PhD Thesis

First-Order Logic Investigation of Relativity Theory with an Emphasis on Accelerated Observers

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Chapter 1

Introduction

This work is a continuation of the works by Andréka, Madarász, Németi and their coauthors, e.g., [1], [2], [3], [4], [33]. Our research is directly related to Hilbert's sixth problem of axiomatization of physics. Moreover, it goes beyond this problem since its general aim is not only to axiomatize physical theories but to investigate the relationship between basic assumptions (axioms) and predictions (theorems). Another general aim of ours is to provide a foundation for physics similar to that of mathematics.

For good reasons, the foundation of mathematics was performed strictly within first-order logic (FOL). One of the reasons is that staying within FOL helps to avoid tacit assumptions. Another reason is that FOL has a complete inference system while second-order logic (and thus any higher-order logic) cannot have one, see, e.g., [20, §IX. 1.6]. For further reasons for staying within FOL, see, e.g., Chap. 11 and [10], [2, §Why FOL?], [77], [81].

Why is it useful to apply the axiomatic method to relativity theory? For one thing, this method makes it possible for us to understand the role of any particular axiom. We can check what happens to our theory if we drop, weaken or replace an axiom. For instance, it has been shown by this method that the impossibility of faster than light motion is not independent from the other assumptions of special relativity, see [2, §3.4], [3]. More boldly: it is superfluous as an axiom because it is provable as a theorem from much simpler and more convincing basic assumptions. The linearity of the transformations between observers (reference frames) can also be proven from some plausible assumptions, therefore it need not be assumed as an axiom, see Thm. 3.2.2 and [2], [3]. Getting rid of unnecessary axioms of a physical theory is important because we do not know whether an axiom is true or not, we just assume so. We can only be sure of experimental facts but they typically correspond not to axioms but to (preferably existentially quantified) intended theorems.

Not only can we get rid of superfluous assumptions by applying the axiomatic method, but we can discover new, interesting and physically relevant theories. That happened in the case of the axiom of parallels in Euclid's geometry; and this kind of investigation led to the discovery of hyperbolic geometry. Our FOL theory of accelerated observers (AccRel), which nicely fills the gap between special and general relativity theories, is also a good example of such a theory.

Moreover, if we have an axiom system, we can ask which axioms are responsible for a certain consequence of our theory. This kind of reverse thinking can help us to answer the why-type questions of relativity. For example, we can take the twin paradox theorem and check which axiom of special relativity was and which one was not needed to derive it. The weaker an axiom system is, the better answer it offers to the question: "Why is the twin paradox true?". The twin paradox is investigated in this manner in Chap. 7, while its inertial approximation (called the clock paradox) in Chap. 4. For details on answering why-type questions of relativity by the methodology of the present work, see [73]. We hope that we have given good reasons why we use the axiomatic method in our research into spacetime theories. For more details and further reasons, see, e.g., Guts [29], Schutz [61], Suppes [66].

This work is structured in the following way: in Chap. 2 we introduce our FOL frame and our basic notation; then, in Chap. 3, we recall a FOL axiomatization of special relativity by our research group. Based on this axiomatization first we investigate the logical connection between the clock paradox theorem and the axiom system in Chap. 4. First we give a geometrical characterization theorem for the clock paradox, see Thm. 4.3.6; then we prove some surprising consequences for both Newtonian and relativistic kinematics. Thm. 4.5.3 answers Question 4.2.17 of Andréka–Madarász–Németi [2].

In Chap. 5 we extend our geometrical approach to relativistic dynamics and investigate the relations between our purely geometrical key axioms of dynamics and the conservation postulates of the standard approaches. For example, we show that the conservation postulates are not needed to prove the relativistic mass increase theorem $m_0 = \sqrt{1 - v^2/c^2} \cdot m$, which is the first step to capture Einstein's insight $E = mc^2$.

In Chap. 6 we extend the theory introduced in Chap. 3 to accelerated observers by introducing our aforementioned theory AccRel, which is the main subject of this thesis. In Chap. 7 we investigate the twin paradox within AccRel; we show that a nontrivial assumption is required if we want the twin paradox to be a consequence of our theory AccRel. In Chap. 8 we prove two formulations of the gravitational time dilation from a streamlined and small set of axioms (AccRel), by using Einstein's equivalence principle.

In Chap. 9 we "derive" a FOL axiom system of general relativity from our theory AccRel in one natural step. The technical parts of the proofs and the development of the necessary tools are presented in Chap. 10. And in Chap. 11 we go into the details of the reasons for choosing FOL in our investigation.

CONVENTION 1.0.1. Throughout this work, there appear "highlighted" statements, such as $\mathsf{AxCenter}^+$ in Chap. 5, which associate the name $\mathsf{AxCenter}^+$ with a formula of our FOL language. It is important to note that these formulas are not automatically elevated to the rank of axiom. Instead, they serve as potential axioms or even as potential statements to appear in theorems, hence they are nothing more than formulas distinguished in our language.

We try to be as self-contained as possible. First occurrences of concepts used in this work are set in boldface to make them easier to find. We also use colored text and boxes to help the reader to find the axioms, notations, etc. Throughout this work, if-and-only-if is abbreviated as **iff**. We hope that the Index at the end of this thesis also helps find the individual definitions and notations introduced.

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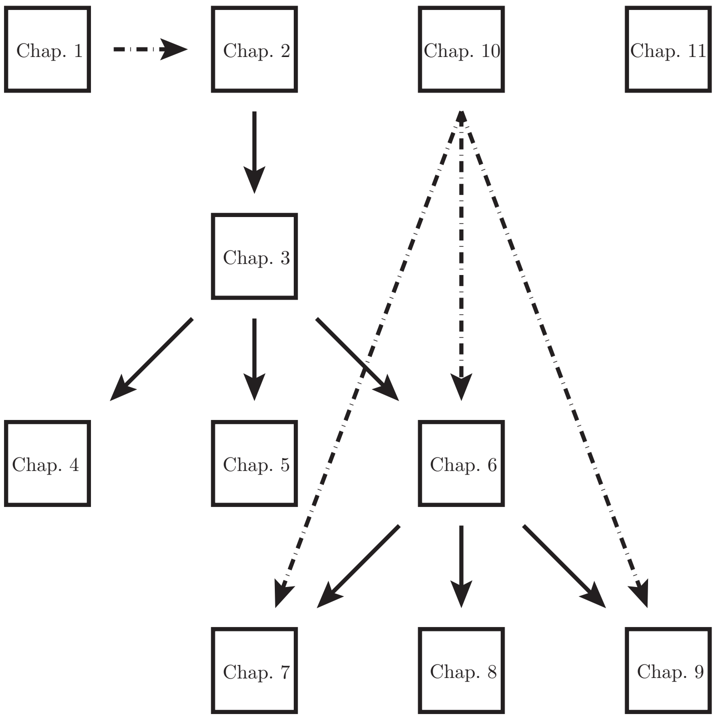

Figure 1.1: Illustration of the connection of the Chapters

Chapter 2

FOL frame

In this chapter we specify the FOL frame within which we will work.

2.1 Frame language

Our basic concepts are explained as follows. This thesis mainly deals with the kinematics of relativity, i.e., with the motion of *bodies* (test-particles). However, we briefly discuss dynamics in Chap. 5, and our co-authored papers [6], [7] and [38] are fully devoted to dynamics. We represent motion as the changing of spatial location in time. Thus we use reference frames for coordinatizing events (sets of bodies). *Quantities* are used for marking time and space. The structure of quantities is assumed to be an ordered field in place of the field of real numbers. For simplicity, we associate reference frames with certain bodies called *observers*. The coordinatization of events by observers is formulated by means of the *world-view relation*. We visualize an observer as "sitting" at the origin of the space part of its reference frame, or equivalently, "living" on the time-axis of the reference frame. We distinguish *inertial* and *noninertial* observers. For the time being, inertiality is only a label on observers which will be defined later by our axioms. We also use another special kind of bodies called *photons*. We use photons only for labeling light paths, so here we do not consider any of their quantum dynamical properties.

In an axiomatic approach to relativity, it is more natural to take relations of bodies (particles) instead of events as basic concepts. That is not uncommon in the literature, see, e.g., Ax [10], Benda [14]. However, a large variety of basic concepts occur in the different axiomatizations of special relativity, see, e.g., Goldblatt [28], Mundy [45, 46], Pambuccian [48], Robb [52], [53] Suppes [67], Schutz [61], [63], [62].

Allowing ordered fields in place of the field of real numbers increases the flexibility of our theory and minimizes the amount of our mathematical presuppositions. For further motivation in this direction, see, e.g., Ax [10]. Similar remarks apply to our flexibility-oriented decisions below, e.g., to treat the dimension of spacetime as a variable.

Using observers in place of coordinate systems or reference frames is only a matter of didactic convenience and visualization. There are many reasons for using observers (or coordinate systems, or reference frames) instead of a single observer-independent spacetime structure. One is that it helps to weed unnecessary axioms from our theories. Nevertheless, we state and emphasize the equivalence between observer-oriented and observer-independent approaches to relativity theory, see [4, §3.6], [33, §4.5].

Keeping the foregoing in mind, let us now set up the FOL language of our axiom systems. First we fix a natural number $d \geq 2$ for the dimension of spacetime. Our language contains the following nonlogical symbols:

- unary relation symbols B (**bodies**), Ob (**observers**), IOb (**inertial observers**), Ph (**photons**) and Q (**quantities**);
- binary function symbols $+$, $\cdot$ and a binary relation symbol $<$ (field operations and ordering on Q); and
- a $2+d$-ary relation symbol W (**world-view relation**).

$\mathrm{B}(x)$, $\mathrm{Ob}(x)$, $\mathrm{IOb}(x)$, $\mathrm{Ph}(x)$ and $\mathrm{Q}(x)$ are translated as "x is a body," "x is an observer," "x is an inertial observer," "x is a photon" and "x is a quantity," respectively. We use the world-view relation W to speak about coordinatization by translating $\mathrm{W}(x, y, z_1, \ldots, z_d)$ as "observer x coordinatizes body y at spacetime location $\langle z_1, \ldots, z_d\rangle$," (i.e., at space location $\langle z_2, \ldots, z_d\rangle$ and at instant z_1).

$\mathrm{B}(x)$, $\mathrm{Ob}(x)$, $\mathrm{IOb}(x)$, $\mathrm{Ph}(x)$, $\mathrm{Q}(x)$, $\mathrm{W}(x, y, z_1, \ldots, z_d)$, $x = y$ and $x < y$ are the so-called atomic formulas of our FOL language, where x, y, $z_1, \ldots, z_d$ can be arbitrary variables or terms built up from variables by using the field operations. The **formulas** of our FOL language are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$), *if-and-only-if* ($\leftrightarrow$), and the quantifiers *exists* x ($\exists x$) and *for all* x ($\forall x$) for every variable x. To abbreviate formulas of FOL we often omit parentheses according to the following convention: quantifiers bind as long as they can, and $\wedge$ binds stronger than $\rightarrow$. For example, we write $\forall x\ \ \varphi \wedge \psi \rightarrow \exists y\ \ \delta \wedge \eta$ instead of $\forall x\big((\varphi \wedge \psi) \rightarrow \exists y(\delta \wedge \eta)\big)$.

We use the notation $\mathrm{Q}^n := \mathrm{Q} \times \ldots \times \mathrm{Q}$ (n-times) for the set of all n-tuples of elements of Q. If $\vec{p} \in \mathrm{Q}^n$, we assume that $\vec{p} = \langle p_1, \ldots, p_n\rangle$, i.e., $p_i \in \mathrm{Q}$ denotes the i-th component of the n-tuple $\vec{p}$. Specially, we write $\mathrm{W}(m,b,\vec{p})$ in place of $\mathrm{W}(m,b,p_1,\ldots,p_d)$, and we write $\forall\vec{p}$ in place of $\forall p_1\ldots\forall p_d$, etc.

To abbreviate formulas, we also use bounded quantifiers in the following way: $\exists x\ \varphi(x)\wedge\psi$ and $\forall x\ \varphi(x)\rightarrow\psi$ are abbreviated to $\exists x\in\varphi\ \psi$ and $\forall x\in\varphi\ \psi$, respectively. For example, to formulate that every observer observes a body somewhere, we write

$$\forall m\in\mathrm{Ob}\ \exists b\in\mathrm{B}\ \exists\vec{p}\in\mathrm{Q}^d\quad W(m,b,\vec{p})$$

instead of

$$\forall m\ \mathrm{Ob}(m)\rightarrow\exists b\ \mathrm{B}(b)\wedge\exists\vec{p}\ \ \mathrm{Q}(p_1)\wedge\ldots\wedge\mathrm{Q}(p_d)\wedge W(m,b,\vec{p}).$$

We use FOL set theory as a metatheory to speak about model theoretical concepts, such as models, validity, etc.

The **models** of this language are of the form

$$\mathfrak{M}=\langle U;\mathrm{B}_{\mathfrak{M}},\mathrm{Ob}_{\mathfrak{M}},\mathrm{IOb}_{\mathfrak{M}},\mathrm{Ph}_{\mathfrak{M}},\mathrm{Q}_{\mathfrak{M}},+_{\mathfrak{M}},\cdot_{\mathfrak{M}},<_{\mathfrak{M}},\mathrm{W}_{\mathfrak{M}}\rangle,$$

where U is a nonempty set, and $\mathrm{B}_{\mathfrak{M}}$, $\mathrm{Ob}_{\mathfrak{M}}$, $\mathrm{IOb}_{\mathfrak{M}}$, $\mathrm{Ph}_{\mathfrak{M}}$ and $\mathrm{Q}_{\mathfrak{M}}$ are unary relations on U, etc. Formulas are interpreted in $\mathfrak{M}$ in the usual way.

Let Σ and Γ be sets of formulas, and let φ and ψ be formulas of our language. Then Σ **logically implies** φ, in symbols $\Sigma\models\varphi$, iff φ is true in every model of Σ, (i.e., φ is a logical consequence of Σ). $\Sigma\not\models\varphi$ denotes that there is a model of Σ in which φ is not true. To simplify our notations, we use the plus sign between formulas and sets of formulas in the following way: $\Sigma+\Gamma:=\Sigma\cup\Gamma$, $\varphi+\psi:=\{\varphi,\psi\}$ and $\Sigma+\varphi:=\Sigma\cup\{\varphi\}$.

Remark 2.1.1. Let us note that the fewer axioms Σ contains, the stronger the logical implication $\Sigma\models\varphi$ is, and similarly the more axioms Σ contains the stronger the counterexample $\Sigma\not\models\varphi$ is.

Remark 2.1.2. By Gödel's completeness theorem, all the theorems of this thesis remain valid if we replace the relation of logical consequence ($\models$) by the deducibility relation of FOL ($\vdash$).

2.2 Frame axioms

Here we introduce two axioms that are going to be treated as part of our logic frame. Our first axiom expresses very basic assumptions, such as: both photons and observers are bodies, inertial observers are also observers, etc.

AxFrame $\mathrm{Ob} \cup \mathrm{Ph} \subseteq \mathrm{B}$, $\mathrm{IOb} \subseteq \mathrm{Ob}$, $\mathrm{W} \subseteq \mathrm{Ob} \times \mathrm{B} \times \mathrm{Q}^d$, $\mathrm{B} \cap \mathrm{Q} = \emptyset$; $+$ and $\cdot$ are binary operations, and $<$ is a binary relation on Q[1].

Instead of using this axiom we could also use many-sorted FOL language as in [2] and [4], and only assume that $\mathrm{IOb} \subseteq \mathrm{Ob}$.

To be able to add, multiply and compare measurements of observers, we provide an algebraic structure for the set of quantities with the help of our next axiom.

AxEOF The **quantity part** $\langle \mathrm{Q}; +, \cdot, < \rangle$ is a Euclidean ordered field, i.e., a linearly ordered field in which positive elements have square roots.

For the FOL definition of linearly ordered field, see, e.g., [15]. We use the usual field operations $0, 1, -, /, \sqrt{\ }$ and binary relation $\leq$, definable within FOL. We also use the vector-space structure of Q^n, i.e., $\vec{p} + \vec{q}, -\vec{p}, \lambda \cdot \vec{p} \in \mathrm{Q}^n$ if $\vec{p}, \vec{q} \in \mathrm{Q}^n$ and $\lambda \in \mathrm{Q}$; and $\vec{o} := \langle 0, \ldots, 0 \rangle$ denotes the **origin**.

Convention 2.2.1. We treat AxFrame and AxEOF as part of our logic frame. Hence without any further mentioning, they are always assumed and will be part of each axiom system we propose herein, except in some of the theorems of Chap. 10.

2.3 Basic definitions and notations

Let us collect here the basic definitions and notations that are going to be used in the following chapters.

Remark 2.3.1. In our formulas we seek to use only FOL definable concepts. So we will always warn the reader whenever we introduce a concept which is not FOL definable in our language.

The ordered field of real numbers, which is not FOL definable in our language, is denoted by $\mathbb{R}$. The **composition** of binary relations R and S is defined as:

$$R \circ S := \{ \langle a, c \rangle : \exists b \ \ \langle a, b \rangle \in R \ \wedge \ \langle b, c \rangle \in S \}.$$

The **domain** and the **range** of a binary relation R are denoted by

$$Dom\, R := \{ a \ : \ \exists b \ \ \langle a, b \rangle \in R \} \quad \text{and} \quad Ran\, R := \{ b \ : \ \exists a \ \ \langle a, b \rangle \in R \},$$

respectively. R^{-1} denotes the **inverse** of R, i.e.,

$$R^{-1} := \{ \langle b, a \rangle \ : \ \langle a, b \rangle \in R \}.$$

[1]These statements can easily be translated to our FOL language, e.g., formula $\forall xy \ x < y \rightarrow \mathrm{Q}(x) \wedge \mathrm{Q}(y)$ means that "$<$ is a binary relation on Q."

Remark 2.3.2. We think of a **function** as a special binary relation. Notation $f : A \to B$ denotes that f is a function from A to B, i.e., $Dom\, f = A$ and $Ran\, f \subseteq B$. Note that if f and g are functions, then

$$\boxed{(f \circ g)(x) = g\big(f(x)\big)}$$

for all $x \in Dom\, f \circ g$. Notation $f : A \xrightarrow{\circ} B$ denotes that f is a partial function on A, i.e., $Dom\, f \subseteq A$ and $Ran\, f \subseteq B$.

The **identity map** on $H \subseteq \mathrm{Q}^d$ is defined as:

$$Id_H := \{\, \langle \vec{p}, \vec{p} \rangle \in \mathrm{Q}^d \times \mathrm{Q}^d \;:\; \vec{p} \in H \,\},$$

and the **restriction** of a function f to a set H is defined as:

$$f\big|_H := \{\, \langle x, y \rangle \;:\; x \in Dom\, f \cap H \;\wedge\; f(x) = y \,\}.$$

The set of positive elements of Q is denoted by

$$\mathrm{Q}^+ := \{x \in \mathrm{Q} : 0 < x\},$$

and the different kinds of interval between $x, y \in \mathrm{Q}$ are defined as:

$$\begin{aligned}
(x,y) &:= \{\, t \in \mathrm{Q} \;:\; x < t < y \;\text{ or }\; y < t < x \,\},\\
[x,y] &:= \{\, t \in \mathrm{Q} \;:\; x \le t \le y \;\text{ or }\; y \le t \le x \,\},\\
[x,y) &:= \{\, t \in \mathrm{Q} \;:\; x \le t < y \;\text{ or }\; y < t \le x \,\}, \text{ and}\\
(x,y] &:= \{\, t \in \mathrm{Q} \;:\; x < t \le y \;\text{ or }\; y \le t < x \,\}.
\end{aligned}$$

We use this nonstandard but convenient notion of intervals to avoid inconveniences of empty intervals, such as $(1,0)$ in the standard notion. By our definition $(1,0)$ is not the empty set but the interval $(0,1)$.
For any $n \ge 1$, the **Euclidean length** of $\vec{p} \in \mathrm{Q}^n$ is defined as:

$$|\vec{p}\,| := \sqrt{p_1^2 + \ldots + p_n^2}.$$

Hence $|x|$ is the absolute value of x if $x \in \mathrm{Q}$. The **(open) ball** with center $\vec{p} \in \mathrm{Q}^d$ and radius $r \in \mathrm{Q}^+$ is defined as:

$$B_r(\vec{p}\,) := \{\, \vec{q} \in \mathrm{Q}^n \;:\; |\vec{p} - \vec{q}\,| < r \,\},$$

A set $G \subseteq \mathrm{Q}^n$ is called **open** iff for all $\vec{p} \in G$ there is an $\varepsilon \in \mathrm{Q}^+$ such that $B_\varepsilon(\vec{p}\,) \subset G$. A set $H \subseteq \mathrm{Q}$ is called **connected** iff $(x,y) \subseteq H$ for all $x, y \in H$. We say that a function $\gamma : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^d$ is a **curve** if $Dom\,\gamma$ is connected and has at least two distinct elements. The **standard basis vectors** of Q^d are denoted by $\vec{1}_i$, i.e.,

$$\vec{1}_i := \langle 0, \ldots, \overset{i}{1}, \ldots, 0 \rangle$$

for all $1 \le i \le d$. We also use notations $\vec{1}_t$, $\vec{1}_x$, $\vec{1}_y$ and $\vec{1}_z$ instead of $\vec{1}_1$, $\vec{1}_2$, $\vec{1}_3$, and $\vec{1}_4$, respectively. The line passing through $\vec{p}$ and $\vec{q}$ is defined as:

$$line(\vec{p}, \vec{q}) := \{\, \vec{p} + \lambda(\vec{p} - \vec{q}) \ :\ \lambda \in \mathrm{Q} \,\}.$$

Let us note that $line(\vec{p}, \vec{p}) = \{\vec{p}\}$ by this definition. It is practical to introduce a notation for the tx-plane:

$$tx\text{-Plane} := \left\{\, \vec{p} \in \mathrm{Q}^d \ :\ p_3 = \ldots = p_d = 0 \,\right\}.$$

2.4 Some fundamental concepts related to relativity

Let us gather here some fundamental definitions and notations which are used in the following chapters. The set Q^d is called the **coordinate system** and its elements are referred to as **coordinate points**. We use the notations

$$\vec{p}_\sigma := \langle p_2, \ldots, p_d \rangle \quad \text{and} \quad p_\tau := p_1$$

for the **space component** and for the **time component** of $\vec{p} \in \mathrm{Q}^d$, respectively. The **event** $ev_m(\vec{p})$ is defined as the set of bodies observed by observer m at coordinate point $\vec{p}$, i.e.,

$$ev_m(\vec{p}) := \{\, b \ :\ \mathrm{W}(m, b, \vec{p}) \,\}.$$

The function that maps $\vec{p}$ to $ev_m(\vec{p})$ is also denoted by ev_m. Event e is said to be **encountered** by observer k if there is a coordinate point $\vec{q}$ such that $k \in ev_k(\vec{q}) = e$. Let Ev_m denote the set of nonempty events coordinatized by observer m, i.e.,

$$Ev_m := \left\{\, e \ :\ \exists \vec{p} \in \mathrm{Q}^d \ \ ev_m(\vec{p}) = e \neq \emptyset \,\right\},$$

and let Ev denote the set of all observed events, i.e.,

$$Ev := \{\, e \ :\ \exists m \in \mathrm{Ob} \ \ e \in Ev_m \,\}.$$

We say that events e_1 and e_2 are **simultaneous** for observer m, in symbols $e_1 \sim_m e_2$, iff there are coordinate points $\vec{p}$ and $\vec{q}$ such that $ev_m(\vec{p}) = e_1$, $ev_m(\vec{q}) = e_2$, and $p_\tau = q_\tau$.

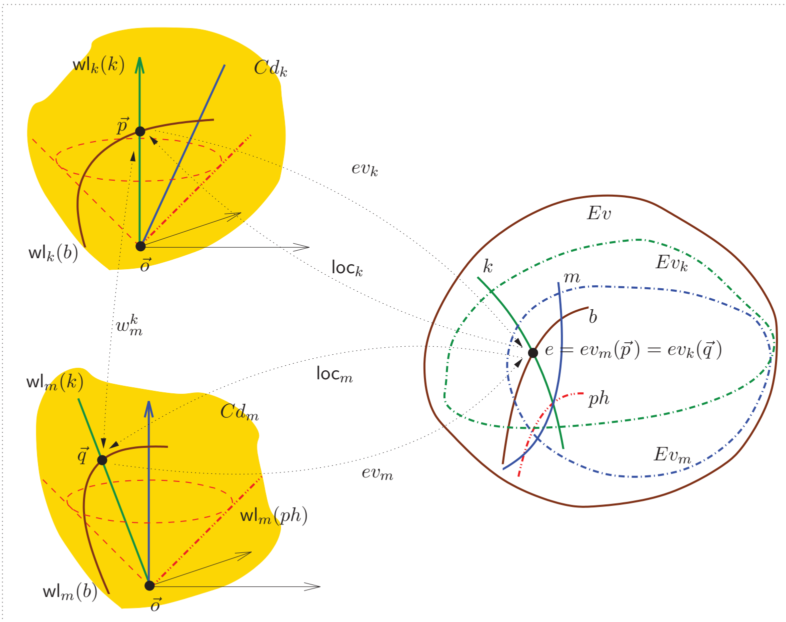

Figure 2.1: Illustration of the basic definitions

Remark 2.4.1. It is easy to see that $\sim_m$ is a reflexive and symmetric relation for every observer m; however, it is not an equivalence relation unless we assume further axioms.

The **coordinate-domain** of observer m, in symbols Cd_m, is the set of coordinate points where m observes something (a nonempty event), i.e.,

$$Cd_m := \{ \vec{p} \in \mathrm{Q}^d \ : \ ev_m(\vec{p}) \neq \emptyset \}.$$

The **world-view transformation** between the coordinate-systems of observers k and m is defined as the set of the pairs of coordinate points in which k and m coordinatize the same nonempty event, i.e.,

$$w^k_m := \{ \langle \vec{p}, \vec{q} \rangle \in \mathrm{Q}^d \times \mathrm{Q}^d \ : \ ev_k(\vec{p}) = ev_m(\vec{q}) \neq \emptyset \}.$$

Let us note that by this definition world-view transformations are only binary relations but axiom $\mathsf{AxPh_0}$, defined below on p.19, turns them into functions, see Prop. 3.1.3.

Convention 2.4.2. Whenever we write $w^k_m(\vec{p})$, we mean there is a unique $\vec{q} \in \mathrm{Q}^d$ such that $\langle \vec{p}, \vec{q} \rangle \in w^k_m$, and $w^k_m(\vec{p})$ denotes this unique $\vec{q}$. That is, if we talk about the value $w^k_m(\vec{p})$ of w^k_m at $\vec{q}$, we postulate that it exists and is unique (by the present convention).

Since in axiomatic approaches we only assume what is explicitly stated by the axioms, we have to prove every other statement, even the plausible ones. So let us prove a proposition here about the basic properties of world-view transformations.

Proposition 2.4.3. Let m and k be observers. Then

(1) $w^k_k \supseteq Id_{Cd_k}$, and

(2) $w^k_k = Id_{Cd_k}$ iff k does not see any nonempty event twice, i.e., w^k_k is a function.

(3) $w^k_m \circ w^m_k \supseteq Id_{Dom\, w^k_m}$, and

(4) $w^k_m \circ w^m_k = Id_{Dom\, w^k_m}$ iff w^k_m is injective.

(5) $w^k_h \circ w^h_m \supseteq w^k_m$, and

(6) $w^k_h \circ w^h_m = w^k_m$ iff $Ev_k \cap Ev_m \subseteq Ev_h$.

Proof. Items (1) and (2) can be easily proved by checking the respective definitions. To prove Item (3), let $\vec{p} \in Dom\, w^k_m$. Then, by our definitions, there is a $\vec{q} \in \mathrm{Q}^d$ such that $ev_k(\vec{p}) = ev_m(\vec{q}) \neq \emptyset$, i.e., $\langle \vec{p}, \vec{q} \rangle \in w^k_m$. Then, by our definition of world-view transformation, $\langle \vec{q}, \vec{p} \rangle \in w^m_k$. Consequently, $\langle \vec{p}, \vec{p} \rangle \in w^k_m \circ w^m_k$, which was to be proved.

Let us now prove Item (4). If w^k_m is not injective, there are $\vec{p}_1, \vec{p}_2, \vec{q} \in \mathrm{Q}^d$ such that $\vec{p}_1 \neq \vec{p}_2$ and $\langle \vec{p}_1, \vec{q} \rangle, \langle \vec{p}_2, \vec{q} \rangle \in w^k_m$. Then $\langle \vec{p}_1, \vec{p}_2 \rangle \in w^k_m \circ w^m_k$. So w^k_m has to be injective if $w^k_m \circ w^m_k = Id_{Dom\, w^k{}_m}$.
To prove the converse implication, let $\langle \vec{p}, \vec{r} \rangle \in w^k_m \circ w^m_k$. Then there is a $\vec{q} \in \mathrm{Q}^d$ such that $\langle \vec{p}, \vec{q} \rangle \in w^k_m$ and $\langle \vec{q}, \vec{r} \rangle \in w^m_k$. So $\langle \vec{r}, \vec{q} \rangle \in w^k_m$, and thus we get that $\vec{p} = \vec{r}$ by the injectivity of w^k_m. So $\langle \vec{p}, \vec{r} \rangle \in Id_{Dom\, w^k_m}$ as it was required.

Items (5) and (6) can be easily proved by checking the respective definitions. ■

The **world-line** of body b according to observer m is defined as the set of coordinate points where b was observed by m, i.e.,

$$\mathsf{wl}_m(b) := \{\, \vec{p} \in \mathrm{Q}^d \ : \ \mathrm{W}(m, b, \vec{p}\,) \,\}.$$

Let us note here that both $\vec{p} \in \mathsf{wl}_k(b)$ and $b \in ev_k(\vec{p}\,)$ represent the atomic formula $\mathrm{W}(k, b, \vec{p}\,)$, but from slightly different aspects.

The **location** $\mathsf{loc}_m(e)$ of event e according to observer m is defined as $\vec{p}$ if $ev_m(\vec{p}\,) = e$ and there is only one such $\vec{p} \in \mathrm{Q}^d$; otherwise $\mathsf{loc}_m(e)$ is undefined. Event e is called **localized** by observer m if it has a unique coordinate according to m, i.e., $\mathsf{loc}_m(e)$ is defined. To express that in our formulas, we use $\mathsf{Loc}_m(e)$ as an abbreviation for the following formula:

$$\exists \vec{p} \in \mathrm{Q}^d \quad ev_m(\vec{p}\,) = e \ \wedge \ \forall \vec{q} \in \mathrm{Q}^d \quad ev_m(\vec{q}\,) = e \ \ \rightarrow \ \ \vec{p} = \vec{q}\,.$$

CONVENTION 2.4.4. We use the equation sign "=" in the sense of existential equality, i.e., $\alpha = \beta$ denotes that both α and β are defined and they are equal. We also use the same convention for other relations (e.g., for "<"). See [33, Conv.2.3.10, p.31] and [2, Conv.2.3.10, p.61].

Remark 2.4.5. Let us note that $\mathsf{loc}_k(e) = \vec{p}$ means that $ev_k(\vec{p}\,) = e$ and $\vec{p} = \vec{q}$ for all $\vec{q}$ for which $ev_k(\vec{q}\,) = e$ by Conv. 2.4.4.

Remark 2.4.6. Let us note that $\mathsf{loc}_m\big(ev_k(\vec{p}\,)\big)$ is defined iff $w^k_m(\vec{p}\,)$ is so and in this case they are the same, i.e.,

$$w^k_m(\vec{p}\,) = \mathsf{loc}_m\big(ev_k(\vec{p}\,)\big).$$

The **time of event** e according to observer m is defined as:

$$\mathsf{time}_m(e) := \mathsf{loc}_m(e)_\tau$$

if e is localized by m; otherwise $\mathsf{time}_m(e)$ is undefined. The **elapsed time** between events e_1 and e_2 measured by observer m is defined as:

$$\mathsf{time}_m(e_1, e_2) := |\mathsf{time}_m(e_1) - \mathsf{time}_m(e_2)|$$

if e_1 and e_2 are localized by m; otherwise $\mathsf{time}_m(e_1, e_2)$ is undefined. $\mathsf{time}_m(e_1, e_2)$ is called the **proper time** measured by m between e_1 and e_2 if $m \in e_1 \cap e_2$. The **spatial location** of event e according to observer m is defined as:

$$\mathsf{space}_m(e) := \mathsf{loc}_m(e)_\sigma$$

if e is localized by m; otherwise $\mathsf{space}_m(e)$ is undefined. The **spatial distance** between events e_1 and e_2 according to observer m is defined as:

$$\mathsf{dist}_m(e_1, e_2) := |\mathsf{space}_m(e_1) - \mathsf{space}_m(e_2)|$$

if e_1 and e_2 are localized by m; otherwise $\mathsf{dist}_m(e_1, e_2)$ is undefined.

Spacetime vector $\vec{r} \in \mathrm{Q}^d$ is called **spacelike** iff $|\vec{r}_\sigma| > |r_\tau|$, **lightlike** iff $|\vec{r}_\sigma| = |r_\tau|$, and **timelike** iff $|\vec{r}_\sigma| < |r_\tau|$. Spacetime vectors $\vec{p}$ and $\vec{q}$ are called **spacelike-separated**, in symbols $\vec{p} \; \sigma \; \vec{q}$, iff $\vec{p} - \vec{q}$ is a spacelike vector; **lightlike-separated**, in symbols $\vec{p} \; \lambda \; \vec{q}$, iff $\vec{p} - \vec{q}$ is a lightlike vector; and **timelike-separated**, in symbols $\vec{p} \; \tau \; \vec{q}$, iff $\vec{p} - \vec{q}$ is a timelike vector. Events e_1 and e_2 which are localized by every *inertial* observer are called spacelike-separated (lightlike-separated; timelike-separated), in symbols $e_1 \, \sigma \, e_2$ ($e_1 \, \lambda \, e_2$; $e_1 \, \tau \, e_2$), iff $\mathsf{loc}_m(e_1)$ and $\mathsf{loc}_m(e_2)$ are such for every *inertial* observer m. A curve γ is called **timelike** iff it is differentiable (see p.105), and $\gamma'(t)$ is timelike for all $t \in Dom\,\gamma$.

Coordinate points $\vec{p}$ and $\vec{q}$ are called **Minkowski orthogonal**, in symbols $\vec{p} \perp_\mu \vec{q}$, iff the following holds: $p_\tau \cdot q_\tau = p_2 \cdot q_2 + \ldots + p_d \cdot q_d$. The (signed) **Minkowski length** of $\vec{p} \in \mathrm{Q}^d$ is

$$\mu(\vec{p}) := \begin{cases} \sqrt{p_\tau^2 - |\vec{p}_\sigma|^2} & \text{if} \quad p_\tau^2 \geq |\vec{p}_\sigma|^2, \\ -\sqrt{|\vec{p}_\sigma|^2 - p_\tau^2} & \text{in other cases,} \end{cases}$$

and the **Minkowski distance** between $\vec{p}$ and $\vec{q}$ is $\mu(\vec{p}, \vec{q}) := \mu(\vec{p} - \vec{q})$. We use the signed version of the Minkowski length because it contains two kinds of information: (i) the length of $\vec{p}$, and (ii) whether it is spacelike, lightlike or timelike. A map $f : \mathrm{Q}^d \to \mathrm{Q}^d$ is called a **Poincaré transformation** iff it is a transformation preserving the Minkowski distance, i.e., $\mu\big(f(\vec{p}), f(\vec{q})\big) = \mu(\vec{p}, \vec{q})$ for all $\vec{p}, \vec{q} \in \mathrm{Q}^d$. Like transformations preserving Euclidean distance, Poincaré transformations are also affine ones. Linear transformations preserving the Minkowski distance are called **Lorentz transformations**.

Chapter 3

Special relativity

In this chapter we axiomatize special relativity within our FOL frame. The axiom system SpecRel, which we introduce in this chapter, is a kind of basic axiom system that we will extend and transform in the forthcoming chapters. Here we also discuss some important properties of SpecRel, such as its completeness with respect to Minkowskian geometries over Euclidean ordered fields or the possible world-view transformations between *inertial* observers.

3.1 Special relativity in four simple axioms

Here we formulate four simple and plausible axioms that capture special relativity. We seek to formulate easily understandable axioms in FOL. We present each axiom at two levels. First we give an intuitive formulation, then a precise formalization using our logical notations. We give the pure FOL version of the first three axioms only. However, all the axioms in this thesis can also be translated easily into FOL formulas by inserting the respective FOL definitions into the formalizations of the axioms.

Let us now formulate our first axiom on observers. Historically, this natural axiom goes back to Galileo Galilei or even to d'Oresme of around 1350, see, e.g., [3, p.23, §5], but it is very probably a prehistoric assumption, see Rem. 3.1.1. It simply states that each observer assumes that it rests at the origin of the space part of its coordinate system.

AxSelf_0 An observer coordinatizes itself at a coordinate point iff it is in the observer's coordinate domain and its space component is the origin:

$$\forall m \in \mathrm{Ob}\, \forall \vec{p} \in \mathrm{Q}^d \quad m \in ev_m(\vec{p}) \;\leftrightarrow\; \vec{p} \in Cd_m \,\wedge\, \vec{p}_\sigma = \vec{o}.$$

A purely FOL formula expressing AxSelf_0 is the following:

$$\forall m\ \forall \vec{p}\quad \mathrm{Ob}(m) \wedge \mathrm{Q}(p_1) \wedge \ldots \wedge \mathrm{Q}(p_d)\ \rightarrow$$
$$\Big(\mathrm{W}(m,m,\vec{p})\ \leftrightarrow\ \exists b\ \mathrm{B}(b) \wedge \mathrm{W}(m,b,\vec{p}) \wedge p_2 = 0 \wedge \ldots\ \wedge\ p_d = 0\Big).$$

Let us also introduce a strengthened version of axiom AxSelf_0:

AxSelf An *inertial* observer coordinatizes itself at a coordinate point iff its space component is the origin:

$$\forall m \in \mathrm{IOb}\ \forall \vec{p} \in \mathrm{Q}^d\quad W(m,m,\vec{p})\ \leftrightarrow\ \vec{p}_\sigma = \vec{o}.$$

Remark 3.1.1. At first glance it is not clear why AxSelf_0 is so natural. As an explanation, let us consider the following simple example. Let us imagine that we are watching sunset. What do we see? We do not see and feel that we are rotating with the Earth but that the Sun is moving towards the horizon; and according to our (the Earth's) reference system, we are absolutely right. But we learned at primary school that "the Earth rotates and goes around the Sun." So why does not this (i.e., the adoption of the heliocentric system) mean that AxSelf_0 and our impression above about the sunset are simply wrong? It is because the debate between geocentric and heliocentric systems was not about AxSelf_0, but about how to choose the best reference frame if we want to study the motions of planets in our solar system. See [57].[1] As reference frames, those of the Earth, the Sun, and even the Moon are equally good. However, if we would like to calculate the motions of the planets, the Sun's reference frame is the most convenient.

Now we formulate our next axiom on the constancy of the speed of photons. For convenience, we choose 1 for this speed. This choice physically means using units of distance compatible with units of time, such as light-year, light-second, etc.

AxPh For every *inertial* observer, there is a photon through two coordinate points $\vec{p}$ and $\vec{q}$ iff the slope of $\vec{p}-\vec{q}$ is 1:

$$\forall m \in \mathrm{IOb}\ \forall \vec{p},\vec{q} \in \mathrm{Q}^d\quad |\vec{p}_\sigma - \vec{q}_\sigma| = |p_\tau - q_\tau|\ \leftrightarrow\ \mathrm{Ph} \cap ev_m(\vec{p}) \cap ev_m(\vec{q}) \neq \emptyset.$$

[1] Here we consider only the basic idea of the two systems (i.e., whether the Earth or the Sun is stationary) and not their details (e.g., epicycles). Of course, Ptolemy's geocentric model was wrong in its details since even if we fix the Earth as a reference frame, the other planets will go around not the Earth but the Sun. It is interesting to note that Tycho Brahe worked out a correct geocentric system in which the Sun and the Moon move around the Earth and the other planets move around the Sun.

A purely FOL formula expressing AxPh is the following:

$$\forall m\, \forall \vec{p}\, \forall \vec{q} \quad \mathrm{IOb}(m) \wedge \mathrm{Q}(p_1) \wedge \mathrm{Q}(q_1) \wedge \ldots \wedge \mathrm{Q}(p_d) \wedge \mathrm{Q}(q_d) \;\rightarrow\; \Big((p_1 - q_1)^2 = (p_2 - q_2)^2 + \ldots + (p_d - q_d)^2 \;\leftrightarrow\; \exists ph\ \mathrm{Ph}(ph) \wedge \mathrm{W}(m, ph, \vec{p}) \wedge \mathrm{W}(m, ph, \vec{q}) \Big).$$

Axiom AxPh is a well-known assumption of Special Relativity, see, e.g., [4], [17, §2.6]. We may weaken AxPh by allowing *inertial* observers to measure different but uniform speeds of light.

$\mathsf{AxPh_0}$ For every *inertial* observer, the speed of light is uniform and positive, and there can be a photon at any point and in any direction with this speed:

$$\forall m \in \mathrm{IOb}\ \exists c_m \in \mathrm{Q}^+\ \forall \vec{p}, \vec{q} \in \mathrm{Q}^d \quad |\vec{p}_\sigma - \vec{q}_\sigma| = c_m \cdot |p_\tau - q_\tau| \;\leftrightarrow\; \mathrm{Ph} \cap ev_m(\vec{p}) \cap ev_m(\vec{q}) \neq \emptyset.$$

The models of our theory SpecRel (see p.20) would change to some extent if we replaced AxPh by $\mathsf{AxPh_0}$; however, they would not be essentially different. We use AxPh for convenience only. Sfarti [64] proves that the principle of relativity and $\mathsf{AxPh_0}$ imply AxPh.

Remark 3.1.2. For convenience, we quantify over events, too. That does not mean abandoning our FOL language. It is just simplifying the formalization of our axioms. Instead of events we could speak about observers and spacetime locations. For example, instead of $\forall e \in Ev_m\ \phi$ we could write $\forall \vec{p} \in Cd_m\ \phi[e \rightsquigarrow ev_m(\vec{p})]$, where none of $p_1 \ldots p_d$ occur free in ϕ, and $\phi[e \rightsquigarrow ev_m(\vec{p})]$ is the formula obtained from ϕ by substituting $ev_m(\vec{p})$ for e in all free occurrences. Similarly, we can replace $\forall e \in Ev\ \phi$ by $\forall m \in \mathrm{Ob}\ \forall e \in Ev_m\ \phi$.

By our next axiom we assume that events observed by *inertial* observers are the same.

AxEv Every *inertial* observer coordinatizes the very same set of events:

$$\forall m, k \in \mathrm{IOb} \quad Ev_m = Ev_k.$$

A purely FOL formula expressing AxEv is the following:

$$\forall m\, \forall k\, \forall \vec{p} \quad \mathrm{IOb}(m) \wedge \mathrm{IOb}(k) \wedge \mathrm{Q}(p_1) \wedge \ldots \wedge \mathrm{Q}(p_d) \;\rightarrow\; \exists \vec{q}\ \ \mathrm{Q}(q_1) \wedge \ldots \wedge \mathrm{Q}(q_d) \wedge \Big(\forall b \quad \mathrm{B}(b) \;\rightarrow\; \big(\mathrm{W}(m, b, \vec{p}) \;\leftrightarrow\; \mathrm{W}(k, b, \vec{q}) \big) \Big).$$

Let us now prove some consequences of the axioms introduced so far.

Proposition 3.1.3. Let h be an observer and let m and k be *inertial* observers. Then

(1) $Cd_m = \mathrm{Q}^d$ and ev_m is injective if $\mathsf{AxPh_0}$ is assumed.

(2) ev_m is a bijection from Cd_m to Ev_m; loc_m is a bijection from Ev_m to Cd_m; and w_m^h is a function from Q^d to Q^d if ev_m is injective on nonempty events.

(3) w_m^k is a bijection from Q^d to Q^d if $\mathsf{AxPh_0}$ and AxEv are assumed.

Proof. To prove Item (1), let $\vec{p} \in \mathrm{Q}^d$. Then by $\mathsf{AxPh_0}$, there is a photon ph such that $ph \in ev_m(\vec{p}) \cap ev_m(\vec{p} + \langle 1, 0, \dots, 0, c_m, 0, \dots, 0\rangle)$. Hence $ev_m(\vec{p}) \neq \emptyset$ for all $\vec{p} \in \mathrm{Q}^d$. So $Cd_m = \mathrm{Q}^d$. Moreover, if $\vec{q} \in \mathrm{Q}^d$ and $\vec{q} \neq \vec{p}$, then it is possible to choose this ph such that $ph \notin ev_m(\vec{q})$ also holds. Thus ev_m is injective.

Item (2) is clear since, if ev_m is injective on nonempty events, both $\mathsf{loc}_m := ev_m^{-1}$ and $w_m^h := ev_h \circ \mathsf{loc}_m$ are functions.

Let us now prove Item (3). By Item (2), we already have that ev_k is a bijection from Cd_k to Ev_k, and loc_m is a bijection from Ev_m to Cd_m. By AxEv, $Ev_k = Ev_m$. Thus $w_m^k = ev_k \circ \mathsf{loc}_m$ is a bijection from Cd_k to Cd_m. But by Item (1), we also have that $Cd_k = Cd_m = \mathrm{Q}^d$. Hence w_m^k is a bijection from Q^d to Q^d. ■

Let us now introduce a symmetry axiom called the symmetric distance axiom, by which we assume that *inertial* observers use the same units of measurement.

$\mathsf{AxSymDist}$ *Inertial* observers m and k agree as to the spatial distance between events e_1 and e_2 if they are simultaneous for both of them:

$$\forall m, k \in \mathrm{IOb}\ \forall e_1, e_2 \in Ev_m \cap Ev_k$$
$$e_1 \sim_m e_2 \ \wedge\ e_1 \sim_k e_2 \ \rightarrow\ \mathsf{dist}_m(e_1, e_2) = \mathsf{dist}_k(e_1, e_2).$$

Let us introduce the following axiom system:

$$\boxed{\mathsf{SpecRel} := \{\, \mathsf{AxSelf_0}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSymDist} \,\}}$$

Now we have a FOL theory of Special Relativity for each natural number $d \geq 2$.

Our symmetry axiom $\mathsf{AxSymDist}$ has many equivalent versions, see [2, §2.8, §3.9, §4.2]. Let us introduce one of them here.

$\mathsf{AxSymTime}$ Any two *inertial* observers see each others' clocks behaving in the same way:

$$\forall k, m \in \mathrm{IOb}\ \forall \lambda \in \mathrm{Q} \quad \left| w_m^k(\lambda \cdot \vec{1}_t)_\tau - w_m^k(\vec{o})_\tau \right| = \left| w_k^m(\lambda \cdot \vec{1}_t)_\tau - w_k^m(\vec{o})_\tau \right|.$$

To prove that AxSymTime is equivalent to AxSymDist, let us introduce a version of SpecRel without this axiom:

$$\boxed{\mathsf{SpecRel}_0 := \{\, \mathsf{AxSelf}_0, \mathsf{AxPh}, \mathsf{AxEv} \,\}}$$

Theorem 3.1.4. Let $d \geq 3$ and assume $\mathsf{AxSpecRel}_0$. Then the following three statements are equivalent:

(1) AxSymDist,

(2) AxSymTime and

(3) $\forall k, m \in \mathrm{IOb}\;\; w_m^k$ is a Poincaré transformation.

On the proof. By using the fact that every Poincaré transformation is the composition of a translation, a space-isomorphism and a Lorentz boost, it is not difficult to prove that Item (3) implies Items (1) and (2).

Item (2) in Thm. 3.2.2 states that Item (1) implies Item (3).

Finally, the implication of Item (3) by Item (2) can be proved analogously to Thm. 3.2.2, i.e., by proving that both the field-automorphism-induced maps and the dilations in the decomposition of w_m^k and w_k^m given by Item (1) in Thm. 3.2.2 are the identity map. ■

3.2 World-view transformations in special relativity

To prove a theorem that characterizes the world-view transformations between *inertial* observers if only AxPh and AxEv are assumed, we need one more definition. A map $\tilde{\varphi} : \mathrm{Q}^d \to \mathrm{Q}^d$ is called a **field-automorphism-induced map** iff there is an automorphism φ of the field $\langle \mathrm{Q}, \cdot, + \rangle$ such that $\tilde{\varphi}(\vec{p}) = \langle \varphi(p_1), \ldots, \varphi(p_d) \rangle$ for every $\vec{p} \in \mathrm{Q}^d$. Now we can state the Alexandrov-Zeeman theorem generalized for fields.

Theorem 3.2.1 (Alexandrov-Zeeman)**.** Let be F a field and $d \geq 3$. Every bijection from F^d to F^d that transforms lines of slope 1 to lines of slope 1 is a Poincaré transformation composed with a dilation and a field-automorphism-induced map.

For the proof of this theorem, see, e.g., [78], [79]. From this theorem we derive the following characterization of world-view transformations.

Theorem 3.2.2. Let $d \geq 3$. Let m and k be *inertial* observers. Then

(1) If AxPh and AxEv are assumed, w_m^k is a Poincaré transformation composed with a dilation D and a field-automorphism-induced map $\tilde{\varphi}$.

(2) If AxPh, AxEv and AxSymDist are assumed, w_m^k is a Poincaré transformation.

On the proof It is not hard to see that AxPh and AxEv imply that w_m^k is a bijection from Q^d to Q^d that preserves lines of slope 1, see Prop. 3.1.3. Hence Item (1) is a consequence of the Alexandrov-Zeeman theorem generalized for fields.

Now let us see why Item (2) is true. By using Item (1), it is easy to see that there is a line l such that both l and its w_m^k image are orthogonal to the time-axis. Thus by AxSymDist, w_m^k restricted to l is distance-preserving. Consequently, both the dilation D and the field-automorphism-induced map $\tilde{\varphi}$ in Item (1) have to be the identity map. Hence w_m^k is a Poincaré transformation. ■

Thm. 3.2.2 shows that SpecRel is a good axiom system for Special Relativity if we restrict our interest to *inertial* motion. It also implies that the most frequently quoted predictions of Special Relativity are provable from SpecRel:

(i) "moving clocks slow down,"

(ii) "moving meter-rods shrink" and

(iii) "moving pairs of clocks get out of synchronism."

Even if we only assume AxPh and AxEv, we can prove qualitative versions of the predictions above; AxSymDist is needed if we want to prove the quantitative versions, too. And AxSelf is only a simplifying axiom; it makes formulating the above predictions easier. For more detail. See, e.g., [2, §2.5], [3, §1], [4, §2].

The following consequence of Thm. 3.2.2 is the starting point for building Minkowski geometry, which is the "geometrization" of Special Relativity. It shows how time and space are intertwined in Special Relativity.

Theorem 3.2.3. Let $d \geq 3$. Assume SpecRel. Then

$$\mathsf{time}_m(e_1, e_2)^2 - \mathsf{dist}_m(e_1, e_2)^2 = \mathsf{time}_k(e_1, e_2)^2 - \mathsf{dist}_k(e_1, e_2)^2$$

for any *inertial* observers m and k and events e_1 and e_2 coordinatized by both of them.

Let us finally state a corollary here about the slowing down of moving clocks.

