PhD Thesis

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2008.

Quantum Groupoid Symmetry, with an Application to Low Dimensional Algebraic Quantum Field Theory

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The Thesis was prepared under auspices of the
Research Institute for Particle- and Nuclear Physics
of the Hungarian Academy of Sciences.

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CHAPTER 1

Overview of this Thesis

In the Introduction, we survey the existing mathematical literature on, and leading upto, quantum groupoids and the algebraic quantum field theory literature on the symmetry reconstruction problem with an emphasis on investigations of quantum symmetry in low dimensions.

Chapter two, Quantum groupoids, provides the basic mathematical background on bialgebroids, Hopf algebroids, distrubutive double algebras (Frobenius Hopf algebroids) and their categories of modules and comodules. All three structures are generalizations of Hopf algebras in that the base field is replaced by a noncommutative ring; they are discussed in their order of appearance in the literature, which is roughly the order of increasingly complex structure.

Chapter three, Galois theory, is an account of the quantum groupoid Galois theory developed in [**3**]. By way of introduction, the classical Galois theory of field extensions and the Hopf Galois theory of ring extensions is reviewed with applications in other areas of mathematics, notably noncommutative geometry. A section is devoted to the important result that bialgebroid Galois extensions ([**47**], generalized to quantum groupoids in [**3**]) may be characterized as depth 2 extensions. This will be exploited in the final chapter to prove that field algebra extensions in AQFT are in general Galois.

In chapter five, Scalar extension, we define and examine the generalization of the scalar extension construction to quantum groupoids and apply it to quantum groupoid Galois theory. We first explain the motivation, coming from Hopf Galois theory and a precursor result of Brzeziński and Militaru. Mathematical preliminaries to the construction are Yetter–Drinfel'd category and braided commutative algebras, which we introduce first for Hopf algebras and then generalize to quantum groupoids. We prove a few fundamental properties of the quantum groupoid scalar extension and also look at it from a monadic point of view. We close the chapter with a result from [**3**] connecting the scalar extension to Galois theory.

Chapter six on Bicoalgebroids is a summary of results in [**4**]. Bicoalgebroids are the categorical dual structure to bialgebroids, and haven't yet received much attention save for their defintion in [**21**]. We construct the monoidal category of comodules and prove a comonadic version of Schauenburg's theorem. The main result is a dualization of the scalar extension to bicoalgebroids, to which end we also define the Yetter–Drinfel'd category over a bicoalgebroid. Anticipating the application of bicoalgebroids to coextensions of coalgebras, we discuss various definitions for the cocentralizer of a coalgebra coextension and the cocenter of a bicomodule.

Chapter seven contains An application to Algebraic Quantum Field Theory. We review the basic postulates of AQFT, the Doplicher–Haag–Roberts theory of superselection sectors, and sketch the Doplicher–Roberts reconstruction theorem. So far, an extension of the full DR theorem to low dimensions is lacking. Working in a pure algebraic setting, we define a field algebra extension of the observable algebra from a fiber functor on the DHR category and prove that it is depth 2, hence Galois. In the case of a semisimple DHR category, the field algebra is shown to be a generalization of the reduced field bundle construction of Fredenhagen, Rehren and Schroer.

We collect our results in chapter eight, the Summary.

CHAPTER 2

Introduction

In the 1990's, results coming from diverse areas of mathematics and mathematical physics pointed in the direction of a new notion of symmetry which subsumes both quantum groups[1], which have been known for some time, and groupoid algebras which have been known for even longer. For now, we refer to this structure, or in fact family of related structures loosely as *quantum groupoids*, but we shall make our usage of this term precise later on. Slightly different definitions appeared in work related to low dimensional quantum field theory in [**8, 9**], noncommutative [**55**] and Poisson [**52**] geometry and operator algebras [**89**]. The most important feature that these definitions share is the existence of two canonical anti–isomorphic subalgebras, which reduce to the base field in the special case of a Hopf algebra. In a vague sense, it can be said that quantum groupoids generalize quantum groups through the replacement of the base field with a noncommutative algebra.

Generalizing Hopf algebras was proposed already in 1974 by Sweedler [**77**], whose construction was generalized by Takeuchi [**86**]. Crucial to their construction was the introduction of a new tensor product $\underset{R}{\times}$ over a noncommutative ring R, lending the name to $\underset{R}{\times}$–bialgebras, which were analyzed further e.g. in [**72**] and [**73**].

A special case of $\times_R$–bialgebras was introduced by Ravenel [**70**] in algebraic geometry, under the name (commutative) Hopf algebroid. Originally, in the context of quantum groupoids of geometric origin, the term 'Hopf algebroid' was reserved for cogroupoid objects in the category of commutative algebras. In present usage, Hopf algebroid refers to a more general structure, over a noncommutative base algebra. Two definitions of bialgebroid appeared in Poisson geometry, Lu's bialgebroid [**52**] and Xu's [**93**] bialgebroid *with an anchor*. This state of affairs was simplified somewhat in [**21**], where it was proven that the definitions of Takeuchi's $\underset{R}{\times}$–bialgebras, Lu's bialgebroid and Xu's bialgebroid with an anchor are in fact equivalent. The latter structures are in fact generalizations of bialgebras and not Hopf algebras, since there is no notion of antipode.

Böhm, Nill and Szlachányi introduced weak bialgebras, weak Hopf algebras and C^*–weak Hopf algebras in [**8, 9, 10, 63**] with applications to inclusions of unital C^*–algebras [**84, 82**]. Weak bialgebras were shown to be a special case of Lu's bialgebroids in [**36**]. The Hopf bimodules of Enock and Vallin introduced in [**35**] are variants of C^*–weak Hopf algebras in the context of von Neumann algebras.

[1]We use 'quantum group' in a general sense, meaning noncommutative, non–cocommutative Hopf algebra which is not necessarily a deformed enveloping algebra of a Lie group

Being more specific, a *weak bialgebra* (WBA for short) $\langle B, \gamma, \pi, m, i\rangle$ consists of an algebra structure $\langle B, m, i\rangle$ and a coalgebra structure $\langle B, \gamma, \pi\rangle$ on the underlying k–module B, such that multiplication and comultiplication are compatible in the bialgebra sense but comultiplicativity of the unit and multiplicativity of the counit are replaced by weaker axioms. For future reference, we reproduce them here:

$$(\gamma(1) \otimes 1)(1 \otimes \gamma(1)) = 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = (1 \otimes \gamma(1))(\gamma(1) \otimes 1) \tag{2.0.1}$$

$$\pi(b\,1_{(1)})\pi(1_{(2)}\,b') = \pi(bb') = \pi(b\,1_{(2)})\pi(1_{(1)}\,b'), \tag{2.0.2}$$

for all $b, b' \in B$. We used the notation $1_{(1)} \otimes 1_{(2)} = \gamma(1) \neq 1 \otimes 1$, and $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = (\gamma \otimes 1) \circ \gamma(1) = (1 \otimes \gamma) \circ \gamma(1)$. The maps

$$\pi^R : B \to B, \;\; b \mapsto 1_{(1)}\pi(b\,1_{(2)}) \text{ and } \pi^L : B \to B, \;\; b \mapsto \pi(1_{(1)}\,b)1_{(2)}$$

turn out to be idempotent– they are the projections onto the 'source' and 'target' subalgebras $A^L = \Pi^L(B)$ and $A^R = \Pi^R(B)$, respectively. A weak Hopf algebra (WHA for short) $\langle B, \gamma, \pi, m, i, S\rangle$ is a weak bialgebra together with an antipode map $S : H \to H$ satisfying the three axioms

$$\begin{aligned} &[\text{WHA1}] \;\; h_{(1)}S(h_{(2)}) = \Pi^L(h) \\ &[\text{WHA2}] \;\; S(h_{(1)})h_{(2)} = \Pi^R(h) \\ &[\text{WHA3}] \;\; S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h) \end{aligned}$$

Finitely generated projective weak Hopf algebras share the property of self–duality with Hopf algebras, i.e. for a f.g.p. weak Hopf algebra $\langle H, \mu, \eta; \Delta, \varepsilon; S\rangle$, there is a WHA structure (the 'transpose') $\langle \hat{H}, \hat{\mu}, \hat{\eta}; \hat{\Delta}, \hat{\varepsilon}; \hat{S}\rangle$ on the k–dual $\hat{H} = \mathrm{Hom}_k(H, k)$. Although weak Hopf algebras are defined over a field k, there are several indications that in fact the source (target) subalgebra should be regarded as the base field. Most importantly, for a weak Hopf algebra H, the trivial left H–module in ${}_H\mathsf{M}$ is the representation on the k–module A^L (or A^R). Also, the dual weak Hopf algebra $\hat{H}$ has canonical subalgebras $\hat{A}^L$ and $\hat{A}^R$ that are isomorphic to A^L and A^R.

Approaching quantum groupoids from classical groupoid algebras gives valuable insights. Recall that a groupoid is simply a category in which every arrow is invertible, viz. a group is a category on one single object with every arrow invertible. Groupoids simultaneously generalize groups and equivalence relations and have found many applications in topology, differential, and Poisson geometry, as evidenced by the monograph [**26**]. Topological and geometric examples include the fundamental groupoid in homotopy theory and the holonomy groupoid of foliations in differential geometry. We denote a groupoid $\mathcal{G}^1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}^0$, with s and t the source and target maps from the set of arrows to the set of objects. The groupoid algebra $k[\mathcal{G}]$ is then a special case of the quantum groupoid such that the canonical subalgebra is $k[\mathcal{G}^0]$, the algebra of functions on the units. An example related directly to weak Hopf algebras is the groupoid of matrix units $\{e_{ij}\}_{i,j=1}^N$, where $\mathcal{G}^1 = \{e_{ij}\}_{i\neq j}$, $\mathcal{G}^0 = \{e_{ii}\}_{i=1}^N$ and the projections onto the left (right) canonical subalgebras are $\pi^L : e_{ij} \to e_{ii}$ and $\pi^R : e_{ij} \to e_{jj}$.

One motivation that research in groupoids and quantum groups have shared is to develop new notions of symmetry, which are applicable in situations where no conventional (i.e. group–) symmetry is present (see [**90**] for an informal introduction to the subject). In some instances, groupoids are more natural to consider than groups, as in the case of the gauge groupoid of a (principal) fiber bundle. Groupoids with additional structure include Lie– and Poisson–Lie groupoids, which are generalized to the noncommutative setting by Lu's bialgebroids.

A useful way of thinking about how groupoids extend the conventional notion of symmetry is that groupoids give a unified description of symmetry together with the action of the symmetry. Considering a classical setup with a geometric (discrete, topological, smooth, etc.) space $\mathcal{M}$ acted on by a (finite, topological, Lie) group G with action $\gamma : \mathcal{M} \times G \to \mathcal{M}$, the action groupoid is $\mathcal{G}_\gamma = \{(m, g, m') \in M \times G \times M \,|\, m' = m \cdot g\}$. In fact, the scalar extension construction to be discussed in this thesis can be seen as a noncommutative cousin of the action groupoid.

The main motivation behind weak Hopf algebras came from quantum field theory and operator algebras. The problem is one of symmetry reconstruction: for an extension $N \subseteq M$ of (C^*– , von Neumann) algebras, find the 'symmetry' G (uniquely if possile) acting on M such that $N = M^G$. The dual problem is to find the dual 'symmetry' $\hat{G}$ acting on N, such that $M = N \rtimes \hat{G}$ is obtained as a cross–product of N and $\hat{G}$. The main difficulty is that determining what kind of mathematical object the parenthesized 'symmetry' is, is part of the problem.

It was shown by Longo in [**50**] that if $N \subset M$ is an irreducible depth 2 extension of von Neumann factors, then the appropriate symmetry is a finite dimesnional C^*–Hopf algebra. In [**64**], it was proven that if one allows the extension to be reducible and N, M are not factors but have arbitrary finite dimensional centers, then the symmetry is a C^*–weak Hopf algebra. This approach was worked out in detail for II_1 type factors in [**61, 62**]. The work of Enock and Vallin fall into this line of work in that their Hopf bimodules are used to describe depth 2 extensions of arbitrary index.

The bearing of quantum groupoids on quantum field theory is through their representation category [**10**]. The representation category of a C^*–weak Hopf algebra is a semisimple monoidal C^*–category with isomorphic left and right duals. This comprises much of the structure of the Doplicher–Roberts category of superselection sectors in Algebraic Quantum Field Theory. In terms of this analogy, *sectors* are the equivalence classes of irreducible morphisms.

In a series of papers [**32, 33, 34**], Doplicher and Roberts established a reconstruction theorem stating that an abstract symmetric monoidal C^*–category with subobjects and direct sums (both guaranteed, if the category is semisimple) with irreducible monoidal unit is equivalent to the representation category of a compact group. These requirements are precisely satisfied by the Doplicher–Roberts category in spacetime dimensions higher than three, but no analogue of the reconstruction theorem is known for two spacetime dimensions.

Model studies in $2d$ conformal field theory and one dimensional quantum lattice systems reveal two related phenomena which fall outside the purview of the DR theorem. First, it often happens that superselection sectors satisfy fusion rules $p \otimes q = \sum_r N_{pq}^r\, r$ with integer fusion coefficients N_{pq}^r that preclude integer dimensions, i.e. the relation $d_p d_q = \sum_r N_{pq}^r\, d_r$ for the dimensions of sectors admits no integer solutions. It should be noted that the dimensions d_p are the intrinsic (quantum) dimensions in the sense of [**51**] which, however, coincide with the integer dimensions of the representation spaces in the case of groups and even Hopf algebras. Secondly, it will happen that the monoidal unit is reducible, allowing for inequivalent irreducible vacuum representations reminiscent of solitonic sectors. The representation categories of quantum groupoids provide both features, making them candidates for the generic symmetry in low dimensional quantum field theory.

In [**47**], Kadison and Szlachányi connected bialgebroids to the theory of depth 2 extensions. The key advancement was a general definition of depth 2 extensions for rings in a purely algebraic setting, divorced from the original context of operator algebras. Earlier results on depth 2 extensions were recovered as special cases, notably ring extensions with separable centralizer entail weak Hopf algebra symmetry and for trivial centralizer (i.e. irreducible extensions) one obtains Hopf algebra symmetry. Also, the inherent one–sidedness resulting from the noncommutative base algebra was acknowledged by defining both left and right bialgebroids. The convention adopted by Lu corresponds to left bialgebroids in this terminology.

It is generally agreed that bialgebroids are the appropriate quantum groupoid generalization of bialgebras but it is less straightforward to determine what the appropriate notion of Hopf algebroid is? Böhm and Szlachányi proposed a definition of Hopf algebroid containing one left–handed and one right–handed constituent bialgebroid in [**11**]. Here, the antipode was defined to be bijective, a requirement that was later relaxed in [**7**]. In [**79**], Szlachányi formulated an equivalent description of Frobenius Hopf algebroids in terms of two compatible multiplicative structures termed Distributive Double Algebras.

In a different line of development, Militaru and Brzeziński examined the structure obtained by taking the abstract dual of a bialgebroid in [**21**], calling it bicoalgebroid. Bicoalgebroids were further investigated in [**4**].

Applications to Galois theory have been a unifying theme in the research on quantum groups and quantum groupoids. Galois theory, in such generality, is as old as the concept of group symmetry itself but even Hopf Galois theory has a history of four decades, dating back to the work of Chase and Sweedler [**28**]. Hopf Galois theory accomodates the classical Galois theory of field extensions, strongly graded group algebras and is the algebraic counterpart of principle bundles in the spirit of noncommutative geometry. For a survey of Hopf Galois theory see e.g. the monograph [**58**]. The appearance of novel symmetry structures and examples from other areas of mathematics, most notably noncommutative geometry, which did not fit seamlessly into Hopf Galois theory also spawned various generalizations.

Two avenues of research unfolded: on the one hand, replacing Hopf algebras by only coalgebras as first suggested in [**76**] and later developed in [**17**] and [**15**] and on the other, to allow bialgebroids instead of Hopf algebras. The former approach has its origins in noncommutative geometry, the latter was initiated by Kadison and Szlachányi in [**47**], in relation to depth 2 ring extensions. Depth 2 ring extension were shown to be the finitary quantum groupoid–Galois extensions in [**83**]. A Galois theory was developed for Hopf–algebroids in [**6**] and in [**3**], the latter taking a double algebraic point of view. The equivalence of depth 2 extensions and bialgebroid–Galois extensions was extended to the equivalence of depth 2 Frobenius extensions and extensions which are Galois over some Frobenius Hopf algebroid.

CHAPTER 3

Quantum groupoids

Bialgebroids are a generalization of bialgebras that capture some of the features expected of a quantum symmetry, in that their categories of modules (and comodules) are monoidal and they possess a Galois theory generalizing the classical Hopf Galois theory. Several definitions of bialgebroid have appeared in the literature, which have eventually turned out to be equivalent. In contrast, there is as yet no consensus on how to adjoin an antipode to bialgebroids, i.e. on the appropriate definition of a Hopf algebroid. Here, we shall stick to the axioms proposed in [**11**]. The particular special case of *Frobenius* Hopf algebroids merits special attention. For one, this structure enjoys the property of being self–dual, just like Hopf algebras, whereas the dual of a general Hopf algebroid admits no natural Hopf algebroid structure. Also, the Frobenius property may be seen as a finiteness condition and Frobenius Hopf algebroid symmetry may be seen as a (vast) generalization of finite group symmetry. The looser term 'quantum groupoid' will be meant to refer to a Frobenius Hopf algebroid. In [**79**], the notion of *distributive double algebras* algebras (DDAs) was introduced and was shown to be equiavelent to Frobenius Hopf algebroid. The word 'double' refers to that the ring and coring structures of the Hopf algebroid are represented by two distinct multiplicative structures in a DDA. We shall use the double algebraic formalism in the remaining part of this Thesis, with the exception of the Chapter on bicoalgebroids.

The Chapter is organized as follows: algebraic structure are discussed, roughly in the order of increasing complexity, as Bialgebroid → Hopf algebroid → Frobenius Hopf algebroid → Distributive Double Algebra (DDA). In each case, we define modules and comodules and discuss the appropriate module (comodule) categories.

1. Bialgebroids

A bialgebra B over a field k is an k-algebra $\langle B, \mu, \eta \rangle$ and a k–coalgebra $\langle B, \Delta, \varepsilon \rangle$ over the same k–module B, subject to the compatibility condition that μ and η should be coalgebra maps (or equivalently, Δ and ε are algebra maps). A generalization to the case where k is replaced by a noncommutative algebra R requires at least two new ingredients, formulated in the following two definitions.

Definition 3.1.1. An R–ring A over a k–algebra R is a triple $\langle A, \mu, \eta \rangle$, where A is an R–bimodule and $\mu : A \underset{R}{\otimes} A \to A$ is an associative multiplication with unit $\eta : R \to A$ such that both μ and η are R–bimodule maps. Equivalently, an R–ring A is a monoid in the category of R–bimodules.

Furthermore, there is a one–to–one correspondence between R–rings A and k–algebra morphisms $j : R \to A$. We shall often think of R–rings in this latter sense.

Definition 3.1.2. An R–coring C over a k–algebra R is a triple $\langle C, \delta, \varepsilon \rangle$, where C is an R–bimodule and $\Delta : C \to C \underset{R}{\otimes} C$ a coassociative comultiplication with counit $\varepsilon : C \to R$ such that both Δ and ε are R–bimodule maps. Equivalently, an R–coring is a comonoid in the category of R–bimodules.

A direct generalization of bialgebras by simply replacing the algebra (coalgebra) structure with an R–ring (R–coring) is impossible. The reason is that the compatibility condition

$$\Delta \circ \mu = (\mu \otimes \mu) \circ \mathsf{tw}_{23} \circ (\Delta \otimes \Delta) \tag{3.1.3}$$
$$(bb')_{(1)} \otimes (bb')_{(2)} = b_{(1)}b'_{(1)} \otimes b_{(2)}b'_{(2)}$$

makes use of the flip map $\mathsf{tw} : M \underset{k}{\otimes} N \to N \underset{k}{\otimes} M$, which is a symmetry of the category M_k but is not well–defined in the category ${}_R\mathsf{M}_R$ of R–bimodules. In fact, ${}_R\mathsf{M}_R$ is in general not even braided.[1]

The way out is that the ring and coring structures of a bialgebroid are defined over different base rings, making a bialgebroid respectively a monoid and a comonoid in two different categories. This allows one to formulate the compatibility condition 3.1.3. Recall also that the *enveloping* algebra of a ring R is defined as $R^e = R \otimes R^{op}$. R^e–rings are in bijective correspondence with pairs of k–algebra homomorphisms $s : R \to A$, $t : R^{op} \to A$ whose images commute in A.

Definition 3.1.4. For a ring R over k, a *right* bialgebroid B over R consists of a k–algebra $\langle B, \mu, \eta \rangle$ and

- an R^e–ring structure $s \otimes t : R \otimes R^{op} \to B$ on B
- an R–coring structure $\langle B, \Delta, \varepsilon \rangle$ on B

such that the following compatibilty axioms are satisfied:

(1) the R–bimodule structure is related to the R^e–ring structure as follows:

$$r \cdot a \cdot r' = as(r')t(r)$$

(2) the image of Δ lies in the sub–R-bimodule of $B \underset{R}{\otimes} B$ defined by

$$B \underset{R}{\times} B = \{\sum_i b_i \underset{R}{\otimes} b'_i \in B \underset{R}{\otimes} B | \sum_i s(r)b_i \underset{R}{\otimes} b'_i = \sum_i b_i \underset{R}{\otimes} t(r)b'_i,\ \forall r \in R\}, \tag{3.1.5}$$

which is a ring with factorwise multiplication, so we may require that

$$\Delta : B \to B \underset{R}{\times} B$$

[1] in a braided category, replacing the tw–map with the braiding leads to the concept of *braided* bialgebras, see e.g. [**57**].

be a ring homorphism. A particular consequence is the unitalness of the coproduct, $\Delta(1) = 1 \otimes 1$.

(3) the counit is compatible with the ring structure in the sense

$$\varepsilon(t(\varepsilon(a)b)) = \varepsilon(ab) = \varepsilon(s(\varepsilon(a)b)), \ \ \forall a, b \in B \tag{3.1.6}$$

(4) the counit preserves the unit,

$$\varepsilon(1) = 1_R \tag{3.1.7}$$

REMARK 3.1.8. The crucial point in the definition is the introduction of the Takeuchi $\underset{R}{\times}$–*product*. For a non–commutative ring R, $B \underset{R}{\otimes} B$ does not have a well–defined ring structure, but $B \underset{R}{\times} B$ is actually an R^e–ring:

$$\hat{\mu} : (B \underset{R}{\times} B) \otimes (B \underset{R}{\times} B) \to B \underset{R}{\times} B, \quad (a \underset{R}{\times} a')(b \underset{R}{\times} b') = (aa' \underset{R}{\times} bb')$$
$$\hat{\eta} : R \otimes R^{op} \to B, \quad r \otimes r' \mapsto t(r') \otimes s(r)$$

Put differently, the Takeuchi submodule is the center of the bimodule ${}_R(B \underset{R}{\otimes} B)_R$, where the left– and right R–actions are given by $r \cdot (b \underset{R}{\otimes} b') \cdot r' = s(r)b \underset{R}{\otimes} t(r')b'$.

The definition is inherently asymmetrical, since the compatibility we imposed on the coring and ring structure was to make B an R–coring via the *right* R^e–action coming from the R^e–ring structure (note that ${}_R\mathsf{M}_R \simeq \mathsf{M}_{R^e}$). Choosing the other possibility, we obtain the definition of *left* bialgebroids:

DEFINITION 3.1.9. A *left* bialgebroid B over L consists of a k–algebra $\langle B, \mu, \eta\rangle$ and

- an R^e–ring structure $s \otimes t : L \otimes L^{op} \to B$ on B
- an R–coring structure $\langle B, \Delta, \varepsilon\rangle$ on B

such that the following compatibilty axioms are satisfied:

(1) the R–bimodule structure is related to the R^e–ring structure in the sense

$$l \cdot a \cdot l' = s(l)t(l')a$$

(2) the image of Δ lies in the sub–L-bimodule of $B \underset{L}{\otimes} B$ defined by

$$B \underset{L}{\times} B = \{\sum_i b_i \underset{L}{\otimes} b'_i \in B \underset{R}{\otimes} B | \sum_i b_i t(l) \underset{R}{\otimes} b'_i = \sum_i b_i \underset{R}{\otimes} b'_i s(l), \ \forall l \in L\}, \tag{3.1.10}$$

and

$$\Delta : B \to B \underset{L}{\times} B$$

is a ring homorphism. In particular, the coproduct is unital: $\Delta(1) = 1 \otimes 1$.

(3) the counit is compatible with the ring structure in the sense

$$\varepsilon(at(\varepsilon(b))) = \varepsilon(ab) = \varepsilon(a(\varepsilon(b))), \ \ \forall a, b \in B \tag{3.1.11}$$

(4) the counit preserves the unit,

$$\varepsilon(1) = 1_L \tag{3.1.12}$$

For any bialgebra, taking the opposite algebra structure or co–opposite coalgebra structure will yield the opposite (respectively, the co–opposite) bialgebra. For bialgebroids, care has to be taken because of the 'handedness' of the $(\,)^{op}$ and $(\,)_{cop}$ operations: for a right bialgebroid $\langle B, R, s, t, \Delta, \varepsilon\rangle$, passing to the opposite ring structure yields the *left* bialgebroid $\langle B^{op}, R^{op}, t^{op}, s^{op}, \Delta, \varepsilon\rangle$. On the other hand, taking the co–opposite coring structure, $\langle B_{cop}, R^{op}, t, s, \Delta_{cop}, \varepsilon\rangle$ is also a right bialgebroid.

For a finite dimensional bialgebra $\langle H, \mu, \eta, \Delta, \varepsilon\rangle$ over k, the k–dual $H^* = \mathrm{Hom}(H, k)$ is also a bialgebra. The structure maps of H^* are the transposes of the structure maps of H, with respect to the canonical pairing $\langle\,,\,\rangle : H^* \otimes H \to k$ afforded by the evaluation $\langle \varphi, h\rangle = \varphi(h)$. Choosing a dual basis $\{h_i, \xi^i\}_{i=1}^N \in H \otimes H^*$, the identities $h = \sum_i h_i\, \xi^i(h)$ and $\varphi = \sum_i \varphi(h_i)\, \xi^i$ hold, independently of the choice of the $\{h_i, \xi^i\}$. The algebra structure is given by the multiplication

$$\mu^* : H^* \otimes H^* \to H^*, \quad \varphi\psi(h) = \varphi(h_{(1)})\psi(h_{(2)})$$

and unit $\eta^* : k \to H^*$, $1^* = \varepsilon_H$. The coalgebra structure of H^* is given in terms of the dual basis, with comultiplication

$$\Delta^* : H^* \to H^* \otimes H^*, \quad \varphi \mapsto \sum_i \varphi(h_i\, \_) \otimes \xi^i$$

and counit $\varepsilon^* : H^* \to k$, $\varphi \mapsto \varphi(1_H)$.

Defining duals of bialgebroids is more complicated, since instead of the k–module structure (or the symmetric k–bimodule structure), we have two inequivalent (left and right) R–module structures, leading to the notions of left and right dual for both left and right bialgebroids, four cases to consider in total. The assumption taking the place of finite dimensionality is that the bialgebroid be *finitely generated projective* over the base ring. A systematic treatment of bialgebroid duality can be found in [**47**], here we only consider one of the cases, for reference and later use.

Consider a right bialgebroid $\langle B, s_R, t_R\rangle$, finitely generated projective as a left R–module. Introducing the notation ${}^*B = \mathrm{Hom}_{R-}(B, R)$ for the left R–dual, this is equivalent to the existence of a dual bases $\{\beta^i\}_{i=1}^N \in B$ and $\{b_i\}_{i=1}^N \in {}^*B$, such that for all $b \in B$, $\beta \in {}^*B$ the identities

$$b = \sum_i \beta^i(b) \cdot b_i, \quad \beta = \sum_i \beta^i \cdot \beta(b_i) \tag{3.1.13}$$

hold.

The R–bimodule structure of ${}^*B$ is given by the left dual of the bimodule $B^{(t)}$, which carries the R–actions $r \cdot b \cdot r' = t(r')bt(r)$. As left and right R–modules, respectively, ${}_R({}^*B) =$

$\mathrm{Hom}_R(B_R^{(t)}, R)$ and $({}^*B)_R = \mathrm{Hom}_R(B^{(t)}, R_R)$. The R–actions on a $\beta \in^* B$ are

$$(r \cdot \beta)(b) = \beta(t(r)b), \quad (\beta \cdot r)(b) = \beta(b)r \tag{3.1.14}$$

We introduce multiplication on ${}^*B$ via the formula

$$(\beta\beta')(b) = \beta(b_{(1)}s(\beta'(b_{(2)}))) \tag{3.1.15}$$

${}^*B$ becomes a ring with unit ${}^*1 = \varepsilon_B$. If ${}^*B$ is to be a (left) bialgebroid, the source and target maps should be expressed by the left and right actions on the unit:

$$ {}^*s : R \to {}^*B, \quad {}^*s = r \cdot {}^*1 = \varepsilon_B(t(r)\,\_) \tag{3.1.16}$$

$$ {}^*t : R \to {}^*B, \quad {}^*t = {}^*1 \cdot r = \varepsilon_B(\_)r \tag{3.1.17}$$

It is then verified that ${}^*s$ and ${}^*t$ are indeed ring homomorphisms, whose images commute in ${}^*B$, making $\langle {}^*B, {}^*s, {}^*t\rangle$ an R^e–ring, and that the R–bimodule structure of ${}^*B$ is given by

$$({}^*s(r)\beta)(b) = (r \cdot \beta)(b) = \beta(t(r)b) \tag{3.1.18}$$

$$({}^*t(r)\beta)(b) = (\beta \cdot r)(b) = \beta(b)r \tag{3.1.19}$$

as it should be for a left bialgebroid. It remains to define the coring structure on ${}^*B$. It is checked that the comultiplication

$$\begin{aligned} {}^*\Delta : {}^*B &\to {}^*B \underset{R}{\otimes} {}^*B \\ \beta &\mapsto \sum_i \beta^i \underset{R}{\otimes} \beta(\_b_i) \end{aligned}$$

is a coassociative R–bimodule map with respect to the R–actions $r \cdot \beta \cdot r' = {}^*s(r){}^*t(r')\beta$ and the dual counit is given by evaluation on the unit, ${}^*\varepsilon : {}^*B \to R$, $\beta \mapsto \beta(1_B)$. The remaining left bialgebroid axioms (compatibility of the ring and coring structures) can be checked by direct computation.

2. Modules & comodules over bialgebroids

In the following, $\langle B, \mu, \eta, \Delta, \varepsilon, s, t\rangle$ shall denote a right bialgebroid over R. We shall define modules and comodules of bialgebroids and show that both the module– and comodule categories are monoidal. The former statement actually characterizes bialgebroids among R^e–rings, and is known as Schauenburg's theorem.

2.1. Modules over a bialgebroid.

Definition 3.2.1. A module M_B over B is a module over the underlying k–algebra of B, i.e. a pair $\langle M, \gamma\rangle$, where

- M is a k–module
- $\gamma : M \underset{k}{\otimes} B \to M$ is a k–module map such that

$$\gamma \circ (\gamma \underset{k}{\otimes} B) = \gamma \circ (M \underset{k}{\otimes} \mu) \text{ and } \gamma \circ (M \underset{k}{\otimes} \eta) = id_M$$

A right B–module map $f : \langle M, \gamma_M \rangle \to \langle M', \gamma_{M'} \rangle$ is a k–module map $f : M \to M'$ that intertwines the actions γ and γ':

$$\begin{array}{ccc} M \otimes B & \xrightarrow{f \otimes B} & M' \otimes B \\ {\scriptstyle \gamma_M} \downarrow & & \downarrow {\scriptstyle \gamma_{M'}} \\ M & \xrightarrow[f]{} & M' \end{array}$$

The category M_B has objects the right B–modules and arrows the right B–module maps.

Recall that a bialgebroid is not only a k–algebra, but an R^e–ring with $s \otimes t : R \otimes R^{op} \to B$. This endows M_B with an additional piece of structure, namely a forgetful functor $U : \mathsf{M}_B \to \mathsf{M}_{R^e} \simeq {}_R\mathsf{M}_R$: a right B–module M_B is a right R^e–module via $m \cdot (r \otimes \bar{r}) = m \triangleleft s(r)t(\bar{r})$.

A salient feature of bialgebras is that their category of modules is monoidal, such that the coalgebra structure is responsible for the monoidality of the module category. More precisely, a classic result of Pareigis (see [**66**]) states that a bialgebra structure on a k–algebra A (for k a commutative ring) is equivalent to a monoidal structure on M_A such that the forgetful functor $U : \mathsf{M}_A \to \mathsf{M}_k$ is strict monoidal. This result generalizes to bialgebroids – we only sketch the proof.

Theorem 3.2.2. *Let R be an algebra over a commutative ring k, then the following data are equivalent on an R^e–ring $\langle B, s, t \rangle$:*

- *a right bialgebroid structure on $\langle B, s, t \rangle$*
- *a monoidal structure on M_B, such that $U : \mathsf{M}_B \to {}_R\mathsf{M}_R$ is strict monoidal*

Proof (sketch). On the one hand, the R–coring structure of a bialgebroid B induces a monoidal structure on the category of B–modules: for $(m \otimes n) \in M \underset{R}{\otimes} N$ define the right B–action on the R–tensor product of modules with $(m \otimes n) \triangleleft b = m \triangleleft b_{(1)} \otimes n \triangleleft b_{(2)}$ and let the B–module structure on R be $r \triangleleft b = \varepsilon(s(r)b)$. Due to the bialgebroid axioms, these maps are well–defined. This is exactly the monoidal structure that renders $U : \mathsf{M}_B \to {}_R\mathsf{M}_R$ strict monoidal.

On the other hand, if M_B is monoidal, define maps $\Delta' : B \to B \underset{R}{\otimes} B$, $b \mapsto (1_B \otimes 1_B) \triangleleft b$ and $\varepsilon' : B \to R$, $b \mapsto 1_R \triangleleft b$. Then it can be shown that $\langle B, \Delta', \varepsilon' \rangle$ defines an R–coring, making B a bialgebroid. □

Remark 3.2.3. For a *left* bialgebroid B over L, ${}_B\mathsf{M}$ is monoidal such that the forgetful functor $U : {}_B\mathsf{M} \to {}_{L^e}\mathsf{M}$ is strict monoidal. The proof of theorem 3.2.2 carries over to this case by taking the opposite *right* bialgebroid B^{op} and noting that ${}_B\mathsf{M}$ is isomorphic to $\mathsf{M}_{B^{op}}$. In principle, one could consider *left* modules over *right* bialgebroids (or *vice versa*), but in this case, there is no natural monoidal structure on the category of modules. In contrast, left (or right) bialgebroids have both left *and* right comodule categories (see below).

2.2. Comodules over a bialgebroid.

DEFINITION 3.2.4. Let R be a k–algebra and B a right bialgebroid over R. A right B–comodule X^B is a pair $\langle X, \delta\rangle$, where

- X is a right R–module
- $\delta : X \to X \underset{R}{\otimes} A$ is a right R–module map such that

$$(\delta \underset{R}{\otimes} B) \circ \delta = (X \underset{R}{\otimes} \Delta) \circ \delta \text{ and } (X \underset{R}{\otimes} \varepsilon) \circ \delta = X$$

  (suppressing natural isomorphisms $X \underset{R}{\otimes} R \simeq X$)

A morphism of B–comodules $g : \langle X, \delta\rangle \to \langle Y, \omega\rangle$ is a right R–module map g, intertwining the coactions δ and ω:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \delta\downarrow & & \downarrow\omega \\ X \underset{R}{\otimes} B & \xrightarrow[g \underset{R}{\otimes} B]{} & Y \underset{R}{\otimes} B \end{array}$$

The category M^B has objects the right B–comodules and arrows the right B–comodule maps.

For a *bialgebra* over $\langle B, \mu, \eta, \Delta, \varepsilon\rangle$ over k, not only M_B but also M^B is a monoidal category such that the forgetful functor $U : \mathsf{M}^B \to \mathsf{M}_k$ is strict monoidal. One would like to extend this result to bialgebroids, but here an obstacle presents itself already in choosing the appropriate forgetful functor. A right B–comodule X^B is *a priori* only a right R–module, which leads to a forgetful functor $U' : \mathsf{M}^A \to \mathsf{M}_R$. Unlike ${}_R\mathsf{M}_R$, however, M_R is not monoidal. The following Lemma (Prop. 1.1 of [**3**]) allows us to construct a functor $U : \mathsf{M}^B \to {}_R\mathsf{M}_R$ which will indeed turn out to be strict monoidal.

LEMMA 3.2.5. *Let $\langle X, \delta\rangle$ be a right B–comodule. Then there is a* left *R–module structure on X such that*

(1) *X is an R–bimodule*
(2) *δ is an R–bimodule map*
(3) $\delta(X) \subseteq X \times_R B$
(4) *every arrow $g \in \mathsf{M}^B$ is an R–bimodule map*

where we used the notation

$$X \times_R B = \{\sum_i x_i \underset{R}{\otimes} b_i \in X \underset{R}{\otimes} B | \\ | \sum_i r \cdot x_i \underset{R}{\otimes} b_i = \sum_i x_i \underset{R}{\otimes} t(r) b_i, \text{ for all } r \in R\}$$

for the Takeuchi–product.

PROOF. The key observation is that although X is not a left R–module, $X \underset{R}{\otimes} B$ is, with the left R–action $r \cdot (x \underset{R}{\otimes} b) = x \underset{R}{\otimes} s(r) b$. If X were a left R-module and δ a left R-module map then by the identity

$$\begin{aligned} r \cdot x &= (r \cdot x)_{(0)} \cdot \varepsilon((r \cdot x)_{(1)}) \\ &= x_{(0)} \cdot \varepsilon(s(r)x_{(1)}), \end{aligned}$$

we've expressed the left R–action in terms of the coaction. This proves uniqueness. If we use the above formula to define $r \cdot x$ then we find that it is indeed a left action because $s : R \to A$ is an algebra homomorphism. It commutes with the right R-action

$$r \cdot (x \cdot r') = x_{(0)} \cdot \varepsilon(s(r)as(r')) = (r \cdot x) \cdot r'$$

so X is an R-R-bimodule and the coaction is a bimodule map,

$$\delta(r \cdot x \cdot r') = x_{(0)} \underset{R}{\otimes} s(r)x_{(1)}s(r').$$

Now the Takeuchi property (3) holds automatically,

$$\begin{aligned} r \cdot x_{(0)} \underset{R}{\otimes} x_{(1)} &= x_{(0)} \cdot \varepsilon(s(r)x_{(1)}) \underset{R}{\otimes} x_{(2)} \\ &= x_{(0)} \underset{R}{\otimes} \varepsilon(t(r)x_{(1)}) \cdot x_{(2)} \\ &= x_{(0)} \underset{R}{\otimes} t(r)x_{(1)} . \end{aligned}$$

If $g : X \to Y$ is a comodule morphism then

$$\begin{aligned} g(r \cdot x) &= g(x_{(0)}) \cdot \varepsilon(s(r)x_{(1)}) = g(x)_{(0)} \cdot \varepsilon(s(r)g(x)_{(1)}) \\ &= r \cdot g(x) . \end{aligned}$$

□

We can now state the following theorem[2]

Theorem 3.2.6. *For a right bialgebroid B over R, the category M^B is monoidal such that the forgetful functor $U : \mathsf{M}^B \to {}_R\mathsf{M}_R$ is strict monoidal.*

Proof (sketch). The base ring R is a right B–comodule via the source map $s : R \to R \underset{R}{\otimes} B \simeq B$. The Theorem is proved by showing that the map

$$\begin{aligned} X \underset{R}{\otimes} Y &\to (X \underset{R}{\otimes} Y) \underset{R}{\otimes} B \\ x \otimes y &\mapsto x_{\langle 0\rangle} \otimes y_{\langle 0\rangle} \otimes x_{\langle 1\rangle}y_{\langle 1\rangle} \end{aligned} \tag{3.2.7}$$

is a well–defined coaction on the R–tensor product of $M, N \in \mathsf{M}^B$ (in particular, an R–bimodule map) and that the coherence isomorphisms in ${}_R\mathsf{M}_R$ lift to M^B–maps. □

The above result amounts to only half of Schauenburg's theorem, i.e. it is not known (and is not expected to be) whether a monoidal structure on the category M^B, such that the forgetful

[2]This is essentially due to Schauenburg, except for the Lemma 3.2.5

functor to ${}_R\mathsf{M}_R$ is strict monoidal, implies a bialgebroid structure on the coring B. We shall find an analogous result for bi*co*algebroids, however, in Section 3 of Chapter 4.

Remark 3.2.8 (Left comodules over a right bialgebroid). As mentioned earlier, the category of *left* comodules of right bialgebroids is also monoidal. The results obtained so far for right comodules go over effortlessly, we only summarize the main points. A left comodule Y over a right bialgebroid B is a pair $\langle Y, \bar{\delta} \rangle$, where Y is a left R–module and $\bar{\delta} : Y \to B \underset{R}{\otimes} Y$ is a map satisfying coassociativity and counitality. By the analogue of Lemma 3.2.5, there is a right R–module structure on Y such that $\bar{\delta}$ is an R–bimodule map and ${}^B\mathsf{M}$ is monoidal with a strict monoidal forgetful functor into ${}_R\mathsf{M}_R$. The monoidal unit is R, with B–comodule structure furnished by the target map $t : R \to B \simeq B \underset{R}{\otimes} R$ and the monoidal product of left B–comodules is

$$
\begin{aligned}
X \underset{R}{\otimes} Y &\to B \underset{R}{\otimes} (X \underset{R}{\otimes} Y) \\
x \otimes y &\mapsto y_{\langle -1 \rangle} x_{\langle -1 \rangle} \otimes x_{\langle 0 \rangle} \underset{R}{\otimes} y_{\langle 0 \rangle}
\end{aligned}
\tag{3.2.9}
$$

Note the reversed order of multiplication with respect to 3.2.7! Needless to say, all previous results hold just as well for left bialgebroids, *mutatis mutandis*.

3. Hopf algebroids

A Hopf algebroid should be to a bialgebroid as a Hopf algebra is to a bialgebra. Translating this intuition into a definition is not straightfroward, however. The antipode for a Hopf algebra H over k is a bialgebra map $S : H \to H^{op}_{cop}$, i.e. it is simultaneously an anti–algebra and an anti–coalgebra map. Replacing H with a (right, say) bialgebroid, H^{op}_{cop} will be a left bialgebroid. However, any notion of a map of bialgebroids should respect the handedness of a bialgebroid: hence, if H and H^{op}_{cop} are both to be right (left) bialgebroids, then a Hopf algebroid should carry both a left and a right bialgebroid structure. We present the definition of [**11**] below – other definitions can also be found in the literature which, unlike the different definitions for bialgebroids, are in general not equivalent.

Definition 3.3.1. For k–algebras R and L, a Hopf algebroid is a triple $H = (H_L, H_R, S)$ consisting of a left bialgebroid structure H_L and a right bialgebroid structure H_R (the two constituent bialgebroids) on the same k–algebra, and the antipode S which is a k–module map. Structure maps pertaining to H_L will carry a subscript L and likewise for H_R, e.g. H_L has the $L \otimes L^{op}$–ring structure $\langle H, s_L, t_L \rangle$ and the L–coring structure $\langle H, \Delta_L, \varepsilon_L \rangle$. The following compatibilities are satisfied:

(1) $s_L \circ \varepsilon_L \circ t_R = t_R, \;\; t_L \circ \varepsilon_L \circ s_R = s_R, \;\; s_R \circ \varepsilon_R \circ t_L = t_L, \;\; t_R \circ \varepsilon_R \circ s_L = s_L$

(2) $(\Delta_L \underset{R}{\otimes} H) \circ \Delta_R = (H \underset{L}{\otimes} \Delta_R) \circ \Delta_L$ and $(\Delta_R \underset{L}{\otimes} H) \circ \Delta_L = (H \underset{R}{\otimes} \Delta_L) \circ \Delta_R$

(3) $S(t_L(l)\, h\, t_R(r)) = s_R(r)\, S(h)\, s_L(l)$, for all $l \in L, r \in R$ and $h \in H$

(4) $\mu_L \circ (S \underset{L}{\otimes} H) \circ \Delta_L = s_R \circ \varepsilon_R, \;\; \mu_R \circ (H \underset{R}{\otimes} S) \circ \Delta_R = s_L \circ \varepsilon_L.$

Care has to be taken in interpreting these axioms: one has to make use of all module structures coming from the $L \otimes L^{op}$–ring structure of H_L and the $R \otimes R^{op}$–ring structure of H_R. Introduce the following notation for the four L–modules of H_L. Let ${}_LH$ stand for the module $l \cdot h = s_L(l)h$, H_L for $h \cdot l = t_L(l)h$, ${}^LH$ for $l \cdot h = ht_L(l)$, H^L for $h \cdot l = hs_L(l)$ and likewise for R–modules, the rule being that left (right) indices stand for left (right) actions and lower (upper) indices stand for actions through left (right) H–multiplications. Then for example the left and right hand sides of the axiom $(\Delta_L \underset{R}{\otimes} H) \circ \Delta_R = (H \underset{L}{\otimes} \Delta_R) \circ \Delta_L$ are $H \to H_L \otimes {}_LH^R \otimes {}^RH$ maps, where we have indicated precisely which L– and R–actions are amalgamated. We have to introduce a generalization of Sweedler's notation to differentiate between the comultiplication of H_L and H_R. The H_R–coproduct will be denoted with upper indices $\Delta_R(h) = h^{(1)} \underset{R}{\otimes} h^{(2)}$ and the H_L–coproduct with lower indices $\Delta_L(h) = h_{(1)} \underset{L}{\otimes} h_{(2)}$. The mixed co-commutativity of the two comultiplications reads

$$h^{(1)}{}_{(1)} \otimes h^{(1)}{}_{(2)} \otimes h^{(2)} = h_{(1)} \otimes h_{(2)}{}^{(1)} \otimes h_{(2)}{}^{(2)} \tag{3.3.2}$$

$$h_{(1)}{}^{(1)} \otimes h_{(1)}{}^{(2)} \otimes h_{(2)} = h^{(1)} \otimes h^{(2)}{}_{(1)} \otimes h^{(2)}{}_{(2)} \tag{3.3.3}$$

4. Modules & comodules over Hopf algebroids

For a Hopf algebroid (H_L, H_R, S), a right module is just an H_R–module and a left module is an H_L–module. However, since bialgebroids of both handedness have right (and left) *comodules*, a right comodule of a Hopf algebroid should be both a right H_L and a right H_R–comodule in a compatible way. It turns out that this compatibility is automatic, by the following proposition.

