each can be projected to a line in such a way that the image has Hausdorff dimension less than 1. The proof contains geometric, combinatorial, algebraic and probabilistic arguments.

Összefoglalás

Az értekezés olyan problémákat vizsgál a geometriai mértékelmélet területéről, amelyek megoldásánál különböző diszkrét módszerek rendkívül hasznosnak bizonyultak. Röviden ismertetjük az értekezés főbb eredményeit.

A második fejezetben a következő, Keleti Tamástól származó kérdést járjuk körül. Mekkora lehet (Hausdorff dimenzió szempontjából) egy $A \subset \mathbb{R}^n$ kompakt halmaz, amely nem tartalmaz valamilyen adott α szöget, azaz nem tartalmaz különböző $P, Q, R \in A$ pontokat, melyekre $\angle PQR = \alpha$? Avagy másik megfogalmazásban: mekkora dimenzió garantálja, hogy a halmazunk biztosan tartalmaz α szöget?

Az értekezés vizsgálja a fenti probléma egy approximatív változatát is. Ahelyett, hogy azt akarnánk garantálni, hogy a halmazban található pontosan α szög, ez esetben megelégszünk azzal, ha α-hoz közeli szöget találunk. Ez a probléma jelentősen különbözik az eredetitől, amit legjobban az $\alpha = \pi/2$ eset illusztrál. Ha a dimenzió 1-nél nagyobb, akkor a halmaz biztosan tartalmaz $\pi/2$-höz akármilyen közeli szögeket. Azonban csak $n/2$-nél nagyobb dimenzió esetén lehetünk abban biztosak, hogy a halmazunk tartalmazza $\pi/2$-t.

Ugyancsak meglepő, hogy más-más α szögek esetén a fenti kérdésekre egészen más válaszokat kapunk. Az approximatív változatban a $\pi/3$, $\pi/2$ és $2\pi/3$ szögeknek különleges szerepük van, míg az eredeti problémában $\pi/2$ mutat a többi szögtől eltérő viselkedést.

A fent vázolt problémák tanulmányozása vezetett el az úgynevezett hegyes halmok vizsgálatához. Egy véges $\mathcal{H} \subset \mathbb{R}^n$ halmazt hegyes halomnak nevezünk, ha bármely három pontja által meghatározott szög hegyesszög. Az értekezés harmadik fejezete az n-dimenziós hegyes halmok $\alpha(n)$ maximális számosságát vizsgálja. Ennek a pontos értéke csak $n \leq 3$ esetén ismert. Az értekezésben minden $n \geq 4$ esetén javítunk a legjobb ismert alsó becslésen. Két megközelítést mutatunk be. Egyrészt valószínűségi módszerrel bebizonyítjuk, hogy $\alpha(n) > c \cdot 1.2^n$ (javítva ezzel Erdős és Füredi egy véletlen konstrukcióján). Másrészt egy teljesen konstruktív eljárást is ismertetünk, ami alacsony dimenzióban ($n \leq 250$) nagyobb hegyes halmot ad, mint a véletlen módszer. Mindkét megközelítésben használjuk azokat a kis dimenziós hegyes halmokat, amiket részben kézzel konstruáltunk ($n = 4, 5$), részben pedig számítógéppel találtunk ($6 \leq n \leq 10$).

Végül a negyedik fejezetben belátjuk, hogy a Koch görbe tubus-nulla, azaz lefedhető akármilyen kis összszélességű sávokkal. Valójában a következő erősebb állítást bizonyítjuk: a Koch görbe felosztható három részre, melyek mindegyikére fennáll, hogy alkalmas egyenesre vetítve a vetület Hausdorff dimenziója kisebb, mint 1. A bizonyítás geometriai, kombinatorikai, algebrai és valószínűségszámítási eszközöket is használ.