Corollary 3.2.4. Assume SpecRel, $d \geq 3$. Let $m, k \in \mathrm{IOb}$, $e_1, e_2 \in Ev_k$, and assume $k \in e_1 \cap e_2$, $\mathsf{dist}_m(e_1, e_2) \neq 0$. Then

$$\mathsf{time}_m(e_1, e_2) > \mathsf{time}_k(e_1, e_2).$$

In the above corollary, a "moving clock" is represented by observer k; the fact that it is moving relative to observer m is expressed by $\mathsf{dist}_m(e_1, e_2) \neq 0$ and $k \in e_1 \cap e_2$; and that k's time is slowing down relative to m's is expressed by $\mathsf{time}_m(e_1, e_2) > \mathsf{time}_k(e_1, e_2)$. This "clock slowing down" is only a relative effect, i.e., "clocks moving relative to m slow down relative to m." But this relative effect leads to a new kind of gravitation-oriented "absolute slowing down of time" effect, as Chap. 8 will show.

We can summarize the results of this chapter (that standard special relativity is provable from SpecRel) as a kind of completeness theorem of SpecRel with respect to its "intended models":

Corollary 3.2.5. Assume $d \geq 3$. Then SpecRel is complete with respect to Minkowskian geometries over Euclidean ordered fields.

The formal meaning of Cor. 3.2.5 is completely analogous to that of Thm. 9.0.6 (about general relativity) and is explained under Thm. 9.0.6. For further details, see [33, §4], too.

Chapter 4

Clock paradox

As one of our main aims is to trace back the surprising predictions of relativity to some convincing axioms, first we investigate an axiomatic basis of the clock paradox[1] (CP), which is an inertial approximation of the famous twin paradox. A similar logical investigation of the twin paradox needs a more complex mathematical apparatus, see [34], [72] and Chap. 7. The results of this chapter are based on [71], [72] and [69].

CP is one of the most famous predictions of special relativity. It concerns three *inertial* observers: one of them is the stay-at-home twin and the other two simulate the accelerated twin in the twin paradox. This simulation is done by replacing the accelerated twin by a leaving *inertial* observer and a returning one that synchronizes its clock with the leaving one's when they meet.

In this chapter we mainly concentrate on the relation of CP to the axioms and other consequences of special relativity, but we also formulate and characterize variants of CP, one where the stay-at-home twin will be the younger one (Anti-CP) and another where no differential aging will take place (No-CP).

In Section 4.1 we introduce a very basic axiom system Kinem_0 of kinematics in which no relativistic effect is assumed. Kinem_0 is a subtheory of Newtonian kinematics and special relativity. In Section 4.3 we formulate and prove a geometrical characterization of CP, Anti-CP and No-CP each within the models of Kinem_0, see Cor. 4.3.5 and

[1]Unfortunately, it is still not uncommon for people misinterpreting the word 'paradox' to look for contradictions in relativity theory, that is why we think it important to note here that its original meaning is "a statement that is seemingly contradictory and yet is actually true," i.e., it has nothing to do with logical contradiction. Having the nearly century long fruitless debate in view, perhaps it would be better to call the paradoxes of relativity theory simply effects, thus saying "clock effect" instead of "clock paradox," but for the time being it appears to be a hopeless effort to have this usage generally accepted. Anyway, we would like to emphasize that it is absolutely pointless to try to find a logical contradiction in relativity theory, as its consistency has been proved, see [2, p.77],[4, Cor.2.2].

Thm. 4.3.6. In Secs. 4.4 and 4.5 we prove some surprising logical consequences of our characterization. In Thm. 4.4.1 we show that the absoluteness of time (in the Newtonian sense) is not equivalent to the lack of the clock paradox (No-CP) without assuming a strong theoretical axiom. Similarly, in Thm. 4.5.2 we show that the slowing down of moving clocks is not equivalent to CP. In Thm. 4.5.3 we show that a symmetry axiom of special relativity is strictly stronger than CP.

4.1 A FOL axiom system of kinematics

We characterize the CP under some very mild assumptions about kinematics. To introduce this weak axiom system ($\mathsf{Kinem_0}$) we formulate some further axioms. Let us recall that $\vec{1}_t = \langle 1, 0, \ldots, 0\rangle$; and let us define the **time-unit vector** of k according to m to be

$$1^k_m := w^k_m(\vec{1}_t) - w^k_m(\vec{o}).$$

$\mathsf{AxLinTime}$ The world-lines of *inertial* observers are lines and time is elapsing uniformly on them:

$$\forall m,k \in \mathrm{IOb}\ \ \mathsf{wl}_m(k) = \left\{\, w^k_m(\vec{o}) + \lambda \cdot 1^k_m \ :\ \lambda \in \mathrm{Q} \,\right\} \land$$
$$\forall \vec{p}, \vec{q} \in \mathsf{wl}_m(k) \quad \mathsf{time}_k\big(ev_m(\vec{p}), ev_m(\vec{q})\big) \cdot \left|1^k_m\right| = |\vec{p} - \vec{q}|.$$

Let us now introduce the aforementioned axiom system of kinematics:

$$\boxed{\mathsf{Kinem_0} := \{\, \mathsf{AxSelf}, \mathsf{AxLinTime}, \mathsf{AxEv} \,\}}$$

Let us note that $\mathsf{Kinem_0}$ is a very weak axiom system of kinematics. By using Item (1) of Prop. 3.1.3 and Thm. 3.2.2, it not difficult to show that AxSelf and $\mathsf{AxLinTime}$ are consequences of $\mathsf{SpecRel}$. So $\mathsf{Kinem_0}$ is weaker than $\mathsf{SpecRel}$.

4.2 Formulating the clock paradox

To formulate CP, first we formulate the situations in which it can occur. We say that *inertial* observer m observes *inertial* observers a, b and c in a **clock paradox situation** at events e, e_a and e_c iff $a \in e_a \cap e$, $b \in e_a \cap e_c$, $c \in e \cap e_c$, $b \notin e$ and $\mathsf{time}_m(e_a) < \mathsf{time}_m(e) < \mathsf{time}_m(e_c)$ or $\mathsf{time}_m(e_a) > \mathsf{time}_m(e) > \mathsf{time}_m(e_c)$, see Fig. 4.1. This situation is denoted by $\mathsf{CP}_m(\widehat{ac}, b)(e_a, e, e_c)$.

Let $a, b, c \in \mathrm{IOb}$ and $e_a, e, e_b \in Ev$. Let $\mathsf{time}(\widehat{ac} < b)(e_a, e, e_b)$ be an abbreviation for $\mathsf{time}_a(e_a, e) + \mathsf{time}_c(e, e_c) < \mathsf{time}_b(e_a, e_c)$. The definitions of $\mathsf{time}(\widehat{ac} = b)(e_a, e, e_b)$

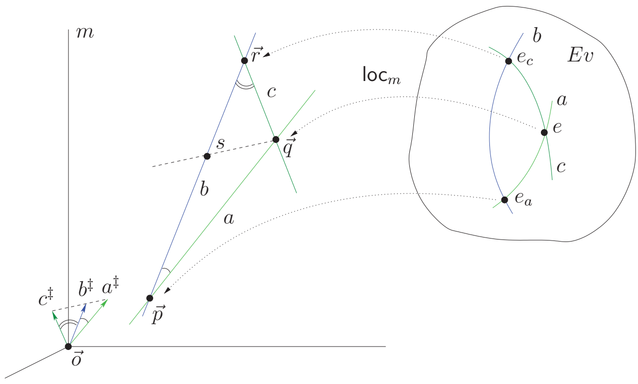

Figure 4.1: Illustration of relation $\mathsf{CP}_m(\widehat{ac}, b)(e_a, e, e_c)$ and the proof of Prop. 4.3.1

and $\mathsf{time}(\widehat{ac} > b)(e_a, e, e_b)$ are analogous. Using this notation, we can formulate the clock paradox as follows:

CP Every *inertial* observer m observes the clock paradox in every clock paradox situation:

$$\forall m, c, a, b \in \mathrm{IOb}\ \forall e, e_a, e_c \in Ev_m \quad \mathsf{CP}_m(\widehat{ac}, b)(e_a, e, e_c) \ \rightarrow\ \mathsf{time}(\widehat{ac} < b)(e_a, e, e_c).$$

We define formulas NoCP and AntiCP by replacing '$<$' by '$=$' and '$>$' in the formula CP, respectively.

4.3 Geometrical characterization of CP

We say that $\vec{q} \in \mathrm{Q}^d$ is (strictly) **between** $\vec{p} \in \mathrm{Q}^d$ and $\vec{r} \in \mathrm{Q}^d$ iff there is a $\lambda \in \mathrm{Q}$ such that $\vec{q} = \lambda\vec{p} + (1-\lambda)\vec{r}$ and $0 < \lambda < 1$. This situation is denoted by $\mathsf{Bw}(\vec{p}, \vec{q}, \vec{r})$.

Let $\vec{p}, \vec{q}, \vec{r} \in \mathrm{Q}^d$ and $\mu \in \mathrm{Q}$ such that $\mathsf{Bw}(\vec{p}, \mu\vec{q}, \vec{r})$. In this case we use notations $\mathsf{Conv}(\vec{p}, \vec{q}, \vec{r})$ and $\mathsf{Conc}(\vec{p}, \vec{q}, \vec{r})$ if $1 < \mu$ and $0 < \mu < 1$, respectively.

For convenience, we introduce the following notation:

$$^\ddagger\vec{p} := \begin{cases} \vec{p} & \text{if} \quad p_t \geq 0, \\ -\vec{p} & \text{if} \quad p_t < 0. \end{cases}$$

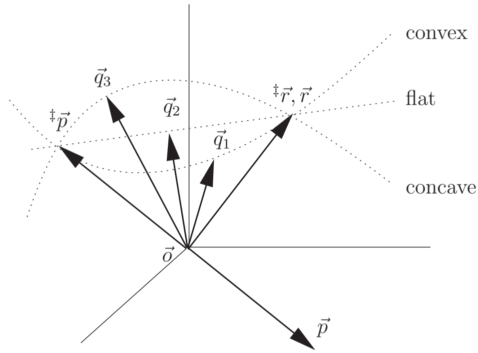

Figure 4.2: Illustration of relations $\mathsf{Conv}({}^{\ddagger}\vec{p}, \vec{q}_1, \vec{r})$, $\mathsf{Bw}({}^{\ddagger}\vec{p}, \vec{q}_2, \vec{r})$ and $\mathsf{Conc}({}^{\ddagger}\vec{p}, \vec{q}_3, \vec{r})$

Proposition 4.3.1. Assume Kinem_0. Let m, a, b, and c be *inertial* observers and e, e_a and e_c events such that $\mathsf{CP}_m(\widehat{ac}, b)(e_a, e, e_c)$. Then

$$\begin{aligned}
\mathsf{time}(\widehat{ac} < b)(e_a, e, e_c) &\iff \mathsf{Conv}({}^{\ddagger}1^a_m, {}^{\ddagger}1^b_m, {}^{\ddagger}1^c_m),\\
\mathsf{time}(\widehat{ac} = b)(e_a, e, e_c) &\iff \mathsf{Bw}({}^{\ddagger}1^a_m, {}^{\ddagger}1^b_m, {}^{\ddagger}1^c_m),\\
\mathsf{time}(\widehat{ac} > b)(e_a, e, e_c) &\iff \mathsf{Conc}({}^{\ddagger}1^a_m, {}^{\ddagger}1^b_m, {}^{\ddagger}1^c_m).
\end{aligned}$$

Proof. Let m, a, b, and c be *inertial* observers and e, e_a and e_c events such that $\mathsf{CP}_m(\widehat{ac}, b)(e_a, e, e_c)$. Let us abbreviate time-unit vectors ${}^{\ddagger}1^k_m$ as $k^{\ddagger}$ throughout this proof. Let $\vec{p} = \mathsf{loc}_m(e_a)$, $\vec{q} = \mathsf{loc}_m(e)$ and $\vec{r} = \mathsf{loc}_m(e_c)$. We have that $\vec{p} \neq \vec{r}$ since $p_\tau < r_\tau$ or $r_\tau < p_\tau$. Therefore, by $\mathsf{AxLinTime}$, the triangle $\vec{p}\,\vec{q}\,\vec{r}$ is nondegenerate since $\vec{p}, \vec{r} \in \mathsf{wl}_m(b)$ but $\vec{q} \notin \mathsf{wl}_m(b)$. Let us first show that b measures the same length of time between e_a and e_c as a and c together if $\mathsf{Bw}(a^{\ddagger}, b^{\ddagger}, c^{\ddagger})$ holds. Let $\vec{s}$ be the intersection of $line(\vec{p}, \vec{r})$ and the line parallel to $line(a^{\ddagger}, c^{\ddagger})$ through $\vec{q}$, see Fig. 4.1. Then the triangles $\vec{o}\,a^{\ddagger}b^{\ddagger}$ and $\vec{p}\,\vec{q}\,\vec{s}$ are similar; and the triangles $\vec{o}\,b^{\ddagger}c^{\ddagger}$ and $\vec{r}\,\vec{s}\,\vec{q}$ are similar. Thus

$$\frac{|\vec{p} - \vec{q}\,|}{|a^{\ddagger}|} = \frac{|\vec{p} - \vec{s}\,|}{|b^{\ddagger}|} \quad \text{and} \quad \frac{|\vec{q} - \vec{r}\,|}{|c^{\ddagger}|} = \frac{|\vec{s} - \vec{r}\,|}{|b^{\ddagger}|}$$

hold. Thus by $\mathsf{AxLinTime}$, we have that

$$\begin{aligned}
\Big|\mathsf{time}_a(e_a, e)\Big| + \Big|\mathsf{time}_c(e, e_c)\Big| &= \frac{|\vec{p} - \vec{q}\,|}{|a^{\ddagger}|} + \frac{|\vec{q} - \vec{r}\,|}{|c^{\ddagger}|}\\
&= \frac{|\vec{p} - \vec{s}\,| + |\vec{s} - \vec{r}\,|}{|b^{\ddagger}|} = \frac{|\vec{r} - \vec{p}\,|}{|b^{\ddagger}|} = \Big|\mathsf{time}_c(e_a, e_c)\Big|.
\end{aligned}$$

Hence $\mathsf{time}(\widehat{ac} = b)(e_a, e, e_c)$ holds if $\mathsf{Bw}(a^\ddagger, b^\ddagger, c^\ddagger)$. By AxLinTime, b measures more (less) time between e_a and e_c iff its time-unit vector is shorter (longer). Thus we get that $\mathsf{time}(\widehat{ac} < b)(e_a, e, e_c)$ holds if $\mathsf{Conv}(a^\ddagger, b^\ddagger, c^\ddagger)$, and $\mathsf{time}(\widehat{ac} > b)(e_a, e, e_c)$ holds if $\mathsf{Conc}(a^\ddagger, b^\ddagger, c^\ddagger)$. The converse implications also hold since one of the relations Conv, Bw and Conc holds for $a^\ddagger$, $b^\ddagger$ and $c^\ddagger$, and only one of the relations $\mathsf{time}(\widehat{ac} < b)$, $\mathsf{time}(\widehat{ac} = b)$ and $\mathsf{time}(\widehat{ac} > b)$ can hold for events e_a, e and e_c. This completes the proof. ∎

A set $H \subseteq \mathrm{Q}^d$ is called **convex** iff $\mathsf{Conv}(\vec{p}, \vec{q}, \vec{r})$ for all $\vec{p}, \vec{q}, \vec{r} \in H$ for which there is a $\mu \in \mathrm{Q}^+$ such that $\mathsf{Bw}(\vec{p}, \mu\vec{q}, \vec{r})$ holds. We call H **flat** or **concave** if $\mathsf{Conv}(\vec{p}, \vec{q}, \vec{r})$ is replaced by $\mathsf{Bw}(\vec{q}, \vec{r}, \vec{p})$ or $\mathsf{Conc}(\vec{r}, \vec{p}, \vec{q})$, respectively.

Remark 4.3.2. If there are no $\vec{p}, \vec{q}, \vec{r} \in H$ for which there is a $\mu \in \mathrm{Q}^+$ such that $\mathsf{Bw}(\vec{p}, \mu\vec{q}, \vec{r})$ holds, then H is convex, flat and concave at the same time. To avoid these undesired situations, let us call H **nontrivial** if there are $\vec{p}, \vec{q}, \vec{r} \in H$ such that $\mathsf{Bw}(\vec{p}, \mu\vec{q}, \vec{r})$ holds for a $\mu \in \mathrm{Q}^+$. By the respective definitions, it is easy to see that any nontrivial convex (flat, concave) set intersects a halfline at most once.

Let us define the **Minkowski sphere** as $MS^\ddagger_m := \left\{ \,{}^\ddagger 1^k_m \;:\; k \in \mathrm{IOb} \,\right\}$.

Remark 4.3.3. Convexity as used here is not far from convexity as understood in geometry or in the case of functions. For example, in the models of $\mathsf{Kinem_0} + \mathsf{AxThExp}^+$ or $\mathsf{SpecRel_0} + \mathsf{AxThExp}$ the Minkowski Sphere $MS^\ddagger_m$ is convex in our sense iff the set of points above it (i.e., $\{\vec{p} \in \mathrm{Q}^d \;:\; \exists \vec{q} \in MS^\ddagger_m \;\, p_\tau \geq q_\tau\}$) is convex in the geometrical sense. Axioms $\mathsf{AxThExp}^+$ and AxThExp are introduced on pp. 29 and 32, respectively.

Remark 4.3.4. By Rem. 4.3.2, if $MS^\ddagger_m$ is a nontrivial convex (flat, concave) set, then it intersects a line at most once.

Now we can state the following corollary of Prop. 4.3.1.

Corollary 4.3.5. Assume $\mathsf{Kinem_0}$. Then

$$\begin{aligned}
\forall m \in \mathrm{IOb}\;\, MS^\ddagger_m \text{ is convex} \quad &\Longrightarrow \quad \mathsf{CP},\\
\forall m \in \mathrm{IOb}\;\, MS^\ddagger_m \text{ is flat} \quad &\Longrightarrow \quad \mathsf{NoCP},\\
\forall m \in \mathrm{IOb}\;\, MS^\ddagger_m \text{ is concave} \quad &\Longrightarrow \quad \mathsf{AntiCP}.
\end{aligned}$$

The implications in Cor. 4.3.5 cannot be reversed because there may be *inertial* observers that are not part of any clock paradox situation. We can solve this problem by using the following axiom to shift *inertial* observers in order to create clock paradox situations.

AxShift Any *inertial* observer observing another *inertial* observer with a certain time-unit vector also observes still another *inertial* observer, with the same time-unit vector, at each coordinate point of its coordinate domain:

$$\forall m,k \in \mathrm{IOb}\ \forall \vec{p} \in Cd_m\ \exists h \in \mathrm{IOb} \quad h \in ev_m(\vec{p}) \ \wedge\ 1^k_m = 1^h_m.$$

Now we can reverse the above implications.

Theorem 4.3.6. Assume Kinem_0 and AxShift. Then

$$\begin{aligned} \mathsf{CP} &\iff \forall m \in \mathrm{IOb}\ MS^{\ddagger}_m \text{ is convex,}\\ \mathsf{NoCP} &\iff \forall m \in \mathrm{IOb}\ MS^{\ddagger}_m \text{ is flat,}\\ \mathsf{AntiCP} &\iff \forall m \in \mathrm{IOb}\ MS^{\ddagger}_m \text{ is concave.} \end{aligned}$$

Proof. By Cor. 4.3.5, we have to prove the "$\Longrightarrow$" part only. For that, let us take three points a', b' and c' from $MS^{\ddagger}_m$ for which there is a $\mu \in \mathrm{Q}$ satisfying $\mathsf{Bw}({}^{\ddagger}a', \mu b', {}^{\ddagger}c')$. If there are no such points, $MS^{\ddagger}_m$ is convex, flat and concave at the same time, see Rem. 4.3.2. Otherwise, by AxShift, there are *inertial* observers a, b and c in a clock paradox situation such that $1^a_m = a'$, $1^b_m = b'$ and $1^c_m = c'$. Thus from Prop. 4.3.1 we get that $MS^{\ddagger}_m$ has the desired property. ■

4.4 Consequences for Newtonian kinematics

Let us investigate the connection of No-CP and the Newtonian assumption of the absoluteness of time.

AbsTime All *inertial* observers measure the same elapsed time between any two events:

$$\forall m,k \in \mathrm{IOb}\ \forall e_1,e_2 \in Ev \quad \mathsf{time}_m(e_1,e_2) = \mathsf{time}_k(e_1,e_2).$$

To strengthen our axiom system, we introduce two axioms that ensure the existence of several *inertial* observers.

AxThExp$^+$ *Inertial* observers can move in any direction at any finite speed:

$$\forall m \in \mathrm{IOb}\ \forall \vec{p},\vec{q} \in \mathrm{Q}^d \quad p_\tau \neq q_\tau \ \rightarrow\ \exists k \in \mathrm{IOb} \quad k \in ev_m(\vec{p}) \cap ev_m(\vec{q}).$$

Let us also introduce a less theoretical version of this axiom.

AxThExp* *Inertial* observers can move in any direction at a speed which is arbitrarily close to any finite speed:

$$\forall m \in \mathrm{IOb}\ \forall \vec{p}, \vec{q} \in \mathrm{Q}^d\ \forall \varepsilon \in \mathrm{Q}^+ \quad p_\tau \neq q_\tau \\ \rightarrow\ \exists k \in \mathrm{IOb}\ \exists \vec{q}\,' \in \mathrm{Q}^d \quad |\vec{q} - \vec{q}\,'| < \varepsilon\ \wedge\ k \in ev_m(\vec{p}) \cap ev_m(\vec{q}\,').$$

By the following theorem, NoCP logically implies AbsTime if AxThExp$^+$ (and some auxiliary axioms) are assumed; however, if we assume the more experimental axiom AxThExp* instead of AxThExp$^+$, AbsTime does not follow from NoCP, which is an astonishing fact since it means that without the theoretical assumption AxThExp$^+$ we would not be able to conclude that time is absolute in the Newtonian sense even if there were no clock paradox in our world.

Theorem 4.4.1.

$$\mathsf{AbsTime} \models \mathsf{NoCP}, \text{ and} \tag{4.1}$$

$$\mathsf{Kinem_0} + \mathsf{AxShift} + \mathsf{AxThExp}^+ + \mathsf{NoCP} \models \mathsf{AbsTime}, \text{ but} \tag{4.2}$$

$$\mathsf{Kinem_0} + \mathsf{AxShift} + \mathsf{AxThExp}^* + \mathsf{NoCP} \not\models \mathsf{AbsTime}. \tag{4.3}$$

Proof. Item (4.1) is easy.

To prove (4.2), let us note that $MS^\ddagger_m$ is flat by Thm. 4.3.6 since Kinem$_0$, AxShift and NoCP are assumed. By axiom AxThExp$^+$, $MS^\ddagger_m$ intersects any nonhorizontal line. So $MS^\ddagger_m$ has to be a horizontal hyperplane containing $\langle 1, 0, \ldots, 0\rangle$. Hence the time components of time-unit vectors are the same for every *inertial* observer. So AbsTime follows from the assumptions.

To prove (4.3), we construct a model in which Kinem$_0$, AxShift, AxThExp* and NoCP hold, but AbsTime does not. Let $\langle \mathrm{Q}; +, \cdot, < \rangle$ be any Euclidean ordered field. Let $B := \mathrm{Q}^d \times \mathrm{Q}^d$. Let $\mathrm{IOb} := \{\langle \vec{p}, \vec{q}\rangle \in \mathrm{B} \ :\ p_\tau \neq q_\tau\ \wedge\ p_\tau - q_\tau \neq p_2 - q_2\}$. Let

$$MS^\ddagger_{\langle 1,0\rangle} := \left\{\, x \in \mathrm{Q}^d\ :\ x_\tau - x_2 = 1\ \wedge\ x_\tau > 0 \,\right\}.$$

Let $W(\langle 1,0\rangle, \langle \vec{p}, \vec{q}\rangle, \vec{r})$ hold iff $\vec{r}$ is in $line(\vec{p}, \vec{q})$. Now the world-view relation is given for *inertial* observer $\langle 1, 0\rangle$. For any other *inertial* observer $\langle \vec{p}, \vec{q}\rangle$, let $w^{\langle \vec{p}, \vec{q}\rangle}_{\langle 1,0\rangle}$ be an affine transformation that takes $\vec{o}$ to $\vec{p}$ while its linear part takes $\vec{1}_t$ to $MS^\ddagger_{\langle 1,0\rangle} \cap \{\lambda(\vec{p} - \vec{q}) : \lambda \in \mathrm{Q}\}$ and fixes the other basis vectors. From these world-view transformations, it is easy to define the world-view relations of other *inertial* observers, hence our model is given. It is not difficult to see that Kinem$_0$, AxShift and AxThExp* are true in this model. Since $MS^\ddagger_{\langle 1,0\rangle}$ is flat and the world-view transformations are affine ones, it is clear that $MS^{\ddagger}_m$ is flat for all $m \in \mathrm{IOb}$. Hence NoCP is also true in this model by Cor. 4.3.5. It is easy to see that AbsTime implies that $(1^k_m)_\tau = \pm 1$ for all $m, k \in \mathrm{IOb}$. Hence AbsTime is not true in this model, as claimed. ■

4.5 Consequences for special relativity theory

Now we investigate the consequences of our characterization for special relativity. To do so, let us first note that if $d \geq 3$, our theory $\mathsf{SpecRel}_0$ is strong enough to prove the most important predictions of special relativity, such as that moving clocks get out of synchronism, see Section 3.2; however, $\mathsf{SpecRel}_0$ is also weak enough not to prove every prediction of special relativity. For example, it does not entail CP or the slowing down of relatively moving clocks. Thus it is possible to compare these predictions within models of $\mathsf{SpecRel}_0$. To investigate the logical connection between them, let us formulate the slowing down effect on moving clocks within our FOL framework.

SlowTime Relatively moving *inertial* observers' clocks slow down:

$$\forall m, k \in \mathrm{IOb} \quad \mathsf{wl}_m(k) \neq \mathsf{wl}_m(m) \;\rightarrow\; \left|(1^k_m)_\tau\right| > 1.$$

To prove a theorem about the logical connection between SlowTime and CP, we need the following lemma, which states that the fact that three *inertial* observers are in clock paradox situation does not depend on the *inertial* observer who watches them.

Lemma 4.5.1. Let $d \geq 3$. Assume AxPh, AxEv and AxLinTime. Let $m, a, b, c \in \mathrm{IOb}$ and let $e_a, e, e_b \in Ev$. Then

$$\mathsf{CP}_m(\widehat{ac}, b)(e_a, e, e_c) \;\leftrightarrow\; \mathsf{CP}_b(\widehat{ac}, b)(e_a, e, e_c).$$

Proof. By (1) of Thm. 3.2.2, AxPh and AxEv imply that w^b_m is a composition of a Poincaré transformation, a dilation and a field-automorphism-induced map. By AxLinTime, the field-automorphism is trivial. Hence $\mathsf{time}_m(e)$ is between $\mathsf{time}_m(e_a)$ and $\mathsf{time}_m(e_c)$ iff $\mathsf{time}_b(e)$ is between $\mathsf{time}_b(e_a)$ and $\mathsf{time}_b(e_c)$. This completes the proof since the other parts of our definition of CP do not depend on *inertial* observers m and b. ■

We cannot consistently extend $\mathsf{SpecRel}_0$ by axiom $\mathsf{AxThExp}^+$ since $\mathsf{SpecRel}_0$ implies the impossibility of faster than light motion of *inertial* observers if $d \geq 3$, see, e.g., [3]. That is why we have to weaken this axiom.

AxThExp *Inertial* observers can move in any direction at any speed slower than 1, i.e., the speed of light:

$$\forall m \in \mathrm{IOb}\ \forall \vec{p}, \vec{q} \in \mathrm{Q}^d \quad |\vec{p}_\sigma - \vec{q}_\sigma| < |p_\tau - q_\tau| \ \rightarrow\ \exists k \in \mathrm{IOb} \quad k \in ev_m(\vec{p}) \cap ev_m(\vec{q}).$$

The following theorem shows that the slowing down of moving clocks (SlowTime) is logically stronger than CP.

Theorem 4.5.2. Let $d \geq 3$. Then

$$\mathsf{SpecRel}_0 + \mathsf{AxLinTime} + \mathsf{SlowTime} \models \mathsf{CP}, \text{ but} \tag{4.4}$$

$$\mathsf{SpecRel}_0 + \mathsf{AxLinTime} + \mathsf{AxShift} + \mathsf{AxThExp} + \mathsf{CP} \not\models \mathsf{SlowTime}. \tag{4.5}$$

Proof. Item (4.4) is clear by Lem. 4.5.1.

To prove Item (4.5), let us construct a model in which $\mathsf{SpecRel}_0$, AxLinTime, AxShift, AxThExp and CP hold, but SlowTime does not. Let $\langle \mathrm{Q}; +, \cdot, < \rangle$ be any Euclidean ordered field. Let $B{:=}\mathrm{Q}^d \times \mathrm{Q}^d$. Let $\mathrm{IOb}{:=}\{\langle \vec{p}, \vec{q} \rangle \in \mathrm{B} : |\vec{p}_\sigma - \vec{q}_\sigma| < |p_\tau - q_\tau|\}$. It is easy to see that there is a nontrivial convex subset M of Q^d such that $\vec{1}_t \in M$ and $|p_\tau| < 1$ for some $\vec{p} \in M$. Let $MS^\ddagger_{\langle 1,0\rangle}$ be such a convex subset of Q^d. Let $W(\langle 1,0\rangle, \langle \vec{p}, \vec{q}\rangle, \vec{r})$ hold iff $\vec{r}$ is in $line(\vec{p}, \vec{q})$. Now the world-view relation is given for *inertial* observer $\langle 1, 0\rangle$. By Rem. 4.3.4, $MS^\ddagger_{\langle 1,0\rangle}$ intersects a line at most once. For any other *inertial* observer $\langle \vec{p}, \vec{q}\rangle$, let $w^{\langle \vec{p},\vec{q}\rangle}_{\langle 1,0\rangle}$ be such a composition of a Lorentz transformation, a dilation and a translation which takes $\vec{o}$ to $\vec{p}$ while its linear part takes $\vec{1}_t$ to the unique element of $MS^\ddagger_{\langle 1,0\rangle} \cap \{\lambda(\vec{p} - \vec{q}) : \lambda \in \mathrm{Q}\}$ and fixes the other basis vectors. It is easy to see that there is such a transformation. From these world-view transformations, it is easy to define the world-view relations of the other *inertial* observers. So the model is given. It is not difficult to see that $\mathsf{SpecRel}_0$, AxShift, AxLinTime and AxThExp are true in this model. Since $MS^\ddagger_{\langle 1,0\rangle}$ is convex and the world-view transformations are affine ones, it is clear that $MS^\ddagger_m$ is convex for all $m \in \mathrm{IOb}$. Hence CP is also true in this model by Cor. 4.3.5. It is clear that SlowTime is not true in this model since there is a $\vec{p} \in MS^\ddagger_{\langle 1,0\rangle}$ such that $|p_\tau| < 1$ (i.e., there is $k \in \mathrm{IOb}$ such that $|(1^k_{\langle 1,0\rangle})_\tau| < 1$); that completes the proof. ■

Like the similar results of [71] and [72], Thm. 4.5.3 answers Question 4.2.17 of Andréka–Madarász–Németi [2]. It shows that CP is logically weaker than the symmetric distance axiom of SpecRel.

Theorem 4.5.3. Let $d \geq 3$. Then

$$\mathsf{SpecRel}_0 + \mathsf{AxSymDist} \models \mathsf{CP}, \text{ but} \tag{4.6}$$

$$\mathsf{SpecRel}_0 + \mathsf{AxLinTime} + \mathsf{AxShift} + \mathsf{AxThExp} + \mathsf{CP} \not\models \mathsf{AxSymDist}. \tag{4.7}$$

Proof. By (2) of Thm. 3.2.2, $\mathsf{SpecRel}_0$ and $\mathsf{AxSymDist}$ imply that w_m^k is a Poincaré transformation for all $m, k \in \mathrm{IOb}$. Hence

$$MS_m^\ddagger \subseteq \left\{ \vec{p} \in \mathrm{Q}^d \ :\ p_\tau^2 - |\vec{p}_\sigma|^2 = 1 \ \wedge\ p_\tau > 0 \right\}.$$

Consequently, $MS_m^\ddagger$ is convex. So by Cor. 4.3.5, CP follows from $\mathsf{SpecRel}_0$ and $\mathsf{AxSymDist}$.

Since $\mathsf{SpecRel}_0$ and $\mathsf{AxSymDist}$ imply $\mathsf{SlowTime}$ if $d \geq 3$, Item (4.7) follows from Thm. 4.5.2. ■

It is interesting that, if the quantity part is the field of real numbers, $\mathsf{AxSymDist}$ and $\mathsf{SlowTime}$ are equivalent in the models of $\mathsf{SpecRel}_0$ and some auxiliary axioms. However, that the quantity part is the field of real numbers, and thus Thm. 4.5.4, cannot be formulated in any FOL language of spacetime theories. Hence Thm. 4.5.4 cannot be formulated and proved within our FOL frame either.

Theorem 4.5.4. Assume $\mathsf{SpecRel}_0$, $\mathsf{AxLinTime}$, $\mathsf{AxShift}$, $\mathsf{AxThExp}$, and that Q is the field of real numbers. Then

$$\mathsf{SlowTime} \iff \mathsf{AxSymDist}.$$

For proof of Thm. 4.5.4, see [72, §3]. This theorem is interesting because it shows that assuming only that all the moving clocks slow down to some degree implies the exact ratio of the slowing down of moving clocks (since $\mathsf{SpecRel}_0 + \mathsf{AxSymDist}$ implies that the world-view transformations are Poincaré ones, see Thm. 3.2.2).

Question 4.5.5. Does Thm. 4.5.4 retain its validity if the assumption that Q is the field of real numbers is removed? If not, is it still possible to replace it by a FOL assumption, e.g., by axiom schema CONT used in [34], [35], [72] and Chaps. 7, 8 and 10?

We have seen that (the inertial approximation of) CP can be characterized geometrically within a weak axiom system of kinematics. We have seen some consequences of this characterization; in particular, CP is logically weaker than the assumption of the slowing down of moving clocks or the $\mathsf{AxSymDist}$ axiom of special relativity. A future task is to explore the logical connections between other assumptions and predictions of relativity theories. For example, in [34], [72] and Chap. 7, $\mathsf{SpecRel}_0^d + \mathsf{AxSymDist}$ is extended to an axiom system AccRel logically implying the twin paradox (the accelerated version of CP), but the natural question below, raised by Thm. 4.5.3, has not been answered yet.

Question 4.5.6. Is it possible to weaken $\mathsf{AxSymDist}$ to CP in AccRel without losing the twin paradox as a consequence? See [34, Que.3.8].

Chapter 5

Extending the axioms of special relativity for dynamics

Another surprising prediction of relativity theory is the equivalence of mass and energy. To find an axiomatic basis to this prediction, we have to extend our approach to dynamics. The results of this chapter are based on [6] and [7].

The idea is that we use collisions for measuring relativistic mass. We could say that the relativistic mass of a body is a quantity that shows the magnitude of its influence on the state of motion of the other bodies it collides with. The bigger the relativistic mass of a body is, the more it changes the motion of the bodies colliding with it. To be able to formulate this idea, let us extend our FOL language by a new $(d+3)$-ary relation M for relativistic mass. We use this relation to speak about the relativistic masses of bodies according to observers by translating $\mathrm{M}(b, \vec{p}, x, k)$ as "the relativistic mass of body b at coordinate point $\vec{p}$ is x according to observer k." Since there can be more than one x which is M-related to b, $\vec{p}$ and k, we introduce the following definition: **the relativistic mass of body** b **at** $\vec{p} \in \mathrm{Q}^d$ **according to observer** k, in symbols $\mathsf{m}_k(b, \vec{p})$, is defined as x if $\mathrm{M}(b, \vec{p}, x, k)$ holds and there is only one such $x \in \mathrm{Q}$; otherwise $\mathsf{m}_k(b, \vec{p})$ is undefined.

5.1 Axioms of dynamics

In this section we introduce a FOL axiomatic theory of special relativistic dynamics. In our first axiom on relativistic mass, we assume that it is positive in meaningful and zero in meaningless situations.

AxMass According to any observer, the relativistic mass of a body b at any coordinate point $\vec{p}$ is defined and nonnegative, and it is zero iff b is not present at $\vec{p}$:

$$\forall k \in \mathrm{Ob}\ \forall b \in \mathrm{B}\ \forall \vec{p} \in \mathrm{Q}^d \quad \mathsf{m}_k(b,\vec{p}) \geq 0 \ \wedge\ \big(\mathsf{m}_k(b,\vec{p}) = 0 \ \leftrightarrow\ b \notin ev_k(\vec{p})\big).$$

In our co-authored papers [6] and [7], this axiom was built into the logic frame.

To formulate our other axioms on relativistic mass, first we have to define collisions. To do so, we introduce the following concepts: the set of incoming bodies $in_k(\vec{q})$ and that of outgoing bodies $out_k(\vec{q})$ of a collision at coordinate point $\vec{q}$ according to observer k are defined as bodies whose world-lines "end" and "start" at $\vec{q}$, respectively (see Fig. 5.1):

$$\begin{aligned} in_k(\vec{q}) &:= \{\, b \in \mathrm{B} : \vec{q} \in \mathsf{wl}_k(b) \ \wedge\ \forall \vec{p} \in \mathsf{wl}_k(b) \quad p_\tau < q_\tau \ \vee\ \vec{p} = \vec{q}\,\}, \\ out_k(\vec{q}) &:= \{\, b \in \mathrm{B} : \vec{q} \in \mathsf{wl}_k(b) \ \wedge\ \forall \vec{p} \in \mathsf{wl}_k(b) \quad p_\tau > q_\tau \ \vee\ \vec{p} = \vec{q}\,\}. \end{aligned}$$

Bodies $b_1, \ldots, b_n$ collide originating bodies $d_1, \ldots, d_m$ according to observer k, in symbols $\mathsf{coll}_k(b_1, \ldots, b_n : d_1, \ldots, d_m)$, iff $b_i \neq b_j$ and $d_i \neq d_j$ whenever $i \neq j$ and there is a coordinate point $\vec{q}$ such that $in_k(\vec{q}) = \{b_1, \ldots, b_n\}$ and $out_k(\vec{q}) = \{d_1, \ldots, d_m\}$. Inelastic collisions are just collisions in which only one body is originated. So in this case, we write $\mathsf{inecoll}_k(b_1, \ldots, b_n : d)$ in place of $\mathsf{coll}_k(b_1, \ldots, b_n : d)$ and say that bodies $b_1, \ldots, b_n$ **collide inelastically** originating body d according to observer k. For the illustration of these concepts, see Fig. 5.1.

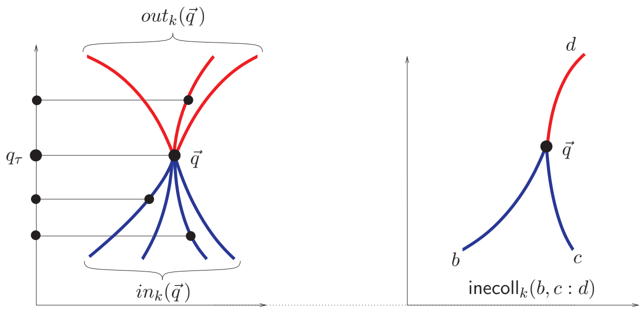

Figure 5.1: Illustration of $in_k(\vec{q})$, $out_k(\vec{q})$ and $\mathsf{inecoll}_k(b, c : d)$

The **spacetime location** $\mathsf{loc}^b_k(t)$ of body b at time instance $t \in \mathrm{Q}$ according to observer k is defined as the coordinate point $\vec{p}$ for which $\vec{p} \in \mathsf{wl}_k(b)$ and $p_\tau = t$ hold if there is such a unique $\vec{p}$; otherwise $\mathsf{loc}^b_k(t)$ is undefined, see Fig. 5.2.

The **center of mass** $\mathsf{cen}_k^{b_1,\ldots,b_n}(t)$ of bodies $b_1,\ldots,b_n$ according to $k \in \mathrm{Ob}$ at time instance t is defined by:

$$\sum_{i=1}^{n} \mathsf{m}_k(b_i, \mathsf{loc}_k^{b_i}(t)) \cdot \left(\mathsf{cen}_k^{b_1,\ldots,b_n}(t) - \mathsf{loc}_k^{b_i}(t)\right) = 0$$

if $\mathsf{loc}_k^{b_i}(t)$ and $\mathsf{m}_k(b_i, \mathsf{loc}_k^{b_i}(t))$ are defined for all $1 \le i \le n$; otherwise $\mathsf{cen}_k^{b_1,\ldots,b_n}(t)$ is undefined. Let us note that the following is an explicit definition for $\mathsf{cen}_k^{b_1,\ldots,b_n}(t)$:

$$\mathsf{cen}_k^{b_1,\ldots,b_n}(t) = \sum_{i=1}^{n} \frac{\mathsf{m}_k(b_i, \mathsf{loc}_k^{b_i}(t))}{\mathsf{m}_k(b_1, \mathsf{loc}_k^{b_1}(t)) + \ldots + \mathsf{m}_k(b_n, \mathsf{loc}_k^{b_n}(t))} \cdot \mathsf{loc}_k^{b_i}(t)$$

if $\mathsf{loc}_k^{b_i}(t)$ and $\mathsf{m}_k(b_i, \mathsf{loc}_k^{b_i}(t))$ are defined for all $1 \le i \le n$. The **center-line of the masses** of bodies $b_1,\ldots,b_n$ according to observer k is defined as:

$$\mathsf{cen}_k(b_1,\ldots,b_n) := \left\{ \mathsf{cen}_k^{b_1,\ldots,b_n}(t) \ :\ t \in \mathrm{Q} \text{ and } \mathsf{cen}_k^{b_1,\ldots,b_n}(t) \text{ is defined} \right\},$$

i.e., the center-line of mass is the world-line of the center of mass.

Remark 5.1.1. Let us note that $\mathsf{cen}_k^b(t) = \mathsf{loc}_k^b(t)$ for all $k \in \mathrm{Ob}$, $b \in \mathrm{B}$ and $t \in \mathrm{Q}$, and thus $\mathsf{cen}_k(b) = \mathsf{wl}_k(b)$ for every $k \in \mathrm{Ob}$ and $b \in \mathrm{B}$ if $\mathsf{m}_k(b, \mathsf{loc}_k^b(t))$ is defined and nonzero for all $t \in \mathit{Dom}\, \mathsf{loc}_k^b$ (e.g., if AxMass is assumed).

The segment determined by $\vec{p}, \vec{q} \in \mathrm{Q}^d$ is defined as:

$$[\vec{p}, \vec{q}] := \{ \lambda\vec{p} + (1-\lambda)\vec{q} \ :\ \lambda \in \mathrm{Q},\ 0 \le \lambda \le 1 \}.$$

Let us call $H \subseteq \mathrm{Q}^d$ a **line segment** if

- it is connected, (i.e., $[\vec{p}, \vec{q}] \subseteq H$ for all $\vec{p}, \vec{q} \in H$),
- it is a subset of a line, and
- it has at least two elements.

Bodies whose world-lines are line segments according to every *inertial* observer are called **inertial bodies**, and their set is defined as:

$$\mathrm{IB} := \{ b \in \mathrm{B} \ :\ \forall k \in \mathrm{IOb} \quad \mathsf{wl}_k(b) \text{ is a line segment} \}.$$

Proposition 5.1.2. Let k be an *inertial* observer and $b_1,\ldots,b_n$ *inertial* bodies such that, for all $1 \le i \le n$, $\vec{p}, \vec{q} \in \mathsf{wl}_k(b_i)$ imply $\mathsf{m}_k(b_i, \vec{p}) = \mathsf{m}_k(b_i, \vec{q}) > 0$. Then the following hold:

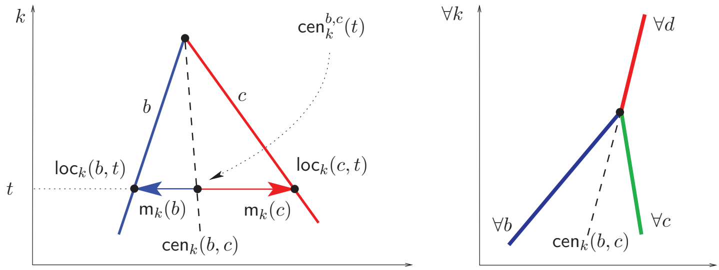

Figure 5.2: Illustration of $\mathsf{cen}_k^{b,c}(t)$, $\mathsf{cen}_k(b,c)$ and of axiom AxCenter

(1) $\mathsf{cen}_k(b_1,\ldots,b_n)$ is a line segment, a point or empty,

(2) $\mathsf{cen}_k(b_1,\ldots,b_n)$ is nonhorizontal, i.e., $\vec{r}=\vec{s}$ if $\vec{r},\vec{s}\in\mathsf{cen}_k(b_1,\ldots,b_n)$ and $r_\tau=s_\tau$,

(3) $\mathsf{wl}_k(b_1)\cap\ldots\cap\mathsf{wl}_k(b_n)\subseteq\mathsf{cen}_k(b_1,\ldots,b_n)$,

(4) $\mathsf{cen}_k(b_1,\ldots,b_n)$ is a line segment if $\mathsf{coll}_k(b_1,\ldots,b_n:d_1,\ldots,d_m)$ or $\mathsf{coll}_k(d_1,\ldots,d_m:b_1,\ldots,b_n)$ for some (not necessarily *inertial*) bodies $d_1,\ldots,d_m$.

Here we omit the easy proof.

Now we are ready to formalize that the relativistic mass of a body is a quantity that shows the magnitude of its influence on the state of motion of any other body it collides with.

AxCenter The world-line of the *inertial* body originated by an inelastic collision of two *inertial* bodies is the continuation of the center-line of the masses of the colliding *inertial* bodies according to every *inertial* observer (see Fig. 5.2):

$$\forall k\in\mathrm{IOb}\;\forall b,c,d\in\mathrm{IB}\quad \mathsf{inecoll}_k(b,c:d)\;\rightarrow\;\mathsf{cen}_k(b,c)\cup\mathsf{wl}_k(d)\subseteq\ell\text{ for some line }\ell.$$

The main axiom of SpecRelDyn is AxCenter which, in a certain sense, can be taken as a definition of relativistic mass. The other axioms of our axiom system will be simplifying or auxiliary ones to make life simpler. We could only get rid of them at the expense of sacrificing the simplicity of expressions.

AxCenter is an axiom in Newtonian Dynamics, too, where the mass $\mathsf{m}_k(b,\vec{p})$ of a body b does not depend on observer k and coordinate point $\vec{p}$. However, in special relativity, AxCenter implies that the mass of a body necessarily depends on the observer.

The reason for this fact is that the simultaneities of different observers in special relativity may differ from one another, and this implies that the proportions involved in AxCenter change, too. See [7, Prop.4.1].

The **velocity** $\mathbf{v}_k^b(t)$ and **speed** $v_k^b(t)$ of body b at instant $t \in \mathrm{Q}$ according to observer k are defined as:

$$\mathbf{v}_k^b(t) := \big((\mathsf{loc}_k^b)_\sigma\big)'(t) \qquad \text{and} \qquad v_k^b(t) := |\mathbf{v}_k^b(t)|$$

if $\mathsf{loc}_k^b(t)$ is defined and loc_k^b is differentiable at t; otherwise they are undefined. (For the FOL definition of $f'(t)$, see Section 10.3.) Let us note that

$$\big((\mathsf{loc}_k^b)_\sigma\big)'(t) = \big((\mathsf{loc}_k^b)'(t)\big)_\sigma \qquad \text{and} \qquad \big(\mathsf{loc}_k^b(t)_\tau\big)' = \big((\mathsf{loc}_k^b)'\big)_\tau(t) = 1$$

if $\mathsf{loc}_k^b(t)$ is defined and differentiable.

The **rest mass** $\mathsf{m}_0(b)$ of body b is defined as $\lambda \in \mathrm{Q}$ if (1) there is an observer according to which b is at rest and the relativistic mass of b is λ, and (2) the relativistic mass of b is λ for every observer according to which b is at rest. That is, $\mathsf{m}_0(b) = \lambda$ if

$$\begin{aligned} &\exists k \in \mathrm{Ob}\ \ \forall t \in Dom\, v_k^b \quad v_k^b(t) = 0 \quad \land \quad \forall \vec{p} \in \mathsf{wl}_k(b) \quad \mathsf{m}_k(b, \vec{p}) = \lambda \\ \land\ &\forall k \in \mathrm{Ob}\ \ \forall t \in Dom\, v_k^b \quad v_k^b(t) = 0 \ \rightarrow\ \forall \vec{p} \in \mathsf{wl}_k(b) \quad \mathsf{m}_k(b, \vec{p}) = \lambda \end{aligned}$$

if there is such λ; otherwise $\mathsf{m}_0(b)$ is undefined.

We have seen that AxCenter implies that the relativistic mass depends on both b and k. Our next axiom states that the relativistic mass of a body depends on its rest mass and velocity at the most.

AxSpeed According to any *inertial* observer, the relativistic masses of two *inertial* bodies are the same if both of their rest masses and speeds are equal:

$$\begin{aligned} &\forall k \in \mathrm{IOb}\ \ \forall b, c \in \mathrm{B}\ \ \forall \vec{p}, \vec{q} \in \mathrm{Q}^d \quad b \in ev_k(\vec{p}) \ \land\ c \in ev_k(\vec{q}) \\ &\qquad\qquad \land\ \mathsf{m}_0(b) = \mathsf{m}_0(c) \ \land\ v_k^b(p_\tau) = v_k^c(q_\tau) \ \rightarrow\ \mathsf{m}_k(b, \vec{p}) = \mathsf{m}_k(c, \vec{q}). \end{aligned}$$

Let B_0 be the set of bodies having rest mass, i.e., $\mathrm{B}_0 := \{\, b \in \mathrm{B} \ :\ \mathsf{m}_0(b) \text{ is defined} \,\}$, and let IB_0 be the set of *inertial* bodies having rest mass, i.e., $\mathrm{IB}_0 := \mathrm{IB} \cap \mathrm{B}_0$.

By the following proposition, AxSpeed implies that the relativistic mass of an *inertial* body having rest mass does not change in time according to *inertial* observers.

Proposition 5.1.3.

$$\mathsf{AxSpeed} \models \forall k \in \mathrm{IOb}\ \forall b \in \mathrm{IB}_0\ \ \forall \vec{p}, \vec{q} \in \mathsf{wl}_k(b) \quad \mathsf{m}_k(b, \vec{p}) = \mathsf{m}_k(b, \vec{q}).$$

Proof. By the respective definitions, it is easy to see that $v_k^b(p_\tau) = v_k^b(q_\tau)$ for all $\vec{p}, \vec{q} \in \mathsf{wl}_k(b)$. Hence by AxSpeed, $\mathsf{m}_k(b, \vec{p}) = \mathsf{m}_k(b, \vec{q})$ if $\vec{p}, \vec{q} \in \mathsf{wl}_k(b)$, k is an *inertial* observer, and b is an *inertial* body having rest mass. ■

Prop. 5.1.3 leads us to introduce the following definition: $\mathsf{m}_k(b)$ is defined as $\mathsf{m}_k(b, \vec{p})$ if $\mathsf{m}_k(b, \vec{p}) = \mathsf{m}_k(b, \vec{q})$ for all $\vec{p}, \vec{q} \in \mathsf{wl}_k(b)$; otherwise $\mathsf{m}_k(b)$ is undefined. So by Prop. 5.1.3 $\mathsf{m}_k(b)$ is defined if $b \in \mathrm{IB}_0$, $k \in \mathrm{IOb}$ and AxSpeed is assumed. Similarly, we use notations $\mathbf{v}_k(b)$ and $v_k(b)$ instead of $\mathbf{v}_k^b(t)$ and $v_k^b(t)$ when b and k are *inertial*, as in this case $\mathbf{v}_k^b(t_1) = \mathbf{v}_k^b(t_2)$ for all $t_1, t_2 \in Dom\, \mathbf{v}_k^b$.