Proposition 3.4.1. *(H_L, H_R, S) be a Hopf algebroid and $\langle M, \delta_M \rangle$ a right H_R–comodule. In particular then, $M \in {}_R\mathsf{M}_R$ with $r \cdot m \cdot r' = m^{\langle 0 \rangle} \cdot \varepsilon_R(s_R(r)m^{\langle 1 \rangle}s_R(r'))$. Then M is also an $L - L$–bimodule via*

$$l \cdot m \cdot l' = \varepsilon_R(t_L(l'))\, m\, \varepsilon_R(t_L(l)) \tag{3.4.2}$$

and

(1) *there is a unique H_L–coaction δ_L on M such that δ_L is H_R–colinear and δ_R is H_L–colinear, meaning*

$$(\delta_R \underset{L}{\otimes} H) \circ \delta_L = (M \underset{R}{\otimes} \Delta_L) \circ \delta_R \tag{3.4.3}$$

$$(\delta_L \underset{R}{\otimes} H) \circ \delta_R = (M \underset{L}{\otimes} \Delta_R) \circ \delta_L \tag{3.4.4}$$

and also, the H_L–coinvariants coincide with the H_R–coinvariants.

(2) *any H_R–colinear map $\langle M, \delta_R^M \rangle \to \langle N, \delta_R^N \rangle$ is also an H_L–colinear map with respect to the H_L–coactions defined above*

PROOF (SKETCH). The desired right H_L–coaction is given by

$$\delta_L : M \to M \underset{L}{\otimes} H \tag{3.4.5}$$

$$m \mapsto m^{\langle 0\rangle} \cdot \varepsilon_R({m^{\langle 1\rangle}}_{(1)}) \underset{L}{\otimes} {m^{\langle 1\rangle}}_{(2)} \tag{3.4.6}$$

□

Strictly speaking, this proposition implies that comodules of a Hopf algebroid can be defined in terms of only one of the constituent bialgebroids. Nevertheless, we keep both H_L and H_R, treating them symmetrically.

DEFINITION 3.4.7. A right comodule of a Hopf algebroid (H_L, H_R, S) is a triple $\langle M, \delta_L, \delta_R\rangle$, where $\langle M, \delta_R\rangle$ is a H_R–comodule, $\langle M, \delta_L\rangle$ is a H_L–comodule such that the 3.4.2 relates the $R-R$– and $L-L$–bimodule structures and the compatibility 3.4.3 holds.

We do not discuss the Galois theory of Hopf algebroids, which was worked out in detail in [**6**]. Instead, we turn immediately to a special case, that of Frobenius Hopf algebroids.

5. Frobenius Hopf algebroids and DDAs

A Hopf algebroid possesses two R– and two L–ring structures, via the source and target maps s_R, t_R and s_L, t_L, respectively. In this section we introduce the additional requirement that these four algebra extensions be *Frobenius* extensions. One motivation for this is that even though the (left and right) duals of finitely generated projective bialgebroids are also bialgebroids, this is not known to be true for Hopf algebroids.The Frobenius property, however, is manifestly self–dual and Frobenius Hopf algebroids will dualize to Frobenius Hopf algebroids. Also, the Frobenius property of an algebra extension $R \subseteq A$ can be viewed as a kind of finiteness of A over R. Turning to the theory of depth 2 extensions of algebras, we shall find that the appropriate symmetry object for depth 2 Frobenius extensions is precisely Frobenius Hopf algebroids. After recalling definitions and basic results about Frobenius Hopf algebroids, we shall introduce the equivalent formalism of *Distributive Double Algebras* developed in [**79**] which we shall use in the following.

DEFINITION 3.5.1. An R–ring (A, j) is said to be *Frobenius* if one of the following equivalent conditions hold:

(1) ${}_RA$ is finitely generated projective and ${}_RA \simeq \mathrm{Hom}_{R-}(A, R)$ as $A-R$ modules
(2) A_R is finitely generated projective and $A_R \simeq \mathrm{Hom}_{-R}(A, R)$ as $R-A$ modules
(3) there exists an R–bimodule map $\Psi : A \to R$ (called the Frobenius functional) and a dual basis $\sum_i e_i \underset{R}{\otimes} f_i \in A \underset{R}{\otimes} A$ such that

$$\sum_i e_i \Psi(f_i a) = a = \sum_i \Psi(a e_i) f_i \tag{3.5.2}$$

for all $a \in A$

The classic theorem of Larson and Sweedler states that every finite dimensional Hopf algebra over a field is Frobenius. More generally, the following theorem holds.

THEOREM 3.5.3. *A Hopf algebra H over k is a Frobenius algebra over k if and only if it possesses a non–degenerate integral.*

An left (right) integral *in* a Hopf algebra is an invariant of the left (right) regular module, i.e. $l \in H$ is a left integral if $hl = \varepsilon_H(h)l$. The integral is said to be non–degenerate if the maps $\xi : \hat{H} \to H$, $\varphi \mapsto \varphi \rightharpoonup l$ and $\tilde{\xi} : \hat{H} \to H$, $\varphi \mapsto l \leftharpoonup \varphi$ are bijections (for Hopf algebras, ξ is a bijection if and only if $\tilde{\xi}$ is). On the other hand a left (right) integral *on* a Hopf algebra H is an invariant of the dual left (right) regular module, i.e. an element of $\hat{H}$. For a left integral $\lambda \in \hat{H}$ on H, $\lambda \rightharpoonup h = 1\,\lambda(h)$.

REMARK 3.5.4. The general idea behind the double algebraic formalism can be explained in the restricted context of Frobenius Hopf algebras (it is discussed in more detail under 8.9. of [**79**]). A Frobenius Hopf algebra is a Hopf algebra with a Frobenius left integral, i.e. a left integral $i \in H$ such that the mapping $\hat{H} \to k$, $\varphi \mapsto \varphi(i)$ is a Frobenius homomorphism on the dual Hopf algebra $\hat{H}$. This is equivalent to H being a Frobenius algebra with a Frobenius homomorphism $\lambda \in \hat{H}$ which is a left integral (the dual integral) *on* H. The dual integrals are connected by the duality relation $\lambda \rightharpoonup i = 1$, or equivalently, $i \rightharpoonup \lambda = \varepsilon$. There is also a dual pair of right integrals, afforded by the antipode, $j = S(i)$ and $\rho = \lambda \circ S^{-1}$. The four integrals satisfy the following relations:

Left	Right
$hi = \varepsilon(h)i$	$jh = j\varepsilon(h)$
$ih = i\sigma(h)$	$hj = \tau(h)j$
$\lambda \rightharpoonup h = 1\,\lambda(h)$	$h \leftharpoonup \rho = 1\,\rho(h)$
$\lambda(i) = 1$	$\rho(j) = 1$

FIGURE 1. Left and Right Frobenius integrals

where $\sigma = \lambda \leftharpoonup i$ is the 'distinguished grouplike element' and $\tau = \sigma \circ S^{-1}$. Furthermore, λ is a Frobenius homomorphism with dual basis $i_{(2)} \otimes S^{-1}(i_{(1)})$ and ρ is a Frobenius homorphism with dual basis $i_{(1)} \otimes S(i_{(2)})$.

Using the dual pair of integrals, we define the Fourier transform $\mathcal{F} : H \to \hat{H}$ with $\mathcal{F}(h) = h \rightharpoonup \lambda$ and the inverse transform $\mathcal{F}^{-1} : \hat{H} \to H$ with $\mathcal{F}^{-1}(\psi) = i \rightharpoonup \psi \circ S^{-1}$. The convolution product $h \star h' = \mathcal{F}^{-1}(\mathcal{F}(h)\mathcal{F}(h'))$ transfers the algebra structure of $\hat{H}$ (the transpose of the coalgebra structure of H) to H. The unit for the new algebra structure is the left integral i. Thus, a Frobenius Hopf algebra admits an equivalent discription in terms of two multiplicative structures $\langle H, \cdot, 1, \star, i\rangle$. Multiplication with the wrong unit produces 4 Frobenius homomorphisms, $\varphi_L(h) = h \star 1 = 1\,\lambda(h)$, $\varphi_R(h) = 1 \star h = 1\,\rho(h)$, $\varphi_B(h) = h \cdot i = \varepsilon(h)i$ and $\varphi_T(h) = i \cdot h = i\sigma(h)$.

A general double algebra will be a Frobenius algebra over four isomorphic base algebras, which are however, in general noncommutative rings. The double algebra derived from a Frobenius Hopf algebra as above is, in addition, a distributive double algebra: the two multiplicative structures satisfy compatibilities of which we list only one, $a \cdot (a' \star a'') = a_{(1)}a' \otimes a_{(2)}a''$.

We recall the Double Algebra formalism based on [**3**] (see also the original reference, [**79**]). A double algebra is a k–module A equipped with two associative algebra structures denoted $H = \langle A, \star, i\rangle$ (the horizontal algebra) and $V = \langle A, \circ, e\rangle$, respectively. The horizontal and vertical algebras satisfy 8 compatibility axioms ([**79**], Def. 1.1). The axioms ensure that multiplication on the left (right) with the wrong units maps onto four subalgebras of A, as follows

$$\varphi_L : A \to L,\ a \mapsto a \star e \qquad \varphi_R : A \to R,\ a \mapsto e \star a \tag{3.5.5}$$

$$\varphi_B : A \to B,\ a \mapsto a \circ i \qquad \varphi_T : A \to T,\ a \mapsto i \circ a \tag{3.5.6}$$

The four subalgebras turn out to be strongly related, namely $L \simeq B \simeq R^{op} \simeq T^{op}$. In terms of the maps φ_X, the defining axioms of a double algebra take on the following form:

$$\begin{array}{llll} (A1) & \varphi_L(a) \circ b = \varphi_B\varphi_L(a) \star b & (A5) & a \circ \varphi_R(b) = a \star \varphi_T\varphi_R(b) \\ (A2) & a \circ \varphi_L(b) = \varphi_T\varphi_L(b) \star a & (A6) & \varphi_R(a) \circ b = b \star \varphi_B\varphi_R(a) \\ (A3) & \varphi_B(a) \star b = \varphi_L\varphi_B(a) \star b & (A7) & a \star \varphi_T(b) = a \circ \varphi_R\varphi_T(b) \\ (A4) & a \star \varphi_B(b) = \varphi_R\varphi_B(b) \star a & (A8) & \varphi_T(a) \star b = b \circ \varphi_L\varphi_T(a) \end{array}$$

A double algebra is a *Frobenius double algebra* if all four algebra extensions $L, R, T, B \subseteq A$ are Frobenius. As mentioned above, a Frobenius algebra extension $X \subseteq A$ implies an X coring structure on A, with comultiplication given by the Frobenius dual bases. The four Frobenius algebra structures on a double algebra, in the bimodule categories ${}_X\mathsf{M}_X$ for $X = L, R, B, T$, respectively, imply four comultiplications:

$\langle A, \Delta_B, \varphi_B\rangle$ is a comonoid in ${}_B\mathsf{M}_B$, where $\Delta_B(a) \equiv a_{(1)} \underset{B}{\otimes} a_{(2)} = a \star u_k \underset{B}{\otimes} v_k$,
$\langle A, \Delta_L, \varphi_L\rangle$ is a comonoid in ${}_L\mathsf{M}_L$, where $\Delta_L(a) \equiv a_{[1]} \underset{L}{\otimes} a_{[2]} = a \circ x_j \underset{L}{\otimes} y_j$,
$\langle A, \Delta_T, \varphi_T\rangle$ is a comonoid in ${}_T\mathsf{M}_T$, where $\Delta_T(a) \equiv a^{(1)} \underset{T}{\otimes} a^{(2)} = a \star u^k \underset{B}{\otimes} v^k$,
$\langle A, \Delta_R, \varphi_R\rangle$ is a comonoid in ${}_R\mathsf{M}_R$, where $\Delta_R(a) \equiv a^{[1]} \underset{R}{\otimes} a^{[2]} = a \circ x^j \underset{R}{\otimes} y^j$,

where we have introduced special notations for the dual bases of the base homomorphisms φ_X, and variants of the Sweedler notation to distinguish between the different comultiplications. The coalgebra structure arising from a Frobenius algebra satisfies a markedly different compatibility condition with multiplication than we expect from a quantum groupoid, specifically, it is not multiplicative. However, we can recover Hopf algebroids from a Frobenius double algebra if we also postulate the following *distributivity* rules:

(3.5.7) $$a \circ (a' \star a'') = (a_{(1)} \circ a') \star (a_{(2)} \circ a'')$$

(3.5.8) $$a \star (a' \circ a'') = (a_{[1]} \star a') \circ (a_{[2]} \star a'')$$

(3.5.9) $$(a' \star a'') \circ a = (a' \circ a^{(1)}) \star (a'' \circ a^{(2)})$$

(3.5.10) $$(a' \circ a'') \star a = (a' \star a^{[1]}) \circ (a'' \star a^{[2]})$$

In this case we say that $\langle A, \circ, e, \star, i\rangle$ is a distributive double algebra (DDA for short). In [**79**] it is shown that in a DDA, the vertical and horizontal algebras, V and H, are Hopf algebroids such that H^{op} is the dual of V. We will sometimes refer to the vertical and horizontal Hopf algebroids of A. The constituent bialgebroids of H are

(3.5.11) $$\langle H, L, \varphi_{B|L}, \varphi_{T|L}, \Delta_L, \varphi_L\rangle \quad \text{and} \quad \langle H, R, \varphi_{T|R}, \varphi_{B|R}, \Delta_R, \varphi_R\rangle$$

and those of V are

(3.5.12) $$\langle V, B, \varphi_{L|B}, \varphi_{R|B}, \Delta_B, \varphi_B\rangle \quad \text{and} \quad \langle V, T, \varphi_{R|T}, \varphi_{L|T}, \Delta_T, \varphi_T\rangle$$

The notation means e.g. that V over T has source map $t \mapsto \varphi_R(t)$, target map $t \mapsto \varphi_L(t)$ and counit φ_T. Or, H over R has source map $r \mapsto \varphi_T(r)$, target map $r \mapsto \varphi_B(r)$, and counit φ_R. The antipode of V – called the antipode of the double algebra – is an anti-automorphism S which is also an anti-automorphism of H but the antipode of H is S^{-1}. It is given by

(3.5.13) $$S(a) = u^k \star \varphi_T(\varphi_L(a \circ v^k)).$$

The unit of the horizontal Hopf algebroid i is a Frobenius integral in the vertical Hopf algebroid V and *vice versa*, e is a Frobenius integral in the horizontal Hopf algebroid H.

6. Modules & comodules over DDAs

We have defined modules and comodules over Hopf algebroids to be consistent with the fact that a Hopf algebroid 'contains' two constituent bialgebroids. We have to do the same with DDA's and define modules and comodules as those of the underlying Hopf algebroids in a consistent manner.

6.1. Modules over DDAs. Let $\langle A, \circ, e, \star, i\rangle$ be a double algebra. A *right* A-module is a right module over the horizontal *right* bialgebroid H over R, i.e. a k-module M together with an associative unital action $M \underset{R}{\otimes} H \to M$ of the horizontal algebra $H = \langle A, \star, i\rangle$ denoted $m \underset{R}{\otimes} h \mapsto m \triangleleft h$. A right A-module morphism $f : X \to Y$ is a right R–module map which is a right module morphism for the bialgebroid H over R. The category of right A-modules will be denoted M_H.

A *left* A-module is a left module over the horizontal *left* bialgebroid H over L. Using the extra structure of a DDA, we could also define 'top' and 'bottom' modules over A as left and right modules over the vertical algebra $V = \langle V, \circ, e\rangle$. In terms of the constituent bialgebroids, these are left modules over the left bialgebroid V over B and right modules over the right bialgebroid V over T, respectively.

6.2. Comodules over DDAs. A *right* A-comodule over a Frobenius double algebra is a right comodule over the vertical Hopf algebroid. It consists of an object M and two arrows $\delta_M\colon M \to M \underset{B}{\otimes} A$, $\delta^M\colon M \to M \underset{T}{\otimes} A$ in $\mathsf{M}_{B\otimes T}$ such that

- $\langle M_B, \delta_M\rangle$ is a right comodule over the left bialgebroid V over B,
- $\langle M_T, \delta^M\rangle$ is a right comodule over the right bialgebroid V over T
- such that the two coactions satisfy the mixed coassociativity conditions

$$m^{(0)}{}_{(0)} \underset{B}{\otimes} m^{(0)}{}_{(1)} \underset{T}{\otimes} m^{(1)} = m_{(0)} \underset{B}{\otimes} m_{(1)}{}^{(1)} \underset{T}{\otimes} m_{(1)}{}^{(2)} \tag{3.6.1}$$

$$m_{(0)}{}^{(0)} \underset{T}{\otimes} m_{(0)}{}^{(1)} \underset{B}{\otimes} m_{(1)} = m^{(0)} \underset{T}{\otimes} m^{(1)}{}_{(1)} \underset{B}{\otimes} m^{(1)}{}_{(2)} \tag{3.6.2}$$

where we used the notation

$$\delta_M(m) = m_{(0)} \underset{B}{\otimes} m_{(1)}$$
$$\delta^M(m) = m^{(0)} \underset{T}{\otimes} m^{(1)}$$

for $m \in M$.

A *right A-comodule morphism* $\tau : X \to Y$ is a right $B \otimes T$-module map which is a right comodule morphism for both the left bialgebroid V_B and the right bialgebroid V_T. The category of right A-comodules is denoted by M^V.

We saw that for Hopf algebroid comodules, mixed coassociativity of the coactions of the underlying bialgebroids leads to an identification of their comodule categories. It turns out that for a DDA A, further identification is possible, namely of the comodule category M^V and the module category M_H.

Lemma 3.6.3. *Let A be a DDA and let δ_M and δ^M be two coactions of V_B, respectively V_T, on M. They then determine two right H-actions on M,*

$$m \underset{B}{\triangleleft} h = m_{(0)} \,.\, \varphi_B(m_{(1)} \star h) \tag{3.6.4}$$

$$m \underset{T}{\triangleleft} h = m^{(0)} \cdot \varphi_T(m^{(1)} \star h)\,. \tag{3.6.5}$$

The two actions coincide if and only if the two coactions satisfy the mixed coassociativity condition (3.6.1) and (3.6.2).

Proof. The inverses of (3.6.4) and (3.6.5) can be given in terms of the dual bases of φ_B and φ_T as

$$m^{(0)} \underset{T}{\otimes} m^{(1)} = m \underset{T}{\triangleleft} u^k \underset{T}{\otimes} v^k \tag{3.6.6}$$

$$m_{(0)} \underset{B}{\otimes} m_{(1)} = m \underset{B}{\triangleleft} u_k \underset{T}{\otimes} v_k \tag{3.6.7}$$

Therefore if $\underset{B}{\triangleleft} = \underset{T}{\triangleleft}$ then

$$\begin{aligned} m^{(0)}{}_{(0)} \underset{B}{\otimes} m^{(0)}{}_{(1)} \underset{T}{\otimes} m^{(1)} &= (m \underset{T}{\triangleleft} u^k) \underset{B}{\triangleleft} u_l \underset{B}{\otimes} v_l \underset{T}{\otimes} v^k = \\ &= m \underset{B}{\triangleleft} (u^k \star u_l) \underset{B}{\otimes} v_l \underset{T}{\otimes} v^k = m \underset{B}{\triangleleft} u_l \underset{B}{\otimes} v_l \star u^k \underset{T}{\otimes} v^k = \\ &= m_{(0)} \underset{B}{\otimes} m_{(1)}{}^{(1)} \underset{T}{\otimes} m_{(1)}{}^{(2)}. \end{aligned}$$

and similarly for (3.6.2). On the other hand, if mixed coassociativity holds then

$$\begin{aligned} m \underset{T}{\triangleleft} h = (m \underset{T}{\triangleleft} h)_{(0)} \cdot \varphi_B((m \underset{T}{\triangleleft} h)_{(1)}) &= m^{(0)}{}_{(0)} \cdot \varphi_B(m^{(0)}{}_{(1)} \star \varphi_T(m^{(1)} \star h)) \\ &= m_{(0)} \cdot \varphi_B(m_{(1)}{}^{(1)} \star \varphi_T(m_{(1)}{}^{(2)} \star h)) = m_{(0)} \cdot \varphi_B(m_{(1)} \star h) \\ &= m \underset{B}{\triangleleft} h. \end{aligned}$$

□

If M is a right module over the DDA A then it is a right V-comodule M^V and a right H-module M_H at the same time. The invariants of M_H,

$$M^H := \{n \in M | n \triangleleft h = n \triangleleft \varphi_T\varphi_R(h),\ h \in H\} \tag{3.6.8}$$

$$= \{n \in M | n \triangleleft h = n \triangleleft \varphi_B\varphi_R(h),\ h \in H\} \tag{3.6.9}$$

and the coinvariants of M^V,

$$\begin{aligned} M^{\text{co-}V} &:= \{n \in M | n^{(0)} \underset{T}{\otimes} n^{(1)} = n \underset{T}{\otimes} e\} \\ &= \{n \in M | n_{(0)} \underset{B}{\otimes} n_{(1)} = n \underset{B}{\otimes} e\}, \end{aligned} \tag{3.6.10}$$

yield one and the same k-submodule of M. This is an instance of the more general identification between the categories of H-modules, V_B-comodules, and V_T-comodules. Since φ_T and φ_B restrict to algebra isomorphisms $R \to T$ and $R^{\mathrm{op}} \to B$, respectively [**79**, Lemma 2.2], the identifications between H-modules and V-comodules provide a monoidal category isomorphism $\mathsf{M}^{V_T} \cong \mathsf{M}_H$ and the antimonoidal category isomorphism $\mathsf{M}^{V_B} \cong \mathsf{M}_H$. We can use these isomorphisms to introduce $\underset{R}{\otimes}$ both in M^{V_T} and M^{V_B} as the monoidal product while keeping $\underset{T}{\otimes}$ and $\underset{B}{\otimes}$ to appear in the coactions. One advantage of this convention is that the difference between (3.2.9) and (3.2.7) disappears, and we can define the monoidal product of both V_B– and V_T–comodules with the same order of multiplication. Now the R becomes a monoidal unit in three senses: As a right ideal in H it is the trivial right H-module, $r \triangleleft h = r \star h$. But it is also a right comodule over V_T via $r^{(0)} \underset{T}{\otimes} r^{(1)} = e \underset{T}{\otimes} r$ and a right comodule over V_B via $r_{(0)} \underset{B}{\otimes} r_{(1)} = e \underset{B}{\otimes} r$.

CHAPTER 4

Galois theory

We begin by recalling the most important results from the classical Galois theory of field extensions in a nutshell. Noncommutative Galois theory is largely self–contained so this is meant only as an aside, to give an idea of the sorts of results one could expect in a more general theory. Hopf Galois theory is understood to mean the theory of noncommutative algebra extensions with Hopf algebra symmetry, and has by now also achieved classical status. Fundamental references are [**28**], [**49**], [**76**]; a review, containing further references, is [**74**]. It is well established that Hopf Galois theory may be regarded as the dual picture to principal bundles in the sense of noncommutative geometry. Consequently, it has geometric applications and geometry, in turn, has been a constant source of motivation for Hopf Galois theory. We shall give some references on the noncommutative geometric view, but our concern here is the generalization of Hopf Galois theory from Hopf algebras to quantum groupoids and ultimately, the application of this theory to algebra extensions coming from Algebraic Quantum Field Theory.

Consider an extension of fields $E : F$ ('E over F') which is in particular an extension of the additive and multiplicative groups of the respective fields, hence E and F share the same unit, zero and characteristic. The extended field E may be considered as a vector space over F (the module E_F) and the F–dimension of this vector space is called the degree of the extension, $[E : F] = \dim E_F$. Standard examples of Galois extensions are:

- $\mathbf{Q}(\sqrt{2})|\mathbf{Q}$, where $\mathbf{Q}(\sqrt{2}) = \{a + b\sqrt{2}\,|\,a, b \in \mathbf{Q}\}$. The degree is 2, since $\{1, \sqrt{2}\}$ are a basis for E_F.
- $\mathbf{C}|\mathbf{R}$ (the complex numbers over the reals), $[\mathbf{C} : \mathbf{R}] = 2$ with $\{1, i\}$ a basis
- for any field K, denote $K[X]$ the polynomial ring with variable X and coefficients in K. For an irreducible polynomial $f(X) \in K[X]$, the roots of f may not lie in K. However, $K[X]/\langle f(X)\rangle$ (the polynomial ring modded by the ideal generated by the polynomial) is an extension field of K and contains the roots of f. We say that we have added the roots of f to K. This example subsumes the previous two: simply take the polynomials $X^2 = 2$ and $X^2 = -1$ in $\mathbf{Q}[X]$ and $\mathbf{R}[X]$, respectively
- for an algebraic variety V, the field of rational functions on V, denoted $K(V)$ is an extension of K

Elements of E that are roots of polynomials in $F[X]$ are called algebraic. An extension such that all elements of E not in F are algebraic, is called an algebraic extension. An element of the extended field that is not a root is called transcendental, and non–algebraic extensions are called transcendental extensions (purely transcendental if there are no algebraic elements at all in $E\ F$). The smallest extension of a field F which contains all roots of polynomials in $F[X]$ is called the algebraic closure. Algebraicity may seem like a special property to require (important and historical examples notwithstanding), but in fact, an extension is algebraic if and only if it is the union of finite sub–extensions. This is corroborated by our examples: the finite–index extensions we listed are indeed algebraic, the rest are transcendental.

The fundamental construction in Galois theory is the automorphism group $\mathrm{Aut}(E|F)$, consisiting of those automorphisms of E, which fix F point–wise, i.e. for $\alpha \in \mathrm{Aut}(E|F)$, $\alpha(x) = x$ for all $x \in F$. Note that for a field, it only makes sense to talk about automorphisms, i.e. endomorphisms with inverse.

Theorem & Definition 4.0.11. *A finite field extension $E\,|\,F$ is Galois if any of the following hold*

- *$E\,|\,F$ is a normal extension and a separable extension*
- *E is the splitting field of a separable polynomial with coefficients in F*
- *$[E : F] = |\,\mathrm{Aut}(E|F)|$, i.e. the degree of the field extension is equal to the order of the automorphism group of $E|F$.*

A field extension is called *normal* if every irreducible polynomial in $F[X]$ that has a root in L completely factors into linear factors over L. Every algebraic extension $E|F$ admits a normal closure L, which is an extension field of F such that $L|F$ is normal and which is minimal with this property. An algebraic extension $E|F$ is called *separable*, if the minimal polynomial of every element of E over F is separable, i.e. it has no repeated roots in E.

If any one of the above conditions hold, $G = \mathrm{Aut}(E|F)$ is said to be the *Galois group* of the extension $E|F$, usually denoted $\mathsf{Gal}(E|F)$. If it exists, it is unique. The *Fundamental Theorem of Galois Theory* states:

Theorem 4.0.12. *There is a one-to-one relationship between subgroups of $H \leq G = \mathrm{Aut}(E|F)$ and intermediate extensions $F \subseteq K \subseteq E$, as follows:*

- *$H \mapsto \mathsf{Fix}(H)$; to any subgroup $H \leq \mathsf{Gal}(E|F)$, associate the intermediate field $\mathsf{Fix}(H) = E^H$ consisting of the elements of E fixed by H,*
- *$K \mapsto \mathrm{Aut}(E|K)$; to any intermediate field K of $E|F$, the corresponding subgroup is just $\mathrm{Aut}(E|K)$, the set of those automorphisms in Gal$(E|F)$ which fix every element of K.*

This bijection between subextensions and subfields of the Galois group constitute an *order–reversing Galois connection*[1] between the lattices of subgroups of $\mathsf{Gal}(E|F)$ and subextensions of $E|F$. It is clearly order–reversing: for subgroups, $H_1 \subseteq H_2$ holds if and only if the inclusion of fields $E^{H_1} \supseteq E^{H_2}$ holds.

[1] In abstract terms, an order–reversing Galois connection between posets $(A, \leq)$ and $(B, \leq)$ is a pair of maps $F : A \to B$ and $G : B \to A$ such that $b \leq F(a)$ if and only if $a \leq G(b)$. In categorical terms, it is an adjoint pair of contravariant functors between the posets A and B.

The degrees of extensions are related to orders of groups, in a manner consistent with the inclusion–reversing property. Specifically, if H is a subgroup of $\mathsf{Gal}(E|F)$, then $|H| = [E|E^H]$ and $[\mathsf{Gal}(E|F) : H] = [E^H : F]$. The field E^H is a normal extension of F if and only if H is a normal subgroup of $\mathsf{Gal}(E|F)$. In this case, the restriction of the elements of $\mathsf{Gal}(E|F)$ to E^H induces an isomorphism between $\mathsf{Gal}(E^H|F)$ and the quotient group $\mathsf{Gal}(E|F)/H$.

1. Hopf Galois theory

The history of Hopf Galois theory goes back to the work of Chase, Harrison and Rosenberg on the Galois theory of groups acting not on fields, but on commutative rings ([**27**]). This was extended, by Chase and Sweedler ([**28**]), to Hopf algebras coacting on commutative algebras over (commutative) rings. The general definition, featuring Hopf algebras coacting on noncommutative rings appeared in the work of Kreimer and Takeuchi ([**49**]). This approach is quite different in flavour from the previous summary of the Galois theory of fields, and the abstract definition is best understood through the classic examples.

First, a few preliminary definitions. A (right) H–comodule algebra A is a monoid in M^H, i.e. for $a, b \in A$, $(ab)_{\langle 0\rangle} \otimes (ab)_{\langle 1\rangle} = a_{\langle 0\rangle} b_{\langle 0\rangle} \otimes a_{\langle 1\rangle} b_{\langle 1\rangle}$ and $(1_A)_{\langle 0\rangle} \otimes (1_A)_{\langle 1\rangle} = 1_A \otimes 1_H$. We define the subalgebra of *coinvariants* $A^{co-H} := \{a \in A \,|\, a_{\langle 0\rangle} \otimes a_{\langle 1\rangle} = a \otimes 1_H\} \subseteq A$.

Definition 4.1.1. Let H be a Hopf algebra and A an H–comodule algebra in M^H with coaction $\rho : A \to A \otimes H$. Then the extension of algebras $A^{co-H} \subseteq A$ is said to be right H–Galois if the map

$$\begin{aligned} \beta : A \otimes_{A^{co-H}} A &\to A \otimes H \\ a \otimes b &\mapsto (a \otimes 1)\, \rho(b) = ab_{\langle 0\rangle} \otimes b_{\langle 1\rangle} \end{aligned} \tag{4.1.2}$$

is bijective.

The first question to consider is how this definition relates to classical field extensions?

Example 4.1.3. Consider the classical setup of a finite group G acting on a field $E \supset k$ as automorphisms with $F = E^G$ the invariant subfield. The action of G extends linearly to the group algebra kG, inducing a coaction of dual $H = k(G) = (kG)^*$ on E. We apply the definition 4.1.2 to H.

$E|F$ is classically Galois with Galois group G if G acts faithfully on E, or equivalently, $[E : F] = |G|$. Let $|G| = n$ and denote $\{x_1, x_2, ..., x_n\}$ the elements of G and $\{b_1, b_2, ..., b_n\}$ the basis of E over F. If $\{p^1, p^2, ..., p^n\}$ is the dual basis of $\{x_i\}_{i=1}^n$ in $k(G)$ (i.e. $p^i(x_j) = \delta_{i,j}$), then the $k(G)$–coaction corresponding to the kG–action on E is given by

$$\begin{aligned} \rho : E &\to E \underset{k}{\otimes} k(G) \\ a &\mapsto \sum_{i=1}^{n} x_i \triangleright a \otimes p^i. \end{aligned} \tag{4.1.4}$$

The Galois map $\beta : E \underset{F}{\otimes} E \to E \otimes k(G)$ is then $\beta(a \otimes b) = \sum_i a(x_i \triangleright b) \otimes p^i$.

EXAMPLE 4.1.5. An important example which does not involve fields is that of graded rings. Let A be a G–graded algebra, i.e. $A = \oplus_{g\in G} A_g$ is a G–graded vector space and for all $g, h \in G$, $A_g A_h \subseteq A_{gh}$ and $1_A \in A_e$. Equivalently, A is a kG–comodule algebra with the coaction $\rho(a_g) = a_g \otimes g$ for $a_g \in A_g$ and extended linearly. Indeed, $\rho(a_g b_h) = a_g b_h \otimes gh$ so that $a_g b_h \in A_{gh}$ and $\rho(1_A) = 1_A \otimes e$ implies $1_A \in A_e$. The kG–coinvariants are precisely the unit component, $A^{co-H} = A_e$. A theorem of Ulbrich describes those extensions $A_e \subseteq A$ which are Galois.

THEOREM 4.1.6. *Let A be a G–graded algebra; then the following are equivalent:*

(1) *$A_e \subseteq A$ is kG–Galois*
(2) *A is* strongly *graded, meaning that $A_x A_y = A_{xy}$ for all $x, y \in G$.*

See [**58**], p. 126 for a proof.

EXAMPLE 4.1.7. A more geometric example is (the dual of) that of a finite group acting on a set. Consider a finite group G acting on a set X from the right by the action $\mu : X \times G \to X$, denoted $(x, g) \mapsto x \triangleleft g$. There is a natural map

$$\begin{aligned} \alpha : X \times G &\to X \times X \\ (x, g) &\mapsto (x, x \triangleleft g), \end{aligned} \tag{4.1.8}$$

often called the *fundamental map* which encodes several properties of the action. To begin with, α maps $X \times G$ to $X \times_Y X$, the fibered product[2] of X with itself over the orbit space $Y = X/G$ (since x and $x \triangleleft g$ are in the same G–orbit for any $x \in X$ and $g \in G$). The surjectivity of α, considered as a map to $X \times_Y X$, is equivalent to the action being transitive; α is injective if and only if μ is free, i.e. there are no fixed points $x \neq x \triangleleft g$ for $g \neq e$.

The connection to Definition 4.1.1 is revealed by considering the dual picture. Denote $A = k(X)$ the algebra of k–valued functions on X under pointwise addition and multiplication. The right G–action on X induces a left kG–action on $k(X)$; for $\alpha \in k(X)$, $g \triangleright \alpha(x) = \alpha(x \triangleleft g)$. This, in turn, induces a right $k(G)$–coaction $\mu^* : A \to A \otimes k(G)$. With the dual bases $\{x_i, p^i\}_{i=1}^n$, the coaction is given by $\mu^* : \alpha \mapsto x_i \triangleright \alpha \otimes p^i$. Denote $B = k(Y)$ the functions from X to k that are constant on G–orbits. Then $B = A^{kG} = A^{co-k(G)}$ is the subalgebra of kG–invariants, or equivalently, of the $k(G)$–coinvariants. Since $k(X \times X) = k(X) \otimes k(X)$, the fundamental map dualizes to

$$\begin{aligned} \alpha^* : A \underset{B}{\otimes} A &\to A \otimes H \\ a \otimes a' &\mapsto (a \otimes 1)\mu^*(a'), \end{aligned} \tag{4.1.9}$$

which is exactly the Galois map of 4.1.2, i.e. $\beta = \alpha^*$. The Galois map β is bijective if and only if α is bijective, which is equivalent to the action μ being free and transitive.

[2]In the present case, $X \times_Y X$ is the categorical pullback of two copies of the projection map $\pi : X \to X/G$, sending each point $x \in X$ to the orbit it is contained in. It is the subset $X \times_Y X = \{(x, y) \in X \times X | \pi(x) = \pi(y)\}$

EXAMPLE 4.1.10. A truly geometric example is that of *quantum pricipal bundles*. In classical geometry, a principal bundle $P(M,G)$ over base M with structure (gauge) group G is defined as follows. P is a smooth manifold and G a Lie group such that there is a smooth right action of G on P, $P \times G \to P$, $(u,g) \mapsto R_g(u) = u \triangleright g$ which is *free*. Just as in the case of finite groups acting on sets, this is equivalent to the injectivity of the map

$$\beta : P \times G \to P \times P, \quad (u,g) \mapsto (u, u \triangleright g). \tag{4.1.11}$$

The orbit space is isomorphic to the base space, $P/G \cong M$, and the canonical projection $\pi : P \to M$ is a smooth map. The bundle P is locally isomorphic to $M \times G$, meaning that for an open subset $U \subseteq M$ contained in one chart, there is a map $\phi_U : \pi^{-1}(U) \to G$ such that the map $\pi^{-1}(U) \to M \times G$, $u \mapsto (\phi_U(u), \pi(u))$ is an isomorphism.

By the standard practice of noncommutative geometry, the algebraic picture is obtained by considering the algebra of functions on the geometric spaces. Hence, a quantum principle bundle is first of all an H–comodule algebra A with coaction $\rho : A \to A \otimes H$ such that the induced Galois map $(\mu_A \otimes H) \circ (A \otimes \rho) : A \otimes A \to A \otimes H$ is a surjection. Denote the subalgebra of coinvariants $B = A^{co-H}$, then the Galois map factorizes through $A \underset{B}{\otimes} A$ and $\beta : A \underset{B}{\otimes} A \to A \otimes H$ is an isomorphism. An example is provided by a classical principle bundle with $A = k(P)$, $B = k(M)$ and $H = k(G)$. The inclusion of the subalgebra $B \hookrightarrow A$ is the analogue of the projection $\pi : P \to M$ in the classical case.

Much work has been done to elaborate this connection between (principal) bundles and Hopf Galois extensions, we only point out a handful of references. A differential calculus on quantum groups (in the sense of compact matrix pseudogroups) has been developed in [**92**], generalizing the calculus of exterior differential forms; differential calculi on noncommutative bundles were considered in [**67**]. Approaching principles bundles from a quantum field theory point of view, a quantum group gauge theory over noncommutative spaces was proposed in [**18**]. Cleft extensions are roughly the analogues of trivial bundles. Their study was initiated [**31**] and [**5**]. A topological invariant of bundles, the Chern character, have been given an algebraic interpretation in [**15**]. Also, concrete geometric constructions like the Hopf fibration also have algebraic counterparts, interpreted as noncommutative monopoles or instantons, see e.g. [**19**], [**20**].

Comparing Hopf Galois theory to the classical Galois theory of field extensions, two differences stand out. First, classical Galois extensions can be characterized as the normal, separable field extensions without explicitly mentioning the Galois group. No analogue result is known in Hopf Galois theory. Also, a Galois field extension determines the Galois group uniquely. A stronger statement is the Fundamental Theorem of Galois theory, which does not hold in the Hopf Galois setting. Instead, there is a weaker result due to Chase and Sweedler ([**28**]):

THEOREM 4.1.12. *Let $K|k$ be an H–Galois extension; for a Hopf subalgebra H', define* $\mathsf{Fix}(H') = \{x \in K \mid \forall h \in H' : \; h \triangleleft x = \varepsilon(h)x\}$. *Then the mapping from Hopf subalgebras to* *intermediate subalgebras*

$$\mathsf{Fix} : \{H' \subseteq H \,|\, H'\, Hopf subalgebra\} \to \{L \,|\, k \subseteq L \subseteq K subalgebra\}$$

is inclusion–reversing and injective.

In Hopf Galois theory, explicit counterexamples are known (to be discussed in Chapter 3), where a Hopf Galois extension exhibits two non–isomorphic Hopf algebras both of which make the extension Galois– little is known in general about the relation of different possible choices of Hopf Galois symmetry. Both of the above shortcomings of Hopf Galois theory can be partially remedied by allowing quantum groupoids instead of Hopf algebras. It is the Galois theory of quantum groupoids which take up in the remaining chapters.

2. Bialgebroid Galois extensions

It was shown in Chapter 1 that for a right bialgebroid $\langle B, s_R, t_R \rangle$ over R, the categories M_B, M^B and ${}^B\mathsf{M}$ are monoidal with a strict monoidal forgetful functor into ${}_R\mathsf{M}_R$. This allows us to define both (right) B–module and B–comodule algebras as monoids in M_B and M^B, respectively. This leads to two notions of bialgebroid extension and bialgebroid–Galois extension, accordingly called the action– and coaction pictures.

2.1. Module– and comodule algebras.

Definition 4.2.1. A right B–module algebra is a monoid $\langle M, \mu_M, \eta_M \rangle$ in M_B, i.e. $M \in \mathsf{M}_B$, with μ and η arrows in M_B (B–comodule maps):

$$\begin{array}{ccc} (M \underset{R}{\otimes} M) \underset{R}{\otimes} B & \xrightarrow{\gamma_{M \otimes M}} & M \underset{R}{\otimes} M \\ \downarrow{\scriptstyle \mu_M \underset{R}{\otimes} B} & & \downarrow{\scriptstyle \mu_M} \\ M \underset{R}{\otimes} B & \xrightarrow[\gamma_M]{} & M \end{array} \qquad \begin{array}{ccc} R & \xleftarrow{\gamma_R} & R \underset{R}{\otimes} B \\ \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \eta_M \underset{R}{\otimes} B} \\ M & \xleftarrow[\gamma_M]{} & M \underset{R}{\otimes} B \end{array}$$

or, in terms of formulæ,

(4.2.2) $$(m \triangleleft b_{(1)})(m' \triangleleft b_{(2)}) = (mm') \triangleleft b$$

(4.2.3) $$1_M \triangleleft b = 1_M \triangleleft s(\varepsilon(b))$$

The strict monoidal forgetful functor on $U : \mathsf{M}_B \to {}_R\mathsf{M}_R$ induces on M the R–bimodule structure

(4.2.4) $$M \in {}_R\mathsf{M}_R, \quad r \cdot m \cdot r' = m \triangleleft (t(r)s(r'))$$

The subring of *invariants* is denoted M^B and is defined as

(4.2.5) $$M^B = \{n \in M \,|\, n \triangleleft b = n \triangleleft s(\varepsilon(b))\}$$

A useful characterization of the invariants is the following

LEMMA 4.2.6. *For a right B–module algebra M,*

$$\mathrm{Hom}_{-B}(R,M) \simeq M^B$$

as k–algebras, where the algebra structure on $\mathrm{Hom}_{-B}(R,M)$ *is the convolution algebra.*

PROOF. It is easily checked that the k–module maps

$$\mathrm{Hom}_{-B}(R,M) \to M^H, \qquad \varphi \mapsto \varphi(1_R)$$

and

$$M^H \to \mathrm{Hom}_{-B}(R,M), \qquad n \mapsto \{r \mapsto n \cdot r\}$$

are inverses of each other. Being the monoidal unit in M_B, R has a coalgebra structure $R \to R \underset{R}{\otimes} R$ given by the natural isomorphism $r \mapsto r \otimes 1_R$. The corresponding convolution product on $\mathrm{Hom}_{-B}(R,M)$ is given by $\varphi\psi(r) = \varphi(r_{(1)})\psi(r_{(2)}) = \varphi(r)\psi(1_R)$. The isomorphism extends to an isomorphism of algebras. □

For any right B–module algebra M, $B \underset{R}{\otimes} M$ has an M–ring structure, the *smash product*, denoted $B\#M$. Multiplication is given by

$$\hat{\mu} : B\#M \underset{M}{\otimes} B\#M \to B\#M$$
$$(b\#m)(b'\#m') = bb'_{(1)}\#(m \triangleleft b'_{(2)})m'$$

and the unit map is $\hat{\eta} : M \to B\#M$, $m \to 1_B \underset{R}{\otimes} m$.

All the above constructions may be applied to a *left* bialgebroid $\langle B, s_L, t_L\rangle$ over L. In particular, a left B–module M is a monoid $\langle M, \mu_M, \eta_M\rangle$ in ${}_B\mathsf{M}$. The multiplication and unit maps are arrows in ${}_B\mathsf{M}$,

$$(b_{(1)} \triangleright m)(b_{(2)} \triangleright m') = b \triangleright (mm') \tag{4.2.7}$$

$$b \triangleright 1_M = s(\varepsilon(b)) \triangleright 1_M \tag{4.2.8}$$

and M is an R–bimodule via $r \cdot m \cdot r' = (s(r)t(r')) \triangleright m$. The subring of *invariants* is denoted M^B and is defined as

$$M^B = \{n \in M |\, b \triangleright n = s_L(\varepsilon(b)) \triangleright n\} \tag{4.2.9}$$

We also have the isomorphism $\mathrm{Hom}_{B-}(R,M) \simeq M^B$. For all B–module algebras in ${}_B\mathsf{M}$, the smash product defines an M–ring structure on $M \underset{L}{\otimes} B$ with multiplication $\hat{\mu} : M\#B \underset{M}{\otimes} M\#B \to M\#B$, $(m\#b)(m'\#b') = m(b_{(1)} \triangleright m')\#b_{(2)}b'$ and unit map $\hat{\eta} : M \to M \underset{L}{\otimes} B$, $m \mapsto m \underset{L}{\otimes} 1_B$.

Consider again a right bialgebroid $\langle B, s_R, t_R\rangle$ over R. Turning to the comodule category M^B:

DEFINITION 4.2.10. A right B–comodule algebra is a monoid in M^B, i.e. a B–comodule $\langle M, \delta_M \rangle \in \mathsf{M}_B$, and B–comodule maps $\mu_M : M \underset{R}{\otimes} M \to M$ and $\eta_M : R \to M$,

$$\begin{array}{ccc} (M \underset{R}{\otimes} M) & \xrightarrow{\delta_{M \otimes M}} & (M \underset{R}{\otimes} M) \underset{R}{\otimes} B \\ \downarrow{\scriptstyle \mu_M} & & \downarrow{\scriptstyle \mu_M \underset{R}{\otimes} B} \\ M & \xrightarrow[\delta_M]{} & M \underset{R}{\otimes} B \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\delta_R} & R \underset{R}{\otimes} B \\ \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \eta_M \underset{R}{\otimes} B} \\ M & \xrightarrow[\delta_M]{} & M \underset{R}{\otimes} B \end{array}$$

or, on elements,

$$(mm')_{\langle 0\rangle} \underset{R}{\otimes} (mm')_{\langle 1\rangle} = m_{\langle 0\rangle} m'_{\langle 0\rangle} \underset{R}{\otimes} m_{\langle 1\rangle} m'_{\langle 1\rangle} \tag{4.2.11}$$

$$(\eta_M(r))_{\langle 0\rangle} \underset{R}{\otimes} (\eta_M(r))_{\langle 1\rangle} = \eta_M(r) \underset{R}{\otimes} 1_B \tag{4.2.12}$$

The strict monoidal forgetful functor on $U : \mathsf{M}_B \to {}_R\mathsf{M}_R$ induces on M the R–bimodule structure

$$M \in {}_R\mathsf{M}_R, \quad r \cdot m \cdot r' = (m \cdot r')_{\langle 0\rangle} \varepsilon_H(t(r)(m \cdot r')_{\langle 0\rangle}) \tag{4.2.13}$$

We define the subring of *coinvariants*, denoted M^{co-B}, as

$$M^{co-B} = \{n \in M \,|\, \delta_M(n) = n \underset{R}{\otimes} 1_B\} \tag{4.2.14}$$

The coinvariants also admit an abstract definition similar to the invariant subring,

LEMMA 4.2.15. *For a right B–comodule algebra M,*

$$\mathrm{Hom}^{-B}(R, M) \simeq M^{co-B}$$

as k–algebras, where the algebra structure on $\mathrm{Hom}^{-B}(R, M)$ *is the convolution algebra.*

For any right B–comodule algebra M, $M \underset{R}{\otimes} B$ has an M–coring structure, given as follows. $M \underset{R}{\otimes} B \in {}_M\mathsf{M}_M$ with left– and right M–actions $m' \cdot (m \otimes b) \cdot m'' = m'\, m\, m''_{\langle 0\rangle} \underset{R}{\otimes} bm''_{\langle 1\rangle}$. With respect to this bimodule structure, the comultiplication and counit

$$\hat{\Delta} : (M \underset{R}{\otimes} B) \to (M \underset{R}{\otimes} B) \underset{M}{\otimes} (M \underset{R}{\otimes} B), \;\; m \otimes b \mapsto m \underset{R}{\otimes} b_{(1)} \underset{M}{\otimes} 1_M \underset{R}{\otimes} b_{(2)}$$
$$\hat{\varepsilon} : M \underset{R}{\otimes} B \to M, \;\; m \underset{R}{\otimes} b \mapsto m\varepsilon_B(b)$$

are M–bimodule maps and make $M \underset{R}{\otimes} B$ an M–coring. Defining left comodule algebras analogously for a right bialgebroid is straightforward.

For *left* bialgebroids we consider the case of *right* comodule algebras for later reference. Let $\langle B, s_L, t_L \rangle$ be a left bialgebroid over L, then a right B–comodule algebra is a monoid in M^B, i.e. a B–comodule $\langle M, \delta_M \rangle \in \mathsf{M}_B$, and B–comodule maps $\mu_M : M \underset{R}{\otimes} M \to M$ and

$\eta_M : R \to M$, i.e.

$$(mm')^{\langle 0\rangle} \underset{L}{\otimes} (mm')^{\langle 1\rangle} = m^{\langle 0\rangle} m'^{\langle 0\rangle} \underset{L}{\otimes} m'^{\langle 1\rangle} m^{\langle 1\rangle} \tag{4.2.16}$$

$$(\eta_M(r))_{\langle 0\rangle} \underset{L}{\otimes} (\eta_M(r))_{\langle 1\rangle} = \eta_M(r) \underset{L}{\otimes} 1_B \tag{4.2.17}$$

The R–bimodule structure induced by the forgetful functor $U : \mathsf{M}_B \to {}_R\mathsf{M}_R$ is $r \cdot m \cdot r' = (m \cdot r')_{\langle 0\rangle}\, \varepsilon_H(s_L(r)(m \cdot r')_{\langle 0\rangle})$ The subring of coinvariants is given by $M^{co-B} = \{n \in M | \delta_M(n) = n \underset{L}{\otimes} 1_B\}$ and the isomorphism $\mathrm{Hom}^{-B}(L, M) \simeq M^{co-B}$ holds. For a right B–comodule algebra M, $M \underset{L}{\otimes} B$ has an M–coring structure, via the left– and right M–actions $m' \cdot (m \otimes b) \cdot m'' = m'^{\langle 0\rangle}\, m\, m'' \underset{L}{\otimes} bm''^{\langle 1\rangle}$ and the M–bimodule maps

$$\hat{\Delta} : (M \underset{L}{\otimes} B) \to (M \underset{L}{\otimes} B) \underset{M}{\otimes} (M \underset{R}{\otimes} B), \;\; m \underset{L}{\otimes} b \mapsto m \underset{L}{\otimes} b^{(2)} \underset{M}{\otimes} 1_M \underset{R}{\otimes} b^{(1)}$$
$$\hat{\varepsilon} : M \underset{L}{\otimes} B \to M, \;\; m \underset{R}{\otimes} b \mapsto m\, \varepsilon_B(b)$$

which make make $M \underset{L}{\otimes} B$ a coalgebra in ${}_M\mathsf{M}_M$.

Relations between module– and comodule algebras are to be expected in the case that the bialgebroid is finitely generated projective over the base ring. More specifically, we have that

Proposition 4.2.18. *For a right bialgebroid $\langle B, s_R, t_R\rangle$, fintely generated projective over R and let ${}^*B = \mathrm{Hom}_{R-}(B, R)$ denote the left R–dual bialgebroid. Then we have that*

1. *there is an isomorphism of categories $\mathsf{M}^B \simeq \mathsf{M}_{{}^*B^{op}}$*
2. *for a B–comodule algebra $M \in \mathsf{M}^B$, the B–coinvariant subalgebra M^{co-B} coincides with the ${}^*B^{op}$–invariant subalgebra $M^{{}^*B^{op}}$*

Proof. (1) Every right B–comodule $\langle M, \delta_M\rangle$ is a right ${}^*B^{op}$–module via

$$m \triangleleft \beta = m_{\langle 0\rangle} \cdot \beta(m_{\langle 1\rangle}), \;\; m \in M, \varphi \in {}^*B \tag{4.2.19}$$

Indeed, for $m \in M$ and $\beta, \beta' \in {}^*B^{op}$, using 4.2.19 we calculate

$$\begin{aligned}(m \triangleleft \beta) \triangleleft \beta' &= (m_{\langle 0\rangle} \cdot \beta(m_{\langle 1\rangle}))_{\langle 0\rangle}\, \beta'((m_{\langle 0\rangle} \cdot \beta(m_{\langle 1\rangle}))_{\langle 1\rangle}) = \\ &= m_{\langle 0\rangle\langle 0\rangle}\, \beta'(m_{\langle 0\rangle\langle 1\rangle}\, s(\beta(m_{\langle 1\rangle}))) = m \triangleleft (\beta'\beta)\end{aligned}$$

Let $f \in \mathsf{M}^B$ be a B–comodule map, satisfying $f(m_{\langle 0\rangle}) \underset{R}{\otimes} m_{\langle 1\rangle} = f(m)_{\langle 0\rangle} \underset{R}{\otimes} f(m)_{\langle 1\rangle}$. It is easily seen that f is then also an ${}^*B^{op}$–module map:

$$f(m) \triangleleft \beta = f(m)_{\langle 0\rangle}\, \beta(f(m)_{\langle 1\rangle}) = f(m_{\langle 0\rangle})\, \beta(m_{\langle 1\rangle}) = f(m \triangleleft \beta)$$

i.e. we have a functor $\mathsf{M}^B \to \mathsf{M}_{{}^*B^{op}}$. For the inverse functor of the equivalence, note that a ${}^*B^{op}$–module M is a B–comodule with the coaction

$$\begin{aligned}\delta_M : M &\to M \underset{R}{\otimes} B \\ m &\mapsto m \triangleleft b_i \underset{R}{\otimes} \beta^i\end{aligned}$$

where $\sum_i \beta^i \underset{R}{\otimes} b_i \in {}^*B \underset{R}{\otimes} B$ is the (left) finitely generated projective basis for ${}_RB$. Coassociativity follows easily,

$$(\delta_M \underset{R}{\otimes} B) \circ \delta_M(m) = (m \triangleleft b_i) \triangleleft b_j \underset{R}{\otimes} b^j \underset{R}{\otimes} \beta^i = m \triangleleft b_j b_i \underset{R}{\otimes} \beta^j \underset{R}{\otimes} \beta^i =$$
$$= m \triangleleft b_i \underset{R}{\otimes} (\beta^i)_{(1)} \underset{R}{\otimes} (\beta^i)_{(2)} = (M \underset{R}{\otimes} \Delta_B) \circ \delta_M(m)$$

The equivalence follows from the defining property of the dual basis.

(2) Let $M \in \mathsf{Alg} - \mathsf{M}^B$ be a B–comodule algebra. Then $n \in M^{co-B}$ iff $\delta_M(n) = n \underset{R}{\otimes} 1_B$. Since

$$n \triangleleft \beta = n \triangleleft \beta = n_{\langle 0 \rangle} \cdot \beta(n_{\langle 1 \rangle}) = n \cdot \beta(1_B) = n\, s({}^*\varepsilon(\beta)),$$

n is an element of the ${}^*B^{op}$–invariant subring. □

Having both module– and comodule algebras at our disposal, we can define the class of bialgebroid Galois extensions from both points of view. In Hopf Galois theory, usually comodule algebras over a Hopf algebra are considered, viewed as an algebra extension of the co–invariant subring. This is called the 'coaction picture', and is predominant in the existing literature. However, some applications, notably scalar extensions arising from Galois extensions, follow from the action picture which is also perhaps closer to physicists' notion of a symmetry *acting* on, e.g. the field algebra. We follow closely the definitions and exposition of [**3**].

2.2. Action picture. Let $\langle B, s_R, t_R \rangle$ be a right bialgebroid and $M \in \mathsf{M}_B$ a right B–module algebra. We call an algebra homomorphism $\hat{\eta} : N \to M$ a (right) B–extension, if $\hat{\eta}$ factorizes through an isomorphism $N \to M^B$. In a more concrete sense, we could require $N = M^B$.

For a B–extension $N \simeq M^B \to M$, N (M^B) acts on M from the left by multiplication; call this action λ. Also, for any extension $N \to M$, there is a right action of $\mathrm{End}({}_NM)$ (the left N–module endomorphisms) on M that commutes, by definition, with λ making M an $N - \mathrm{End}({}_NM)$–bimodule. Here we have to view the endomorphisms in $\mathrm{End}({}_NM)$ as composing to the *right*. We shall denote $\mathcal{E} = \mathrm{End}({}_NM)$, hence $M = {}_NM_{\mathcal{E}}$. Furthermore, there is a homomorphism $H\#M \to \mathrm{End}({}_NM)$, defined by $m' \cdot (h\#m) = (m' \triangleleft h)m$, which makes M an $N - H\#M$–bimodule.

LEMMA 4.2.20. *For a B–extension $N \to M$, we have the following equalities:*

$$\lambda(N) = \mathrm{End}(M_{\mathcal{E}}) = \mathrm{End}(M_{B\#M}) = \lambda(M^B) \tag{4.2.21}$$

PROOF. We write the action of a $\gamma \in \mathrm{End}_{N-}(M)$ on $m \in M$ from the right as $m \mapsto (m)\overleftarrow{\gamma}$. The inclusions $\lambda(N) \subseteq \mathrm{End}(M_{\mathcal{E}}) \subseteq \mathrm{End}(M_{B\#M})$ are easily proved. For any $n \in N$, $\gamma \in \mathrm{End}_{N-}(M)$ we have $(n \cdot m)\overleftarrow{\beta} = n \cdot (m)\overleftarrow{\beta}$ by definition, hence $\lambda(N) \subseteq \mathrm{End}(M_{\mathcal{E}})$. The inclusion $\mathrm{End}(M_{\mathcal{E}}) \subseteq \mathrm{End}(M_{B\#M})$ follows from the existence of the homomorphism $H\#M \to \mathrm{End}({}_NM) = \mathcal{E}$. The Lemma is then a consequence of $N = M^B$ and the equality $\mathrm{End}(M_{B\#M}) = \lambda(M^B)$. The latter is proved entirely analogously to Lemma 8.3.2 of [**58**]. The map λ :

$M^B \to \mathrm{End}(M_{B\#M})$ is obviously injective. To see that it is also surjective, observe that for $\sigma \in \mathrm{End}(M_{B\#M})$ and $m \in M$, $\sigma(m) = \sigma(1_M)m$, i.e. $\sigma = \lambda_{\sigma(1)}$. But $\sigma(1) \in M^B$, since for all $b \in B$, $\sigma(1)b = \sigma(1 \cdot b) = \sigma(1)\varepsilon_B(b)$.

□

We introduce some basic notions for bimodules, allowing us to rephrase in abstract terms what we have just proved. A more detailed treatment is found in [**2**]. Let R and S be rings, then note that $M \in {}_R\mathsf{M}_S$ is an $R-S$–bimodule if and only $\lambda : R \to \mathrm{End}(M_S)$ (left action by R) and $\rho : S \to \mathrm{End}({}_RM)$ (right action by S) are ring homomorphisms.

Definition 4.2.22. Consider R, S and $M \in {}_R\mathsf{M}_S$ as above. With respect to the homomorphisms $\lambda : R \to \mathrm{End}(M_S)$ and $\rho : S \to \mathrm{End}({}_RM)$, the bimodule ${}_RM_S$ is *balanced* if both λ and ρ are *surjective* and the bimodule ${}_RM_S$ is *faithfully balanced* if both λ and ρ are *isomorphisms.*

To every left module ${}_RM$, there is a canonical bimodule ${}_RM_{\mathcal{E}}$ where $\mathcal{E} = \mathrm{End}_{R-}(M)$ is the ring of left R–endomorphisms, viewed as acting from the right. Continuing, we can introduce the *biendomorphism ring* of the original module ${}_RM$ as $\mathrm{BiEnd}({}_RM) = \mathrm{End}(M_{\mathcal{E}})$. Applying our bimodule–terminology to the canonical bimodule of a left module just defined, we define:

Definition 4.2.23. A module ${}_RM$ over a ring R is *balanced* if the bimodule ${}_RM_{\mathcal{E}}$ is balanced ($\mathcal{E} = \mathrm{End}_{N-}(M)$), i.e. $\lambda : R \to \mathrm{End}(M_{\mathcal{E}}) \equiv \mathrm{BiEnd}({}_RM)$ is surjective.

With this terminology, we collect these results in the following

Lemma 4.2.24. *Let $N \to M$ be a right B–extension; then*

(1) ${}_NM_{\mathcal{E}}$ *is balanced, i.e.* $\mathrm{BiEnd}({}_NM) \equiv \mathrm{End}(M_{\mathcal{E}}) = \lambda(N)$, *and*
(2) *the bimodule ${}_NM_{B\#M}$ is faithfully balanced if and only if the map $B\#M \to \mathcal{E}$ is an isomorphism.*

As we shall see, this map $H\#M \to \mathcal{E}$ plays a fundamental role in Galois theory.

Definition 4.2.25. For a right bialgebroid B over R, a right B–extension $N = M^B \subseteq M$ is said to be a Galois extension if the M–bimodule map (the *canonical map*)

$$\Gamma_M : B \underset{R}{\otimes} M \to \mathrm{End}_{N-}(M) \tag{4.2.26}$$
$$b \underset{R}{\otimes} m \mapsto \{m' \mapsto (m' \triangleleft b)m\}$$

is an isomorphism.

Recall that for any B–extension, $B \underset{R}{\otimes} M$ has an M–ring structure given by the smash product $B\#M$. $\mathrm{End}_{N-}(M)$ is an M–ring with composition (to the right) of endomorphisms as multiplication and unit map $M \to \mathrm{End}_{N-}(M)$ sending $m \in M$ to ρ_m. The canonical map extends to a morphism of M–rings.

Now let $\langle \bar{B}, s_L, t_L\rangle$ be a left bialgebroid and $M \in {}_{\bar{B}}\mathsf{M}$ a left $\bar{B}$–module algebra. A left $\bar{B}$–extension $N = M^{\bar{B}} \subseteq M$ is said to be a Galois extension if the $M - M$–bimodule map

$$\begin{aligned} \Gamma^M : M \underset{R}{\otimes} \bar{B} &\to \mathrm{End}_{-N}(M) \\ m \underset{R}{\otimes} b &\mapsto \{m' \mapsto m(b \triangleleft m')\} \end{aligned} \tag{4.2.27}$$

is an isomorphism. Again, Γ^M extends to an M–ring morphism if the M–ring structure on $\mathrm{End}_{-N}(M)$ is given by the composition of endomorphisms (to the left) as multiplication and left multiplication by M as the unit map. We shall only be considering Galois extensions (in the action picture) of right bialgebroids in what follows. Nevertheless, the above left–sided definition can be put to use by noting that for a right bialgebroid $\langle B, s_L, t_L\rangle$, the opposite $\langle B^{op}, t_L^{op}, s_L^{op}\rangle$ is a left bialgebroid. From the categorical isomorphism ${}_B\mathsf{M} \simeq \mathsf{M}_{B^{op}}$, it follows that an equivalent formulation of the Galois condition is that the M–bimodule map

$$\begin{aligned} \Gamma^M : M \underset{R}{\otimes} B^{op} &\to \mathrm{End}_{-N}(M) \\ m \underset{R}{\otimes} b &\mapsto \{m' \mapsto m(m' \triangleleft b)\} \end{aligned} \tag{4.2.28}$$

is an isomorphism.

2.3. Coaction picture. As discussed earlier, right (left) bialgebroids may coact both to the left and to the right. Correspondingly, there are in principle four cases of bialgebroid extensions (and Galois extensions) to consider. We collect these in two definitions

Definition 4.2.29. Let $\langle B, s_R, t_R\rangle$ be a right bialgebroid. For a *right* B–comodule algebra $M \in \mathsf{M}^B$, the B–extension $N = M^{co-B} \subseteq M$ is Galois if the M–bimodule map

$$\begin{aligned} \gamma_M : M \underset{N}{\otimes} M &\to M \underset{R}{\otimes} B \\ m \underset{N}{\otimes} m' &\mapsto mm'_{\langle 0\rangle} \underset{R}{\otimes} m'_{\langle 1\rangle} \end{aligned} \tag{4.2.30}$$

is an isomorphism. For a *left* B–comodule algebra $M \in {}^B\mathsf{M}$, $N = M^{co-B} \subseteq M$ is Galois if the M–bimodule map

$$\begin{aligned} \delta_M : M \underset{N}{\otimes} M &\to B \underset{R}{\otimes} M \\ m \underset{N}{\otimes} m' &\mapsto m'_{\langle -1\rangle} \underset{R}{\otimes} mm'_{\langle 0\rangle} \end{aligned} \tag{4.2.31}$$

is an isomorphism.

Recall that both $M \underset{R}{\otimes} B$ and $B \underset{R}{\otimes} M$ have an M–coring structure. The coring structure on $M \underset{N}{\otimes} M$ is called the Sweedler coring, and has the comultiplication

$$\begin{aligned} \Delta : M \underset{N}{\otimes} M &\to (M \underset{N}{\otimes} M) \underset{M}{\otimes} (M \underset{N}{\otimes} M) \simeq M \underset{N}{\otimes} M \underset{N}{\otimes} M \\ m \otimes m' &\mapsto m \otimes 1_M \otimes m' \end{aligned} \tag{4.2.32}$$

and counit $\varepsilon = \mu_M : M \underset{N}{\otimes} M \to M$, simply the ring multiplication, $m \otimes m' \mapsto mm'$. As expected, both canonical maps γ_M and δ_M extend to maps of M–corings.

The left handed twin of the previous definition goes as follows.

DEFINITION 4.2.33. Let $\langle \bar{B}, s_L, t_L \rangle$ be a left bialgebroid. For a *left* $\bar{B}$–comodule algebra $M \in {}^{\bar{B}}\mathsf{M}$, the $\bar{B}$–extension $N = M^{co-\bar{B}} \subseteq M$ is Galois if the M–bimodule map

$$\begin{aligned} \delta^M : M \underset{N}{\otimes} M &\to \bar{B} \underset{R}{\otimes} M \\ m \underset{N}{\otimes} m' &\mapsto m_{\langle -1 \rangle} \underset{R}{\otimes} m_{\langle 0 \rangle} m' \end{aligned} \tag{4.2.34}$$

is an isomorphism. For a *right* $\bar{B}$–comodule algebra $M \in \mathsf{M}^{\bar{B}}$, the $\bar{B}$–extension $N = M^{co-\bar{B}} \subseteq M$ is Galois if the M–bimodule map

$$\begin{aligned} \gamma^M : M \underset{N}{\otimes} M &\to M \underset{R}{\otimes} \bar{B} \\ m \underset{N}{\otimes} m' &\mapsto m_{\langle 0 \rangle} m' \underset{R}{\otimes} m_{\langle 1 \rangle} \end{aligned} \tag{4.2.35}$$

is an isomorphism.

For bialgebroids that are finitely generated projective over their base ring, we found a relation between comodule algebras and module algebras over the dual. It is natural to ask whether this relation extends to Galois extensions?

THEOREM 4.2.36. *Let $\langle B, s_R, t_R \rangle$ be a right bialgebroid which is finitely generated projective as an R–module and let $N = M^B \to M \in \mathsf{M}^B$ be a right B–extension. Then*

$$\gamma_M : M \underset{N}{\otimes} M \to M \underset{R}{\otimes} B \tag{4.2.37}$$

is an isomorphism if and only if

$$\Gamma_N : M \underset{R}{\otimes} {}^*B \to \mathrm{End}_{-N}(M) \tag{4.2.38}$$

is an isomorphism.

PROOF. Applying $\mathrm{Hom}_{M-}(M, \_)$ to 4.2.37, we obtain the isomorphism

$$\mathrm{Hom}_{M-}(\gamma_M, M) : \mathrm{Hom}_{M-}(M \underset{R}{\otimes} B, M) \to \mathrm{Hom}_{M-}(M \underset{N}{\otimes} M, M) \tag{4.2.39}$$

We shall denote the left M–dual of ${}_M X$ as $\mathrm{Hom}_{M-}(X, M) = {}^*X$. First, we have that

$${}^*(M \underset{N}{\otimes} M) \simeq \mathrm{End}_{-N}(M),$$

by the inverse equivalences

$$\begin{aligned} &\mathrm{Hom}_{M-}(M \underset{N}{\otimes} M, M) \to \mathrm{End}_{-N}(M), \quad \Theta \mapsto \{\alpha_\Theta : m' \mapsto \Theta(m', \_)\} \\ &\mathrm{End}_{-N}(M) \to \mathrm{Hom}_{M-}(M \underset{N}{\otimes} M, M), \quad \alpha \mapsto \{\Theta_\alpha : m \otimes m' \mapsto m\alpha(m')\} \end{aligned}$$

Secondly, ${}^*(M \underset{R}{\otimes} B) = M \underset{R}{\otimes} {}^*B$ by the inverse equivalences

$$\mathrm{Hom}_{M-}(M \underset{R}{\otimes} B, M) \to M \underset{R}{\otimes} {}^*B, \quad \Upsilon \mapsto \Upsilon(1_M \underset{R}{\otimes} b_i) \underset{R}{\otimes} \beta^i$$
$$M \underset{R}{\otimes} {}^*B \to \mathrm{Hom}_{M-}(M \underset{R}{\otimes} B, M), \quad m \underset{R}{\otimes} \varphi \mapsto \{m' \underset{R}{\otimes} b' \mapsto m'm\varphi(b')\},$$

using the finitely generated projective basis $\sum_i b_i \underset{R}{\otimes} \beta^i \in B \underset{R}{\otimes} {}^*B$. □

3. Quantum groupoid Galois extensions

The next task is to formulate what we mean by Galois extension in the case of a DDA. Following the same strategy that we had when defining modules and comodules, we would like the definition to imply that the extension is Galois with respect to the constituent bialgebroids in a consistent way. The different notions of bialgebroid Galois extensions (in the action and coaction pictures) would thus be unified. This is in fact the main result of this section. The standing assumption is that A is a DDA with vertical and horizontal Hopf algebroids V and H. Put equivalently, V is a Frobenius Hopf algebroid and H^{op} is its dual. We begin with the action picture.

3.1. Action picture. Let A be a DDA, M a right A–module algebra and $N = M^A$ the subring of invariants. We define the canonical maps

$$\Gamma^M : M \underset{R}{\otimes} H \to \mathrm{End}(M_N) \qquad (4.3.1)$$
$$m \otimes h \mapsto \{m' \mapsto m(m' \triangleleft h)\}$$

and

$$\Gamma_M : H \underset{R}{\otimes} M \to \mathrm{End}({}_N M) \qquad (4.3.2)$$
$$h \otimes m \mapsto \{m' \mapsto (m' \triangleleft h)m\}$$

A right A–module is a right module over the horizontal right bialgebroid and indeed, 4.3.1 and 4.3.2 are nothing but the two canonical maps belonging to the right horizontal bialgebroid H over R, as defined in 4.2.28 and 4.2.26. We recall the facts established there, that Γ^M extends to a map of M–rings $\Gamma^M : M \underset{R}{\otimes} H^{op} \to \mathrm{End}^l(M_N)$ and Γ_M extends to a map $\Gamma_M : H \underset{R}{\otimes} m \to \mathrm{End}^r(M_N)$, where $\mathrm{End}^l(M_N)$ means that we regard endomorphisms as acting from the left and accordingly, the elements of $\mathrm{End}^r(M_N)$ act from the right.

3.2. Coaction picture. Now let M be an A–comodule algebra, $N = M^{co-A}$ the subring of coinvariants (which, by our earlier results, coincides with the A–invariant subring). The canonical maps in the coaction picture are

$$\gamma^M : M \underset{R}{\otimes} M \to M \underset{T}{\otimes} V \qquad (4.3.3)$$
$$m \otimes m' \mapsto mm'^{(0)} \underset{T}{\otimes} m'^{(1)}$$

and

$$\gamma_M : M \underset{R}{\otimes} M \to M \underset{B}{\otimes} V \qquad (4.3.4)$$

$$m \otimes m' \mapsto m_{(0)} m' \underset{B}{\otimes} m_{(1)}$$

A right A–comodule is exactly a right comodule over the vertical Hopf algebroid V, and 4.3.3 and 4.3.4 are precisely the canonical maps for the left vertical bialgebroid V over B and the right vertical bialgebroid V over T, respectively. Recall that both γ^M and γ_M are M–bimodule maps and furthermore, both extend to M–coring maps. We re-write the M–bimodule structure on $M \underset{T}{\otimes} V$ and $M \underset{B}{\otimes} V$ in the double algebraic notation,

$$m' \cdot (m \underset{T}{\otimes} v) \cdot m'' = m' m m''^{(0)} \underset{T}{\otimes} v \circ m''^{(1)} \qquad (4.3.5)$$

$$m' \cdot (m \underset{B}{\otimes} v) \cdot m'' = m'_{(0)} m m'' \underset{B}{\otimes} m'_{(1)} \circ v \qquad (4.3.6)$$

We saw in the discussion of Hopf algebroids that the existence of a bijective antipode leads to the equivalence of the comodule categories over the constituent (left and right) bialgebroids. This removes some of the ambiguity in defining a Galois extension. The content of the next Lemma is that for a right V–comodule algebra M, it is immaterial whether we define the Galois property of $N = M^{co-V} \subseteq M$ as being bialgebroid Galois with respect to the left or to the right constituent bialgebroid.

Lemma 4.3.7. *Let M be a right V-comodule algebra over the Hopf algebroid V. Then γ^M is an isomorphism if and only if γ_M is.*

Proof. Let ϕ denote the composite

$$M \underset{T}{\otimes} V \xrightarrow{\delta_M \underset{T}{\otimes} V} M \underset{B}{\otimes} V \underset{T}{\otimes} V \xrightarrow{M \underset{B}{\otimes} V \otimes S} M \underset{B}{\otimes} V \underset{R}{\otimes} V \xrightarrow{M \underset{B}{\otimes} \mu_V} M \underset{B}{\otimes} V$$

i.e., $m \underset{T}{\otimes} v \mapsto m_{(0)} \underset{T}{\otimes} m_{(1)} S(v)$, where S is the antipode of the Hopf algebroid V.

Then ϕ has inverse

$$\phi^{-1}(m \underset{B}{\otimes} v) = m^{(0)} \underset{T}{\otimes} S^{-1}(v) m^{(1)} .$$

and one obtains that $\phi \circ \gamma^M = \gamma_M$. □

A module over a DDA is simultaneously a a module over the horizontal Hopf algebroid and a comodule over the vertical Hopf algebroid (such that the respective invariants and co-invariants coincide). This means that in a sense, the DDA is a large enough structure to encompass both the action and coaction pictures of Hopf algebroid (bialgebroid) extension. The next theorem shows that the notion of 'double algebra Galois extension' is essentially unique.

Theorem & Definition 4.3.8. *Let A be a distributive double algebra and M a right H-module algebra, equivalently a right V-comodule algebra, over the horizontal, resp. vertical Hopf algebroid of A. Let $N = M^H \equiv M^{\text{co-}V}$. Then $N \subseteq M$ is called an A-Galois extension if anyone of the following equivalent conditions hold:*

(1) γ^M *is iso.* *(2)* γ_M *is iso.*
(3) Γ^M *is iso and* M_N *is fgp.* *(4)* Γ_M *is iso and* ${}_N M$ *is fgp.*

PROOF. (1) ⇒ (3) Considering it as a right M-module map, γ^M induces the isomorphism (of left M-modules)

$$\gamma^{M*} \colon \operatorname{Hom}_{-M}(M \underset{T}{\otimes} V, M) \xrightarrow{\sim} \operatorname{Hom}_{-M}(M \underset{N}{\otimes} M, M) .$$

If $\chi \in \operatorname{Hom}_{-M}(M \underset{T}{\otimes} V, M)$ then $\chi(1 \underset{T}{\otimes} \_) \in \operatorname{Hom}(V_T, M_T)$ because

$$\begin{aligned}\chi(1 \underset{T}{\otimes} v \star t) &= \chi(1 \underset{T}{\otimes} v \circ \varphi_R(t)) \\ &= \chi\left(j(\varphi_R(t))^{(0)} \underset{T}{\otimes} v \circ j(\varphi_R(t))^{(1)} \right) \\ &= \chi(1 \underset{T}{\otimes} v) j(\varphi_R(t)) .\end{aligned}$$

Thus we have a well defined map (of left M-modules)

$$\begin{aligned}\operatorname{Hom}_{-M}(M \underset{T}{\otimes} V, M) &\to \operatorname{Hom}(V_T, M_T) \\ \chi &\mapsto \chi(1 \underset{T}{\otimes} \_)\end{aligned} \tag{4.3.9}$$

We claim that this map is an isomorphism with inverse

$$\kappa \mapsto \{ m \underset{T}{\otimes} v \mapsto \kappa(v \circ S^{-1}(m_{(1)})) m_{(0)} \}$$

This follows from the computation

$$\begin{aligned}\chi(m \underset{T}{\otimes} v) &= \chi(m^{(0)} \underset{T}{\otimes} \varphi_T(m^{(1)}) \star v) = \chi(m^{(0)} \underset{T}{\otimes} v \circ \varphi_L \varphi_T(m^{(1)})) \\ &= \chi(m^{(0)} \underset{T}{\otimes} v \circ S^{-1}({m^{(1)}}_{(2)}) \circ {m^{(1)}}_{(1)}) \\ &= \chi({m_{(0)}}^{(0)} \underset{T}{\otimes} v \circ S^{-1}(m_{(1)}) \circ {m_{(0)}}^{(1)}) \\ &= \chi(1 \underset{T}{\otimes} v \circ S^{-1}(m_{(1)}))\, m_{(0)}\end{aligned}$$

on the one hand and on the other hand from $\delta_M(1) = 1 \underset{B}{\otimes} e$. Composing the map (4.3.9) with the isomorphism

$$\begin{aligned}\operatorname{Hom}(V_T, M_T) &\to M \underset{R}{\otimes} H \\ \kappa &\mapsto \kappa(x^j) \underset{R}{\otimes} y^j\end{aligned} \tag{4.3.10}$$

where $x^j \otimes_R y^j \equiv \Delta_R(e)$ is the dual basis of φ_R, we obtain the left vertical arrow in the diagram

(4.3.11)
$$\begin{array}{ccc} \mathrm{Hom}_{-M}(M \otimes_T V, M) & \xrightarrow{\gamma^{M*}} & \mathrm{Hom}_{-M}(M \otimes_N M, M) \\ \downarrow & & \downarrow \\ M \otimes_R H & \xrightarrow[\Gamma^M]{} & \mathrm{End}(M_N) \end{array}$$

The vertical arrow on the right is the isomorphism $\sigma \mapsto \sigma(\_ \otimes_N 1)$ therefore the composite along the top and right is $\chi \mapsto \chi(\_ \otimes_T e)$. The other two compose to give

$$\chi \mapsto \chi(1 \otimes_T x^j) \otimes_R y^j \mapsto \chi(1 \otimes_T x^j)(\_ \triangleleft y^j)\,.$$

In order to see commutativity of the diagram we need a calculation.

$$\begin{aligned} \chi(1 \otimes_T x^j)(m \triangleleft y^j) &= \chi(1 \otimes_T x^j)\, m^{(0)} \cdot \varphi_T(m^{(1)} \star y^j) \\ &= \chi(m^{(0)} \otimes_T x^j \circ m^{(1)} \circ \varphi_R\varphi_T(m^{(2)} \star y^j)) \\ &= \chi(m^{(0)} \otimes_T x^j \circ (m^{(1)} \star \varphi_T(m^{(2)} \star y^j)) \\ &= \chi(m^{(0)} \otimes_T x^j \circ (m^{(1)} \star y^j) \\ &= \chi(m^{(0)} \otimes_T \varphi_L\varphi_T(m^{(1)})) \\ &= \chi(m^{(0)} \cdot \varphi_T(m^{(1)}) \otimes_T e) \;=\; \chi(m \otimes_T e)\,, \end{aligned}$$

where in the fifth equality we used [**79**, Equation (4.16)]. So (4.3.11) is commutative and therefore Γ^M is an isomorphism.

The proof of (2) ⇒ (4) goes similarly by proving commutativity of the diagram

(4.3.12)
$$\begin{array}{ccc} \mathrm{Hom}_{M-}(M \otimes_B V, M) & \xrightarrow{\gamma_M^*} & \mathrm{Hom}_{M-}(M \otimes_N M, M) \\ \downarrow & & \downarrow \\ H \otimes_R M & \xrightarrow[\Gamma_M]{} & \mathrm{End}({}_N M) \end{array}$$

with the left hand side arrow being the isomorphism $\chi \mapsto x^j \otimes_R \chi(1 \otimes_B y^j)$ and the one on the right hand side being $\sigma \mapsto \sigma(1 \otimes_N \_)$.

(3) ⇒ (2) Consider the diagram

(4.3.13)
$$\begin{array}{ccc} M \otimes_N M & \xrightarrow{\gamma_M} & M \otimes_B V \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{M-}(\mathrm{End}(M_N), M) & \xrightarrow[\Gamma^{M*}]{} & \mathrm{Hom}_{M-}(M \otimes_R H, M) \end{array}$$

The lower horizontal arrow is an isomorphism since Γ^M is. The vertical arrow on the left, mapping $m \underset{N}{\otimes} m'$ to the homomorphism $\alpha \mapsto \alpha(m)m'$, is an isomorphism because M_N is fgp. The other vertical arrow is the composite of two maps,

$$\mathrm{Hom}_{M-}(M \underset{R}{\otimes} H, M) \longrightarrow \mathrm{Hom}({}_RH, {}_RM) \longrightarrow M \underset{B}{\otimes} V$$

where the second one is the isomorphism $\kappa \mapsto \kappa(u^k) \underset{B}{\otimes} v^k$ with $u^k \underset{B}{\otimes} v^k \equiv \Delta_B(i)$ denoting the dual basis of φ_B. The first one, $\chi \mapsto \chi(1 \underset{R}{\otimes} \_)$, is obviously invertible (in contrast to the similar map in the (3) $\Rightarrow$ (5) part) because the left M-module structure of $M \underset{R}{\otimes} H$ we need here is the trivial one. It remains to show commutativity of (4.3.13). So we compute the action of the lower three arrows,

$$\begin{aligned} m \underset{N}{\otimes} m' &\mapsto \{\alpha \mapsto \alpha(m)m'\} \mapsto \{m'' \underset{R}{\otimes} h \mapsto m''(m \triangleleft h)m'\} \\ &\mapsto \{h \mapsto (m \triangleleft h)m'\} \mapsto (m \triangleleft u^k)m' \underset{B}{\otimes} v^k \end{aligned}$$

which is indeed γ_M if we compare the right H-action with the right V-coaction δ_M. This proves that γ_M is invertible.

The proof of the implication (4) $\Rightarrow$ (1) can be done similarly by using the diagram

(4.3.14)
$$\begin{array}{ccc} M \underset{N}{\otimes} M & \xrightarrow{\quad \gamma^M \quad} & M \underset{T}{\otimes} V \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{-M}(\mathrm{End}({}_NM), M) & \xrightarrow[\Gamma^*_M]{} & \mathrm{Hom}_{-M}(H \underset{R}{\otimes} M, M) \end{array}$$

where on the left hand side we have the map $m \underset{N}{\otimes} m' \mapsto \{\alpha \mapsto m\alpha(m')\}$ which is an isomorphism because ${}_NM$ is fgp. □

Ever since the classic paper [**49**] in Hopf Galois theory, there has been interest in results that show that the Galois map is an isomorphism already if it is epi. This is important for applications, since the latter statement is usually easier to prove. The next result is an immediate generalization of [**58**, Theorem 8.3.1].

PROPOSITION 4.3.15. *Assume that V is a Frobenius Hopf algebroid and M is a right V-comodule algebra with coinvariant subalgebra N. Then γ^M being epi implies that γ^M is an isomorphism and M_N is finitely generated projective.*

PROOF. Let V and H be the vertical and horizontal Hopf algebroid of a distributive double algebra $\langle A, \circ, e, \star, i\rangle$. Then M is a right H-module algebra and e, the unit of V, is an integral for H, therefore $m \triangleleft e \in N$, $m \in M$. By the hypothesis there exists $\sum_j m_j \underset{N}{\otimes} m'_j \in M \underset{N}{\otimes} M$ such that

$$\sum_j m_j m'^{(0)}_j \underset{T}{\otimes} m'^{(1)}_j = 1 \underset{T}{\otimes} i\,.$$

Therefore we can write for arbitrary $m \in M$ that

$$\begin{aligned}\sum_j m_j((m'_j m) \triangleleft e) &= \sum_j m_j(m'_j \triangleleft e^{[1]})(m \triangleleft e^{[2]}) \\ &= \sum_j m_j \left(m'^{(0)}_j \cdot \varphi_T(m'^{(1)}_j \star e^{[1]}) \right) (m \triangleleft e^{[2]}) \\ &= (1 \triangleleft (i \circ e^{[1]}))(m \triangleleft e^{[2]}) = (1 \triangleleft i^{[1]})(m \triangleleft i^{[2]}) \\ &= m \triangleleft i \;=\; m\end{aligned}$$

proving that $(m'_j \_) \triangleleft e$ is dual basis of m_j for M_N, thus M_N is fgp.

Next we show that γ_M is mono. Let $\sum_i z_i \underset{N}{\otimes} w_i \in \operatorname{Ker} \gamma_M$. Then

$$\sum_i z_{i(0)} w_i \underset{B}{\otimes} z_{i(1)} \;=\; 0\,.$$

Using the dual bases for M_N we find that

$$\begin{aligned}\sum_i z_i \underset{N}{\otimes} w_i &= \sum_i \sum_j m_j((m'_j z_i) \triangleleft e) \underset{N}{\otimes} w_i \\ &= \sum_j m_j \underset{N}{\otimes} \sum_i \left(m'_{j(0)} z_{i(0)} \,.\, \varphi_B((m'_{j(1)} \circ z_{i(1)}) \star e) \right) w_i \\ &= \sum_j m_j \underset{N}{\otimes} \sum_i m'_{j(0)} z_{i(0)} w_i \,.\, \varphi_B(m'_{j(1)} \circ z_{i(1)}) \\ &= 0\,.\end{aligned}$$

Therefore γ_M is mono. But it is also epi because γ^M is. Therefore γ_M is iso, and so is γ^M. □

We can thus enlarge Theorem 4.3.8 with two further equivalent conditions:

COROLLARY 4.3.16. *Let A be a distributive double algebra and M a right H-module algebra, equivalently a right V-comodule algebra, over the horizontal, resp. vertical Hopf algebroid of A. Let $N = M^H \equiv M^{\text{co-}V}$. Then $N \subseteq M$ is called an A-Galois extension if anyone of the following equivalent conditions hold:*

(1) γ^M *is epi.*	*(2)* γ_M *is epi.*
(3) γ^M *is iso.*	*(4)* γ_M *is iso.*
(5) Γ^M *is iso and* M_N *is fgp.*	*(6)* Γ_M *is iso and* ${}_N M$ *is fgp.*

4. Depth 2 extensions

In [**47**] a general, purely algebraic notion of depth 2 was introduced for ring extensions and it was proven that for any balanced, depth 2 extension $N \subseteq M$ there is a dual pair of bialgebroids acting, respectively coacting on M such that N is the subalgebra of invariants, respectively coinvariants and making the extension Galois. This result may be compared to the fundamental Theorem of classical Galois theory which states that a normal and separable extension of fields is a Galois extension. The analogy is that 'depth 2 and balanced' is an intrisic characterization of a ring extension which makes no reference to an action of a symmetry, just like 'normal and separable' is an intrisic characterization of a field extension which makes no reference to any Galois group.

We recall the necessary definitions from [**47**].

DEFINITION 4.4.1. A ring extension $N \subseteq M$ is called *left* depth 2 if

$$_N M \underset{N}{\otimes} M_M \oplus * \cong \oplus^n_N M_M$$

for some positive integer n; it is called *right* depth 2 if

$$_M M \underset{N}{\otimes} M_N \oplus * \cong \oplus^m_M M_N$$

for some integer m. It is simply depth 2 if it is both left– and right depth 2.

A more useful equivalent definition is in terms of the depth 2 quasibasis. Introducing the notation $A = \mathrm{End}\,_N M_N$ and $B = (M \underset{N}{\otimes} M)^N$, the following Lemma holds.

LEMMA 4.4.2. *An extension $N \subseteq M$ is left depth 2 if and only if there exist $b_i \in B$ and $\beta_i \in A$ such that*

$$\sum_i b_i^1 \otimes b_i^2 \beta_i(m) = m \otimes 1 \ \text{ for all } \ m \in M,$$

and it is left depth 2 if and only if there exist $c_i \in B$ and $\gamma_i \in A$ such that

$$\sum_i \gamma_i(m) c_i^1 \otimes c_i^2 = m \otimes 1 \ \text{ for all } \ m \in M.$$

$\{b_i^1, \beta_i^2\}$ is the left D2 quasibasis and $\{c_i^1, \gamma_i^2\}$ is the right D2 quasibasis.

THEOREM 4.4.3. *Let $N \to M$ be a depth 2 extension of rings; then $A = \mathrm{End}\,_N M_N$ is a left bialgebroid over R with a left action of A on M. If M_N is a balanced module, then the subring of invariants is N. The base ring is the centralizer $R = C_M(N)$ and the structure maps of the bialgebroid $\langle A, R, s_A, t_A, \Delta_A, \varepsilon_A \rangle$ are*

(1) $s_A(r) = \lambda(r) : m \mapsto rm, \ \ t_A(r) = \rho(r) : m \mapsto mr$
(2) $r \cdot \alpha \cdot r' = \lambda(r)\rho(r')\alpha : m \mapsto r\alpha(m)r'$
(3) $\Delta_A(\alpha) = \sum_i \gamma_i \underset{R}{\otimes} c_i^1 \alpha(c_i^2 \text{-})$
(4) $\varepsilon_A(\alpha) = \alpha(1_M)$

Tha action of A on M is the canonical action of endomorphisms, $\alpha \triangleright m = \alpha(m)$. The map

$$\Gamma^M : M \rtimes A \to \mathrm{End}(M_N) \qquad (4.4.4)$$

$$m \rtimes \alpha \mapsto \lambda(m)\alpha$$

is an isomorphism, making $N \subseteq M$ an A–Galois extension.