Our last axiom on dynamics states that every observer can make experiments in which they make *inertial* bodies of arbitrary rest masses and velocities collide inelastically:

Ax∀inecoll For any *inertial* observer, any possible kind of inelastic collision of *inertial* bodies can be realized:

$$\forall k \in \mathrm{Ob}\ \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathrm{Q}^{d-1}\ \forall m_1, m_2 \in \mathrm{Q} \quad |\mathbf{v}_1| < 1 \ \wedge\ |\mathbf{v}_2| < 1 \\ \wedge\ \mathbf{v}_1 \neq \mathbf{v}_2 \ \wedge\ m_1 > 0 \ \wedge\ m_2 > 0 \ \rightarrow\ \exists b, c, d \in \mathrm{IB} \quad \mathsf{inecoll}_k(b, c : d) \\ \wedge\ \mathbf{v}_k(b) = \mathbf{v}_1 \ \wedge\ \mathbf{v}_k(c) = \mathbf{v}_2 \ \wedge\ \mathsf{m}_0(b) = m_1 \ \wedge\ \mathsf{m}_0(c) = m_2.$$

We often add axioms to SpecRel which do not change the spacetime structure, but are useful as auxiliary axioms. For example, AxThExp$^\uparrow$ below states that every observer can make thought experiments in which they assume the existence of "slowly moving" observers (see, e.g., [4, p.622 and Thm.2.9(iii)]):

AxThExp$^\uparrow$ For any *inertial* observer, in any spacetime location, in any direction, at any speed slower than that of light it is possible to "send out" an *inertial* observer whose time flows "forwards:"

$$\forall k \in \mathrm{IOb}\ \forall \vec{p}, \vec{q} \in \mathrm{Q}^d \quad |(\vec{p} - \vec{q})_\sigma| < (\vec{p} - \vec{q})_\tau \\ \rightarrow\ \exists h \in \mathrm{IOb} \quad h \in ev_k(\vec{p}) \cap ev_k(\vec{q}) \ \wedge\ w_h^k(\vec{q})_\tau < w_h^k(\vec{p})_\tau.$$

Let us extend SpecRel by AxThExp$^\uparrow$ and the axioms of dynamics above:

SpecRelDyn:= { AxMass, AxCenter, AxSpeed, Ax∀inecoll, AxThExp$^\uparrow$ } ∪ SpecRel

Let us note that SpecRelDyn is provably consistent. Moreover, it has nontrivial models, see Prop. 5.3.7.

The following theorem provides the connection between the rest mass and the relativistic mass of an *inertial* body. Its conclusion is a well-known result of special relativity. We will see that our theorem is stronger than the corresponding result in the literature since it contains fewer assumptions.

Theorem 5.1.4. Let $d \geq 3$. Assume SpecRelDyn and let k be an *inertial* observer and b be an *inertial* body having rest mass. Then

$$\mathsf{m}_0(b) = \sqrt{1 - v_k(b)^2} \cdot \mathsf{m}_k(b).$$

A purely geometrical proof of Thm. 5.1.4 can be found in [7].

Remark 5.1.5. Assuming AxPh, photons cannot have rest masses since their speed is 1 according to any *inertial* observer. However, by Thm. 5.1.4, it is natural to extend our rest mass concept for photons as $\mathsf{m}_0(ph){:=}0$ for all $ph \in \mathrm{Ph}$. After this extension photons may be regarded as "pure energy" as they have zero rest masses.

Remark 5.1.6. The conclusion of Thm. 5.1.4 fails if we omit any of the axioms AxMass, AxCenter, AxSpeed, Ax∀inecoll, AxThExp$^{\uparrow}$ from SpecRelDyn. However, it remains true if we weaken Ax∀inecoll and AxThExp$^{\uparrow}$ to the following two axioms, respectively:

Ax∃inecoll According to every observer, for every *inertial* body a having rest mass, there are *inertial* bodies b and c colliding inelastically such that a, b and c have the same rest masses, a and b have the same speeds and the speed of c is 0 (see the left-hand side of Fig. 5.3):

$$\forall k \in \mathrm{IOb}\, \forall a \in \mathrm{IB}_0\ \exists b, c, d \in \mathrm{IB} \quad \mathsf{m}_0(a) = \mathsf{m}_0(b) = \mathsf{m}_0(c)$$
$$\wedge\ v_k(b) = v_k(a)\ \wedge\ v_k(c) = 0\ \wedge\ \mathsf{inecoll}_k(b, c : d);$$

AxMedian For every two *inertial* bodies colliding inelastically, there is an *inertial* observer for which these two *inertial* bodies have opposite velocities and collide inelastically (see the right-hand side of Fig. 5.3):

$$\forall k \in \mathrm{IOb}\, \forall b, c, d \in \mathrm{IB} \quad \mathsf{inecoll}_k(b, c : d)$$
$$\rightarrow\ \exists h \in \mathrm{IOb} \quad \mathbf{v}_h(b) = -\mathbf{v}_h(c)\ \wedge\ \mathsf{inecoll}_h(b, c : d).$$

On Einstein's $E = mc^2$: The conclusion $\mathsf{m}_0(b) = \sqrt{1 - v_k(b)^2} \cdot \mathsf{m}_k(b)$ of our Thm. 5.1.4 above is used in Rindler's relativity textbook [51, pp.111-114] to explain the discovery

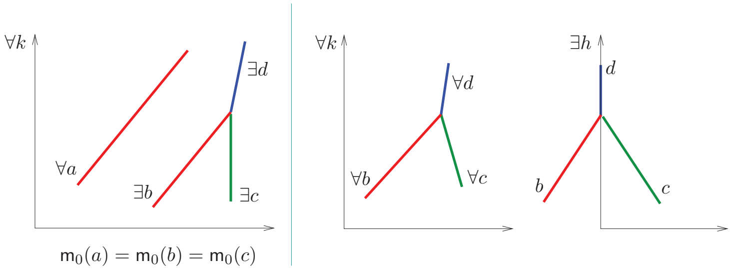

Figure 5.3: Illustration of axioms Ax∃inecoll and AxMedian

and meaning of Einstein's famous insight $E = mc^2$. We could literally repeat this part of the text of [51] to arrive at $E = mc^2$ in the framework of our theory SpecRelDyn based on the axiom AxCenter. We postpone this to a later point, because then we will have developed more "ammunition," hence the didactics can be more inspiring.

5.2 Conservation of relativistic mass and linear momentum

In a certain sense AxCenter states that the center of mass of an isolated system consisting of two *inertial* bodies moves along a line regardless whether the two bodies collide or not. It is natural to generalize AxCenter to more than two bodies (but permitting only two-by-two inelastic collisions). Let $\mathsf{AxCenter_n}$ denote, temporarily, a version of AxCenter concerning any isolated system consisting of n bodies. Thus AxCenter is just $\mathsf{AxCenter_2}$ in this series of increasingly stronger axioms. We will see that it does not imply $\mathsf{AxCenter_3}$; thus $\mathsf{AxCenter_3}$ is strictly stronger than AxCenter if certain auxiliary axioms are assumed, see Cor. 5.3.3 and Prop. 5.3.4. However, it can be proved that the rest of the axioms in this series are all equivalent to $\mathsf{AxCenter_3}$ if AxCenter is assumed, see Cor. 5.4.4. That motivates us to introduce $\mathsf{SpecRelDyn}^+$ by replacing AxCenter in SpecRelDyn by the stronger $\mathsf{AxCenter_3}$. Our theory $\mathsf{SpecRelDyn}^+$ is still very geometric and observation-oriented in spirit. Let us now introduce $\mathsf{AxCenter_3}$, and denote it as $\mathsf{AxCenter}^+$.

AxCenter$^+$ If a is an *inertial* body and *inertial* bodies b and c collide inelastically originating *inertial* body d, the center-line of the masses of a and d is the continuation of the center-line of the masses of a, b and c, see Fig. 5.4:

$$\forall k \in \mathrm{Ob}\ \forall a, b, c, d \in \mathrm{IB}$$
$$\mathsf{inecoll}_k(b, c : d) \ \rightarrow\ \mathsf{cen}_k(a, b, c) \cup \mathsf{cen}_k(a, d) \subseteq \ell \text{ for some line } \ell.$$

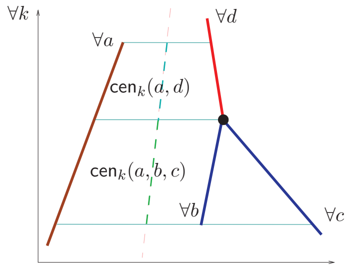

Figure 5.4: Illustration of AxCenter$^+$

Let us replace AxCenter by AxCenter$^+$ in SpecRelDyn:

$$\boxed{\mathsf{SpecRelDyn}^+ := \{\, \mathsf{AxMass}, \mathsf{AxCenter}^+, \mathsf{AxSpeed}, \mathsf{Ax\forall inecoll}, \mathsf{AxThExp}^{\uparrow} \,\} \cup \mathsf{SpecRel}}$$

Let us note that SpecRelDyn$^+$ is also consistent. Moreover, it has nontrivial models, see Prop. 5.3.7.

AxCenter determines the velocity of the body emerging from an inelastic collision, and we will see that AxCenter$^+$ also determines the relativistic mass of the body emerging from the collision.

Let us now formulate the conservation of relativistic mass in our FOL language.

ConsMass If *inertial* bodies b and c collide inelastically originating *inertial* body d, the relativistic mass of d is the sum of the relativistic masses of b and c:

$$\forall k \in \mathrm{IOb}\ \forall b, c, d \in \mathrm{IB} \quad \mathsf{inecoll}_k(b, c : d) \ \rightarrow\ \mathsf{m}_k(b) + \mathsf{m}_k(c) = \mathsf{m}_k(d).$$

The **linear momentum** of *inertial* body b according to *inertial* observer k is defined as $\mathsf{m}_k(b)\mathbf{v}_k(b)$ if $\mathbf{v}_k(b)$ and $\mathsf{m}_k(b)$ are defined, otherwise it is undefined. Now we can formulate the conservation of linear momentum in our FOL language.

ConsMomentum If *inertial* bodies b and c collide inelastically originating *inertial* body d, the linear momentum of d is the sum of the linear momentum of b and c:

$$\forall k \in \mathrm{IOb}\ \forall b, c, d \in \mathrm{IB} \quad \mathsf{inecoll}_k(b, c : d) \ \rightarrow\ \mathsf{m}_k(b)\mathbf{v}_k(b) + \mathsf{m}_k(c)\mathbf{v}_k(c) = \mathsf{m}_k(d)\mathbf{v}_k(d).$$

To state a theorem on the connection of AxCenter, ConsMass and ConsMomentum, we need the following auxiliary axiom.

AxInMass According to any *inertial* observer, the relativistic mass of every *inertial* body is constant:

$$\forall k \in \mathrm{IOb}\ \forall b \in \mathrm{IB}\ \ \forall \vec{p}, \vec{q} \in \mathsf{wl}_k(b) \quad \mathsf{m}_k(b, \vec{p}) = \mathsf{m}_k(b, \vec{q}).$$

That is a consequence of AxSpeed for *inertial* bodies having rest mass, see Prop. 5.1.3. In [6] and [7], this axiom was also built in our logic frame.

The following theorem states that axiom AxCenter$^+$ is equivalent to the conjunction of ConsMass and either of the two formulas AxCenter and ConsMomentum if certain auxiliary axioms are assumed. That means in a sense that ConsMass represents the "difference" between AxCenter and AxCenter$^+$, and the same holds if AxCenter is replaced by ConsMomentum.

Theorem 5.2.1. Let us assume AxMass, AxInMass, AxSelf and that $\mathrm{IOb} \subseteq \mathrm{IB}$. Then:

$$\mathsf{AxCenter}^+ \iff \mathsf{ConsMass} \wedge \mathsf{ConsMomentum} \iff \mathsf{ConsMass} \wedge \mathsf{AxCenter}.$$

The proof of Thm. 5.2.1 is in [6].

Corollary 5.2.2. Let us assume SpecRelDyn$^+$. Let k be an *inertial* observer and b, c and d *inertial* bodies such that $\mathsf{inecoll}_k(b, c : d)$ holds. Then

$$\begin{aligned} \mathsf{m}_k(d) &= \mathsf{m}_k(b) + \mathsf{m}_k(c), && \text{but} \\ \mathsf{m}_0(d) &> \mathsf{m}_0(b) + \mathsf{m}_0(c), && \text{whenever } \mathbf{v}_k(b) \neq \mathbf{v}_k(c). \end{aligned}$$

The proof itself is in [6], here we are only concerned with the idea of the proof.

Returning to $E = mc^2$: Cor. 5.2.2 above can be used to arrive at Einstein's insight $E = mc^2$ in the same way as it is done in Rindler's [51] and d'Inverno's [17] relativity textbooks. Namely, we have seen above that rest mass can be created under appropriate conditions. Created from what? Well, from kinetic energy (energy of motion). That points in the direction of Einstein's connecting mass with energy. In more detail, let us start with two bodies b_1 and b_2 of rest mass m_0. Let us accelerate the two bodies towards each other and let them collide inelastically, so that they stick together forming the new body "$b_1 + b_2$" (deliberately sloppy notation). Let us assume $b_1 + b_2$ is at rest relative to the observer conducting the experiment. Then the rest mass $\mathsf{m}_0(b_1 + b_2)$ is the sum of relativistic masses $\mathsf{m}_k(b_1)$ and $\mathsf{m}_k(b_2)$. Assuming that at collision the speed of both b_1 and b_2 were v, we have $\mathsf{m}_0(b_1 + b_2) = \mathsf{m}_0(b_1)/\sqrt{1 - v^2} + \mathsf{m}_0(b_2)/\sqrt{1 - v^2}$, which is definitely greater than $\mathsf{m}_0(b_1) + \mathsf{m}_0(b_2)$ if $v \neq 0$. So, rest mass was created from the kinetic energy supplied to our test bodies b_1 and b_2 when they were accelerated towards each other. So far, we have a qualitative argument (based on our $\mathsf{SpecRelDyn}^+$) in the direction that energy (in our example kinetic) can be "transformed" to "create" mass. A quantitatively (and physically) more detailed analysis of $E = mc^2$ in terms of Thm. 5.1.4 is given in [51, pp.111-114] where we refer the reader for more detail and for the "second part" of the argument. The "first part" was provided by Thm. 5.1.4 and Cor. 5.2.2.

Proposition 5.2.3.

$$\begin{aligned} \mathsf{SpecRelDyn} &\not\models \mathsf{ConsMass}, \quad \text{and} \\ \mathsf{SpecRelDyn} &\not\models \mathsf{ConsMomentum}. \end{aligned}$$

The proof of Prop. 5.2.3 is in [6].

In the literature, the conservation of relativistic mass and that of linear momentum are used to derive the conclusion of Thm. 5.1.4. By Prop. 5.2.3 above, our axiom system SpecRelDyn implies neither ConsMass nor ConsMomentum. By Thm. 5.2.1, ConsMass and ConsMomentum together imply the key axiom AxCenter of SpecRelDyn. So Thm. 5.1.4 is stronger than the corresponding result in the literature since it requires fewer assumptions.

Thm. 5.2.1 also states that the conservation axioms can be replaced by the natural, purely geometrical symmetry postulate $\mathsf{AxCenter}^+$ without loss of predictive power or expressive power. Since the conservation axioms ConsMass and ConsMomentum are not "purely geometrical" and they are less observation-oriented than $\mathsf{AxCenter}^+$, we think that it may be more convincing to use AxCenter or $\mathsf{AxCenter}^+$ in an axiom system when we introduce the basics of relativistic dynamics. See [68, p.22 footnote 22].

5.3 Four-momentum

Neither relativistic mass nor linear momentum is Lorentz-covariant. However, they can be "put together" to obtain a Lorentz-covariant quantity called four-momentum, as follows. Let $k \in \mathrm{Ob}$ and $b \in \mathrm{IB}$. The **four-momentum** $\vec{\mathbf{p}}_k(b)$ of *inertial* body b according to *inertial* observer k is defined as the element of Q^d whose time component and space component are the relativistic mass and linear momentum of b according to k, respectively, see Fig. 5.5. i.e.,

$$\vec{\mathbf{p}}_k(b)_\tau = \mathsf{m}_k(b) \quad \text{and} \quad \vec{\mathbf{p}}_k(b)_\sigma = \mathsf{m}_k(b)\mathbf{v}_k(b).$$

It is not difficult to prove that $\vec{\mathbf{p}}_k(b)$ is parallel to the world-line of b and its Minkowski length is $\mathsf{m}_0(b)$, see Prop. 5.3.1. Hence, it is indeed a Lorentz-covariant quantity. The **four-velocity** $\vec{\mathbf{v}}_k(b)$ of *inertial* body b according to *inertial* observer k is defined as $\vec{q} \in \mathrm{Q}^d$ if $q_\tau > 0$, $\mu(\vec{q}) = 1$ and $\vec{q}$ is parallel to $\mathsf{wl}_k(b)$. Let us note that $\vec{\mathbf{v}}_k(b)$ is defined iff $\mathbf{v}_k(b)$ is defined; and

$$\vec{\mathbf{v}}_k(b)_\tau = \frac{1}{\sqrt{1 - v_k(b)^2}} \quad \text{and} \quad \vec{\mathbf{v}}_k(b)_\sigma = \frac{\mathbf{v}_k(b)}{\sqrt{1 - v_k(b)^2}},$$

see Fig. 5.5.

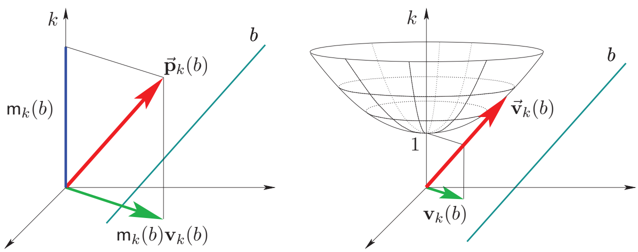

Figure 5.5: Illustration of four-momentum $\vec{\mathbf{p}}_k(b)$ and four-velocity $\vec{\mathbf{v}}_k(b)$

Proposition 5.3.1. Let $d \geq 3$. Assume SpecRelDyn and let k be an *inertial* observer and b an *inertial* body having rest mass. Then

$$\vec{\mathbf{p}}_k(b) = \mathsf{m}_0(b)\vec{\mathbf{v}}_k(b).$$

Proof. By the definition of $\vec{\mathbf{p}}_k(b)$, it is easy to see that

$$\mu\big(\vec{\mathbf{p}}_k(b)\big) = \sqrt{1 - v_k(b)^2} \cdot \mathsf{m}_k(b).$$

Hence by Thm. 5.1.4, $\mu\big(\vec{\mathbf{p}}_k(b)\big) = \mathsf{m}_0(b)$. Thus $\vec{\mathbf{p}}_k(b) = \mathsf{m}_0(b)\vec{\mathbf{v}}_k(b)$. ■

ConsFourMoment Conservation of four-momentum:

$$\forall k \in \mathrm{IOb}\ \forall b, c, d \in \mathrm{IB} \quad \mathsf{inecoll}_k(b, c : d) \ \rightarrow\ \vec{\mathbf{p}}_k(b) + \vec{\mathbf{p}}_k(c) = \vec{\mathbf{p}}_k(d).$$

The following can be easily proved from the definition of four-moment.

Proposition 5.3.2. $\mathsf{ConsFourMoment} \leftrightarrow \mathsf{ConsMass} \land \mathsf{ConsMoment}$.

Hence the following is an immediate corollary of Thm. 5.2.1.

Corollary 5.3.3.

$$\mathsf{AxMass} + \mathsf{AxInMass} + \mathsf{AxSelf} + \mathrm{IOb} \subseteq \mathrm{IB} \models \mathsf{AxCenter}^{+} \leftrightarrow \mathsf{ConsFourMoment}.$$

Let us return to discussing the merits of using $\mathsf{AxCenter}^{+}$ in place of the more conventional preservation principles. In the context of Cor. 5.2.2, $\mathsf{ConsFourMoment}$ has the advantage that it is computationally direct and simple, while $\mathsf{AxCenter}^{+}$ has the advantage that it is more observational, more geometrical, and more basic in some intuitive sense.

The following proposition shows the relation of $\mathsf{ConsFourMoment}$ and $\mathsf{AxCenter}$. By Cor. 5.3.3, it also shows that $\mathsf{AxCenter}^{+}$ is a strictly stronger axiom than $\mathsf{AxCenter}$.

Proposition 5.3.4. Assume AxMass and $\mathsf{AxInMass}$. Then $\mathsf{AxCenter}$ is equivalent to the following formula

$$\forall k \in \mathrm{Ob}\ \ \forall b, c, d \in \mathrm{IB} \quad \mathsf{inecoll}_k(b, c : d) \ \ \rightarrow\ \ \exists \lambda \in \mathrm{Q} \quad \vec{\mathbf{p}}_k(b) + \vec{\mathbf{p}}_k(c) = \lambda \vec{\mathbf{p}}_k(d).$$

Prop. 5.4.8 on p.51 is an extension of this proposition. The following is a consequence of Props. 5.3.2 and 5.3.4.

Corollary 5.3.5.

$$\begin{aligned}
\mathsf{AxMass} + \mathsf{AxInMass} + \mathsf{ConsMass} &\models \mathsf{AxCenter} \leftrightarrow \mathsf{ConsMoment}.\\
\mathsf{AxMass} + \mathsf{AxInMass} + \mathsf{ConsMoment} &\models \mathsf{ConsMass} \rightarrow \mathsf{AxCenter}.\\
\mathsf{AxMass} + \mathsf{AxInMass} + \mathsf{AxCenter} &\models \mathsf{ConsMass} \rightarrow \mathsf{ConsMoment}.
\end{aligned}$$

Remark 5.3.6. Let us, however, note that the two implications in the corollary above cannot be reversed since it is possible to construct a model in which there are an *inertial* observer k and *inertial* bodies b, c and d such that $\mathsf{inecoll}_k(b, c : d)$, $\mathbf{v}_k(b) = -\mathbf{v}_k(c)$, $v_k(d) = 0$ and $m_k(b) = m_k(c) = m_k(d)$; and in this model both $\mathsf{AxCenter}$ and $\mathsf{ConsMoment}$ hold while $\mathsf{ConsMass}$ does not hold.

Let us finally state a theorem about the existence of nontrivial models of our axiom systems. The proof of Thm. 5.3.7 can be found in [6].

Theorem 5.3.7. $\mathsf{SpecRelDyn}^{+} \cup \{\mathrm{IOb} \neq \emptyset\}$ is consistent.

5.4 Some possible generalizations

Let us formulate AxCenter in a more general setting. To do so, we introduce the set of bodies whose world-lines can be parametrized by differentiable curves according to any *inertial* observer:

$$\mathrm{DB} := \left\{ b \in \mathrm{B} \ : \ \forall k \in \mathrm{IOb} \quad \mathsf{loc}_k^b \text{ is a differentiable curve} \right\}.$$

For the FOL definition of differentiability, see Section 10.3.

AxCenterDiff If bodies $b, c \in \mathrm{DB}$ collide inelastically originating body $d \in \mathrm{DB}$, the world-line of d is a differentiable continuation of the center-line of the masses of b and c according to any *inertial* observer (see Fig. 5.6):

$$\forall k \in \mathrm{IOb}\ \forall b, c, d \in \mathrm{DB}\ \forall t \in \mathrm{Q} \quad \mathsf{inecoll}_k(b, c : d)$$
$$\wedge\ \mathsf{loc}_k^b(t) = \mathsf{loc}_k^c(t) = \mathsf{loc}_k^d(t) \ \rightarrow\ (\mathsf{cen}_k^{b,c})'(t) = (\mathsf{loc}_k^d)'(t).$$

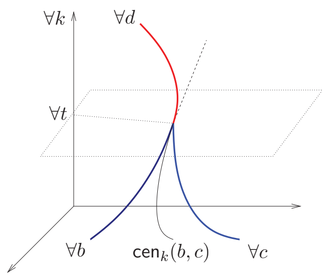

Figure 5.6: Illustration of axiom AxCenterDiff

By the following proposition, AxCenterDiff is an extension of AxCenter.

Proposition 5.4.1. AxMass + AxInMass $\models$ AxCenterDiff $\rightarrow$ AxCenter.

On the proof. The proof is based on the following two facts: (1) $\mathrm{IB} \subseteq \mathrm{DB}$, and (2) AxMass and AxInMass imply that $\mathsf{cen}_k(b, c)$ is a line segment if $k \in \mathrm{IOb}$, $b, c \in \mathrm{IB}$ such that $\mathsf{inecoll}_k(b, c : d)$, see Prop. 5.1.2. ■

It is also natural to generalize AxCenter to more than two bodies. Now we formulate some of the possible generalizations.

$\mathsf{AxCenter_n}$ After some two-by-two inelastic collisions of *inertial* bodies $b_1, \ldots, b_n$, their center-line of mass and the center-line of the masses of the last originated *inertial* body and the noncolliding bodies are in one line:

$$\forall k \in \mathrm{Ob}\ \ \forall b_1, \ldots, b_n \in \mathrm{IB}$$
$$\bigwedge_{j=1}^{n-1} \Big[\forall d_1, \ldots, d_{j+1} \in \mathrm{IB}\ \ d_1 = b_1\ \wedge\ \bigwedge_{i=1}^{j} \mathsf{inecoll}_k(d_i, b_{i+1} : d_{i+1})$$
$$\rightarrow\ \mathsf{cen}_k(b_1, \ldots, b_n) \cup \mathsf{cen}_k(d_{j+1}, b_{j+2}, \ldots, b_n) \subseteq \ell \text{ for some line } \ell \Big].$$

$\mathsf{AxCenter^*_n}$ During all successive two-by-two inelastic collisions of *inertial* bodies $b_1, \ldots, b_n$, the center-line of mass after each collision is the continuation of the center-line of mass before the collision:

$$\forall k \in \mathrm{Ob}\ \forall b_1, \ldots, b_n \in \mathrm{IB}$$
$$\bigwedge_{j=1}^{n-1} \Big[\forall d_1, \ldots, d_{j+1} \in \mathrm{IB}\ \ d_1 = b_1\ \wedge\ \bigwedge_{i=1}^{j} \mathsf{inecoll}_k(d_i, b_{i+1} : d_{i+1})\ \ \rightarrow$$
$$\mathsf{cen}_k(b_1, \ldots, b_n) \cup \bigcup_{s=1 \le j} \mathsf{cen}_k(d_{s+1}, b_{s+2}, \ldots, b_n) \subseteq \ell \text{ for some line } \ell \Big].$$

$\mathsf{AxCenter}$ is just $\mathsf{AxCenter_2}$ or $\mathsf{AxCenter^*_2}$ in these two series of axioms. Let us now decompose $\mathsf{AxCenter_n}$ and $\mathsf{AxCenter^*_n}$ into the following fragments:

$\mathsf{AxCenter_{n,j}}$ After j two-by-two inelastic collisions of *inertial* bodies $b_1, \ldots, b_n$, their center-line of mass and the center-line of the masses of the last originated *inertial* body and the noncolliding bodies are in one line:

$$\forall k \in \mathrm{Ob}\ \forall b_1, \ldots, b_n, d_1, \ldots, d_{j+1} \in \mathrm{IB}\quad d_1 = b_1\ \wedge\ \bigwedge_{i=1}^{j} \mathsf{inecoll}_k(d_i, b_{i+1} : d_{i+1})$$
$$\rightarrow\ \mathsf{cen}_k(b_1, \ldots, b_n) \cup \mathsf{cen}_k(d_{j+1}, b_{j+2}, \ldots, b_n) \subseteq \ell \text{ for some line } \ell.$$

The definition of $\mathsf{AxCenter^*_{n,j}}$ is analogous. Then $\mathsf{AxCenter_n}$ and $\mathsf{AxCenter^*_n}$ are equivalent to

$$\bigwedge_{j=1}^{n-1} \mathsf{AxCenter_{n,j}} \quad \text{and} \quad \bigwedge_{j=1}^{n-1} \mathsf{AxCenter^*_{n,j}},$$

respectively. Let us now see some of the logical connections between the above two series of axioms.

Proposition 5.4.2. Let x, y, n and m be natural numbers such that $1 \le x < y < n < m$. Then

(1) $\mathsf{AxCenter} \models \mathsf{AxCenter_{n,y}} \leftrightarrow \mathsf{AxCenter_{n,x}}$, and
$\mathsf{AxCenter} \models \mathsf{AxCenter^*_{n,y}} \leftrightarrow \mathsf{AxCenter^*_{n,x}}$.

(2) a.) $\mathsf{AxCenter^*_{n,y}} \models \mathsf{AxCenter^*_{n,x}}$, but
b.) $\mathsf{AxCenter_{n,y}} \not\models \mathsf{AxCenter_{n,x}}$.

(3) a.) $\mathsf{AxCenter^*_{n,x}} \models \mathsf{AxCenter_{n,x}}$, but
b.) $\mathsf{AxCenter_{n,x}} \not\models \mathsf{AxCenter^*_{n,x}}$.

(4) a.) $\mathsf{AxCenter_{n,x}} \leftrightarrow \mathsf{AxCenter_{m,x}}$, and
b.) $\mathsf{AxCenter^*_{n,x}} \leftrightarrow \mathsf{AxCenter^*_{m,x}}$.

Proof. Item (1) can be proved by induction on $y - x$.
Item (2a) is true since

$$\bigcup_{s=1\leq x} \mathsf{cen}_k(d_{s+1}, b_{s+2}, \ldots, b_n) \subseteq \bigcup_{s=1\leq y} \mathsf{cen}_k(d_{s+1}, b_{s+2}, \ldots, b_n).$$

To prove (2b), let $\mathfrak{M}$ be a model such that IB:=$\{b_1, \ldots, b_n, d_1, \ldots d_{x+1}\}$, IOb:=$\{k\}$, $wl_k(k) = \emptyset$, $b_1 = d_1$ and $\mathsf{AxCenter_{n,x}}$ is not valid. It is not difficult to see that there is such a model $\mathfrak{M}$. Since there are no $n + y - 1$ pieces of distinct bodies in the required collision situation, $\mathsf{AxCenter_{n,y}}$ is valid (its condition is empty).

Item (3a) is true since

$$\mathsf{cen}_k(d_{x+1}, b_{x+2}, \ldots, b_n) \subseteq \bigcup_{s=1\leq x} \mathsf{cen}_k(d_{s+1}, b_{s+2}, \ldots, b_n).$$

Item (3b) is true since its opposite together with (2a) and (3a) states that $\mathsf{AxCenter_{n,x+1}} \models \mathsf{AxCenter^*_{n,x+1}} \models \mathsf{AxCenter^*_{n,x}} \models \mathsf{AxCenter_{n,x}}$ which contradicts (2b).

Item (4) is true since noncolliding bodies cannot affect the center-line of mass. ■

Let us now introduce an axiom about the most general situations of two-by-two inelastic collisions. Let Sq_n be the set of at most n-long sequences of natural numbers between 1 and n. Let us denote the concatenation of sequences $a, b \in Sq_n$ by $a\hat{\ }b$. We say that GI_n is a **generalized index set** of basis n if

- $\{1, \ldots, n\} \subseteq GI_n \subseteq Sq_n$;
- $a_1 < \ldots < a_k$ if $a_1\hat{\ }\ldots\hat{\ }a_k \in GI_n$ and $a_1, \ldots, a_k \in \{1, \ldots, n\}$; and
- sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are disjoint whenever $a_1\hat{\ }\ldots\hat{\ }a_k, b_1\hat{\ }\ldots\hat{\ }b_k \in GI_n$ and $a_1, \ldots, a_k, b_1, \ldots, b_k \in \{1, \ldots, n\}$.

$\mathsf{AxCenter^*GI_n}$ During a collision situation of type GI_n of *inertial* bodies $b_1, \ldots, b_n$, the center-line of mass after each collision is the continuation of the center-line of mass before the collision:

$$\forall k \in \mathrm{Ob}\ (\forall b_i \in \mathrm{IB}\ \ i \in GI_n) \bigwedge_{i_1,\ldots,i_s,i_1\hat{}\ldots\hat{}i_s \in GI_n} \mathsf{inecoll}_k(b_{i_1},\ldots,b_{i_s} : b_{i_1\hat{}\ldots\hat{}i_s})$$
$$\rightarrow \bigcup_{j_1,\ldots,j_m \in GI_n\ j_1\hat{}\ldots\hat{}j_m = 1\hat{}\ldots\hat{}n} \mathsf{cen}_k(b_{j_1}, b_{j_2}, \ldots, b_{j_m}) \subseteq \ell \text{ for some line } \ell.$$

The following theorem shows that $\mathsf{AxCenter}$ and $\mathsf{AxCenter^+}$ imply all the generalizations above.

Theorem 5.4.3. Let n be a natural number and let GI_n be a generalized index set. Then

$$\mathsf{AxCenter} + \mathsf{AxCenter^+} \models \mathsf{AxCenter^*GI_n}.$$

On the proof The theorem can be proved by induction on the complexity of the generalized index set GI_n. ■

Corollary 5.4.4. Let n be a natural number. Then

$$\mathsf{AxCenter} \models \mathsf{AxCenter^+} \rightarrow \mathsf{AxCenter_n}, \quad \text{and}$$
$$\mathsf{AxCenter} \models \mathsf{AxCenter^+} \rightarrow \mathsf{AxCenter^*_n}.$$

Let us finally generalize $\mathsf{AxCenter}$ to noninelastic collisions, too.

$\mathsf{AxCenter_{n:m}}$ If *inertial* bodies $b_1, \ldots, b_n$ collide originating *inertial* bodies $d_1, \ldots, d_m$, the center-line of the masses of $d_1, \ldots, d_m$ is the continuation of the center-line of the masses of $b_1, \ldots, b_n$:

$$\forall k \in \mathrm{Ob}\ \forall b_1,\ldots,b_n,d_1,\ldots,d_m \in \mathrm{IB}\quad \mathsf{coll}_k(b_1,\ldots,d_n : d_1,\ldots,d_m)$$
$$\rightarrow \mathsf{cen}_k(b_1,\ldots,b_n) \cup \mathsf{cen}_k(d_1,\ldots,d_m) \subseteq \ell \text{ for some line } \ell.$$

$\mathsf{AxCenter}$ is just $\mathsf{AxCenter_{2:1}}$ in this series of axioms since $\mathsf{inecoll}_k(b, c : d)$ is the same formula as $\mathsf{coll}_k(b, c : d)$.

Remark 5.4.5. $\{\mathsf{AxCenter_{n:m}} : n, m \in \omega\}$ is an independent axiom system.

Let us now extend the formula expressing the conservation of four-momentum of two inelastically colliding *inertial* bodies ($\mathsf{ConsFourMomentum}$) to general collision situations.

ConsFourMomentum$_{n:m}$ If *inertial* bodies $b_1, \ldots, b_n$ collide and originate *inertial* bodies $d_1, \ldots, d_m$, the sum of four-momentums of $d_1, \ldots, d_m$ and $b_1, \ldots, b_n$ is the same:

$$\forall k \in \mathrm{IOb}\ \forall b_1, \ldots, b_n, d_1, \ldots, d_m \in \mathrm{IB}$$
$$\mathsf{coll}_k(b_1, \ldots, b_n : d_1, \ldots, d_m) \ \rightarrow\ \sum_{i=1}^{n} \vec{\mathbf{p}}_k(b_i) = \sum_{j=1}^{m} \vec{\mathbf{p}}_k(d_j).$$

Remark 5.4.6. It is suggested by Rindler to assume all the formulas above as axioms of relativistic dynamics, see [51, p.109]. However, the formulas ConsFourMoment$_{n:m}$ are not natural and observational enough assumptions to regard them as axioms. By Cor. 5.4.9, we can offer a list of more natural formulas to be assumed, which is equivalent to the list above.

Remark 5.4.7. The following theory is consistent and independent:

$$\mathsf{SpecRel} \cup \{\, \mathrm{IOb} \neq \emptyset, \mathsf{AxThExp} \,\} \cup \{\, \mathsf{AxInMass}, \mathsf{AxMass} \,\}$$
$$\cup \{\, \mathsf{Ax\forall coll}_{n:m}, \mathsf{ConsFourMoment}_{n:m} \ :\ n, m \in \omega \,\},$$

where Ax∀coll$_{n:m}$ is a generalization of Ax∀inecoll which ensures the realization of every possible collision of type "$n : m$."

The following proposition extends Prop. 5.3.4. It also shows the logical connection between AxCenter$_{n:m}$ and ConsFourMoment$_{n:m}$.

Proposition 5.4.8. Assume AxMass and AxInMass. Let k be an *inertial* observer and let $b_1, \ldots, b_n$ be *inertial* bodies. Then the following is equivalent to AxCenter$_{n:m}$:

$$\forall k \in \mathrm{IOb}\ \ \forall b_1, \ldots, b_n, d_1, \ldots, d_m \in \mathrm{IB}$$
$$\mathsf{coll}_k(b_1, \ldots, b_n : d_1, \ldots, d_m) \ \rightarrow\ \exists \lambda \in \mathrm{Q} \quad \sum_{i=1}^{n} \vec{\mathbf{p}}_k(b_i) = \lambda \sum_{j=1}^{m} \vec{\mathbf{p}}_k(d_j).$$

Proof. By Lem. 5.4.10, $\mathsf{cen}_k(b_1, \ldots, b_n)$ and $\mathsf{cen}_k(d_1, \ldots, d_n)$ are parallel iff $\sum_{i=1}^{n} \vec{\mathbf{p}}_k(b_i)$ and $\sum_{j=1}^{m} \vec{\mathbf{p}}_k(d_j)$ are parallel. We have that $\mathsf{cen}_k(b_1, \ldots, b_n) \cap \mathsf{cen}_k(d_1, \ldots, d_n) \neq \emptyset$ if $\mathsf{coll}_k(b_1, \ldots, b_n : d_1, \ldots, d_m)$. Thus AxCenter$_{n:m}$ holds iff

$$\sum_{i=1}^{n} \vec{\mathbf{p}}_k(b_i) = \lambda \sum_{j=1}^{m} \vec{\mathbf{p}}_k(d_j)$$

for some $\lambda \in \mathrm{Q}$. ■

Let us now introduce a list of axioms which are formulated in the spirit of AxCenter and whose elements are equivalent to the corresponding $\mathsf{ConsFourMomentum}_{\mathsf{n:m}}$ formulas.

$\mathsf{AxCenter}^{+}_{\mathsf{n:m}}$ If a is an *inertial* body and *inertial* bodies $b_1, \ldots, b_n$ collide originating *inertial* bodies $d_1, \ldots, d_m$, the center-line of the masses of $a, b_1, \ldots, b_n$ is the continuation of the center-line of the masses of $a, d_1, \ldots, d_m$, i.e., there is a line that contains both of them:

$$\forall k \in \mathrm{IOb}\ \forall b_1, \ldots, b_n, d_1, \ldots, d_m \in \mathrm{IB} \quad \mathsf{coll}_k(b_1, \ldots, b_n : d_1, \ldots, d_m)$$
$$\rightarrow\ \mathsf{cen}_k(a, b_1, \ldots, b_n) \cup \mathsf{cen}_k(a, d_1, \ldots, d_m) \subseteq \ell \text{ for some line } \ell.$$

The following corollary can be proved from Prop. 5.4.8 in a strictly analogous way to the proof of Cor. 5.3.3.

Corollary 5.4.9.

$$\mathsf{AxInMass} + \mathsf{AxMass} + \mathsf{AxSelf} + \mathrm{IOb} \subseteq \mathrm{IB} \models \mathsf{Center}^{+}_{\mathsf{n:m}} \leftrightarrow \mathsf{ConsFourMomentum}_{\mathsf{n:m}}$$

Lemma 5.4.10. Assume AxMass and AxInMass. Then for all $k \in \mathrm{IOb}$ and $b_1, \ldots, b_n \in \mathrm{IB}$,

$$\mathsf{cen}_k(b_1, \ldots, b_n) \quad \text{is parallel to} \quad \sum_{i=1}^{n} \vec{\mathbf{p}}_k(b_i)$$

if $\mathsf{cen}_k(b_1, \ldots, b_n) \neq \emptyset$.

Proof. We prove the statement by induction on n. It is clear if $n = 1$ since $\vec{\mathbf{p}}_k(b)$ is parallel to $\mathsf{cen}_k(b) = \mathsf{wl}_k(b)$ if AxMass and AxInMass are assumed, see Prop. 5.1.2. Now we have to prove that if it is true for n, then it is also true for $n+1$. Let $b_1, \ldots, b_n, b_{n+1} \in \mathrm{IB}$. Since $\mathsf{cen}_k(b_1, \ldots, b_{n+1}) \neq \emptyset$ and translations preserve parallelism, we can assume that $\vec{o} \in \mathsf{cen}_k(b_1, \ldots, b_{n+1})$. By the induction hypothesis, we can expand our model with *inertial* body c such that $\mathsf{wl}_k(c) = \mathsf{cen}_k(b_2, \ldots, b_{n+1})$ and $\vec{\mathbf{p}}_k(c) = \sum_{i=2}^{n+1} \vec{\mathbf{p}}_k(b_i)$. Since $\mathsf{cen}_k(b_1, \ldots, b_n) = \mathsf{cen}_k(b_1, c)$, we have to prove that $\mathsf{cen}_k(b_1, c)$ is parallel to $\vec{\mathbf{p}}_k(b_1) + \vec{\mathbf{p}}_k(c)$. Without losing generality, we can assume that $\mathsf{wl}_k(b_1)$ and $\mathsf{wl}_k(c)$ are lines. Then

$$\vec{\mathbf{p}}_k(b_1) \cdot \frac{\mathsf{m}_k(b_1) + \mathsf{m}_k(c)}{\mathsf{m}_k(b_1)} \quad \text{and} \quad \vec{\mathbf{p}}_k(c) \cdot \frac{\mathsf{m}_k(b_1) + \mathsf{m}_k(c)}{\mathsf{m}_k(c)}$$

have the same time component $\mathsf{m}_k(b_1) + \mathsf{m}_k(c)$; and they are in $\mathsf{wl}_k(b_1)$ and $\mathsf{wl}_k(c)$, respectively. So $\mathsf{cen}_k^{b_1,c}\big(\mathsf{m}_k(b_1) + \mathsf{m}_k(c)\big) = \vec{\mathbf{p}}_k(b_1) + \vec{\mathbf{p}}_k(c)$ since $\vec{o} \in \mathsf{cen}_k(b_1, c)$. Hence $\mathsf{cen}_k(b_1, c)$ and $\vec{\mathbf{p}}_k(b_1) + \vec{\mathbf{p}}_k(c)$ are parallel; and that is what was to be proved. ■

5.5 Collision duality

To every formula φ in the language of SpecRelDyn, we can define its **collision dual** $\varphi^{\otimes}$ in which the subformulas of the form $\mathsf{coll}_k(b_1, \ldots, b_n : d_1, \ldots, d_m)$ are replaced by $\mathsf{coll}_k(d_1, \ldots, d_m : b_1, \ldots, b_n)$, i.e., the incoming and the outgoing bodies in the collisions are interchanged. By this definition of collision dual, it is clear that $(\varphi^{\otimes})^{\otimes}$ and φ are the same. Let Σ be a set of formulas. The set of collision duals of the formulas of Σ is denoted by $\Sigma^{\otimes}$.

Formula φ is called **self-dual** if φ and $\varphi^{\otimes}$ are the same. Every formula which is not about collisions is self-dual. So most of the axioms of SpecRelDyn are self-dual. There are also self-dual axioms about collisions, e.g., $\mathsf{AxCenter}_{\mathsf{n:n}}$ is such for all natural number n.

Proposition 5.5.1. Let Σ be a set of formulas and let φ be formula in the language of SpecRelDyn. Then

$$\Sigma \models \varphi \quad \text{iff} \quad \Sigma^{\otimes} \models \varphi^{\otimes}.$$

Proof. By Gödel's completeness theorem, $\Sigma \models \varphi$ holds iff $\Sigma \vdash \varphi$, i.e., there is a formal proof of φ from Σ. It is clear that $\Sigma \vdash \varphi$ iff $\Sigma^{\otimes} \vdash \varphi^{\otimes}$ since we get a formal proof of $\varphi^{\otimes}$ from Σ by replacing every formula by its collision in the formal proof of φ from Σ. From this, we get $\Sigma^{\otimes} \models \varphi^{\otimes}$ by Gödel's completeness theorem. ■

By applying Prop. 5.5.1 to Thm. 5.1.4, we get the following as its conclusion is self-dual.

Corollary 5.5.2. Let $d \geq 3$ and assume $\mathsf{SpecRelDyn}^{\otimes}$. Let k be an *inertial* observer and b be an *inertial* body having rest mass. Then

$$\mathsf{m}_0(b) = \sqrt{1 - v_k(b)^2} \cdot \mathsf{m}_k(b).$$

Remark 5.5.3. It is natural to interpret inecoll as nuclear fusion. By this interpretation $\mathsf{inecoll}^{\otimes}$ becomes nuclear fission. In this case axioms $\mathsf{Ax\forall Center}^{\otimes}$ and $\mathsf{Ax\forall Center}$ (and even $\mathsf{Ax\exists Center}^{\otimes}$ and $\mathsf{Ax\exists Center}$) are too strong since they require the existence of several fusions and fissions which might not exist in nature. However, it is not a problem since all the theorems which use these axioms can be reformulated without them by building them into the statements. So instead of assuming that certain fusion/fission situations exist and proving a statement from that, we can omit this assumption and prove the statement only for the bodies which appear in the corresponding fusion/fission situations.

5.6 Concluding remarks on dynamics

We have introduced a purely geometrical axiom system of special relativistic dynamics which is strong enough to prove the formula connecting relativistic and rest masses of bodies. We have also studied the connection of our key axioms AxCenter and AxCenter$^+$ and the usual axioms about the conservation of mass, momentum and four-momentum. We saw that the conservation postulates are not needed to prove the relativistic mass increase theorem $m_0 = \sqrt{1 - v^2/c^2} \cdot m$, see Prop. 5.2.3 at (p.44) and Thm. 5.1.4 at (p.40). See also [68, p.22 footnote 22]. Connections with Einstein's insight $E = mc^2$ have also been discussed. The contents of the present chapter represent only the first steps towards a logical conceptual analysis of relativistic dynamics. A glimpse into Chap. 6 (pp.108-130) "Relativistic particle mechanics" of the textbook by Rindler [51] suggests the topics to be covered by future work in this line. In another direction, looking at the logical issues in [2] and [4] suggests questions and investigations to be carried out in the future about the logical analysis of relativistic dynamics.
Let us mention here two tasks that should be done in the future:

Question 5.6.1. Analyzing the possibility/impossibility of faster than light motion of colliding bodies within axiomatic special relativistic dynamics similarly to what was done for observers within special relativistic kinematic, see, e.g., [4, Thm.1.7], [37, Thm.3, Thm.5].

Question 5.6.2. Extending the axiomatization of relativistic dynamics for accelerated observers.

A work related to this chapter with somewhat different aims is [55].

Chapter 6

Extending the axioms of special relativity for accelerated observers

In this chapter we extend our axiomatization of special relativity to non-*inertial* observers, too. Non-*inertial* observers are also going to be called **accelerated observers**. We have two reasons for extending our aproach to accelerated observers: to take a step towards a FOL axiomatization of general relativity (see Chap. 9) and to provide an axiomatic basis of the twin paradox and other surprising predictions of special relativity extended to non-*inertial* observers. The results of this chapter are based on [5], [34] and [72]. A further aim is to prove predictions of general relativity from our theory of accelerated observers, by using Einstein's equivalence principle. Compare our interpretation of answering why-questions at p.4 or [73].

6.1 The key axiom of accelerated observers

It is clear that SpecRel is too weak to answer any nontrivial question about acceleration since AxSelf_0 is its only axiom that mentions non-*inertial* observers. To extend SpecRel, we now formulate the key axiom about accelerated observers. It will state that the world-views of accelerated and *inertial* observers are locally the same.

To connect the world-views of the accelerated and the *inertial* observers, we formulate the statement that, at each moment of its world-line, each accelerated observer coordinatizes the nearby world for a short while as an *inertial* observer does. To formalize that, first we introduce the relation of being a co-moving observer. Observer m is a **co-moving observer** of observer k at $\vec{q} \in \mathrm{Q}^d$, in symbols $m \succ_{\vec{q}} k$, iff $\vec{q} \in Dom\, w^k_m$ and the following holds:

$$\forall \varepsilon \in \mathrm{Q}^+\ \exists \delta \in \mathrm{Q}^+\ \forall \vec{p} \in B_\delta(\vec{q}) \cap Dom\, w^k_m \quad \left| w^k_m(\vec{p}) - \vec{p} \right| \le \varepsilon \cdot |\vec{p} - \vec{q}|.$$

Behind the definition of co-moving observers is the following intuitive image: as we zoom in the neighborhood of the coordinate point, the world-views of the two observers are getting more and more similar.

Remark 6.1.1. Let us note that $\vec{q} \in Cd_m$ and $ev_m(\vec{q}) = ev_k(\vec{q})$ if $m \succ_{\vec{q}} k$. It can be proved by choosing $\vec{p}$ as $\vec{q} \in B_\delta(\vec{q}) \cap Dom\, w_m^k$.

Remark 6.1.2. By Conv. 2.4.2, there is a $\delta \in Q^+$ such that w_m^k is a function on $B_\delta(\vec{q}) \cap Dom\, w_m^k$ if $m \succ_{\vec{q}} k$. So w_m^k is a function on a small enough neighborhood of $\vec{q}$ if $m \succ_{\vec{q}} k$ and $Dom\, w_m^k$ is open.

The relation $\succ_{\vec{q}}$ is transitive but it is neither reflexive nor symmetric. It is not reflexive because if k is an observer such that $ev_k(\langle 0, \frac{1}{n}, 0 \ldots, 0\rangle) = ev_k(\vec{o})$ for all $n \in \omega$, then w_k^k is not a function on any neighborhood of $\vec{o}$. Thus $k \not\succ_{\vec{o}} k$ see Rem. 6.1.2. Example 6.2.4 shows that $\succ_{\vec{q}}$ is not symmetric. The relation $\succ_{\vec{q}}$ becomes an equivalence relation, e.g., if w_m^k is a function and defined in a small enough neighborhood of $\vec{q}$ for each $k, m \in \mathrm{Ob}$. This will be the case in our last axiom system GenRel in Chap. 9.