The DDA generalization of this Theorem is related to the special case of Frobenius D2 extensions.

THEOREM 4.4.5. *For an algebra extension $N \subseteq M$ the following conditions are equivalent.*

(1) *There is a Frobenius Hopf algebroid V and a coaction of V on M such that $N \subseteq M$ is V-Galois.*

(2) *$N \subseteq M$ is of depth 2 and Frobenius and M_N is balanced.*

PROOF. (1) $\Rightarrow$ *$N \subseteq M$ is Frobenius*: Consider the composite

$$M \underset{N}{\otimes} M \xrightarrow{\gamma^M} M \underset{T}{\otimes} V \xrightarrow{M \otimes S} M \underset{R}{\otimes} H \xrightarrow{\Gamma^M} \mathrm{End}(M_N) \tag{4.4.6}$$

where the middle arrow is meaningful in the double algebraic picture because V and H have the same underlying k-module A and $S(t \star a) = S(a) \star \varphi_B \varphi_R(t) = \varphi_R(t) \circ a$ holds for all $a \in A$, $t \in T$, see [**79**, Lemma 5.4]. Computing the value of the map (4.4.6) on $m \underset{N}{\otimes} m'$ we obtain

$$\begin{aligned} m{m'}^{(0)}(m'' \triangleleft S({m'}^{(1)})) &= m{m'}^{(0)}{m''}^{(0)} \cdot \varphi_T({m''}^{(1)} \star S({m'}^{(1)})) \\ &= m{m'}^{(0)}{m''}^{(0)} \cdot \varphi_T \varphi_L({m'}^{(1)} \circ {m''}^{(1)}) \\ &= m(m'm'')^{(0)} \cdot \varphi_T((m'm'')^{(1)} \star e) \\ &= m((m'm'') \triangleleft e) \end{aligned}$$

Therefore (4.4.6) has the familiar form $m \underset{N}{\otimes} m' \mapsto m\psi m'$ in terms of the N-N-bimodule map $\psi = \_ \triangleleft e$ from M into N. Since (4.4.6) is isomorphism it follows that ψ is a Frobenius homomorphism with dual basis obtained from id_M by applying the inverse of (4.4.6) .

(1) $\Rightarrow$ *$N \subseteq M$ is D2*: Since ${}_TV$ is fgp and γ^M provides an M-N-bimodule isomorphism $M \underset{N}{\otimes} M \overset{\sim}{\to} ({}_MM_N) \underset{T}{\otimes} V$, it follows that $N \subseteq M$ is right D2. Similarly, the existence of the isomorphism γ_M and the ${}_BV$ being fgp imply that $N \subseteq M$ is left D2.

(1) $\Rightarrow$ *$N \subseteq M$ is balanced*: This follows from that every V-extension is balanced, see Lemma 4.2.25.

(2) $\Rightarrow$ (1): The endomorphism algebra $H^{\mathrm{op}} := \mathrm{End}({}_NM_N)$ has a natural structure of a Frobenius Hopf algebroid, see [**79**, Subsection 8.6] or [**12**]. Moreover, the natural action of H^{op} on M makes it a left H^{op}-module algebra and the corresponding smash product $M \# H^{\mathrm{op}}$ is isomorphic to $\mathrm{End}(M_N)$ via Γ^M by [**47**, Corollary 4.5]. So $N \subseteq M$ will be V-Galois, for V the dual of H^{op}, provided $N = M^H$. But this is equivalent to M_N being balanced. □

Note that in the presence of the Frobenius condition left D2 is equivalent to right D2 and in the presence of the D2 Frobenius condition M_N is balanced iff ${}_NM$ is balanced.

CHAPTER 5

Scalar extension

In the most basic sense, 'extension of scalars' refers to replacing the field over which an algebraic structure is defined with another. The simplest such construction is perhaps the complexification of a real vector space: let $V \simeq \mathbf{R}^n$ be an n–dimensional real vector space. Then the complexification of V is the complex vector space $V^C = \mathbf{C} \underset{R}{\otimes} V$ of complex dimension n.

A similar construction exists for Hopf algebras over a field. For a Hopf algebra H over k and an extension field $k \subseteq k'$ of k, $k' \underset{k}{\otimes} H$ has the structure of a Hopf algebra over the extended field k'. Both constructions are clearly functorial. Underlying both is tensor functor,

$$\mathbf{C} \underset{R}{\otimes} \_ : {}_{\mathbf{R}}\mathsf{M} \to {}_{\mathbf{C}}\mathsf{M}$$

in the first case and

$$k' \underset{k}{\otimes} \_ : {}_{k}\mathsf{M} \to {}_{k'}\mathsf{M}$$

in the second. Replacing Hopf algebra, above, with Hopf algebroid and the fields k and k' whith noncommutative rings R' and R, there is no obvious way that $R' \underset{R}{\otimes} H$ carries a Hopf algebroid structure. In the present section, we shall show that in a more involved sense, the scalar extension construction *does* carry over to Hopf algebroids if we assume additional structure on R', namely that it is a *braided commutative algebra* over H.

The motivation for such a construction is a phenomenon in Hopf Galois theory which we have already mentioned in Section 1 of Chapter 2. In [**44**], Greither and Pareigis show that there are separable field extensions that are H–Galois for two non–isomorphic Hopf algebras H and H' (note that this is a Hopf Galois extension, despite that it is an extension of *fields*), which however, become isomorphic after an appropriate extension of scalars $k \subseteq K$, i.e. the Hopf algebras H and H' are 'forms' of each other.

Example 5.0.7 (Forms of Hopf algebras in Hopf Galois extensions). The first surprising result is that it is possible that a separable field extension $E|F$ is H–Galois for some Hopf algebra H, yet it is not Galois in the classical sense. The example of Greither and Pareigis is the following. Let the base field be the rationals, $k = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt[4]{2})$ the extension by the real fourth root of 2 (which will be abbreviated $\omega = \sqrt[4]{2}$). It is known that $\mathbf{Q}(\sqrt[4]{2}) \,|\, \mathbf{Q}$ is not Galois in the classical sense. It is however $H^*_{\mathbf{Q}}$–Galois, where $H^*_{\mathbf{Q}}$ is the *circle Hopf algebra* over base field $H^*_{\mathbf{Q}}$. Over a field k, H_k is defined as an algebra with a presentation on generators

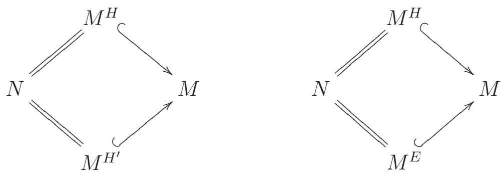


FIGURE 1. H, H' are non–isomorphic Hopf algebroids; $E = \mathrm{End}_{N-N}(M)$ is the endomorphism Hopf algebroid

and relations as

$$H = k[c, s]/\langle c^2 + s^2 - 1,\ cs\rangle. \tag{5.0.8}$$

The coalgebra structure is given on the generators by $\Delta(c) = c \otimes c - s \otimes s$, $\Delta(s) = c \otimes s + s \otimes c$, $\varepsilon(c) = 1$, $\varepsilon(s) = 0$, $S(c) = c$ and $S(s) = -s$. The action of H on K is given by the action on ω, which is

	1	ω	ω^2	ω^3
c	1	0	$-\omega^2$	0
s	0	$-\omega$	0	ω^3

From the table above, it is clear that if $k(\omega) \mid k$ *were* Galois in the classical sense for some field k, then the Galois group would be cyclic of order four (presented here redundantly on two generators). In fact, extending the base field from $\mathbf{Q}$ to $\mathbf{Q}(i)$, the extension $\mathbf{Q}(i) \otimes \mathbf{Q}(\omega) \mid \mathbf{Q}(i)$ becomes Galois with $H_{\mathbf{Q}}$ extended to the Galois *group* $\mathbf{Q}C_4$. Hence, $\mathbf{Q}$ and the group algebra $\mathbf{Q}C_4$ are $\mathbf{Q}(i)$–forms of each other.

Interestingly, if the base field is further extended to $\mathbf{Q}(i, \sqrt[4]{2})$, then the extended ring $\mathbf{Q}(i, \sqrt[4]{2}) \otimes \mathbf{Q}(\sqrt[4]{2})$ becomes isomorphic to the dual of the extended group algebra, $(\mathbf{Q}(i, \sqrt[4]{2})C_4)^*$, as $H^*_{\mathbf{Q}}$–comodules. In other words, the $H^*_{\mathbf{Q}}$–comodule algebra $K = \mathbf{Q}(\omega)$ is a form of the trivial $H^*_{\mathbf{Q}}$–comodule algebra $H^*_{\mathbf{Q}}$.

There is also a second Hopf algebra H' which acts on $K = \mathbf{Q}(\sqrt[4]{2})$ such that the extension $\mathbf{Q}(\sqrt[4]{2}) \mid \mathbf{Q}$ is Galois. As an algebra, it is given on generators and relations as

$$H' = \mathbf{Q}[c, s]/\langle s^2 - 2c^2 + 2,\ cs\rangle. \tag{5.0.9}$$

The comultiplication on the generators is $\Delta(c) = c \otimes c - \frac{1}{2} s \otimes s$, $\Delta(s) = c \otimes s + s \otimes c$. The action of H on K is given by the action on ω, which is

	1	ω	ω^2	ω^3
c	1	0	$-\omega^2$	0
s	0	ω^3	0	-2ω

This Hopf algebra turns out to be a $\mathbf{Q}(\sqrt[2]{2})$–form of $\mathbf{Q}(\mathbf{Z}_2 \times \mathbf{Z}_2)$.

This example illustrates that it is possible that a (separable) extension is Hopf Galois with two different Hopf algebras (the previous example shows that this also holds for extensions which are classicaly Galois). This ambiguity of Hopf Galois extensions is borne out by the following theorem (Theorem 4.3 of [**65**]).

THEOREM 5.0.10. *Any classical Galois extension $K|k$ can be endowed with an H–Galois structure such that the Fundamental Theorem holds in the following form: there is a canonical bijection between Hopf subalgebras of H and normal intermediate fields $k \subseteq E \subseteq K$.*

The main result of this chapter can now be stated as follows: for any Hopf algebroid–Galois extension $N = M^H \subseteq M$, the endomorphism Hopf algebroid E is a Hopf algebroid scalar extension (to be defined below) of H. Since any Galois extension is also Galois with the endomorphism Hopf algebroid, we can say that any Galois Hopf–algebroid is a form of the endomorphism Hopf algebroid.

The Hopf algebroid scalar extension is modelled on the construction of the Breziński–Militaru theorem [**21**]. Briefly, it states that for a Hopf algebra H and a braided commutative algebra A in the Yetter–Drinfel'd category ${}_H\mathcal{YD}^H$ over H, there is a Hopf algebroid structure (over A) on the smash product $A\#H$ and may be regarded as a source of examples for Hopf algebroids. We shall now state the necessary definitions and the theorem without proof to lend substance to the preceding remarks and as an introduction to this chapter. Note that we follow the conventions and notations of [**21**].

Let H be a Hopf algebra over k. A Yetter–Drinfel'd module $\langle M, \triangleright, \rho^M \rangle$ over H is simultaneously a module and a comodule over H with the left action $\triangleright : H \otimes M \to M$ and right coaction $\rho^M : M \to M \otimes H$, such that the action and coaction satisfy the following compatibility condition:

$$h_{(1)} \triangleright m_{\langle 0 \rangle} \otimes h_{(2)} m_{\langle 1 \rangle} = (h_{(2)} \triangleright m)_{\langle 0 \rangle} \otimes (h_{(2)} \triangleright m)_{\langle 1 \rangle} h_{(1)} \tag{5.0.11}$$

An algebra $\langle A, \mu_A, \eta_A \rangle$ is called a braided commutative algebra (BCA) over H if it is an H–module H–comodule algebra such that for all $a, b \in A$

$$b_{\langle 0 \rangle}(b_{\langle 1 \rangle} \triangleright a) = ab \tag{5.0.12}$$

is satisfied. Now, Theorem 4.1 of [**21**] states

THEOREM 5.0.13. *(Breziński – Militaru) Let H be a bialgebra, $\langle A, \triangleleft \rangle$ a left H–module algebra and $\langle A, \rho^A \rangle$ a right H–comodule. Then $\langle A, \triangleleft, \rho^A \rangle$ is a BCA in ${}_H\mathcal{YD}^H$ if and only if $\langle A\#H, s, t, \Delta, \varepsilon \rangle$ is a bialgebroid over A with source, target, comultiplication and counit given by*

- $s(a) = a\#1_H$, $t(a) = a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle}$
- $\Delta(a\#h) = a\#h_{(1)} \underset{A}{\otimes} 1_A\#h_{(2)}$
- $\varepsilon(a\#h) = \varepsilon_H(h)a$

for all $a \in A$, $h \in H$, where $A\#H$ denotes the smash product algebra.

In [**3**], this construction was generalized to bialgebroids, Hopf algebroids and Frobenius Hopf algebroids in the sense that from a bialgebroid (resp. (Frobenius) Hopf algebroid) H and a BCA Q, the scalar extension $Q\#H$ is also a bialgebroid (resp. (Frobenius) Hopf algebroid) with base algebra replaced by Q. It was argued that this construction plays the rôle of the extension of scalars. In fact, it is even more useful: if $N \subseteq M$ is a Galois extension for a (Frobenius) Hopf algebroid, then the center $C_M(N)$ of the extension is always a braided commutative algebra for H and the corresponding scalar extension $C_M(N)\#H$ is the endomorphism Hopf algebroid for the extension.

1. The $\mathcal{YD}$ category and the monoidal center

To introduce the basic ideas and relate them to standard and well–known constructions, we first restrict ourselves to Hopf algebras. Generalizing the definitions to Hopf algebroids will then be relatively straightforward. So, for a Hopf algebra $\langle H, \mu, \eta; \Delta, \varepsilon; S\rangle$, we define

Definition 5.1.1. The category of Yetter–Drinfel'd modules, ${}^H\mathcal{YD}_H$ has objects $\langle Z, \triangleleft, \tau\rangle$, such that

- $\langle Z, \triangleleft\rangle$ is a right H–module,
- $\langle Z, \tau\rangle$ is a left H–comodule, and
- the Yetter–Drinfel'd *compatibility condition* holds:

$$h_{(2)}(z \triangleleft h_{(1)})_{\langle -1\rangle} \otimes (z \triangleleft h_{(1)})_{\langle 0\rangle} = z_{\langle -1\rangle}h_{(1)} \otimes z_{\langle 0\rangle} \triangleleft h_{(2)} \tag{5.1.2}$$

the arrows $f : Z \to Z'$ in ${}^H\mathcal{YD}_H$ are simultaneously H–module and H–comodule maps.

The Yetter–Drinfel'd category ${}^H\mathcal{YD}_H$ is monoidal and –more surprisingly– braided. To obtain the monoidal product of two Yetter–Drinfel'd modules Z and Z', we equip $Z \otimes Z'$ with the following action and coaction:

$$(z \otimes z') \triangleleft h = (z \triangleleft h_{(1)}) \otimes (z' \triangleleft h_{(2)}) \tag{5.1.3}$$

$$(z \otimes z')_{\langle -1\rangle} \otimes (z \otimes z')_{\langle 0\rangle} = z'_{\langle -1\rangle}z_{\langle -1\rangle} \otimes (z_{\langle 0\rangle} \otimes z'_{\langle 0\rangle}) \tag{5.1.4}$$

where the alternate order of z and z' in the second line should be noted. The monoidal unit is k with the (trivial) action and coaction of k as monoidal unit of M_H and ${}^H\mathsf{M}$, respectively.

The importance of the Yetter–Drinfel'd category stems from the fact that it is not only monoidal but also braided. In terms of the action and coaction, the braiding is given by

$$\begin{aligned} \beta_{Z,Z'} : Z \otimes Z' &\to Z' \otimes Z \\ z \otimes z' &\mapsto z' \triangleleft z_{\langle -1\rangle} \otimes z_{\langle 0\rangle} \end{aligned} \tag{5.1.5}$$

The existence of the antipode guarantees that the map β has an inverse, given by

$$\beta^{-1}_{Z,Z'}(z \otimes z') = z \triangleleft S(z'_{\langle -1\rangle}) \otimes z'_{\langle 0\rangle}.$$

In the absence of antipode (i.e. for a bialgebra) β generally has no inverse and we only have a pre–braiding. Recall that for $\langle {}^H\mathcal{YD}_H, \otimes, k; \beta\rangle$ to be a braided category, $\beta : \otimes \dot{\to} \otimes^{op}$ should be natural and it should satisfy the compatibilities

$$\beta_{V\otimes W,Z} = \beta_{V,Z} \circ \beta_{W,Z} \ \ \& \ \ \beta_{V,W\otimes Z} = \beta_{V,Z} \circ \beta_{V,W} \tag{5.1.6}$$

In particular, $\beta_{V,W}$ is an H–comodule map

$$\begin{array}{ccc} V \otimes W & \xrightarrow{\beta_{V,W}} & W \otimes V \\ \downarrow{\scriptstyle \delta_{V\otimes W}} & & \downarrow{\scriptstyle \delta_{W\otimes V}} \\ H \otimes V \otimes W & \xrightarrow[\beta_{V,W}]{} & H \otimes W \otimes V \end{array}$$

evaluating the diagram on $v \otimes w \in V \otimes W$ yields

$$\delta_{W\otimes V} \circ \beta_{V,W}(v \otimes w) = v_{\langle 0\rangle\langle -1\rangle}(w \triangleleft v_{\langle -1\rangle})_{\langle -1\rangle} \otimes (w \triangleleft v_{\langle -1\rangle})_{\langle 0\rangle} v_{\langle 0\rangle\langle 0\rangle} =$$
$$= w_{\langle -1\rangle} v_{\langle 0\rangle} \otimes w_{\langle 0\rangle} \triangleleft v_{\langle 0\rangle\langle -1\rangle} \otimes v_{\langle 0\rangle\langle 0\rangle} = (H \otimes \beta_{V,W}) \circ \delta_{V,W}(v \otimes w),$$

where the second equality holds because of the Yetter–Drinfel'd compatibility condition. It is similarly proved that $\beta_{V,W}$ is an H–module map. The relations 5.1.6 hold 'automatically', by the H–module and H–comodule properties alone – we omit the simple proof.

EXAMPLE 5.1.7. Crossed kG–modules Let kG be the group algebra of a finite group G. A Yetter–Drinfel'd module V is a kG–module that is simultaneously G–graded such that

$$|v \triangleleft g| = g^{-1}|v|g \tag{5.1.8}$$

is satisfied, where $|\cdot| : V \to G$ defines the grading. Indeed, kG–comodule is nothing but a G–graded module, and the $\mathcal{YD}$–compatibility condition reduces to 5.1.8. The braiding, for $v \in V$, $w \in W$ is given by

$$\beta_{V,W}(v \otimes w) = w \triangleleft |v| \otimes v \tag{5.1.9}$$

The most well–known example of this sort is undoubtedly the category of $\mathbf{Z}_{/2}$–graded ('super') vector spaces.

The paradigmatic example of a braided monoidal category is the module category of a quasi–triangular Hopf algebra, where the braiding is given by the quasi–triangular structure (the $\mathcal{R}$–matrix). The celebrated Drinfel'd (or quantum–) double construction associates to any Hopf algebra H a quasi–triangular Hopf algebra structure $\mathcal{D}(H)$ on $H^* \otimes H$. As it turns out, there is an intimate connection with the Yetter–Drinfel'd category.

PROPOSITION 5.1.10. *Let H be a Hopf algebra, and $\mathcal{D}(H) = H^{*op} \bowtie H$ the associated Drinfel'd double. Then there is an equivalence of categories ${}^H\mathcal{YD}_H \simeq \mathsf{M}_{\mathcal{D}(H)}$.*

Proof. Recall that the Drinfel'd double $\mathcal{D}(H) = H^{*op} \bowtie H$ has underlying k–module $H^* \otimes H$, with multiplication $\mu_{\mathcal{D}}$ given by

$$(\varphi \otimes h)(\psi \otimes g) = \psi_{(2)}\varphi \otimes h_{(2)}g \, \langle Sh_{(1)}, \psi_{(1)}\rangle\langle h_{(3)}, \psi_{(3)}\rangle \tag{5.1.11}$$

for all $\varphi, \psi \in H^*$ and $h, g \in H$ with unit $1_{\mathcal{D}} = \varepsilon \otimes 1$ and comultiplication

$$\Delta_{\mathcal{D}}(\varphi \otimes h) = (\varphi_{(1)} \otimes h_{(1)}) \otimes (\varphi_{(2)} \otimes h_{(2)}) \tag{5.1.12}$$

with counit $\varepsilon_{\mathcal{D}}(\varphi \otimes h) = \hat{\varepsilon}(\varphi)\varepsilon(h)$. If H is finite dimensional (or at least f.g.p.), then $\mathcal{D}(H)$ is quasi–triangular with $\mathcal{R}$–matrix given by

$$\mathcal{R} = \sum_a (\xi^a \otimes 1) \otimes (\hat{1} \otimes e_a) \tag{5.1.13}$$

where $\{\xi^a, e_a\}_{a=1}^N$ is a dual basis.

Clearly, H and H^{*op} are subalgebras of $\mathcal{D}(H)$ with inclusions given by $\iota : H \to \mathcal{D}(H)$, $\iota(h) = \hat{1} \otimes h$ and $\iota : H^{*op} \to \mathcal{D}(H)$, $\iota(\varphi) = \varphi \otimes 1$. We then have $\iota(\varphi)\iota(h) = \varphi \otimes h$ and

$$\iota(h)\iota(\varphi) = \iota(\varphi_{(2)})\iota(h_{(2)})\langle Sh_{(1)}, \varphi_{(1)}\rangle\langle h_{(3)}, \varphi_{(3)}\rangle \tag{5.1.14}$$

A right $\mathcal{D}(H)$–module is then a right H–module by restriction of the action to $\iota(H)$ and a left H–comodule by restriction to $\iota(H^{*op})$ and the isomorphism ${}^H\mathsf{M} \cong \mathsf{M}_{H^{*op}}$. The Yetter–Drinfel'd compatibility of the action and coaction is equivalent to the commutation relation of the subalgebras H and H^{*op} given by 5.1.14.

Looking at ${}^H\mathcal{YD}_H$ this way, the braided structure is due to the canonical $\mathcal{R}$–matrix of the Drinfel'd double. □

The monoidal center construction was introduced originally by Majid ([**54**], see also [**41**] and [**48**]). It associates to any abstract monoidal category a braided monoidal category and reproduces the Yetter–Drinfel'd category in the case of the module (or comodule) category M_H (M^H). For now, let $\langle \mathcal{C}, \otimes, \iota\rangle$ be a monoidal category. The center $\mathcal{Z}(\mathcal{C})$ is a category with objects the pairs $\langle Z, \theta\rangle$, where $Z \in \mathcal{C}$ and $\theta : Z \otimes \_ \dot{\to} \_ \otimes Z$ is a natural isomorphism (with components $\theta_Y : Z \otimes Y \to Y \otimes Z$) satisfying

$$\theta_{X \otimes Y} = (X \otimes \theta_Y) \circ (\theta_X \otimes Y) \quad \text{and} \quad \theta_\iota = Z \tag{5.1.15}$$

An arrow $\langle Z, \theta\rangle \to \langle Z', \theta'\rangle$ is an arrow $\alpha : Z \to Z'$ in $\mathcal{C}$ such that

$$(Y \otimes \alpha) \circ \theta_Y = \theta'_Y \circ (\alpha \otimes Y) \tag{5.1.16}$$

for all objects $Y \in \mathcal{C}$.

This category is monoidal. For objects, the monoidal product is defined by

$$\langle Z, \theta\rangle \otimes \langle Z', \theta'\rangle = \langle Z \otimes Z', (\theta_\_ \otimes Z') \circ (Z \otimes \theta'_\_)\rangle \tag{5.1.17}$$

For arrows, it is simply the ordinary monoidal product in $\mathcal{C}$. The category $\mathcal{Z}(\mathcal{C})$ is braided with

$$\beta_{\langle Z,\theta\rangle,\langle Z',\theta'\rangle} = \theta_{Z'} \,. \tag{5.1.18}$$

REMARK 5.1.19. A natural relaxation of the definition of the monoidal center is to let the natural isomorphism θ be only a natural transformation (without inverse). This leads to the notion of left (right) *weak* center, which seems to appear in [**73**], Definition 4.3 (see also [**25**], Section 1.3 and [**3**]).

The left weak center $\overrightarrow{\mathcal{Z}}(\mathcal{C})$ has objects $\langle Z,\theta\rangle$ with θ a (not necessarily invertible) natural transformation $\theta_Y : Z \otimes Y \to Y \otimes Z$ which satisfies 5.1.15. The right weak center $\overleftarrow{\mathcal{Z}}(\mathcal{C})$ has objects $\langle Z,\bar\theta\rangle$ with $\bar\theta_X : X \otimes Z \to Z \otimes X$. Of course, the relation 5.1.15 has to be modified accordingly, i.e. $\bar\theta_X$ satisfies

$$\bar\theta_{X\otimes Y} = (\bar\theta_X \otimes Y) \circ (X \otimes \bar\theta_Y) \quad \text{and} \quad \bar\theta_\iota = Z \tag{5.1.20}$$

This has the immediate consequence that the left and right weak centers are only pre–braided, with the pre–braiding defined as $\overrightarrow{\beta}_{\langle Z,\theta\rangle,\langle Z',\theta'\rangle} = \theta_{Z'}$ for the left weak center and $\overleftarrow{\beta}_{\langle Z,\bar\theta\rangle,\langle Z',\bar\theta'\rangle} = \bar\theta_Z$ for the right weak center. Taking $\mathcal{C} = \mathsf{M}_H$, with H a Hopf algebra, we shall see that the existence of antipode guarantees that the center is always braided, not just pre–braided. For bialgebras (bialgebroids) however, the weak center is the appropriate notion.

The center $\mathcal{Z}(\mathsf{M}_H)$ is the full subcategory of $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ in which the objects $\langle Z,\theta\rangle$ have invertible θ. For such objects $\langle Z,\theta^{-1}\rangle$ is an object in $\overleftarrow{\mathcal{Z}}(\mathsf{M}_H)$ in which $\bar\theta$ is invertible. The center is braided monoidal.

Next, we would like to connect this abstract construction with Yetter–Drinfel'd modules. To be in line with later generalizations to bialgebroids we consider bialgebras and accordingly, the one–sided weak center for the reason explained above. We find the following

PROPOSITION 5.1.21. *For a bialgebra H, there is an isomorphism of categories $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H) \simeq {}^H\mathcal{YD}_H$ which is prebraided.*

PROOF. First, consider the functor $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H) \to {}^H\mathcal{YD}_H$. Let $\langle Z,\theta\rangle \in \overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$. The crucial step is to introduce the left coaction

$$\tau : Z \to H \otimes Z, \qquad \tau(z) = \theta_H(z \otimes i) = z_{\langle -1\rangle} \otimes z_{\langle 0\rangle} \tag{5.1.22}$$

Before checking that this coaction makes Z a Yetter–Drinfel'd module, we note the following important fact: the natural transformation θ may be re–expressed in terms of this coaction as

$$\theta_X(z \otimes x) = x \triangleleft z_{\langle -1\rangle} \otimes z_{\langle 0\rangle} \tag{5.1.23}$$

This is proved by evaluating the following diagram expressing the naturality of θ_X on $z \otimes i$:

$$\begin{array}{ccccc}
Z \otimes H & & \xrightarrow{\quad\theta_H\quad} & & H \otimes Z \\
\Big\downarrow{\scriptstyle Z\otimes\alpha_x} & z \otimes i & \longrightarrow & z_{\langle -1 \rangle} \otimes z_{\langle -1 \rangle} & \Big\downarrow{\scriptstyle \alpha_x\otimes Z} \\
 & \big\downarrow & & \big\downarrow & \\
 & z \otimes x & \longrightarrow & \theta_X(z \otimes x) = x \triangleleft z_{\langle -1 \rangle} \otimes z_{\langle -1 \rangle} & \\
Z \otimes X & & \xrightarrow[\quad\theta_X\quad]{} & & X \otimes Z
\end{array}$$

where the map $\alpha_x : H \to X$ is defined as the evaluation of the action on x, $\alpha_x(h) = x \triangleleft h$. Composing the upper and right sides of the diagram yields $z \otimes i \mapsto z_{\langle -1 \rangle} \otimes z_{\langle -1 \rangle} \mapsto$
$\mapsto x \triangleleft z_{\langle -1 \rangle} \otimes z_{\langle -1 \rangle}$, by 5.1.22 and the definition of α_x. The left and lower sides compose to give $z \otimes i \mapsto z \otimes x \mapsto \theta_X(z \otimes x)$, proving 5.1.22. It is now easy to see that the coassocitivity and counitality of the coaction τ follow from 5.1.15. $\square$

The right weak center $\overleftarrow{\mathcal{Z}}(\mathsf{M}_H) = \overrightarrow{\mathcal{Z}}(\mathsf{M}_H^{\mathrm{coop}}) = \overrightarrow{\mathcal{Z}}(\mathsf{M}_{H^{\mathrm{coop}}})$ is the co-opposite of the left weak center, and consists of objects $\langle Z, \bar\theta \rangle$ where the natural transformation $\bar\theta_Y : Y \underset{R}{\otimes} Z \to Z \underset{R}{\otimes} Y$ is subject to 5.1.20. $\bar\theta$ determines a *right* coaction

$$\bar\tau : Z \to Z \underset{R}{\otimes} H\,, \qquad z \mapsto z^{\langle 0 \rangle} \underset{R}{\otimes} z^{\langle 1 \rangle} = \bar\theta_H(i \underset{R}{\otimes} z)$$

and can be expressed with this coaction as

$$\bar\theta_Y(y \underset{R}{\otimes} z) = z^{\langle 0 \rangle} \underset{R}{\otimes} y \triangleleft z^{\langle 1 \rangle}\,. \tag{5.1.24}$$

In the language of Yetter-Drinfeld modules the objects of the center are two-sided Yetter-Drinfeld modules $\langle Z, \triangleleft, \tau, \bar\tau \rangle \in {}^H\mathcal{YD}_H^H$ in which the two coactions are inverse to each other, i.e.,

$$z^{\langle 0 \rangle \langle 0 \rangle} \underset{R}{\otimes} z^{\langle -1 \rangle} \star z^{\langle 0 \rangle \langle 1 \rangle} = z \underset{R}{\otimes} i \tag{5.1.25}$$

$$z^{\langle 1 \rangle} \star z^{\langle 0 \rangle \langle -1 \rangle} \underset{R}{\otimes} z^{\langle 0 \rangle \langle 0 \rangle} = i \underset{R}{\otimes} z\,. \tag{5.1.26}$$

Combined with proposition 5.1.10, we have the following three–way isomorphism.

Theorem 5.1.27. *For a bialgebra H, the following categories are isomorphic:*

(1) *the Yetter–Drinfel'd category* ${}^H\mathcal{YD}_H$
(2) *the left weak center* $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$
(3) *the module category of the Drinfel'd double* $\mathsf{M}_{\mathcal{D}(H)}$

It would be desirable that the various generalizations of bialgebras (Hopf algebras) preserve Theorem 5.1.27.

We shall see that the isomorphism ${}^H\mathcal{YD}_H \simeq \overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ holds true for bialgebroids. In the case of bicoalgebroids, for example, the reasoning in the proof of Proposition 5.1.21 breaks down, but ${}^H\mathcal{YD}_H$ is still a subcategory of $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$.

2. Yetter–Drinfel'd modules over quantum groupoids

The constructions of the previous section generalize in a more or less straightforward manner to bialgebroids. The difficulties are due to the fact that we are working over the underlying category $\langle {}_R\mathsf{M}_R, \underset{R}{\otimes}, R\rangle$ instead of M_k, as we have done for bialgebras. From now on, M_H shall mean the module category over a bialgebroid H. We begin with the monoidal center: our previous definition may be repeated almost verbatim.

Definition 5.2.1. For a right bialgebroid H over R, the left weak center $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ is the category which has

- for objects, the pairs $\langle Z, \theta\rangle$, where $Z \in \mathsf{M}_H$ and $\theta_Y : Z \underset{R}{\otimes} Y \to Y \underset{R}{\otimes} Z$ a natural transformation satisfying

$$\theta_{X \underset{R}{\otimes} Y} = (X \underset{R}{\otimes} \theta_Y) \circ (\theta_X \underset{R}{\otimes} Y) \quad \text{and} \quad \theta_R = Z \tag{5.2.2}$$

- for arrows, $\alpha : \langle Z, \theta\rangle$ the H–module maps $\alpha : Z \to Z'$ which satisfy

$$(Y \underset{R}{\otimes} \alpha) \circ \theta_Y = \theta'_Y \circ (\alpha \underset{R}{\otimes} Y) \tag{5.2.3}$$

  for all objects $Y \in \mathsf{M}_H$.

$\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ is monoidal, with the monoidal product on objects given by

$$\langle Z, \theta\rangle \underset{R}{\otimes} \langle Z', \theta'\rangle = \langle Z \underset{R}{\otimes} Z', (\theta_{\_} \underset{R}{\otimes} Z') \circ (Z \underset{R}{\otimes} \theta'_{\_})\rangle \tag{5.2.4}$$

and on arrows, by the ordinary monoidal product of M_H, namely $\underset{R}{\otimes}$. The category $\mathcal{Z}(\mathsf{M}_H)$ is pre–braided with

$$\beta_{\langle Z,\theta\rangle, \langle Z',\theta'\rangle} = \theta_{Z'} \tag{5.2.5}$$

We now define the Yetter–Drinfel'd category over a bialgebroid.

Definition 5.2.6. For a right bialgebroid $\langle H, \star, i, R, \varphi_T, \varphi_B, \Delta_R, \varphi_R\rangle$ the category ${}^H\mathcal{YD}_H$ has objects $\langle Z, \triangleleft, \tau\rangle$ where

(1) $\langle Z, \triangleleft\rangle$ is a right H-module
(2) $\langle Z, \tau\rangle$ is a left H-coaction
(3) The action and coaction satisfy the Yetter-Drinfeld condition

$$h^{[2]} \star (z \triangleleft h^{[1]})^{\langle -1\rangle} \underset{R}{\otimes} (z \triangleleft h^{[1]})^{\langle 0\rangle} = z^{\langle -1\rangle} \star h^{[1]} \underset{R}{\otimes} z^{\langle 0\rangle} \triangleleft h^{[2]}\,. \tag{5.2.7}$$

The arrows are the H-module H-comodule maps $Z \to Z'$. The monoidal product of two Yetter-Drinfeld modules Z and Z' is $Z \underset{R}{\otimes} Z'$ equipped with

$$(z \underset{R}{\otimes} z') \triangleleft h = (z \triangleleft h^{[1]}) \underset{R}{\otimes} (z' \triangleleft h^{[2]})$$
$$(z \underset{R}{\otimes} z')^{\langle -1\rangle} \underset{R}{\otimes} (z \underset{R}{\otimes} z')^{\langle 0\rangle} = z'^{\langle -1\rangle} \star z^{\langle -1\rangle} \underset{R}{\otimes} (z^{\langle 0\rangle} \underset{R}{\otimes} z'^{\langle 0\rangle})$$

The monoidal unit is R with $r \triangleleft h = r \star h$ and $r^{\langle -1\rangle} \underset{R}{\otimes} r^{\langle 0\rangle} = \varphi_B(r) \underset{R}{\otimes} e$. The prebraiding is defined by

$$\beta_{Z,Z'} : Z \underset{R}{\otimes} Z' \to Z' \underset{R}{\otimes} Z, \qquad z \underset{R}{\otimes} z' \mapsto z' \triangleleft z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} \tag{5.2.8}$$

The definition is essentially identical to the bialgebra case, however one should bear in mind the R-R-bimodule properties that are implied. We note, for reference, that $\langle Z, \triangleleft\rangle$ being a right H-module means it is also an R-R-bimodule via $r \cdot z \cdot r' = z \triangleleft (\varphi_B(r) \star \varphi_T(r'))$. Furthermore, $\langle Z, \tau\rangle$ being a left H-coaction means

(1) $\tau : Z \to H \underset{R}{\otimes} Z$ is an R-R-bimodule map in the sense of

$$(r \cdot z \cdot r')^{\langle -1\rangle} \underset{R}{\otimes} (r \cdot z \cdot r')^{\langle 0\rangle} = \varphi_B(r') \star z^{\langle -1\rangle} \star \varphi_B(r) \underset{R}{\otimes} z^{\langle 0\rangle}, \tag{5.2.9}$$

(2) τ is coassociative and counital,

$$z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle \langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle \langle 0\rangle} = z^{\langle -1\rangle [1]} \underset{R}{\otimes} z^{\langle -1\rangle [2]} \underset{R}{\otimes} z^{\langle 0\rangle}$$
$$\varphi_R(z^{\langle -1\rangle}) \cdot z^{\langle 0\rangle} = z$$

(3) τ factorizes through $H \underset{R}{\times} Z \subseteq H \underset{R}{\otimes} Z$, i.e. the Takeuchi property holds:

$$\varphi_T(r) \star z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} = z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} \triangleleft \varphi_T(r) \tag{5.2.10}$$

The proof of the isomorphism $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H) \cong {}^H\mathcal{YD}_H$ is along the lines of the proof for bialgebras. We indicate here the finer points. For an object $\langle Z, \theta\rangle \in \overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$, introduce again the coaction $\tau(z) = \theta_H(z \underset{R}{\otimes} i) = z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle}$ which is the composite map:

$$\tau :\ Z \longrightarrow Z \underset{R}{\otimes} R \xrightarrow{Z \underset{R}{\otimes} \varphi_B} Z \underset{R}{\otimes} H \xrightarrow{\theta_H} H \underset{R}{\otimes} Z$$

τ is an R-R-bimodule map. The left R-module structure is clearly preserved, since all maps in the composite are left R-module maps. As for the right R-module structure, we have

$$\begin{aligned}\varphi_B(r) \star z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} &= \varphi_B(r) \triangleleft z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} = \theta_H(z \underset{R}{\otimes} \varphi_B(r)) \\ &= \theta_H(z \triangleleft \varphi_T(r) \underset{R}{\otimes} i) = \tau(z \triangleleft \varphi_T(r)) = \tau(z \cdot r).\end{aligned}$$

In particular, this means that the right R–action we could construct from the H–coaction, as we did in Lemma 3.2.5, coincides with the original R–action.

For τ to be a coaction, the Takeuchi property 5.2.10 must also hold, i.e. for $r \in R$, $z \in Z$:

$$\begin{aligned}\varphi_T(r) \star z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} &= \varphi_T(r) \triangleleft z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} = \theta_H(z \underset{R}{\otimes} \varphi_T(r)) \\ &= \theta_H((z \underset{R}{\otimes} i) \triangleleft \varphi_T(r)) = \tau(z) \triangleleft \varphi_T(r) \\ &= z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} \triangleleft \varphi_T(r).\end{aligned}$$

Again, the naturality of θ allows us to re-express θ_X in terms of the coaction τ for all X:

$$\theta_X(z \underset{R}{\otimes} x) = x \triangleleft z^{\langle -1\rangle} \underset{R}{\otimes} z^{\langle 0\rangle} \tag{5.2.11}$$

The coassociativity and counitality of τ is proved as in Proposition 5.1.21, and the condition for 5.2.11 to be an H-module map is equivalent to the Yetter–Drinfel'd condition 5.2.7. Summing up, we have

THEOREM 5.2.12. *For a right bialgebroid H over R, there is a monoidal pre–braided isomorphism of categories $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H) \cong {}^H\mathcal{YD}_H$.*

DEFINITION 5.2.13. For a right bialgebroid H the commutative monoids in $\mathcal{Z}(\mathsf{M}_H)$ are called BCAs (braided commutative algebras) over H. The commutative monoids in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ and $\overleftarrow{\mathcal{Z}}(\mathsf{M}_H)$ are called left and right pre-BCAs over H, respectively.

Therefore a left pre-BCA consists of an algebra Q with an algebra map $\eta : R \to Q$ and a Yetter-Drinfeld module structure $\langle Q, \triangleleft, \tau\rangle \in {}^H\mathcal{YD}_H$ such that

$$\eta(r)\, q\, \eta(r') = r \cdot q \cdot r' \tag{5.2.14}$$

$$(qq') \triangleleft h = (q \triangleleft h^{[1]})(q' \triangleleft h^{[2]}) \tag{5.2.15}$$

$$1 \triangleleft h = \eta\, \varphi_R(h) \tag{5.2.16}$$

$$(qq')^{\langle -1\rangle} \underset{R}{\otimes} (qq')^{\langle 0\rangle} = q'^{\langle -1\rangle} \star q^{\langle -1\rangle} \underset{R}{\otimes} q^{\langle 0\rangle} q'^{\langle 0\rangle} \tag{5.2.17}$$

$$\eta(r)^{\langle -1\rangle} \underset{R}{\otimes} \eta(r)^{\langle 0\rangle} = \varphi_B(r) \underset{R}{\otimes} 1 \tag{5.2.18}$$

and the prebraided commutativity

$$(q' \triangleleft q^{\langle -1\rangle})q^{\langle 0\rangle} = qq' \tag{5.2.19}$$

holds. If Q is a BCA then there exists also a right coaction $\bar{\tau}$ with which $\langle Q, \triangleleft, \bar{\tau}\rangle \in \mathcal{YD}_H^H$ and which is inverse to τ in the sense of equations (5.1.25), (5.1.26).

We note that the ground ring R of the bialgebroid is always a BCA with the structure $\langle R, \mu_R, R\rangle$ that comes from R being the monoidal unit of $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$.

3. Scalar extension of quantum groupoids

We shall first state the generalization of 5.0.13 to bialgebroids and give a straightforward proof. After presenting a more categorical approach we prove some important properties of the construction and discuss the extension to Hopf algebroids and Frobenius Hopf algebroids.

THEOREM 5.3.1. *Let H be a right bialgebroid and let Q be a left pre-BCA over H. Then the smash product $G := H\#Q$ is a right bialgebroid over Q with structure maps*

$$s_G(q) = i\#q \tag{5.3.2}$$

$$t_G(q) = q^{\langle -1\rangle}\#q^{\langle 0\rangle} \tag{5.3.3}$$

$$\Delta_G(h\#q) = (h^{[1]}\#1) \underset{Q}{\otimes} (h^{[2]}\#q) \tag{5.3.4}$$

$$\varepsilon_G(h\#q) = \eta(\varepsilon_H(h))q \tag{5.3.5}$$

where $\eta : R \to Q$ is the unit of Q. Moreover, $h \mapsto h\#1$ is a bialgebroid map $\iota : H \to G$.

PROOF. First note that s_G (t_G) is an algebra morphism (anti–algebra morphism) from Q to $Q\#H$, respectively, and the Q-Q-bimodule structure on $H\#Q$ is given by:

$$q' \cdot (h\#q) = (h\#q)t_G(q') = hq'^{\langle -1\rangle}\#q'^{\langle 0\rangle}q, \quad \text{and} \tag{5.3.6}$$

$$(h\#q) \cdot q' = (h\#q)s_G(q') = h\#qq' \tag{5.3.7}$$

(recall that $H\#Q$ is a right bialgebroid). The comultipliction is a map

$$\Delta_G : H\#Q \to H\#Q \underset{Q}{\times} H\#Q \tag{5.3.8}$$

in other words, the Takeuchi property holds:

$$\begin{aligned}
s_G(q')(h\#q)^{[1]} \underset{Q}{\otimes} (h\#q)^{[2]} &= i \star h^{[1][1]}\#q' \triangleleft h^{[1][2]} \underset{Q}{\otimes} h^{[2]}\#q \\
&= h^{[1]}\#1 \underset{Q}{\otimes} h^{[2]}(q' \triangleleft h^{[1][2]})^{\langle -1\rangle}\#(q' \triangleleft h^{[1][2]})^{\langle 0\rangle}q \\
&= h^{[1]}\#1 \underset{Q}{\otimes} q'^{\langle -1\rangle}h^{[2][1]}\#(q'^{\langle 0\rangle} \triangleleft h^{[2][2]})q \\
&= h^{[1]}\#1 \underset{Q}{\otimes} (q'^{\langle -1\rangle}\#q'^{\langle 0\rangle})(h^{[2]}\#q) = (h\#q)^{[1]} \underset{Q}{\otimes} t_G(q')(h\#q)^{[2]}.
\end{aligned}$$

Δ_G is easily seen to be comultiplicative, and also multiplicative by the following calculation:

$$\begin{aligned}
\Delta_G((h\#q)(h'\#q')) &= \Delta_G(h \star h'^{[1]})\#(q \triangleleft h'^{[2]})q') \\
&= (h \star h'^{[1]})^{[1]}\#1 \underset{Q}{\otimes} (h \star h'^{[1]})^{[2]}\#(q \triangleleft h'^{[2]})q' \\
&= h \star h'^{[1][1]}\#1 \triangleleft h'^{[1][2]} \underset{Q}{\otimes} h^{[2]} \star h'^{[2][1]}\#(q \triangleleft h'^{[2][2]})q' \\
&= \Delta_G((h\#q)\Delta_G(h'\#q').
\end{aligned}$$

The rest of the bialgebroid axioms are easily checked. □

We can look at the bialgebroid structure on $H\#Q$ more abstractly as follows. Recall that for any H-module algebra Q, the category of modules over the smash product algebra $\mathsf{M}_{H\#Q}$ can be identified with the category of internal Q-modules in M_H, which we denote $(\mathsf{M}_H)_Q$. To any action $X \underset{R}{\otimes} Q \to X$, $x \otimes q \mapsto x \cdot q$ in M_H, we can associate the $H\#Q$–action

$$X \otimes (H\#Q) \to X, \;\; x \otimes (h\#q) \mapsto (x \triangleleft h) \cdot q$$

Vice versa, any $H\#M$–module is both an H–module and an M–module, and the M–action is an H–module map. If Q is moreover a BCA, then pre–braided commutativity ensures that right Q-modules in M_H are also left Q-modules. This defines a categorical embedding

(5.3.9) $$I : \mathsf{M}_{H\#Q} = (\mathsf{M}_H)_Q \hookrightarrow {}_Q(\mathsf{M}_H)_Q$$

into the category of internal Q-Q-bimodules. Composing with the forgetful functor $U : {}_Q(\mathsf{M}_H)_Q \to {}_Q\mathsf{M}_Q$, we have:

$$\begin{array}{ccc} \mathsf{M}_{H\#Q} & \xhookrightarrow{I} & {}_Q(\mathsf{M}_H)_Q \\ & \searrow & \downarrow U \\ & & {}_Q\mathsf{M}_Q \end{array}$$

Both I and U are strong monoidal, hence also the composite $UI : \mathsf{M}_{H\#Q} \to {}_Q\mathsf{M}_Q$. By Schauenburg's theorem, this is equivalent to a Q-bialgebroid structure on $H\#Q$ such that the monoidal structure of $\mathsf{M}_{H\#Q}$ is that of the module category of the bialgebroid! The forgetful functor UI is associated to the algebra morphism

(5.3.10) $$Q^{op} \otimes Q \to H\#Q$$

(5.3.11) $$q \otimes q' \mapsto q^{\langle -1\rangle}\#q^{\langle 0\rangle}q'$$

Compared with 5.3.1, this is just the map $t_G \otimes s_G : Q^{op} \otimes Q \to H\#Q$.

In the motivating example of Hopf algebra scalar extensions, the tensor functor $K' \underset{K}{\otimes} \_$ extended to Hopf algebra map $H \to K' \underset{K}{\otimes} H$. For the present case of bialgebroid scalar extensions, we have the following

Proposition 5.3.12. *If H is a right bialgebroid over R and Q is a left pre-BCA over H then the functor $\_ \underset{R}{\otimes} Q : \mathsf{M}_H \to (\mathsf{M}_H)_Q$ is strong monoidal.*

Proof. The monoidal structure of the functor $\_ \underset{R}{\otimes} Q$ is given by the invertible natural transformation:

$$\mathsf{Q}_{Y,Y'} : (Y \underset{R}{\otimes} Q) \underset{Q}{\otimes} (Y' \underset{R}{\otimes} Q) \to (Y \underset{R}{\otimes} Y') \underset{R}{\otimes} Q$$
$$(y \underset{R}{\otimes} q) \underset{Q}{\otimes} (y' \underset{R}{\otimes} q') \mapsto (y \underset{R}{\otimes} y' \triangleleft q^{\langle -1\rangle}) \underset{R}{\otimes} q^{\langle 0\rangle}q'$$

the inverse is given by $(y \underset{R}{\otimes} y') \underset{R}{\otimes} q \mapsto (y \underset{R}{\otimes} 1) \underset{Q}{\otimes} (y' \underset{R}{\otimes} q)$. The unit part of the monoidal structure is the $H\#Q$-module map

$$\mathsf{Q}_0 : Q \to R \underset{R}{\otimes} Q\,, \qquad q \mapsto e \underset{R}{\otimes} q$$

which is obviously also invertible. □

In fact, this leads to yet another proof of the Theorem: consider the comonoid $\langle H, \Delta_H, \varepsilon_H\rangle$ in M_H. The strong monoidal functor $\_ \underset{R}{\otimes} Q$ maps H precisely to the comonoid $\langle G, \Delta_G, \varepsilon_G\rangle$ in

M_G (with $G = H\#Q$ and the notations of 5.3.1), which is a strong comonoid. By the results of [**80**], $\mathrm{End}_G\, G \simeq G$ has a bialgebroid structure.

We now prove two important properties of the scalar extension. First, it is transitive in the following sense.

PROPOSITION 5.3.13. *If Q is a BCA over H and P is a BCA over $H\#Q$ then P is a BCA over H, too. Furthermore, $(H\#Q)\#_Q P \cong H\#_R P$.*

PROOF. The unit of P as a monoid in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ will be the composition of the unit of Q as an H-module and the unit of P as an $H\#Q$-module: $\eta = \eta^P \circ \eta^Q : R \to P$. The H-module structure on P is the restriction of the $H\#Q$-action: $p \triangleleft h = p \triangleleft (h\#1)$. The H-comodule structure on P is slightly more complicated. For $p \in P$, it will be denoted $p \mapsto p^{\{-1\}} \underset{R}{\otimes} p^{\{0\}}$ (note the curly brackets!) and is defined as

$$p^{\{-1\}} \underset{R}{\otimes} p^{\{0\}} = (H \underset{R}{\otimes} \eta^P)(p^{\langle -1\rangle})p^{\langle 0\rangle} \equiv p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle 0\rangle Q} \cdot p^{\langle 0\rangle} = p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle 0\rangle} \triangleleft t_{H\#Q}(p^{\langle -1\rangle Q}). \tag{5.3.14}$$

Recall that $p^{\langle -1\rangle} \underset{R}{\otimes} p^{\langle 0\rangle}$ denotes the $H\#Q$-coaction on P. In the second equality, we introduced the notation $g = g^H \underset{R}{\otimes} g^Q$ for elements $g \in H\#Q$, to be used throughout.

It is simple to check counitality:

$$\begin{aligned}\varepsilon_H(p^{\{-1\}}) \cdot p^{\{0\}} &= \varepsilon_H(p^{\langle -1\rangle H}) \cdot (p^{\langle -1\rangle Q} \cdot p^{\langle 0\rangle}) \\ &= (\varepsilon_H(p^{\langle -1\rangle H}) \cdot p^{\langle -1\rangle Q}) \cdot p^{\langle 0\rangle} = \varepsilon_{H\#Q}(p^{\langle -1\rangle}) \cdot p^{\langle 0\rangle} = p.\end{aligned}$$

Coassociativity requires a longer, but straightforward calculation:

$$\begin{aligned}&p^{\{-1\}} \underset{R}{\otimes} p^{\{0\}\{-1\}} \underset{R}{\otimes} p^{\{0\}\{0\}} \\ &= p^{\langle -1\rangle H} \underset{R}{\otimes} (p^{\langle 0\rangle\langle -1\rangle} t_{H\#Q}(p^{\langle -1\rangle Q}))^H \underset{R}{\otimes} (p^{\langle 0\rangle\langle -1\rangle} t_{H\#Q}(p^{\langle -1\rangle Q}))^Q \cdot p^{\langle 0\rangle\langle 0\rangle} \\ &= p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle 0\rangle\langle -1\rangle H} \star p^{\langle -1\rangle Q\langle -1\rangle} \underset{R}{\otimes} p^{\langle -1\rangle Q\langle 0\rangle} p^{\langle 0\rangle\langle -1\rangle Q} \cdot p^{\langle 0\rangle\langle 0\rangle} \\ &= p^{\langle -2\rangle H} \underset{R}{\otimes} p^{\langle -1\rangle H} \star p^{\langle -2\rangle Q\langle -1\rangle} \underset{R}{\otimes} p^{\langle -2\rangle Q\langle 0\rangle} p^{\langle -1\rangle Q} \cdot p^{\langle 0\rangle} \\ &= p^{\langle -2\rangle H} \underset{R}{\otimes} p^{\langle -2\rangle Q} \cdot (p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle -1\rangle Q}) \cdot p^{\langle 0\rangle} \\ &= p^{\langle -1\rangle H[1]} \underset{R}{\otimes} p^{\langle -1\rangle H[2]} \underset{R}{\otimes} p^{\langle -1\rangle Q} \cdot p^{\langle 0\rangle} = p^{[-1][1]} \underset{R}{\otimes} p^{[-1][2]} \underset{R}{\otimes} p^{[0]},\end{aligned}$$

where we have used the definition 5.3.14 in the first equality, 5.3.3 in the third equality, the Q-Q-bimodule structure 5.3.6 in the third equality and the definition of the coproduct of G 5.3.4 in the fifth.

To prove the Takeuchi property, we use 5.3.6 and the fact that $s_{H\#Q}(Q)$ and $t_{H\#Q}$ commute

$$\begin{aligned} p^{\{-1\}} \underset{R}{\otimes} p^{\{0\}} \triangleleft s_H(r) &= p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle -1\rangle Q} \cdot (p^{\langle 0\rangle} \triangleleft (s_H(r)\#1)) \\ &= p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle -1\rangle Q} \cdot (p^{\langle 0\rangle} \triangleleft (s_{H\#Q}(\eta^Q(r))) = s_H(r) \star p^{\langle -1\rangle} \underset{R}{\otimes} p^{\langle 0\rangle}. \end{aligned}$$

With the Yetter–Drinfel'd condition,

$$\begin{aligned} & h^{[2]} \star (p \triangleleft h^{[1]})^{\langle -1\rangle} \underset{R}{\otimes} (p \triangleleft h^{[1]})^{\langle 0\rangle} \\ &= h^{[2]} \star (p \triangleleft (h^{[1]}\#1))^{\langle -1\rangle H} \underset{R}{\otimes} (p \triangleleft (h^{[1]}\#1))^{\langle -1\rangle Q} \cdot (p \triangleleft (h^{[1]}\#1))^{\langle 0\rangle} \\ &= p^{\langle -1\rangle H} \star h^{[1]} \underset{R}{\otimes} (p^{\langle -1\rangle Q} \triangleleft h^{[2]}) \cdot (p^{\langle 0\rangle} \triangleleft h^{[3]}) = p^{[-1]} \star h^{[1]} \underset{R}{\otimes} p^{[0]} \star h^{[2]}, \end{aligned}$$

we have then established that P is an object of $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$. It remains to show that it is also a commutative monoid in the weak center. The bimodule property 5.2.14 holds:

$$\begin{aligned} r \cdot p \cdot r' &= p \triangleleft (t_H(r) \star s_H(r')\#1) = p \triangleleft t_{H\#Q}(\eta^Q(r)) s_{H\#Q}(\eta^Q(r')) \\ &= \eta^P(\eta^Q(r)) p \eta^P(\eta^Q(r')) = \eta(r) p \eta(r') \end{aligned}$$

P is obviously an H–module algebra by restriction of the $H\#Q$–action. It is also an H–comodule algebra, by the following calculation.

$$\begin{aligned} (pp')^{\{-1\}} \underset{R}{\otimes} (pp')^{\{-1\}} &= p'^{\langle -1\rangle H} \star p^{\langle -1\rangle H[1]} \underset{R}{\otimes} (p'^{\langle -1\rangle Q} \triangleleft p^{\langle -1\rangle H[2]}) p^{\langle -1\rangle Q} \cdot p^{\langle 0\rangle} p'^{\langle 0\rangle} \\ &= p'^{\langle -1\rangle H} \star p^{\langle -2\rangle H} \underset{R}{\otimes} p^{\langle -2\rangle Q} (p'^{\langle -1\rangle Q} \triangleleft p^{\langle -1\rangle H}) p^{\langle -1\rangle Q} \cdot p^{\langle 0\rangle} p'^{\langle 0\rangle} \\ &= p'^{\langle -1\rangle H} \star p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle -1\rangle Q} \cdot \eta^P(p'^{\langle -1\rangle Q} \triangleleft p^{\langle 0\rangle\langle -1\rangle}) p^{\langle 0\rangle\langle 0\rangle} p'^{\langle 0\rangle} \\ &= p'^{\langle -1\rangle H} \star p^{\langle -1\rangle H} \underset{R}{\otimes} \eta^P(p^{\langle -1\rangle Q}) p^{\langle 0\rangle} \eta^P(p'^{\langle -1\rangle Q}) p'^{\langle 0\rangle} \\ &= p'^{\{-1\}} \star p^{\{-1\}} \underset{R}{\otimes} p^{\langle 0\rangle} p'^{\langle 0\rangle} \end{aligned}$$

In the second equality, we used the coassociativity of the $H\#Q$–coaction and 5.3.4. In the fourth, we made use of the braiding formula 5.2.8. The H–coaction is unital, $1^{\{-1\}} \underset{R}{\otimes} 1^{\{0\}} = i \underset{R}{\otimes} 1$, and pre–braided commutativity also holds,

$$(p' \triangleleft p^{\{-1\}}) p^{\{0\}} = (p' \triangleleft p^{\langle -1\rangle}) p^{\langle 0\rangle} = pp'$$

Summing up, we have shown that P is a BCA in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$. The isomorphism between the iterated smash product and $H\#P$ is provided by the map $h \underset{R}{\otimes} q \underset{Q}{\otimes} p \mapsto h \underset{R}{\otimes} q \cdot p$. It is easily seen to be an invertible bialgebroid map. □

Remark 5.3.15 (A note on coring extensions). Answering a question of Gabriella Böhm in the affirmative, we show that the coaction 5.3.14 is in fact derived from the (left) coring extension H of $H\#Q$. Coring extensions were defined in [**14**]; in this paper, the author only considers *right* coring extensions, observing that the case of left extensions follow from an obvious left–right correspondence. We shall quote the left–sided versions of definitions and results on coring extensions, but refer the reader to [**14**] for details.

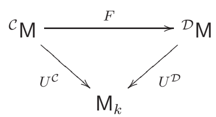
Definition 5.3.16. Let $A, B \in k - \mathsf{Alg}$, $\mathcal{C}$ an A–coring and $\mathcal{D}$ a B–coring; then $\mathcal{D}$ is a *left* coring extension of C iff $\mathcal{C}$ is an $\mathcal{D}$-$\mathcal{C}$ bicomodule with right coaction $\Delta_{\mathcal{C}}$.

The bialgebroids H and $H\#Q$ provide an example of a coring extension. Indeed, H is an R–coring and $H\#Q$ is a Q–coring; moreover, $H\#Q \in {}^{H}\mathsf{M}^{H\#Q}$ with $\Delta_{H\#Q}$ as right coaction: the bicomodule property follows trivially from the coassociativity of Δ_H.

Proposition 5.3.13 then follows from the following fundamental result on coring extensions (cf. theorem 2.6 of [**14**]).

Theorem 5.3.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be corings over A and B, respectively. Then the following are equivalent:*

(1) *$\mathcal{D}$ is a left extension of $\mathcal{C}$*
(2) *there exists a k–additive functor $F : {}^{\mathcal{C}}\mathsf{M} \to {}^{\mathcal{D}}\mathsf{M}$ with the factorization property*



where $U^{\mathcal{C}}$ and $U^{\mathcal{D}}$ are the canonical forgetful functors.

For the full proof, cf. [**14**]. We note, however, that the functor F is constructed as follows. Denote $\sigma : \mathcal{C} \to \mathcal{D} \otimes \mathcal{C}$ the $\mathcal{D}$–coaction on $\mathcal{C}$; for a left $\mathcal{C}$–comodule $M \in {}^{\mathcal{C}}\mathsf{M}$, define a $\mathcal{D}$–comodule structure on M with

$$M \xrightarrow{\sim} \mathcal{C}\,\Box_C\, M \xrightarrow{\sigma\Box M} (\mathcal{D} \underset{B}{\otimes} \mathcal{C})\,\Box_C\, M \xrightarrow{\sim} \mathcal{D} \underset{B}{\otimes} M$$
$$m \mapsto m^{\langle -1\rangle\{-1\}} \underset{C}{\otimes} \varepsilon_{\mathcal{C}}(m^{\langle -1\rangle\{0\}}) \cdot m^{\langle 0\rangle}, \tag{5.3.18}$$

where we used the notation $\sigma(c) = c^{\{-1\}} \otimes c^{\{0\}}$.

Setting $\mathcal{C} = H\#Q$ and $\mathcal{D} = H$, it follows that we have a functor $F : {}^{H\#Q}\mathsf{M} \to {}^{H}\mathsf{M}$, which is essentially the content of proposition 5.3.13. The H–coaction on $H\#Q$ is given by $\sigma : H\#Q \to H \underset{R}{\otimes} (H\#Q)$, $h\#q \mapsto h^{[1]} \otimes (h^{[2]} \otimes q)$. For a comodule $P \in {}^{H\#Q}\mathsf{M}$, the H–coaction (from 5.3.18) is given by

$$p \mapsto p^{\langle -1\rangle H} \otimes p^{\langle -1\rangle Q} \otimes p^{\langle 0\rangle} \mapsto (p^{\langle -1\rangle H})^{[1]} \otimes (p^{\langle -1\rangle H})^{[2]} \otimes p^{\langle -1\rangle Q} \otimes p^{\langle 0\rangle} \mapsto$$
$$\mapsto (p^{\langle -1\rangle H})^{[1]} \otimes (\varepsilon_H((p^{\langle -1\rangle H})^{[2]})p^{\langle -1\rangle Q}) \cdot p^{\langle 0\rangle} = p^{\langle -1\rangle H} \otimes p^{\langle -1\rangle Q} \cdot p^{\langle 0\rangle},$$

which is exactly the coaction 5.3.14.

The next proposition is, in a sense, a converse to the transitivity of scalar extension.

PROPOSITION 5.3.19. *If $\eta : Q \to P$ is a monoid morphism in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ between commutative monoids (pre–BCAs over H) then there is a unique pre–BCA structure on P over $H\#Q$ which returns the original pre–BCA over H when Proposition 5.3.13 is applied to it*

PROOF. The $H\#Q$–action on P is

$$p \triangleleft (h\#q) = (p \triangleleft h)\eta(q) \tag{5.3.20}$$

The action 5.3.20 is the unique $H\#Q$–action on P which restricts to the original H–action. It is clearly an associative and unital action, since η is both an H–module map and an algebra morphism, and makes P an $H\#Q$–module algebra. The Q–bimodule structure induced by the $H\#Q$–action is the same as the one coming from the map η. It is obvious from the definitions that $p \triangleleft s_{H\#Q} = p\eta(q)$. For the left action, we use braided commutativity in M_H to obtain

$$\eta(q)p = (p \triangleleft \eta(q)^{\{-1\}})\eta(q)^{\{0\}} = (p \triangleleft q^{\langle -1\rangle})\eta(q^{\langle 0\rangle}) = p \triangleleft t_{H\#Q}(q). \tag{5.3.21}$$

The $H\#Q$–coaction on P is

$$p^{\langle -1\rangle} \underset{Q}{\otimes} p^{\langle 0\rangle} = (p^{\langle -1\rangle H} \underset{R}{\otimes} 1_Q) \underset{Q}{\otimes} p^{\langle 0\rangle H} \tag{5.3.22}$$

This is the unique $H\#Q$–coaction which projects to the original H–coaction. From 5.3.14, we should have $p^{\langle -1\rangle H} \underset{R}{\otimes} p^{\langle -1\rangle Q} \cdot p^{\langle 0\rangle} = p^{\{-1\}} \underset{R}{\otimes} p^{\{0\}}$. Applying the inverse of the natural isomorphism $Q \underset{Q}{\otimes} \_ \stackrel{\cdot}{\to} id$ to both sides, we have

$$(p^{\{-1\}}\#1_Q) \underset{Q}{\otimes} p^{\{0\}} = (p^{\langle -1\rangle H}\#p^{\langle -1\rangle Q}) \underset{Q}{\otimes} p^{\langle 0\rangle} = p^{\langle -1\rangle} \underset{Q}{\otimes} p^{\langle 0\rangle} \tag{5.3.23}$$

The bimodule property of 5.2.9 is obtained as follows.

$$\begin{aligned}
(q\cdot p \cdot q')^{\langle -1\rangle} \underset{Q}{\otimes} (q \cdot p \cdot q')^{\langle 0\rangle} &= (\eta(q)p\eta(q'))^{\{-1\}}\#1 \underset{Q}{\otimes} (\eta(q)p\eta(q'))^{\{0\}} \\
&= \eta(q')^{\{-1\}} \star p^{\{-1\}} \star \eta(q)^{\{-1\}}\#1 \underset{Q}{\otimes} \eta(q)^{\{0\}}p^{\{0\}}\eta(q')^{\{0\}} \\
&= q'^{\langle -1\rangle} \star p^{\{-1\}} \star q^{\langle -1\rangle}\#1 \underset{Q}{\otimes} \eta(q^{\langle 0\rangle})p^{\{0\}}\eta(q'^{\langle 0\rangle}) \\
&= q'^{\langle -1\rangle} \star p^{\{-1\}[1]} \star q^{\langle -1\rangle}\#1 \underset{Q}{\otimes} \eta(q^{\langle 0\rangle})(\eta(q'^{\langle 0\rangle}) \triangleleft p^{\{-1\}[2]})p^{\{0\}} \\
&= q'^{\langle -1\rangle} \star p^{\{-1\}[1]} \star q^{\langle -1\rangle}\#q^{\langle 0\rangle}(q'^{\langle 0\rangle} \triangleleft p^{\{-1\}[2]}) \underset{Q}{\otimes} p^{\{0\}} \\
&= q'^{\langle -1\rangle} \star p^{\{-1\}[1]} \star q^{\langle -2\rangle}\#(q'^{\langle 0\rangle} \triangleleft p^{\{-1\}[2]} \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{Q}{\otimes} p^{\{0\}} \\
&= (q'^{\langle -1\rangle}\#q'^{\langle 0\rangle})(p^{\{-1\}}\#1)(q^{\langle -1\rangle}\#q^{\langle 0\rangle}) = t_{H\#Q}(q')p^{\langle -1\rangle}t_{H\#Q}(q) \underset{Q}{\otimes} p^{\langle 0\rangle}.
\end{aligned}$$

Coassociativity and counitality of 5.3.22 follow simply from the coalgebra structure 5.3.4 and 5.3.5 of the smash product. For P to be a Yetter–Drinfel'd module in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$, we need the Takeuchi property,

$$p^{\langle -1\rangle} \underset{Q}{\otimes} p^{\langle 0\rangle} \lhd (i\#q) = p^{\{-1\}}\#1 \underset{Q}{\otimes} p^{\{0\}}\eta(q) = p^{\{-2\}}\#1 \underset{Q}{\otimes} (\eta(q) \lhd p^{\{-1\}})p^{\{0\}}$$
$$= p^{\{-1\}[1]}\#q \lhd p^{\{-1\}[2]} \underset{Q}{\otimes} p^{\{0\}} = (i\#q)p^{\langle -1\rangle} \underset{Q}{\otimes} p^{\langle 0\rangle},$$

and Yetter–Drinfel'd compatibility:

$$(h^{[2]}\#q)(p \lhd h^{[1]})^{\langle -1\rangle} \underset{Q}{\otimes} (p \lhd h^{[1]})^{\langle 0\rangle} = h^{[2]} \star (p \lhd h^{[1]})^{\{-2\}}\#q \lhd (p \lhd h^{[1]})^{\{-1\}} \underset{Q}{\otimes} (p \lhd h^{[1]})^{\{0\}}$$
$$= h^{[2]} \star (p \lhd h^{[1]})^{\{-1\}}\#1 \underset{Q}{\otimes} (p \lhd h^{[1]})^{\{0\}}\eta(q) = p^{\{-1\}} \star h^{[1]}\#1 \underset{Q}{\otimes} (p^{\{0\}} \lhd h^{[2]})\eta(q)$$
$$= p^{\langle -1\rangle}(h\#q)^{[1]} \underset{Q}{\otimes} p^{\langle 0\rangle} \lhd (h\#q)^{[2]}.$$

We used 5.2.8 in the second equality and the Yetter–Drinfel'd condition for M_H, 5.2.7 in the third equality.

Now to see that P is also a BCA. The multiplication of P is an $H\#Q$–module map:

$$(p \lhd (h^{[1]}\#q)^{[1]})(p' \lhd (h^{[2]}\#q)^{[2]}) = (p \lhd (h^{[1]}\#1)) \quad (p' \lhd (h^{[2]}\#1)) = (p \lhd h^{[1]})(p' \lhd h^{[2]})\eta(q)$$
$$= ((pp') \lhd h)\,\eta(q) = (pp') \lhd (h\#q)$$

and also an $H\#Q$–comodule map:

$$(pp')^{\langle -1\rangle} \underset{Q}{\otimes} (pp')^{\langle 0\rangle} = (pp')^{\{-1\}}\#1 \underset{Q}{\otimes} (pp')^{\{0\}} = p'^{\{-1\}} \star p^{\{-1\}}\#1 \underset{Q}{\otimes} p^{\{0\}}p'^{\{0\}}$$
$$= p'^{\langle -1\rangle}p^{\langle -1\rangle} \underset{Q}{\otimes} p^{\langle 0\rangle}p'^{\langle 0\rangle}$$

The unit η is both an H–module map and H–comodule map and pre–braided commutativity holds. □

The previous two results imply that the study of successive scalar extensions reduces to the study commutative monoids in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$. This is a loose statement of the functoriality of the scalar extension construction, in the following sense.

COROLLARY 5.3.24. *Let H be a (right) bialgebroid over R, then there is a functor*

$$\mathcal{S} : (R \downarrow \mathsf{BCA}(\mathsf{M}_H)) \to (H \downarrow \mathsf{Bgd})$$

which associates to each object $R \to Q$ the scalar extension of H by Q.

Until now, we looked at the scalar extension as a map of bialgebroids. We now show that it extends to a map of Hopf algebroids and Frobenius Hopf algebroids (double algebras). Let $\langle H, R, s_R, t_R, \Delta_R, \varepsilon_R, S\rangle$ be a Hopf algebroid, represented as right constituent bialgebroid +

antipode, i.e. the antipode $S : H \to H^{op}_{cop}$ is a (co–) algebra anti–isomorphism satisfying

$$S(h^{[2]})^{[1]} \underset{R}{\otimes} h^{[1]} S(h^{[2]})^{[2]} = S(h) \underset{R}{\otimes} 1 \tag{5.3.25}$$

$$h^{[2]} S^{-1}(h^{[1]})^{[1]} \underset{R}{\otimes} S^{-1}(h^{[1]})^{[2]} = 1 \underset{R}{\otimes} S^{-1}(h) \tag{5.3.26}$$

$$S \circ t_R = s_R \tag{5.3.27}$$

Proposition 5.3.28. *Let (H_R, H_L, S_H) be a Hopf algebroid as above, and Q a BCA over the right constituent bialgebroid H_R. Then the scalar extension $H\#Q$ is a Hopf algebroid over Q with the inverse antipode*

$$S^{-1} : H\#Q \to (H\#Q)^{op}_{cop} \quad , \quad h\#q \mapsto q^{\langle -1\rangle} S_H^{-1}(h)^{[1]} \# q^{\langle 0\rangle} \triangleleft S_H^{-1}(h)^{[2]} \tag{5.3.29}$$

Furthermore, if e is a Frobenius integral in H, then $e\#1$ is a Frobenius integral in $H\#Q$.

Proof. Observe that the antipode on $H\#Q$ can be expressed with its restrictions $S|_{H\#1}$ and $S|_{i\#Q}$ to the subalgebras H and Q, respectively. Any element $h\#q \in H\#Q$ can be written $h\#q = (h\#1)(i\#q)$; using that S and S^{-1} are algebra anti–isomorphisms, we have

$$S(h\#q) = S(i\#q)S(h\#1) = S_Q(s(q))(S_H(h)\#1), \tag{5.3.30}$$

and the same for the iverse antipode:

$$S^{-1}(h\#q) = S^{-1}(i\#q)S^{-1}(h\#1) = S_Q^{-1}(s(q))(S_H^{-1}(h)\#1) \tag{5.3.31}$$

(introducing the notation S_H and S_Q). The antipode and its inverse are anti–isomorphisms when restricted to the image of s or t. Axiom 5.3.27 fixes the restriction of S to $\mathrm{Im}\, t$,

$$S(t(q)) = s(q) \tag{5.3.32}$$

and the restriction to $\mathrm{Im}\, s$ can be written

$$S(s(q)) = \nu \circ t(q) := t(q') \tag{5.3.33}$$

which defines the isomorphism ν, or equivalently, the element q'. Using 5.3.32 and 5.3.33, we can express ν from $S(s(q)) = S^2 \circ t(q) = \nu \circ t(q)$, i.e. $\nu = S^2|_{\mathrm{Im}\, t}$. For $H\#Q$ Frobenius over Q, the map ν is then the Nakayama automorphism. For reference, the definition of the amalgamated product over Q, to be used below:

$$\begin{aligned}(h'\#q')s(q) \underset{Q}{\otimes} h''\#q'' = h'\#q'q \underset{Q}{\otimes} h''\#q'' &= h'\#q' \underset{Q}{\otimes} h'' q^{\langle -1\rangle} \# q^{\langle 0\rangle} q'' \\ &= h'\#q' \underset{Q}{\otimes} (h''\#q'')t(q)\end{aligned} \tag{5.3.34}$$

Collecting the above results, the Ansatz for the antipode can be written

$$S(h\#q) = t(q')(S_H(h)\#1) = q'^{\langle -1\rangle} S_H(h)^{[1]} \# q'^{\langle 0\rangle} \triangleleft S_H(h)^{[2]} \tag{5.3.35}$$

$$S^{-1}(h\#q) = t(q')(S_H^{-1}(h)\#1) = q'^{\langle -1\rangle} S_H^{-1}(h)^{[1]} \# q'^{\langle 0\rangle} \triangleleft S_H^{-1}(h)^{[2]} \tag{5.3.36}$$

We have used 5.3.27 in deriving the Ansatz, so it remains to prove 5.3.25 and 5.3.26. It will only be necessary to check 5.3.26:

$$
\begin{aligned}
&(h\#q)^{[2]}S^{-1}((h\#q)^{[1]})^{[1]} \underset{Q}{\otimes} S^{-1}((h\#q)^{[1]})^{[2]} = (h^{[2]}\#q)S^{-1}(h^{[1]}\#1)^{[1]} \underset{Q}{\otimes} S^{-1}(h^{[1]}\#1)^{[2]} \\
&\quad = (h^{[2]}\#q)(i \star S_H^{-1}(h^{[1]})^{[1]}\#1 \triangleleft S_H^{-1}(h^{[1]})^{[2]})^{[1]} \underset{Q}{\otimes} (i \star S_H^{-1}(h^{[1]})^{[1]}\#1 \triangleleft S_H^{-1}(h^{[1]})^{[2]})^{[2]} \\
&\quad = (h^{[2]}\#q)(S_H^{-1}(h^{[1]})^{[1]}\#1) \underset{Q}{\otimes} (S_H^{-1}(h^{[1]})^{[2]}\#1 \triangleleft S_H^{-1}(h^{[1]})^{[3]}) \\
&\quad \overset{\mathcal{YD}}{=} (h^{[2]}\#q)(S_H^{-1}(h^{[1]})^{(1)}\#1) \underset{Q}{\otimes} S_H^{-1}(h^{[1]})^{[2][2]}(1 \triangleleft S_H^{-1}(h^{[1]})^{[2][1]})^{\langle -1\rangle}\#(1 \triangleleft S_H^{-1}(h^{[1]})^{[2][1]})^{\langle 0\rangle} \\
&\quad \overset{\underset{Q}{\otimes}}{=} (h^{[2]}\#q)(S_H^{-1}(h^{[1]})^{[1]}\#1 \triangleleft S_H^{-1}(h^{[1]})^{[2][1]}) \underset{Q}{\otimes} S_H^{-1}(h^{[1]})^{[2][2]}\#1 \\
&\quad = h^{[2]}S_H^{-1}(h^{[1]})^{[1]}\#(q \triangleleft S_H^{-1}(h^{[1]})^{[2]})(1 \triangleleft S_H^{-1}(h^{[1]})^{[3]}) \underset{Q}{\otimes} S_H^{-1}(h^{[1]})^{[4]}\#1 \\
&\quad = h^{[2]}S_H^{-1}(h^{[1]})^{[1]}\#q \triangleleft S_H^{-1}(h^{[1]})^{[2]} \underset{Q}{\otimes} S_H^{-1}(h^{[1]})^{[3]}\#1 \\
&\quad \overset{S_H^{-1}}{=} i\#q \triangleleft S_H^{-1}(h^{[1]})^{[1]} \underset{Q}{\otimes} S_H^{-1}(h^{[1]})^{[2]}\#1 \\
&\quad = i\#1 \underset{Q}{\otimes} S_H^{-1}(h^{[1]})^{[2]}(q \triangleleft S_H^{-1}(h^{[1]})^{[1]})^{\langle -1\rangle}\#(q \triangleleft S_H^{-1}(h^{[1]})^{[1]})^{\langle 0\rangle} \\
&\quad \overset{\mathcal{YD}}{=} i\#1 \underset{Q}{\otimes} q^{\langle -1\rangle}S_H^{-1}(h^{[1]})^{[1]}\#q^{\langle 0\rangle} \triangleleft S_H^{-1}(h^{[1]})^{[2]} = i\#1 \underset{Q}{\otimes} S_H^{-1}(q\#h).
\end{aligned}
$$

Some comments on the calculation: we used the Ansatz 5.3.36 for the inverse antipode in the second equality, Yetter–Drinfel'd compatibility for the BCA $Q \in {}^H\mathcal{YD}_H$ in the fourth, equation 5.3.34 in the fifth, axiom 5.3.26 for S_H^{-1} in the eighth and $\mathcal{YD}$–compatibility again in the tenth. Since 5.3.35 and 5.3.36 only differ in that q is replaced by q', the proof of 5.3.35 is exactly the same. □

We could have arrived at this result by amending the proof of Theorem 5.3.1. If H is a Frobenius Hopf algebroid then it has a distributive double algebra structure [**79**]. Therefore we may assume that H is the horizontal Hopf algebroid of $\langle A, \circ, e, \star, i\rangle$. Then $\langle H, \Delta_R, \varphi_R, \circ, R \hookrightarrow H\rangle$ is a Frobenius algebra in M_H, so mapped by the strong monoidal functor of Proposition 5.3.12 to a Frobenius algebra in M_G. The comonoid part of this Frobenius algebra has already been determined to be $\langle G, \Delta_G, \varepsilon_G\rangle$. The monoid part will provide a convolution product with unit on G which, together with the smash product algebra structure, will make G a distributive double algebra. This convolution product (vertical multiplication) is obtained as the composite

$$(h\#q) \underset{Q}{\otimes} (h'\#q') \mapsto (h \underset{R}{\otimes} h' \star q^{\langle -1\rangle}) \underset{R}{\otimes} q^{\langle 0\rangle}q' \mapsto h \circ (h' \star q^{\langle -1\rangle})\#q^{\langle 0\rangle}q'$$

and its unit element e_G is the image of $1 \in Q$ under the map

$$Q \xrightarrow{\sim} R \underset{R}{\otimes} Q \to H\#Q\,.$$

So $e_G = e\#1$ is a two-sided Frobenius integral in G.

REMARK 5.3.37. The construction of a vertical multiplication on $H\#Q$ suggests the new interpretation of the smash product as a double algebraic one. If $\langle A, \circ, e, \star, i\rangle$ is a DDA and Q is a BCA over the bialgebroid H over R then there is a smash product double algebra $A\#Q$ with

- underlying k-module $A \underset{R}{\otimes} Q$,
- horizontal multiplication $(a\#q) \star (a'\#q') = a \star a'^{[1]}\#(q \triangleleft a'^{[2]})q'$,
- horizontal unit $i\#1$,
- vertical multiplication $(a\#q) \circ (a'\#q') = a \circ (a' \star q^{\langle -1\rangle})\#q^{\langle 0\rangle}q'$,
- and vertical unit $e\#1$.

As a byproduct of the double algebraic picture we obtain the following result.

PROPOSITION 5.3.38. *For Frobenius Hopf algebroids H the prebraiding of the left weak center $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ is a braiding. Therefore $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H) = \mathcal{Z}(\mathsf{M}_H) = \overleftarrow{\mathcal{Z}}(\mathsf{M}_H)$ and every pre-BCA is a BCA over H.*

PROOF. We claim that the inverse braiding encoded in the right coaction $\bar{\tau}$ by (5.1.24) is given by

$$q^{\langle 0\rangle} \underset{R}{\otimes} q^{\langle 1\rangle} = \eta_Q\varphi_R\varphi_T(x^j \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{R}{\otimes} y^j\,. \tag{5.3.39}$$

The proof is motivated by the double algebraic structure on $H\#Q$ given in the above Remark but we do not use that the given structure maps satisfy the axioms of a DDA. Let us compute the would-be φ_R of $H\#Q$. It is

$$\Phi_R(h\#q) = (e\#1) \star (h\#q) = e\#\eta_Q\varphi_R(h)q\,.$$

One conjectures $(x^j\#1) \underset{Q}{\otimes} (y^j\#1)$ to be its dual basis. Instead of proving that we prove its special case

$$\begin{aligned}
\Phi_R((i\#q) \circ (x^j\#1)) \circ (y^j\#1) &= (e\#\eta_Q\varphi_R\varphi_T(x^j \star q^{\langle -\rangle})q^{\langle 0\rangle}) \circ (y^j\#1)\\
&= y^j \star (\varphi_R\varphi_T(x^j \star q^{\langle -2\rangle}) \circ q^{\langle -1\rangle})\#q^{\langle 0\rangle}\\
&= y^j \star q^{\langle -1\rangle} \star \varphi_B\varphi_R\varphi_T(x^j \star q^{\langle -2\rangle})\#q^{\langle 0\rangle}\\
&= \varphi_R(i \circ \varphi_R(q^{\langle -1\rangle})^{[1]}) \circ \varphi_R(q^{\langle -1\rangle})^{[2]}\#q^{\langle 0\rangle}\\
&= i \circ \varphi_R(q^{\langle -1\rangle})\#q^{\langle 0\rangle}\\
&= i\#q\,.
\end{aligned}$$

Comparing the first row with the Ansatz (5.3.39) and then using the vertical multiplication of $H\#Q$ we arrive to

$$\begin{aligned}
i\#q &= (e\#q^{\langle 0\rangle}) \circ (q^{\langle 1\rangle}\#1)\\
&= q^{\langle 1\rangle} \star q^{\langle 0\rangle\langle -1\rangle}\#q^{\langle 0\rangle\langle 0\rangle}
\end{aligned}$$

which is equation (5.1.26). The verification of (5.1.25) is a bit longer,

$$\begin{aligned}
q^{\langle 0\rangle\langle 0\rangle} \underset{R}{\otimes} q^{\langle -1\rangle} \star q^{\langle 0\rangle\langle 1\rangle} &= \eta_Q\varphi_R\varphi_T(x^j \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{R}{\otimes} q^{\langle -2\rangle} \star y^j \\
&= \eta_Q\varphi_R\varphi_T(S^{-1}(q^{\langle -2\rangle}) \star x^j \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{R}{\otimes} y^j \\
&= \eta_Q\varphi_R\varphi_T(S^{-1}(x^k) \star x^j \star (y^k \circ q^{\langle -1\rangle}))q^{\langle 0\rangle} \underset{R}{\otimes} y^j \\
&= \eta_Q\varphi_R\varphi_T((S^{-1}(y^k) \circ (S^{-1}(x^k) \star x^j)) \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{R}{\otimes} y^j
\end{aligned}$$

$$\begin{aligned}
&= \eta_Q\varphi_R\varphi_T((x_k \circ (y_k \star x^j)) \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{R}{\otimes} y^j \\
&= \eta_Q\varphi_R\varphi_T(\varphi_R\varphi_T(x^j) \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{R}{\otimes} y^j \\
&= \eta_Q\varphi_R(\varphi_T(x^j) \star q^{\langle -1\rangle})q^{\langle 0\rangle} \underset{R}{\otimes} y^j \\
&= \eta_Q\varphi_R(q^{\langle -1\rangle})q^{\langle 0\rangle} \lhd \varphi_T(x^j) \underset{R}{\otimes} y^j \\
&= q \underset{R}{\otimes} \varphi_R\varphi_T(x^j) \circ y^j \\
&= q \underset{R}{\otimes} i\,.
\end{aligned}$$

□

4. A monadic look at the scalar extension

In [**81**] a monadic characterization of bialgebroids was given, leading to the definition of a 2–category of bialgebroids. Without going into the details of this construction, we explain how scalar extension fits in this picture.

For a right bialgebroid H over R, the forgetful functor $U : \mathsf{M}_H \to \mathsf{M}_{R^e}$ is strong monoidal (even strict monoidal, as dictated by Schauenburg's theorem) and is right adjoint to the induction functor $I = \_ \underset{R^e}{\otimes} H : \mathsf{M}_{R^e} \to \mathsf{M}_H$. By the Eilenberg–Moore construction (see [**53**]) the adjunction $I \dashv U$ gives rise to a monad $\mathbf{T} = \langle T, \mu, \eta\rangle$ on the category M_{R^e} with underlying endofunctor $T = UI : \mathsf{M}_{R^e} \dot{\to} \mathsf{M}_{R^e}$ and a comonad $\mathbf{G} = \langle G, \Delta, \varepsilon\rangle$ on the category M_H with underlying endofunctor $G = IU : \mathsf{M}_H \to \mathsf{M}_H$. The monad multiplication is $\mu = U\varepsilon I : TT \dot{\to} T$ and the monad unit $\eta : \mathsf{M}_{R^e} \dot{\to} T$ is the unit of the adjunction. The comultiplication on the comonad is $\Delta = I\eta U : G \dot{\to} GG$ and the counit $\varepsilon : G \dot{\to} \mathsf{M}_H$ is the counit of the adjunction. Denote $\mathsf{M}_{\mathbf{T}}$ the Eilenberg–Moore category of $\mathbf{T}$–algebras, then $\mathsf{M}_{\mathbf{T}}$ can be identified with M_H, since $T = \_ \underset{R^e}{\otimes} H_{R^e}$. Also, the canonical forgetful functor $U_T : \mathsf{M}_{\mathbf{T}} \to \mathsf{M}_{R^e}$ can be identified with $U : \mathsf{M}_H \to \mathsf{M}_{R^e}$. The situation is summarized in the Figure 2.

By Prop. 2.1. of [**81**], the (strong) monoidal structure on U implies an opmonoidal structure on the left adjoint I, and the adjunction is in the category of monoidal categories. This implies that the unit and counit are monoidal natural transformations. The following definition is essentially from [**81**] (see also [**56**]):

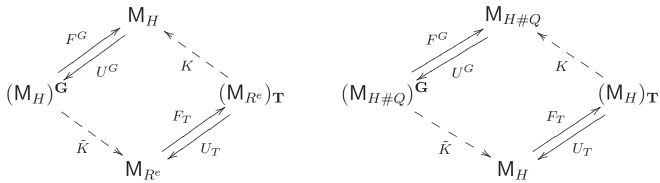


FIGURE 2. Eilenberg monad and comonad

DEFINITION 5.4.1. Let $\langle \mathsf{M}, \otimes, i\rangle$ be a monoidal category. Then a bimonad on M is a monoid in the category of opmonoidal endofunctors from M to M. Thus, it is an endofunctor $T : \mathsf{M} \to \mathsf{M}$, furnished with:

- a natural transformation $\mathsf{T}^{X,Y} : T\,(X \otimes Y) \to T\,X \otimes T\,Y$, and
- an arrow $\mathsf{T}^0 : T\,i \to i$

such that $\langle T, \mathsf{T}^{X,Y}, \mathsf{T}^0\rangle$ is an opmonoidal functor;

- a natural transformation $\mu_X : TTX \to TX$ and
- a natural transformation $\eta_X : X \to TX$

such that $\langle T, \mu, \eta\rangle$ is a monoid in M^{M}, and four compatibility axioms stating that μ is opmonoidal,

$$\mathsf{T}^{X,Y} \circ \mu_{X\otimes Y} = (\mu_X \otimes \mu_Y) \circ \mathsf{T}^{TX,TY} \circ T\mathsf{T}^{X,Y} \tag{5.4.2}$$

$$\mathsf{T}^0 \circ T\mathsf{T}^0 = \mathsf{T}^0 \circ \mu i \tag{5.4.3}$$

and that η is opmonoidal

$$\eta_X \otimes \eta_Y = \mathsf{T}^{X\otimes Y} \circ \eta_{X\otimes Y} \tag{5.4.4}$$

$$\mathsf{T}^0 \circ \eta_i = i \tag{5.4.5}$$

We state the following proposition without proof.