Now we can formulate the key axiom of accelerated observers, called the co-moving axiom. This axiom is about the connection between the world-views of *inertial* and accelerated observers:

AxCmv For every observer and event encountered by it, there is a co-moving *inertial* observer:

$$\forall k \in \mathrm{Ob}\ \forall \vec{q} \in Q^d \quad k \in ev_k(\vec{q}) \ \rightarrow\ \exists m \in \mathrm{IOb}\ \ m \succ_{\vec{q}} k.$$

Remark 6.1.3. Let us note that from AxCmv and AxEv follows that *inertial* observers coordinatize every event encountered by an observer, i.e., $e \in Ev_m$ for all event e and *inertial* observer m whenever $k \in e \in Ev_k$ for some observer k. That is true since *inertial* observers coordinatize the same events by AxEv; there is an $m \in \mathrm{IOb}$ such that $m \succ_{\vec{q}} k$ and $ev_k(\vec{q}) = e$ by AxCmv; and $\vec{q} \in Cd_m$ if $m \succ_{\vec{q}} k$, see Rem. 6.1.1.

Before we go on building our theory of accelerated observers, let us prove a proposition reformulating the co-moving relation. For the notion of differentiability in our framework, see Section 10.3. Let us note that in our framework a function differentiable at $\vec{q}$ may have several derivatives at $\vec{q}$.

Proposition 6.1.4. Let m and k be observers and $\vec{q}$ be a coordinate point. The following two statements are equivalent:

(a) $m \succ_{\vec{q}} k$, i.e., m is a co-moving observer of k at $\vec{q}$.

(b) $w^k_m(\vec{q}) = \vec{q}$, w^k_m is differentiable at $\vec{q}$, and one of its derivatives at $\vec{q}$ is the identity map.

Proof. To prove $(a) \implies (b)$, let $m \succ_{\vec{q}} k$. Then $\vec{q} \in Dom\, w^k_m$ by definition. By Rem. 6.1.2, we have that w^k_m is a function on $B_{\delta_0}(\vec{q}) \cap Dom\, w^k_m$ for some $\delta_0 \in Q^+$. Thus $w^k_m(\vec{q})$ is defined, and it is a $\vec{q}$ by Rem. 6.1.1. By the definition of $m \succ_{\vec{q}} k$, for all $\varepsilon \in Q^+$, there is a $\delta \in Q^+$ such that the inequality $|w^k_m(\vec{p}) - \vec{p}| \leq \varepsilon|\vec{p} - \vec{q}|$ holds for all $\vec{p} \in B_\delta(\vec{q}) \cap Dom\, w^k_m$. Since $w^k_m(\vec{q}) = \vec{q}$, this inequality can be rewritten as $|w^k_m(\vec{p}) - w^k_m(\vec{q}) - Id(\vec{p} - \vec{q})| \leq \varepsilon|\vec{p} - \vec{q}|$. So w^k_m is differentiable at $\vec{q}$ and one of its derivatives at $\vec{q}$ is the identity map.

To prove the converse implication, let w^k_m be differentiable at $\vec{q}$ such that one of its derivatives at $\vec{q}$ is the identity map, and $w^k_m(\vec{q}) = \vec{q}$. Then for all $\varepsilon \in Q^+$, there is a $\delta \in Q^+$ such that $|w^k_m(\vec{p}) - w^k_m(\vec{q}) - Id(\vec{p} - \vec{q})| \leq \varepsilon|\vec{p} - \vec{q}|$ holds for all $\vec{p} \in B_\delta(\vec{q}) \cap Dom\, w^k_m$. And, since $w^k_m(\vec{q}) = \vec{q}$, this last inequality is the same as $|w^k_m(\vec{p}) - \vec{p}| \leq \varepsilon|\vec{p} - \vec{q}|$. Thus $m \succ_{\vec{q}} k$. ■

The world-line of an observer represents the set of coordinate points where the observer is during its life but it does not tell how "old" the observer is at a certain event. So let us define the **life-curve** lc^k_m of observer k according to observer m as the world-line of k according to m *parametrized by the time measured by* k, formally:

$$\mathsf{lc}^k_m := \left\{ \langle t, \vec{p} \rangle \in Q \times Q^d \ : \ \exists \vec{q} \in Q^d \quad k \in ev_k(\vec{q}) = ev_m(\vec{p}) \ \wedge \ q_\tau = t \right\}.$$

For the most important properties of life-curves, see Prop. 6.1.6.

Let the natural embedding $\iota : Q \to Q^d$ be defined as $\iota(x) := \langle x, 0, \ldots, 0 \rangle$ for all $x \in Q$.

Lemma 6.1.5. Assume AxSelf_0. Let k and m be observers. Then $\mathsf{lc}^k_m := \iota \circ w^k_m$.

Proof. By our definitions,

$$\iota \circ w^k_m = \left\{ \langle t, \vec{p} \rangle \in Q \times Q^d \ : \ \exists \vec{q} \in Q^d \quad ev_k(\vec{q}) = ev_m(\vec{p}) \neq \emptyset \ \wedge \ q_\tau = t \ \wedge \ \vec{q}_\sigma = \vec{o} \right\}.$$

By AxSelf_0 and the fact that $ev_k(\vec{q}) \neq \emptyset$, we have $\vec{q}_\sigma = \vec{o}$ iff $k \in ev_k(\vec{q})$. So $\mathsf{lc}^k_m = \iota \circ w^k_m$. ■

Let us introduce here a very natural axiom about observers, which is going to be used in the following proposition.

AxEvTr Every observer encounters the events in which it has been observed:

$$\forall m \in \mathrm{Ob} \ \forall e \in Ev \quad m \in e \ \rightarrow \ e \in Ev_m.$$

Proposition 6.1.6. Let m, k and h be observers. Then

(1) lc_m^k is a function iff

(i) event e has a unique coordinate in Cd_m whenever $k \in e \in Ev_m \cap Ev_k$, and

(ii) $ev_k(\vec{q}) = ev_k(\vec{q}\,')$ if $\vec{q}, \vec{q}\,' \in Cd_k$ such that $ev_k(\vec{q}), ev_k(\vec{q}\,') \in Ev_m$, $k \in ev_k(\vec{q}) \cap ev_k(\vec{q}\,')$ and $q_\tau = q'_\tau$.

(2) lc_m^k is a function if m is *inertial*, and $\mathsf{AxPh_0}$ and $\mathsf{AxSelf_0}$ are assumed.

(3) $\mathsf{lc}_m^h \supseteq \mathsf{lc}_k^h \circ w_m^k$ always holds, and
$\mathsf{lc}_m^h = \mathsf{lc}_k^h \circ w_m^k$ holds if we assume $Ev_m \subseteq Ev_k$.

(4) $\{q_\tau : k \in ev_k(\vec{q})\} = Dom\, \mathsf{lc}_k^k \supseteq Dom\, \mathsf{lc}_m^k$ always holds, and
$Dom\, \mathsf{lc}_m^k = Dom\, \mathsf{lc}_k^k$ holds if we assume AxCmv and $m \in \mathrm{IOb}$.

(5) $Ran\, \mathsf{lc}_m^k \subseteq \mathsf{wl}_m(k)$ always holds, and
$Ran\, \mathsf{lc}_m^k = \mathsf{wl}_m(k)$ if we assume AxEvTr.

Proof. Item (1) is a straightforward consequence of the definition of lc_m^k. To see that, let $R{:=}\{\langle t, \vec{q}\rangle \in \mathrm{Q} \times Cd_k \,:\, k \in ev_k(\vec{q}) \,\wedge\, q_\tau = t\}$. Then $\mathsf{lc}_m^k = R \circ w_m^k = R \circ ev_k \circ \mathsf{loc}_m$. Since ev_k is a function and loc_m is an inverse of a function, it is easy to see that lc_m^k is a function iff loc_m is a function on $Ran\,(R \circ ev_k)$ and $R \circ ev_k$ is a function to $Dom\, \mathsf{loc}_m = Ev_m$. It is clear that loc_m is a function on $Ran\,(R \circ ev_k)$ iff (i) holds; and it is also clear that R is a function to $Dom\, \mathsf{loc}_m = Ev_m$ iff (ii) holds. Hence lc_m^k is a function iff both (i) and (ii) hold.

To prove Item (2), we should check (i) and (ii) of Item (1). By Item (1) of Prop. 3.1.3, (i) is true. By $\mathsf{AxSelf_0}$ if $k \in ev_k(\vec{q}) \cap ev_k(\vec{q}\,')$, then $\vec{q}_\sigma = \vec{o} = \vec{q}\,'_\sigma$. Thus if $q_\tau = q'_\tau$ also holds, then $\vec{q} = \vec{q}\,'$. Hence (ii) is also true.

To prove Item (3), let $\langle t, \vec{p}\rangle \in \mathsf{lc}_k^h \circ w_m^k$. That means $\exists \vec{c} \in Cd_k$ such that $\langle t, \vec{c}\rangle \in \mathsf{lc}_k^h$ and $\langle \vec{c}, \vec{p}\rangle \in w_m^k$, which is equivalent to $\exists \vec{q} \in Cd_h$ such that $h \in ev_h(\vec{q}) = ev_k(\vec{c})$, $q_\tau = t$ and $ev_k(\vec{c}) = ev_m(\vec{p})$. Thus $\langle t, \vec{p}\rangle \in \mathsf{lc}_m^h$. To prove the converse inclusion, let $\langle t, \vec{p}\rangle \in \mathsf{lc}_m^h$. That means that there is a coordinate point $\vec{q} \in Cd_h$ such that $h \in ev_h(\vec{q}) = ev_m(\vec{p})$ and $q_\tau = t$. By the assumption $Ev_m \subseteq Ev_k$, we have that $\exists \vec{c} \in Cd_k$ such that $ev_k(\vec{c}) = ev_m(\vec{p})$. Thus $\langle t, \vec{p}\rangle \in \mathsf{lc}_k^h \circ w_m^k$. That proves Item (3).

To prove Item (4), let us recall that $t \in Dom\, \mathsf{lc}_m^k$ iff there are $\vec{p} \in Cd_m$ and $\vec{q} \in Cd_k$ such that $k \in ev_m(\vec{p}) = ev_k(\vec{q})$ and $q_\tau = t$. From that, it easily follows that $t \in Dom\, \mathsf{lc}_k^k$ iff there is a coordinate point $\vec{q} \in Cd_k$ such that $q_\tau = t$ and $k \in ev_k(\vec{q})$. Thus $\{q_\tau : k \in ev_k(\vec{q})\} = Dom\, \mathsf{lc}_k^k \supseteq Dom\, \mathsf{lc}_m^k$ is clear; and if we assume AxCmv and

$m \in \mathrm{IOb}$, then $Dom\,\mathsf{lc}_k^k \subseteq Dom\,\mathsf{lc}_m^k$ is also clear since *inertial* observers coordinatize every event encountered by observers, see Rem. 6.1.3.

To prove Item (5), let us recall that $\vec{p} \in Ran\,\mathsf{lc}_m^k$ iff $\vec{p} \in Cd_m$ and there are $t \in \mathrm{Q}$ and $\vec{q} \in Cd_k$ such that $k \in ev_m(\vec{p}) = ev_k(\vec{q})$ and $q_\tau = t$. Thus $Ran\,\mathsf{lc}_m^k \subseteq \mathsf{wl}_m(k) := \{\vec{p} \in Cd_m : k \in ev_m(\vec{p})\}$ is clear. If $\vec{p} \in \mathsf{wl}_m(k)$, then $k \in ev_m(\vec{p})$. Therefore, by AxEvTr, we have that $ev_m(\vec{p}) \in Ev_k$. Thus there is a coordinate point $\vec{q} \in Cd_k$ such that $ev_m(\vec{p}) = ev_k(\vec{q})$. Hence $Ran\,\mathsf{lc}_m^k = \mathsf{wl}_m(k)$. ■

We call a timelike curve α **well-parametrized** if $\mu\big(\alpha'(t)\big) = 1$ for all $t \in Dom\,\alpha$. For the FOL definition of α', see Section 10.3.

Assume $\mathfrak{Q} = \mathbb{R}$. Then curve f is well-parametrized iff f is parametrized according to the Minkowski length, i.e., for all $x, y \in Dom\,f$, the Minkowski length of f restricted to $[x, y]$ is $y - x$. (By the Minkowski length of a curve we mean length according to the Minkowski metric, e.g., in the sense used by Wald [80, p.43, (3.3.7)]). If the *proper time* is defined as the Minkowski length of a timelike curve, see, e.g., Wald [80, p.44, (3.3.8)], Taylor-Wheeler [74, 1-1-2] or d'Inverno [17, p.112, (8.14)], a curve defined on a subset of $\mathbb{R}$ is well-parametrized iff it is parametrized according to proper time (see, e.g., [17, p.112, (8.16)]). Hence for well-parametrized curves, our definition of proper time (see p.16) coincides with the definition of the literature.

Example 6.1.7. Let us list some examples of well-parametrized curves here:

(1) $\gamma(t) = 1/2 \cdot \langle t^3/3 - 1/t, t^3/3 + 1/t, 0, \ldots, 0\rangle$ for all $t \in \mathrm{Q}^+$.

(2) $\gamma(t) = \langle \sqrt{t^3}/3 + \sqrt{t}, \sqrt{t^3}/3 - \sqrt{t}, 0, \ldots, 0\rangle$ for all $t \in \mathrm{Q}^+$.

(3) $\gamma(t) = \langle a \cdot sh(t/a), a \cdot ch(t/a), 0, \ldots, 0\rangle$ for all $a \in \mathbb{R}^+$ and $t \in \mathbb{R}$.

(4) $\gamma(t) = \langle \sqrt{a^2+1} \cdot t, \cos(a \cdot t), \sin(a \cdot t), 0, \ldots, 0\rangle$ for all $a \in \mathbb{R}^+$ and $t \in \mathbb{R}$.

Let us note that examples (1) and (2) do not have well-parametrized extensions. Definability in Prop. 6.1.8 is meant in the same way as in Sec. 7.2.

Proposition 6.1.8. The vertical timelike unit-hyperbola

$$Hyp := \left\{\, \vec{p} \in \mathrm{Q}^d \ : \ p_2^2 - p_t^2 = 1, p_3 = \ldots = p_d = 0 \,\right\}$$

can be well-parametrized by a definable curve iff an **exponential function** is definable over Q, i.e., there is a definable well-parametrized curve $\gamma : \mathrm{Q} \to \mathrm{Q}^d$ such that $Ran\,\gamma = Hyp$ iff there is a definable function $e : \mathrm{Q} \to \mathrm{Q}$ such that $e'(t) = e(t)$ and $e(-t) = 1/e(t)$ for all $t \in \mathrm{Q}$.

Proof. Using the fact that the Minkowski distance of $\vec{1}_x$ and the points of Hyp tend to infinity in both directions, it can be proved that $Dom\,\gamma = \mathrm{Q}$ for any well-parametrization γ of Hyp. Let $\gamma = \langle \gamma_1, \gamma_2, 0 \ldots, 0\rangle$ be a definable well-parametrization of Hyp. Then

$$\gamma_2(t)^2 - \gamma_1(t)^2 = 1 \quad \text{for all} \quad t \in \mathrm{Q}.$$

By differentiating both sides of this equation, we get that (see Sec. 10.3)

$$\gamma_2(t)\gamma_2'(t) - \gamma_1(t)\gamma_1'(t) = 0 \quad \text{for all} \quad t \in \mathrm{Q},$$

which means that $\gamma(t)$ and $\gamma'(t)$ are Minkowski orthogonal since $\gamma' = \langle \gamma_1', \gamma_2', 0 \ldots, 0\rangle$. Since γ is well-parametrized, $\gamma'(t)$ is of Minkowski length 1 for all $t \in \mathrm{Q}$. Hence $\gamma_1'(t)^2 - \gamma_2'(t)^2 = 1$ for all $t \in \mathrm{Q}$. So $\gamma(t)$ and $\gamma'(t)$ are two Minkowski orthogonal vectors of the tx-Plane, for which $\mu\big(\gamma(t)\big) = -1$ and $\mu\big(\gamma'(t)\big) = 1$. Thus there are two possibilities: either (1) $\gamma_1(t) = \gamma_2'(t)$ and $\gamma_2(t) = \gamma_1'(t)$, or (2) $\gamma_1(t) = -\gamma_2'(t)$ and $\gamma_2(t) = -\gamma_1'(t)$. From the differentiability of γ it follows that only one of these two cases can hold for all $t \in \mathrm{Q}$. Let $e(t) := \gamma_1(t) + \gamma_2(t)$ in case (1) and let $e(t) := \gamma_1(-t) + \gamma_2(-t)$ in case (2). Then $e : \mathrm{Q} \to \mathrm{Q}$ is a definable differentiable function for which $e'(t) = e(t)$ and $e(-t) = 1/e(t)$ for all $t \in \mathrm{Q}$.

To prove the other direction, let $e : \mathrm{Q} \to \mathrm{Q}$ be a definable differentiable function such that $e(t)' = e(t)$ and $e(-t) = 1/e(t)$ for all $t \in \mathrm{Q}$. Then let us define functions ch and sh as follows:

$$ch(t) := \frac{e(t) + e(-t)}{2} \quad \text{and} \quad sh(t) := \frac{e(t) - e(-t)}{2} \quad \text{for all} \quad t \in \mathrm{Q}.$$

Then the following can be shown by a straightforward calculation:

$$ch'(t) = sh(t), \quad sh'(t) = ch(t) \quad \text{and} \quad ch(t)^2 - sh(t)^2 = 1 \quad \text{for all} \quad t \in \mathrm{Q}.$$

From these equations it is not difficult to prove that the following curve is a definable well-parametrization of Hyp:

$$\gamma(t) := \langle sh(t), ch(t), 0 \ldots, 0\rangle \quad \text{for all} \quad t \in \mathrm{Q}.$$

That completes the proof of Prop. 6.1.8. ■

Remark 6.1.9. It is well known that uniformly accelerated motion and hyperbolic motion are the same, see [17, §3.8]. Thus according to *inertial* observers, the world-line of a uniformly accelerated observer is the unit-hyperbola Hyp distorted by a Poincaré transformation and a dilation. Thus Prop. 6.1.8 implies that there can be uniformly accelerated observers iff an exponential function of the quantities is definable. See Question 8.3.3.

Let us introduce an axiom here that we will use to strengthen AxSelf_0:

AxSelf_0^+ The set of time-instances in which an observer encounters an event is connected and has at least two distinct elements, i.e.,

$$\forall k \in \mathrm{Ob}\ \exists \vec{p}, \vec{q} \in \mathrm{Q}^d \ \ p_\tau \neq q_\tau \wedge k \in ev_k(\vec{p}) \cap ev_k(\vec{q}) \wedge \{r_\tau : k \in ev_k(\vec{r})\} \text{ is connected.}$$

Let us note here that axioms AxSelf_0 and AxSelf_0^+ together are still weaker than AxSelf.

Let now introduce an axiom system which is the extension of $\mathsf{SpecRel}$ by AxCmv and some simplifying axioms:

$$\boxed{\mathsf{AccRel}_0 := \left\{ \mathsf{AxSelf}_0, \mathsf{AxSelf}_0^+, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxEvTr}, \mathsf{AxSymDist}, \mathsf{AxCmv} \right\}}$$

Remark 6.1.10. AccRel_0 is an extension of $\mathsf{SpecRel}$ since AxSelf is implied by AxSelf_0, AxPh and AxEv, see Prop. 3.1.3. Moreover, AccRel_0 is a conservative extension of $\mathsf{SpecRel}$ with respect to accelerated observers.

Our next theorem states that life-curves of accelerated observers in the models of AccRel_0 are well-parametrized. That implies that in the models of AccRel_0, accelerated clocks behave as expected. And Rem. 6.1.12 states a kind of "completeness theorem" for life-curves of accelerated observers.

Theorem 6.1.11. Let $d \geq 3$. Assume AccRel_0. Let k be an observer and m be an *inertial* observer. Then lc_m^k is a well-parametrized timelike curve.

Proof. By (2) in Prop. 6.1.6 we have that lc_m^k is a function. To prove that lc_m^k is also a curve, we need to show that $Dom\, \mathsf{lc}_m^k$ is connected and has at least two distinct elements. That is so because by Item (4) in Prop. 6.1.6, $Dom\, \mathsf{lc}_m^k = \{q_\tau : k \in ev_k(\vec{q})\}$ and the latter is connected and has at least two distinct elements by AxSelf_0^+. Hence lc_m^k is a curve.

To complete the proof, we have to show that lc_m^k is also timelike and well-parametrized. Let $t \in Dom\, \mathsf{lc}_m^k$. We have to prove that lc_m^k is differentiable at t and its derivative at t is of Minkowski length 1. By (4) of Prop. 6.1.6, there is a $\vec{q} \in Cd_k$ such that $k \in ev_k(\vec{q})$ and $q_\tau = t$. Thus, by AxCmv, there is a co-moving *inertial* observer of k at $\vec{q}$. By AxSelf_0, $\iota(t) = \vec{q}$. By Prop. 10.3.13, we can assume that m is a co-moving *inertial* observer of k at $\vec{q}$, i.e., $m \succ_{\vec{q}} k$, because of the following three statements. By (3) of Prop. 6.1.6 and AxEv, for every $h \in \mathrm{IOb}$, both lc_m^k and lc_h^k can be obtained from the other by composing it by a world-view transformation between *inertial* observers. By Thm. 3.2.2, world-view transformations between *inertial* observers are Poincaré-transformations in the models of SpecRel. Poincaré-transformations are affine and preserve the Minkowski distance.

So let us assume that m is a co-moving *inertial* observer of k at $\vec{q} = \iota(t)$. We prove that lc^k_m is differentiable at t and $\vec{1}_t = \langle 1, 0, \dots, 0\rangle$ is its derivative. This will complete the proof since $\vec{1}_t$ is a timelike vector of Minkowski length 1. By Lem. 6.1.5, $\mathsf{lc}^k_m = \iota \circ w^k_m$. So by Chain Rule, the derivative of lc^k_m at t is the derivative of w^k_m at $\iota(t)$ evaluated on the derivative of ι at t, i.e., $(\mathsf{lc}^k_m)'(t) = d_{\vec{q}} w^k_m\big(\iota'(t)\big)$. By Prop. 6.1.4, $d_{\vec{q}} w^k_m = Id$ since $m \succ_{\vec{q}} k$. It is clear that $\iota'(t) = \vec{1}_t$. Thus $(\mathsf{lc}^k_m)'(t) = \vec{1}_t$ as it was stated. ■

Let us note that we have not used AxEvTr in the proof of Thm. 6.1.11.

Remark 6.1.12. Well-parametrized curves are exactly the life-curves of accelerated observers in the models of AccRel_0, by which we mean the following. Let $\mathfrak{Q}$ be an Euclidean ordered field and let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^d$ be well-parametrized. Then there are a model $\mathfrak{M}$ of AccRel_0, observer k and *inertial* observer m such that $\mathsf{lc}^k_m = f$ and the quantity part of $\mathfrak{M}$ is $\mathfrak{Q}$. That is not difficult to prove by using the methods of the present work, see Thm. 6.2.2.

The co-moving relation $\succ_{\vec{q}}$ is not symmetric while the intuitive image behind it is. Therefore, let us introduce a symmetric version, too. We say that observers m and k are **strong co-moving observers** at $\vec{q}$, in symbols $m \rtimes\!\!\!\!\!\ltimes_{\vec{q}} k$, iff both $m \succ_{\vec{q}} k$ and $k \succ_{\vec{q}} m$ hold. The following axiom gives a stronger connection between the world-views of *inertial* and accelerated observers:

AxSCmv For every observer and event encountered by it, there is a strong co-moving *inertial* observer:

$$\forall k \in \mathrm{Ob}\ \forall \vec{q} \in \mathrm{Q}^d \quad k \in ev_k(\vec{q}) \ \rightarrow\ \exists m \in \mathrm{IOb}\ \ m \rtimes\!\!\!\!\!\ltimes_{\vec{q}} k.$$

Theorem 6.1.13. Let $d \geq 3$. Assume AxSCmv and SpecRel. Let h and k be observers and let $\vec{q}$ be a coordinate point such that $\vec{q} \in \mathsf{wl}_k(k) \cap \mathsf{wl}_k(h)$. Then the world-view transformation w^k_h is differentiable at $\vec{q}$ and one of its derivatives is a Lorentz transformation.

Proof. By axiom AxSCmv, there are *inertial* observers h_0 and k_0 such that $h_0 \succ_{\vec{q}} h$ and $k \succ_{\vec{q}} k_0$. By Prop. 6.1.4, $w^h_{h_0}(\vec{q}) = \vec{q} = w^{k_0}_k(\vec{q})$, and $w^h_{h_0}$ and $w^{k_0}_k$ are differentiable at $\vec{q}$ and one of their derivatives at $\vec{q}$ is the identity map. By Thm. 3.2.2, $w^{h_0}_{k_0}$ is a Poincaré transformation. So $w^{h_0}_{k_0}$ is differentiable and its derivative is a Lorentz transformation.

Hence, by Thm. 10.3.6, the composition of $w^h_{h_0}$, $w^{h_0}_{k_0}$ and $w^{k_0}_k$ is differentiable at $\vec q$ and one of its derivatives is a Lorentz transformation. By Prop. 2.4.3 this composition extends w^k_h. So w^k_h is also differentiable at $\vec q$ and one of its derivatives is a Lorentz transformation, see Rem. 10.3.2. ■

6.2 Models of the extended theory

First let us note that it is easy to construct nontrivial models of AccRel_0, for example, the construction in Misner – Thorne – Wheeler [42, §6, especially pp.172-173 and §13.6 on pp.327-332] can be used for constructing models for AccRel_0.

To characterize the world-view transformations between *inertial* and accelerated observers in the models of AccRel_0, let us introduce the following definition. A function $f : \mathrm{Q}^d \overset{\circ}{\longrightarrow} \mathrm{Q}^d$ is called **world-view compatible** iff $\{\, p_\tau : \vec p \in Dom\, f \wedge \vec p_\sigma = \vec o \,\}$ is connected and has at least two distinct elements, f is differentiable at every $\vec p \in \mathrm{Q}^d$ for which $\vec p \in Dom\, f$ and $\vec p_\sigma = \vec o$, and one of its derivatives is a Lorentz transformation at $\vec p$ in this case.

Remark 6.2.1. The world-view transformation w^k_m between observer k and *inertial* observer m is world-view compatible if $d \geq 3$ and AccRel_0 is assumed. This can be proved by using Thms. 6.1.11 and 3.2.2 and the fact that $\{p_\tau : \vec p \in Dom\, w^k_m \wedge \vec p_\sigma = \vec o\} = \{p_\tau : k \in ev_k(\vec p)\}$, which follows by Rem. 6.1.3.

Theorem 6.2.2. Let $f : \mathrm{Q}^d \overset{\circ}{\longrightarrow} \mathrm{Q}^d$ be a world-view compatible function. Then there is a model of AccRel_0, and there are an observer k and an *inertial* observer m in this model such that $w^k_m = f$.

Proof. We construct a model of AccRel_0 over the field Q. Let

$$\mathrm{Ph} := \{\, line(\vec p, \vec q) \;:\; \vec p, \vec q \in \mathrm{Q}^d \;\wedge\; |\vec p_\sigma - \vec q_\sigma| = |p_\tau - q_\tau| \,\},$$

$$\mathrm{IOb} := \{\, m_{\vec r} \;:\; \vec r \in Dom\, f \;\wedge\; \vec r_\sigma = \vec o \} \cup \{m\}, \quad \mathrm{Ob} := \mathrm{IOb} \cup \{k\} \quad \mathrm{B} := \mathrm{Ob} \cup \mathrm{Ph}.$$

To finish the construction of the model, we should give the world-view relation W, too. Instead, it is enough to give the event functions of all observers or to give the event function of one particular observer and the world-view transformations that define the event functions of the other observers. So let us first give the event function of observer m. For all $\vec r \in Dom\, f$ if $\vec r_\sigma = \vec o$, let $d_{\vec r} f$ be the Lorentz transformation which is a derivative of f at $\vec r$ (since f is world-view compatible, there is such a Lorentz transformation). Let

$$ph \in ev_m(\vec p) \quad \text{iff} \quad \vec p \in ph, \qquad m \in ev_m(\vec p) \quad \text{iff} \quad \vec p_\sigma = \vec o,$$

$$k \in ev_m(\vec{p}) \quad \text{iff} \quad \vec{p} = f(\vec{r}) \text{ for some } \vec{r} \in Dom\, f \text{ for which } \vec{r}_\sigma = \vec{o},$$

$$m_{\vec{r}} \in ev_m(\vec{p}) \quad \text{iff} \quad \vec{p} = f(\vec{r}) + \lambda \cdot d_{\vec{r}} f(\vec{1}_t) \text{ for some } \lambda \in \mathrm{Q}, \text{ i.e.,}$$

iff the $line(\vec{p}, f(\vec{r}))$ is the tangent line of $\mathsf{wl}_m(k)$. Now we have arranged every body in the events observed by m, thus we have given the event function ev_m. Since $d_{\vec{r}} f$ is a Lorentz transformation so is its inverse $\big[d_{\vec{r}} f\big]^{-1}$. Let

$$w^m_{m_{\vec{r}}}(\vec{p}) := \big[d_{\vec{r}} f\big]^{-1}(\vec{p} - f(\vec{r})) + \vec{r},$$

which is a Poincaré transformation since $\big[d_{\vec{r}} f\big]^{-1}$ is a Lorentz transformation. And let $w^k_m := f$. Now we have given the model since the world-view relation W can be defined from the world-view transformations and ev_m. Let us check the axioms. It is easy to see that AxSelf_0 is valid by the definition of ev_m. AxSelf_0^+ is valid since $\{p_\tau : \vec{p} \in Dom\, f \wedge \vec{p}_\sigma = \vec{o}\}$ is connected and has at least two distinct elements. AxEv, AxPh and AxSymDist are valid by the definition of ev_m and the fact that $w^m_{m_{\vec{r}}}$ are Poincaré transformations. AxEvTr is valid by the definition of ev_m and w^k_m. To prove that AxCmv is valid, we show that $m_{\vec{r}} \succ_{\vec{r}} k$, i.e., $m_{\vec{r}}$ is a co-moving inertial observer of k at $\vec{r}$. By Prop. 6.1.4, we have to check two things, (1) $w^k_{m_{\vec{r}}}(\vec{r}) = \vec{r}$ and (2) $w^k_{m_{\vec{r}}}$ is differentiable at $\vec{r}$ and the identity map is one of its derivatives.

$$w^k_{m_{\vec{r}}}(\vec{r}) = \big[d_{\vec{r}} f\big]^{-1}(f(\vec{r}) - f(\vec{r})) + \vec{r} = \vec{r}$$

since $[d_{\vec{r}} f]^{-1}(\vec{o}) = \vec{o}$ by the linearity of $[d_{\vec{r}} f]^{-1}$. By Thm. 10.3.6,

$$d_{\vec{r}} w^k_{m_{\vec{r}}}(\vec{r}) = d_{\vec{r}} f \circ d_{f(\vec{r})} w^m_{m_{\vec{r}}} = d_{\vec{r}} f \circ [d_{\vec{r}} f]^{-1} = Id_{\mathrm{Q}^d}$$

since the derivative of $w^k_{m_{\vec{r}}}$ at $f(\vec{r})$ is its linear part $[d_{\vec{r}} f]^{-1}$. That completes the proof of the theorem. ■

Remark 6.2.3. Let f be a world-view compatible transformation. It is not hard to see, by the proof above, that we can extend any model of AccRel_0 and $m \in \mathrm{IOb}$ such that $w^k_m = f$ for some $k \in \mathrm{Ob}$ in the extended model.

Example 6.2.4. The function

$$f(\vec{p}) := \begin{cases} \langle 0, \vec{p}_\sigma \rangle & \text{iff} \quad |p_\tau| \le |\vec{p}_\sigma|^2 \\ \langle \frac{|p_\tau| - |\vec{p}_\sigma|^2}{|\vec{p}_\sigma|^2}, \vec{p}_\sigma \rangle & \text{iff} \quad |\vec{p}_\sigma|^2 < |p_\tau| < 2|\vec{p}_\sigma|^2 \\ \vec{p} & \text{iff} \quad 2|\vec{p}_\sigma|^2 \le |p_\tau|, \end{cases}$$

see Fig. 6.1, is world-view compatible, thus it can define a world-view of an accelerated observer. Nevertheless, f is not injective in any neighborhood of the origin.

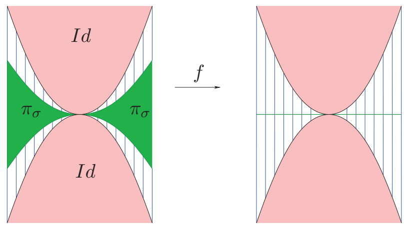

Figure 6.1: Illustration for Example 6.2.4.

Remark 6.2.5. AccRel_0 is flexible enough to allow an accelerated observer's coordinate domain to be a subset of the time-axis, i.e., there can be an observer h such that $\vec{p}_\sigma = \vec{o}$ for all $\vec{p} \in Cd_h$. Observers of this kind behave as accelerated clocks because they only use the time coordinate of their coordinate systems, so we can use them to define **accelerated clocks** within AccRel_0.

Chapter 7

The twin paradox

The results of this chapter are based on [34] and [72]. Here we investigate the logical connection of our accelerated relativity theory and the twin paradox, which is the accelerated version of the clock paradox, see Chap. 4. According to the twin paradox (TwP), if a twin makes a journey into space (accelerates), he will return to find that he has aged less than his twin brother who stayed at home (did not accelerate). However surprising TwP is, it is not a contradiction. It is only a fact that shows that the concept of time is not as simple as it seems to be. To do logical investigation on TwP, first we have to formulate it in our FOL language.

7.1 Formulating the twin paradox

To formulate TwP, let us denote the **set of events encountered by observer** m **between events** e_1 **and** e_2 localized by m as

$$\mathsf{Enc}_m(e_1,e_2) := \{\, e \in Ev_m : m \in e \wedge \exists \vec{p} \in \mathrm{Q}^d ev_m(\vec{p}) = e \wedge \mathsf{time}_m(e_1) \leq p_\tau \leq \mathsf{time}_m(e_2) \,\}.$$

Then TwP in our FOL setting can be formulated as follows:

TwP Every *inertial* observer m measures at least as much time as any other observer k between any two events e_1 and e_2 in which they meet and which are localized by both of them; and they measure the same time iff they have encountered the very same events between e_1 and e_2:

$$\begin{aligned} \forall m \in \mathrm{IOb}\ \forall k \in \mathrm{Ob}\ \forall e_1, e_2 \in Ev \quad & \mathsf{Loc}_m(e_1) \wedge \mathsf{Loc}_m(e_2) \wedge \mathsf{Loc}_k(e_1) \\ \wedge\ \mathsf{Loc}_k(e_2) \wedge k, m \in e_1 \cap e_2 \ &\rightarrow\ \mathsf{time}_k(e_1,e_2) \leq \mathsf{time}_m(e_1,e_2) \\ \wedge\ \big(\mathsf{time}_m(e_1,e_2) = \mathsf{time}_k(e_1,e_2) \ &\leftrightarrow\ \mathsf{Enc}_m(e_1,e_2) = \mathsf{Enc}_k(e_1,e_2)\big). \end{aligned}$$

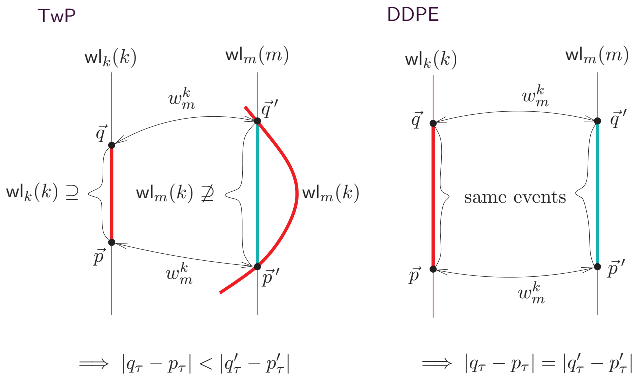

Figure 7.1: Illustration of TwP and DDPE

Let us also formulate a property of clocks which we call the Duration Determining Property of Events (DDPE). This property states that the clocks of any two observers with the same world-line are synchronized, i.e., they measure the same amount of time between any two events that they encounter. DDPE is such a basic property of clocks that it is a possible candidate for assuming it as an axiom (if it is not provable from the other axioms).

DDPE If each of two observers encounters the very same (nonempty) events between two given events, they measure the same time between these two events:

$$\forall k, m \in \mathrm{Ob}\ \forall e_1, e_2 \in Ev \quad m, k \in e_1 \cap e_2 \\ \wedge\ \mathsf{Enc}_m(e_1, e_2) = \mathsf{Enc}_k(e_1, e_2) \ \rightarrow\ \mathsf{time}_m(e_1, e_2) = \mathsf{time}_k(e_1, e_2),$$

see the right hand side of Fig. 7.1.

Theorem 7.1.1. For every Euclidean ordered field $\mathfrak{Q}$ not isomorphic to $\mathbb{R}$, there is a model $\mathfrak{M}$ of AccRel_0 such that the quantity part of $\mathfrak{M}$ is $\mathfrak{Q}$ and $\mathfrak{M} \not\models$ Twp; moreover, $\mathfrak{M} \not\models$ DDPE.

Thm. 7.1.1 is rather surprising since stationary *inertial* clocks are synchronized by SpecRel, and AxCmv states that accelerated clocks locally behave like *inertial* ones. The proof of this theorem is at p.69.

Thm. 7.1.1 also has strong consequences, it implies that to prove the Twin Paradox or even DDPE, it does not suffice to add all the FOL formulas valid in $\mathbb{R}$ to AccRel_0. Let $Th(\mathbb{R})$ denote the set of all FOL formulas valid in $\mathbb{R}$. The following corollary formulates this strong consequence.

Corollary 7.1.2. $Th(\mathbb{R}) + \mathsf{AccRel}_0 \not\models \mathsf{TwP}$ and $Th(\mathbb{R}) + \mathsf{AccRel}_0 \not\models \mathsf{DDPE}$.

Proof of Cor. 7.1.2. Let $\mathfrak{Q}$ be a field elementarily equivalent to $\mathbb{R}$, i.e., all FOL formulas valid in $\mathbb{R}$ are valid in $\mathfrak{Q}$, too. Assume that $\mathfrak{Q}$ is not isomorphic to $\mathbb{R}$. For example, the field of the real algebraic numbers is such. Let $\mathfrak{M}$ be a model of AccRel_0 with quantity part $\mathfrak{Q}$ in which neither TwP nor DDPE is true. Such an $\mathfrak{M}$ exists by Thm. 7.1.1. That shows that $Th(\mathbb{R}) + \mathsf{AccRel}_0 \not\models \mathsf{TwP} \lor \mathsf{DDPE}$ since $\mathfrak{M} \models Th(\mathbb{R})$ by assumption. ■

An ordered field is called **non-Archimedean** if it has an element a such that, for every positive integer n,

$$-1 < \underbrace{a + \ldots + a}_{n} < 1.$$

We call these elements **infinitesimally small**. These are not FOL definable concepts in our language; however, that is not a problem since we will not use them in formulas.

The following theorem says that, for countable or non-Archimedean Euclidean ordered fields, there are quite sophisticated models of AccRel_0 in which TwP and DDPE are false.

Theorem 7.1.3. For every Euclidean ordered field $\mathfrak{Q}$ which is non-Archimedean or countable, there is a model $\mathfrak{M}$ of AccRel_0 such that $\mathfrak{M} \not\models \mathsf{TwP}$, $\mathfrak{M} \not\models \mathsf{DDPE}$, the quantity part of $\mathfrak{M}$ is $\mathfrak{Q}$ and (i)–(iv) below also hold in $\mathfrak{M}$.

(i) Every observer uses the whole coordinate system as coordinate-domain:

$$\forall m \in \mathrm{Ob} \quad Cd_m = \mathrm{Q}^d.$$

(ii) At any point in Q^d, there is a co-moving *inertial* observer of any observer:

$$\forall k \in \mathrm{Ob}\ \forall q \in \mathrm{Q}^d\ \exists m \in \mathrm{IOb} \quad m \succ_q k.$$

(iii) All observers coordinatize the same set of events:

$$\forall m, k \in \mathrm{Ob}\ \forall \vec{p} \in \mathrm{Q}^d\ \exists \vec{q} \in \mathrm{Q}^d \quad ev_m(\vec{p}) = ev_k(\vec{q}).$$

(iv) Every observer coordinatizes every event only once:

$$\forall m \in \mathrm{Ob}\ \forall \vec{p}, \vec{q} \in \mathrm{Q}^d \quad ev_m(\vec{p}) = ev_m(\vec{q}) \ \rightarrow\ p = q.$$

Proofs of Thms. 7.1.1 and 7.1.3. We construct four models. Before the constructions let us introduce a definition. For every $\vec{p} \in \mathrm{Q}^d$, let $m_{\vec{p}} : \mathrm{Q}^d \to \mathrm{Q}^d$ denote the translation by vector $\vec{p}$, i.e., $m_{\vec{p}} : \vec{q} \mapsto \vec{q} + \vec{p}$. Function $f : \mathrm{Q}^d \to \mathrm{Q}^d$ is called **translation-like** iff for all $\vec{q} \in \mathrm{Q}^d$, there is a $\delta \in \mathrm{Q}^+$ such that $f(\vec{p}) = m_{f(\vec{q})-\vec{q}}(\vec{p})$ for all $\vec{p} \in B_\delta(\vec{q})$, and $f(\vec{p}) = f(\vec{q})$ and $\vec{p}_\sigma = \vec{o}$ imply that $\vec{q}_\sigma = \vec{o}$ for all $\vec{p}, \vec{q} \in \mathrm{Q}^d$.

Let $\mathfrak{Q} = \langle \mathrm{Q}; +, \cdot, < \rangle$ be an Euclidean ordered field and let $k : \mathrm{Q}^d \to \mathrm{Q}^d$ be a translation-like map. First we construct a model $\mathfrak{M}_{(\mathfrak{Q},k)}$ of AccRel_0 and (i) and (ii) of Thm. 7.1.3, which will be a model of (iii) and (iv) of Thm. 7.1.3 if k is a bijection. Then we choose $\mathfrak{Q}$ and k appropriately to get the desired models in which DDPE and TwP are false.

Let us now construct the model $\mathfrak{M}_{(\mathfrak{Q},k)}$. Let IOb:=$\{m_{\vec{p}} : \vec{p} \in \mathrm{Q}^d\}$, Ob:=IOb $\cup \{k\}$, Ph:=$\{l : \exists \vec{p}, \vec{q} \in \mathrm{Q}^d\ l = line(\vec{p}, \vec{q}) \ \wedge\ |\vec{p}_\sigma - \vec{q}_\sigma| = |p_\tau - q_\tau|\}$, and B:=Ob $\cup$ Ph. Recall that $\vec{o}$ is the origin, i.e., $\langle 0, \ldots, 0 \rangle$. First we give the world-view of $m_{\vec{o}}$, then we give the world-view of an arbitrary observer h by giving the world-view transformation between h and $m_{\vec{o}}$. Let $wl_{m_{\vec{o}}}(ph)$:=ph and $wl_{m_{\vec{o}}}(h)$:=$\{h(\vec{x}) : \vec{x}_\sigma = \vec{o}\}$ for all $ph \in$ Ph and $h \in$ Ob. And let $ev_{m_{\vec{o}}}(\vec{p})$:=$\{b \in \mathrm{B} : \vec{p} \in wl_{m_{\vec{o}}}(b)\}$ for all $\vec{p} \in \mathrm{Q}^d$. Let $w^h_{m_{\vec{o}}}$:=h for all $h \in$ Ob. From these world-view transformations, we can obtain the world-view of each observer h in the following way: $ev_h(\vec{p})$:=$ev_{m_{\vec{o}}}\big(h(\vec{p})\big)$ for all $\vec{p} \in \mathrm{Q}^d$. And from the world-views, we can obtain the W relation as follows: for all $h \in$ Ob, $b \in$ B and $\vec{p} \in \mathrm{Q}^d$, let $\mathrm{W}(h, b, \vec{p})$ iff $b \in ev_h(\vec{p})$. Thus we have given the model $\mathfrak{M}_{(\mathfrak{Q},k)}$. Let us note that $w^m_h = m \circ h^{-1}$ and $m_{h(\vec{q})-\vec{q}} \succ_{\vec{q}} h$ for all $m, h \in$ Ob and $\vec{q} \in \mathrm{Q}^d$. It is easy to check that the axioms of AccRel_0 and (i) and (ii) of Thm. 7.1.3 are true in $\mathfrak{M}_{(\mathfrak{Q},k)}$ and that (iii) and (iv) of Thm. 7.1.3 are also true in $\mathfrak{M}_{(\mathfrak{Q},k)}$ if k is a bijection.

To construct the first model, we choose $\mathfrak{Q}$ and k such that TwP falls in $\mathfrak{M}_{(\mathfrak{Q},k)}$. Let $\mathfrak{Q}$ be an Euclidean ordered field different from $\mathbb{R}$. To define k let $\{I_1, I_2, I_3, I_4, I_5\}$ be a partition[1] of Q such that every I_i is open, $x \in I_2 \ \leftrightarrow\ x+1 \in I_3 \ \leftrightarrow\ x+2 \in I_4$, and for all $y \in I_i$ and $z \in I_j$, $y \le z \ \leftrightarrow\ i \le j$. Such a partition can be easily constructed.[2]

[1] i.e., I_i's are disjoint and $\mathrm{Q} = I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$.

[2] Let $H \subset \mathrm{Q}$ be a nonempty bounded set that does not have a supremum. Let I_1:=$\{x \in \mathrm{Q} : \exists h \in H \quad x < h\}$, I_2:=$\{x+1 \in \mathrm{Q} : x \in I_1\} \setminus I_1$, I_3:=$\{x+1 \in \mathrm{Q} : x \in I_2\}$, I_4:=$\{x+1 \in \mathrm{Q} : x \in I_3\}$ and I_5:=$\mathrm{Q} \setminus (I_1 \cup I_2 \cup I_3 \cup I_4)$.

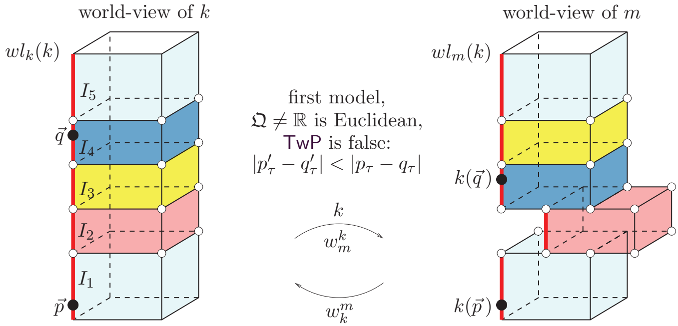

Figure 7.2: Illustration for the proofs of Thms. 7.1.1 and 7.1.3

Let

$$k(\vec{p}) := \begin{cases} \vec{p} & \text{if} \quad p_\tau \in I_1 \cup I_5, \\ \vec{p} - \vec{1}_t & \text{if} \quad p_\tau \in I_4, \\ \vec{p} + \vec{1}_t & \text{if} \quad p_\tau \in I_3, \\ \vec{p} + \vec{1}_x & \text{if} \quad p_\tau \in I_2 \end{cases}$$

for every $\vec{p} \in \mathrm{Q}^d$, see Fig. 7.2. It is easy to see that k is a translation-like bijection. Let $\vec{p}, \vec{q} \in \mathrm{Q}^d$ be coordinate points such that $\vec{p}_\sigma = \vec{q}_\sigma = \vec{o}$ and $p_\tau \in I_1$, $q_\tau \in I_4$; and let $m := m_{\vec{o}}$, $e_1 := ev_k(\vec{p})$, $e_2 := ev_k(\vec{q})$. It is easy to see that TwP is false in $\mathfrak{M}_{(\mathfrak{Q},k)}$ for k, m, and e_1, e_2 since

$$\mathsf{time}_m(e_1, e_2) = |k(\vec{p})_\tau - k(\vec{q})_\tau| < |p_\tau - q_\tau| = \mathsf{time}_k(e_1, e_2),$$

see Fig. 7.2.

To construct the second model, let $\mathfrak{Q}$ be an arbitrary Euclidean ordered field different from $\mathbb{R}$ and let $\{I_1, I_2\}$ be a partition of Q such that $x < y$ for all $x \in I_1$ and $y \in I_2$. Let

$$k(\vec{p}) := \begin{cases} \vec{p} & \text{if} \quad p_\tau \in I_1, \\ \vec{p} - \vec{1}_t & \text{if} \quad p_\tau \in I_2 \end{cases}$$

for every $\vec{p} \in \mathrm{Q}^d$, see Fig. 7.3. It is easy to see that k is translation-like. Let $\vec{p}, \vec{q} \in \mathrm{Q}^d$ such that $\vec{p}_\sigma = \vec{q}_\sigma = \vec{o}$; $p_\tau, p_\tau + 1 \in I_1$; and $q_\tau, q_\tau - 1 \in I_2$. And let $m := m_{\vec{o}}$, $e_1 := ev_k(\vec{p})$, $e_2 := ev_k(\vec{q})$. It is also easy to see that DDPE is false in $\mathfrak{M}_{(\mathfrak{Q},k)}$ for k, m and e_1, e_2 since m and k encounter the very same events between e_1 and e_2, however,

$$\mathsf{time}_k(e_1, e_2) = |p_\tau - q_\tau| \neq |k(\vec{p})_\tau - k(\vec{q})_\tau| = \mathsf{time}_m(e_1, e_2),$$

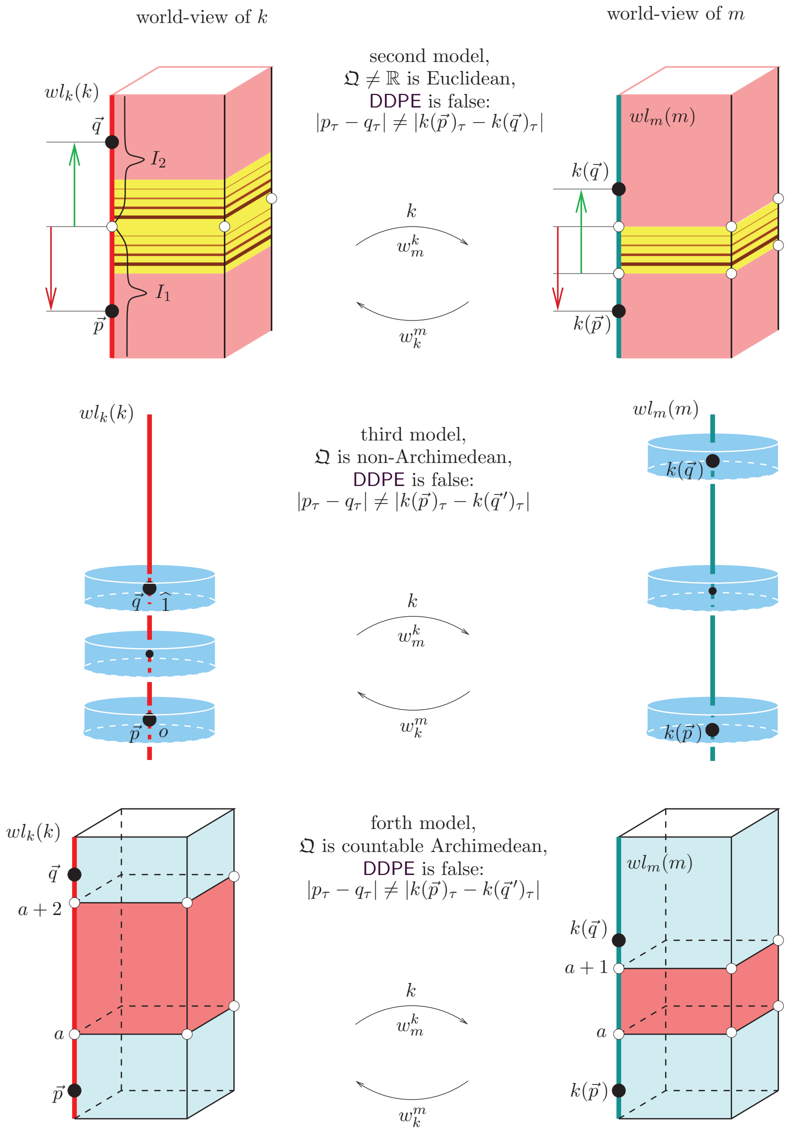

Figure 7.3: Illustration for the proofs of Thms. 7.1.1 and 7.1.3

see Fig. 7.3. This completes the proof of Thm. 7.1.1.