PROPOSITION 5.4.6. *The endofunctor* $T = UI = \_ \underset{R^e}{\otimes} H : \mathsf{M}_{R^e} \to \mathsf{M}_{R^e}$ *is an opmonoidal monad with the structure maps:*

$$\mu_X : X \underset{R^e}{\otimes} H \underset{R}{\otimes} H \to X \underset{R^e}{\otimes} H,\ \ x \otimes h \otimes h' \mapsto x \otimes hh' \tag{5.4.7}$$

$$\eta_X : X \to X \underset{R^e}{\otimes} H,\ \ x \mapsto x \otimes 1_H \tag{5.4.8}$$

$$\begin{aligned}\gamma_{X,Y} &: (X \underset{R}{\otimes} Y) \underset{R^e}{\otimes} H \to (X \underset{R^e}{\otimes} H) \underset{R}{\otimes} (Y \underset{R^e}{\otimes} H)\\ &(x \otimes y) \otimes h \mapsto (x \otimes h^{[1]}) \otimes (y \otimes h^{[2]})\end{aligned} \tag{5.4.9}$$

$$\pi : R \underset{R^e}{\otimes} H \to R,\ \ r \otimes h \mapsto \varepsilon(s(r)h) \tag{5.4.10}$$

For a scalar extension $H\#Q$ of a right bialgebroid H over R, the inclusion $\iota : H \hookrightarrow H\#Q$ induces a monoidal forgetful functor $U : \mathsf{M}_{H\#Q} \to \mathsf{M}_H$. The *left* adjoint $I \dashv U$ is

$$I : \mathsf{M}_H \to \mathsf{M}_{H\#Q}, \;\; X \mapsto X \underset{H}{\otimes} (H\#Q)$$

with unit

$$\begin{aligned} \eta : X &\to X \underset{H}{\otimes} (H\#Q)_H \\ x &\mapsto x \otimes (i\#1_Q) \end{aligned} \tag{5.4.11}$$

and counit

$$\begin{aligned} \varepsilon : Y \underset{H}{\otimes} (H\#Q)_{H\#Q} &\to Y \\ y \otimes (h \otimes q) &\mapsto y \triangleleft (h\#q) = (y \triangleleft h) \cdot q \end{aligned} \tag{5.4.12}$$

As $R-R$–bimodules, $H\#Q = H \underset{R^e}{\otimes} Q$, so the induction functor $\_ \underset{H}{\otimes} (H\#Q)$ is isomorphic to $\_ \underset{R}{\otimes} Q$. By Lemma 5.3.12, this functor is strong monoidal, hence also monoidal. It will remain monoidal upon composition with the monoidal forgetful functor, making the canonical monad $\mathbf{T} = \langle UF, U\varepsilon F, \eta\rangle$ a monoidal endofunctor. The compatibility of the monoidal and monadic structure make $\mathbf{T}$ a monoidal monad, as opposed to the opmonoidal monad we called bimonad earlier.

Definition 5.4.13. A monoidal monad in a monoidal category $\langle \mathsf{M}, \otimes, i\rangle$ a monoid in the category of monoidal endofunctors $\mathsf{M} \to \mathsf{M}$, i.e. a monoidal endofunctor $\langle T, \mathsf{T}_{X,Y}, \mathsf{T}_0\rangle$, with

- a natural map $\mathsf{T}_{X,Y} : TX \otimes TY \to T(X \otimes Y)$, and
- an arrow $\mathsf{T}_0 : T\,i \to i$

satisfying the appropriate coherence laws, and monad structure maps consisting of

- a natural transformation $\mu_X : TTX \to TX$ and
- a natural transformation $\varepsilon_X : X \to TX$

making $\langle T, \mu, \eta\rangle$ a monoid in M^{M}. In particular, the compatibility of the monoidal and monad structures imply the following four diagrams:

$$\begin{array}{ccc} T^2X \otimes T^2Y & \xrightarrow{T\mathsf{T}_{X,Y} \circ \mathsf{T}_{TX,TY}} & T^2(X \otimes Y) \\ {\scriptstyle \mu_X \otimes \mu_Y}\downarrow & & \downarrow{\scriptstyle \mu_{X \otimes Y}} \\ TX \otimes TY & \xrightarrow[\mathsf{T}_{X,Y}]{} & T(X \otimes Y) \end{array} \qquad \begin{array}{ccc} i & \xrightarrow{T\eta \circ \eta} & T^2 i \\ \| & & \downarrow{\scriptstyle \mu_i} \\ i & \xrightarrow{\eta} & Ti \end{array}$$

for the monoidality of $\mu : TT \dot{\to} T$, and

$$\begin{array}{ccc} X \otimes Y & = & X \otimes Y \\ {\scriptstyle \eta_X \otimes \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_{X \otimes Y}} \\ TX \otimes TY & \xrightarrow[\mathsf{T}_{X,Y}]{} & T(X \otimes Y) \end{array} \qquad \begin{array}{ccc} i & = & i \\ \| & & \downarrow{\scriptstyle \eta_i} \\ Ti & \xrightarrow[\mathsf{T}_0]{} & i \end{array}$$

for the monoidality of $\eta : \mathrm{id} \dot{\to} T$.

Braided commutative algebras can now be characterized as follows.

PROPOSITION 5.4.14. *Let Q be a BCA over the right bialgebroid H. Then the endofunctor $T = \_ \underset{R}{\otimes} Q$ is a monoidal monad on M_H.*

PROOF. Recall from 5.3.12 that $\mathsf{Q}_{X,Y} : (X \underset{R}{\otimes} Q) \underset{R}{\otimes} (Y \underset{R}{\otimes} Q) \to (X \underset{R}{\otimes} Y) \underset{R}{\otimes} Q$ reads, on elements: $(y \underset{R}{\otimes} q) \underset{Q}{\otimes} (y' \underset{R}{\otimes} q') \mapsto (y \underset{R}{\otimes} y' \triangleleft q^{\langle -1 \rangle}) \underset{R}{\otimes} q^{\langle 0 \rangle} q'$ and $\mathsf{Q}_0 : Q \to R \underset{R}{\otimes} Q$ is the natural isomorphism $q \mapsto e \otimes q$. The monad structure follows trivially from the algebra structure of Q; we only check the compatibility of the monad and monoidal structure, i.e. the four commutative diagrams above. For monoidality of multiplication,

$$\begin{aligned}
&\mu_{X,Y} \circ (Q\mathsf{Q}_{X,Y} \circ \mathsf{Q}_{QX,QY})\,[\,(x \otimes q \otimes q') \otimes (y \otimes q'' \otimes q''')\,] \\
&\quad = \mu_{X,Y} \circ Q\mathsf{Q}_{X,Y} \left[(x \otimes q \otimes (y \otimes q'') \triangleleft q'^{\langle -1 \rangle}) \otimes q'^{\langle 0 \rangle} q''' \right] \\
&\quad = \mu_{X,Y} \circ Q\mathsf{Q}_{X,Y} \left[(x \otimes q) \otimes (y \triangleleft q'^{\langle -1 \rangle (1)} \otimes q'' q'^{\langle -1 \rangle (2)}) \otimes q'^{\langle 0 \rangle} q''' \right] \\
&\quad = \mu_{X,Y} \left[x \otimes (y \triangleleft q'^{\langle -1 \rangle (1)} q^{\langle -1 \rangle}) \otimes q^{\langle 0 \rangle} q'' q'^{\langle -1 \rangle (2)} \otimes q'^{\langle 0 \rangle} q''' \right] \\
&\quad = x \otimes (y \triangleleft q'^{\langle -1 \rangle (1)} q^{\langle -1 \rangle}) \otimes q^{\langle 0 \rangle} q'' q'^{\langle -1 \rangle (2)} q'^{\langle 0 \rangle} q''' \\
&\quad = x \otimes y \triangleleft (qq')^{\langle -1 \rangle} \otimes (qq')^{\langle 0 \rangle} q'' q''' \\
&\quad = \mathsf{Q}_{X,Y} \circ (\mu_X \otimes \mu_Y)\,[\,(x \otimes q \otimes q') \otimes (y \otimes q'' \otimes q''')\,],
\end{aligned}$$

where we have used braided commutativity in the fifth equality, and the second diagram reduces to a triviality. We check one of the diagrams corresponding to monoidality of the unit map,

$$\begin{aligned}
\mathsf{Q}_{X,Y} \circ (\eta_X \otimes \eta_Y)(x \otimes y) &= \mathsf{Q}_{X,Y}(x \otimes 1_Q \otimes y \otimes 1_Q) \\
&= x \otimes y \triangleleft i \otimes 1_Q = \eta_{X \otimes Y}(x \otimes y)
\end{aligned}$$

the other being even simpler. □

5. Application to Galois extensions

Our most important application of scalar extension by BCAs appears in Galois extensions. We shall be working in the action picture.

PROPOSITION 5.5.1. *Let M be a monoid in M_H over the right bialgebroid H and let $N = M^H$. Assume that H_R is fgp and that the canonical map $\Gamma_M : H\#M \to \mathrm{End}({}_N M)$ is an isomorphism. Then the centralizer $M^N = \{c \in M \,|\, nc = cn,\ n \in N\}$ of the extension $N \subseteq M$*

is a left pre-BCA over H with H-module algebra structure inherited from $M^N \subseteq M$ (the Miyashita-Ulbrich action) and with left coaction $\tau(c) := \Gamma_M^{-1}(\lambda_M(c))$ where $\lambda_M(c) = \{m \mapsto cm\}$.

Proof. For each $h \in H$ the action $\_ \triangleleft h$ is an N-N-bimodule map. Therefore $M^H \subseteq M$ is a sub-H-module algebra. As such the unit $\eta : R \to M$ has image in M^H. Since Γ_M is an N-N-bimodule map, it restricts to an isomorphism $(H \underset{R}{\otimes} M)^N \xrightarrow{\sim} \mathrm{End}({}_N M_N)$ between the centralizers. The H_R being fgp we have $(H \underset{R}{\otimes} M)^N = H \underset{R}{\otimes} M^N$. Since $\lambda(c)$ for $c \in M^N$ belongs to $\mathrm{End}({}_N M_N)$, the τ is a map $M^N \to H \underset{R}{\otimes} M^N$. The τ is uniquely determined by the equation

$$(m \triangleleft c^{\langle -1\rangle})c^{\langle 0\rangle} = cm\,, \qquad m \in M \tag{5.5.2}$$

from which the bimodule property (5.2.9) and the centrality (5.2.10) easily follow. The calculation

$$c(mm') = ((mm') \triangleleft c^{\langle -1\rangle})c^{\langle 0\rangle} = (m \triangleleft c^{\langle -1\rangle(1)})(m' \triangleleft c^{\langle -1\rangle(2)})c^{\langle 0\rangle}$$
$$(cm)m' = (m \triangleleft c^{\langle -1\rangle})c^{\langle 0\rangle}m' = (m \triangleleft c^{\langle -1\rangle})(m' \triangleleft c^{\langle 0\rangle\langle -1\rangle})c^{\langle 0\rangle\langle 0\rangle}$$

will imply coassociativity after verifying the next

Lemma 5.5.3. *Under the assumptions of the Proposition and with the notations $E := \mathrm{End}({}_N M_N)$, $C := M^N$ the maps*

$$\begin{aligned} E \underset{C}{\otimes} E &\to \mathrm{Hom}_{N\text{-}N}(M \underset{N}{\otimes} M, M) \\ \alpha \underset{C}{\otimes} \alpha' &\mapsto \{m \underset{N}{\otimes} m' \mapsto \alpha(m)\alpha'(m')\} \end{aligned} \tag{5.5.4}$$

$$\begin{aligned} H \underset{R}{\otimes} H \underset{R}{\otimes} C &\to \mathrm{Hom}_{N\text{-}N}(M \underset{N}{\otimes} M, M) \\ h \underset{R}{\otimes} h' \underset{R}{\otimes} c &\mapsto \{m \underset{N}{\otimes} m' \mapsto (m \triangleleft h)(m' \triangleleft h')c\} \end{aligned} \tag{5.5.5}$$

are isomorphisms.

Proof. Using both the isomorphism Γ_M and its restriction $H\#C \xrightarrow{\sim} E$ we have a sequence of isomorphisms

$$\begin{aligned} E \underset{C}{\otimes} E &\xrightarrow{\sim} (H \underset{R}{\otimes} C) \underset{C}{\otimes} E \xrightarrow{\sim} H \underset{R}{\otimes} E = H \underset{R}{\otimes} \mathrm{Hom}_{N\text{-}N}(M, M) \\ &\xrightarrow{\sim} \mathrm{Hom}_{N\text{-}N}(M, H \underset{R}{\otimes} M) \xrightarrow{\sim} \mathrm{Hom}_{N\text{-}N}(M, \mathrm{Hom}_{N\text{-}}(M, M)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{N\text{-}N}(M \underset{N}{\otimes} M, M) \end{aligned}$$

The action of these isomorphisms can be computed by inserting $\alpha = (\_ \triangleleft h)c$ and $\alpha' = (\_ \triangleleft h')c'$:

$$\begin{aligned} \alpha \underset{C}{\otimes} \alpha' &\mapsto (h \underset{R}{\otimes} c) \underset{C}{\otimes} \alpha' \mapsto h \underset{R}{\otimes} c\,\alpha'(\_) \mapsto \{m \mapsto h \underset{R}{\otimes} c\,\alpha'(m)\} \\ &\mapsto \{m \mapsto \{m' \mapsto \alpha(m')\alpha'(m)\}\} \mapsto \{m' \underset{N}{\otimes} m \mapsto \alpha(m')\alpha'(m)\} \end{aligned}$$

This proves that (5.5.4) is an isomorphism. The map in (5.5.5) is the composite

$$\begin{array}{ccccc}
H \underset{R}{\otimes} H \underset{R}{\otimes} C & & & & \mathrm{Hom}_{N-N}(M \underset{N}{\otimes} M, M) \\
\downarrow{\scriptstyle H \underset{R}{\otimes} \Gamma} & & & & \uparrow{\scriptstyle \cong} \\
H \underset{R}{\otimes} E & \xrightarrow{\ \cong\ } & H \underset{R}{\otimes} C \underset{C}{\otimes} E & \xrightarrow{\ \Gamma \underset{C}{\otimes} E\ } & E \underset{C}{\otimes} E
\end{array}$$

of isomorphisms. □

Returning to the proof of the Proposition counitality of τ can be seen as

$$\varphi_R(c^{\langle -1\rangle}) \cdot c^{\langle 0\rangle} = (1 \triangleleft c^{\langle -1\rangle})c^{\langle 0\rangle} = c1 = c\,.$$

As for the Yetter-Drinfeld compatibility condition it suffices to verify the equality

$$\begin{aligned}(m \triangleleft c^{\langle -1\rangle} \star h^{(1)})(c^{\langle 0\rangle} \triangleleft h^{(2)}) &= \left((m \triangleleft c^{\langle -1\rangle})c^{\langle 0\rangle}\right) \triangleleft h = (cm) \triangleleft h \\ &= (c \triangleleft h^{(1)})(m \triangleleft h^{(2)}) = \left(m \triangleleft h^{(2)} \star (c \triangleleft h^{(1)})^{\langle -1\rangle}\right)(c \triangleleft h^{(1)})^{\langle 0\rangle}\end{aligned}$$

In order to see compatibility of τ with multiplication and unit in C it suffices to check

$$c'cm = c'(m \triangleleft c^{\langle -1\rangle})c^{\langle 0\rangle} = (m \triangleleft c^{\langle -1\rangle} \star c'^{\langle -1\rangle})c'^{\langle 0\rangle}c^{\langle 0\rangle}\,.$$

Finally, braided commutativity $(c' \triangleleft c^{\langle -1\rangle})c^{\langle 0\rangle} = cc'$ follows from the more general relation (5.5.2). □

Corollary 5.5.6. *If $N \subseteq M$ is a right A-Galois extension for a distributive double algebra A then M^N is a BCA over the horizontal Hopf algebroid H.*

Proof. It suffices to prove that the prebraiding is invertible. Define the right coaction $\bar\tau(c) := (\Gamma^M)^{-1}(\rho_M(c))$ where ρ_M is right multiplication on M. This is equivalent to $\bar\tau(c) = c^{\langle 0\rangle} \underset{R}{\otimes} c^{\langle 1\rangle}$ satisfying

$$c^{\langle 0\rangle}(m \triangleleft c^{\langle 1\rangle}) = mc\,, \qquad m \in M\,. \tag{5.5.7}$$

Applying (5.5.2) to (5.5.7) we obtain

$$(m \triangleleft i)c = mc = (m \triangleleft c^{\langle -1\rangle} \star c^{\langle 0\rangle\,\langle -1\rangle})\, c^{\langle 0\rangle\,\langle 0\rangle}$$

from which equation (5.1.26) follows. Equation (5.1.25) can be seen similarly. □

Notice that this proof does not use very much from the Hopf algebroid structure. Therefore the Corollary holds true for any right bialgebroid for which both H_R and ${}_RH$ are fgp and for all extensions for which both Γ^M and Γ_M are invertible.

For any Galois extension $N = M^H \subseteq M$, the centralizer M^N is a canonical BCA over H. This begs the question what is the scalar extension of M by the centralizer?

Proposition 5.5.8. *Let $N \subseteq M$ be a Galois extension over the Frobenius Hopf algebroid H. Then the restriction of the Galois map Γ_M provides an isomorphism of Hopf algebroids $H\#C \cong E$ where E is the endomorphism Hopf algebroid of the extension.*

PROOF. The structure maps (5.3.2), (5.3.3), (5.3.4) and (5.3.5) of the smash product are mapped by Γ_M to

$$s_E : C \to E \qquad c \mapsto \{m \mapsto mc\} \tag{5.5.9}$$

$$t_E : C^{\text{op}} \to E \qquad c \mapsto \{m \mapsto cm\} \tag{5.5.10}$$

$$\Delta_E : E \to E \underset{C}{\otimes} E \quad \text{such that } \alpha^{[1]}(m)\alpha^{[2]}(m') = \alpha(mm') \tag{5.5.11}$$

$$\varepsilon_E : E \to C \qquad \alpha \mapsto \alpha(1) \tag{5.5.12}$$

respectively, where note that multiplicativity of Δ_E uniquely fixes it by Lemma 5.5.3, (5.5.4). Now it is easy to check that $\Gamma_M : H\#C \to E$ satisfies the axioms of bialgebroid maps. □

This result gives partial control over the ambiguity in quantum groupoid extensions. If H and H' are non–isomorphic quantum groupoids such that $N = M^H \subseteq M$ is Galois over both (i.e. H and H' are forms of each other in the terminology of Greither and Pareigis), then 5.5.8 implies that $H\#C \cong H'\#C \cong E$, meaning that both H and H' are forms of the endomorphism Hopf algebroid E. This leaves open the question whether H and H' are perhaps scalar extensions of each other?

CHAPTER 6

Bicoalgebroids

Bicoalgebroids are a dualization of bialgebroids in the categorical sense (in the sense of 'reversing arrows') and were proposed in [**21**]. This notion of dual is not to be confused with the different kinds of bialgebroid–duals that were later introduced in [**47**], and that we have mentioned in Chapter 1. The motivation for studying bicoalgebroids is two–fold. First, it is well established that a bialgebroid may be thought of as a non–commutative analogue of the *algebra of functions* on a groupoid, the latter being a bialgebroid over commutative base, namely the algebra of functions on the 0–cells. It follows that a bicoalgebroid, in turn, should be regarded as a non–commutative analogue of the groupoid itself. One can hope then that geometric constructions on groupoids have direct non–commutative generalizations in the context of bicoalgebroids. Secondly, just as bialgebroids play a fundamental rôle in depth–two extensions of algebras, it is expected that bicoalgebroids are just as important in extensions of coalgebras [1]. A dual Hopf Galois theory for extensions of coalgebras was put forward in [**76**]. Since Hopf algebra is a self–dual structure, a Hopf–Galois theory of coalgebra extensions involves Hopf algebras acting (coacting) on coalgebras. A similar dualization of bialgebroid–Galois theory however, should involve bicoalgebroids.

We begin by defining the module– and comodule categories over bicoalgebroids (Section 1). The construction of the monoidal category of bicomodules over a ring is presented in Appendix A. A Schauenburg–type result, which characterizes bicoalgebroids with the monoidality of their comodule category, is proven in Section 2. Various notions of the cocenter of a coalgebra and the cocentralizer of a coalgebra extension are discussed in Section 3. Section 4 is devoted to a central result of this Chapter, the scalar extension for bicoalgebroids. We prove results analogous to those proven for bialgebroids and supply a few examples. Finally, Section 5 rephrases scalar extension in comonadic terms, in analogy with Sect. 4 of Chap. 4.

1. The Definition

Throughout this chapter, k will be a field and the category $\mathsf{M} = \mathsf{M}_k$ of k–modules will serve as our underlying category. The unadorned $\otimes$ will always mean $\otimes_k$. Just as a bialgebroid H over R is a comonoid in the category ${}_R\mathsf{M}_R$, a bicoalgebroid H over C is a monoid in the bicomodule category over the base coalgebra C (see App. A). The frequently occuring C–coactions will be denoted with square brackets, e.g. $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ to avoid confusion with coactions of the bicoalgebroid H.

[1]From a different approach, in [**46**] Kadison constructs *bialgebroids* from depth 2 extensions of coalgebras

Definition 6.1.1. A *left* bicoalgebroid $\langle H, \Delta, \varepsilon, \mu, \eta, \alpha, \beta, C\rangle$ consists of

- a k–coalgebra $\langle H, \Delta_H, \varepsilon_H\rangle$
- two coalgebra maps $\alpha : H \to C$ and $\beta : H \to C_{cop}$, such that α and β 'cocommute', i.e. $\alpha(h_{(1)}) \otimes \beta(h_{(2)}) = \alpha(h_{(2)}) \otimes \beta(h_{(1)})$. These maps furnish H with a $(C \otimes C)$–bicomodule structure, such that $(H; \lambda_L, \lambda_R; \rho_L, \rho_R) \in {}^{C\otimes C}\mathsf{M}^{C\otimes C}$. The four C–coactions are:

$$\lambda_L(h) = \alpha(h_{(1)}) \otimes h_{(2)}\,, \quad \rho_L(h) = h_{(2)} \otimes \beta(h_{(1)})$$
$$\lambda_R(h) = \beta(h_{(2)}) \otimes h_{(1)}\,, \quad \rho_R(h) = h_{(1)} \otimes \alpha(h_{(2)})$$

- C–bicomodule maps $\mu_H : H \,\Box_C\, H \to H$ and $\eta_H : C \to H$ (multiplication & unit) making (H, λ_L, ρ_L) an algebra in ${}^{C}\mathsf{M}^{C}$,

subject to the following axioms:

(1) The multiplication map $\mu : H \,\Box_C\, H \to H$ satisfies:

$$\sum_i \mu(g^i \otimes h^i_{(1)}) \otimes \alpha(h^i_{(2)}) = \mu(g^i_{(1)} \otimes h^i) \otimes \beta(g^i_{(2)}) \tag{6.1.2}$$

(2) and it is comultiplicative:

$$\Delta \circ \mu(\sum_i g^i \otimes h^i) = \sum_i \mu(g^i_{(1)} \otimes h^i_{(1)}) \otimes \mu(g^i_{(2)} \otimes h^i_{(2)}) \tag{6.1.3}$$

(3) Furthermore, the product is counital (note that this axiom seems to be missing in Ref. [**21**]):

$$\varepsilon(g)\varepsilon(h) = \varepsilon \circ \mu(g \otimes h) \tag{6.1.4}$$

(4) The unit map $\eta : C \to H$ satisfies the unit axiom:

$$\mu \circ (\eta \,\Box\, H) \circ \lambda_L = H = \mu \circ (H \,\Box\, \eta) \circ \rho_L \tag{6.1.5}$$

(5) The unit map is compatible with the coalgebra structure in the following sense:

$$\Delta(\eta(c)) = \eta(c)_{(1)} \otimes \eta(\alpha(\eta(c)_{(2)})) = \eta(c)_{(1)} \otimes \eta(\beta(\eta(c)_{(2)})) \tag{6.1.6}$$

$$\varepsilon(\eta(c)) = \varepsilon(c) \tag{6.1.7}$$

In the original reference [**21**], it is first proved that the condition 6.1.2 makes sense, i.e. the two sides of the equation are well–defined maps. This, in turn, implies that 6.1.3 makes sense, which boils down to $(\mu \,\Box\, \mu) \circ \mathsf{tw}_{23} \circ (\Delta \,\Box\, \Delta)$ being a well–defined map. The condition 6.1.2 on the multiplication map may be rephrased by saying that μ factorizes through the *cocenter* of the C–bicomodule ${}^{C}H \,\Box\, H^{C}$, where the two coactions are λ_R and ρ_R. The notion of cocenter will be discussed further in Section 4.

Definition 6.1.8. Let $M \in {}^C\mathsf{M}^C$ a C–bicomodule. Define the map

$$\Phi : M \otimes C^* \to M$$
$$m \otimes \varphi \mapsto m_{[0]}\, \varphi(m_{[1]}) - m_{[0]}\, \varphi(m_{[-1]})$$

where C^* denotes the k-dual of the coalgebra C. Then, the *cocenter* of M is defined by the cokernel map $\zeta : M \to \mathcal{Z}(M)$, where

$$M \otimes C^* \xrightarrow{\Phi} M \xrightarrow{\zeta} \mathcal{Z}(M) \longrightarrow 0$$

Introduce also the epi–mono factorization $\Phi : M \otimes C^* \xrightarrow{e} J_M \xrightarrow{i} M$

The cocenter satisfies the following universal property. Let

$$W_M = \{m_{[0]} \otimes m_{[1]} - m_{[0]} \otimes m_{[-1]} \mid m \in M\} \subseteq M \otimes C$$

then for all k–module maps $f : M \to N$ which satisfy

$$(f \otimes C)(W_M) = 0 \tag{6.1.9}$$

i.e. $f(m_{[0]}) \otimes m_{[1]} = f(m_{[0]}) \otimes m_{[-1]}$, there is a unique $f' : \mathcal{Z}(M) \to N$ such that $f = f' \circ \zeta$:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle\zeta}\downarrow & \nearrow{\scriptstyle f'} & \\ \mathcal{Z}(M) & & \end{array}$$

Indeed, applying $(N \otimes \varphi)$ to 6.1.9, we find that $(f \otimes \varphi)(W_M) = 0$ for all $\varphi \in C^*$, i.e. f annihilates J_M.

If the coalgebra C is locally projective as a k–module (see [**22**], **42.9**), then $(\zeta \otimes C)(W_M) = 0$. To see this, note that for C locally projective, $(\zeta \otimes C)(W_M) = 0$ if and only if $(\mathrm{id} \otimes \varphi) \circ (\zeta \otimes C)(W_M) = 0$ for all $\varphi \in C^*$. This, however, holds by the definiton of ζ.

Thus, for locally projective C, a k–module map $f : M \to N$ factorizes through $\zeta : M \to \mathcal{Z}(M)$ *if and only if* $(f \otimes C)\,(W_M) = 0$. Since, throughout this paper, we are working over a field, it is in fact unnecessary to explicitly assume local projectivity: modules over a field are always free, hence they are projective. A projective module is also locally projective.

We apply the above definition to the bicomodule ${}^C H \,\square\, H^C$ afforded by the coactions λ_R and ρ_R. For reference, the bicomodule structure is

$$(\lambda_R \,\square\, H) : \sum_i g^i \,\square\, h^i \mapsto \sum_i \beta(g^i_{(2)}) \,\square\, g^i_{(1)} \,\square\, h^i \tag{6.1.10}$$

$$(H \,\square\, \rho_R) : \sum_i g^i \,\square\, h^i \mapsto \sum_i g^i \,\square\, h^i_{(1)} \,\square\, \alpha(h^i_{(2)}) \tag{6.1.11}$$

We make the following

Definition 6.1.12. $H \boxtimes H$ is the cocenter of the bicomodule ${}^C H \,\Box\, H^C$,

$$H \,\Box_C\, H \otimes C^* \xrightarrow{\Phi_2} H \,\Box_C\, H \xrightarrow{\zeta_2} H \boxtimes H$$

where $\Phi_2(g^i \,\Box\, h^i \otimes \varphi) = g^i \,\Box\, h^i_{(1)}\, \varphi(\alpha(h^i_{(2)})) - g^i_{(1)} \,\Box\, h^i\, \varphi(\beta(g^i_{(2)}))$

Using 6.1.10 and 6.1.11, the multiplication map $\mu : H \,\Box_C\, H \to H$ factorizes through $H \boxtimes H$, i.e.

$$\begin{array}{ccc} H \,\Box_C\, H & \xrightarrow{\mu} & H \\ {\scriptstyle \zeta}\downarrow & \nearrow_{f'} & \\ H \boxtimes H & & \end{array}$$

precisely if $\sum_i \mu(g^i \otimes h^i_{(1)}) \otimes \alpha(h^i_{(2)}) = \mu(g^i_{(1)} \otimes h^i) \otimes \beta(g^i_{(2)})$ (condition 6.1.2) holds.

This construction can be seen as dual to that of the Takeuchi product $\times_R$. For a left bialgebroid A, the submodule $A \times_R A \hookrightarrow A \underset{R}{\otimes} A$ is the center of the R–bimodule $r \cdot (A \otimes A) \cdot r' = At(r) \otimes As(r)$. It is well–known that there is no well–defined multiplication on $A \underset{R}{\otimes} A$, but $A \times_R A$ is a ring with component–wise multiplication. The dual result is that even though comultiplication is not well-defined on $H \,\Box_C\, H$, the factor $H \boxtimes H$ becomes a well–defined coalgebra. This ensures that 6.1.3 is well-defined.

Remark 6.1.13. The reader may easily convince herself that the axioms 6.1.1 are dual to those of a left bialgebroid $\langle A, \mu_A, \eta_A, \Delta_A, \varepsilon_A, s, t, R\rangle$ in the sense of reversing the direction of maps and making the following substitutions: $\langle A, \mu_A, \eta_A\rangle \leftrightarrow \langle H, \Delta_H, \varepsilon_H\rangle$, $\{\Delta_A, \varepsilon_A\} \leftrightarrow \{\mu_H, \eta_H\}$, $\{s, t\} \leftrightarrow \{\alpha, \beta\}$, $R \leftrightarrow C$. A *right* bicoalgebroid is a C–bicomodule algebra with the coactions λ_R and ρ_R, i.e. we require (H, λ_R, ρ_R) to be a monoid in the category of C–bicomodules. The axioms dualize those of a *right* bialgebroid (cf. the Example below).

Example 6.1.14. The simplest right bialgebroid over a ring R is the enveloping algebra $R^e = R \otimes R^{op}$ (its opposite is a left bialgebroid). The co–enveloping coalgebra $C^e = C \otimes C_{cop}$ provides our first example of a bicoalgebroid. The source– and target maps are given by

$$\alpha : C \otimes C_{cop} \to C,\ c \otimes \bar{c} \mapsto c\,\varepsilon(\bar{c}) \ \text{ and} \tag{6.1.15}$$

$$\beta : C \otimes C_{cop} \to C_{cop},\ c \otimes \bar{c} \mapsto \varepsilon(c)\,\bar{c} \tag{6.1.16}$$

Multiplication is given by

$$\mu^e : (C \otimes C_{cop}) \,\Box\, (C \otimes C_{cop}) \to C \otimes C_{cop} \tag{6.1.17}$$

$$(c \otimes \bar{c}) \,\Box\, (d \otimes \bar{d}) \mapsto d\varepsilon(c)\,\varepsilon(\bar{d}) \otimes \bar{c} \tag{6.1.18}$$

and the unit map is Δ_{cop}, $\eta^e : C \to C \otimes C_{cop}$, $\eta^e(c) = c_{(2)} \otimes c_{(1)}$. Just as in the dual case (where $R^{e,op} = R^{op} \otimes R$ is a left bialgebroid), we also have that $C^e_{cop} = C_{cop} \otimes C$ is a left bicoalgebroid.

2. Modules & comodules over bicoalgebroids

Based on experience with bialgebroids and dualization arguments, it may be expected that a categorical approach to bicoalgebroids leads to the study of its category of comodules.

2.1. Comodules over bicoalgebroids. Recall that a left module $\langle M, \triangleright \rangle$ over a bialgebroid B is a left R^e–module ${}_{R^e}M$ together with an associative and unital action $\triangleright : B \otimes_{R^e} M \to M$. Note that this is equivalent to the underlying k–module M being a left module of the underlying k–algebra of B, since the k–algebra action factorizes as

$$\triangleright : B \otimes M \xrightarrow{\pi} B \otimes_{R^e} M \longrightarrow M \ ,$$

i.e. the left R^e–action of M is exactly the one induced by the left B–action. Dualizing to the case of bicoalgebroids, one can say that comodules over a bicoalgebroid are simply comodules over the underlying coalgebra, or equivalently:

Definition 6.2.1. A left H–comodule over a left bicoalgebroid H is a pair $\langle M, \delta_M \rangle$, where $M \in {}^{C^e}\mathsf{M}$, and $\delta_M : M \to H \,\Box_{C^e} M$ is a left C^e–comodule map for which

$$\begin{array}{ccc} M & \xrightarrow{\delta_M} & H \,\Box_{C^e} M \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \Delta \,\Box_{C^e} M} \\ H \,\Box_{C^e} M & \xrightarrow[H \,\Box_{C^e} \delta_M]{} & H \,\Box_C H \,\Box_{C^e} M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\delta_M} & H \,\Box_{C^e} M \\ & {\scriptstyle l_M}\searrow & \downarrow{\scriptstyle \varphi \,\Box_{C^e} M} \\ & & C^e \,\Box_{C^e} M \end{array}$$

where $\varphi = (\alpha \otimes \beta) \circ \Delta$. This makes δ_M a coassociative & counital coaction.

The category of H–comodules ${}^H\mathsf{M}$ has objects the left H–comodules, and the arrows $f : \langle M, \delta_M \rangle \to \langle N, \delta_N \rangle$ are the C^e–bicomodule maps $f : M \to N$ such that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \delta_N} \\ H \,\Box_{C^e} M & \xrightarrow[H \,\Box_{C^e} f]{} & H \,\Box_{C^e} N \end{array}$$

2.2. Modules over a bicoalgebroid. We now define modules over a bicoalgebroid, which will be necessary to construct Yetter–Drinfel'd modules in Section 5.

Definition 6.2.2. A right module over a left bicoalgebroid H (over C) is a pair $\langle X, \triangleleft \rangle$, where $X \in \mathsf{M}^C$ is a right C–comodule and the action is a right C–comodule map $\triangleleft : X \,\Box_C H^C \to X^C$. Similarly, a left module is a pair $\langle Y, \triangleright \rangle$ with $Y \in {}^C\mathsf{M}$ and $\triangleright : {}^C H \,\Box_C Y \to Y$ a left C–comodule map. H is a C–bicomodule through the coactions λ_L and ρ_L.

The module category of a bicoalgebroid is expected to be monoidal as well, coming with an embedding into ${}^C\mathsf{M}^C$. The above definition doesn't seem to allow for this, but luckily, a dual of Prop. 1.1. of [**3**] holds:

Proposition 6.2.3. *Let $\langle X, \triangleleft\rangle$ be a right module over the bicoalgebroid H. Then X has a unique left C–comodule structure such that*

(1) *X is a C–bicomodule*
(2) *the action is a C–bicomodule map*
(3) *$\triangleleft : X \,\square_C\, H \to X$ factorizes through $X \boxtimes H$*

Proof. Note that the action being a right C–comodule map means

$$(x \triangleleft h)_{[0]} \otimes (x \triangleleft h)_{[1]} = x \triangleleft h_{(2)} \otimes \beta(h_{(1)}) \tag{6.2.4}$$

The left comodule structure in question will be denoted $\tau(x) = x_{[-1]} \otimes x_{[0]}$. In fact, τ is uniquely determined by demanding that the right H action be also a left C–comodule map w.r.t τ. Note that $X \,\square_C\, H$ is a left C–comodule through the left C–coaction $\lambda_R(h) = \beta(h_{(2)}) \otimes h_{(1)}$ on H, i.e. we impose:

$$(x \triangleleft h)_{[-1]} \otimes (x \triangleleft h)_{[0]} = \beta(h_{(2)}) \otimes x \triangleleft h_{(1)} \tag{6.2.5}$$

The identity $x = x_{[0]} \triangleleft \eta(x_{[1]})$ and 6.2.5 yield an explicit formula for the left coaction τ:

$$\begin{aligned} x_{[-1]} \otimes x_{[0]} &= (x_{[0]} \triangleleft \eta(x_{[1]}))_{[-1]} \otimes (x_{[0]} \triangleleft \eta(x_{[1]}))_{[0]} = \\ &= \beta(\eta(x_{[1]})_{(2)}) \otimes x_{[0]} \triangleleft \eta(x_{[1]})_{(1)} \end{aligned}$$

This is indeed a coaction, i.e. $(C \otimes \tau) \circ \tau = (\Delta_C \otimes X) \circ \tau$. Inserting definitions, the

$$\begin{aligned} LHS = \beta(\eta(x_{[1]})_{(2)}) \otimes \beta\{\eta[(x_{\langle 0\rangle} \triangleleft \eta(x_{\langle 1\rangle})_{(1)})_{[1]}]_{(2)}\} \otimes \\ \otimes (x_{\langle 0\rangle} \triangleleft \eta(x_{\langle 1\rangle})_{(1)})_{[0]} \triangleleft \eta[(x_{\langle 0\rangle} \triangleleft \eta(x_{\langle 1\rangle})_{(1)})_{[1]}]_{(1)} \end{aligned}$$

Using 6.2.4, we find:

$$\begin{aligned} LHS = \beta(\eta(x_{[1]})_{(2)}) \otimes \beta\{\eta[\alpha(\eta(x_{[1]})_{(1)(2)})]_{(2)}\} \otimes \\ \otimes (x_{[0]} \triangleleft \eta(x_{[0]})_{(1)(1)}) \triangleleft \eta[\alpha(\eta(x_{[1]})_{(1)(2)})]_{(1)} \end{aligned}$$

which, by the bicoalgebroid axiom 6.1.6, is further equal:

$$\begin{aligned} LHS &= \beta(\eta(x_{[1]})_{(2)}) \otimes \beta(\eta(x_{[1]})_{(1)(2)})_{(2)} \otimes x_{[0]} \triangleleft \eta(x_{[1]})_{(1)(1)}\eta(x_{[1]})_{(1)(2)(1)} = \\ &= \beta(\eta(x_{[1]})_{(3)}) \otimes \beta(\eta(x_{[1]})_{(2)}) \otimes x_{[0]} \triangleleft \eta(x_{[1]})_{(1)} = \beta(\eta(x_{[1]})_{(2)})_{(1)} \otimes \\ &\otimes \beta(\eta(x_{[1]})_{(2)})_{(2)} \otimes x_{[0]} \triangleleft \eta(x_{[1]})_{(1)} = RHS. \end{aligned}$$

In the first equality, we used comultiplicativity of the unit and coassociativity. In the second, the fact that β is an anti–coalgebra map.

As for (1), the coaction τ makes X a bicomodule. Using the definition of the left coaction, and that the H–action is a right C comodule map:

$$x_{[-1]} \otimes x_{[0][0]} \otimes x_{[0][1]} = \beta(\eta(x_{[1]})_{(2)}) \otimes (x_{[0]} \triangleleft \eta(x_{[1]})_{(1)})_{[0]} \otimes \\ \otimes (x_{[0]} \triangleleft \eta(x_{[1]})_{(1)})_{[1]} = \beta(\eta(x_{[1]})_{(2)}) \otimes x_{[0]} \triangleleft \eta(x_{[1]})_{(1)(2)} \otimes \beta(\eta(x_{[1]})_{(1)(1)})$$

Using that $\eta : C^C \to H^C$ is a C–bicomodule map,

(6.2.6) $$\eta(c_{(1)}) \otimes c_{(2)} = \eta(c)_{(2)} \otimes \beta(\eta(c)_{(1)})$$

and the coassociativity of the coaction:

$$x_{[-1]} \otimes x_{[0][0]} \otimes x_{[0][1]} = \beta(\eta(x_{[1]})_{(2)(2)}) \otimes x_{[0]} \triangleleft \eta(x_{[1]})_{(2)(1)} \otimes \beta(\eta(x_{[1]})_{(1)}) = \\ = \beta(\eta(x_{[1](1)})_{(2)}) \otimes x_{[0]} \triangleleft \eta(x_{[1](1)})_{(1)} \otimes x_{[1](2)} = \beta(\eta(x_{[0][1]})_{(2)}) \otimes \\ \otimes x_{[0][0]} \triangleleft \eta(x_{[0][1]})_{(1)} \otimes x_{[1]} = x_{[0][-1]} \otimes x_{[0][0]} \otimes x_{[1]}$$

(we apply 6.2.6 to $c = x_{[1]}$ in the second equality). The action will then (by construction) be a C–bicomodule map, proving (2). It remains to see that the action factorizes through the cocenter of $X \,\Box_C\, H$, meaning:

(6.2.7) $$(x_{[0]} \triangleleft \eta(x_{[1]})_{(1)}) \triangleleft h \otimes \beta(\eta(x_{[1]})_{(2)}) = x_{[0]} \triangleleft h \otimes x_{[-1]}$$

This is a simple consequence of 6.2.5:

$$LHS = (x_{[0]} \triangleleft \eta(x_{[1]}))_{[0]} \triangleleft h \otimes (x_{[0]} \triangleleft \eta(x_{[1]}))_{[-1]} = RHS.$$

□

3. A dual of Schauenburg's theorem

The forgetful functor associated to the map $\varphi : H \to C^e$,

(6.3.1) $$F : {}^H\mathsf{M} \to {}^{C^e}\mathsf{M} \simeq {}^C\mathsf{M}^C$$
$$\langle M, \delta_M \rangle \to \langle M, (\varphi \otimes M) \circ \delta_M \rangle$$

is faithful and *left* adjoint to $H \,\Box_{C^e}\, \_ : {}^{C^e}\mathsf{M} \to {}^H\mathsf{M}$. In the dual case, a left bialgebroid A over R is an R^e–ring with $s \otimes t : R \otimes R^{op} \to A$, i.e. a monoid in ${}_{R^e}\mathsf{M}_{R^e}$. The forgetful functor $U : {}_A\mathsf{M} \to {}_{R^e}\mathsf{M}$ is *right* adjoint to $A \otimes_{R^e} \_ :_{R^e} \mathsf{M} \to {}_A\mathsf{M}$. Furthermore, Schauenburg's theorem states that bialgebroid structures on the R^e–ring A are in one-to-one correspondence with monoidal structures on the category ${}_A\mathsf{M}$ such that the forgetful functor U is strict monoidal.

This raises the question whether a dual of this theorem holds for bicoalgebroids, namely: is there a one-to-one correspondence between bicoalgebroid structures on the coalgebra $\langle H, \Delta, \varepsilon \rangle$ and monoidal structures on the category ${}^H\mathsf{M}$ such that $F : {}^H\mathsf{M} \to {}^C\mathsf{M}^C$ is strict monoidal? The next theorem gives the forward implication.

Theorem 6.3.2. *Let H be a left bicoalgebroid over C. Then there is a monoidal structure on ${}^H\mathsf{M}$ making the forgetful functor $F : {}^H\mathsf{M} \to {}^{C^e}\mathsf{M} \simeq {}^C\mathsf{M}^C$ strict monoidal. Identifying*

H–comodules with their underlying C–bicomodules, the monoidal product is $\Box_C$, the cotensor product over C and C is the monoidal unit.

Proof. Assume there is a monoidal structure $\langle {}^H\mathsf{M}, \odot, I\rangle$ on ${}^H\mathsf{M}$ such that the forgetful functor is strict monoidal, meaning that we have a triple $\langle F, F^2, F^0\rangle$, where the maps $F^{M,N} : F(M \odot N) \to F(M) \Box_C F(N)$ and $F^0 : F(I) \to C$ are identities. This amounts to specifying

- an H–comodule structure on C,
$$\delta_C : C \to H \Box_{C^e} C, \text{ and}$$
- an H–comodule structure on the cotensor product of objects $M, N \in {}^C\mathsf{M}^C$,
$$\delta_{M \Box N} : M \Box_C N \to H \Box_{C^e} (M \Box_C N),$$
natural in M and N

The bicoalgebroid structure on H allows us to construct such maps δ_C and $\delta_{M\Box N}$.

The unit map $\eta : C \to H$ provides the desired H–comodule structure on C:

$$\delta_C = (H \otimes \alpha) \circ \Delta \circ \eta, \quad \delta_C(c) = \eta(c)_{(1)} \otimes \alpha(\eta(c)_{(2)})$$

This is indeed a coaction,

$$\begin{aligned}(H \otimes \delta_C) \circ \delta_C(c) &= \eta(c)_{(1)} \otimes \eta(\alpha(\eta(c_{(2)})))_{(1)} \otimes \alpha((\eta(\alpha(\eta(c_{(2)}))))_{(1)})_{(2)}) = \\ &= \eta(c)_{(1)} \otimes \eta(c)_{(2)(1)} \otimes \alpha(\eta(c)_{(2)(2)}) = \eta(c)_{(1)(1)} \otimes \eta(c)_{(1)(2)} \otimes \alpha(\eta(c)_{(2)}) = \\ &= (\Delta_H \otimes C) \circ \delta_C(c),\end{aligned}$$

applying 6.1.6 in the second equality and coassociativity in the third.

For $M, N \in {}^H\mathsf{M}$, define the coaction $\delta_{M\Box N} : M \Box_C N \to H \Box_{C^e} (M \Box_C N)$ as the composite map:

$$\delta_{M\Box N} : \; M \Box_C N \xrightarrow{\delta_M \Box_C \delta_N} (H \Box_{C^e} M) \Box_C (H \Box_{C^e} N) \xrightarrow{\kappa} H \Box_{C^e} (M \Box_C N)$$

Implicit in this definition is the map

$$\kappa : (H \Box_{C^e} M) \Box_C (H \Box_{C^e} N) \to H \Box_{C^e} (M \Box_C N) \tag{6.3.3}$$

which we define as the unique arrow in the following diagram

$$\begin{array}{ccc}
(H \Box_{C^e} M) \Box_C (H \Box_{C^e} N) & \overset{\kappa}{\dashrightarrow} & H \Box_{C^e} (M \Box_C N) \\
\downarrow{\scriptstyle \iota_{H\Box M, H\Box N}} & & \downarrow{\scriptstyle \bar\iota_{H, M\Box N}} \\
(H \Box_{C^e} M) \otimes (H \Box_{C^e} N) & & H \otimes (M \Box_C N) \\
\downarrow{\scriptstyle \bar\iota_{H,M} \otimes \bar\iota_{H,N}} & & \downarrow{\scriptstyle H \otimes \iota_{M,N}} \\
(H \otimes M) \otimes (H \otimes N) & \xrightarrow[(\mu_H \otimes M \otimes N) \circ \mathsf{tw}_{23}]{} & H \otimes (M \otimes N)
\end{array}$$

By the definition of the kernel maps $\iota_{M,N} : M \,\square_C\, N \to M \otimes N$ and $\bar{\iota}_{U,V} : U \,\square_{C^e}\, V \to U \otimes V$, $(h \otimes m) \otimes (h' \otimes n) \in (H \,\square_{C^e}\, M) \,\square_C\, (H \,\square_{C^e}\, N)$ if and only if the following identities hold:

$$(h_{(1)} \otimes \alpha(h_{(2)}) \otimes m) \otimes (h' \otimes n) = (h \otimes m_{[-1]} \otimes m_{[0]}) \otimes (h' \otimes n) \tag{6.3.4}$$

$$(h_{(1)} \otimes \beta(h_{(2)}) \otimes m) \otimes (h' \otimes n) = (h \otimes m_{[1]} \otimes m_{[0]}) \otimes (h' \otimes n) \tag{6.3.5}$$

$$(h \otimes m) \otimes (h'_{(1)} \otimes \alpha(h'_{(2)}) \otimes n) = (h \otimes m) \otimes (h' \otimes n_{[-1]} \otimes n_{[0]}) \tag{6.3.6}$$

$$(h \otimes m) \otimes (h'_{(1)} \otimes \beta(h'_{(2)}) \otimes n) = (h \otimes m) \otimes (h' \otimes n_{[1]} \otimes n_{[0]}) \tag{6.3.7}$$

and

$$h_{(2)} \otimes m \otimes \beta(h_{(1)}) \otimes h' \otimes n = h \otimes m \otimes \alpha(h'_{(1)}) \otimes h'_{(2)} \otimes n \tag{6.3.8}$$

The arrow κ is defined by the universal property of the composite kernel map $(H \otimes \iota_{M,N}) \circ \bar{\iota}_{H,M \,\square\, N}$, provided

$$(\mu_H \otimes M \otimes N) \circ \mathsf{tw}_{23} \circ (\bar{\iota}_{H,M} \otimes \bar{\iota}_{H,N}) \circ \iota_{H \,\square\, M, H \,\square\, N}((h \otimes m) \otimes (h' \otimes n)) = \\ = (hh') \otimes (m \otimes n) \in H \,\square_{C^e}\, (M \,\square_C\, N)$$

This leads to the following equations:

$$(hh') \otimes m_{[0]} \otimes m_{[1]} \otimes n = (hh') \otimes m \otimes n_{[-1]} \otimes n_{[0]} \tag{6.3.9}$$

$$(hh')_{(1)} \otimes \alpha((hh')_{(2)}) \otimes m \otimes n = (hh') \otimes m_{[-1]} \otimes m_{[0]} \otimes n \tag{6.3.10}$$

$$(hh')_{(1)} \otimes \beta((hh')_{(2)}) \otimes m \otimes n = (hh') \otimes n_{[1]} \otimes m \otimes n_{[0]} \tag{6.3.11}$$

Observe that by the multiplicativity of the coproduct and because (H, λ_L, ρ_L) is a monoid in ${}^C\mathsf{M}^C$, we have the following identities:

$$\alpha(hh') = \alpha(h)\varepsilon(h') \tag{6.3.12}$$

$$\beta(hh') = \varepsilon(h)\beta(h') \tag{6.3.13}$$

To show 6.3.12, compute

$$\alpha(hh') = \alpha((hh')_{(1)})\varepsilon((hh')_{(2)}) = \alpha(h_{(1)})\varepsilon(h_{(2)})\varepsilon(h') = \alpha(h)\varepsilon(h'),$$

and analagously for 6.3.13. Note that 6.3.12 and 6.3.13 are dual to the relations $\Delta_A(t(r)) = 1_A \otimes t(r)$ and $\Delta_A(s(r)) = s(r) \otimes 1_A$, which hold for a left bialgebroid A over R.

To prove 6.3.10, use 6.3.12 in the first equality and 6.3.4 in the second:

$$(hh')_{(1)} \otimes \alpha((hh')_{(2)}) \otimes m \otimes n = h_{(1)}h' \otimes \alpha(h_{(2)}) \otimes m \otimes n = \\ = (hh') \otimes m_{[-1]} \otimes m_{[0]} \otimes n$$

Similarly, 6.3.11 is by proved by using 6.3.13 in the first equality and 6.3.6 in the second:

$$(hh')_{(1)} \otimes \beta((hh')_{(2)}) \otimes m \otimes n = hh'_{(1)} \otimes \beta(h'_{(2)}) \otimes m \otimes n =$$
$$= (hh') \otimes n_{[1]} \otimes m \otimes n_{[0]},$$

To prove 6.3.9, applying 6.3.5 and 6.3.6 to the left– and right hand sides, respectively, yields

$$h_{(1)}h' \otimes m \otimes \beta(h_{(2)}) \otimes n = hh'_{(1)} \otimes m \otimes \alpha(h'_{(2)}) \otimes n$$

which holds precisely because multiplication satisfies the property 6.1.2.

Let us check that $\delta_{M,N} : M \,\Box_C\, N \to H \,\Box_{C^e}\, (M \,\Box_C\, N)$ is indeed a coaction. Expressed on elements, $\delta_{M,N}(m \otimes n) = m_{[-1]}n_{[-1]} \otimes m_{[0]} \otimes n_{[0]}$ (we think of the domain and range of $\delta_{M,N}$ as embedded into $M \otimes N$ and $H \otimes (M \otimes N)$, respectively).

$$\begin{aligned}(H \,\Box\, \delta_{M,N}) \circ \delta_{M,N}(m \otimes n) &= m_{[-1]}n_{[-1]} \otimes m_{[0][-1]}n_{[0][-1]} \otimes m_{[0]} \otimes n_{[0]} = \\ &= m_{[-1](1)}n_{[-1](1)} \otimes m_{[-1](2)}n_{[-1](2)} \otimes m_{[0]} \otimes n_{[0]} = \\ &= (m_{[-1]}n_{[-1]})_{(1)} \otimes (m_{[-1]}n_{[-1]})_{(2)} \otimes m_{[0]} \otimes n_{[0]} = \\ &= (\Delta \otimes M) \circ \delta_{M,N}(m \otimes n)\end{aligned}$$

where we used the comultiplicativity of the multiplication on H in the third equality.

For $\langle {}^H\mathsf{M}, \Box, C\rangle$ to be a monoidal category, we have still to define the natural isomorphisms $\alpha_{M,N,P} : (M \,\Box\, N) \,\Box\, P \to M \,\Box\, (N \,\Box\, P)$ (the associator), $\lambda_M : C \,\Box\, M \to M$ and $\rho_N : N \,\Box\, C \to N$. Due to the strict monoidality of F, these maps may be defined as the lifting of the respective coherence morphisms of ${}^C\mathsf{M}^C$ to ${}^H\mathsf{M}^H$, provided they induce H–comodule maps. This, however, follows from the associativity and unit property of the multiplication and unit on H. □

For the converse direction, assume the existence of a strict monoidal forgetful functor $F : {}^H\mathsf{M} \to {}^{C^e}\mathsf{M}$ on the comodule category of the coalgebra $\langle H, \Delta, \varepsilon\rangle$. We construct a bicoalgebroid structure on H as follows. Define a multiplication map $\mu : H \,\Box_C\, H \to H$ as the composite

$$H \,\Box_C\, H \xrightarrow{\delta_{H,H}} H \,\Box_{C^e}\, (H \,\Box_C\, H) \hookrightarrow H \otimes (H \,\Box_C\, H) \xrightarrow{H \otimes (\varepsilon \,\Box_C\, \varepsilon)} H$$

and define a unit map $\eta : C \to H$ with

$$\eta = (H \,\Box_{C^e}\, \varepsilon_C) \circ \delta_C : C \to H \,\Box_{C^e}\, C \to C.$$

Note that these maps exactly dualize $\Delta : H \to H \underset{R}{\otimes} H$, $h \mapsto h \triangleright (1_H \underset{R}{\otimes} 1_H)$ and $\varepsilon_H : H \to R$, $h \mapsto h \triangleright 1_H$ which produce a bialgebroid structure on the underlying algebra, given a strong monoidal forgetful functor $U : {}_H\mathsf{M} \to {}_{R^e}\mathsf{M}$ (cf. [**72**] and the discussion of Theorem 3.2.2). Instead of directly proving the bicoalgebroid axioms satisfied by μ and η, we reformulate the problem in terms of comonads and apply a result on 'liftings of functors'. We start with a monadic formulation of the bicoalgebroid structure.

DEFINITION 6.3.14. Let $\langle \mathsf{M}, \Box, I\rangle$ be a monoidal category. Then a bicomonad on M is a comonoid in the category of monoidal endofunctors from M to M. Thus, it is an endofunctor $G : \mathsf{M} \to \mathsf{M}$, furnished with:

- a natural transformation $\kappa_{X,Y} : (G\,X) \Box (G\,Y) \to G\,(X \Box Y)$, and
- an arrow $\xi : C \to G\,C$

such that $\langle G, \kappa_{X,Y}, \xi\rangle$ is a monoidal functor;

- a natural transformation $\delta_X : G\,X \to GGX$ and
- a natural transformation $\varepsilon_X : G\,X \to X$

such that $\langle G, \delta, \varepsilon\rangle$ is a comonoid in M^{M}, and four compatibility axioms stating that δ is monoidal,

$$\delta_{X\otimes Y} \circ \kappa_{X,Y} = (G\,\kappa_{X,Y} \circ \kappa_{G\,X,G\,Y}) \circ (\delta_X \otimes \delta_Y) \tag{6.3.15}$$

$$\delta_I \circ \xi = G\xi \circ \xi \tag{6.3.16}$$

and that ε is monoidal

$$\varepsilon_{X\otimes Y} \circ \kappa_{X,Y} = \varepsilon_X \otimes \varepsilon_Y \tag{6.3.17}$$

$$\varepsilon \circ \xi = I \tag{6.3.18}$$

Not surprisingly, there is an analogue of Theorem 5.4.6:

PROPOSITION 6.3.19. *The endofunctor* $G = FI = {}^{C^e}H \Box_{C^e} \_ : {}^{C^e}\mathsf{M} \to {}^{C^e}\mathsf{M}$ *is a monoidal comonad with the structure maps:*

$$\begin{aligned}\delta_X : H \Box_{C^e} X &\to H \Box_{C^e} (H \Box_{C^e} X) \\ h \otimes x &\mapsto h_{(1)} \otimes (h_{(2)} \otimes x)\end{aligned} \tag{6.3.20}$$

$$\begin{aligned}\varepsilon_X : H \Box_{C^e} X &\to X \\ h \otimes x &\mapsto \varepsilon_H(h)x\end{aligned} \tag{6.3.21}$$

$$\begin{aligned}\kappa_{X,Y} : (H \Box_{C^e} X) \Box_C (H \Box_{C^e} Y) &\to H \Box_{C^e} (X \Box_C Y) \\ (h \otimes x) \otimes (h' \otimes y) &\mapsto hh' \otimes (x \otimes y)\end{aligned} \tag{6.3.22}$$

$$\begin{aligned}\xi : C &\to H \Box_{C^e} C \\ c &\mapsto \eta(c)_{(1)} \otimes \alpha(\eta(c)_{(2)})\end{aligned} \tag{6.3.23}$$

PROOF. The associativity of κ corresponds to the associativity of the multiplication μ of H, and ξ is a unit for κ precisely because η is a unit for μ. The monoidality of δ_X and ε_X are due to the multiplicativity and unitalness of Δ_H and ε_H. Finally, G is a comonad because H is a coalgebra. $\Box$

A monoidal structure on ${}^H\mathsf{M}$ such that $F : {}^H\mathsf{M} \to {}^{C^e}\mathsf{M} \simeq {}^C\mathsf{M}^C$ is strict monoidal implies that the monoidal product on ${}^{C^e}\mathsf{M}$ is *lifted* to the Eilenberg–Moore category of **G**-coalgebras in the following sense:

$$\begin{array}{ccc} {}^{\mathbf{G}}\mathsf{M} \times {}^{\mathbf{G}}\mathsf{M} & \xrightarrow{\hat{\square}} & {}^{\mathbf{G}}\mathsf{M} \\ {\scriptstyle F\times F}\downarrow & & \downarrow{\scriptstyle F} \\ {}^{C^e}\mathsf{M} \times {}^{C^e}\mathsf{M} & \xrightarrow[\square_C]{} & {}^{C^e}\mathsf{M} \end{array}$$

This is a special case of the problem of *liftings of functors*, orginally considered by Johnstone ([**39**]). We refer to [**91**], from which we quote part (1) of Theorem 3.3 .

THEOREM 6.3.24. *Let* $\mathbf{G} = \langle G, \delta, \varepsilon\rangle$ *and* $\mathbf{G}' = \langle G', \delta', \varepsilon'\rangle$ *be comonads on the categories* M *and* M'*, respectively, and let* $T : \mathsf{M}' \to \mathsf{M}$ *be a functor. Denote* $U : {}^{\mathbf{G}}\mathsf{M} \to \mathsf{M}$ *and* $U' : {}^{\mathbf{G}'}\mathsf{M} \to \mathsf{M}'$ *the canonical forgetful functors.*

Then, the liftings $\hat{T} : {}^{\mathbf{G}'}\mathsf{M} \to {}^{\mathbf{G}}\mathsf{M}$ *of* T*, in the sense:*

$$\begin{array}{ccc} {}^{\mathbf{G}'}\mathsf{M} & \xrightarrow{\hat{T}} & {}^{\mathbf{G}}\mathsf{M} \\ {\scriptstyle U'}\downarrow & & \downarrow{\scriptstyle U} \\ \mathsf{M}' & \xrightarrow[T]{} & \mathsf{M} \end{array}$$

are in bijective correspondence with natural transformations $\kappa : TG' \to GT$ *for which the following diagrams commute:*

$$\begin{array}{ccccc} TG' & \xrightarrow{T\delta'} & TG'G' & \xrightarrow{\kappa G} & GTG' \\ {\scriptstyle \kappa}\downarrow & & & & \downarrow{\scriptstyle G\kappa} \\ GT & & \xrightarrow[\delta T]{} & & GGT \end{array} \qquad \begin{array}{ccc} TG' & \xrightarrow{T\varepsilon'} & T \\ {\scriptstyle \kappa}\downarrow & \nearrow{\scriptstyle \varepsilon T} & \\ GT & & \end{array}$$

Taking $\mathsf{M}' = {}^{C^e}\mathsf{M} \times {}^{C^e}\mathsf{M}$, $\mathsf{M} = {}^{C^e}\mathsf{M}$ and $T = \_ \square_C \_ : {}^{C^e}\mathsf{M} \times {}^{C^e}\mathsf{M} \to {}^{C^e}\mathsf{M}$, we find that liftings of the monoidal structure to ${}^{\mathsf{G}}\mathsf{M} \simeq {}^{H}\mathsf{M}$ are in bijective correspondence with natural transformations

$$\kappa_{M,N} : (H \square_{C^e} M) \square_C (H \square_{C^e} N) \to H \square_{C^e} (M \square_C N)$$

inducing commutative diagrams

$$\begin{array}{ccccc} G(M) \square_C G(N) & \xrightarrow{\delta_M \square \delta_N} & G^2(M) \square_C G^2(N) & \xrightarrow{\kappa_{G(M),G(N)}} & G(G(M) \square_C G(N)) \\ {\scriptstyle \kappa_{M,N}}\downarrow & & & & \downarrow{\scriptstyle G\kappa_{M,N}} \\ G(M \square_C N) & & \xrightarrow[\delta_{M,N}]{} & & G^2(M \square_C N) \end{array}$$

(6.3.25) $$\delta_{M \square N} \circ \kappa_{M,N} = G\kappa_{M,N} \circ \kappa_{G(M),G(N)} \circ (\delta_M \square_C \delta_N)$$

and

$$\begin{array}{ccc} G(M)\,\square_C\, G(N) & \xrightarrow{\varepsilon\,\square_C\,\varepsilon} & M\,\square_C\, N \\ {\scriptstyle \kappa_{M,N}}\big\downarrow & \nearrow{\scriptstyle \varepsilon_{M\,\square\, N}} & \\ G(M\,\square_C\, N) & & \end{array}$$

(6.3.26) $$\varepsilon_{M\,\square\,N} \circ \kappa_{M,N} = \varepsilon_M \,\square_C\, \varepsilon_N$$

The two diagrams above recover two of the compatibility relations (6.3.15 and 6.3.17) of a bicomonad. If, furthermore, we have an arrow $\xi : C \to G(C)$ making C a $\mathbf{G}$–coalgebra such that the remaining two bicomonad conditions (6.3.16 and 6.3.18) are satisfied, then ${}^{\mathbf{G}}\mathsf{M}$ becomes a (unital) monoidal category. Summarizing, we have the following monadic version of Schauenburg's theorem:

THEOREM 6.3.27. *Let $\langle H, \Delta, \varepsilon\rangle$ be a comonoid in ${}^{C^e}\mathsf{M}$. Then there is a bijective correspondence between*

(1) *monoidal structures on ${}^{H}\mathsf{M}$ such that the forgetful functor $F : {}^{H}\mathsf{M} \to {}^{C^e}\mathsf{M}$ is strict monoidal*
(2) *a map $\kappa_{M,N} : (H\,\square_{C^e}\, M)\,\square_C\,(H\,\square_{C^e}\, N) \to H\,\square_{C^e}\,(M\,\square_C\, N)$, natural in both arguments and a map $\xi : C \to H\,\square_{C^e}\, C$ such that $\langle H\,\square_{C^e}\, \_, \Delta, \varepsilon, \kappa, \xi\rangle$ constitutes a bicomonad, i.e. the compatibilty conditions 6.3.15, 6.3.16, 6.3.17 and 6.3.18 are satisfied.*

4. Cocenter and cocentralizer

We have already had to define the cocenter of a bicomodule in order to be able to define bicoalgebroids in full analogy (or, rather, full duality) with bialgebroids. Recall that the multiplication map of a bicoalgebroid H factorizes through the cocenter of the tensor square $H\,\square_C\, H$, which is dual to Takeuchi's $\times_R$–product. It was stated but wasn't proved, that unlike $H\,\square_C\, H$, the cocenter $H \boxtimes H$ has a coalgebra structure. We shall give the proof now and also explore ways to define the cocenter of a bicomodule and the cocentralizer of a coalgebra extension, beginning with the latter.

4.1. The cocentralizer. A coalgebra coextension is simply a morphism of coalgebras $\pi : C \to X$, and we say that X is a coextension of C. Recall that the map π induces an X–bicomodule structure on C. The left and right X–coactions will be denoted $\delta^l_\pi = (\pi \otimes C) \circ \Delta$ and $\delta^r_\pi = (C \otimes \pi) \circ \Delta$, respectively. We would like to dualize the centralizer construction for ring extensions to the coalgebraic setting.

First, we shall take an abstract, categorical approach and define the cocentralizer as a universal object in an appropriate category. Consider the comma category $(C \downarrow k - \mathsf{Coalg})$, consisting of

- Objects $\langle C \xrightarrow{\pi} X\rangle$, where π is a coalgebra map

- Arrows $\langle C \xrightarrow{\pi} X\rangle \xrightarrow{f} \langle C \xrightarrow{\pi'} X'\rangle$, where $f : X \to X'$ is a coalgebra map $\pi' = f \circ \pi$

Each object of $(C \downarrow k - \mathsf{Coalg})$ corresponds to a coextension of the coalgebra C. In a similar vein, k–algebra extensions can be looked at as objects in $(A \downarrow k - \mathsf{Alg})$. The universal property of the centralizer of an algebra extension $B \subseteq A$ is that it contains all subalgebras commuting with the subalgebra B. As a first step, we define a notion of cocommutation of 'coalgebras under C'.

Definition 6.4.1. Two objects $\langle C \xrightarrow{\pi} X\rangle, \langle C \xrightarrow{\pi'} X'\rangle \in (C \downarrow k - \mathsf{Coalg})$ will be said to *cocommute* if $\delta^r_\pi : C \to C \otimes X$ is an X'–comodule map with respect to the coaction $\delta_{\pi'}$

Remark 6.4.2. This relation is in fact symmetric: $\delta^r_\pi : C \to C \otimes X$ is an X'–comodule map if and only if $\delta^r_{\pi'} : C \to C \otimes X'$ is an C–comodule map.

$$\begin{array}{ccc} C & \xrightarrow{\delta^{op}_{\pi'}} & X'^{cop} \otimes C \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle X'^{cop} \otimes \delta_\pi} \\ C \otimes X & \xrightarrow[\delta^{op}_{\pi'}]{} & X'^{cop} \otimes C \otimes X \end{array}$$

Reading the same diagram horizontally and vertically, we obtain the equivalence of the two statements above.

Now consider the full subcategory of $(C \downarrow k - \mathsf{Coalg})$ consisting of the objects cocommuting with a given object $C \xrightarrow{\pi} X$.

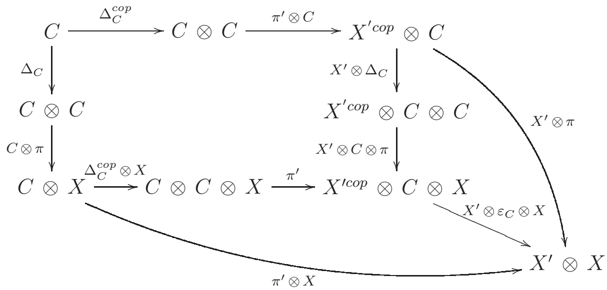
Definition 6.4.3. Let $\mathcal{C}_X(C) \overset{full}{\subseteq} (C \downarrow k - \mathsf{Coalg})$ be the subcategory of objects $\langle C \xrightarrow{\pi'} X'\rangle \in \mathcal{C}_X(C)$ for which $\delta_{\pi'} : C \to C \otimes X'$ is an X–comodule map with respect to the coaction δ_π

The following proposition gives a useful property of $\mathcal{C}_X(C)$, given only in terms of the maps π and π'.

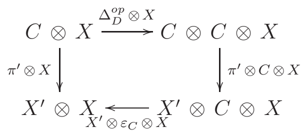
Proposition 6.4.4. *If* $\langle C \xrightarrow{\pi'} X'\rangle$ *is any object in* $\mathcal{C}_C(X)$, *then*

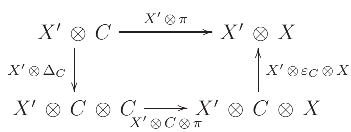
$$(\pi' \otimes \pi) \circ \Delta = (\pi' \otimes \pi) \circ \Delta^{cop} \tag{6.4.5}$$

PROOF. Inserting the definitions of δ_π and $\delta_{\pi'}$, the desired relation is the outermost hexagon:



The inner diagram expresses that $X' \in \mathcal{C}_C(X)$. Consider the lower diagrams,





they commute by the counit property of X and X', since $\pi'(c_{(2)})\varepsilon(c_{(1)}) = \pi'(c)$ and $\varepsilon(c_{(1)})\pi(c_{(2)}) = \pi(c)$. $\square$

Note that this is nothing but the dualization of the following simple reasoning for an algebra extension. A subalgebra $\iota' : B' \to A$ is in the centralizer $C_A(B)$ of anoter subalgebra $\iota : B \to A$ if and only if $\lambda_{B'} : B' \otimes A \to A$, $b' \otimes a \mapsto \iota'(b')a'$ is an ${}_B\mathsf{M}$–map, that is

$$\lambda_{B'}(b')(b \cdot a) = b \cdot \lambda_{B'}(b')(a)$$

Taking $a = 1_A$, we find $bb' = b'b$. For coalgebras, 'if and only if' fails to hold, rather, we use the property 6.4.5 to define a category $\tilde{\mathcal{C}}_C(X)$ as follows.

DEFINITION 6.4.6. Let $\tilde{\mathcal{C}}_X(C) \overset{full}{\subseteq} (C \downarrow k - \mathsf{Coalg})$ be the subcategory of objects $\langle C \overset{\pi'}{\to} X' \rangle \in \mathcal{C}_X(C)$ for which

$$(\pi' \otimes \pi) \circ \Delta = (\pi' \otimes \pi) \circ \Delta^{cop}$$

which leads to our main

DEFINITION 6.4.7. The cocentralizer $\mathcal{C}_C^{co}(X)$ of a coalgebra coextension $\pi : C \to X$ is the initial object in the category $\tilde{\mathcal{C}}_X(C)$

The coalgebra structure of the initial object (if it exists!) comes from the universal property. Denoting the coalgebra structure $\langle \tilde{\mathcal{C}}_C(X), \tilde{\Delta}, \tilde{\varepsilon} \rangle$, the structure maps are defined by the unique arrows

$$\begin{array}{ccc} C & & \\ {\scriptstyle \pi}\downarrow & \searrow^{(\pi\otimes\pi)\circ\Delta} & \\ \tilde{\mathcal{C}}_C(X) & \underset{\tilde{\Delta}}{\dashrightarrow} & \tilde{\mathcal{C}}_C(X)\otimes\tilde{\mathcal{C}}_C(X) \end{array} \qquad \begin{array}{ccc} C & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\varepsilon} & \\ \tilde{\mathcal{C}}_C(X) & \underset{\tilde{\Delta}}{\dashrightarrow} & k \end{array}$$

Since the centralizer of an algebra extension is a subalgebra of the extended algebra, we can hope to define the cocentralizer $\mathcal{C}_C(X)$ *constructively* as a factor coalgebra. We require that X be locally projective as a k–module (see [**22**]). First, a lemma:

LEMMA 6.4.8. *$W = k - \langle c_{(1)}\varphi(\pi(c_{(2)})) - c_{(2)}\varphi(\pi(c_{(1)}))\rangle_{\varphi\in X^*}$ is a Hopf–ideal in C*

PROOF. We need to prove $\Delta_C(W) \subseteq W \otimes C + C \otimes W$ and $\varepsilon_C(W) = 0$. Taking the coproduct of an element of $w = c_{(1)}\varphi(\pi(c_{(2)})) - c_{(2)}\varphi(\pi(c_{(1)})) \in W$, we find:

$$\Delta(w) = c_{(1)} \otimes c_{(2)}\varphi \circ \pi(c_{(3)}) - \varphi \circ \pi(c_{(1)})c_{(2)} \otimes c_{(3)} = c_{(1)} \otimes [c_{(2)}\varphi \circ \pi(c_{(3)})-$$
$$- \varphi \circ \pi(c_{(2)})c_{(3)}] - [c_{(1)}\varphi \circ \pi(c_{(2)}) - \varphi \circ \pi(c_{(1)})c_{(2)}] \otimes c_{(3)} \in W \otimes C + C \otimes W$$

Obviously,

$$\varepsilon(c_{(1)}\varphi(\pi(c_{(2)})) - c_{(2)}\varphi(\pi(c_{(1)}))) = \varphi(\pi(c)) - \varphi(\pi(c)) = 0 \tag{6.4.9}$$

□

Note that by the local projectivity of X, the disappearance of W is equivalent to $c_{(1)} \otimes \pi(c_{(2)}) = c_{(2)} \otimes \pi(c_{(1)})$. We can now define the cocentralizer coalgebra as follows:

DEFINITION 6.4.10. $\zeta : C \to C/W$ is the projection onto the cocentralizer coalgebra of X in C.

Because W is a Hopf–ideal, C/W is a coalgebra. The connection to our previous definition is the following:

PROPOSITION 6.4.11. *For X a locally projective coalgebra, and $\pi : C \to X$ a coalgebra coextension, the factor coalgebra $\zeta : C \to C/W$, with W the Hopf–ideal defined above, is an initial object in the category $\tilde{\mathcal{C}}_C(X)$*

PROOF. Suppose we have a coalgebra morphism $\gamma : C \to D$ that annihilates the coideal W,

$$\gamma(c_{(1)})\,\varphi(\pi(c_{(2)})) = \gamma(c_{(2)})\,\varphi(\pi(c_{(1)})). \tag{6.4.12}$$

By the local projectivity of X, this is equivalent to:

$$\gamma(c_{(1)}) \otimes \pi(c_{(2)}) = \gamma(c_{(2)}) \otimes \pi(c_{(1)}) \tag{6.4.13}$$

which says precisely that $\gamma : C \to D$ is an object in $\tilde{\mathcal{C}}_C(X)$. On the other hand, all morphisms $\langle C \xrightarrow{\gamma} D\rangle \in \tilde{\mathcal{C}}_C(X)$ annihilate W, hence factor uniquely through C/W: $\gamma = \gamma' \circ \zeta$:

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & D \\ {\scriptstyle\zeta}\big\downarrow & \nearrow{\scriptstyle\gamma'} & \\ C/W & & \end{array}$$

This diagram states that $\langle C \xrightarrow{\zeta} C/W\rangle$ is initial in $\tilde{\mathcal{C}}_C(X)$. □

4.2. The cocenter. Our first definition is categorical, similar in spirit to the abstract definition of the cocentralizer.

Definition 6.4.14. For a coalgebra C, and $Y \in {}^C\mathsf{M}^C$ a bicomodule over C, let $\mathcal{C}_Y \overset{full}{\subseteq} (Y \downarrow \mathsf{M}_k)$ be the category consisting of objects $\langle Z, t\rangle$, where $t : Y \to Z$ is an arrow in M_k satisfying $t(m_{\langle 0\rangle}) \otimes m_{\langle 1\rangle} = t(m_{\langle 0\rangle}) \otimes m_{\langle -1\rangle}$ and arrows induced from the comma category $(Y \downarrow \mathsf{M}_k)$

Definition 6.4.15. The cocenter $\langle \mathcal{Z}^{co}(Y), \zeta : Y \to \mathcal{Z}^{co}(Y)\rangle$ of a C–bicomodule $Y \in {}^C\mathsf{M}^C$ is the (unique) initial object in $\mathcal{C}_Y$

Note that $\mathcal{Z}^{co}(Y)$ is a k–module, and has no natural C–comodule structure. It is well–known that for a ring R, and M an R–bimodule, the center of M is $\mathcal{Z}(M) = \mathrm{Hom}_{R-R}(R, M)$. A similar characterization is possible for the cocenter of a bicomodule, using Takeuchi's *cohom* functor. We briefly recall the definition, referring to the original paper [**85**] and the monograph [**22**]. Recall (see [**85**]) that for $X \in {}^C\mathsf{M}^C$ quasi–finite, the left adjoint of $\_ \otimes X : \mathsf{M}_k \to {}^C\mathsf{M}^C$ exists, and it is the cohom functor $h^{C-C}(X, \_) : {}^C\mathsf{M}^C \to \mathsf{M}_k$. The adjunction is completely determined by a natural map $\theta : \mathrm{id}_{{}^C\mathsf{M}^C} \to h^{C-C}(X, \_) \otimes X$, such that for all $Y \in {}^C\mathsf{M}^C$, $\langle h^{C-C}(X, Y), \theta_Y\rangle$ is universal from Y to the functor $\_ \otimes X$, i.e.

$$\begin{array}{ccc} Y & \xrightarrow{\theta_Y} & h^{C-C}(X,Y) \otimes X \\ & {\scriptstyle f}\searrow & \big\downarrow {\scriptstyle f' \otimes X} \\ & & W \otimes X \end{array} \qquad \begin{array}{c} h^{C-C}(X,Y) \\ \big\downarrow {\scriptstyle f' \otimes X} \\ W \end{array}$$

for any $f : Y \to W \otimes X$ in ${}^C\mathsf{M}^C$. Of course, this means precisely that the map $f' \mapsto (f' \otimes X) \circ \theta_Y$ defines a bijection $\Psi : \mathrm{Hom}_k(h^{C-C}(X, Y), W) \simeq \mathrm{Hom}_{C-C}(Y, W \otimes X)$. The universality of $\langle \theta_Y, h^{C-C}(X, Y)\rangle$ can be rephrased (see [**53**]) by saying that it is an initial object in the comma category $(Y \downarrow \_ \otimes X)$.

Theorem 6.4.16. *For $Y \in {}^C\mathsf{M}^C$, and C quasi–finite, $\langle h^{C-C}(C, Y), \theta_Y\rangle$ is initial in $\mathcal{C}_Y$*

Proof. In view of the previous discussion, the proof rests on the following

Lemma 6.4.17. *The categories $\mathcal{C}_Y$ and $(Y \downarrow \_ \otimes C)$ are equivalent*

Proof. The functor $\mathcal{C}_Y \to (Y \downarrow \_ \otimes C)$ has

- Object map:

$$\begin{aligned} \langle Y \xrightarrow{t} Z\rangle &\mapsto \langle Y \xrightarrow{f_t} Z \otimes C\rangle \\ t &\mapsto f_t = (t \otimes C) \circ \delta^R{}_Y = (t \otimes C) \circ \delta^{L,op}{}_Y \end{aligned}$$

- Arrow map:

$$\{\alpha : \langle Y \xrightarrow{t} Z\rangle \to \langle Y \xrightarrow{t'} Z'\rangle\} \mapsto \{\alpha \otimes C : \langle Y \xrightarrow{f_t} Z \otimes C\rangle \to \langle Y \xrightarrow{f_{t'}} Z' \otimes C\rangle\}$$

The functor $(Y \downarrow \_ \otimes X) \to \mathcal{C}_Y$ has

- Object map:

$$\begin{aligned} \langle Y \xrightarrow{f} Z \otimes C\rangle &\mapsto \langle Y \xrightarrow{t_f} Z\rangle \\ f &\mapsto t_f = (Z \otimes \varepsilon_C) \circ f \end{aligned}$$

- Arrow map:

$$\{\beta : \langle Y \xrightarrow{f} Z \otimes C\rangle \to \langle Y \xrightarrow{f} Z' \otimes C\rangle\} \mapsto \{\beta : \langle Y \xrightarrow{t_f} Z\rangle \to \langle Y \xrightarrow{t_{f'}} Z'\rangle\}$$

We have to check that f_t is a C–bicomodule map. First, it is a right C–comodule map:

$$\begin{array}{ccccc} M & \xrightarrow{\delta^R_Y} & M \otimes C & \xrightarrow{t \otimes C} & N \otimes C \\ \downarrow{\scriptstyle \delta^R_Y} & & \downarrow{\scriptstyle M \otimes \Delta_C} & & \downarrow{\scriptstyle N \otimes \Delta_C} \\ M \otimes C & \xrightarrow[\delta^R_Y \otimes C]{} & M \otimes C \otimes C & \xrightarrow[t \otimes C \otimes C]{} & N \otimes C \otimes C \end{array}$$

the outer rectangle states that $f_t = (t \otimes C) \circ \delta^R_Y$ is a bicomodule map. Clearly, it commutes by the coassociativity of the coaction δ^R_Y. We express the left C–comodule structure as a right C^{cop}–comodule structure,

$$\begin{array}{ccccc} M & \xrightarrow{\delta^{L,op}_Y} & M \otimes C & \xrightarrow{t \otimes C} & N \otimes C \\ \downarrow{\scriptstyle \delta^{L,op}_Y} & & \downarrow{\scriptstyle M \otimes \Delta^{cop}_C} & & \downarrow{\scriptstyle N \otimes \Delta^{cop}_C} \\ M \otimes C & \xrightarrow[\delta^{L,op}_Y \otimes C]{} & M \otimes C \otimes C & \xrightarrow[t \otimes C \otimes C]{} & N \otimes C \otimes C \end{array}$$

here, the outer rectangle states that $f_t = (t \otimes C) \circ \delta^{L,op}_Y$ is a right C^{cop} comodule map. The proof is the same *mutatis mutandis*.

In the reverse direction, we check that $t_f = (Z \otimes \varepsilon_C) \circ f$ satisfies

(6.4.18) $$t_f(y_{\langle 0\rangle}) \otimes y_{\langle 1\rangle} = t_f(y_{\langle 0\rangle}) \otimes y_{\langle -1\rangle}$$

By assumption, f is a bicomodule map, i.e.

$$f(y_{\langle 0\rangle}) \otimes y_{\langle 1\rangle} = (Z \otimes \Delta) \circ f(y)$$
$$f(y_{\langle 0\rangle}) \otimes y_{\langle -1\rangle} = (Z \otimes \Delta^{cop}) \circ f(y)$$

Now,

$$t_f(y_{\langle 0\rangle}) \otimes y_{\langle 1\rangle} = (Z \otimes \varepsilon_C \otimes C)(f(y_{\langle 0\rangle}) \otimes y_{\langle 1\rangle}) = (Z \otimes \varepsilon_C \otimes C) \circ (Z \otimes \Delta) \circ f(y)$$
$$= (Z \otimes \varepsilon_C \otimes C) \circ (Z \otimes \Delta^{cop}) \circ f(y) = t_f(y_{\langle 0\rangle}) \otimes y_{\langle -1\rangle},$$

proving 6.4.18.

To see that the arrow map of $\mathcal{C}_Y \to (Y \downarrow \_ \otimes C)$ is well–defined, we check that

$$(\alpha \otimes C) \circ (t \otimes C) \circ \delta^R_Y = (\alpha \circ t \otimes C)\delta^R_Y = (t' \otimes C) \circ \delta^R_Y.$$

Simlarly, for the functor $(Y \downarrow \_ \otimes X) \to \mathcal{C}_Y$,

$$\beta \circ t_f = \beta \circ (Z \otimes \varepsilon_C) \circ f = (Z' \otimes \varepsilon_C) \circ f' = t_{f'}.$$

□

By definition, $\langle h^{C-C}(C, Y), \theta_Y\rangle$ is initial in $\mathcal{C}_Y \to (Y \downarrow \_ \otimes C)$ and by the previous Lemma, it is also initial in $\mathcal{C}_Y$. □

Just as for the cocentralizer, we can take the more concrete approach and define the cocenter as a factor module, as we have done in Definition 6.1.8. Slightly rephrased:

Definition 6.4.19. Let C be a localy projective k–coalgebra. The cocenter of a bicomodule $Y \in {}^X\mathsf{M}^X$ is the k–module $\mathcal{Z}^{co}(Y) = M/W$, where W is the submodule $k - \langle c_{[0]}\, \varphi(c_{[1]}) - c_{[0]}\, \varphi(c_{[-1]})\rangle_{\varphi \in X^*}$.

With the notation of Definition 6.1.8, $\langle Y \xrightarrow{\zeta} \mathcal{Z}^{co}(Y)\rangle$ is an initial object in $\mathcal{C}_Y$. Furthermore, for Clearly, for the C–bicomodule structure induced by a coalgebra map $\pi : D \to C$ on D, the cocenter $\mathcal{Z}^{co}(D)$ of the bicomodule $D \in {}^C\mathsf{M}^C$ is isomorphic to $\mathcal{C}^{co}_D(C)$ as k–modules.

The following phenomenon is well–known from the theory of R–rings. For an R–ring $j : R \to A$, it turns out that the R–tensor square of A, $A \underset{R}{\otimes} A$ has no natural ring structure, but the subring $A \times_R A \subseteq A \underset{R}{\otimes} A$ is a ring with multiplication $(a \otimes a')(b \otimes b') = (ab \otimes b'a')$. Here $A \times_R A$ is the center of the R–bimodule ${}_A A \underset{R}{\otimes} A_A$, known as the Takeuchi product.

The dual result for coalgebra coextensions states that the cocenter of the bicomodule ${}^C(D \,\square_X\, D)^C$ is a well-defined coalgebra. This result was cited in Definition 6.1.12 without proof. First of all, we define the C–bicomodule structure on $D \,\square_C\, D$. Assume that C is flat as a k–module, then the lower line of the following diagram is an equalizer:

$$\begin{array}{ccccc}
D \,\square_C\, D & \xrightarrow{\;i\;} & D \otimes D & \overset{D\otimes\lambda_D}{\underset{\rho_D\otimes D}{\rightrightarrows}} & D \otimes C \otimes D \\
{\scriptstyle \rho_D}\downarrow & & \downarrow{\scriptstyle D\otimes\rho_D} & & \downarrow{\scriptstyle D\otimes C\otimes\rho_D} \\
D \,\square_C\, D \otimes C & \xrightarrow{\;i\otimes C\;} & D \otimes D \otimes C & \overset{D\otimes\lambda_D\otimes C}{\underset{\rho_D\otimes D\otimes C}{\rightrightarrows}} & D \otimes C \otimes D \otimes C
\end{array}$$

$\rho_{D\,\square\,D}$ is defined by the universal property of the equalizer $(i \otimes C)$. Indeed, because the rectangle on the right is serially commutative, $(D \otimes \rho_D) \circ i$ equalizes the parallel maps $D \otimes \lambda_D \otimes C$ and $\rho_D \otimes D \otimes X$. The definition of $\lambda_{D\,\square\,D}$ is entirely similar. In terms of formulæ,

(6.4.20)
$$\lambda_{D\,\square\,D}(d \,\square_C\, d') = d_{(1)} \otimes d_{(2)} \,\square\, d'$$

(6.4.21)
$$\rho_{D\,\square\,D}(d \,\square_C\, d') = d \,\square\, d'_{(1)} \otimes d'_{(2)}$$

We define a coalgebra structure on the cocenter of the bicomodule ${}^{C}D \,\square_C\, D^{C}$,

(6.4.22)
$$\zeta_2 : D \,\square_C\, D \to (D \,\square_C\, D)/W_2,$$

where $W_2 = k - \langle\, d \otimes d'_{(1)}\, \varphi(\pi(d'_{(2)})) - d_{(2)} \otimes d'\, \varphi(\pi(d_{(1)}))\, \rangle_{\varphi\in C^*}$

$$\begin{array}{ccc}
D \,\square_C\, D & \xrightarrow{\;\zeta_2\;} & \mathcal{Z}_2 \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \Delta_2} \\
D \,\square_C\, D \otimes \mathcal{Z}_2 & \xrightarrow[\zeta_2\otimes\mathcal{Z}_2]{} & \mathcal{Z}_2 \otimes \mathcal{Z}_2
\end{array}
\qquad
\begin{array}{ccc}
D \,\square_C\, D & \xrightarrow{\;\zeta_2\;} & \mathcal{Z}_2 \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varepsilon_2} \\
D \,\square_C\, D & \xrightarrow[\varepsilon\otimes\varepsilon]{} & k
\end{array}$$

The map β is defined by the following unique arrow, based on the universal property of the equalizer $(i \otimes \mathcal{Z}_2)$. Of course, we need to assume that $\mathcal{Z}_2$ is flat.

$$\begin{array}{ccccc}
D \,\square_C\, D & \xrightarrow{\;\Delta\,\square\,\Delta\;} & D \otimes D \,\square_C\, D \otimes D & \xrightarrow{\;D\otimes\zeta_2\otimes D\;} & D \otimes \mathcal{Z}_2 \otimes D \\
 & \searrow{\scriptstyle \beta} & & & \downarrow{\scriptstyle \mathsf{tw}_{\mathcal{Z}_2,D}} \\
 & & D \,\square_C\, D \otimes \mathcal{Z}_2 & \xrightarrow[i\otimes\mathcal{Z}_2]{} & D \otimes D \otimes \mathcal{Z}_2
\end{array}$$

we only have to check that the composite map of the upper row satisfies the appropriate equalizer property:

$$\mathsf{tw}_{\mathcal{Z}_2,D} \circ (D \otimes \zeta_2 \otimes D) \circ (\Delta \,\square\, \Delta) : \sum_i d^i \,\square\, d'^i \mapsto d^i_{(1)} \,\square\, d^i_{(2)} \,\square\, d'^i_{(1)} \,\square\, d'^i_{(2)} \mapsto$$
$$\mapsto d^i_{(1)} \,\square\, d'^i_{(2)} \otimes \zeta_2(d^i_{(2)} \,\square\, d'^i_{(1)})$$

This indeed factorizes through $(i \otimes \mathcal{Z}_2)$:

$$d^i_{(1)} \otimes \pi(d^i_{(2)}) \otimes d'^i_{(2)} \otimes \zeta_2(d^i_{(3)} \,\square\, d'^i_{(1)}) = d^i_{(1)} \otimes \pi(d'^i_{(2)}) \otimes d'^i_{(3)} \otimes \zeta_2(d^i_{(2)} \,\square\, d'^i_{(1)})$$

Returning to the definition of Δ_2 and ε_2, it remains to show that $(\zeta_2 \otimes \mathcal{Z}_2) \circ \beta$ and $(\varepsilon \otimes \varepsilon) \circ i$ annihilate W_2, or equivalently:

$$(6.4.23) \qquad (\zeta_2 \otimes \mathcal{Z}_2) \circ \beta(d^i \otimes d'^i_{(1)}) \otimes \pi(d'^i_{(2)}) = (\zeta_2 \otimes \mathcal{Z}_2) \circ \beta(d^i_{(2)} \otimes d'^i) \otimes \pi(d^i_{(1)})$$

$$(6.4.24) \qquad (\varepsilon \otimes \varepsilon) \circ i(d^i \otimes d'^i_{(1)}) \otimes \pi(d'^i_{(2)}) = (\varepsilon \otimes \varepsilon) \circ i(d^i_{(2)} \otimes d'^i) \otimes \pi(d^i_{(1)})$$

As for 6.4.23,

$$LHS = \zeta_2(d^i_{(1)} \otimes d'^i_{(2)}) \otimes \zeta_2(d^i_{(2)} \otimes d'^i_{(1)}) \otimes \pi(d'^i_{(3)}) =$$
$$= \zeta_2(d^i_{(2)} \otimes d'^i_{(2)}) \otimes \zeta_2(d^i_{(3)} \otimes d'^i_{(1)}) \otimes \pi(d^i_{(1)}) = RHS$$

where we used coassociativity and that, by the definition of ζ_2, the left and right coactions are equal on $\zeta_2(d^i_1 \,\square\, d'^i_{(2)})$. 6.4.24 is similarly verified,

$$LHS = (\varepsilon \otimes \varepsilon) \circ i(d \,\square\, d'_{(1)}) \otimes \pi(d'_{(2)}) = \varepsilon(d)\pi(d') = \varepsilon(d')\pi(d) =$$
$$= (\varepsilon \otimes \varepsilon) \circ i(d_{(2)} \,\square\, d') \otimes \pi(d_{(1)}) = RHS$$

the third equality is a simple consequence of the fact that $\sum_i d^i \otimes d'^i \in D \,\square_C\, C$, since $\varepsilon(d)\pi(d') = \varepsilon(d)\pi(d'_{(1)})\varepsilon(d'_{(2)}) = \varepsilon(d_{(1)})\pi(d_{(2)})\varepsilon(d') = \pi(d)\varepsilon(d')$.