To construct the third model, let $\mathfrak{Q}$ be an arbitrary non-Archimedean, Euclidean ordered field. Let $a \sim b$ denote that $a, b \in \mathrm{Q}$ and $a - b$ is infinitesimally small. It is not difficult to see that $\sim$ is an equivalence relation. Let us choose an element from every equivalence class of $\sim$; and let the chosen element equivalent to $a \in \mathrm{Q}$ be denoted by $\tilde{a}$. Let $k(\vec{p}) := \langle p_\tau + \tilde{p}_\tau, \vec{p}_\sigma \rangle$ for every $\vec{p} \in \mathrm{Q}^d$, see Fig. 7.3. It is easy to see that k is a translation-like bijection. Let $p := \vec{o}$, $q := \vec{1}_t$, $k(\vec{p}) = \langle \tilde{0}, 0, \ldots, 0 \rangle$, $k(\vec{q}) = \langle 1 + \tilde{1}, 0, \ldots, 0 \rangle$. And let $m := m_{\vec{o}}$, $e_1 := ev_k(\vec{p})$, $e_2 := ev_k(\vec{q})$. It is also easy to check that DDPE is false in $\mathfrak{M}_{(\mathfrak{Q},k)}$ for k, m and e_1, e_2 since m and k encounter the very same events between e_1 and e_2, however,

$$\mathsf{time}_k(e_1, e_2) = |p_\tau - q_\tau| \neq |k(\vec{p})_\tau - k(\vec{q})_\tau| = \mathsf{time}_m(e_1, e_2),$$

see Fig. 7.3.

To construct the fourth model, let $\mathfrak{Q}$ be an arbitrary countable Archimedean Euclidean ordered field and let $k(\vec{p}) = \langle f(p_\tau), \vec{p}_\sigma \rangle$ for every $\vec{p} \in \mathrm{Q}^d$ where $f : \mathrm{Q} \to \mathrm{Q}$ is constructed as follows, see Figs. 7.3 and 7.4. We can assume that $\mathfrak{Q}$ is a subfield of $\mathbb{R}$ by [27, Thm.1 in §VIII]. Let a be a real number that is not an element of Q. Let us enumerate the elements of $[a, a+2] \cap \mathrm{Q}$ and denote the i-th element by r_i. First we cover $[a, a+2] \cap \mathrm{Q}$ with infinitely many disjoint subintervals of $[a, a+2]$ such that the sum of their lengths is 1, the length of each interval is in Q and the distance of the left endpoint of each interval from a is also in Q. We construct this covering by recursion. In the i-th step, we will use only finitely many new intervals such that the sum of their lengths is $1/2^i$. In the first step, we cover r_1 with an interval of length $1/2$. Let us suppose that we have covered r_i for each $i < n$. Since we have used only finitely many intervals so far, we can cover r_n with an interval that is not longer than $1/2^n$. Since $\sum_{i=1}^{n} 1/2^i < 1$, it is not difficult to see that we can choose finitely many other subintervals of $[a, a+2]$ to be added to this interval such that the sum of their lengths is $1/2^n$. We are given the covering of $[a, a+2]$. Let us enumerate these intervals. Let I_i be the i-th interval, d_i be the length of I_i, $d_0 := 0$ and $a_i \geq 0$ the distance of a and the left endpoint of I_i. $\sum_{i=1}^{\infty} d_i = 1$ since $\sum_{i=1}^{\infty} 1/2^i = 1$. Let

$$f(x) := \begin{cases} x & \text{if} \quad x < a, \\ x - 1 & \text{if} \quad a + 2 \leq x, \\ x - a_n + \sum\limits_{i=0}^{n-1} d_i & \text{if} \quad x \in I_n \end{cases}$$

for all $x \in \mathrm{Q}$, see Fig. 7.4. It is easy to see that k is a translation-like bijection. Let $\vec{p}, \vec{q} \in \mathrm{Q}^d$ such that $p_\tau < a$ and $a + 2 < q_\tau$; and let $m := m_{\vec{o}}$, $e_1 := ev_k(\vec{p})$, $e_2 := ev_k(\vec{q})$. It is also easy to check that DDPE is false in $\mathfrak{M}_{(\mathfrak{Q},k)}$ for k, m and e_1, e_2 since m and k encounters the very same same events between e_1 and e_2, however,

$$\mathsf{time}_k(e_1, e_2) = |p_\tau - q_\tau| \neq |k(\vec{p})_\tau - k(\vec{q})_\tau| = \mathsf{time}_m(e_1, e_2),$$

see Fig. 7.3.

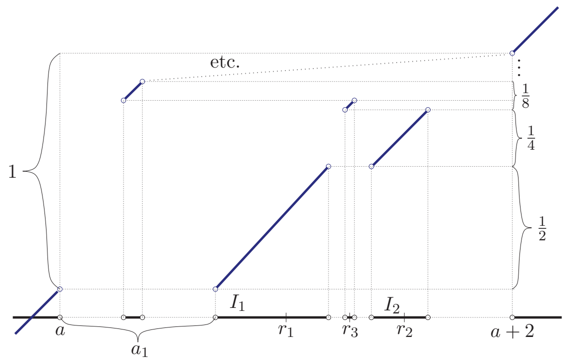

Figure 7.4: Illustration for the proofs of Thms. 7.1.1 and 7.1.3.

■

7.2 Axiom schema of continuity

As it was proved in Section 7.1, AccRel_0 is not strong enough to prove properties of accelerated clocks, such as the twin paradox or even DDPE. The additional property we need is that every bounded nonempty subset of the quantity part has a supremum. That is a second-order logic property (because it concerns all subsets) which we cannot use in a FOL axiom system. Instead, we will use a kind of "induction" axiom schema. It will state that every nonempty, bounded subset of the quantity part which can be defined by a FOL formula (using possibly the extra part of the model, e.g., using the world-view relation) has a supremum. To formulate this FOL axiom schema, we need some more definitions.

If φ is a formula and x is a variable, then we say that x is a **free variable** of φ iff x does not occur under the scope of either $\exists x$ or $\forall x$. Sometimes we introduce a formula φ as $\varphi(\vec{x}\,)$, which means that all the free variables of φ lie in $\vec{x}$.

If $\varphi(x,y)$ is a formula and $\mathfrak{M} = \langle U; \dots \rangle$ is a model, then whether φ is true or false in $\mathfrak{M}$ depends on how we associate elements of U with the free variables x and y. When we associate $a \in U$ with x and $b \in U$ with y, $\varphi(a,b)$ denotes this truth-value; so $\varphi(a,b)$ is either true or false in $\mathfrak{M}$. For example, if φ is $x < y$, then $\varphi(0,1)$ is true while $\varphi(1,0)$ is false in any ordered field. A formula φ is said to be **true** in $\mathfrak{M}$ if φ is true in $\mathfrak{M}$ no matter how we associate elements with the free variables. We say that a **subset** H **of** Q **is** (parametrically) **definable by** $\varphi(y,\vec{x}\,)$ iff there is an $\vec{a} \in U^n$ such that $H = \{b \in \mathrm{Q} : \varphi(b,\vec{a}\,) \text{ is true in } \mathfrak{M}\}$. We say that a subset of Q is **definable** iff it is definable by a FOL formula.

Now we formulate the promised axiom schema. To do so, let $\phi(x,\vec{y}\,)$ be a FOL formula of our language.

AxSup_ϕ Every subset of Q definable by $\phi(x,\vec{y}\,)$ has a supremum if it is nonempty and **bounded**.

A FOL formula expressing AxSup_ϕ can be found in Chap. 10. Our axiom schema CONT below says that every nonempty bounded subset of Q that is definable in our language has a supremum:

$$\mathsf{CONT} := \{\, \mathsf{AxSup}_\varphi \;:\; \varphi \text{ is a FOL formula of our language} \,\}\,.$$

Let us note that CONT is true in any model whose quantity part is $\mathbb{R}$. And let us call the collection of the axioms introduced so far AccRel:

$$\boxed{\mathsf{AccRel} := \mathsf{AccRel}_0 \cup \mathsf{CONT}}$$

Our next theorem states that DDPE can be proved from our FOL axiom system AccRel if $d \geq 3$.

Theorem 7.2.1. $\mathsf{AccRel} \models \mathsf{DDPE}$ if $d \geq 3$.

Proof. Let k and m be observers, and let e_1 and e_2 be events localizable by m and k such that $m,k \in e_1 \cap e_2$ and $\mathsf{Enc}_m(e_1,e_2) = \mathsf{Enc}_k(e_1,e_2)$. We have to prove that $\mathsf{time}_m(e_1,e_2) = \mathsf{time}_k(e_1,e_2)$. Let $\vec{p} := \mathsf{loc}_k(e_1)$ and $\vec{q} := \mathsf{loc}_k(e_2)$, and let $\vec{p}\,' := \mathsf{loc}_m(e_1)$ and $\vec{q}\,' := \mathsf{loc}_m(e_2)$. Then $\mathsf{time}_k(e_1,e_2) = |p_\tau - q_\tau|$ and $\mathsf{time}_m(e_1,e_2) = |p'_\tau - q'_\tau|$. See the right hand side of Fig. 7.1.

We can assume that $p_\tau \leq q_\tau$ and $p'_\tau \leq q'_\tau$. Let $h \in \mathrm{IOb}$. We prove that $\big|q_\tau - p_\tau\big| = \big|q'_\tau - p'_\tau\big|$, by applying Thm. 10.4.3 as follows: let $[a,b] := [p_\tau, q_\tau]$, $[a',b'] := [p'_\tau, q'_\tau]$, $f := \mathsf{lc}^k_h$ and $g := \mathsf{lc}^m_h$. By AxSelf^+_0 and AxCmv, we conclude that $[a,b] \subseteq Dom\, f$ and $[a',b'] \subseteq Dom\, g$ since $h \in \mathrm{IOb}$, see Prop. 6.1.6. From AccRel_0 it follows that f and g

are definable and well-parametrized timelike curves, see Thm. 6.1.11. By AxSelf_0, we have that $\{\, f(r) : r \in [a,b] \,\} = \{\, g(r') : r' \in [a',b'] \,\}$ since $\mathsf{Enc}_k(e_1,e_2) = \mathsf{Enc}_m(e_1,e_2)$. Thus, by Thm. 10.4.3, we conclude that $\big|q_\tau - p_\tau\big| = \big|q'_\tau - p'_\tau\big|$; and that is what we wanted to prove. ■

Now let us prove the following theorem stating that the twin paradox is a logical consequence of AccRel if $d \geq 3$.

Theorem 7.2.2. $\mathsf{AccRel} \models \mathsf{TwP}$ if $d \geq 3$.

Proof. Let $m \in \mathrm{IOb}$ and $k \in \mathrm{Ob}$; and let e_1 and e_2 be events localizable by m and k such that $m, k \in e_1 \cap e_2$. By Thm. 7.2.1, DDPE is provable from AccRel. So we have to prove the following only:

$$\mathsf{time}_m(e_1,e_2) \geq \mathsf{time}_k(e_1,e_2), \quad \text{and} \tag{7.1}$$

$$\mathsf{Enc}_m(e_1,e_2) = \mathsf{Enc}_k(e_1,e_2) \quad \text{if} \quad \mathsf{time}_m(e_1,e_2) = \mathsf{time}_k(e_1,e_2). \tag{7.2}$$

To do so, let $\vec{p}{:=}\mathsf{loc}_k(e_1)$, $\vec{q}{:=}\mathsf{loc}_k(e_2)$; and let $\vec{p}\,'{:=}\mathsf{loc}_m(e_1)$, $\vec{q}\,'{:=}\mathsf{loc}_m(e_2)$. Then

$$\mathsf{time}_k(e_1,e_2) = |p_\tau - q_\tau| \quad \text{and} \quad \mathsf{time}_m(e_1,e_2) = |p'_\tau - q'_\tau|,$$

see Fig. 7.1. Thus we have to prove that $|q_\tau - p_\tau| \leq |q'_\tau - p'_\tau|$, and that $\mathsf{Enc}_m(e_1,e_2) = \mathsf{Enc}_k(e_1,e_2)$ if $|q_\tau - p_\tau| = |q'_\tau - p'_\tau|$. We are going to prove them by applying Thm. 10.4.2 to lc^k_m and $[p_\tau, q_\tau]$. From AccRel_0 we have that

$$\mathsf{lc}^k_m : \mathrm{Q} \overset{\circ}{\rightarrow} \mathrm{Q}^d \text{ is a definable and well-parametrized timelike curve,} \tag{7.3}$$

see Thm. 6.1.11. By AxSelf_0, $\vec{p}_\sigma = \vec{q}_\sigma = \vec{p}\,'_\sigma = \vec{q}\,'_\sigma = \vec{o}$ since $m, k \in e_1 \cap e_2$. By the definition of life-curve,

$$\mathsf{lc}^k_m(p_\tau) = \vec{p}\,' \quad \text{and} \quad \mathsf{lc}^k_m(q_\tau) = \vec{q}\,'. \tag{7.4}$$

By AxSelf^+_0, we have

$$[p_\tau, q_\tau] \subseteq Dom\, \mathsf{lc}^k_m. \tag{7.5}$$

Hence, by applying (i) of Thm. 10.4.2 to lc^k_m and $[p_\tau, q_\tau]$, we get that

$$|q_\tau - p_\tau| \leq |\mathsf{lc}^k_m(q_\tau)_\tau - \mathsf{lc}^k_m(p_\tau)_\tau| = |q'_\tau - p'_\tau|.$$

Consequently, $\mathsf{time}_k(e_1,e_2) \leq \mathsf{time}_m(e_1,e_2)$. So (7.1) is proved.

We prove (7.2) by proving its contraposition. Moreover, we prove that $\mathsf{time}_k(e_1,e_2) < \mathsf{time}_m(e_1,e_2)$ if $\mathsf{Enc}_m(e_1,e_2) \neq \mathsf{Enc}_k(e_1,e_2)$. That will be proved by applying (ii) of

Thm. 10.4.2 to lc_m^k and $[p_\tau, q_\tau]$. To do so, let us assume that $\mathsf{Enc}_m(e_1, e_2) \neq \mathsf{Enc}_k(e_1, e_2)$. Since $\mathsf{Enc}_m(e_1, e_2) \neq \mathsf{Enc}_k(e_1, e_2)$, there are two possibilities: either there is an event e such that $e \in \mathsf{Enc}_k(e_1, e_2)$ and $e \notin \mathsf{Enc}_m(e_1, e_2)$, or there is an event $\bar{e}$ such that $\bar{e} \in \mathsf{Enc}_m(e_1, e_2)$ and $\bar{e} \notin \mathsf{Enc}_k(e_1, e_2)$. If there is such e, there is an $x \in [p_\tau, q_\tau]$ such that $\mathsf{lc}_m^k(x) = \mathsf{loc}_m(e)$. By CONT-Bolzano Theorem, $\mathsf{lc}_m^k(x)_\tau \in [q'_\tau, p'_\tau]$, since lc_m^k is a definable timelike curve and $\mathsf{lc}_m^k(p_\tau) = \vec{p}\,'$, $\mathsf{lc}_m^k(q_\tau) = \vec{q}\,'$. Thus, since $e \notin \mathsf{Enc}_m(e_1, e_2)$, we have $\mathsf{lc}_m^k(x)_\sigma \neq \vec{o}$. If $\bar{e}$ is such that $\bar{e} \in \mathsf{Enc}_m(e_1, e_2)$ and $\bar{e} \notin \mathsf{Enc}_k(e_1, e_2)$, then $\mathsf{lc}_m^k(t) \neq \mathsf{loc}_m(\bar{e})$ for all $t \in [p_\tau, q_\tau]$. By CONT-Bolzano Theorem, there is an $x \in [p_\tau, q_\tau]$ such that $\mathsf{lc}_m^k(x)_\tau = \mathsf{loc}_m(\bar{e})_\tau$. By AxSelf_0, $\mathsf{loc}_m(\bar{e})_\sigma = \vec{o}$. Therefore, $\mathsf{lc}_m^k(x)_\sigma \neq \vec{o}$ since $\bar{e} \notin \mathsf{Enc}_k(e_1, e_2)$. So in both cases there is an $x \in [p_\tau, q_\tau]$ such that $\mathsf{lc}_m^k(x)_\sigma \neq \vec{o} = \mathsf{lc}_m^k(p_\tau)_\sigma$. Consequently, there is an $x \in Dom\, \mathsf{lc}_m^k$ such that

$$x \in [p_\tau, q_\tau] \quad \text{and} \quad \mathsf{lc}_m^k(x)_\sigma \neq \mathsf{lc}_m^k(p_\tau)_\sigma.$$

By (ii) of Thm. 10.4.2, we get that

$$|q_\tau - p_\tau| < |\mathsf{lc}_m^k(q_\tau)_\tau - \mathsf{lc}_m^k(p_\tau)_\tau| = |q'_\tau - p'_\tau|.$$

Consequently, $\mathsf{time}_k(e_1, e_2) < \mathsf{time}_m(e_1, e_2)$ if $\mathsf{Enc}_k(e_1, e_2) \neq \mathsf{Enc}_m(e_1, e_2)$. That completes the proof of the theorem. ■

Question 7.2.3. Can the CONT axiom schema be replaced by some natural assumptions on observers such that the theorem above remains valid?

Remark 7.2.4. The assumption $d \geq 3$ cannot be omitted from Thm. 7.2.2. However, Thms. 7.2.2 and 7.2.1 remain true if we omit the assumption $d \geq 3$ and assume the auxiliary axioms AxThExp of Chap. 4 and AxLine defined below, i.e.,

$$\mathsf{AccRel} + \mathsf{AxThExp} + \mathsf{AxLine} \models \mathsf{TwP} \ \wedge \ \mathsf{DDPE}$$

holds for $d = 2$, too. A proof for the latter statement can be obtained from the proofs of Thms. 7.2.2 and 7.2.1 by [72, Items 4.3.1, 4.2.4, 4.2.5] and [3, Thm.1.4(ii)].

AxLine World-lines of *inertial* observers are lines according to any *inertial* observer:

$$\forall m, k \in \mathrm{IOb}\ \exists \vec{p}, \vec{q} \in \mathrm{Q}^d \quad wl_m(k) = line(\vec{p}, \vec{q}).$$

Question 7.2.5. Can the assumption $d \geq 3$ be omitted from Thm. 7.2.1, i.e., does $\mathsf{AccRel} \models \mathsf{DDPE}$ hold for $d = 2$?

In the next chapter, we discuss how the present methods and in particular AccRel_0 and CONT can be used for introducing gravity via Einstein's equivalence principle and for proving that "gravity causes time to run slow" (also called gravitational time dilation). In this connection we would like to point out that it is explained, in Misner et al. [42, pp.172-173, 327-332], that the theory of accelerated observers (in flat spacetime) is a rather useful first step in building up general relativity by using the methods of that book.

Chapter 8

Simulating gravitation by accelerated observers

Before we derive a FOL axiom system of general relativity from our theory AccRel, let us investigate the strength of AccRel by proving some theorems on gravitation from it. The results of this chapter are based on [36] and [35]. Here we investigate the effect of gravitation on clocks in our FOL setting by proving theorems about gravitational time dilation. This effect roughly means that "gravitation makes time flow slower," that is to say, clocks in the bottom of a tower run slower than clocks in its top. We use Einstein's equivalence principle to treat gravitation in AccRel. This principle says that a uniformly accelerated frame of reference is indistinguishable from a rest frame in a uniform gravitational field, see, e.g., d'Inverno [17, §9.4]. So instead of gravitation we will talk about acceleration and instead of towers we will talk about spaceships. This way the gravitational time dilation will become the following statement: "Time flows more slowly in the back of a uniformly accelerated spaceship than in its front."

One of the reasons why gravitational time dilation is interesting and important is that general relativistic hypercomputing is based on this effect, see [9], [16]. Another reason is that it leads to other surprising effects, such as that "time stops" at the event horizons of huge[1] (ca. 10^{10} solar mass) black holes. That is true because at the event horizon "gravitational force" (meant in the sense of Rindler [51, §11.2 p.230]) tends to infinity. The possibility of the existence of (traversable) wormholes is also related to these ideas, see Novikov [47], Thorne [75] and [50].

Here we concentrate on the general case when the spaceship is not necessarily uniformly accelerated. This case corresponds to the situation when the tower is in a possibly changing gravitational field. At first it is not clear whether the changing

[1]This statement is true for any black hole but it is interesting in the case of huge ones.

gravitational field has any physical relevance. However, every "physical" gravitational field is changing slightly. For example, the source of the gravitation may lose energy by radiation, which might significantly change the gravitational field in the long run. Black holes may radiate by Hawking's radiation hypothesis. Changing gravitational fields also play a key role in the theory of gravitational waves.

8.1 Formulating gravitational time dilation

Let us formulate the sentence "Time flows more slowly in the back of an accelerated spaceship than in its front." in our FOL language.

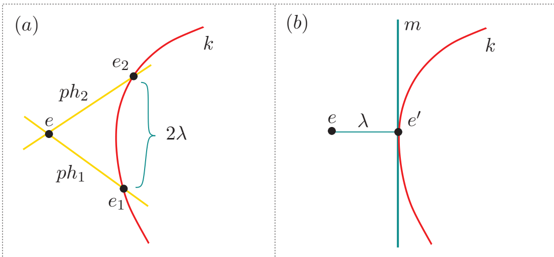

Figure 8.1: Illustrations of the radar distance and the Minkowski distance, respectively

To talk about spaceships, we need a concept of distance between events and observers. We have two natural candidates for that:

- Event e is at **radar distance** $\lambda \in \mathrm{Q}^+$ from observer k iff there are events e_1 and e_2 and photons ph_1 and ph_2 such that $k \in e_1 \cap e_2$, $ph_1 \in e \cap e_1$, $ph_2 \in e \cap e_2$ and $\mathsf{time}_k(e_1, e_2) = 2\lambda$. Event e is at **radar distance** 0 from observer k iff $k \in e$. See (a) of Fig. 8.1.

- Event e is at **Minkowski distance** $\lambda \in \mathrm{Q}$ from observer k iff there is an event e' such that $k \in e'$, $e \sim_m e'$ and $\mathsf{dist}_m(e, e') = \lambda$ for every co-moving *inertial* observer m of k at e'. See (b) of Fig. 8.1.

We say body b is at constant radar distance from observer k according to k iff the radar distance (from k) of every event in which b participates is the same. The notion of constant Minkowski distance is analogous.

To state that the *spaceship does not change its direction*, we need to introduce another concept. We say that observers k and b are **coplanar** iff $\mathsf{wl}_m(k) \cup \mathsf{wl}_m(b)$ is a subset of a vertical plane in the coordinate system of an *inertial* observer m. A plane is called a **vertical plane** iff it is parallel to the time-axis.

Now we introduce two concepts of spaceship. Observers b, k and c form a **radar spaceship**, in symbols $>\!\!|b,k,c\rangle_{rad}$, iff b, k and c are coplanar and b and c are at (not necessarily the same) constant radar distances from k according to k. The definition of the **Minkowski spaceship**, in symbols $>\!\!|b,k,c\rangle_{\mu}$, is analogous.

We say that event e_1 **precedes** event e_2 according to observer k iff $\mathsf{loc}_m(e_1)_\tau \leq \mathsf{loc}_m(e_2)_\tau$ for all co-moving *inertial* observers m of k. In this case we also say that e_2 **succeeds** e_1 according to k. We need these concepts to distinguish the past and the future light cones according to observers. Let us note that no time orientation is definable from AccRel; so we can only speak of orientation according to observers. However, there are several possible axioms which make time orientation possible, e.g.,

$$\forall m,k \in \mathrm{IOb} \quad w_m^k(\vec{o})_\tau < w_m^k(\vec{1}_t)_\tau$$

is such.

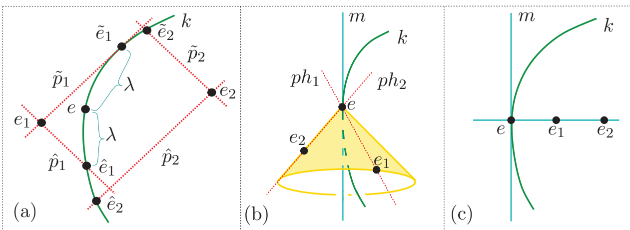

Figure 8.2: Illustrations of relations $e_1 \sim_k^{rad} e_2$, $e_1 \sim_k^{ph} e_2$ and $e_1 \sim_k^{\mu} e_2$, respectively

We also need a concept to decide which events happen at the same time according to an accelerated observer. The following three natural concepts offer themselves:

- Events e_1 and e_2 are **radar simultaneous** for observer k, in symbols $e_1 \sim_k^{rad} e_2$, iff there are events e, $\hat{e}_1$, $\hat{e}_2$, $\tilde{e}_1$, $\tilde{e}_2$ and photons $\tilde{p}_1$, $\tilde{p}_2$, $\hat{p}_1$, $\hat{p}_2$ such that $k \in e \cap \tilde{e}_i \cap \hat{e}_i$, $\hat{p}_i \in e_i \cap \hat{e}_i$, $\tilde{p}_i \in e_i \cap \tilde{e}_i$, ($\tilde{e}_i \neq \hat{e}_i$ or $e_i = e$) and $\mathsf{time}_k(e,\hat{e}_i) = \mathsf{time}_k(e,\tilde{e}_i)$ if $i \in \{1,2\}$, see Fig. 8.2.

- Events e_1 and e_2 are **photon simultaneous** for observer k, in symbols $e_1 \sim_k^{ph} e_2$, iff there are an event e and photons ph_1 and ph_2 such that $k \in e$, $ph_1 \in e \cap e_1$, $ph_2 \in e \cap e_2$ and e_1 and e_2 precede e according to k. See (b) of Fig. 8.2.

- Events e_1 and e_2 are **Minkowski simultaneous** for observer k, in symbols $e_1 \sim_k^\mu e_2$, iff there is an event e such that $k \in e$ and e_1 and e_2 are simultaneous for any co-moving *inertial* observer of k at e. See (c) of Fig. 8.2.

Remark 8.1.1. Let us note that, for *inertial* observers, the concepts of radar simultaneity, Minkowski simultaneity and the concept of simultaneity introduced on p.12 coincide, and any two of these three simultaneity concepts coincide only for *inertial* observers.

Radar simultaneity and Minkowski simultaneity are the two most natural generalizations (for non-*inertial* observers) of the standard simultaneity introduced by Einstein in [21]. In the case of Minkowski simultaneity, the standard simultaneity of co-moving *inertial* observers is rigidly copied, while in the case of radar simultaneity, the standard simultaneity is generalized in a more flexible way. Dolby and Gull calculate and illustrate the radar simultaneity of some coplanar accelerated observers in [18].

Let us note that the Minkowski simultaneity of observer k is an equivalence relation if and only if k does not accelerate. So one can argue against regarding it as a simultaneity concept for non-*inertial* observers, too. We think, however, that it is so straightforwardly generalized from the standard concept of simultaneity that it deserves to be forgiven for its weakness and to be called simultaneity. Let us also note that the Minkowski simultaneity of k is an equivalence relation on a small enough neighborhood of the world-line of k if this world-line is smooth enough.

The concept of photon simultaneity is the least usual and the most naive. It is based on the simple idea that an event is happening right now iff it is seen to be happening right now. Some authors require from a simultaneity concept to be an equivalence relation such that its equivalence classes are smooth spacelike hypersurfaces, see, e.g., Matolcsi [41]. In spite of the fact that equivalence classes of $\sim_k^{ph}$ are neither smooth nor spacelike, we think that it might to be called simultaneity, see, e.g., Hogarth [31] and Malament [39]. This concept occurs as a possible simultaneity concept in some of the papers investigating the question of conventionality/definability of simultaneity, see, e.g., Ben-Yami [13], Rynasiewicz [58], Sarkar and Stachel [60]. Let us also note that all of the introduced simultaneity and distance concepts are experimental ones, i.e., they can be determined by observers by means of experiments with clocks and photons.

We distinguish the front and the back of the spaceship by the direction of the acceleration, so we need a concept for direction. We say that the **directions of $\vec{p} \in \mathrm{Q}^d$ and $\vec{q} \in \mathrm{Q}^d$ are the same**, in symbols $\vec{p} \uparrow\uparrow \vec{q}$, if $\vec{p}$ and $\vec{q}$ are spacelike vectors, and there is a $\lambda \in \mathrm{Q}^+$ such that $\lambda \cdot \vec{p}_\sigma = \vec{q}_\sigma$, see (a) of Fig. 8.3. When $\vec{p}$ and $\vec{q}$ are timelike vectors, we also use this notation if $p_\tau q_\tau > 0$.

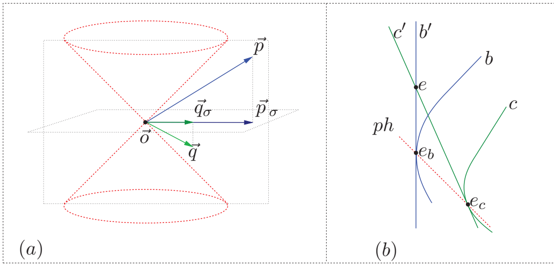

Figure 8.3: (a) illustrates $\vec{p} \uparrow\uparrow \vec{q}$, and (b) illustrates that observer c is approaching observer b, as seen by b by photons.

Now let us focus on the definition of acceleration in our FOL setting. The life-curves of observers and the derivative f' of a given function f are both FOL definable concepts, see pages 57 and 107. Thus if the life-curve of observer k according to observer m is a function, then the following definitions are also FOL ones. The **relative velocity** $\vec{\mathbf{v}}^k_m$ of observer k according to observer m at instant $t \in \mathrm{Q}$ is the derivative of the life-curve of k according to m at t if it is differentiable at t; otherwise it is undefined. The **relative acceleration** $\vec{\mathbf{a}}^k_m$ of observer k according to observer m at instant $t \in \mathrm{Q}$ is the derivative of the relative velocity of k according to m at t if it is differentiable at t; otherwise it is undefined.

We say that **the direction of the spaceship** $>\!\!|b,k,c\rangle$ **is the same as that of the acceleration of** k iff the following holds:

$$\forall m \in \mathrm{IOb}\ \ \forall t \in Dom\,\vec{\mathbf{a}}^k_m\ \ \ \forall \vec{p},\vec{q} \in Cd_m$$
$$c \in ev_m(\vec{p}\,)\ \wedge\ b \in ev_m(\vec{q}\,)\ \wedge\ \vec{p}\,\sigma\,\vec{q}\ \ \rightarrow\ \vec{\mathbf{a}}^k_m(t) \uparrow\uparrow (\vec{p}-\vec{q}\,).$$

The **acceleration** of observer k at instant $t \in \mathrm{Q}$ is defined as the unsigned Minkowski length of the relative acceleration according to any *inertial* observer m at t, i.e.,

$$a_k(t) := -\,\mu\big(\vec{\mathbf{a}}^k_m(t)\big).$$

The reason for the "$-$" sign in this definition is the fact that $\mu\big(\vec{\mathbf{a}}^k_m(t)\big)$ is negative since $\vec{\mathbf{a}}^k_m(t)$ is a spacelike vector, see Thm. 6.1.11 and Prop. 10.5.7. The acceleration is a well-defined concept since it is independent of the choice of the *inertial* observer m, see Thm. 3.2.2 and Prop. 10.5.9. We say that observer k is **positively accelerated** iff

$a_k(t)$ is defined and greater than 0 for all $t \in Dom\, \mathsf{lc}_k^k$. Observer k is called **uniformly accelerated** iff there is an $a \in \mathrm{Q}^+$ such that $a_k(t) = a$ for all $t \in Dom\, \mathsf{lc}_k^k$.

We say that **the clock of b runs slower than the clock of c as seen by k by radar** iff $\mathsf{time}_b(e_b, \bar{e}_b) < \mathsf{time}_c(e_c, \bar{e}_c)$ for all events $e_b, \bar{e}_b, e_c, \bar{e}_c$ for which $b \in e_b \cap \bar{e}_b$, $c \in e_c \cap \bar{e}_c$ and $e_b \sim_k^{rad} e_c$, $\bar{e}_b \sim_k^{rad} \bar{e}_c$. If it is **seen by photons**, we use $\sim_k^{ph}$ instead of $\sim_k^{rad}$. Similarly, if it is **seen by Minkowski simultaneity**, we use $\sim_k^{\mu}$ instead of $\sim_k^{rad}$.

8.2 Proving gravitational time dilation

Let us prove here two theorems about gravitational time dilation. Both theorems state that gravitational time dilation follows from AccRel, they only differ in the formulation of this statement.

Let us first prove a theorem about the clock-slowing effect of gravitation in radar spaceships.

Theorem 8.2.1. Let $d \geq 3$. Assume AccRel. Let $>\!\!|b, k, c\rangle_{rad}$ be a radar spaceship such that:

(i) observer k is positively accelerated,

(ii) the direction of the spaceship is the same as that of the acceleration of observer k.

Then both (1) and (2) hold:

(1) The clock of b runs slower than the clock of c as seen by k by radar.

(2) The clock of b runs slower than the clock of c as seen by each of k, b and c by photons.

Proof. To prove Item (1), let $>\!\!|b, k, c\rangle_{rad}$ be a radar spaceship such that k is positively accelerated and the direction of the spaceship is the same as that of the acceleration of k. Let e_b, $\bar{e}_b$, e_c, $\bar{e}_c$ be such events that $b \in e_b \cap \bar{e}_b$, $c \in e_c \cap \bar{e}_c$ and $e_b \sim_k^{rad} e_c$, $\bar{e}_b \sim_k^{rad} \bar{e}_c$. To prove Item (1), we have to prove that $\mathsf{time}_b(e_b, \bar{e}_b) < \mathsf{time}_c(e_c, \bar{e}_c)$. Since $>\!\!|b, k, c\rangle_{rad}$ is a spaceship, there is an *inertial* observer $m \in \mathrm{IOb}$ such that $\mathsf{wl}_m(b) \cup \mathsf{wl}_m(k) \cup \mathsf{wl}_m(c)$ is a subset of a vertical plane. Let m be such an *inertial* observer. Without losing generality, we can assume that this plane is the tx-Plane. We are going to apply Lem. 10.5.5. To do so, let $\beta = \mathsf{lc}_m^b$, $\gamma = \mathsf{lc}_m^c$ and $\alpha = \mathsf{lc}_m^k$; and let β_* and γ_* be the radar

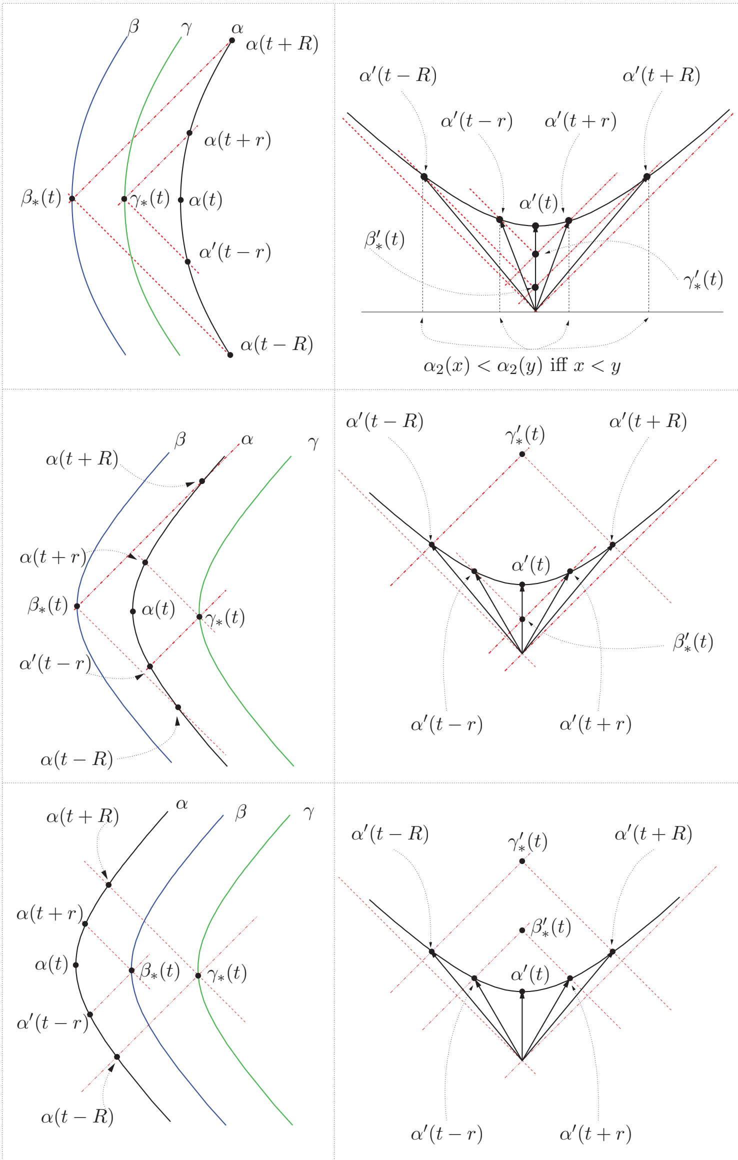

Figure 8.4: Illustration for the proof of Item (1) in Thm. 8.2.1 verifying requirement (iii) in Lem. 10.5.5

reparametrization of β and γ according to α, respectively. By Thm. 6.1.11, β and γ are definable and well-parametrized timelike curves. By Lems. 10.5.8 and 10.5.20, we can assume that α'_2 is increasing and $\alpha' \uparrow\uparrow \vec{1}_t$. By Prop. 10.5.12, β_* and γ_* are definable timelike curves since the photon sum of any two timelike vectors of $Ran\, \alpha'$ is also a timelike one. Requirement (i) in Lem. 10.5.5 is clear by the definition of the radar reparametrization. It is also clear that there are $x_\beta, y_\beta \in Dom\, \beta$, $x_\gamma, y_\gamma \in Dom\, \gamma$ and $x, y \in Dom\, \beta_* \cap Dom\, \gamma_*$ such that $\beta(x_\beta) = \mathsf{loc}_m(e_b) = \beta_*(x)$, $\beta(y_\beta) = \mathsf{loc}_m(\bar{e}_b) = \beta_*(y)$ and $\gamma(x_\gamma) = \mathsf{loc}_m(e_c) = \gamma_*(x)$, $\gamma(y_\gamma) = \mathsf{loc}_m(\bar{e}_c) = \gamma_*(y)$. Hence requirement (ii) in Lem. 10.5.5 also holds. Since the direction of $>\!\!|b, k, c\rangle_{rad}$ is the same as that of the acceleration of k, there are only three possible orders of the observers in the spaceship. All these three cases are illustrated by Fig. 8.4. By Prop. 10.5.19, it is easy to see that $\mu\big(\beta'_*(t)\big) < \mu\big(\gamma'_*(t)\big)$ for all $t \in (x, y)$; and that is requirement (iii) in Lem. 10.5.5. Hence by Lem. 10.5.5, $|x_\beta - y_\beta| < |x_\gamma - y_\gamma|$. Thus $\mathsf{time}_b(e_b, \bar{e}_b) < \mathsf{time}_c(e_c, \bar{e}_c)$ since by Lem. 8.2.7, $\mathsf{time}_i(e_i, \bar{e}_i) = |x_i - y_i|$ for all $i \in \{b, c\}$; and that is what we wanted to prove.

To prove Item (2), there are many cases we should consider resulting from which order is taken by the observers in the spaceship, and which observer is watching the other two. The proof in all the cases is based on the very same ideas and lemmas as the proof of Item (1). The only difference is that we should use photon simultaneity and photon reparametrization instead of radar ones, and we should use Prop. 10.5.12 (and Lem. 10.5.8) when verifying requirement (iii) in Lem. 10.5.5. In Fig. 8.5, we illustrate the proof of requirement (iii) in Lem. 10.5.5 in one of the many cases. In the other cases, this part of the proof can also be attained by means of similar figures without any extra difficulty. ■

To prove a similar theorem for Minkowski spaceships, we need the following concept. We say that observer b **is not too far behind** the positively accelerated observer k iff the following holds:

$$\forall m \in \mathrm{IOb}\ \forall t \in Dom\, \vec{\mathbf{a}}^k_m\ \forall \vec{p}, \vec{q} \in Cd_m \quad k \in ev_m(\vec{p}) \ \wedge\ b \in ev_m(\vec{q})$$

$$\wedge\ ev_m(\vec{p}) \sim^\mu_k ev_m(\vec{q}) \ \wedge\ \vec{\mathbf{a}}^k_m(t) \uparrow\uparrow (\vec{p} - \vec{q}) \ \rightarrow\ \forall \tau \in Dom\, \vec{\mathbf{a}}^k_m \quad \mu(\vec{p}, \vec{q}) < \frac{-1}{a_k(\tau)}.$$

Now we can state and prove our theorem about the clock-slowing effect of gravitation in Minkowski spaceships.

Theorem 8.2.2. Let $d \geq 3$. Assume AccRel. Let $>\!\!|b, k, c\rangle_\mu$ be a Minkowski spaceship such that:

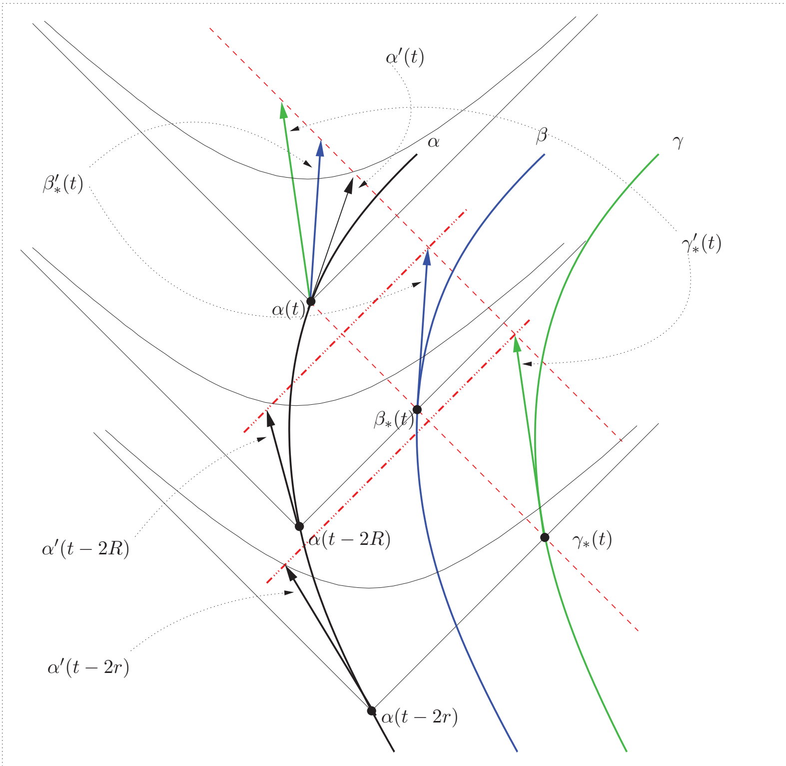

Figure 8.5: Illustration for the proof of Item (2) in Thm. 8.2.1 verifying requirement (iii) in Lem. 10.5.5

(i) observer k is positively accelerated,

(ii) the direction of the spaceship is the same as that of the acceleration of observer k,

(iii) observer b is not too far behind k.

Then both (1) and (2) hold:

(1) The clock of b runs slower than the clock of c as seen by k by Minkowski simultaneity.

(2) The clock of b runs slower than the clock of c as seen by each of k, b and c by photons.

Proof. The proof of this theorem is based on the very same ideas and lemmas as the proof of Thm. 8.2.1. The only difference is that we should use Minkowski simultaneity and Minkowski reparametrization instead of radar ones, and in the proof of Item (1) we should use Prop. 10.5.21 instead of Prop. 10.5.19 when verifying requirement (iii) in Lem. 10.5.5. In the proof of Item (1) of this theorem, we face the same three cases as in the proof of Item (1) in Thm. 8.2.1. By (a), (b) and (c) of Fig. 8.6, we illustrate the proof of requirement (iii) in Lem. 10.5.5 in this three cases. Similarly, in the proof of Item (2) of this theorem, we face the same large number of cases as in the proof of Item (2) in Thm. 8.2.1. By (d) of Fig. 8.6, we illustrate the proof of requirement (iii) in Lem. 10.5.5 in one of these many cases. We do not go into more details here since the rest of the proof can be put together with the help of the hints above. ■

We have seen that gravitation (acceleration) makes "time flow slowly." However, we left the question open which feature of gravitation (its "magnitude" or its "direction") plays a role in this effect. The following theorem shows that two observers, say b and c, can feel the same gravitation while the clock of b runs slower than the clock of c. Thus it is not the "magnitude" of the gravitation that makes "time flow slowly."

Theorem 8.2.3. Let $d \geq 3$. Then there is a model of AccRel, and there are observers b and c in this model such that $a_b(t) = a_c(t) = 1$ for all $t \in Q$, but the clock of b runs slower than the clock of c as seen by b by photons (or by radar or by Minkowski simultaneity).

Proof. To prove the theorem, let Q be the field of real numbers and let

$$\beta(t) := \big(sh(t), ch(t), 0, \dots, 0\big) \quad \text{and} \quad \gamma(t) := \big(sh(t), ch(t) + 1, 0 \dots, 0\big)$$

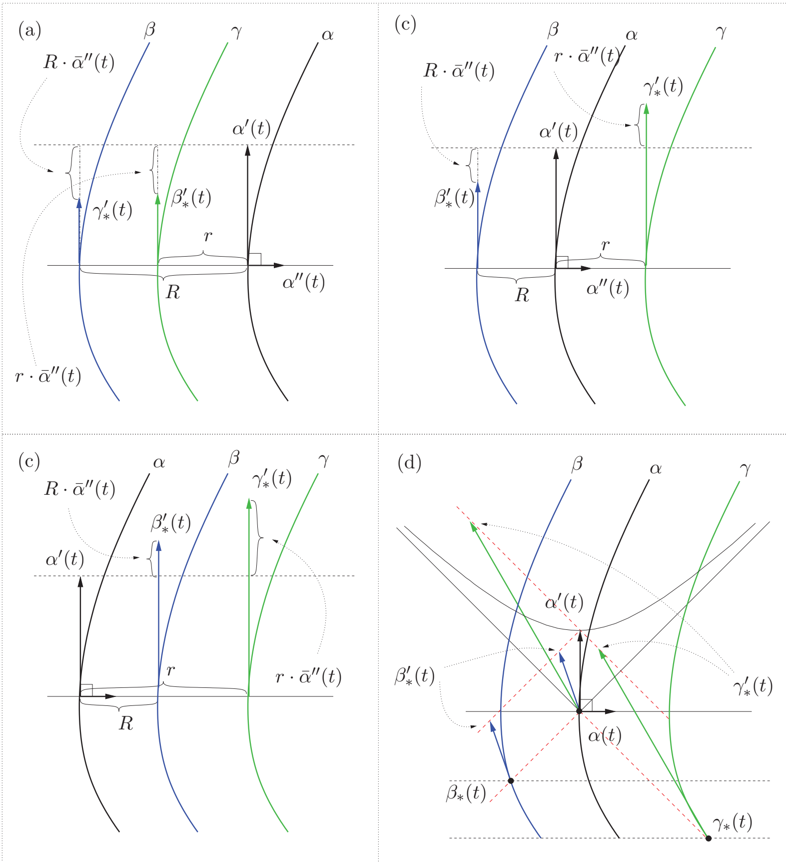

Figure 8.6: Illustration for the proof of Thm. 8.2.2 verifying requirement (iii) in Lem. 10.5.5

where sh and ch are the hyperbolic sine and cosine functions. Since both β and γ are smooth and well-parametrized timelike curves, we can easily build a model of AccRel such that $\mathsf{lc}^b_m = \beta$ and $\mathsf{lc}^c_m = \alpha$ for some $m \in \mathrm{IOb}$. By a straightforward calculation, we can show that $\mu\big(\beta''(t)\big) = \mu\big(\gamma''(t)\big) = -1$ for all $t \in \mathrm{Q}$. Hence $a_b(t) = a_c(t) = 1$ for all $t \in \mathrm{Q}$.

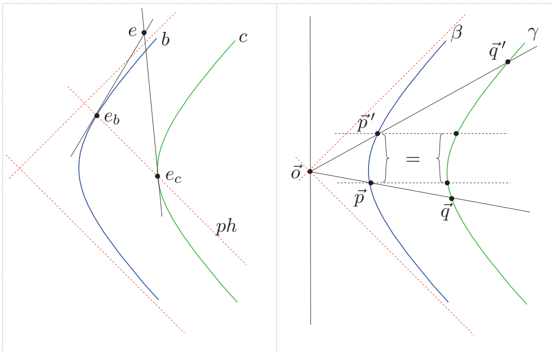

Figure 8.7: Illustration for the proof of Thm. 8.2.3

It is easy to show that c is approaching b as seen by b by photons, see (a) of Fig. 8.7. Thus by Lem. 8.2.6, the clock of b runs slower than the clock of c as seen by b by photons. It is not difficult to show that $ev_m(\vec{p}) \sim^{rad}_b ev_m(\vec{q})$ iff $ev_m(\vec{p}) \sim^{\mu}_b ev_m(\vec{q})$ iff $\vec{o} \in line(\vec{p}, \vec{q})$. Thus the clock of b runs slower than the clock of c as seen by b by both radar simultaneity and Minkowski simultaneity, see (b) of Fig. 8.7. ■

Let us now prove some lemmas that were used in the proofs above. First let us introduce two concepts which are strongly connected to the flow of time as seen by photons, see Lem. 8.2.6. We say that observer c is **approaching** (or **moving away** from) observer b as seen by b by photons at event e_b iff the following hold

- $b \in e_b$,

- for all events e_c for which $c \in e_c$ and $e_b \sim^{ph}_b e_c$ hold, there is an event e such that $b', c' \in e$ for every co-moving *inertial* observers b' and c' of b at event e_b and of c at event e_c, respectively, and

- e_b precedes (succeeds) e according to b, see (b) of Fig. 8.3. We say that c is approaching (moving away from) b as seen by b by photons iff it is so for every event e_b for which $b \in e_b$. The idea behind these definitions is the following: two observers are considered approaching when they would meet if they stopped accelerating at simultaneous events.

Remark 8.2.4. Let us note that coplanar *inertial* observers seen by photons are approaching each other before the event of meeting and moving away from each other after it. This fact explains the words used for these concepts.

Remark 8.2.5. There is no direct connection between the two concepts above. For example, it is not difficult to construct a model of AccRel in which there are (uniformly accelerated) observers b and c such that c is approaching b seen by b by photons while b is moving away from c seen by c by photons, see the proof of Thm. 8.2.3.

Lem. 8.2.6 can be interpreted as a refined version of the Doppler effect.

Lemma 8.2.6. Let $d \geq 3$. Assume AccRel. Let b and c be coplanar observers. Then

(1) If c is approaching b as seen by b by photons, the clock of b runs slower than the clock of c as seen by b by photons.

(2) If c is moving away from b as seen by b by photons, the clock of c runs slower than the clock of b as seen by b by photons.

Proof. To prove Item (1), let b and c be coplanar observers, and let e_b, $\bar{e}_b$, e_c and $\bar{e}_c$ be such events that $b \in e_b \cap \bar{e}_b$, $c \in e_c \cap \bar{e}_c$ and $e_b \sim^{ph}_b e_c$, $\bar{e}_b \sim^{ph}_b \bar{e}_c$. Let us suppose that c is approaching b as seen by b by photons. We have to prove that $\mathsf{time}_b(e_b, \bar{e}_b) < \mathsf{time}_c(e_c, \bar{e}_c)$. Since c and b are coplanar, there is an *inertial* observer $m \in \mathrm{IOb}$ such that $\mathsf{wl}_m(c) \cup \mathsf{wl}_m(b)$ is a subset of a vertical plane. Let m be such an *inertial* observer. We are going to apply Lem. 10.5.5. To do so, let $\beta = \beta_* = \mathsf{lc}^b_m$, $\gamma = \mathsf{lc}^c_m$, and let γ_* be the photon reparametrization of γ according to β. By Thm. 6.1.11, $\beta = \beta_*$ and γ are definable and well-parametrized timelike curves. Without losing generality, we can assume that $\beta' \uparrow\uparrow \vec{1}_t$ and $\gamma' \uparrow\uparrow \vec{1}_t$. It is easy to see that $\mathsf{wl}_m(b) \cap \mathsf{wl}_m(c) = \emptyset$ since c is approaching b as seen by b. Thus $Ran\,\beta \cap Ran\,\gamma = \emptyset$ since $Ran\,\beta = \mathsf{wl}_m(b)$ and $Ran\,\gamma = \mathsf{wl}_m(c)$ by Item (5) in Prop. 6.1.6. Thus γ_* is also a definable timelike curve by Prop. 10.5.12. Requirement (i) in Lem. 10.5.5 is clear by the definition of the photon reparametrization. It is also clear that there are $x_\beta, y_\beta \in Dom\,\beta$, $x_\gamma, y_\gamma \in Dom\,\gamma$ and $x, y \in Dom\,\beta_* \cap Dom\,\gamma_*$ such that $\beta(x_\beta) = \mathsf{loc}_m(e_b) = \beta_*(x)$, $\beta(y_\beta) = \mathsf{loc}_m(\bar{e}_b) = \beta_*(y)$ and $\gamma(x_\gamma) = \mathsf{loc}_m(e_c) = \gamma_*(x)$, $\gamma(y_\gamma) = \mathsf{loc}_m(\bar{e}_c) = \gamma_*(y)$. Hence requirement (ii) in

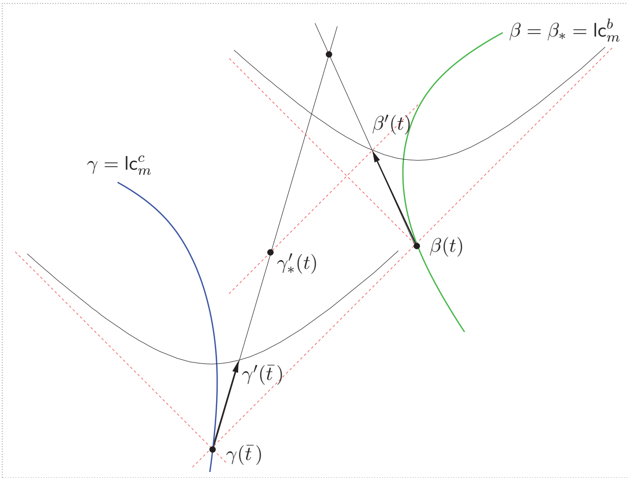

Figure 8.8: Illustration for the proof of Lem. 8.2.6

Lem. 10.5.5 also holds. Since c is approaching b as seen by b by photons, the tangent lines of β_* and γ_* at any $t \in (x, y)$ intersect in the future of $\beta_*(t)$ and $\gamma_*(t)$. Thus $\mu\big(\beta'_*(t)\big) = 1 < \mu\big(\gamma'_*(t)\big)$ for all $t \in (x, y)$ by Prop. 10.5.12, see Fig. 8.8; and that is requirement (iii) in Lem. 10.5.5. Hence by Lem. 10.5.5, we have that $|x_\beta - y_\beta| < |x_\gamma - y_\gamma|$. Consequently, $\mathsf{time}_b(e_b, \bar{e}_b) < \mathsf{time}_c(e_c, \bar{e}_c)$ since by Lem. 8.2.7, $\mathsf{time}_i(e_i, \bar{e}_i) = |x_i - y_i|$ for all $i \in \{b, c\}$. So Item (1) is proved.

The proof of (2) is similar. Hence we omit it. ■

Lem. 8.2.7 states that the time measured according to the parametrization of the life-curve lc^k_m between two parameter points and the time measured by observer k between the corresponding events is the same if AxSelf_0 and AxPh are assumed and $m \in \mathrm{IOb}$.

Lemma 8.2.7. Assume AxSelf_0, AxPh, and let $m \in \mathrm{IOb}$. Let $k \in \mathrm{Ob}$. Let $x, y \in$ *Dom* lc^k_m. Then

$$\mathsf{time}_k\Big(ev_m\big(\mathsf{lc}^k_m(x)\big), ev_m\big(\mathsf{lc}^k_m(y)\big)\Big) = |x - y|. \tag{8.1}$$

Proof. By (2) in Prop. 6.1.6, lc^k_m is a function. Thus $\mathsf{lc}^k_m(x)$ and $\mathsf{lc}^k_m(y)$ are meaningful. We have that $k \in ev_m\big(\mathsf{lc}^k_m(x)\big) \bigcap ev_m\big(\mathsf{lc}^k_m(y)\big)$ by the definition of lc^k_m. Thus by AxSelf_0, both events $ev_m\big(\mathsf{lc}^k_m(x)\big)$ and $ev_m\big(\mathsf{lc}^k_m(y)\big)$ have unique coordinates in Cd_k. Thus the left hand side of equation (8.1) is defined and equal to

$$\left|\mathsf{loc}_k\big(ev_m\big(\mathsf{lc}^k_m(x)\big)\big)_\tau - \mathsf{loc}_k\big(ev_m\big(\mathsf{lc}^k_m(y)\big)\big)_\tau\right|$$

by definition. However, by the definition of lc^k_m,

$$\mathsf{loc}_k\big(ev_m\big(\mathsf{lc}^k_m(x)\big)\big)_\tau = x \quad \text{and} \quad \mathsf{loc}_k\big(ev_m\big(\mathsf{lc}^k_m(y)\big)\big)_\tau = y.$$

Hence equation (8.1) holds. ∎

None of the axioms introduced so far require the existence of accelerated observers. Our following axiom schema says that every definable timelike curve is the world-line of an observer. Since there are many timelike curves that are not lines, that will ensure the existence of many accelerated observers since from AxSelf_0, AxPh and AxEv it follows that the world-lines of *inertial* observers are lines, see, e.g., Thm. 3.2.2.

We say that a **function** f **is** (parametrically) **definable by** $\psi(x,\vec{y},\vec{z}\,)$ iff there is an $\vec{a}\in U^n$ such that $f(b)=\vec{p} \;\leftrightarrow\; \psi(b,\vec{p},\vec{a}\,)$ is true in $\mathfrak{M}$. Let ψ be a FOL formula of our language.

$\mathsf{Ax\exists Ob}_\psi$ If a function that is parametrically definable by ψ is a timelike curve, then there is an observer whose world-line is the range of this function:

$$\mathsf{COMPR} := \{\, \mathsf{Ax\exists Ob}_\psi \;:\; \psi \text{ is a FOL formula of our language}\,\}.$$

A precise formulation of COMPR can be obtained from that of its analogue in [4].

The following three theorems say that the clocks can run arbitrarily slow or fast, as seen by the three different methods.

Theorem 8.2.8. Let $d\geq 3$. Assume AccRel and COMPR. Let m be a positively accelerated observer such that $Dom\,\mathsf{lc}^m_m = \mathrm{Q}$ and let e and e' be two events such that $e\neq e'$ and $m\in e\cap e'$. Then for all $\lambda\in\mathrm{Q}^+$, there are an observer b and events e_b and e'_b such that $b\in e_b\cap e'_b$, $e\sim^{rad}_m e_b$, $e'\sim^{rad}_m e'_b$ and $\mathsf{time}_b(e_b,e'_b)=\lambda\cdot\mathsf{time}_m(e,e')$.

Theorem 8.2.9. Let $d\geq 3$. Assume AccRel and COMPR. Let m be a uniformly accelerated observer and let e and e' be two events such that $e\neq e'$ and $m\in e\cap e'$. Then for all $\lambda\in\mathrm{Q}^+$, there are an observer b and events e_b and e'_b such that $b\in e_b\cap e'_b$, $e\sim^{\mu}_m e_b$, $e'\sim^{\mu}_m e'_b$ and $\mathsf{time}_b(e_b,e'_b)=\lambda\cdot\mathsf{time}_m(e,e')$.

Theorem 8.2.10. Let $d\geq 3$. Assume AccRel and COMPR. Let m be a positively accelerated observer and let e and e' be two events such that $e\neq e'$ and $m\in e\cap e'$. Then for all $\lambda\in\mathrm{Q}^+$, there are an observer b and events e_b and e'_b such that $b\in e_b\cap e'_b$, $e\sim^{ph}_m e_b$, $e'\sim^{ph}_m e'_b$ and $\mathsf{time}_b(e_b,e'_b)=\lambda\cdot\mathsf{time}_m(e,e')$.

8.3 Concluding remarks on gravitational time dilation

We have proved several qualitative versions of gravitational time dilation from axiom system AccRel by the use of Einstein's equivalence principle. It is important to note that the axioms of AccRel and Einstein's equivalence principle have different statuses. Einstein's equivalence principle is not an axiom, it is just a guiding principle.

The theorems of this chapter can be interpreted as saying that observers will experience time dilation in the direction of gravitation by the corresponding measuring methods (photon, radar, Minkowski) if all the axioms of AccRel are "true in our world" and Einstein's equivalence principle is a "good" principle.

Since gravitation can be defined by the acceleration of dropped *inertial* bodies, Einstein's equivalence principle can be formulated within AccRel. It raises the possibility of checking within AccRel how good a principle Einstein's equivalence principle is. That is, we can investigate for what kind of accelerated observers the Einstein's equivalence principle can be proved within AccRel. For a detailed investigation on this subject, see [70].

Remark 8.3.1. By Thm. 6.1.11 and Prop. 6.1.8, it is not difficult to prove that the quantity part of a model of AccRel_0 cannot be the field of real algebraic numbers if we assume that there are uniformly accelerated observers.

Remark 8.3.2. By Prop. 10.1.2, the quantity part of a model of AccRel has to be a real-closed field.

These remarks generate the following three questions, each of which is unanswered yet:

Question 8.3.3. What can be the quantity part of a model of AccRel_0 if we also assume that there are uniformly accelerated observers?

Question 8.3.4. What can be the quantity part of a model of AccRel_0+COMPR?

Question 8.3.5. What can be the quantity part of a model of AccRel+COMPR?

Chapter 9

A FOL axiomatization of General relativity

In this chapter we extend our investigations to general relativity by deriving a its FOL axiomatization from our theory AccRel, see also [8]. The axioms of general relativity are going to be slightly modified versions of the four axioms of special relativity together with one more assumption which is a refinement of the co-moving axiom of AccRel. We are also going to give the connections between models of our axioms and spacetimes that we meet in the literature on general relativity.

We slightly refine the axioms of SpecRel and the strong co-moving axiom of accelerated observers AxSCmv (see p.62) and get an axiomatic theory of general relativity. To do so, we "eliminate the privileged class of inertial reference frames" which was Einstein's original recipe for obtaining general relativity from special relativity, see [26]. So below we realize Einstein's original program formally and literally. We modify the axioms one by one using the following two guidelines:

- let the new axioms not speak about *inertial* observers, and
- let the new axioms be consequences of the old ones and our theory AccRel_0.

To get the modified version of AxSelf, let us note that AxSelf_0 (see p.17) and AxSelf_0^+ (see p.61) satisfy the requirements above. So let AxSelf^- be $\mathsf{AxSelf}_0 \land \mathsf{AxSelf}_0^+$. The localized version of AxEv contains the following two statements: (1) every observer encounters the events in which it is observed, and (2) if observer k coordinatizes event e which is also coordinatized by observer m, then k also coordinatizes the events which are near e according to m. The first statement is already formulated in AxEvTr (see p.57), and the second one can be formulated by saying that $Dom\, w_m^k$ is open for any observers k and m.

AxEv$^-$ Every observer encounters the events in which it is observed; and the domains of world-view transformations are open, i.e.,

$$\mathsf{AxEvTr} \ \wedge \ \forall m, k \in \mathrm{Ob} \quad Dom\, w_m^k \text{ is open.}$$

The localized version of AxPh is the following:

AxPh$^-$ The instantaneous velocity of photons is 1 in the moment when they are sent out according to the observer sending them out, and any observer can send out a photon in any direction with this instantaneous velocity:

$$\forall k \in \mathrm{Ob}\ \forall \vec{p} \in \mathrm{Q}^d\ \ k \in ev_k(\vec{p}\,) \ \rightarrow \ \big(\forall ph \in \mathrm{Ph} \quad ph \in ev_k(\vec{p}\,) \ \rightarrow \ \vec{v}_k^{\,ph}(\vec{p}\,) = 1\big)$$
$$\wedge \ \big(\forall \vec{v} \in \mathrm{Q}^{d-1} \quad |\vec{v}\,| = 1 \ \rightarrow \ \exists ph \in \mathrm{Ph} \quad ph \in ev_k(\vec{p}\,) \ \wedge \ \vec{v}_k^{\,ph}(\vec{p}\,) = \vec{v}\,\big),$$

where $\vec{v}_k^{\,b}(\vec{p}\,)$ is the instantaneous velocity of body b according to observer k at $\vec{p}$.

Our symmetry axiom AxSymDist has many equivalent versions with respect to SpecRel$_0$, see [2, §2.8, §3.9, §4.2]. We can localize any of these versions and use it in our FOL axiom system of general relativity. For aesthetic reasons we use AxSymTime, the version stating that "*inertial* observers see each others' clocks behaving in the same way," see Thm. 3.1.4 at p.21 and [4].

AxSymTime$^-$ Any two observers meeting see each others' clocks behaving in the same way at the event of meeting:

$$\forall k, m \in \mathrm{Ob}\ \forall t_1, t_2 \in \mathrm{Q} \quad k, m \in ev_k(\langle t_1, \vec{o}\,\rangle) \cap ev_m\left(\langle t_2, \vec{o}\,\rangle\right) \ \rightarrow \ |\vec{\mathbf{v}}_k^{\,m}(t_1)| = \left|\vec{\mathbf{v}}_m^{\,k}(t_2)\right|.$$

Now all the four axioms of theory SpecRel are modified according to the requirements above.

Strictly following the guidelines above, AxSCmv$^-$ would state that the world-view transformations between observers are differentiable in their meeting-point. Instead, we introduce a series of axioms, each of which ensures the smoothness of world-view transformations to some degree.

AxDiff$_n$ The world-view transformations are n-times differentiable functions, i.e.,

$$\forall k, m \in \mathrm{Ob} \quad w_m^k \text{ is } n\text{-times differentiable function.}$$

Let us introduce the following axiom systems of general relativity:

$$\boxed{\mathsf{GenRel_n} := \big\{\, \mathsf{AxSelf}^-, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSymTime}^-, \mathsf{AxDiff_n} \,\big\} \cup \mathsf{CONT}}$$

Let us note that every model of $\mathsf{GenRel_m}$ is a model of $\mathsf{GenRel_n}$ if $m \ge n$. Let us also introduce a smooth version:

$$\boxed{\mathsf{GenRel}_\omega := \{\, \mathsf{AxSelf}^-, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSymTime}^- \,\} \cup \{\, \mathsf{AxDiff_n} : n \ge 1 \,\} \cup \mathsf{CONT}}$$

For completeness, let us mention here the localized version of AxSymDist, too. The reader may safely skip this axiom.

$\mathsf{AxSymDist}^-$ Observers meeting each other agree approximately as to the spatial distance of a neighbouring event if this event and the event of meeting are simultaneous approximately enough according to both observers:

$$\forall k, m \in \mathrm{Ob}\ \forall \varepsilon \in \mathrm{Q}^+\ \forall \vec{p} \in wl_k(k) \cap wl_k(m)\ \ \exists \delta \in \mathrm{Q}^+\ \forall \vec{q} \in B_\delta(\vec{p})$$
$$|q_\tau| < \delta \cdot |\vec{q}_\sigma| \ \wedge\ \left|w^k_m(\vec{q})_\tau\right| < \delta \cdot \left|w^k_m(\vec{q})_\sigma\right| \ \ \rightarrow\ \ \Big|\, |\vec{q}_\sigma| - \left|w^k_m(\vec{q})_\sigma\right| \Big| \le \varepsilon \cdot |\vec{p} - \vec{q}|.$$

The definition of Lorentzian manifolds over arbitrary real closed fields is a natural extension of their standard definition over $\mathbb{R}$. By the following theorems, which we are going to prove in a forthcoming paper, the models of $\mathsf{GenRel_n}$ are exactly the n-times differentiable Lorentzian manifolds over real closed fields; and the models of GenRel_ω are exactly the smooth Lorentzian manifolds over real closed fields.

Theorem 9.0.6. Let $d \ge 3$. Then $\mathsf{GenRel_n}$ is complete with respect to n-times differentiable Lorentzian manifolds over real closed fields.

Theorem 9.0.7. Let $d \ge 3$. Then GenRel_ω is complete with respect to smooth Lorentzian manifolds over real closed fields.

The proofs and formal statements of Thms. 9.0.6 and 9.0.7 are analogous to those of Cor. 3.2.5 at p.23. These theorems can be regarded as completeness theorems in the following sense. Let us consider Lorentzian manifolds as intended models of GenRel. How to do that? In our forthcoming paper, we will give a method for constructing a model of GenRel from each Lorentzian manifold; and conversely, we will also show that each model of GenRel is obtained this way from a Lorentzian manifold. By the above, we defined what we mean by a formula φ in the language of GenRel being valid in a Lorentzian manifold, or in all Lorentzian manifolds. Then completeness means that for any formula φ in the language of GenRel, we have $\mathsf{GenRel_n} \vdash \varphi$ iff φ is valid in all n-times differentiable Lorentzian manifolds over real closed fields. That is completely analogous to the way how Minkowskian geometries were regarded as intended models of SpecRel in the completeness theorem of SpecRel, see [33, §4] and [4, Thm.11.28 p.681].

Our theory GenRel was obtained from AccRel by getting rid of the concept of inertiality in the level of axioms. However, we can redefine this concept. We call the world-line of observer m **timelike geodesic**, if its every point has a neighborhood within which this observer measures the most time between any two encountered event, i.e.,

$$\forall \vec{r} \in wl_m(m)\ \exists \delta \in \mathrm{Q}^+\ \ \forall \vec{p}, \vec{q} \in wl_m(m) \cap B_\delta(\vec{r})\ \ \forall k \in \mathrm{Ob} \cap ev_m(\vec{p}) \cap ev_m(\vec{q})$$
$$wl_m(k) \subseteq B_\delta(\vec{r}) \ \ \rightarrow\ \ |p_\tau - q_\tau| \geq |w_k^m(\vec{p})_\tau - w_k^m(\vec{q})_\tau| \,.$$

In this case we also say that observer m is an *inertial* body. This definition is justified by the Twin Paradox theorem of AccRel, see Thm. 7.2.2. This theorem says that in the models of AccRel the world-lines of *inertial* observers are timelike geodesics in the above sense.

We can define lightlike geodesics in a similar fashion: a lightlike geodesic γ is a lightlike curve with the property that each point in the curve has a neighborhood in which γ is the unique lightlike curve through any two points of γ.

The assumption of axiom schema COMPR guarantees that our definition of geodesic coincides with that of the literature on Lorentzian manifolds. Therefore we also introduce the following theory:

$$\boxed{\mathsf{GenRel}_\mathsf{n}^+ := \mathsf{GenRel}_\mathsf{n} \cup \mathsf{COPMR}}$$

So in our theory GenRel^+, our notion of timelike geodesic coincides with its standard notion in the literature on general relativity. All the other key notions of general relativity, such as curvature or Riemannian tensor field, are definable from timelike geodesics. Therefore we can treat all these notions (including the notion of metric tensor field) in our theory GenRel^+ in a natural way.

<u>*Connections with our results on* AccRel</u>: Theorems proved from AccRel (our first approximation of GenRel) can also be reformulated and proved from GenRel, such as the gravitational time dilation, see Thms. 8.2.1 and 8.2.2. For lack of space, we postpone that to a forthcoming paper.

Chapter 10

The tools necessary for proving the main results

This chapter is about the development of the tools that were used in the proofs of the main results of the former chapters. First we have to build a FOL theory of real analysis. The point is to formulate and prove theorems of real analysis staying within FOL. We also seek for using as few assumptions as possible.

A part of real analysis can be generalized for arbitrary ordered fields without any real difficulty. However, a certain fragment of real analysis can only be generalized within FOL for *definable* functions and for proofs we need a version of the CONT axiom schema; and there are some theorems of real analysis which are not provable even by the CONT schema. We refer to the generalizations which cannot be proved without CONT by marking them "CONT-." The FOL generalizations of some theorems, such as Chain Rule can be proved without CONT, so they are naturally referred to without the "CONT-" mark.

Throughout this chapter $\mathcal{L}$ is assumed to be a FOL language that contains the binary relation symbol $<$ and the unary relation symbol Q, such as our frame language or the language of the ordered fields. We use notation $\mathcal{L}_0$ for the language $\{\mathrm{Q}, <\}$. Let the set of FOL formulas in language $\mathcal{L}$ is denoted by $Fm(\mathcal{L})$.

In this chapter we also use the following generalized versions of our field axiom AxEOF:

AxOF $\langle \mathrm{Q}; +, \cdot, < \rangle$ is an ordered field.

AxPOS $\langle \mathrm{Q}; < \rangle$ is a partially ordered set, i.e., $\leq$ is a reflexive, antisymmetric and transitive relation on Q.

Naturally, we do not assume AxEOF in the theorems of this chapter in which AxOF or AxPOS is used, see Conv. 2.2.1.

10.1 The axiom schema of continuity

To prove some of the theorems of real analysis, we need a property of $\mathbb{R}$. This property is that in $\mathbb{R}$ every bounded nonempty set has a **supremum**, i.e., a least upper bound. It is a second-order logic property which cannot be used in a FOL axiom system. Instead, we use an axiom schema stating that every nonempty and bounded subset of the quantity part that can be defined parametrically by a FOL formula has a supremum.

This way of imitating a second-order formula by a FOL formula schema comes from the methodology of approximating second-order theories by FOL ones. Examples are Tarski's replacement of Hilbert's second-order geometry axiom by a FOL axiom schema and Peano's FOL axiom schema of induction replacing the second-order logic induction.

Let $\{Q\} \subseteq \mathcal{L}$ be a FOL language, $\mathfrak{M}$ an $\mathcal{L}$-model with universe M. We say that a subset H of Q is **(parametrically) $\mathcal{L}$-definable by** $\varphi \in Fm(\mathcal{L})$ iff there are $a_1, \ldots, a_n \in U$ such that

$$H = \{\, d \in \mathrm{Q} \,:\, \mathfrak{M} \models \varphi(d, a_1, \ldots, a_n) \,\}.$$

We say that a subset of Q is **$\mathcal{L}$-definable** iff it is definable by an $\mathcal{L}$-formula. More generally, an n-ary relation $R \subseteq \mathrm{Q}^n$ is said to be **$\mathcal{L}$-definable** in $\mathfrak{M}$ by parameters iff there is a formula $\varphi \in Fm(\mathcal{L})$ with only free variables $x_1, \ldots, x_n, y_1, \ldots, y_k$ and there are $a_1, \ldots, a_k \in U$ such that

$$R = \{\, \langle p_1, \ldots, p_n \rangle \in \mathrm{Q}^n : \mathfrak{M} \models \varphi(p_1, \ldots, p_n, a_1, \ldots, a_k) \,\}.$$

AxSup$_\varphi$ Every subset of Q definable by $\varphi \in Fm(\mathcal{L})$ (when using $a_1, \ldots, a_n$ as fixed parameters) has a supremum if it is nonempty and bounded:

$$\forall y_1, \ldots, y_n \quad [\exists x \in \mathrm{Q} \quad \varphi] \wedge [\exists b \in \mathrm{Q} \quad \forall x \in \mathrm{Q} \quad \varphi \Longrightarrow x \leq b]$$
$$\rightarrow \Big[\exists s \in \mathrm{Q} \ \forall b \in \mathrm{Q} \quad (\forall x \in \mathrm{Q} \quad \varphi \Longrightarrow x \leq b) \iff s \leq b\Big].$$

Our axiom schema CONT$_\mathcal{L}$ below says that every nonempty bounded and $\mathcal{L}$-definable subset of Q has a supremum.

$$\mathsf{CONT}_\mathcal{L} := \{\, \mathsf{AxSup}_\varphi : \varphi \text{ is a FOL formula of the language } \mathcal{L} \,\}.$$

When $\mathcal{L}$ is our frame language, we omit the subscript and write CONT only. When the language is $\mathcal{L}_0$, we write CONT$_0$. The language $\{\mathrm{Q}, +, \cdot, <\}$ is denoted by $\mathcal{OF}$.

Remark 10.1.1. $\mathsf{CONT}_{\mathcal{L}'}$ is stronger than $\mathsf{CONT}_{\mathcal{L}}$ if $\{Q, <\} \subseteq \mathcal{L} \subseteq \mathcal{L}'$.

An ordered field $\mathfrak{Q}$ is called **real closed** if every positive element has a square root and every polynomial of odd degree has a root.

Proposition 10.1.2. Let $\mathfrak{Q}$ be an ordered field. Then

$$\mathfrak{Q} \models \mathsf{CONT}_{\mathcal{OF}} \quad \text{iff} \quad \mathfrak{Q} \text{ is real closed.}$$

Proof. Let $\mathfrak{Q}$ be an ordered field such that $\mathfrak{Q} \models \mathsf{CONT}_{\mathcal{OF}}$. To prove that $\mathfrak{Q}$ is real closed, let $p(y)$ be the odd degree polynomial $a_{2n+1}y^{2n+1} + \ldots + a_1 y + a_0$. It is enough to prove that $p(y)$ has a root when $a_{2n+1} > 0$. Let $H := \{t \in Q : p(t) < 0\}$. It is clear that H is nonempty, bounded and $\mathcal{OF}$-definable. From $\mathsf{CONT}_{\mathcal{OF}}$, it follows that H has a supremum, let us call it s. Both $\{t : p(t) > 0\}$ and $\{t : p(t) < 0\}$ are open sets, since $p(y)$ is continuous. Thus $p(s)$ cannot be negative since s is an upper bound of H, and cannot be positive since s is the smallest upper bound, i.e., $p(s) = 0$ as it was required.

Let a be a positive element of Q and let $H := \{y \in Q : y^2 < a\}$. Then H is nonempty, bounded and $\mathcal{OF}$-definable. From $\mathsf{CONT}_{\mathcal{OF}}$, it follows that H has a supremum and for the same reasons as before this supremum is a square root of a.

If $\mathfrak{Q}$ is real closed field, it is elementary equivalent to $\mathbb{R}$, see [40, Cor.3.3.16.]. Thus $\mathfrak{Q} \models \mathsf{CONT}_{\mathcal{OF}}$ since $\mathbb{R} \models \mathsf{CONT}_{\mathcal{OF}}$. ■

Remark 10.1.3. Let us note that $\mathsf{CONT}_{\mathcal{L}}$ is not strong enough to prove every theorem of real analysis, e.g., the statement that there is a function f such that $f'(x) = f(x)$ and $Ran\, f = Q$ is not provable from $\mathsf{CONT}_{\mathcal{L}}$.

Let f be an $\mathcal{L}$-definable function. Then we denote one of the formulas defining f by ϕ_f, i.e., ϕ_f is a formula in the language $\mathcal{L}$ such that

$$f = \{\, \langle \vec{x}, \vec{y} \rangle \;:\; \phi_f(\vec{x}, \vec{y}) \,\}.$$

Proposition 10.1.4. Let $f, g : Q^n \xrightarrow{\circ} Q^m$ and $h : Q^m \xrightarrow{\circ} Q^k$ be $\mathcal{L}$-definable functions and let $\lambda \in Q$. Then $Dom\, f$ and $Ran\, f$ are $\mathcal{L}$-definable and the following functions are also $\mathcal{L}$-definable ones: $\lambda \cdot f$, $f + g$ and $f \circ h$.

Proof. Let $\phi_f(\vec{x}, \vec{y})$, $\phi_g(\vec{x}, \vec{y})$ and $\phi_h(\vec{x}, \vec{y})$ be formulas defining f, g and h in the language $\mathcal{L}$, respectively. Then we can define $Dom\, f$ and $Ran\, f$ as

$$Dom\, f = \{\, \vec{x} \;:\; \exists \vec{y} \quad \phi_f(\vec{x}, \vec{y}) \,\} \quad \text{and} \quad Ran\, f = \{\, \vec{y} \;:\; \exists \vec{x} \quad \phi_f(\vec{x}, \vec{y}) \,\}.$$

Furthermore,

$$\begin{aligned}
\lambda \cdot f &= \{\, \langle \vec{x}, \vec{y} \rangle \;:\; \phi_f(\vec{x}, \vec{z}) \;\wedge\; \vec{y} = \lambda \cdot \vec{z} \,\}, \\
f + g &= \{\, \langle \vec{x}, \vec{y} \rangle \;:\; \phi_f(\vec{x}, \vec{y}_1) \;\wedge\; \phi_g(\vec{x}, \vec{y}_2) \;\wedge\; \vec{y} = \vec{y}_1 + \vec{y}_2 \,\}, \\
f \circ h &= \{\, \langle \vec{x}, \vec{y} \rangle \;:\; \phi_f(\vec{x}, \vec{z}) \;\wedge\; \phi_h(\vec{z}, \vec{y}) \,\}.
\end{aligned}$$

From these equations, it is easy to recognize the required formulas defining $Dom\, f$, $Ran\, f$, $\lambda \cdot f$, $f + g$ and $f \circ g$. ■

Proposition 10.1.5. Let $f, g : Q^n \xrightarrow{\circ} Q$ be $\mathcal{L}$-definable functions. Then the $f \cdot g$ and $1/f$ functions are also $\mathcal{L}$-definable ones.

Proof. Let $\phi_f(\vec{x}, \vec{y})$ and $\phi_g(\vec{x}, \vec{y})$ be formulas defining f and g in the language $\mathcal{L}$, respectively. Then

$$\begin{aligned}
f \cdot g &= \{\, \langle \vec{x}, \vec{y} \rangle \;:\; \phi_f(\vec{x}, \vec{y}_1) \;\wedge\; \phi_g(\vec{x}, \vec{y}_2) \;\wedge\; \vec{y} = \vec{y}_1 \cdot \vec{y}_2 \,\}, \\
1/f &= \{\, \langle \vec{x}, \vec{y} \rangle \;:\; \phi_g(\vec{x}, \vec{z}) \;\wedge\; \vec{z} \neq 0 \;\wedge\; \vec{y} = 1/\vec{z} \,\}.
\end{aligned}$$

From these equations, it is easy to recognize the required formulas defining $f \cdot g$ and $1/f$. ■

10.2 Continuous functions over ordered fields

In this section we define the concept of continuity within FOL and prove some related theorems which are used in the proofs of the main results.

CONT-Cousin's Lemma. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxOF. Let $a, b \in Q$ such that $a < b$, and let $\mathcal{A}$ be a set of subintervals of $[a, b]$ which has the following properties:

(i) **beginable**: for each $x \in [a, b]$, $\mathcal{A}$ contains any small enough right and left neighborhood of x, i.e.,

$$\forall x \in [a, b]\; \exists c, d \in Q \quad c < x < d \;\wedge\; \forall y \in [c, d] \cap [a, b]$$
$$(y < x \;\rightarrow\; [y, x] \in \mathcal{A}) \;\wedge\; (x < y \;\rightarrow\; [x, y] \in \mathcal{A}),$$

(ii) **connectable**: if $[x, y], [y, z] \in \mathcal{A}$ then $[x, z] \in \mathcal{A}$,

(iii) **$\mathcal{L}$-definable**: the set $\{t \in Q : [a, t] \in \mathcal{A}\}$ is $\mathcal{L}$-definable.

Then $[a, b] \in \mathcal{A}$.

Proof. From $\mathsf{CONT}_{\mathcal{L}}$, it follows that the set

$$H := \{\, x \in \mathrm{Q} \;:\; a < x \;\wedge\; \forall t \in (a,x) \quad [a,t] \in \mathcal{A} \,\}$$

has a supremum since it is an $\mathcal{L}$-definable, nonempty (since $\mathcal{A}$ is beginable) and bounded set. Let us call this supremum s. We complete the proof by proving that $[a,s] \in \mathcal{A}$ and $s = b$.

Since $\mathcal{A}$ is beginable, there is a $c \in [a,s)$ such that $[c,s] \in \mathcal{A}$. Since s is the supremum of H, $[a,t] \in \mathcal{A}$ for all $t \in (a,s)$. Thus $[a,c] \in \mathcal{A}$, so by the connectability of $\mathcal{A}$, we get that $[a,s] \in \mathcal{A}$.

If $s < b$, there is an $e \in (s,b]$ such that $[s,t] \in \mathcal{A}$ for all $t \in (s,e]$ since $\mathcal{A}$ is beginable. Thus we get that for all $t \in (s,e]$ $[a,t] \in \mathcal{A}$ by using the connectability of $\mathcal{A}$ and the fact that $[a,s] \in \mathcal{A}$. Then for all $t \in (a,e]$ $[a,t] \in \mathcal{A}$. This contradicts the fact that s is the supremum of the set, H therefore $s = b$. ■

A set $G \subseteq \mathrm{Q}$ is called **open** if it contains an open interval around its every element, i.e., for all $x \in G$, there are $a, b \in G$ such that $x \in (a,b) \subseteq G$. The open sets of Q form a topology, which is called the **order topology**. A function $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ is called **order-continuous** if the inverse image of any open subinterval of Q is open, i.e., $\{x : f(x) \in (c,d)\}$ is open for all $c, d \in \mathrm{Q}$. It is easy to see that while the order-topology is a second-order logic concept both the openness of a given set or the order-continuousness of a given function are FOL ones.

CONT-order-Bolzano's Theorem. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxPOS. Let $f : \mathrm{Q} \xrightarrow{\circ} Q$ be an $\mathcal{L}$-definable order-continuous function such that $[a,b] \subseteq Dom\, f$. If $f(a) < c < f(b)$, then there is a $t \in [a,b]$ such that $f(t) = c$.

Proof. Let

$$\mathcal{A} := \Big\{\, [x,y] \subseteq [a,b] : \big(\forall t \in [x,y] \;\; f(t) < c\big) \;\vee\; \big(\forall t \in [x,y] \;\; f(t) > c\big) \,\Big\}$$

and assume that there is no such $t \in [a,b]$ that $f(t) = c$. $\mathcal{A}$ is $\mathcal{L}$-definable since f is such. $\mathcal{A}$ is beginable since f is order-continuous. The connectability of $\mathcal{A}$ is also clear. Thus from CONT-Cousin's lemma we get that $f(t) < c$ for all $t \in [a,b]$ or $f(t) > c$ for all $t \in [a,b]$. So if $f(a) < c$ and $f(b) > c$, then there must be a t where $f(t) = c$. This completes the proof of the theorem. ■

Theorem 10.2.1. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxPOS. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be an $\mathcal{L}$-definable order-continuous function such that $[a,b] \subseteq Dom\, f$. Then $sup\,\{f(x) : x \in [a,b]\}$ exists and there is a $t \in [a,b]$ where $f(t) = sup\,\{f(x) : x \in [a,b]\}$.

Proof. Let $H{:=}\{f(x) : x \in [a,b]\}$ and

$$\mathcal{A}{:=}\{\,[x,y] \subseteq [a,b] \;:\; \exists c \in \mathrm{Q} \;\; \forall t \in [x,y] \;\; f(t) < c\,\}\,.$$

Since $\mathcal{A}$ is $\mathcal{L}$-definable, beginable and connectable, H is bounded by CONT-Cousin's Lemma. Thus from $\mathsf{CONT}_{\mathcal{L}}$ it follows that $sup\,H$ exists since H is nonempty, $\mathcal{L}$-definable and bounded. If there is no $t \in [a,b]$ such that $f(t) = sup\,H$, then

$$\mathcal{A}{:=}\{\,[x,y] \subseteq [a,b] \;:\; \exists q \in \mathrm{Q} \;\; \forall t \in [x,y] \;\; f(t) < q < sup\,H\,\}$$

is also $\mathcal{L}$-definable, beginable and connectable. Thus $[a,b] \in \mathcal{A}$ by Cousin's lemma, therefore there is a $q < sup\,H$ such that $f(t) < q$ for all $t \in [a,b]$ and this contradicts the supremum property. This completes the proof of the theorem. ■

A function $f : \mathrm{Q}^n \xrightarrow{\circ} \mathrm{Q}^m$ is called **continuous** at $\vec{q} \in Dom\,f$ if the usual formula of continuity holds for f, i.e.:

$$\forall \varepsilon \in \mathrm{Q}^+ \; \exists \delta \in \mathrm{Q}^+ \;\; \forall \vec{p} \in Dom\,f \quad |\vec{p} - \vec{q}| < \delta \;\; \rightarrow \;\; |f(\vec{p}) - f(\vec{q})| < \varepsilon.$$

The function f is called continuous if it is continuous at every $\vec{q} \in Dom\,f$. Let us note that if $f : \mathrm{Q}^n \xrightarrow{\circ} \mathrm{Q}^m$ is a continuous function, $f\big|_H$ is also continuous for all $H \subseteq \mathrm{Q}^n$.

CONT-Bolzano's Theorem. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxOF. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be an $\mathcal{L}$-definable and continuous function such that $[a,b] \subseteq Dom\,f$. If c is between $f(a)$ and $f(b)$, there is an $s \in [a,b]$ such that $f(s) = c$.

Proof. Let c be between $f(a)$ and $f(b)$. We can assume that $f(a) < f(b)$. Let $H{:=}\{x \in [a,b] : f(x) < c\}$. Then H is $\mathcal{L}$-definable, bounded and nonempty. Thus by $\mathsf{CONT}_{\mathcal{L}}$, the supremum of H exists. Let us call it s. Both $\{x \in (a,b) : f(x) < c\}$ and $\{x \in (a,b) : f(x) > c\}$ are nonempty open sets since f is continuous on $[a,b]$. Thus $f(s)$ cannot be less than c since s is an upper bound of H and cannot be greater than c since s is the least upper bound. Hence $f(s) = c$ as it was required. ■

Theorem 10.2.2. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxOF. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be an $\mathcal{L}$-definable and continuous function such that $[a,b] \subset Dom\,f$. Then the supremum s of $\{f(x) : x \in [a,b]\}$ exists and there is a $y \in [a,b]$ such that $f(y) = s$.

Proof. The supremum of $H{:=}\{y \in [a,b] : \exists c \in \mathrm{Q} \;\; \forall x \in [a,y] \quad f(x) < c\}$ exists by $\mathsf{CONT}_{\mathcal{L}}$ since H is $\mathcal{L}$-definable, nonempty and bounded. This supremum has to be b and $b \in H$ since f is continuous on $[a,b]$. Thus $Ran(f){:=}\{f(x) : x \in [a,b]\}$ is bounded. Thus by $\mathsf{CONT}_{\mathcal{L}}$, it has a supremum, say s, since it is $\mathcal{L}$-definable and nonempty. We can assume that $f(a) \neq s$. Let $A{:=}\{y \in [a,b] : \exists c \in \mathrm{Q}\ \forall x \in [a,y] \quad f(x) < c < s\}$. By $\mathsf{CONT}_{\mathcal{L}}$, A has a supremum. At this supremum, f cannot be less than s since f is continuous on $[a,b]$ and s is the supremum of $Ran(f)$. ■

We call function f **monotonous** if it preserves or reverses the relation $<$, i.e., $f(x) < f(y)$ [or $f(x) > f(y)$] for all $x, y \in Dom\, f$ if $x < y$.

Lemma 10.2.3. If $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ is monotonous and $Ran\, f$ is connected, f is continuous. ■

Lemma 10.2.4. Assume CONT. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be definable and continuous such that $Dom\, f$ is connected. Then

(1) $Ran\, f$ is also connected.

(2) If f is injective, it is also monotonous. Moreover, f^{-1} is also a definable monotonous and continuous function.

Proof. Item (1) is a consequence of CONT-Bolzano theorem. To prove Item (2), let us first note that if f were not monotonous, it would not be injective by CONT-Bolzano theorem. It is clear that f^{-1} is definable and monotonous since f is such. Thus by Lem. 10.2.3, f^{-1} is continuous. ■

The following can be easily proved without any of the CONT axiom schemas.

Proposition 10.2.5. Assume AxOF. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be a function. Then f is continuous iff it is order-continuous. ■

Proposition 10.2.6. Assume AxOF. Let $f, g : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be continuous functions. Then $f + g$, $f \cdot g$ and $f \circ g$ are also continuous ones. ■

Example 10.2.7. Let $exp : \mathbb{R} \to \mathbb{R}$ be the exponential map. Then exp is a continuous function but it is not $\mathcal{OF}$-definable.

We call a set $Z \subseteq \mathrm{Q}^n$ **closed** iff $\mathrm{Q}^n \setminus Z$ is open. Let us note that $\{p\}$ is closed for all $\vec{p} \in \mathrm{Q}^n$. The following can be easily proved without any CONT schema.

Proposition 10.2.8. Let Q be an ordered field. Let $f : \mathrm{Q}^n \to \mathrm{Q}^m$. The following three statements are equivalent:

(i) f is continuous.

(ii) The f^{-1}-image of an open set is open.

(iii) The f^{-1}-image of a closed set is closed. ■

We say that $f : \mathrm{Q}^n \xrightarrow{\circ} \mathrm{Q}^k$ **tends to** $\vec{q} \in \mathrm{Q}^k$ while $\vec{x} \in Dom\, f$ tends to $\vec{p} \in \mathrm{Q}^n$ if the usual formula for the limit of a function holds for f:

$$\forall \varepsilon \in \mathrm{Q}^+\ \exists \delta \in \mathrm{Q}^+\ \forall \vec{x} \in Dom\, f\ \ 0 < |\vec{x} - \vec{p}| < \delta\ \ \rightarrow\ \ |f(\vec{x}\,) - \vec{q}\,| < \varepsilon.$$

This $\vec{q}$ is unique iff $\vec{p}$ is not isolated from the set $Dom\, f \setminus \{\vec{p}\}$, i.e., $B_\varepsilon(\vec{p}) \cap Dom\, f \setminus \{\vec{p}\} \neq \emptyset$ for all $\varepsilon \in \mathrm{Q}^+$. In this case we call $\vec{q}$ the **limit** of the function f at $\vec{p}$ and we write that

$$\lim_{\vec{x} \to \vec{q}} f(\vec{x}\,) = \vec{q}.$$

10.3 Differentiable functions over ordered fields

In this section we define the concept of differentiability within FOL and prove some theorems about it which are used in the proofs of the main theorems.

We say that a function $f : \mathrm{Q}^n \xrightarrow{\circ} \mathrm{Q}^m$ is **differentiable** at $\vec{q} \in Dom\, f$ if the usual formula

$$\forall \varepsilon \in \mathrm{Q}^+\ \exists \delta \in \mathrm{Q}^+\ \forall \vec{p} \in Dom\, f \cap B_\delta(\vec{q})\quad |f(\vec{p}) - f(\vec{q}) - L(\vec{p} - \vec{q})| \leq \varepsilon \cdot |\vec{p} - \vec{q}|$$

holds for a linear map $L : \mathrm{Q}^n \rightarrow \mathrm{Q}^m$. In this case L is called **a derivative** of f at $\vec{q}$. The **set of derivative maps** of f at $\vec{q}$ is denoted by $\mathsf{Der}_{\vec{q}} f$ and any derivative of f at $\vec{q}$ is denoted by $d_{\vec{q}} f$. Function f is called **uniquely differentiable** at $\vec{q}$ if it has one and only one derivative at $\vec{q}$. In this case, $d_{\vec{q}} f$ is called **the derivative** of f at $\vec{q}$.

Remark 10.3.1. We say that a binary relation is differentiable at $\vec{q}$ if it is equal to a differentiable function on a small enough neighbourhood of $\vec{q}$.

Remark 10.3.2. If f extends f_0 (i.e., $f \supseteq f_0$) and f is differentiable at $\vec{q}$, then f_0 is also differentiable at $\vec{q}$ and every derivative of f at $\vec{q}$ is also a derivative of f_0 at $\vec{q}$.

Several theorems can be proved about differentiable functions without using any CONT axiom schema. Here we prove some of them. To do so, we will use the following easily provable and well-known fact about linear maps.

Lemma 10.3.3. Every linear map L is bounded in the following sense: there is a bound $M \in \mathrm{Q}^+$ such that $|L(\vec{x}\,) - L(\vec{y}\,)| \leq M \cdot |\vec{x} - \vec{y}\,|$ for all $\vec{x}, \vec{y} \in Dom\, L$.

Theorem 10.3.4. Let f be differentiable at $\vec{x} \in Dom\, f$. Then there are $\delta, K \in \mathrm{Q}^+$ such that $|f(\vec{x}\,) - f(\vec{y}\,)| \leq K \cdot |\vec{x} - \vec{y}\,|$ for all $\vec{y} \in Dom\, f \cap B_\delta(\vec{x}\,)$.

Proof. We have to choose δ and K appropriately. Since f is differentiable at $\vec{x} \in$ *Dom* f, there is a linear map L and δ such that

$$|f(\vec{y}) - f(\vec{x}) - L(\vec{y} - \vec{x})| \leq |\vec{y} - \vec{x}|$$

for all $\vec{y} \in Dom\, f \cap B_\delta(\vec{x})$. Let M be a bound of L which exists by Lem. 10.3.3, and let K be $M + 1$. Then, by the triangle inequality and the linearity of L,

$$|f(\vec{y}) - f(\vec{x})| \leq |f(\vec{y}) - f(\vec{x}) - L(\vec{y} - \vec{x})| + |L(\vec{y}) - L(\vec{x})|.$$

Thus, by Lem. 10.3.3,

$$|f(\vec{y}) - f(\vec{x})| \leq (1 + M) \cdot |\vec{y} - \vec{x}| = K \cdot |\vec{y} - \vec{x}|$$

for all $\vec{y} \in Dom\, f \cap B_\delta(\vec{x})$. ■

Corollary 10.3.5. If f is differentiable at $\vec{x} \in Dom\, f$, then f is also continuous at $\vec{x}$. ■

Theorem 10.3.6. Let Q be an ordered field. Let $g : \mathrm{Q}^n \overset{\circ}{\rightarrow} \mathrm{Q}^m$ and $f : \mathrm{Q}^m \overset{\circ}{\rightarrow} \mathrm{Q}^k$ such that g is differentiable at $\vec{q} \in \mathrm{Q}^n$ and f is differentiable at $g(\vec{q})$. Let $L_g \in \mathsf{Der}(\vec{q}, g)$ and $L_f \in \mathsf{Der}(g(\vec{q}), f)$. Then $L_g \circ L_f \in \mathsf{Der}(\vec{q}, g \circ f)$.

Proof. Let $\varepsilon \in \mathrm{Q}^+$ be fixed. Let L_g be a derivative of g at $\vec{q}$ and L_f be a derivative of f at $g(\vec{q})$. Let K, δ_0 be the bounding constants of g given by Thm. 10.3.4 and let M be the bounding constant of L_f given by Lem. 10.3.3. Then there is a $\delta \in \mathrm{Q}^+$ such that $\delta \leq \delta_0$,

$$|g(\vec{p}) - g(\vec{q}) - L_g(\vec{p} - \vec{q})| \leq \frac{\varepsilon}{2M} \cdot |\vec{p} - \vec{q}| \text{ and}$$

$$\big|f\big(g(\vec{p})\big) - f(g(\vec{q})) - L_f\big(g(\vec{p}) - g(\vec{q})\big)\big| \leq \frac{\varepsilon}{2K} \cdot |g(\vec{p}) - g(\vec{q})|$$

for all $\vec{p} \in Dom\, g \cap B_\delta(\vec{q})$ and $g(\vec{p}) \in Dom\, f \cap B_\delta(\vec{q})$. By the triangle inequality and the linearity of L,

$$\begin{aligned}
&\big|f\big(g(\vec{p})\big) - f\big(g(\vec{q})\big) - L_f\big(L_g(\vec{p} - \vec{q})\big)\big| \\
&\quad \leq \big|f\big(g(\vec{p})\big) - f\big(g(\vec{q})\big) - L_f\big(g(\vec{p}) - g(\vec{q})\big)\big| + \big|L_f\big(g(\vec{p}) - g(\vec{q}) - L_g(\vec{p} - \vec{q})\big)\big| \\
&\quad \leq \frac{\varepsilon}{2K} \cdot |g(\vec{p}) - g(\vec{q})| + M \cdot |g(\vec{p}) - g(\vec{q}) - L_g(\vec{p} - \vec{q})| \\
&\quad \leq \frac{\varepsilon}{2K} \cdot K \cdot |\vec{p} - \vec{q}| + M \cdot \frac{\varepsilon}{2M} \cdot |\vec{p} - \vec{q}| = \varepsilon \cdot |\vec{p} - \vec{q}|
\end{aligned}$$

for all $\vec{p} \in Dom\, f \circ g \cap B_\delta(\vec{q})$; and that is what we wanted to prove. ■

Example 10.3.7. Let $g : \mathrm{Q} \to \mathrm{Q}$ and $f : [-1, 0] \to \mathrm{Q}$ be defined as $g(x) = x^2$ for all $x \in \mathrm{Q}$ and $f(x) = 0$ for all $x \in [-1, 0]$. Then $g \circ f = \{\langle 0, 0\rangle\}$. So f, g and $g \circ f$ are differentiable at 0 by definition, but derivative of $g \circ f$ is not unique though the derivatives of f and g are such.

The following Thms. 10.3.8 and 10.3.9 can be proved by the usual textbook proofs of the uniqueness of derivatives.

Theorem 10.3.8 (uniqueness of derivatives)**.** Assume AxOF. If $f : \mathrm{Q}^n \xrightarrow{\circ} \mathrm{Q}^m$ is differentiable at $\vec{q}$ and there is a $\delta \in \mathrm{Q}^+$ such that $B_\delta(\vec{q}) \subseteq Dom\, f$ then the derivative of f at $\vec{q}$ is unique.

When f is a function from a subset of Q to Q^k, a derivative of f can be defined as a limit of the function $h \mapsto \frac{f(x+h)-f(x)}{h}$, i.e., as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. In this situation the derivatives of f at x are not linear maps but vectors. In this case, we use the notation $f'(x)$ for a derivative and we call it a **derivative vector** of f at x. The connection between the two definitions is the following: $d_x f(t) = t \cdot f'(x)$ for all $t \in \mathrm{Q}$. By the following theorem, the derivatives of a differentiable curve are unique.

Theorem 10.3.9 (uniqueness of derivative vectors)**.** Assume AxOF. The derivative of f at t is unique, if $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^k$ is a curve (i.e., $Dom\, f$ is connected and has at least two elements) and f is differentiable at t.

In the case when $f : Q \xrightarrow{\circ} \mathrm{Q}^k$, we define the derivative function f' of f as the binary relation that relates the derivatives of f at x to $x \in Dom\, f$. Of course, f' is a function only if f is uniquely differentiable.

Proposition 10.3.10. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^k$ be a $\mathcal{L}$-definable function. Then $f' \subseteq \mathrm{Q} \times \mathrm{Q}^k$ is also $\mathcal{L}$-definable.

Proof. Let $\phi_f(x, \vec{y})$ be a formula defining f in the language $\mathcal{L}$. Then

$$f' = \big\{ \langle x_0, \vec{z}\rangle \ : \ \phi_f(x_0, \vec{y}_0) \ \land \ \forall \varepsilon \in \mathrm{Q}^+ \ \exists \delta \in \mathrm{Q}^+ \ \forall x \in \mathrm{Q} \\ |x - x_0| \ \land \ \phi_f(x, \vec{y}) \ \to \ |\vec{y} - \vec{y}_0 - (x - x_0) \cdot \vec{z}\,| \le \varepsilon \cdot |x - x_0| \,\big\}\,;$$

and from this equation, it is easy to recognize the formula defining f'. ■

Corollary 10.3.11. If $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^k$ is a uniquely differentiable and $\mathcal{L}$-definable function, $f' : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^k$ is an $\mathcal{L}$-definable function.

Let $h : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ and let $H \subseteq Dom\, h$. Then h is said to be **increasing** on H iff $h(x) < h(y)$ for all $x, y \in H$ for which $x < y$; and h is said to be **decreasing** on H iff $h(y) < h(x)$ for all $x, y \in H$ for which $x < y$. The proof of the following theorem also uses only the ordered field property of the real numbers, see, e.g., [56], [32].

Proposition 10.3.12. Let Q be an ordered field. Let $f, g : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^n$ and $h : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$. Then (i)–(v) below hold.

(ii) Let $\lambda \in \mathrm{Q}$. If f is differentiable at x, then $\lambda \cdot f$ is also differentiable at x and $(\lambda \cdot f)'(x) = \lambda \cdot f'(x)$.

(iii) If f and g are differentiable at x and x is an accumulation point of $Dom\, f \cap Dom\, g$, then $f + g$ is differentiable at x and $(f + g)'(x) = f'(x) + g'(x)$.

(v) If h is increasing (or decreasing) on (a, b), differentiable at $x \in (a, b)$ and $h'(x) \neq 0$, then h^{-1} is differentiable at $h(x)$.

On the proof Since the proofs of the statements are based on the same calculations and ideas as in real analysis, we omit the proof, see [54, Thms. 28.2, 28.3, 28.4 and 29.9]. ■

Chain Rule. Assume AxOF. Let $g : \mathrm{Q}^n \xrightarrow{\circ} \mathrm{Q}^m$ and $f : \mathrm{Q}^m \xrightarrow{\circ} \mathrm{Q}^k$. If g is differentiable at $\vec{p} \in \mathrm{Q}^n$ and f is differentiable at $g(\vec{p})$, then $g \circ f$ is differentiable at $\vec{p}$ and $d_{\vec{p}}\, g \circ d_{g(\vec{p})} f$ is one of its derivatives, i.e.,

$$d_{\vec{p}}(g \circ f) = d_{\vec{p}}\, g \circ d_{g(\vec{p})} f.$$

In particular, if $g : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^m$, and g is differentiable at $t \in \mathrm{Q}$ and f is differentiable at $g(t)$, then

$$(g \circ f)'(t) = d_{g(t)} f\big(g'(t)\big).$$

Proposition 10.3.13. Assume AxOF. The derivative of an affine map is its linear part; i.e., if $A : \mathrm{Q}^n \to \mathrm{Q}^m$ is an affine map, then $(d_{\vec{q}} A)(\vec{p}) = A(\vec{p}) - A(\vec{o})$, where $\vec{p}, \vec{q} \in \mathrm{Q}^n$ and $\vec{o}$ is the origin of Q^n.

Proof. The proof is straightforward from our definitions. ■

Corollary 10.3.14. Let Q be an ordered field. If $g : \mathrm{Q} \to \mathrm{Q}^n$ and $A : \mathrm{Q}^n \to \mathrm{Q}^m$ is an affine map, then $(g \circ A)'(t) = A\big(g'(t)\big) - A(\vec{o})$. ■

We say that function $f : \mathrm{Q}^n \to \mathrm{Q}$ is **locally maximal** at $x \in Dom\, f$ iff there is a $\delta \in \mathrm{Q}^+$ such that $f(y) \leq f(x)$ for all $y \in B_\delta(x)$. The **local minimality** is analogously defined.

Proposition 10.3.15. Assume AxOF. If $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ is differentiable on (a, b) and locally maximal or minimal at $x \in (a, b)$, its derivative is 0 at x, i.e., $f'(x) = 0$.

On the proof The proof is the same as in real analysis, see e.g., [56, Thm.5.8]. ■

Function f is said to be **differentiable on set** H if $H \subseteq Dom\, f$ and f is differentiable at x for all $x \in H$.

CONT-Mean–Value Theorem. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxOF. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be an $\mathcal{L}$-definable function which is differentiable on (a, b) and continuous on (a, b). If $a \neq b$, there is an $s \in (a, b)$ such that $f'(s) = \frac{f(b)-f(a)}{b-a}$. ■

Proof. Let $h(t) := \big(f(b) - f(a)\big) \cdot t - (b - a) \cdot f(t)$. Then h is continuous on $[a, b]$, differentiable on (a, b) and $h(a) = f(b) \cdot a - b \cdot f(a) = h(b)$. If h is constant then $h'(t) = 0$ for all $t \in (a, b)$. Otherwise, by Thm. 10.2.2, there is a maximum/minimum of h different from $h(a) = h(b)$ in a $t \in (a, b)$. Hence $h'(t) = 0$ by Prop. 10.3.15. This completes the proof since $h'(t) = f(b) - f(a) - (b - a) \cdot f'(t)$. ■

CONT-Rolle's Theorem. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxOF. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be definable and $\mathcal{L}$-differentiable function which is differentiable on (a, b) and continuous on (a, b). If $a \neq b$ and $f(a) = f(b)$, there is an $s \in (a, b)$ such that $f'(s) = 0$. ■

Corollary 10.3.16. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume $\mathsf{CONT}_{\mathcal{L}}$ and AxOF. Let $\gamma : \mathrm{Q} \to \mathrm{Q}^n$ be an $\mathcal{L}$-definable and differentiable curve. Then for all distinct $a, b \in \mathrm{Q}$ and for every $(n-1)$-dimensional subspace H that contains $\gamma(a) - \gamma(b)$, there is at least one c between a and b such that $\gamma'(c) \in H$.

Proof. The derivative vector of a curve γ composed with a linear map A at $t \in \mathrm{Q}$ is the A-image of $\gamma'(t)$ by Cor. 10.3.14. Since any $(n-1)$-dimensional subspace of Q^n can be taken to $\{0\} \times \mathrm{Q}^{n-1}$ by a linear transformation, we can assume that $H = \{0\} \times \mathrm{Q}^{n-1}$. Recall that the function $\pi_t : \mathrm{Q}^n \to \mathrm{Q}$ is defined as $p \mapsto p_t$. Then $\gamma \circ \pi_t(a) = \gamma \circ \pi_t(b)$ since $\gamma(a) - \gamma(b) \in H$. By applying Rolle's Theorem to $\gamma \circ \pi_t$, we get that there is a $c \in \mathrm{Q}$ such that $(\gamma \circ \pi_t)'(c) = 0$. Thus $\gamma'(c)$ is an element of H since $(\gamma \circ \pi_t)'(c) = \pi_t\big(\gamma'(c)\big) = \gamma'(c)_t$ by Cor. 10.3.14. ■

Proposition 10.3.17. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume AxOF and $\mathsf{CONT}_{\mathcal{L}}$. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be an $\mathcal{L}$-definable differentiable function $(a, b) \subseteq Dom\, f$ for some $a, b \in \mathrm{Q}$. If $f'(t) = 0$ for all $t \in (a, b)$ then there is a $c \in \mathrm{Q}$ such that $f(t) = c$ for all $t \in (a, b)$.

Proof. If there are $x, y \in (a, b)$ such that $f(x) \neq f(y)$ and $x \neq y$, then from CONT-Mean-Value Theorem there is a t between x and y such that $f'(t) \cdot (y - x) = f(y) - f(x) \neq 0$ and this contradicts that $f'(t) = 0$. ■

CONT-Darboux's Theorem. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume AxOF and $\mathsf{CONT}_{\mathcal{L}}$. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be an $\mathcal{L}$-definable and differentiable function such that $(a,b) \subseteq Dom\, f$ for some $a, b \in \mathrm{Q}$. If $c \in \big(f'(a), f'(b)\big)$, there is an $s \in (a,b)$ such that $f'(s) = c$. ■

Proof. We can assume that $f'(a) > d > f'(b)$. Let $g(t) := f(t) - t \cdot d$. Then g is differentiable and $g'(a) > 0$, $g'(b) < 0$. Thus g cannot be maximal at a or b by Prop. 10.3.15. Thus, from Thm. 10.2.2, we get that there is a point, say c, between a and b where g is maximal. Thus from Prop. 10.3.15, we also get that $g'(c) = f'(c) - d = 0$. ■

Let $i \leq n$. $\pi_i : \mathrm{Q}^n \to \mathrm{Q}$ denotes the i-th projection function, i.e., $\pi_i : p \mapsto p_i$. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^n$. We denote the i-th coordinate function of f by f_i, i.e., $f_i := f \circ \pi_i$. Sometimes f_τ is used instead of f_1. A function $A : \mathrm{Q}^n \to \mathrm{Q}^j$ is said to be an **affine map** if it is a linear map composed with a translation.[1]

The following proposition says that the derivative of a function f composed with an affine map A at x is the image of the derivative $f'(x)$ taken by the linear part of A.

Proposition 10.3.18. Assume AxOF. Let $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^n$ be differentiable at x and let $A : \mathrm{Q}^n \to \mathrm{Q}^j$ be an affine map. Then $f \circ A$ is differentiable at x and $(f \circ A)'(x) = A\big(f'(x)\big) - A(o)$. In particular, $f'(x) = \langle f_1'(x), \ldots, f_n'(x)\rangle$, i.e., $f_i'(x) = f'(x)_i$.

On the proof The statement follows straightforwardly from the respective definitions. ■

Proposition 10.3.19. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume AxOF and $\mathsf{CONT}_{\mathcal{L}}$. Let $f, g : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$ be $\mathcal{L}$-definable and differentiable functions on (a,b). If $f'(x) = g'(x)$ for all $x \in (a,b)$, then there is a $c \in \mathrm{Q}$ such that $f(x) = g(x) + c$ for all $x \in (a,b)$.

Proof. Assume that $f'(x) = g'(x)$ for all $x \in (a,b)$. Let $h := f - g$. Then $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a,b)$ by (ii) and (iii) of Prop. 10.3.12. If there are $y, z \in (a,b)$ such that $h(y) \neq h(z)$ and $y \neq z$, then by the CONT-Mean-Value Theorem, there is an x between y and z such that $h'(x) = \frac{h(z)-h(y)}{z-y} \neq 0$ and this contradicts $h'(x) = 0$. Thus $h(y) = h(z)$ for all $y, z \in (a,b)$. Hence there is a $c \in \mathrm{Q}$ such that $h(x) = c$ for all $x \in (a,b)$. ■

[1] That is, A is an affine map if there are $L : \mathrm{Q}^n \to \mathrm{Q}^j$ and $a \in \mathrm{Q}^j$ such that $A(\vec{p}) = L(\vec{p}) + a$, $L(p+q) = L(\vec{p}) + L(\vec{q})$ and $L(\lambda \cdot p) = \lambda \cdot L(\vec{p})$ for all $\vec{p}, \vec{q} \in \mathrm{Q}^n$ and $\lambda \in \mathrm{Q}$.

10.4 Tools used for proving the Twin Paradox

In this section we develop the tools which were used in Chap. 7. To do so, let us first introduce a notation. We say that $\vec{p} \in \mathrm{Q}^d$ is **vertical** iff $\vec{p}_\sigma = \vec{o}$.

Lemma 10.4.1. Let $f : \mathrm{Q} \overset{\circ}{\longrightarrow} \mathrm{Q}^d$ be a well-parametrized timelike curve. Then the following hold:

(i) Let $x \in \mathit{Dom}\, f$. Then f_τ is differentiable at x and $1 \le |f'_\tau(x)|$. Furthermore, $|f'_\tau(x)| = 1$ iff $f'(x)$ is vertical.

(ii) Assume CONT and let f be definable. Then f_τ is increasing or decreasing. Moreover, $1 \le f'_\tau(x)$ for all $x \in \mathit{Dom}\, f$ if f_τ is increasing.

Proof. As f is a well-parametrized curve, $f'(x)$ is of Minkowski length 1. By Prop. 10.3.13, f_τ is differentiable at x and $f'_\tau(x) = f'(x)_\tau$. Now, Item (i) follows from the fact that the absolute value of the time component of a vector of Minkowski length 1 is always at least 1 and it is 1 iff the vector is vertical.

Let us now prove Item (ii). From Item (i), we have $f'_\tau(x) \neq 0$ for all $x \in \mathit{Dom}\, f$. Thus by CONT-Rolle's Theorem, f_τ is injective. Consequently, by CONT-Bolzano's Theorem, f_τ is increasing or decreasing since f_τ is continuous and injective. Let us now assume that f_τ is increasing. Then $0 \le f'_\tau(x)$ for all $x \in \mathit{Dom}\, f$ by our definition of the derivative. Hence by Item (i), $1 \le f'_\tau(x)$ for all $x \in \mathit{Dom}\, f$; and that is what we wanted to prove. ■

Theorem 10.4.2. Assume CONT. Let $f : \mathrm{Q} \overset{\circ}{\longrightarrow} \mathrm{Q}^d$ be a definable well-parametrized timelike curve, and let $a, b \in \mathit{Dom}\, f$ such that $a < b$. Then the following hold:

(i) $b - a \le |f_\tau(b) - f_\tau(a)|$, and

(ii) $b - a < \big|f_\tau(b) - f_\tau(a)\big|$ if $f(x)_\sigma \neq f(a)_\sigma$ for any $x \in [a, b]$.

Proof. For every $i \le d$, we have that f_i is definable and differentiable by Prop. 10.3.13. Hence by the CONT-Mean-Value Theorem, there is an $s \in (a, b)$ such that $f'_\tau(s) = \frac{f_\tau(b) - f_\tau(a)}{b-a}$. By Item (i) of Lem. 10.4.1, we have $1 \le |f'_\tau(s)|$. Then $b - a \le \big|f_\tau(b) - f_\tau(a)\big|$. That completes the proof of Item (i).

To prove Item (ii), let $x \in [a, b]$ such that $f(x)_\sigma \neq f(a)_\sigma$. Then there is an $i > 1$ such that $f_i(x) \neq f_i(a)$. Hence by the CONT-Mean-Value Theorem, there is a $y \in (a, b)$ such that $f'_i(y) = \frac{f_i(x) - f_i(a)}{x-a} \neq 0$. Thus $f'(y)$ is not vertical. Therefore, by Item (i) of

Lem. 10.4.1, we have $1 < |f'_\tau(y)|$. Thus by our definition of the derivative, there is a $z \in (y, b)$ such that $1 < \frac{|f_\tau(z)-f_\tau(y)|}{z-y}$. Hence we have

$$z - y < |f_\tau(z) - f_\tau(y)|.$$

Let us note that $a < y < z < b$. By applying Item (i) to $[a, y]$ and $[z, b]$ we get

$$y - a \le \big|f_\tau(y) - f_\tau(a)\big| \quad \text{and} \quad b - z \le \big|f_\tau(b) - f_\tau(z)\big|.$$

f_τ is increasing or decreasing by Item (ii) of Lem. 10.4.1. Thus $f_\tau(a) < f_\tau(y) < f_\tau(z) < f_\tau(b)$ or $f_\tau(a) > f_\tau(y) > f_\tau(z) > f_\tau(b)$. Therefore, by adding up the last three inequalities, we get $b - a < \big|f_\tau(b) - f_\tau(a)\big|$. ■

Theorem 10.4.3. Assume CONT. Let $f, g : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^d$ be definable well-parametrized timelike curves. Let $a, b, a', b' \in \mathrm{Q}$ such that

- $a \le b$ and $a' \le b'$,
- $[a, b] \subseteq Dom\, f$ and $[a', b'] \subseteq Dom\, g$, and
- $\{f(t) : t \in [a, b]\} = \{g(t') : t' \in [a', b']\}$.

Then $b - a = b' - a'$.

Proof. By (ii) of Lem. 10.4.1, f_τ is increasing or decreasing on $[a, b]$ and so is g_τ on $[a', b']$. Without losing generality, we can assume that $Dom\, f = [a, b]$, $Dom\, g = [a', b']$ and that f_τ and g_τ are increasing on $[a, b]$ and $[a', b']$, respectively.[2] Then $Ran\,(f) = Ran\,(g)$ by the assumptions of the theorem. Furthermore, f and g are injective since f_τ and g_τ are such. Since $Ran\,(f) = Ran\,(g)$ and g_τ is injective, $f \circ g^{-1} = f_\tau \circ g_\tau^{-1}$. Let $h := f \circ g^{-1} = f_\tau \circ g_\tau^{-1}$. Since $Ran\,(f_\tau) = Ran\,(g_\tau)$ and f_τ and g_τ are increasing, h is an increasing bijection between $[a, b]$ and $[a', b']$. Hence $h(a) = a'$ and $h(b) = b'$. We prove that $b - a = b' - a'$ by proving that there is a $c \in \mathrm{Q}$ such that $h(x) = x + c$ for all $x \in [a, b]$. We can assume that $a \neq b$ and $a' \neq b'$. By Lem. 10.4.1, f_τ and g_τ are differentiable on $[a, b]$ and $[a', b']$, respectively, and $f'_\tau(x) > 0$ for all $x \in [a, b]$ and $g'_\tau(x') > 0$ for all $x' \in [a', b']$. By Chain Rule and (v) of Prop. 10.3.12, $h = f_\tau \circ g_\tau^{-1}$ is also differentiable on (a, b). By $h = f \circ g^{-1}$, we have $f = h \circ g$. Thus $f'(x) = h'(x) g'\big(h(x)\big)$ for all $x \in (a, b)$ by Chain Rule. Since both $f'(x)$ and $g'\big(h(x)\big)$ are of Minkowski length 1 and their time-components are positive[3] for all $x \in (a, b)$, we conclude that $h'(x) = 1$

[2] It can be assumed that f_τ is increasing on $[a, b]$ because the assumptions of the theorem remain true when f and $[a, b]$ are replaced by $-Id \circ f$ and $[-b, -a]$, respectively, and f_τ is decreasing on $[a, b]$ iff $(-Id \circ f)_\tau$ is increasing on $[-b, -a]$.

[3] That is, $f'_\tau(x) > 0$ and $g'_\tau\big(h(x)\big) > 0$.

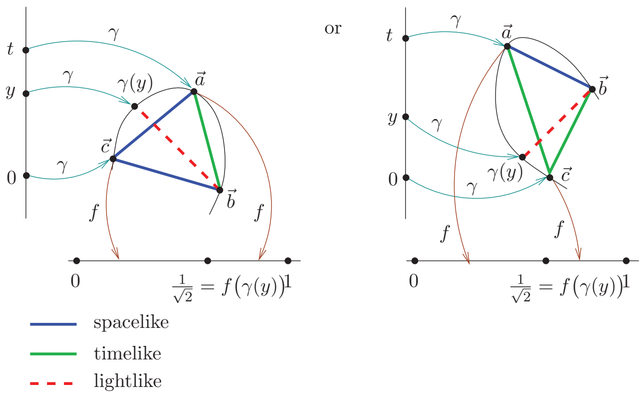

Figure 10.1: Illustration for the proof of Prop. 10.4.4

for all $x \in (a, b)$. By Prop. 10.3.19, we get that there is a $c \in \mathrm{Q}$ such that $h(x) = x + c$ for all $x \in (a, b)$ and thus for all $x \in [a, b]$ since h is an increasing bijection between $[a, b]$ and $[a', b']$. ■

A curve is called slower than light (STL) and faster than light (FTL) iff any of its chords is timelike and spacelike, respectively.

Proposition 10.4.4. Let $\mathcal{L} \supseteq \mathcal{OF}$. Assume AxOF and $\mathsf{CONT}_{\mathcal{L}}$. Let $\gamma : \mathrm{Q} \overset{\circ}{\to} \mathrm{Q}^d$ be an $\mathcal{L}$-definable and continuous curve. Then (i) and (ii) below hold:

(i) γ is timelike $\Longrightarrow$ γ is STL.

(ii) γ is STL, FTL or it has a lightlike chord.

Proof. To prove the first statement, let us assume that γ is not STL. Then it has a lightlike or spacelike chord, say $\{\vec{p}, \vec{q}\}$. Let H be a $(d-1)$-dimensional subspace that contains $\vec{p} - \vec{q}$ and does not contain timelike vectors. Thus by Cor. 10.3.16, we get that there is a $t \in \mathrm{Q}$ such that $\gamma'(t)$ is in H. Since H does not contain timelike vectors, $\gamma'(t)$ is not timelike. Thus γ is not timelike.

To prove the second statement, let us assume that γ is not STL or FTL and does not have a lightlike chord. Then γ has both timelike and spacelike chords. Then there are distinct points $\vec{a}, \vec{b}, \vec{c} \in Ran(\gamma)$ such that the triangle $\{\vec{a}, \vec{b}, \vec{c}\}$ determines two timelike and one spacelike or two spacelike and one timelike chords of γ. We can assume that $\gamma(0) = \vec{c}$ and $\vec{c}$ is the intersection of the chords of same type. See Fig. 10.1.

For every $\vec{p} \in Ran\,(\gamma)$ by CONT, there is a closest $t \in Q$ to 0 such that $\gamma(t) = \vec{p}$, i.e., the set $H{:=}\{\,|x| \;:\; \gamma(x) = \vec{p}\,\}$ has a minimal element.[4] Thus there is a $t \in Q$ such that $\gamma(t)$ is $\vec{a}$ or $\vec{b}$ and there is no t' between 0 and t such that $\gamma(t')$ is $\vec{a}$ or $\vec{b}$. We can assume that $\gamma(t) = \vec{a}$ and $t > 0$.

Let $f : Q^d \setminus \{\vec{b}\,\} \to Q$ be the function defined as $\vec{p} \mapsto \frac{|p_\tau - b_\tau|}{|\vec{p}-\vec{b}\,|}$. It is easy to see that f is continuous and for all $\vec{p} \in Q^d \setminus \{\vec{b}\,\}$

$$\begin{aligned} f(\vec{p}) = 1/\sqrt{2} &\iff \vec{p} - \vec{b} \text{ is lightlike,} \\ f(\vec{p}) > 1/\sqrt{2} &\iff \vec{p} - \vec{b} \text{ is timelike,} \\ f(\vec{p}) < 1/\sqrt{2} &\iff \vec{p} - \vec{b} \text{ is spacelike.} \end{aligned} \tag{10.1}$$

Consider the function $g{:=}\gamma\big|_{[0,t]} \circ f$. It is a continuous function. Furthermore, $Dom\, g = [0,t]$ since there is no $t' \in [0,t]$ such that $\gamma(t') = b$. By (10.1) above and by the fact that $\gamma(0) = \vec{c}$ and $\gamma(t) = \vec{a}$, we have that

$$\big(g(0) > 1/\sqrt{2} \text{ and } g(t) < 1/\sqrt{2}\big) \quad \text{or} \quad \big(g(0) < 1/\sqrt{2} \text{ and } g(t) > 1/\sqrt{2}\big)$$

since one of the chords $\{\vec{b}, \vec{c}\,\}$ and $\{\vec{b}, \vec{a}\,\}$ is timelike and the other is spacelike. However, by CONT-Bolzano's Theorem, there is a $y \in [0,t]$ such that $g(y) = 1/\sqrt{2}$. Hence by (10.1) above, we have that $\gamma(y) - \vec{b}$ is lightlike for this y. Consequently, $\{\vec{b}, \gamma(y)\}$ is a lightlike chord of γ. This contradiction proves our proposition. ∎

10.5 Tools used for simulating gravity by accelerated observers

In this section we develop the tools which were used in Chap. 8. To do so, let us first introduce the following convenient notation. We say that $\alpha : Q \xrightarrow{\circ} Q$ is a **nice map** if it is differentiable such that $0 \notin Ran\,\alpha'$, and $Dom\,\alpha$ is connected.

Lemma 10.5.1. Let α be a timelike curve. Then α_τ is a nice map.

Proof. Since α is a timelike curve, $Dom\,\alpha$ is connected and $\alpha'(x)_\tau \neq 0$ for all $x \in Dom\,\alpha$. But $Dom\,\alpha_\tau = Dom\,\alpha$ and $(\alpha_\tau)' = (\alpha')_\tau$. Thus α_τ is a nice map. ∎

Lemma 10.5.2. Assume CONT. Let α be a definable nice map. Then α is injective. Moreover, α is monotonous.

[4] That is so because of the following. Let s be the supremum of the nonempty bounded definable set $\{\,-|x| \;:\; \gamma(x) = \vec{p}\}$. By the continuity of γ, one of $\gamma(s)$ and $\gamma(-s)$ must be $\vec{p}$. Then $-s$ is the minimal element of H.

Proof. If α were not injective, then $\alpha'(x)$ would be 0 for some x by CONT-Rolle's Theorem. But $\alpha'(x)$ cannot be 0 since α is a nice map. Thus α is injective. Then α is also monotonous by (2) in Lem. 10.2.4. ■

Lemma 10.5.3. Assume CONT. If α and δ are nice maps, δ^{-1} and $\alpha \circ \delta$ are also nice maps. ■

Lemma 10.5.4. Assume CONT. Let α and δ be definable timelike curves such that $\mathit{Ran}\,\alpha \subseteq \mathit{Ran}\,\delta$ (or $\mathit{Ran}\,\delta \subseteq \mathit{Ran}\,\alpha$), and let $h{:=}\alpha \circ \delta^{-1}$. Then h is a nice map and

$$|h'(x)| = \frac{\mu\big(\alpha'(x)\big)}{\mu\big(\delta'(h(x))\big)} \quad \text{for all} \quad x \in \mathit{Dom}\,h. \tag{10.2}$$

Proof. First we show that $h = \alpha_\tau \circ \delta_\tau^{-1}$. Since α and δ are definable timelike curves, α_τ and δ_τ are definable nice maps by Lem. 10.5.1. Thus α_τ and δ_τ are injective by Lem. 10.5.2. Consequently, α and δ are also injective. Therefore, $\langle x, y\rangle \in \alpha_\tau \circ \delta_\tau^{-1}$ iff $\alpha_\tau(x) = \delta_\tau(y)$ and $\langle x, y\rangle \in \alpha \circ \delta^{-1}$ iff $\alpha(x) = \delta(y)$. Since $\alpha(x) = \delta(y) \rightarrow \alpha_\tau(x) = \delta_\tau(y)$ is clear, we have to show the converse implication only. By symmetry, we can assume that $\mathit{Ran}\,\alpha \subseteq \mathit{Ran}\,\delta$. Then there is a $z \in \mathit{Dom}\,\delta$ such that $\delta(z) = \alpha(x)$, so $\delta_\tau(z) = \alpha_\tau(x) = \delta_\tau(y)$. Thus $z = y$ since δ is injective, so $\alpha(x) = \delta(y)$. That proves $h = \alpha_\tau \circ \delta_\tau^{-1}$.

By Lem. 10.5.3, h is a nice map, so $\mathit{Dom}\,h$ is an interval. We have that $\alpha \supseteq h \circ \delta$ since $h = \alpha \circ \delta^{-1}$. Thus by Chain Rule, $\alpha'(x) = h'(x) \cdot \delta'\big(h(x)\big)$ for all $x \in \mathit{Dom}\,h$. Since $\mu(\lambda \vec{p}) = |\lambda| \cdot \mu(\vec{p})$ for all $\lambda \in Q$ and $\vec{p} \in Q^d$, we have that $\mu\big(\alpha'(x)\big) = |h'(x)| \cdot \mu\big(\delta'(h(x))\big)$ for all $x \in \mathit{Dom}\,h$. We have that $\mu\big(\delta'(h(x))\big) \neq 0$ since δ is timelike. Hence equation (10.2) holds. ■

Lemma 10.5.5. Assume CONT. Let β and γ be definable and well-parametrized timelike curves; let β_* and γ_* be definable timelike curves; let $x_\beta, y_\beta \in \mathit{Dom}\,\beta$, $x_\gamma, y_\gamma \in \mathit{Dom}\,\gamma$ and $x, y \in \mathit{Dom}\,\beta_* \cap \mathit{Dom}\,\gamma_*$ such that

(i) $\mathit{Ran}\,\beta_* \subseteq \mathit{Ran}\,\beta$ and $\mathit{Ran}\,\gamma_* \subseteq \mathit{Ran}\,\gamma$.

(ii) $\beta(x_\beta) = \beta_*(x)$, $\beta(y_\beta) = \beta_*(y)$, $\gamma(x_\gamma) = \gamma_*(x)$, $\gamma(y_\gamma) = \gamma_*(y)$.

(iii) $x \neq y$ and $\mu\big(\gamma_*'(z)\big) > \mu\big(\beta_*'(z)\big)$ for all $z \in (x, y)$.

Then $\big|x_\gamma - y_\gamma\big| > \big|x_\beta - y_\beta\big|$.

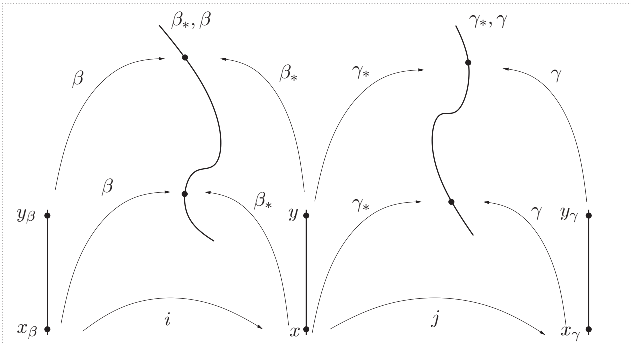

Figure 10.2: Illustration for the proof of Lem. 10.5.5

Proof. Since β, β_*, γ and γ_* are definable timelike curves, they are injective by Lems. 10.5.1 and 10.5.2. Thus $x_\beta \neq y_\beta$ and $x_\gamma \neq y_\gamma$ since $x \neq y$. Let

$$i{:=}\beta \circ \beta_*^{-1} \quad \text{and} \quad j{:=}\gamma_* \circ \gamma^{-1},$$

see Fig. 10.2. Then i, j and $i \circ j$ are nice maps by Lem. 10.5.3 and 10.5.4. Furthermore,

$$i(x_\beta) = x, \;\; i(y_\beta) = y, \;\; j(x) = x_\gamma, \;\; j(y) = y_\gamma, \;\; (i \circ j)(x_\beta) = x_\gamma \;\text{ and }\; (i \circ j)(y_\beta) = y_\gamma.$$

Since $x_\beta, y_\beta \in Dom\, i \circ j$, and $i \circ j$ is a nice map, we have that $(x_\beta, y_\beta) \subseteq Dom\, i \circ j$.

Now we will show that

$$\forall t \in (x_\beta, y_\beta) \quad \big|(i \circ j)'(t)\big| > 1.$$

To prove this statement, let $t \in (x_\beta, y_\beta)$. Since i is a nice map, it is monotonous by Lem. 10.5.2, thus $i(t) \in (x, y)$. By Lem. 10.5.4 and the fact that β and γ are well-parametrized, we have that

$$\big|i'(t)\big| = \frac{\mu\big(\beta'(t)\big)}{\mu\big(\beta_*'(i(t))\big)} = \frac{1}{\mu\big(\beta_*'(i(t))\big)} \tag{10.3}$$

and

$$\big|j'\big(i(t)\big)\big| = \frac{\mu\big(\gamma_*'(i(t))\big)}{\mu\big(\gamma'\big(j(i(t))\big)\big)} = \mu\big(\gamma_*'(i(t))\big). \tag{10.4}$$

From equations (10.3), (10.4) and Item (iii) by Chain Rule, we have that

$$\big|(i \circ j)'(t)\big| = \big|i'(t) \cdot j'\big(i(t)\big)\big| = \frac{\mu\big(\gamma_*'(i(t))\big)}{\mu\big(\beta_*'(i(t))\big)} > 1.$$

This completes the proof of (10.5).

By CONT-Mean–Value Theorem there is a $z \in (x_\beta, y_\beta)$ such that

$$(i \circ j)'(z) = \frac{(i \circ j)(x_\beta) - (i \circ j)(y_\beta)}{x_\beta - y_\beta} = \frac{x_\gamma - y_\gamma}{x_\beta - y_\beta}.$$

By this and (10.5), we conclude that $\left|\frac{x_\gamma - y_\gamma}{x_\beta - y_\beta}\right| > 1$. Hence $|x_\gamma - y_\gamma| > |x_\beta - y_\beta|$, as it was required. ■

Remark 10.5.6. Lem. 10.5.5 remains true even if we substitute "=" or "≥" for ">". The proof can be achieved by the same substitution in the original proof.

Proposition 10.5.7.

(1) Let α be a well-parametrized timelike curve. If α is twice differentiable at $t \in Dom\,\alpha$, then $\alpha'(t) \perp_\mu \alpha''(t)$.

(2) Let $d \geq 3$. Assume $\mathsf{AccRel_0}$. Let $k \in \mathrm{Ob}$ and $m \in \mathrm{IOb}$. Then $\vec{\mathbf{v}}^k_m(t) \perp_\mu \vec{\mathbf{a}}^k_m(t)$ for all $t \in Dom\,\vec{\mathbf{a}}^k_m$.

Proof. To prove Item (1), let $t \in Dom\,\alpha$ such that α is twice differentiable at t. Since α is a well-parametrized timelike curve, we have that

$$\left(\alpha_1'(t)\right)^2 - \left(\alpha_2'(t)\right)^2 - \ldots - \left(\alpha_d'(t)\right)^2 = 1. \tag{10.5}$$

By derivation of both sides of equation (10.5) we have that

$$2\alpha_1'(t) \cdot \alpha_1''(t) - 2\alpha_2'(t) \cdot \alpha_2''(t) - \ldots - 2\alpha_d'(t) \cdot \alpha_d''(t) = 0.$$

Thus $\alpha'(t) \perp_\mu \alpha''(t)$, which is what we wanted to prove.

Item (2) is a consequence of Item (1) since $\vec{\mathbf{v}}^k_m = (\mathsf{lc}^k_m)'$, $\vec{\mathbf{a}}^k_m = (\mathsf{lc}^k_m)''$, lc^k_m is a well-parametrized timelike curve by Thm. 6.1.11, and lc^k_m is twice differentiable at t iff $t \in Dom\,\vec{\mathbf{a}}^k_m$. ■

If $f : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}$, we abbreviate $f(t) > 0$ for all $t \in Dom\,f$ to $f > 0$. We also use the analogous notation $f < 0$.

Lemma 10.5.8.

(1) Assume CONT. Let α be a definable and twice differentiable timelike curve such that $Ran\,\alpha \subset tx\text{-}\mathsf{Plane}$. If $\alpha'' \circ \mu < 0$, then α_2' is increasing or decreasing.

(2) Let $d \geq 3$. Assume AccRel. Let $k \in \mathrm{Ob}$ and $m \in \mathrm{IOb}$ such that $\mathsf{wl}_m(k) \subset tx\text{-}\mathsf{Plane}$ and $Dom\,\vec{\mathbf{a}}^k_m = Dom\,\mathsf{lc}^k_m$. If k is positively accelerated, $(\vec{\mathbf{v}}^k_m)_2$ is increasing or decreasing.

Proof. To prove Item (1), let $t \in Dom\,\alpha$. By Prop. 10.5.7, $\alpha''(t)$ is a spacelike vector since it is Minkowski orthogonal to a timelike one. Therefore, $\mu(\alpha''(t)) < 0$ iff $|\alpha''_\sigma(t)| \neq 0$. Thus, since $Ran\,\alpha \subset tx$-Plane, we have that $\mu(\alpha''(t)) < 0$ iff $\alpha''_2(t) \neq 0$. Thus by CONT-Darboux's Theorem, $\alpha'' \circ \mu < 0$ iff $\alpha''_2 > 0$ or $\alpha''_2 < 0$ since α'_2 is definable and $Dom\,\alpha' = Dom\,\alpha$ is connected. Then by CONT-Mean-Value Theorem, α'_2 is increasing or decreasing.

Item (2) is a consequence of Item (1) because of the following. Let $\alpha = \mathsf{lc}^k_m$. Then by Thm. 6.1.11, α is definable (well-parametrized) timelike curve. α twice differentiable since $Dom\,\alpha'' = Dom\,\vec{\mathbf{a}}^k_m = Dom\,\mathsf{lc}^k_m = Dom\,\alpha$; and $Ran\,\alpha \subset tx$-Plane since by (5) in Prop. 6.1.6, $\mathsf{wl}_m(k) = Ran\,\mathsf{lc}^k_m$. Then k is positively accelerated iff $\alpha'' \circ \mu < 0$. Hence by Item (1), if k is positively accelerated, $(\vec{\mathbf{v}}^k_m)_2 = \alpha'_2$ is increasing or decreasing. ■

Let us introduce the following notation:

$$dw^k_m(\vec{p}) := w^k_m(\vec{p}) - w^k_m(\vec{o}).$$

Proposition 10.5.9. Let $d \geq 3$. Assume $\mathsf{SpecRel}$. Let $m, k \in \mathrm{IOb}$ and $h \in \mathrm{Ob}$. Then

(1) $\vec{p} \perp_\mu \vec{q}$ iff $dw^k_m(\vec{p}) \perp_\mu dw^k_m(\vec{q})$.

(2) $\vec{\mathbf{v}}^h_m = \vec{\mathbf{v}}^h_k \circ dw^k_m$ and $Dom\,\vec{\mathbf{v}}^h_m = Dom\,\vec{\mathbf{v}}^h_k$.

(3) $\vec{\mathbf{a}}^h_m = \vec{\mathbf{a}}^h_k \circ dw^k_m$ and $Dom\,\vec{\mathbf{a}}^h_m = Dom\,\vec{\mathbf{a}}^h_k$.

Proof. To prove Item (1), observe that $\mu\big(dw^k_m(\vec{p})\big) = \mu(\vec{p})$ by Thm. 3.2.2. The statement $\vec{p} \perp_\mu \vec{q}$ iff $\mu(\vec{p} + \vec{q})^2 = \mu(\vec{p})^2 + \mu(\vec{q})^2$ can be proved by straightforward calculation. Thus Item (1) is clear since dw^k_m is linear by Thm. 3.2.2.

To prove Items (2) and (3), let us note that lc^h_m and lc^h_k are functions by Item (2) in Prop. 6.1.6. Thus $\vec{\mathbf{v}}^h_m = \vec{\mathbf{v}}^h_k \circ dw^k_m$ follows by Chain Rule because $\mathsf{lc}^h_k = \mathsf{lc}^h_m \circ w^m_k$ (by (3) in Prop. 6.1.6), the derivative of w^k_m is dw^k_m (since w^k_m is affine transformation by Thm. 3.2.2), and $\vec{\mathbf{v}}^h_x = (\mathsf{lc}^h_x)'$ (by definition). Hence $Dom\,\vec{\mathbf{v}}^h_m = Dom\,\vec{\mathbf{v}}^h_k$ also holds since dw^k_m is a bijection. $\vec{\mathbf{a}}^h_m = \vec{\mathbf{a}}^h_k \circ dw^k_m$ follows from (2) of this proposition by Chain Rule because the derivative of dw^k_m is dw^k_m (since dw^k_m is a linear transformation), and $\vec{\mathbf{a}}^h_x = (\vec{\mathbf{v}}^h_x)'$ (by definition). Hence $Dom\,\vec{\mathbf{a}}^h_m = Dom\,\vec{\mathbf{a}}^h_k$ also holds since dw^k_m is a bijection. ■

The **light cone** of $\vec{p} \in \mathrm{Q}^d$ is defined as $\Lambda_{\vec{p}} := \{\vec{q} \in \mathrm{Q}^d : \vec{p}\,\lambda\,\vec{q}\}$. The **past light cone** of $\vec{p} \in \mathrm{Q}^d$ is defined as $\Lambda^-_{\vec{p}} := \{\vec{q} \in \mathrm{Q}^d : \vec{p}\,\lambda\,\vec{q} \;\wedge\; q_\tau \leq p_\tau\}$. The **future light cone** of $\vec{p} \in \mathrm{Q}^d$ is defined as $\Lambda^+_{\vec{p}} := \{\vec{q} \in \mathrm{Q}^d : \vec{p}\,\lambda\,\vec{q} \;\wedge\; q_\tau \geq p_\tau\}$. We say that $\vec{p} \in \mathrm{Q}^d$ **chronologically precedes** $\vec{q} \in \mathrm{Q}^d$, in symbols $\vec{p} \ll \vec{q}$, iff $\vec{p}\,\tau\,\vec{q}$ and

$p_\tau < q_\tau$. The **chronological past** of $\vec{p} \in \mathrm{Q}^d$ is defined as $I^-_{\vec{p}} := \{\, \vec{q} \in \mathrm{Q}^d \,:\, \vec{q} \ll \vec{p} \,\}$. The **chronological future** of $\vec{p} \in \mathrm{Q}^d$ is defined as $I^+_{\vec{p}} := \{\, \vec{q} \in \mathrm{Q}^d \,:\, \vec{p} \ll \vec{q} \,\}$. The **chronological interval** between $\vec{p} \in \mathrm{Q}^d$ and $\vec{q} \in \mathrm{Q}^d$ is defined as $\langle\!\langle \vec{p}, \vec{q} \rangle\!\rangle := \{\vec{r} \in \mathrm{Q}^d : \vec{p}\,\tau\,\vec{r} \,\wedge\, \vec{q}\,\tau\,\vec{r} \,\wedge\, r_\tau \in (p_\tau, q_\tau)\}$. We also use the notation $I_{\vec{p}} := I^-_{\vec{p}} \cup I^+_{\vec{p}} \cup \{\vec{p}\}$.

Lemma 10.5.10. Let $\vec{p}, \vec{q} \in \mathrm{Q}^d$. Then

(1) If $\vec{p}\,\tau\,\vec{q}$, then $\Lambda^-_{\vec{p}} \cap \Lambda^-_{\vec{q}} = \Lambda^+_{\vec{p}} \cap \Lambda^+_{\vec{q}} = \emptyset$.

(2) If $\vec{p} \ll \vec{q}$, then $\Lambda^-_{\vec{q}} \cap I^-_{\vec{p}} = \emptyset$, and $\Lambda^-_{\vec{p}} \cup I^-_{\vec{p}} \subset I^-_{\vec{q}}$.

(3) $\vec{p} \ll \vec{q}$ iff $I^+_{\vec{p}} \cap I^-_{\vec{q}} \neq \emptyset$. ■

Lemma 10.5.11. Assume CONT. Let γ be a definable timelike curve, and let $x, y \in Dom\,\gamma$ such that $x \neq y$. Then

(1) All the chords of γ are timelike, i.e., $\gamma(x)\,\tau\,\gamma(y)$.

(2) If $\gamma(x) \in I^-_{\vec{p}}$ and $\gamma(y) \notin I^-_{\vec{p}}$, there is a $z \in [x, y]$ such that $\gamma(z) \in \Lambda^-_{\vec{p}}$.

(3) If $\gamma(x) \in I^+_{\vec{p}}$ and $\gamma(y) \notin I^+_{\vec{p}}$, there is a $z \in [x, y]$ such that $\gamma(z) \in \Lambda^+_{\vec{p}}$.

(4) If γ_τ is increasing (decreasing), $\gamma(x) \ll \gamma(y)$ iff $x < y$ ($y < x$).

(5) $z \in (x, y)$ iff $\gamma(z) \in \langle\!\langle \gamma(x), \gamma(y) \rangle\!\rangle$.

Proof. Item (1) follows from Prop. 10.4.4. To prove Item (2), let

$$H := \{\, t \in [x, y] \,:\, \mu\big(\gamma(t), \vec{p}\big) < 0 \,\wedge\, \gamma(t)_\tau < p_\tau \,\}\,.$$

It is clear that $H \subseteq Dom\,\gamma$ is definable, bounded and nonempty. Let $z := sup\,H$ which exists by CONT. Thus by continuity of $t \mapsto \mu(\gamma(t), \vec{p})$ and γ_τ, we have that $\gamma(z) \notin I^-_{\vec{p}}$ since z is an upper bound of H. Furthermore, $\mu(\gamma(t), \vec{p}) \leq 0$ and $\gamma(t)_\tau \leq p_\tau$ since z is the least upper bound of H. But $\gamma(t)_\tau = p_\tau$ and $\mu(\gamma(t), \vec{p}) < 0$ is impossible. Thus $\gamma(t)_\tau \leq p_\tau$ and $\mu(\gamma(t), \vec{p}) = 0$. Hence $\gamma(\vec{p}) \in \Lambda^-_{\vec{p}}$.

Item (3) is clear from Item (2) since the continuous bijection $\vec{p} \mapsto -\vec{p}$ takes $I^+_{\vec{p}}$ to $I^-_{\vec{p}}$ and $\Lambda^+_{\vec{p}}$ to $\Lambda^-_{\vec{p}}$.

Item (4) is clear by Item (1).

Item (5) is a consequence of Item (4) since γ_τ is either increasing or decreasing by Lems. 10.5.2 and 10.5.3. ■

We use the following notations:

$$Cone_\varepsilon(\vec{p};\vec{q}) := \bigcup_{\vec{r}\in B_\varepsilon(\vec{q})} line(\vec{p},\vec{r}) \quad \text{and} \quad \Lambda^-[H] := \bigcup_{\vec{p}\in H} \Lambda^-_{\vec{p}}.$$

Let α and β be timelike curves. We say that β_* is the **photon reparametrization of β according to α** if

$$\beta_* = \{\langle t,\vec{p}\rangle \in Dom\,\alpha \times Ran\,\beta \ : \ \vec{p} \in \Lambda^-_{\alpha(t)}\}.$$

Proposition 10.5.12. Assume CONT. Let α and β be definable timelike curves. Let β_* be the photon reparametrization of β according to α.

(1) Then β_* is a definable, continuous and injective curve.

(2) If $Ran\,\alpha \cap Ran\,\beta = \emptyset$, and $Ran\,\alpha \cup Ran\,\beta$ is in a vertical plane, β_* is a timelike curve, and $\beta_*(t_0) + \beta_*'(t_0) \in \Lambda^-_{\alpha(t_0)+\alpha'(t_0)}$.

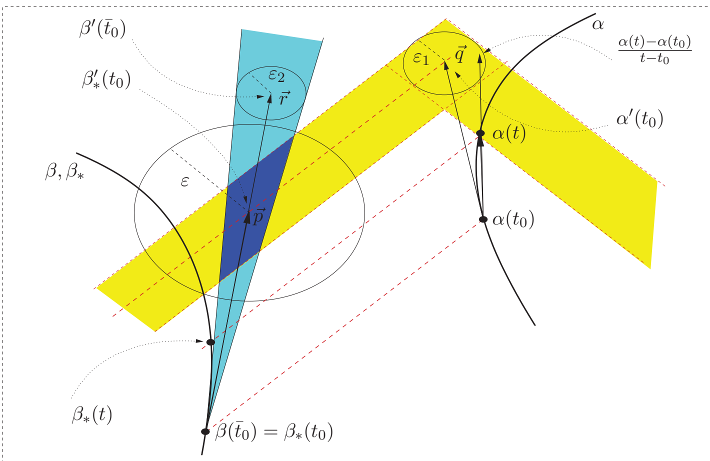

Figure 10.3: Illustration for the proof of Prop. 10.5.12

Proof. It is clear that β_* is definable.

To show that β_* is a function, we need to prove that $\Lambda^-_{\alpha(t)} \cap Ran\,\beta$ has one element at the most for all $t \in Dom\,\alpha$. It is clear by Lem. 10.5.11 since if it had two distinct elements, say $\vec{p}$ and $\vec{q}$, then $\{\vec{p},\vec{q}\}$ would not be a timelike chord of β, but a lightlike one.

For all $t \in Dom\,\beta_*$, let $\bar{t} \in Dom\,\beta$ such that $\beta(\bar{t}\,) = \beta_*(t)$, and let $f : t \mapsto \bar{t}$ be the reparametrization map, i.e., $f{:=}\beta_* \circ \beta^{-1}$. First we show that

$$t \in (t_1, t_2) \;\; \leftrightarrow \;\; \alpha(t) \in \langle\!\langle \alpha(t_1), \alpha(t_2) \rangle\!\rangle \;\overset{(*)}{\leftrightarrow}\; \beta_*(t) \in \langle\!\langle \beta_*(t_1), \beta_*(t_2) \rangle\!\rangle \;\; \leftrightarrow \;\; \bar{t} \in (\bar{t}_1, \bar{t}_2)$$

if $t, t_1, t_2 \in Dom\,\beta_*$. The first and the last equivalence are clear by (5) in Lem. 10.5.11 since α, β are timelike curves and $\beta(\bar{t}\,) = \beta_*(t)$ for all $t \in Dom\,\beta_*$. To prove $(*)$, we can assume that $\alpha(t_1) \ll \alpha(t) \ll \alpha(t_2)$. Thus $\beta_*(t) \ll \alpha(t_2)$ since $\Lambda^-_{\alpha(t)} \subset I^-_{\alpha(t_2)}$ (2) of by Lem. 10.5.10. Therefore, $\beta_*(t) \ll \beta_*(t_2)$ since $\beta_*(t) \in I_{\beta_*(t_2)}$ by (1) in Lem. 10.5.11, but $I^+_{\beta_*(t_2)} \cap I^-_{\alpha(t_2)} = \emptyset$ (3) by Lem. 10.5.10. A similar argument can show that $\beta_*(t_1) \ll \beta_*(t)$, so $(*)$ is proved. Now we have that f preserves betweenness, so it is monotonous.

To show that $Dom\,\beta_*$ is connected, let $x, y \in Dom\,\beta_*$, and let $z \in (x, y)$. Then $z \in Dom\,\alpha$ since $x, y \in Dom\,\alpha$ and $Dom\,\alpha$ is connected. Since α is a timelike curve, $\alpha(z) \in \langle\!\langle \alpha(x), \alpha(y) \rangle\!\rangle$. Without losing generality, we can assume that $\alpha(x) \ll \alpha(z) \ll \alpha(y)$. Then $\beta_*(x) \in I^-_{\alpha(z)}$ since $\beta_*(x) \in \Lambda^-_{\alpha(x)} \subset I^-_{\alpha(z)}$; and $\beta_*(y) \notin I^-_{\alpha(z)}$ since $\beta_*(y) \in \Lambda^-_{\alpha(y)}$ and $\Lambda^-_{\alpha(y)} \cap I^-_{\alpha(z)} = \emptyset$, see Lem. 10.5.10. Then by (2) in Lem. 10.5.11, there is a $\widehat{z} \in Dom\,\beta$ such that $\beta(\widehat{z}) \in \Lambda^-_{\alpha(z)}$ since $\beta(\bar{x}) \in I^-_{\alpha(z)}$ and $\beta(\bar{y}) \notin I^-_{\alpha(z)}$. Thus $\langle z, \beta(\widehat{z}) \rangle \in \beta_*$. Consequently, $z \in Dom\,\beta_*$. Hence $Dom\,\beta_*$ is connected.

Now using a similar argument, we show that $Ran\,f \subseteq Dom\,\beta$ is also connected. To do so, let $\bar{x}, \bar{y} \in Ran\,f$ and $\widehat{z} \in (\bar{x}, \bar{y})$. Then $\widehat{z} \in Dom\,\beta$. We can assume that $\beta(\bar{x}) \ll \beta(\widehat{z}) \ll \beta((\bar{y})$. Then $\alpha(x) \in I^+_{\beta(\widehat{z})}$ and $\alpha(y) \notin I^+_{\beta(\widehat{z})}$. Thus there is a $z \in Dom\,\alpha$ such that $\alpha(z) \in \Lambda^+_{\beta(\widehat{z})}$. Consequently, $\beta(\widehat{z}) \in \Lambda^-_{\alpha(z)}$, so $\langle z, \beta(\widehat{z}) \rangle \in \beta_*$. Therefore, $\widehat{z} \in Ran\,f$, and hence $Ran\,f$ is connected.

Since $Ran\,f$ is connected and f is monotonous, f must be continuous by Lem. 10.2.3. Hence $\beta_* = f \circ \beta$ is also continuous and β_* injective since both β and f are such. So Item (1) is proved.

To prove Item (2), let $\vec{q} = \alpha'(t_0) + \alpha(t_0)$, $\vec{r} = \beta'(\bar{t}_0) + \beta(\bar{t}_0)$, and let $\vec{p}$ be the unique element of $\Lambda^-_{\vec{q}} \cap line(\beta(\bar{t}_0), \vec{r})$, see Fig. 10.3. We will show that $\beta'_*(t_0) = \vec{p} - \beta_*(t_0)$. To do so, let $\varepsilon \in \mathrm{Q}^+$ be fixed. We have to show that there is a $\delta \in \mathrm{Q}^+$ such that $\frac{\beta_*(t) - \beta_*(t_0)}{t - t_0} \in B_\varepsilon(\vec{p})$ if $t \in Dom\,\beta_* \cap B_\delta(t_0)$. It is clear that we can choose ε_1 and ε_2 such that

$$\Lambda^-[B_{\varepsilon_1}(\vec{q})] \cap Cone_{\varepsilon_2}(\beta_*(t_0); \vec{r}\,) \subset B_\varepsilon(\vec{p}). \tag{10.6}$$

Since α is differentiable at t_0, there is a $\delta_1 \in \mathrm{Q}^+$ such that

$$\frac{\alpha(t) - \alpha(t_0)}{t - t_0} + \alpha(t_0) \in B_{\varepsilon_1}(\vec{q}) \tag{10.7}$$

if $t \in Dom\,\alpha \cap B_{\delta_1}(t_0)$. Since $Ran\,\beta \cap Ran\,\alpha = \emptyset$, and $Ran\,\beta \cup Ran\,\alpha$ is in a vertical plane, $line\big(\beta_*(t),\alpha(t)\big)$ and $line\big(\beta_*(t_0),\alpha(t_0)\big)$ are parallel. Hence

$$\frac{\beta_*(t)-\beta_*(t_0)}{t-t_0}+\beta_*(t_0)\in\Lambda^-_{\frac{\alpha(t)-\alpha(t_0)}{t-t_0}+\alpha(t_0)}.$$

Thus by (10.7), we have that

$$\frac{\beta_*(t)-\beta_*(t_0)}{t-t_0}+\beta_*(t_0)\in\Lambda^-[B_{\varepsilon_1}(\vec{q})] \tag{10.8}$$

if $t\in Dom\,\beta_*\cap B_{\delta_1}(t_0)$. Since β is differentiable at $\bar{t}_0$, there is a $\bar{\delta}_2\in\mathrm{Q}^+$ such that

$$\frac{\beta(\bar{t})-\beta(\bar{t}_0)}{\bar{t}-\bar{t}_0}+\beta(\bar{t}_0)\in B_{\varepsilon_2}(\vec{r}) \tag{10.9}$$

if $\bar{t}\in Dom\,\beta\cap B_{\bar{\delta}_2}(\bar{t}_0)$. Since $f:t\mapsto\bar{t}$ is continuous, there is a $\delta_2\in\mathrm{Q}^+$ such that (10.9) holds if $t\in Dom\,\beta_*\cap B_{\delta_2}(t_0)$. Since

$$\frac{\beta_*(t)-\beta_*(t_0)}{t-t_0}=\frac{\beta(\bar{t})-\beta(\bar{t}_0)}{\bar{t}-\bar{t}_0}\cdot\frac{\bar{t}-\bar{t}_0}{t-t_0},$$

we have that

$$\frac{\beta_*(t)-\beta_*(t_0)}{t-t_0}+\beta_*(t_0)\in Cone_{\varepsilon_2}(\beta_*(t_0);\vec{r}) \tag{10.10}$$

if $t\in Dom\,\beta_*\cap B_{\delta_2}(t_0)$. Let $\delta=\min(\delta_1,\delta_2)$. Therefore, by equations (10.8) and (10.10), we have that

$$\frac{\beta_*(t)-\beta_*(t_0)}{t-t_0}+\beta_*(t_0)\in\Lambda^-[B_{\varepsilon_1}(\vec{q})]\cap Cone_{\varepsilon_2}(\beta_*(t_0);\vec{r})$$

if $t\in Dom\,\beta_*\cap B_\delta(t_0)$. But the latter is a subset of $B_\varepsilon(\vec{p})$ by equation (10.6). Consequently,

$$\frac{\beta_*(t)-\beta_*(t_0)}{t-t_0}+\beta_*(t_0)\in B_\varepsilon(\vec{p})$$

if $t\in Dom\,\beta_*\cap B_\delta(t_0)$. Hence β_* is differentiable at t_0 and $\beta_*'(t_0)=\vec{p}-\beta_*(t_0)$, as it was required. ■

The following example shows that the assumption $Ran\,\alpha\cap Ran\,\beta=\emptyset$ is necessary in item 2 in Prop. 10.5.12.

Example 10.5.13. Let $\lambda\in\mathrm{Q}$ such that $1<\lambda$ or $\lambda<-1$. Let timelike curves α and β be defined as $\alpha(t)=\langle t,0,\dots,0\rangle$ and $\beta(t)=\langle\lambda\cdot t,t,0,\dots,0\rangle$ for all $t\in\mathrm{Q}$. Then the photon reparametrization of β according to α is:

$$\beta_*=\begin{cases}\langle\frac{\lambda}{\lambda+1}\cdot t,\frac{1}{\lambda+1}\cdot t,0,\dots,0\rangle & t\ge 0\\ \langle\frac{\lambda}{\lambda-1}\cdot t,\frac{1}{\lambda-1}\cdot t,0,\dots,0\rangle & t\le 0,\end{cases}$$

see Fig. 10.4. Therefore, β_* is continuous, but it is not differentiable at $t=0$.

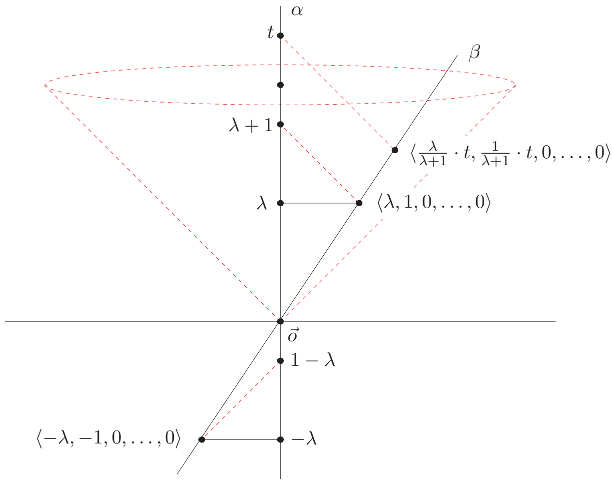

Figure 10.4: Illustration of Example 10.5.13

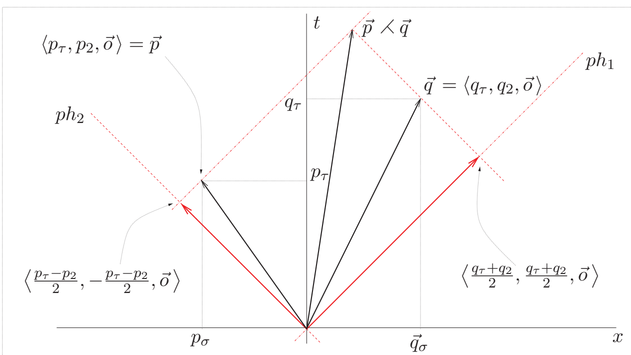

Figure 10.5: Illustration of the photon sum $\vec{p} \curlywedge \vec{q}$, and for the proof of Lem. 10.5.14

Let $\vec{p}, \vec{q} \in tx\text{-Plane}$. Then the **photon sum** of $\vec{p}$ and $\vec{q}$, in symbols $\vec{p} \curlywedge \vec{q}$, is the intersection of the two photon lines $\{\vec{p} + \langle A, A, 0, \dots, 0\rangle : A \in \mathrm{Q}\}$ and $\{\vec{q} + \langle B, -B, 0, \dots, 0\rangle : B \in \mathrm{Q}\}$.

Lemma 10.5.14. Let $\vec{p}, \vec{q} \in tx\text{-Plane}$, and let $a = \frac{q_\tau + q_2}{2}$ and $b = \frac{p_\tau - p_2}{2}$. Then $\vec{p} \curlywedge \vec{q} = \langle a+b, a-b, 0, \dots, 0\rangle$.

Proof. The proof is straightforward by the respective definitions, see Fig. 10.5. ■

Lemma 10.5.15. Assume CONT. Let β be a definable timelike curve. Then $\beta^{-1} : \mathit{Ran}\,\beta \to \mathit{Dom}\,\beta$ is definable, injective and continuous.

Proof. It is clear that β^{-1} is definable and injective.

Since by Lems. 10.5.1 and 10.5.2 β is injective, β^{-1} is a function from $\mathit{Ran}\,\beta$ to $\mathit{Dom}\,\beta$. To prove that it is also continuous, let $t_0 \in \mathit{Dom}\,\beta$. We have to show that for all $\varepsilon \in \mathrm{Q}^+$, there is a $\delta \in \mathrm{Q}^+$ such that if $t \in \mathit{Dom}\,\beta$ and $|\beta(t) - \beta(t_0)| < \delta$, then $|t - t_0| < \varepsilon$. By Lem. 10.5.11, $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ iff $\beta(t) \in \langle\!\langle \beta(t_0 - \varepsilon), \beta(t_0 + \varepsilon)\rangle\!\rangle$. Thus, since $\langle\!\langle \beta(t_0 - \varepsilon), \beta(t_0 + \varepsilon)\rangle\!\rangle$ is an open set, there is a good δ. ■

Lemma 10.5.16. Assume CONT. Let β be a definable timelike curve and β_* a definable continuous curve such that $\mathit{Ran}\,\beta_* \subseteq \mathit{Ran}\,\beta$, and let $f := \beta_* \circ \beta^{-1}$.

(1) Then f is a definable and continuous function.

(2) If β_* is injective, f is also injective. Moreover, $\mathit{Dom}\,f$ and $\mathit{Ran}\,f$ are connected and f^{-1} is also a definable, monotonous and continuous function.

(3) If β_* is differentiable such that $\beta_*'(t) \neq \vec{o}$ for all $t \in \mathit{Dom}\,\beta$, then f is injective and differentiable, and $f'(t) \neq 0$. Hence f^{-1} is also a differentiable function.

Proof. Item (1) is clear by Lem. 10.5.15.

Item (2) is clear by Item (1) and Lem. 10.2.4 since $\mathit{Dom}\,f = \mathit{Dom}\,\beta_*$ which is connected.

To prove Item (3), let $t_0 \in \mathit{Dom}\,f$. Since $\mathit{Ran}\,\beta_* \subseteq \mathit{Ran}\,\beta$, we have that there is a $\lambda \in \mathrm{Q}$ such that $\lambda \cdot \beta'(t_0) = \beta_*'(f(t_0))$. Since $\big(f(t) - f(t_0)\big)/(t - t_0)$ is the ratio of parallel vectors

$$\frac{\beta(t) - \beta(t_0)}{t - t_0} \quad \text{and} \quad \frac{\beta_*\big(f(t)\big) - \beta_*\big(f(t_0)\big)}{f(t) - f(t_0)},$$

we have that $\big(f(t) - f(t_0)\big)/(t - t_0)$ tends to $\beta'(t_0)/\beta_*'(f(t_0)) = 1/\lambda$ if t tends to t_0. Thus f is differentiable, and $f'(t_0) = 1/\lambda$. ■

Lemma 10.5.17. Assume CONT. Let α be a definable timelike curve. Let $t \in Dom\,\alpha$ and $x = \alpha_\tau(t)$. Let $f_\alpha := \alpha_\tau^{-1} \circ \alpha_\sigma$.

(1) Then f_α is a differentiable curve, and $f'_\alpha(x) = \alpha'_\sigma(t)/\alpha'_\tau(t)$.

(2) If α is twice differentiable at t, then so is f_α at x, and

$$f''_\alpha(x) = \frac{\alpha'_\tau(t)\alpha''_\sigma(t) - \alpha''_\tau(t)\alpha'_\sigma(t)}{\alpha'_\tau(t)^3}.$$

Proof. Let us first prove Item (1). We have that α_τ is injective by Lems. 10.5.1 and 10.5.2. Hence f_α is a function. $Dom\, f_\alpha$ is connected since $Dom\, f_\alpha = Ran\,\alpha_\tau$ and $Ran\,\alpha_\tau$ is connected by Lem. 10.2.4. Thus f_α is a curve. Since α_τ is an injective differentiable curve, α_τ^{-1} is also such and $(\alpha_\tau^{-1})'(x) = 1/\alpha'_\tau(t)$. Thus by Chain Rule, we have that $f'_\alpha(x) = \alpha'_\sigma(t)/\alpha'_\tau(t)$.

Now let us prove Item (2). If α is twice differentiable at t, then so are α_σ and α_τ. By Item (1), $f'_\alpha = \alpha_\tau^{-1} \circ \alpha'_\sigma/\alpha'_\tau$. Thus f_α is twice differentiable at x and a straightforward calculation based on the rules of differential calculus can show that $f''_\alpha(x)$ is what was stated. ■

Lemma 10.5.18. Assume CONT. Let α and β be definable timelike curves such that $Ran\,\alpha \cup Ran\,\beta$ is in a vertical plane. Let $t_1, t_2 \in Dom\,\alpha$ and $\bar{t}_1, \bar{t}_2 \in Dom\,\beta$ such that $\alpha(t_1)\,\sigma\,\beta(\bar{t}_1)$, $\alpha(t_2)\,\sigma\,\beta(\bar{t}_2)$ and $\big(\beta(\bar{t}_1) - \alpha(t_1)\big) \uparrow\uparrow \big(\alpha(t_2) - \beta(\bar{t}_2)\big)$. Then there is a $t \in (t_1, t_2)$ such that $\alpha(t) \in Ran\,\beta$. Hence $Ran\alpha \cap Ran\,\beta \neq \emptyset$.

Proof. Since $Ran\,\alpha \cup Ran\,\beta$ is in a vertical plane, we can assume, without losing generality, that $d = 2$. By CONT-Bolzano's Theorem, we can also assume that $\alpha(t_1)_\tau = \beta(\bar{t}_1)_\tau$ and $\alpha(t_2)_\tau = \beta(\bar{t}_2)_\tau$. Let $x_1 = \alpha(t_1)_\tau$ and $x_2 = \alpha(t_2)_\tau$. Let $f_\alpha := \alpha_\tau^{-1} \circ \alpha_\sigma$ and $f_\beta := \beta_\tau^{-1} \circ \beta_\sigma$. Then f_α and f_β are continuous curves, see Lem. 10.5.17. By the assumption $\big(\beta(\bar{t}_1) - \alpha(t_1)\big) \uparrow\uparrow \big(\alpha(t_2) - \beta(\bar{t}_2)\big)$, we have that $\big(f_\beta(x_1) - f_\alpha(x_1)\big)\big(f_\alpha(x_2) - f_\beta(x_2)\big) < 0$. Thus by CONT-Bolzano's Theorem, there is an $x \in (x_1, x_2)$ such that $f_\alpha(x) = f_\beta(x)$. Let $t := \alpha_\tau^{-1}(x)$. Then $\alpha(t) \in Ran\,\beta$. ■

Let α and β be timelike curves. We say that β_* is the **radar reparametrization of β according to α** if

$$\beta_* = \{\langle t, \vec{p}\rangle \in Dom\,\alpha \times Ran\,\beta \;:\; \exists r \in \mathrm{Q} \quad \vec{p} \in \Lambda^-_{\alpha(t+r)} \cap \Lambda^+_{\alpha(t-r)}\}.$$

We say that β is at constant radar distance r from α iff

$$Ran\,\beta \subseteq \bigcup_{t \pm r \in Dom\,\alpha} \Lambda^-_{\alpha(t+r)} \cap \Lambda^+_{\alpha(t-r)}.$$

Let us note that this r can be negative if α_τ is decreasing since by this definition $\alpha(t-r) \ll \alpha(t+r)$.

Proposition 10.5.19. Assume CONT. Let α and β be definable timelike curves. Let β_* be the radar reparametrization of β according to α.

(1) Then β_* is a definable, injective, and continuous curve.

(2) If $Ran\,\alpha \cup Ran\,\beta$ is in a vertical plane, and β is at constant radar distance r from α, then β_* is differentiable.

(3) Let us further assume that this vertical plane is the tx-Plane. Then

$$\beta_*'(t) = \alpha'(t-r) \curlywedge \alpha'(t+r) \quad \text{iff} \quad \big(\beta_*(t) - \alpha(t)\big) \uparrow\uparrow \ \vec{1}_x,$$
$$\beta_*'(t) = \alpha'(t+r) \curlywedge \alpha'(t-r) \quad \text{iff} \quad \big(\beta_*(t) - \alpha(t)\big) \uparrow\uparrow -\vec{1}_x.$$

Proof. It is clear that β_* is definable. Without losing generality, we can assume that α_τ is increasing, see Lems. 10.5.1 and 10.5.2.

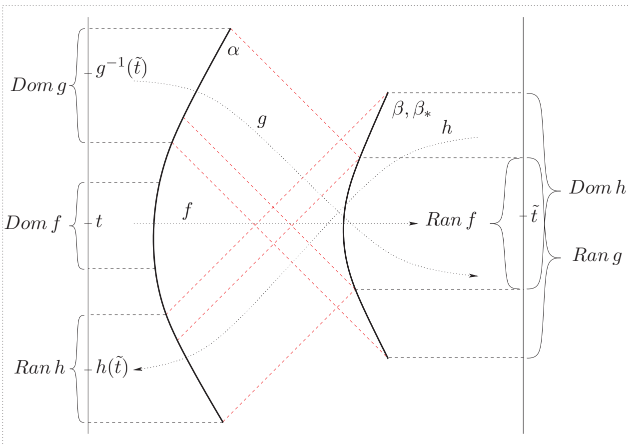

Figure 10.6: Illustration for the proof of Prop. 10.5.19

To show that β_* is a function, let $\langle t, \vec{p}\rangle, \langle t, \vec{q}\rangle \in \beta_*$. Then there are $r, s \in Q$ such that $\vec{p} \in \Lambda^-_{\alpha(t+r)} \cap \Lambda^+_{\alpha(t-r)}$ and $\vec{q} \in \Lambda^-_{\alpha(t+s)} \cap \Lambda^+_{\alpha(t-s)}$. We can assume that $0 \le r \le s$. Since both α and β are timelike curves, $\vec{p} = \vec{q}$ iff $r = s$. Therefore, if $\vec{p} \ne \vec{q}$, $\alpha(t+r) \ll \alpha(t+s)$ and $\alpha(t-s) \ll \alpha(t-r)$. Thus $\vec{q} \notin I^-_{\vec{p}}$ since $I^-_{\vec{p}} \subset I^-_{\alpha(t+r)}$ and $I^-_{\alpha(t+r)} \cap \Lambda^-_{\alpha(t+s)} = \emptyset$;

and $\vec{q} \notin I^+_{\vec{p}}$ since $I^+_{\vec{p}} \subset I^+_{\alpha(t-r)}$ and $I^+_{\alpha(t-r)} \cap \Lambda^+_{\alpha(t-s)} = \emptyset$. Thus $\vec{p} = \vec{q}$ since $\vec{q} \in I_{\vec{p}}$ by Lem. 10.5.11.

For all $t \in Dom\,\beta_*$, let $\tilde{t} \in Dom\,\beta$ such that $\beta(\tilde{t}\,) = \beta_*(t)$, and let $f : t \mapsto \tilde{t}$ be the (radar) reparametrization map, i.e., $f{:=}\beta_* \circ \beta^{-1}$. Then f is injective since if $\Lambda^-_{\alpha(t_1+r)} \cap \Lambda^+_{\alpha(t_1-r)} \cap \Lambda^-_{\alpha(t_2+s)} \cap \Lambda^+_{\alpha(t_2-s)} \neq \emptyset$, then $t_1 = t_2$ and $r = s$, see (1) in Lem. 10.5.10. Let g and h be the photon reparametrization maps of β according to α and of α according to β, respectively. Then g, g^{-1} and h, h^{-1} are monotonous and continuous bijections between connected sets, see Prop. 10.5.12 and Lem. 10.2.4. It is clear by the respective definitions, that

$$f^{-1}(\tilde{t}\,) = t = \frac{g^{-1}(\tilde{t}\,) + h(\tilde{t}\,)}{2}$$

for all $\tilde{t} \in Ran\,f$, see Fig. 10.6. Thus f^{-1} is continuous since both h and g^{-1} are such. It is clear that $Dom\,f^{-1} = Ran\,f = Dom\,h \cap Ran\,g$. Thus $Dom\,f^{-1}$ is connected since both $Dom\,h$ and $Ran\,g$ are such. Therefore, $Dom\,\beta_* = Dom\,f = Ran\,f^{-1}$ is also connected and f is definable and continuous, see Lem. 10.2.4. Hence $\beta_* = f \circ \beta$ is also continuous; and β_* is injective since both β and f are such. So Item (1) is proved.

Now let us prove Item (2). If $r = 0$, then β_* is the restriction of α to $Dom\,\beta_*$ which is connected, thus it is obviously differentiable. If $r \neq 0$, then $Ran\,\alpha \cap Ran\,\beta = \emptyset$. Thus by (2) in Prop. 10.5.12 and Lem. 10.5.16, we have that h and g^{-1} are differentiable. Thus f is also differentiable.

To prove Item (3), let $Ran\,\alpha \cup Ran\,\beta \subset tx$-Plane. By Item (2) of this proposition, β_* is differentiable. It is not difficult to see that

$$\begin{aligned} &\beta_*(t) = \alpha(t-r) \curlywedge \alpha(t+r) \text{ iff } \big(\beta_*(t) - \alpha(t)\big) \upuparrows \vec{1}_x \text{ and} \\ &\beta_*(t) = \alpha(t+r) \curlywedge \alpha(t-r) \text{ iff } \big(\beta_*(t) - \alpha(t)\big) \upuparrows -\vec{1}_x \end{aligned} \tag{10.11}$$

if $t \in Dom\,\beta_*$ since β is at constant radar distance r from α. By Lem. 10.5.18, we have that the direction of $\beta_*(t) - \alpha(t)$ cannot change. Thus it is always the same equation in (10.11) that holds for β_*. Hence Item (3) follows from Lem. 10.5.14 by an easy calculation. ■

If $\alpha : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^d$ and $\vec{p} \in \mathrm{Q}^d$, we abbreviate $\alpha(t) \upuparrows \vec{p}$ for all $t \in Dom\,\alpha$ to $\alpha \upuparrows \vec{p}$. We use analogously the notation $\alpha \upuparrows \beta$ if $\alpha, \beta : \mathrm{Q} \xrightarrow{\circ} \mathrm{Q}^d$. Let $\bar{\alpha}{:=}\langle \alpha_2, \alpha_1, \alpha_3, \ldots, \alpha_d \rangle$ for all $\alpha : \mathrm{Q} \to \mathrm{Q}^d$, i.e., the first two coordinates are interchanged.

Lemma 10.5.20. Assume CONT. Let α be a definable timelike curve.

(1) Then $\alpha' \upuparrows \vec{1}_t$ or $\alpha' \upuparrows -\vec{1}_t$.

(2) If $Ran\,\alpha \subset tx$-Plane, then $\bar{\alpha}' \uparrow\uparrow \vec{1}_x$ iff $\alpha' \uparrow\uparrow \vec{1}_t$ and $\bar{\alpha}' \uparrow\uparrow -\vec{1}_x$ iff $\alpha' \uparrow\uparrow -\vec{1}_t$.

(3) If α is twice differentiable, $Ran\,\alpha \subset tx$-Plane and $\vec{o} \notin Ran\,\alpha''$, then $\alpha'' \uparrow\uparrow \vec{1}_x$ ($\alpha'' \uparrow\uparrow -\vec{1}_x$) iff α_2' is increasing (decreasing).

(4) If α is twice differentiable, $Ran\,\alpha$ is in a vertical plane and $\vec{o} \notin Ran\,\alpha''$, then $\alpha''(t_1) \uparrow\uparrow \alpha''(t_2)$ for all $t_1, t_2 \in Dom\,\alpha$.

(5) If α is twice differentiable and $Ran\,\alpha \subset tx$-Plane, then for all $t \in Dom\,\alpha$, there is a $\lambda_t \in Q$ such that $\lambda_t \alpha'(t) = \alpha''(t)$. Furthermore, if $\vec{o} \notin Ran\,\alpha''$, the sign of λ_t is the same for all $t \in Dom\,\alpha$ and

$$\begin{aligned} \lambda_t > 0 \quad &\text{iff} \quad \bar{\alpha}' \uparrow\uparrow \alpha'' \\ \lambda_t < 0 \quad &\text{iff} \quad -\bar{\alpha}' \uparrow\uparrow \alpha'' \end{aligned} \tag{10.12}$$

Proof. Item (1) is easy to prove since by Lem. 10.5.1, $0 \notin Ran\,\alpha_\tau$. Thus by CONT-Darboux's Theorem, we have that $\alpha_\tau' > 0$ or $\alpha_\tau' < 0$.

To prove Item (2), let us first note that $\alpha = \langle \alpha_\tau, \alpha_2, 0, \ldots, 0\rangle$ since $Ran\,\alpha \subset tx$-Plane. Therefore, $\bar{\alpha}' = \langle \alpha_2', \alpha_\tau', 0, \ldots, 0\rangle$. Hence $\bar{\alpha}' \uparrow\uparrow \vec{1}_x$ iff $\alpha_\tau' > 0$, and $\bar{\alpha}' \uparrow\uparrow -\vec{1}_x$ iff $\alpha_\tau' < 0$.

To prove Item (3), let $t \in Dom\,\alpha$. It is clear that $\alpha''(t)$ is spacelike or $\vec{o}$ since $\alpha''(t) \perp_\mu \alpha'(t)$ by Prop. 10.5.7. Thus $\vec{o} \notin Ran\,\alpha''$ iff $\vec{o} \notin Ran\,\alpha_\sigma''$. We have that $\alpha_\sigma = \langle \alpha_2, 0, \ldots, 0\rangle \in Q^{d-1}$ since $Ran\,\alpha \subset tx$-Plane. Thus $\vec{o} \notin Ran\,\alpha_\sigma''$ iff $0 \notin Ran\,\alpha_2''$. Hence $0 \notin Ran\,\alpha_2''$. Therefore, by CONT-Darboux's Theorem, we have that $\alpha_2'' > 0$ or $\alpha_2'' < 0$. Consequently, $\alpha'' \uparrow\uparrow \vec{1}_x$ iff $\alpha_2'' > 0$, and $\alpha'' \uparrow\uparrow -\vec{1}_x$ iff $\alpha_2'' < 0$. Thus, since $0 \notin Ran\,\alpha_2''$, $\alpha'' \uparrow\uparrow \vec{1}_x$ iff α_2' is increasing, and $\alpha'' \uparrow\uparrow -\vec{1}_x$ iff α_2' is decreasing.

Let us now prove Item (4). Without losing generality, we can assume that the vertical plane is the tx-Plane. By Lem. 10.5.8, we have that α_2' is increasing or decreasing since $\alpha'' \circ \mu < 0$ iff $\vec{o} \notin Ran\,\alpha''$. Thus Item (4) follows by Item (3).

Let us finally prove Item (5). Since both $\bar{\alpha}'(t)$ and $\alpha''(t)$ are Minkowski orthogonal to $\alpha'(t)$ and are in the tx-Plane, there is a $\lambda_t \in Q$ such that $\bar{\alpha}'(t) = \lambda_t \alpha''(t)$. By Items (2) and (3), equation (10.12) is clear. ■

Let α and β be timelike curves. We say that β_* is the **Minkowski reparametrization of β according to α** if

$$\beta_* = \{\langle t, \vec{p}\rangle \in Dom\,\alpha \times Ran\,\beta \,:\, \big(\vec{p} - \alpha(t)\big) \perp_\mu \alpha'(t)\}.$$

We say that β **is at constant Minkowski distance** $r \in Q^+$ **from** α iff for all $\vec{p} \in Ran\,\beta$, there is a $t \in Dom\,\alpha$ such that $-\mu\big(\vec{p}, \alpha(t)\big) = r$.

Proposition 10.5.21. Assume CONT. Let α and β be definable timelike curves such that α is well-parametrized, and let β_* be the Minkowski reparametrization of β according to α such that.

(i) α is twice differentiable, and $\vec{o} \notin Ran\, \alpha''$.

(ii) $Ran\, \alpha \cup Ran\, \beta$ is in a vertical plane.

(iii) If $\langle t, \vec{p} \rangle \in \beta_*$ and $\big(\alpha(t) - \vec{p}\big) \upuparrows \alpha''(t)$, then $-\mu\big(\vec{p}, \alpha(t)\big) < -1/\mu(\alpha''(\tau))$ for all $\tau \in Dom\, \alpha$.

(iv) β is at constant Minkowski distance $r \in \mathrm{Q}^+$ from α.

Then β_* is a definable timelike curve. Furthermore,

$$\begin{aligned} \beta_*'(t) = \alpha'(t) + r \cdot \bar{\alpha}''(t) \text{ iff } \alpha''(t) \upuparrows \big(\beta_*(t) - \alpha(t)\big), \\ \beta_*'(t) = \alpha'(t) - r \cdot \bar{\alpha}''(t) \text{ iff } \alpha''(t) \upuparrows \big(\alpha(t) - \beta_*(t)\big) \end{aligned} \tag{10.13}$$

if $Ran\, \alpha \cup Ran\, \beta \subseteq tx$-Plane, $\alpha' \upuparrows \vec{1}_t$ and $\alpha'' \upuparrows \vec{1}_x$.

Proof. It is clear that β_* is definable.

To see that β_* is a function, let $\langle t, \vec{q} \rangle, \langle t, \vec{p} \rangle \in \beta_*$. Then $(\vec{p} - \vec{q}) \bot_\mu \alpha'(t)$. If $\vec{p} \neq \vec{q}$, they are timelike-separated by Lem. 10.5.11 since $\vec{p}, \vec{q} \in Ran\, \beta$. Thus, since two timelike vectors cannot be Minkowski orthogonal, we have that $\vec{p} = \vec{q}$. Hence β_* is a function.

Without losing generality, we can assume that the vertical plane that contains $Ran\, \alpha \cup Ran\, \beta$ is the tx-Plane, $\alpha' \upuparrows \vec{1}_t$ and $\alpha'' \upuparrows \vec{1}_x$, see Lems. 10.5.8 and 10.5.20.

Since β is at constant Minkowski distance r from α,

$$\begin{aligned} \beta_*(t) = \alpha(t) + r \cdot \bar{\alpha}'(t) \text{ iff } \bar{\alpha}'(t) \upuparrows \big(\beta_*(t) - \alpha(t)\big), \\ \beta_*(t) = \alpha(t) - r \cdot \bar{\alpha}'(t) \text{ iff } \bar{\alpha}'(t) \upuparrows \big(\alpha(t) - \beta_*(t)\big) \end{aligned} \tag{10.14}$$

if $t \in Dom\, \beta_*$.

Since β is at constant Minkowski distance $r \in \mathrm{Q}^+$ from α, we have that $Ran\, \alpha \cap Ran\, \beta = \emptyset$. Hence by Lem. 10.5.18, we have that the direction of $\beta_*(t) - \alpha(t)$ cannot change. Thus it is always the same equation in (10.14) that holds for β_*.

Since α is twice differentiable, so is $\bar{\alpha}$. Thus both $\alpha + r \cdot \bar{\alpha}'$ and $\alpha - r \cdot \bar{\alpha}'$ are definable differentiable curves.

Now we will show that $\alpha + r \cdot \bar{\alpha}'$ is a timelike curve and if $\bar{\alpha}'(t) \upuparrows \big(\alpha(t) - \beta_*(t)\big)$ for some $t \in Dom\, \beta_*$, then $\alpha - r \cdot \bar{\alpha}'$ is also a timelike curve. It is clear that $(\alpha \pm r \cdot \bar{\alpha}')' = \alpha' \pm r \cdot \bar{\alpha}''$. Let $t \in Dom\, \alpha$. By (5) in Lem. 10.5.20, we have that $\mu\big(\alpha'(t) + r \cdot \bar{\alpha}''(t)\big) =$

$\mu\big(\alpha'(t)\big) + r\mu\big(\bar{\alpha}''(t)\big)$ and $\mu\big(\alpha'(t) - r\cdot\bar{\alpha}''(t)\big) = \mu\big(\alpha'(t)\big) - r\mu\big(\bar{\alpha}''(t)\big)$. By Thm. 6.1.11, we have that $\mu\big(\alpha'(t)\big) = 1$. Thus $\mu\big((\alpha + r\cdot\bar{\alpha}')'(t)\big) > 0$. Hence $\alpha + r\cdot\bar{\alpha}'$ is a timelike curve. Since $\alpha' \uparrow\uparrow \vec{1}_t$ and $\alpha'' \uparrow\uparrow \vec{1}_x$, we have that $\alpha''(t) \uparrow\uparrow \bar{\alpha}'(t)$ by Lem. 10.5.20. Thus by assumption (iii) and the fact that β is at constant Minkowski distance r from α, we have that $r < -1/\mu(\alpha''(\tau))$ for all $\tau \in \mathit{Dom}\,\alpha$ if $\bar{\alpha}'(t) \uparrow\uparrow \big(\alpha(t) - \beta_*(t)\big)$ for some $t \in \mathit{Dom}\,\alpha$. Since $\mathit{Ran}\,\alpha \subseteq tx$-Plane, we have that $\mu(\alpha''(t)) = -\mu(\bar{\alpha}''(t))$. Thus $\mu(\bar{\alpha}''(t)) < 1/r$. Consequently, $\mu\big(\alpha'(t) - r\cdot\bar{\alpha}''(t)\big) > 0$. Hence $\alpha - r\cdot\bar{\alpha}'$ is also a timelike curve.

Here we only prove that $\mathit{Dom}\,\beta_*$ is connected when $\bar{\alpha}'(t) \uparrow\uparrow \big(\alpha(t) - \beta_*(t)\big)$ for some $t \in \mathit{Dom}\,\beta_*$ because the proof in the other case is almost the same. Let $t_1, t_2 \in \mathit{Dom}\,\beta_*$, and let $t \in (t_1, t_2)$. Then $t_1, t_2 \in \mathit{Dom}\,\alpha$, and thus $t \in \mathit{Dom}\,\alpha$ since $\mathit{Dom}\,\alpha$ is connected. Since $\alpha - r\cdot\bar{\alpha}'$ is a timelike curve and $\alpha' - r\cdot\bar{\alpha}'' \uparrow\uparrow \vec{1}_t$, we have that

$$\beta_*(t_1) = \alpha(t_1) - r\cdot\bar{\alpha}'(t_1) \ll \alpha(t) - r\cdot\bar{\alpha}'(t) \ll \alpha(t_2) - r\cdot\bar{\alpha}'(t_2) = \beta_*(t_2).$$

Thus by CONT-Bolzano's Theorem, there is a $\bar{t} \in \mathit{Dom}\,\beta$ such that $\big(\beta(\bar{t}) - \alpha(t)\big) \perp_\mu \alpha'(t)$. Since β is at constant Minkowski distance r from α, we have that $\beta(\bar{t}) = \alpha(t) - r\cdot\bar{\alpha}'(t)$. Hence $t \in \mathit{Dom}\,\beta_*$, as it was required.

Since β_* agrees with one of the two timelike curves $\alpha + r\cdot\bar{\alpha}'$ and $\alpha - r\cdot\bar{\alpha}'$ on the connected set $\mathit{Dom}\,\beta_*$, we have that β_* is also a timelike curve. Since $\alpha'' \uparrow\uparrow \vec{1}_x$ and $\alpha \uparrow\uparrow \vec{1}_t$ we have that $\alpha'' \uparrow\uparrow \bar{\alpha}'$. Therefore, by derivation of the equations of (10.14), we have that the derivative of β_* is what was stated in (10.13). ■

Chapter 11

Why do we insist on using FOL for foundation?

In this chapter we are going to give a detailed explanation why FOL is the best logic to be used in foundational works, such as this one.

11.1 On the purposes of foundation

The main purpose of foundation is to get a deeper understanding of fundamental concepts of a theory by stating axioms about them and studying the relationship between the axioms and their consequences. There are three main kinds of question to ask in the course of foundation:

- What are the consequences of the given axioms?
- What axioms are responsible for a certain theorem?
- How do statements independent from the theory relate to one another?

The first one is a usual question of ordinary axiomatic mathematics. The other two are new kinds of question in foundational thinking and reverse mathematics. The third one is meaningful only in the case of incomplete theories; but there are a lot of incomplete theories, e.g., any consistent axiom system containing arithmetic is incomplete by Gödel's first incompleteness theorem. Moreover, it is usually reasonable to weaken a complete theory to make it possible to ask this third type of question. For example, to facilitate studying the role of the axiom of parallels, Euclid's complete axiom system of geometry was weakened to an incomplete one. These three kinds of question are studied in the hope that they will lead to a more refined and deeper understanding of the fundamental concepts and assumptions of the given theory. For more details on the role and importance of foundational thinking, see, e.g., [2, Introduction] and [24].

11.2 The success story of foundation in mathematics

Experience shows that foundational thinking does lead to deeper understanding. For example, in geometry it clarified the status of the axiom of parallels and led to the discovery of hyperbolic geometry. It has been shown by foundation that this axiom is independent from the other basic assumptions of Euclidean geometry.

Foundation also eliminated Russel's antinomy from set theory and thus from mathematics. It helped to gain a deeper understanding of many statements of set theory by providing many other statements which are weaker, equivalent or stronger according to some axiom system of set theory, such as the Zermelo–Fraenkel set theory (ZF). For example, the axiom of choice is equivalent to Zermelo's well-ordering theorem, Zorn's lemma and the existence of basis in every vector space; the Baire Category Theorem, Stone's representation Theorem and the Banach-Tarski paradox are some of its many consequences; and the statement "every subset of real numbers $\mathbb{R}$ is Lebesgue measurable" is stronger than the negation of the axiom of choice. These results tell us more about what it means to postulate the axiom of choice or its negation. And there are lots of other statements of set theory which are investigated in this way.

Second-order arithmetic[1] is also a good example of the successfulness of foundational thinking. The main goal of second-order arithmetic is to investigate how strong a set existence axiom is needed to prove certain theorems of mathematics, such as the Bolzano-Weierstrass Theorem or König's Lemma. For more details, see, e.g., Simpson [65].

The examples above show that foundational thinking has been fruitful in many fields of mathematics. Hence it seems to be a good idea to apply it in a wider range; for example, in certain fields of physics, such as relativity theory as suggested by Harvey Friedman [25], for instance.

There are lots of interesting assumptions, statements and questions of relativity theory (both special and general), such as the possibility/impossibility of faster than light motion, the twin paradox, gravitational time dilation or the existence of closed

[1]Let us note that the name second-order arithmetic is misleading since it is a two-sorted FOL theory, i.e., it is a FOL theory which studies two kinds of individual: sets and numbers.

timelike curves (i.e., the possibility of time travel), to mention only a few. There is much hope that foundation will help to clarify and understand the statuses of these statements and questions as well as the concepts related to them. This is one of the many reasons why the relatively large group led by Andréka Hajnal and István Németi have devoted so much effort and enthusiasm to providing foundation for spacetime theories. See [68, pp.144 footnote 137] for a physicist's reflection on some of that.

11.3 Choosing a logic for foundation

To provide logical foundation of any field of science, we have to choose a formal logic. In the following sections we show that our choosing FOL is the best possible choice in several senses. To do so, we compare it to other logics from different aspects.

Since we would like to treat the physical world as a possible model of our theory in certain physical interpretation, we need a logic with semantical consequence relation. Even after this restriction, there are a great many different logics which we could use for axiomatic foundation. The two most popular candidates are FOL and (standard or full) second-order logic. The main difference between them is that in second-order logic it is possible to quantify over n-ary relations while in FOL we can quantify just over individuals.

Because of its great expressive power, it would be convenient to use second-order logic. However, as it will be showed in the forthcoming paragraphs, its great expressive power is rather a disadvantage.

A main problem with second-order logic is that it contains tacit assumptions about sets. That is so because unary relations and subsets are essentially the same things. Hence if we use second-order logic, we tacitly build set theory into our theory, and that generates several problems. On the other hand, FOL does not contain any assumptions about sets.[2]

Väänänen in [77] says: "First-order set theory and second-order logic are not radically different: the latter is a major fragment of the former." In [49, §Set theory in Sheep's Clothing], Quine also argues that second-order logic is none other than a set theory in disguise. So if we do not want to be burdened (or loaded) by any hidden assumptions about sets, we cannot use second-order logic for foundation. The same argument applies to standard higher-order logic and type theory.

[2]Of course, to prove nontrivial theorems about FOL, we need some basic set theory as a metatheory. However, that does not contradict the fact that FOL is free of any hidden assumptions about sets.

11.4 Completeness

Completeness is also a fundamental property of the logic we choose for foundation since without it we cannot have control over the true statements in the models of our axioms. FOL is complete by Gödel's completeness theorem, but second-order logic is not, see, e.g., [20, §IX.1.]. That means that the semantical consequence relation of second-order logic is vague, which by itself is enough to exclude second-order logic from the list of possible logics for foundation.

Let us, however, dwell on the vagueness of the semantical consequence relation of second-order logic. Not just there is no sound and complete system of derivation rules for second-order logic, but the set of Gödel numbers of second-order logic validities is not definable by any second-order logic formula (in the standard model of arithmetic), see [22, Thm.41C]. Hence it is not just not recursively enumerable, but it is not at any level of the arithmetical hierarchy of FOL definable sets of numbers.

The complexity of second-order validities in a language containing one binary relation symbol is also very high since it cannot be defined by any higher-order logic formula in the language of Peano arithmetic, and a formula of complexity Π_2 is needed to define it in the language of set theory, see Väänänen [77]. These results show that the validity relation of second-order logic is too blurred and vague for our purposes.

In contrast, the set of FOL validities is recursively enumerable, see [20, §X. Prop.1.6], and the set of consequences of any recursive enumerable FOL theory is recursively enumerable by [22, Thm.35I] and [43, Thm.15.1].

11.5 Absoluteness

Naturally, we would like to choose logic L such that its semantical consequence relation ($\models_{\mathrm{L}}$) is as independent from set theory as possible. This property of a logic is called *absoluteness*. Absoluteness of a logic roughly means that the truth or falsity of $\mathfrak{M} \models_{\mathrm{L}} \varphi$ does not depend on the entire set theoretical universe, only on the sets required to exist by some fixed list of axioms (e.g., ZF or a fragment of ZF) and on the transitive closures of the sets $\mathfrak{M}$ and φ under discussion. For exact definition, see, e.g., [12], [76].

Let us now see some examples that show what can happen if we use a non-absolute logic, such as second-order logic. We can formulate the continuum hypothesis (CH) in second-order logic, see, e.g., [20], [44], [59]. Let φ_{CH} be a second-order formula expressing CH and let $\models_2$ be the semantical consequence relation of second-order logic.

Example 11.5.1. The answer to the simple question whether $\mathbb{R} \models_2 \varphi_{CH}$ or $\mathbb{R} \not\models_2 \varphi_{CH}$

holds, depends on the model of set theory we are working in. So it is *unknowable*. Moreover, this dependence is so strong that the answer may alter by moving from the set theoretical universe V we work in to a transitive submodel of V.

Let us now see a more general example.

Example 11.5.2. Let φ_∞ be a formula of second-order logic expressing that its model contains infinitely many elements; it is not difficult to write up such a formula, see [20, §IX. 1.3]. Let ψ be the following formula of second-order logic: $\varphi_\infty \rightarrow \varphi_{CH}$. Then for any infinite structure $\mathfrak{M}$, the question whether $\mathfrak{M} \models_2 \psi$ or $\mathfrak{M} \not\models_2 \psi$ holds is also *unknowable*.

On the basis of the many independent statements of set theory, we can generate a great many *unknowable* sentences of second-order logic.

Let us note here that a statement being *unknowable* and being independent from a theory does not mean the same. *Unknowability* of a statement means that its validity depends on what class model of the metatheory we are working in. So *unknowability* is highly undesirable, while independence is not problematic at all. Moreover, in foundations, it is useful to study incomplete theories, see Section 11.1. Hence independence can be useful.

The examples above show that absoluteness is a desired property of a logic used for foundation. It is important to note that the above situations cannot occur in FOL because it is absolute in a strong sense, i.e., it is absolute in relation to the Kripke–Platek set theory[3] (KP), which is considerably weaker than ZF, see, e.g., [76, Example 2.1.3].

11.6 Categoricity

First of all let us note that, in logical foundation, the fewer axioms a theory contains the better it is; so categoricity (and even completeness) of an axiom system is not a desired property. Moreover, searching for strong (e.g., categorical or complete) axiom systems is a fallacy in foundation of a physical theory since in physics we do not really know whether an axiom is true or not, we just presume so.

Nevertheless, it is often considered as a great advantage of second-order logic that it is possible to axiomatize something categorically within it, i.e., it can capture structures

[3]KP consists only the axioms of extensionality, foundation, pair, union, and the separation and collection schemas restricted to formulas containing only bounded quantifiers, see, e.g., [11]

up to isomorphism. The following examples will show that categoricity is rather a disadvantage as it can obscure things we are interested in.

Example 11.6.1. Let us consider the "nice" (finite and categorical) second-order axiomatization $\mathbf{RCF}_2$ of real numbers. Since $\mathbf{RCF}_2$ is categorical, there is only one model of it ($\mathbb{R}$). Now we can think that we have captured what we wanted and nothing else. However, if we take a closer look, we will see that we can ask many *unanswerable* questions about $\mathbb{R}$. For example, since CH is independent from set theory, we do not know whether there is or there is not an uncountable subset H of $\mathbb{R}$ such that there is no bijection between H and $\mathbb{R}$. That is inconvenient because there is only one model of $\mathbf{RCF}_2$. So either $\mathbb{R} \models_2 \varphi_{CH}$ or $\mathbb{R} \models_2 \neg\varphi_{CH}$ must be valid but we cannot know which one. At first glance it is not clear at all how a concrete yes-or-no question can exist without a definite answer? The problem results from the fact that we have captured one $\mathbb{R}$ in each set theory model. However, since there are several models of set theory, we have several $\mathbb{R}$'s, too.

Let us now see another example which shows that we can lose important information about the model we intend to capture if we use second-order logic.

Example 11.6.2. In second-order logic, thanks to its expressive power, we can formulate an axiom that states that if CH is true, there is an isomorphism between its model and $\mathbb{N}$, and if CH is false, its model is isomorphic to the ordered ring of integers ($\mathbb{Z}$):

$$(\varphi_{CH} \to \mathfrak{M} \cong \mathbb{N}) \wedge (\neg\varphi_{CH} \to \mathfrak{M} \cong \mathbb{Z}).$$

So the axiom system containing the above formula only is categorical, yet it is *unknowable* whether there is a least element of the structure which is captured up to isomorphism.

By the trick of the above example we can provide many categorical axiom systems where some very basic properties (e.g., finiteness/infiniteness) about the unique model are *unknowable*, see Andréka–Madarász–Németi [2, §Why FOL?].

These examples show that categoricity is not at all as good a thing as it seems to be, and sometimes a non-categorical FOL axiomatization can provide more information about its several models than a categorical second-order logic axiomatization can about its unique model. Second-order logic only makes us believe that we have one particular object in hand, but in fact, we have many.

11.7 Henkin semantics of second-order logic

There are also other semantics of second-order logic in addition to standard (or full) semantics where all the relations are present. Henkin further generalized standard semantics and introduced such ones in which just some of the relations (but at least all the definable ones) are present such that these relations satisfy certain requirements, see Henkin [30]. Väänänen in [77] argues that if second-order logic is used for foundation, we cannot meaningfully ask which semantics is being used. So the standard version of second-order logic cannot be used for foundation, only the generalized Henkin second-order logic is suitable for this purpose. The Henkin second-order logic is actually a theory of many-sorted FOL, so one can only pretend using standard second-order logic for foundation. Furthermore, when mathematicians are apparently using higher-order logic, they are actually using Henkin higher-order one.

In the present approach Henkin higher-order logic is considered absolutely acceptable. It has a completeness theorem and is absolute. So we do not hesitate to use it when needed. Hence higher-order logic tools are acceptable and available for us if they are treated with appropriate caution.

11.8 Our choice of logic in the light of Lindström's Theorem

So far we have mainly argued for choosing a complete and absolute logic for foundation, such as FOL. Thus second-order logic and hence any higher-order logic is too strong for our aims. However, there are many model-theoretic logics which are stronger than FOL but weaker than second-order logic, e.g., weak second-order logic, infinitary logics or logics with generalized quantifiers, etc. Can any of these logics be good for our purpose? So our question can be restated as follows: Is there any complete and absolute model-theoretic logic stronger than FOL?[4] To answer this question, let us recall two properties of abstract model-theoretic logics. A logic is *compact* iff every set of sentences Σ of the logic has a model if all finite subsets of Σ have models. A logic has the *Löwenheim–Skolem property* iff every sentences φ of the logic has a countable model if φ has a model. There is a well-known theorem of Lindström that characterizes FOL as the strongest model-theoretic logic with these properties, see, e.g., [23].

Theorem 11.8.1 (Lindström)**.** FOL is the strongest compact model-theoretic logic

[4] For the exact definition of comparing the strength of two logics, see [19].

with Löwenheim–Skolem property.

However, Lindström's theorem does not answer our question by itself since we have required two different properties (absoluteness and completeness) of a logic to judge it *suitable for foundation*. A theorem of Väänänen's comes to our aid, see Cor.2.2.3 in [76].

Theorem 11.8.2 (Väänänen)**.** Every absolute model-theoretic logic has the Löwenheim–Skolem property.

Since completeness implies compactness, by putting Lindström's and Väänänen's theorems together, we get the following:

Corollary 11.8.3. FOL is the strongest abstract model-theoretic logic which is complete and absolute.

This corollary implies that FOL is the strongest abstract model-theoretic logic suitable for foundation.

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