We have shown that the maps Δ_2 and ε_2 are uniquely defined, and satisfy

$$(6.4.25) \qquad \Delta_2(\zeta_2(d^i \otimes d'^i)) = \zeta_2(d^i_{(1)} \otimes d'^i_{(2)}) \otimes \zeta_2(d^i_{(2)} \otimes d'^i_{(1)})$$

$$(6.4.26) \qquad \varepsilon_2(\zeta_2(d^i \otimes d'^i)) = \varepsilon_2(d^i)\varepsilon(d'^i)$$

5. The scalar extension for bicoalgebroids

In this section we dualize the scalar extension construction to bicoalgebroids, and give a few simple examples. We begin by defining the smash coproduct, with a slight variation compared to [**58**].

Definition 6.5.1. Let H be a bicoalgebroid over C and D an H–comodule coalgebra. Then their smash coproduct $D \,\sharp\, H$ is a coalgebra, isomorphic to $D \,\square_C\, H$ as C–bicomodules and with the coalgebra structure:

$$(6.5.2) \qquad \Delta(d \,\sharp\, h) = d_{(1)} \,\sharp\, d_{(2)}{}^{\langle -1\rangle} h_{(1)} \,\square_D\, d_{(2)}{}^{\langle 0\rangle} \,\sharp\, h_{(2)}$$

$$(6.5.3) \qquad \varepsilon(d \,\sharp\, h) = \varepsilon(d)\varepsilon_H(h)$$

That these maps define a coalgebra is easily verified. The category of $(D \,\sharp\, H)$–comodules may also be described as the internal D–comodules in ${}^H\mathsf{M}$, i.e. ${}^D({}^H\mathsf{M}) = {}^{D\,\sharp\, H}\mathsf{M}$. Indeed, assume $X \in {}^D({}^H\mathsf{M})$. To every coaction $\delta_D : X \to D \,\square_C\, X$ in ${}^H\mathsf{M}$, we can associate a coaction of $D \,\sharp\, H$, namely $\delta_{D\,\sharp\, H} = (D \otimes \delta) \circ \delta_D : X \to D \otimes X \to D \otimes (H \otimes X)$, $\delta_{D\,\sharp\, H}(x) = x_{[-1]} \otimes x_{[0]}{}^{\langle -1\rangle} \otimes x_{[0]}{}^{\langle 0\rangle}$. A straightforward calculation proves that $(\Delta_{D\,\sharp\, H} \otimes X) \circ \delta_{D\,\sharp\, H} = ((D \,\sharp\, H) \otimes \delta_{D\,\sharp\, H}) \circ \delta_{D\,\sharp\, H}$, using that $\delta_{D\,\sharp\, H}$ is an H–comodule map. In the reverse direction, an $(D \,\sharp\, H)$–comodule is both an H-comodule and a D–comodule such that the D–coaction is an H–comodule map, which means precisely that it is an internal D–comodule in ${}^H\mathsf{M}$.

5.1. Cocommutative coalgebras over bicoalgebroids. Keeping with the method of reversing arrows, we arrive at the following definition for Yetter–Drinfel'd modules over a bicoalgebroid.

DEFINITION 6.5.4. Let H be a (left–) bicoalgebroid over C. A Yetter–Drinfel'd module over H is a triple $\langle Z, \triangleleft, \delta\rangle$ such that the C–bicomodule Z is simultaneously a right H–module with $\triangleleft : Z \,\square_C\, H \to Z$ and a left H–comodule with $\delta : Z \to H \,\square_{C^e}\, Z$ so that the action and coaction satisfy the compatibility condition

$$d^{\langle -1\rangle} h_{[1]} \,\square_C\, d^{\langle 0\rangle} \triangleleft h_{[2]} = h_{[2]}(d \triangleleft h_{[1]})^{\langle -1\rangle} \,\square_C\, (d \triangleleft h_{[1]})^{\langle 0\rangle} \tag{6.5.5}$$

The Yetter–Drinfel'd category, denoted ${}^H\mathcal{YD}_H$ over H has objects the Yetter–Drinfel'd modules over H and arrows the C–bicomodule maps that are at the same time H–module maps and H–comodule maps.

The category ${}^H\mathcal{YD}_H$ becomes monoidal if we define the monoidal product of two Yetter–Drinfel'd modules Z, Z' as $Z \,\square_C\, Z'$ with action and coaction:

$$(z \,\square_C\, z') \triangleleft h = z \triangleleft h_{(2)} \,\square_C\, z' \triangleleft h_{(1)}$$
$$(z \,\square_C\, z')_{\langle -1\rangle} \,\square_C\, (z \,\square_C\, z')_{\langle 0\rangle} = z_{\langle -1\rangle} z'_{\langle -1\rangle} \,\square_C\, z_{\langle 0\rangle} \,\square_C\, z'_{\langle 0\rangle}$$

The monoidal unit is of course C, with $c \triangleleft h = c\,\varepsilon(h)$ and $c_{\langle -1\rangle} \otimes c_{\langle 0\rangle} = \eta(c_{(1)}) \otimes c_{(2)}$. Moreover, ${}^H\mathcal{YD}_H$ is pre–braided with

$$\tau_{Z,Z'} : Z \,\square_C\, Z' \to Z' \,\square_C\, Z, \quad z \otimes z' \mapsto z'_{\langle 0\rangle} \,\square_C\, z \triangleleft z'_{\langle -1\rangle} \tag{6.5.6}$$

From experience with Hopf algebras, weak Hopf algebras and bialgebroids, we expect that the Yetter–Drinfel'd category over a bicoalgebroid is related to the (weak) center of the category of comodules, as defined in Section 1 of Chapter 3. We proved that the Yetter–Drinfel'd category over a bialgebroid is equivalent to the monoidal weak center. Unfortunately this isn't true for bicoalgebroids in general. Nevertheless, the $\mathcal{YD}$ category over a bicoalgebroid still embeds into the monoidal weak center.

LEMMA 6.5.7. *For a Yetter–Drinfel'd module $\langle Z, \delta, \triangleleft\rangle$, the map*

$$\theta_X : Z \,\square_C\, X \to X \,\square_C\, Z \tag{6.5.8}$$
$$z \otimes x \mapsto x^{\langle 0\rangle} \otimes z \triangleleft x^{\langle -1\rangle}$$

makes $\langle Z, \theta\rangle$ an object in $\overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$ and defines an embedding of categories ${}^H\mathcal{YD}_H \hookrightarrow \overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$.

PROOF. θ_X is natural in X, since the arrows of $\overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$ are H–comodule maps. For $\langle Z, \theta\rangle$ to be an object of the left weak center, θ_X should satisfy

$$\theta_{X\,\Box_C\,Y} = (X \,\Box_C\, \theta_Y) \circ (\theta_X \,\Box_C\, Y) \tag{6.5.9}$$

$$\theta_C = Z. \tag{6.5.10}$$

6.5.10 is trivially satisfied and

$$\begin{aligned}\theta_{X\,\Box\,Y}(x \otimes y) &= x^{\langle 0\rangle} \otimes y^{\langle 0\rangle} \otimes z \triangleleft (x^{\langle -1\rangle} y^{\langle -1\rangle}) = x^{\langle 0\rangle} \otimes y^{\langle 0\rangle} \otimes (z \triangleleft x^{\langle -1\rangle}) \triangleleft y^{\langle -1\rangle}\\ &= (X \,\Box\, \theta_Y)(x^{\langle 0\rangle} \otimes z \triangleleft x^{\langle -1\rangle} \otimes y) = (X \,\Box\, \theta_Y) \circ (\theta_X \,\Box\, Y)(x \otimes y).\end{aligned}$$

□

REMARK 6.5.11. We mention, for the sake of completeness, the analogous result for the *right* weak center. The one–sided Yetter–Drinfel'd category ${}^H_H\mathcal{YD}$ is embedded into $\overleftarrow{\mathcal{Z}}({}^H\mathsf{M})$. The objects of ${}^H_H\mathcal{YD}$ are triples $\langle Z, \delta, \triangleright\rangle$, C–bicomodules which are simultaneously H–modules and H–comodules, satifying the compatibilty condition

$$h_{(1)} z^{\langle -1\rangle} \,\Box_C\, h_{(2)} \triangleright z^{\langle 0\rangle} = (h_{(1)} \triangleright z)^{\langle -1\rangle} h_{(2)} \,\Box_C\, (h_{(1)} \triangleright z)^{\langle 0\rangle} \tag{6.5.12}$$

${}^H_H\mathcal{YD}$ is then a pre–braided monoidal category with the pre–braiding

$$\kappa_{Z',Z}(z' \otimes z) = z'^{\langle -1\rangle} \triangleright z \otimes z'^{\langle 0\rangle}. \tag{6.5.13}$$

The functor ${}^H_H\mathcal{YD} \to \overleftarrow{\mathcal{Z}}({}^H\mathsf{M})$ is given by associating to $\langle Z, \delta, \triangleright\rangle \in {}^H_H\mathcal{YD}$ the map

$$\bar\theta_X : X \,\Box_C\, Z \to Z \,\Box_C\, X, \;\; x^{\langle -1\rangle} \triangleright z \otimes x^{\langle 0\rangle}, \tag{6.5.14}$$

which is easily seen to satisfy 5.1.20.

To construct a functor in the reverse direction, we can associate to every object $\langle Z, \theta\rangle$ of $\overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$ a right action of H as follows:

$$\begin{aligned}\triangleleft : Z \,\Box_C\, H &\to Z\\ z \otimes h &\mapsto (\varepsilon_H \otimes Z) \circ \theta_H(z \otimes h)\end{aligned} \tag{6.5.15}$$

It is easily checked that this is indeed a right H–action, and is the candidate to make Z a Yetter–Drinfel'd module. If θ enjoys the property $\theta_X(z \otimes x) = x^{\langle 0\rangle} \otimes \theta_H(z \otimes x^{\langle -1\rangle})$ for all objects $X \in {}^H\mathsf{M}$, then $\langle Z, \delta, \triangleleft\rangle$ becomes a Yetter–Drinfel'd module and, moreover, ${}^H\mathcal{YD}_H$ and $\overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$ are isomorphic categories. This would mean that the natural map θ can be expressed with it's component θ_H. This is indeed the case with *bialgebroids*, since any bialgebroid is a generator in the category of modules over itself, and natural transformations are determined by their value on the generator (cf. the discussion of Proposition 5.1.21).

Now, a braided cocommutative coalgebra (hereinafter abbreviated BCC) over H is defined as a cocommutative comonoid in ${}^H\mathcal{YD}_H$. Spelled out in detail, we have the

DEFINITION 6.5.16. A BCC over H is a coalgebra D, equipped with a coalgebra map $\varepsilon : D \to C$ and the structure of a Yetter–Drinfel'd module $\langle D, \triangleleft, \delta\rangle \in {}^H\mathcal{YD}_H$ so that the left/right C–comodule structures on D are given by $\varepsilon(d_{(1)}) \otimes d_{(2)}$ and $d_{(1)} \otimes \varepsilon(d_{(2)})$, respectively and the relations stating that D is an H–module and H–comodule coalgebra:

$$(d \triangleleft h)_{(1)} \otimes (d \triangleleft h)_{(2)} = d_{(1)} \triangleleft h_{(1)} \otimes d_{(2)} \triangleleft h_{(2)} \tag{6.5.17}$$

$$\varepsilon(d \triangleleft h) = \varepsilon(d)\varepsilon_H(h) \tag{6.5.18}$$

$$d_{(1)}{}^{\langle -1\rangle} d_{(2)}{}^{\langle -1\rangle} \otimes d_{(1)}{}^{\langle 0\rangle} \otimes d_{(2)}{}^{\langle 0\rangle} = d^{\langle -1\rangle} \otimes d^{\langle 0\rangle}{}_{(1)} \otimes d^{\langle 0\rangle}{}_{(2)} \tag{6.5.19}$$

$$d^{\langle -1\rangle} \otimes \varepsilon(d^{\langle 0\rangle}) = \eta(\varepsilon(d)_{(1)}) \otimes \varepsilon(d)_{(2)} \tag{6.5.20}$$

and braided cocommutativity:

$$d_{(1)} \otimes d_{(2)} = d_{(2)}{}^{\langle 0\rangle} \otimes d_{(1)} \triangleleft d_{(2)}{}^{\langle -1\rangle} \tag{6.5.21}$$

We have the following functorial characterization of BCC's, entirely analogous to 5.3.12:

LEMMA 6.5.22. *If D is a BCC in ${}^H\mathcal{YD}_H$, then the functor $D \,\square_C\, \_ : {}^H\mathsf{M} \to {}^D({}^H\mathsf{M}) = {}^{D\sharp H}\mathsf{M}$ is strong monoidal*

PROOF. Denote the opmonoidal structure $\langle D \,\square_C\, \_, D^2, D^0\rangle$. The natural transformation

$$\begin{aligned}(D^2)_{X,Y} : & D \,\square_C\, (X \,\square_C\, Y) \to (D \,\square_C\, X) \,\square_D\, (D \,\square_C\, Y)\\ & d \otimes x \otimes y \mapsto (d_{(1)} \otimes x_{\langle 0\rangle}) \otimes (d_{(2)} \triangleleft x_{\langle -1\rangle} \otimes y)\end{aligned}$$

has the inverse $(d \otimes x) \otimes (d' \otimes y) \mapsto d\varepsilon_C(\varepsilon(d')) \otimes x \otimes y$. Furthermore, $D^0 : D \,\square_C\, C \to D$ is obviously an isomorphism. □

We now come to the dualization of Theorem 5.3.1, the bicoalgebroid scalar extension.

THEOREM 6.5.23. *Let $\langle H, \Delta, \varepsilon; \mu, \eta; \alpha, \beta; C\rangle$ be a (left–) bicoalgebroid over C and D a BCC over H, then $\langle D \sharp H, \tilde{\Delta}, \tilde{\varepsilon}; \tilde{\mu}, \tilde{\eta}; \tilde{\alpha}, \tilde{\beta}; D\rangle$ is a (left–) bicoalgebroid over D, with the following structure maps:*

$$\tilde{\Delta}(d \sharp h) = d_{(1)} \sharp d_{(2)}{}^{\langle -1\rangle} h_{(1)} \,\square_D\, d_{(2)}{}^{\langle 0\rangle} \sharp h_{(2)} \tag{6.5.24}$$

$$\tilde{\varepsilon}(d \sharp h) = \varepsilon_C(\varepsilon(d))\varepsilon_H(h) \tag{6.5.25}$$

$$\tilde{\mu}(d \sharp h \,\square_D\, d' \sharp h') = d\varepsilon_C(\varepsilon(d')) \sharp hh' \tag{6.5.26}$$

$$\tilde{\eta}(d) = d_{(1)} \sharp \eta(\varepsilon(d_{(2)})) \tag{6.5.27}$$

$$\tilde{\alpha}(d \sharp h) = d\varepsilon_H(h), \quad \tilde{\beta}(d \sharp h) = d \triangleleft h \tag{6.5.28}$$

PROOF. First, we check that $\tilde{\alpha}$ ($\tilde{\beta}$) is a coalgebra (anti–coalgebra) map, respectively. Inserting the definitions, a trivial calculation shows

$$\alpha((d \sharp h)_{(1)}) \otimes \alpha((d \sharp h)_{(2)}) = d_{(1)} \otimes d_{(2)}\varepsilon(h) = (\alpha(d \sharp h))_{(1)} \otimes (\alpha(d \sharp h))_{(2)}$$

As required, β is an anti–coalgebra map:

$$\beta((d\,\sharp\,h)_{(2)})\otimes\beta((d\,\sharp\,h)_{(1)})=\beta(d_{(2)}{}^{\langle 0\rangle}\,\sharp\,h_{(2)})\otimes\beta(d_{(1)}\,\sharp\,d_{(2)}{}^{\langle -1\rangle}h_{(1)})=$$
$$=d_{(2)}{}^{\langle 0\rangle}\triangleleft h_{(2)}\otimes d_{(1)}\triangleleft d_{(2)}{}^{\langle -1\rangle}h_{(1)}=d_{(1)}\triangleleft h_{(2)}\otimes d_{(2)}\triangleleft h_{(1)}=(d\triangleleft h)_{(1)}\otimes(d\triangleleft h)_{(2)}=$$
$$=(\beta(d\,\sharp\,h))_{(1)}\otimes(\beta(d\,\sharp\,h))_{(2)}.$$

where we have used 6.5.21 in the third equality, and the fact that D is an H_{cop}–coalgebra in the fourth.

To prove that $\tilde{\mu}:(D\,\sharp\,H)\,\square_D\,(D\,\sharp\,H)\to D\,\sharp\,H$ factorizes through $(D\,\sharp\,H)\boxtimes(D\,\sharp\,H)$, we calculate the D–comodule structure of $D\,\sharp\,H$:

$$\tilde{\lambda}_L:d\,\sharp\,h\mapsto\alpha((d\,\sharp\,h)_{(1)})\otimes(d\,\sharp\,h)_{(2)}=\alpha(d_{(1)}\,\sharp\,d_{(2)}{}^{\langle -1\rangle}h_{(1)})\otimes d_{(2)}{}^{\langle 0\rangle}\,\sharp\,h_{(2)}=$$
$$=d_{(1)}\otimes d_{(2)}\,\sharp\,h$$

$$\tilde{\rho}_L:d\,\sharp\,h\mapsto(d\,\sharp\,h)_{(2)}\otimes\beta((d\,\sharp\,h)_{(2)})=d_{(2)}{}^{\langle 0\rangle}\,\sharp\,h_{(2)}\otimes d_{(1)}\triangleleft(d_{(2)}{}^{\langle -1\rangle}h_{(1)})=$$
$$=d_{(1)}\,\sharp\,h_{(2)}\otimes d_{(2)}\triangleleft h_{(1)},$$

using (6.5.21) in the last step. The definition of the cotensor product over D then reads: $(d\otimes h)\otimes(d'\otimes h')\in(D\,\sharp\,H)\,\square_D\,(D\,\sharp\,H)$ *iff*

$$(d\,\sharp\,h)_{(2)}\otimes\beta((d\,\sharp\,h)_{(1)})\otimes(d'\,\sharp\,h')=(d\,\sharp\,h)\otimes\alpha((d'\,\sharp\,h')_{(1)})\otimes(d'\,\sharp\,h')_{(2)}$$

or, using (6.5.21):

$$d_{(1)}\,\sharp\,h_{(2)}\otimes d_{(2)}\triangleleft h_{(1)}\otimes d'\,\sharp\,h'=d\,\sharp\,h\otimes d'_{(1)}\otimes d'_{(2)}\,\sharp\,h' \tag{6.5.29}$$

We now prove $(d\,\sharp\,h)(d'\,\sharp\,h')_{(1)}\otimes\alpha((d'\,\sharp\,h')_{(2)})=(d\,\sharp\,h)_{(1)}(d'\,\sharp\,h')\otimes\beta((d\,\sharp\,h)_{(2)})$. Inserting definitions, and using the Yetter–Drinfel'd condition (6.5.5) we find:

$$RHS=d_{(1)}\varepsilon_C(\varepsilon(d'))\,\sharp\,d_{(2)}{}^{\langle -1\rangle}h_{(1)}h'\otimes d_{(2)}{}^{\langle 0\rangle}\triangleleft h_{(2)}=$$
$$=d_{(1)}\varepsilon_C(\varepsilon(d'))\,\sharp\,h_{(2)}(d_{(2)}\triangleleft h_{(1)})^{\langle -1\rangle}h'\otimes(d_{(2)}\triangleleft h_{(1)})^{\langle 0\rangle},$$

using the Yetter–Drinfel'd condition (eq. 6.5.5). Applying (6.5.29), we arrive at

$$RHS=d\varepsilon_C(\varepsilon(d'_{(2)}))\,\sharp\,hd'_{(1)}{}^{\langle -1\rangle}h'\otimes d'_{(1)}{}^{\langle 0\rangle}=d\,\sharp\,hd'^{\langle -1\rangle}h'\otimes d'^{\langle 0\rangle}$$

A quick calculation shows that the

$$LHS=d\varepsilon_C(\varepsilon(d'_{(1)}))\,\sharp\,hd'_{(2)}{}^{\langle -1\rangle}h'\otimes d'_{(2)}{}^{\langle 0\rangle}=d\,\sharp\,hd'^{\langle -1\rangle}h'\otimes d'^{\langle 0\rangle},$$

as claimed.

Comultiplicativity of the product (which makes sense due to our above assertion) means

$$(\tilde{\Delta}\circ\tilde{\mu})[(d\,\sharp\,h)\,\square_D\,(d'\,\sharp\,h')]=(\tilde{\mu}\,\square_D\,\tilde{\mu})\circ\tau_{23}\circ(\tilde{\Delta}\,\square_D\,\tilde{\Delta})[(d\,\sharp\,h)\,\square_D\,(d'\,\sharp\,h')]$$

inserting our definitions, we have:

$$LHS=\tilde{\Delta}(d\varepsilon_C\varepsilon(d')\,\sharp\,hh')=d_{(1)}\varepsilon_C\varepsilon(d')\,\sharp\,d_{(2)}{}^{\langle -1\rangle}(hh')_{(1)}\,\square_D\,d_{(2)}{}^{\langle 0\rangle}\,\sharp\,(hh')_{(2)},$$

on the other hand, the

$$RHS = (d_{(1)} \sharp d_{(2)}{}^{\langle -1\rangle} h_{(1)})(d'_{(1)} \sharp d'_{(2)}{}^{\langle -1\rangle} h'_{(1)}) \Box_D (d_{(2)}{}^{\langle 0\rangle} \sharp h_{(2)})(d'_{(2)}{}^{\langle 0\rangle} \sharp h'_{(2)}) =$$
$$= d_{(1)}\varepsilon_C\varepsilon(d'_{(1)}) \sharp d_{(2)}{}^{\langle -1\rangle} h_{(1)} d'_{(2)}{}^{\langle -1\rangle} h'_{(1)} \Box_D d_{(2)}{}^{\langle 0\rangle} \varepsilon_C\varepsilon(d'_{(2)}{}^{\langle 0\rangle}) \sharp h_{(2)} h'_{(2)} =$$
$$= d_{(1)}\varepsilon_C(\varepsilon(d'_{(1)})) \sharp d_{(2)}{}^{\langle -1\rangle} h_{(1)} \eta(\varepsilon(d'_{(2)}))h'_{(1)} \Box_D d_{(2)}{}^{\langle 0\rangle} \varepsilon_C(\varepsilon(d'_{(3)})) \sharp (hh')_{(2)},$$

where we made use of 6.5.20 and coassociativity in the third equality. Now, $d'_{(1)} \otimes \varepsilon(d'_{(2)}) \otimes h' = d' \otimes \alpha(h'_{(1)}) \otimes h'_{(2)}$, because $d \sharp h \in D \Box_C H$. From this, and the unit property of η, the statement follows.

The product is counital:

$$\tilde{\varepsilon}(d \sharp h)\tilde{\varepsilon}(d' \sharp h') = \varepsilon_C(\varepsilon(d))\varepsilon_C(\varepsilon(d'))\varepsilon_H(hh') = \tilde{\varepsilon}(d\varepsilon_C(\varepsilon(d)) \sharp hh') \tag{6.5.30}$$

The unit map $\tilde{\eta}$ is indeed a unit for $\tilde{\mu}$. The first unit property reads:

$$\tilde{\mu} \circ (\tilde{\eta} \Box D \sharp H) \circ \tilde{\lambda}_L(d \sharp h) = (d_{(1)(1)} \sharp \eta(\varepsilon(d_{(1)(2)})))(d_{(2)} \sharp h) =$$
$$= d_{(1)}\varepsilon_C\varepsilon(d_{(3)}) \sharp \eta(\varepsilon(d_{(2)}))h = d \sharp h, \tag{6.5.31}$$

using $d \sharp h \in D \Box_C H$ and the unit axiom (for H) in the last equality. The second,

$$\tilde{\mu} \circ (D \sharp H \Box \tilde{\eta}) \circ \tilde{\rho}_L(d \sharp h) = (d_{(1)} \sharp h_{(2)})((d_{(2)} \triangleleft h_{(1)})_{(1)} \sharp \eta((d_{(2)} \triangleleft h_{(1)})_{(2)})) =$$
$$= d_{(1)}\varepsilon_C\varepsilon(d_{(2)} \triangleleft h_{(1)}) \sharp h_{(2)}\eta(\varepsilon(d_{(2)} \triangleleft h_{(1)})_{(2)}) = d_{(1)} \sharp \eta(\varepsilon(d_{(2)}))h = d \sharp h$$

is proved using that D is an H_{cop}–algebra in the third equality, and $d \sharp h \in D \Box_C H$ in the last. As a coalgebra, $D \sharp H$ is the smash coproduct. The algebra structure of $\langle D \sharp H, \tilde{\mu}, \tilde{\eta}\rangle$ and the remaining axioms are easily verified. □

Example 6.5.32. *The action groupoid*[2]

In the category Set, there is a unique comultiplication, namely the diagonal coproduct: $x \in X$, $\Delta_X(x) = x \times x$. The counit is just a constant map to a (the) one–element set 1, hence $\varepsilon_X(x) = *$ for all $x \in X$, where $*$ is the unique element of 1. The coaction of a group G on X is completely specified by an arbitrary function $\varphi : X \to G$, via $\delta_\varphi(x) = x_{\langle -1\rangle} \times x_{\langle 0\rangle} = \varphi(x) \times x$. Now, consider a G–Set $\langle X, \triangleleft\rangle$, carrying a right action of G. Choosing a G–coaction δ_φ, the Yetter–Drinfel'd compatibility condition takes the form

$$\varphi(x)g \times x \triangleleft g = g\varphi(x \triangleleft g) \times x \triangleleft g \tag{6.5.33}$$

so $\langle X, \delta_\varphi, \triangleleft\rangle$ is a $\mathcal{YD}$–module in ${}^G\mathcal{YD}_G$ if and only if $g^{-1}\varphi(x)g = \varphi(x \triangleleft g)$. Moreover, X is a BCC if $x \times x = x \times x \triangleleft \varphi(x)$, i.e. iff

$$x \triangleleft \varphi(x) = x \tag{6.5.34}$$

[2] see also [**24**], Prop. 4.2.

6.5.34 implies that the value of φ at a point x must lie in the stabilizer subgroup G^x of the point x, and from 6.5.33 we conclude that it suffices to define φ for a single representative, say x_0 of each G-orbit. Then, if $x_0 \in G^{x_0}$, $\varphi(x) = \varphi(x_0 \triangleleft g) = g^{-1}\varphi(x_0)g \in G^x$.

Choosing a trivial coaction $\varphi(x) \equiv e$, the scalar extension of G by X is nothing but the action groupoid. Indeed, $\tilde{\alpha}(x \sharp g) = x$ and $\tilde{\beta}(x \sharp g) = x \triangleleft g$, so $(X \sharp G) \Box_X (X \sharp G)$ is the set of composable pairs in the action groupoid and the multiplication $\tilde{\mu}$ is the composition of arrows in the action groupoid.

The phenomenon behind this example is that in Set, the fibered product of two parallel maps $\alpha, \beta : X \to Y$, defined by the pullback

$$\begin{array}{ccc} X \times_{\alpha,\beta} X & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow[\beta]{} & Y \end{array}$$

is equivalent to the equalizer

$$X \times_{\alpha,\beta} X \to X \times X \underset{\rho \times X}{\overset{X \times \lambda}{\rightrightarrows}} X \times Y \times X$$

where λ and ρ are the 'coactions' $\lambda = (\alpha \times X) \circ \Delta_{diag}$ and $\rho = (X \times \beta) \circ \Delta_{diag}$. It is in this sense that a groupoid may be regarded as a classical ancestor of a bicoalgebroid.

Example 6.5.35. *The regular BCC for H a Hopf algebra*

k–Hopf algebras (and bialgebras) are examples both of bialgebroids and bicoalgebroids. It is not immaterial, however whether we consider the Yetter-Drinfel'd category ${}^H\mathcal{YD}_H$ as embedded in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ (the 'bialgebroid view', see [**3**]), or in $\overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$ (the 'bicoalgebroid view'). Namely, the braiding is different in the two cases, $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ is pre–braided with $\overrightarrow{\beta}_{Z,Z'} = z' \triangleleft z^{\langle -1 \rangle} \otimes z^{\langle 0 \rangle}$ and $\overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$ is pre–braided with $\overrightarrow{\gamma}_{Z,Z'} = z'^{\langle 0 \rangle} \otimes z \triangleleft z'^{\langle -1 \rangle}$.

A k–Hopf algebra H, with invertible antipode is a Yetter–Drinfel'd module $\langle H, Ad_R, \Delta \rangle$ in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$ (the *regular* module) via the coproduct, considered as left H–coaction and the right adjoint action, $Ad_R : H \otimes H \to H, \;\; h \otimes h' \mapsto S^{-1}(h'_{(2)})hh'_{(1)}$. Furthermore, H^{op} is a BCA in $\overrightarrow{\mathcal{Z}}(\mathsf{M}_H)$, that is $\mu_{op} \circ \beta(h \otimes h') = \mu_{op}(h \otimes h')$. Indeed, $h'h = h' \triangleleft h_{(1)} \otimes h_{(2)} = S(h_{(1)(2)})h'h_{(1)(1)} \otimes h_{(2)}$.

Dually, a k–Hopf algebra H is a Yetter–Drinfel'd module $\langle H, \mu, \tilde{Ad}_L \rangle$ in $\overrightarrow{\mathcal{Z}}({}^H\mathsf{M})$ (the *regular* module) via the multiplication considered as a right action, and the left adjoint coaction,

$$\tilde{Ad}_L : H \to H \otimes H \qquad (6.5.36)$$
$$h \mapsto S^{-1}(h_{(3)})h_{(1)} \otimes h_{(2)}$$

Yetter–Drinfel'd compatibility is easily checked:

$$h'_{(2)}(hh'_{(1)})^{\langle -1\rangle} \otimes (hh'_{(1)})^{\langle 0\rangle} = h'_{(2)}S^{-1}((hh'_{(1)})_{(3)})(hh'_{(1)}) \otimes (hh'_{(2)}) =$$
$$= h'_{(2)}S^{-1}(h'_{(1)(3)})S^{-1}(h_{(3)})h'_{(1)(1)}h_{(1)} \otimes h_{(2)}h'_{(1)(2)} = S^{-1}(h_{(3)})h_{(1)}h'_{(1)} \otimes h_{(2)}h'_{(2)} =$$
$$= h^{\langle -1\rangle}h'_{(1)} \otimes h^{\langle 0\rangle} \triangleleft h'_{(2)}$$

As one might expect from the previous example, H_{cop} is a BCC in $\overrightarrow{\mathcal{Z}}(^{H}\mathsf{M})$,

$$\overrightarrow{\beta} \circ \Delta_{cop}(h) = \overrightarrow{\beta}(h_{(2)} \otimes h_{(1)}) = h_{(1)(2)} \otimes h_{(2)}S^{-1}(h_{(1)(3)})h_{(1)(1)} = h_{(2)} \otimes h_{(1)}$$

To construct an example which does not require the invertibility of the antipode, consider ${}^{H}_{H}\mathcal{YD}$ as being in the *right* weak center $\overleftarrow{\mathcal{Z}}(^{H}\mathsf{M})$. The Yetter–Drinfel'd condition takes the form

$$h_{(1)}z^{\langle -1\rangle} \otimes h_{(2)} \triangleright z^{\langle 0\rangle} = (h_{(1)} \triangleright z)^{\langle -1\rangle}h_{(2)} \otimes (h_{(1)} \triangleright z)^{\langle 0\rangle}, \tag{6.5.37}$$

and the pre–braiding is $\overleftarrow{\beta}_{Z',Z} : z'^{\langle -1\rangle} \triangleright z \otimes z'^{\langle 0\rangle}$. We find that for an arbitrary Hopf algebra, $\langle H, Ad_L, \mu_H\rangle$ is a BCC in ${}^{H}_{H}\mathcal{YD}$, where

$$Ad_L : H \to H \otimes H$$
$$h \mapsto h_{(1)}S(h_{(3)}) \otimes h_{(2)},$$

and $\overleftarrow{\beta}_{H,H} \circ \Delta(h) = h_{(1)(1)}S(h_{(1)(3)})h_{(2)} \otimes h_{(1)(2)} = h_{(1)} \otimes h_{(2)}$.

6. Bicoalgebroids and Weak Bialgebras

It was observed in [**75**] and [**82**] that the base algebra R of a WBA B over R is Frobenius separable. Moreover, [**75**] and [**82**] prove a converse to this result, namely that for a bialgebroid B over R, a Frobenius separability structure on R determines a WBA structure on the underlying k–module of B. In a sense, this result is a characterizes WBAs within bialgebroids, as precisely those bialgebroids which have Frobenius separable base.

As it was shown by G. Böhm[3], a WBA always has a bi**co**algebroid structure. This raises the question whether WBAs could also be characterized as bicoalgebroids whose base coalgebra posesses a dual Frobenius separability structure?

We briefly recapitulate basic results on Frobenius separability –together with the dual notion– and sketch the proof of the case of WBAs and bialgebroids over Frobenius separable base. We shall be working over a field k, although it is possible to be more general.

Definition 6.6.1 (Frobenius separable algebra). For a k–algebra R, a *separability structure* is an R–bimodule map $\delta : {}_RR_R \to {}_RR \otimes R_R$ that splits multiplication, i.e. $\mu_R \circ \delta = \mathrm{id}_R$. The element $\delta(1_R) = \sum_i e_i^{(1)} \otimes e_i^{(2)} \in R \otimes R$ is called the separability idempotent. By definition, it satisfies $\sum_i e_i^{(1)}e_i^{(2)} = 1_R$. A separable algebra R is said to be *Frobenius separable* if it possesses a Frobenius structure with Frobenius functional $\Psi : R \to k$ (recall defintion 3.5.1) such that

[3]Private communication. Note that her construction is not the same as that appearing in [**21**]!

the separability idempotent is a Frobenius quasibasis, i.e. for all $r \in R$,

$$\Psi(re_{(1)})\, e_{(2)} = r = e_{(1)}\, \Psi(e_{(2)}r).$$

In the terminology of Frobenius algebras, the separability property implies that a Frobenius separable algebra is a Frobenius algebra of index 1.

The dual notion structure is that of a Frobenius co–separable coalgebra:

DEFINITION 6.6.2 (Frobenius co–separable coalgebra). For a k–coalgebra R, a *co–separability structure* is a C–bicomodule map $\omega : {}^C C \otimes C^C \to {}^C C^C$ such that $\omega \circ \Delta_C = \mathrm{id}_C$. The functional $\hat{e} = \varepsilon_C \circ \omega \in \mathrm{Hom}(C \otimes C, k)$ is the called the co–separability functional. By definition, it satisfies $\hat{e}(c_{(1)} \otimes c_{(2)}) = \varepsilon_C(c)$ for all $c \in C$. A *Frobenius co–separable* coalgebra has a co–Frobenius element $t : k \to C$ (we mean $t = t(1)$, by abuse of notation) such that for all $c \in C$,

$$t_{(1)}\, \varepsilon_C \circ \omega(t_{(2)} \otimes c) = c = \varepsilon_C \circ \omega(c \otimes t_{(1)}))\, t_{(2)}$$

holds.

The following observation is instrumental in proving that a bialgebroid over separable base possesses a WBA structure. For R–modules M_R and ${}_RN$, a separability structure $\delta : R \to R \otimes R$ gives rise to a natural transformation

$$\delta_{M,N} : M \underset{R}{\otimes} N \to M \underset{k}{\otimes} N, \;\; m \otimes n \mapsto \sum_i m \cdot e_i^{(1)} \otimes e_i^{(2)} \cdot n, \tag{6.6.3}$$

which splits the canonical epimorphism $\pi_{M,N} : M \underset{k}{\otimes} N \to M \underset{R}{\otimes} N$,

$$\pi_{M,N} \circ \delta_{M,N} = \mathrm{id}_{M \underset{R}{\otimes} N}.$$

This follows straightforwardly from the definition of the $\underset{R}{\otimes}$–tensor product and the separability property $e^{(1)}e^{(2)} = 1_R$. We formulate the dual result in the following lemma.

LEMMA 6.6.4. *For C–comodules M^C and ${}^CN$, a co–separability structure on C determines a natural transformation*

$$\omega_{M,N} : M \underset{k}{\otimes} N \to M \,\square_C\, N, \;\; m \otimes n \mapsto m_{\langle 0\rangle}\varepsilon_C \circ \omega(m_{\langle 0\rangle} \otimes n_{\langle -1\rangle}) \otimes n_{\langle 0\rangle}, \tag{6.6.5}$$

such that $\iota_{M,N} \circ \omega_{M,N} = \mathrm{id}_{M \square_C N}$, where $\iota_{M,N} : M \,\square_C\, N \to M \otimes N$ is the canonical embedding of the cotensor product.

PROOF. We need only to show that $\omega_{M,N}$ is indeed an identity on $m \otimes n \in M \,\square_C\, N$:

$$\begin{aligned}
\omega_{M,N}(m \otimes n) &= m_{\langle 0\rangle}\varepsilon_C \circ \omega(m_{\langle 1\rangle} \otimes n_{\langle -1\rangle}) \otimes n_{\langle 0\rangle} \\
&= m_{\langle 0\rangle\langle 0\rangle}\varepsilon_C \circ \omega(m_{\langle 0\rangle\langle 1\rangle} \otimes m_{\langle 1\rangle}) \otimes n_{\langle 0\rangle} \\
&= m_{\langle 0\rangle}\varepsilon_C(\omega(m_{\langle 1\rangle(1)} \otimes m_{\langle 1\rangle(2)})) \otimes n_{\langle 0\rangle} = m \otimes n,
\end{aligned}$$

using the definition of $M \,\square_C\, N$ and coassociativity. □

As observed first by Abrams [**1**], a Frobenius algebra $\langle R, \mu, \eta, \delta, \Psi\rangle$ also has a coalgebra structure (cf. the comultiplications in DDAs, section 5 of chapter 3) with comultiplication and counit supplied by the Frobenius structure as follows:

(6.6.6) $$\Delta_F : R \to R \otimes R,\ \ r \mapsto re^{(1)} \otimes e^{(2)} \quad \text{and} \quad \varepsilon_F : R \to k,\ \ r \mapsto \Psi(r).$$

The algebra and coalgebra structures satisfy the compatibility relations

$$(R \otimes \mu) \circ (\Delta_F \otimes R) = \Delta_F \circ \mu = (\mu \otimes R) \circ (R \otimes \Delta_F)$$

and a Frobenius separable algebra also satisfies $\mu \circ \Delta_F = \mathrm{id}_R$. Likewise, a co–Frobenius coalgebra $\langle C, \Delta, \varepsilon, \omega, t\rangle$ is also an algebra via the co–Frobenius structure. The mutiplication and unit are

(6.6.7) $$\mu_F : C \otimes C \to C,\ \ c \otimes c' \mapsto c_{(1)}\varepsilon_C \circ \omega(c_{(2)} \otimes c') \quad \text{and} \quad \eta_F : k \to C,\ \ k \mapsto t(k)$$

and a Frobenius co–separable coalgebra satisfies $\mu_F \circ \Delta = \mathrm{id}_C$. The following Lemma tells us that Frobenius separable algebra (Frobenius co–separable coalgebra) is a self–dual structure.

LEMMA 6.6.8. *A Frobenius separable algebra $\langle R, \mu, \eta, \delta, \Psi\rangle$ is Frobenius co–separable as a coalgebra, with Frobenius structure given by the multiplication $\omega := \mu : R \otimes R \to R$ and unit $t := \eta : k \to R$. Dually, a Frobenius co–separable coalgebra $\langle C, \Delta, \varepsilon, \omega, t\rangle$ is Frobenius separable as an algebra, with Frobenius structure given by the comultiplication $\delta := \Delta : C \to C \otimes C$ and counit $\Psi := \varepsilon : C \to k$.*

We only sketch the proof of the equivalence of WBAs and bialgebroids over separable base. For details, consult [**82**] and [**75**].

PROPOSITION 6.6.9. *Let $\langle B, \gamma, \pi, \mu, \eta\rangle$ be a weak bialgebra, with structure maps satisfying axioms 2.0.1 – 2.0.2. Then, the subalgebra $R := A^R = \Pi^R(B) \subseteq B$ is Frobenius separable and there are maps $\Delta : B \to B \underset{R}{\otimes} B$ and $\varepsilon : B \to R$ making $\langle B, R, \Delta, \varepsilon, \mu, \eta\rangle$ a right bialgebroid over R.*

Conversely, let $\langle B, R, \Delta, \varepsilon, \mu, \eta\rangle$ be a right bialgebroid over R such that the base algebra R is Frobenius separable over k. Then there are maps $\gamma : B \to B \otimes B$ and $\pi : B \to k$ making $\langle B, \gamma, \pi, \mu, \eta\rangle$ a weak bialgebra.

SKETCH OF PROOF. Consider the first implication. First, B is an $R \otimes R^{op}$–ring with source map the inclusion $R = A^R \hookrightarrow B$ and target map the restriction to R of the map $\Pi'^L : B \to B$, $b \mapsto \varepsilon b 1_{(1)} 1_{(2)}$ (in fact, Π'^L is an anti–isomorphism $A^{R,op} \to A^L$ hence the images of the source and target maps commute, since the subalgebras A^L and A^R commute in a WBA).

Making B an $R - R^{op}$–bimodule via right multiplication by the above source and target maps, the comultiplication gotten by composing the WBA's comultiplication and the canonical projection onto the $\underset{R}{\otimes}$–product,

$$\Delta = \pi_{H,H} \circ \pi : H \to H \otimes H \to H_R \underset{R}{\otimes} H_{R^{op}}$$

is an $R-R$–bimodule map. The counit is $\varepsilon = \Pi^R : H \to R$. The $R \otimes R^{op}$–ring and R–coring structures (which are checked to be compatible) make $\langle B, R; \Delta, \varepsilon; \mu, \eta\rangle$ a right bialgebroid.

The Frobenius separability structure on $R = A^R$ is given as follows: the Frobenius functional is the restriction of the counit, $\Psi = \varepsilon|_R : R \to k$ and the associated quasibasis is $1_{(1)} \otimes \Pi^R(1_2) \in R \otimes R$. Note that by lemma 6.6.8, R may just as well be regarded as a Frobenius co–separable coalgebra.

For the reverse implication, we make use of the natural transformation $\delta_{M,N}$ defined in 6.6.3. Since R is Frobenius separable, we have at our disposal the natural map $\delta_{H,H} : H \underset{R}{\otimes} H \to H \underset{k}{\otimes} H$. The WBA comultiplication is then given by the composition

(6.6.10)
$$\gamma := \delta_{H,H} \circ \Delta : B \to B \underset{R}{\otimes} B \to B \underset{k}{\otimes} B,$$
$$b \mapsto b_{(1)} \underset{R}{\otimes} b_{(2)} \mapsto \sum_i b_{(1)} \cdot e_i^{(1)} \underset{R}{\otimes} e_i^{(2)} \cdot b_{(2)}.$$

The WBA counit, $\pi := \Psi \circ \varepsilon : H \to R \to k$, is the composition of the counit with the Frobenius functional. It is checked that the algebra structure $\langle B, \mu, \eta\rangle$ and coalgebra structure $\langle B, \gamma, \pi\rangle$ are compatible in the sense of weak bialgebras, i.e. axioms 2.0.1 and 2.0.2 are satisfied. $\square$

A dual result relating weak bialgebras and bicoalgebroids rests on the existence of the natural transformation $\omega_{M,N} : M \underset{k}{\otimes} N \to M \,\square_C\, N$ for comodules over a Frobenius co–separable coalgebra and the self–duality of weak bialgebras. We state the result and sketch the proof.

Proposition 6.6.11. *Let $\langle B; \gamma, \pi; m, i\rangle$ be a weak bialgebra, and denote $C := A^R = \Pi^R(B) \subseteq B$ the Frobenius separable subalgebra, considered* as a coalgebra*. Then C is a Frobenius co–separable coalgebra, B is a $C \otimes C_{cop}$–bicomodule and there are maps $\mu : B \,\square_C\, B \to B$ and $\eta : C \to B$ making $\langle B, C; \Delta, \varepsilon; \mu, \eta\rangle$ a right bicoalgebroid over C.*

Conversely, let $\langle B, C; \Delta, \varepsilon; \mu, \eta\rangle$ be a bicoalgebroid over C such that the base coalgebra C is Frobenius co–separable over k. Then there are maps $m : B \otimes B \to B$ and $i : k \to B$ making $\langle B; \Delta, \varepsilon; \mu, \eta\rangle$ a weak bialgebra.

Sketch of proof. To see the first implication, we need to show first that B is a $C \otimes C^{op}$-bicomodule. As shown in the previous proposition, A^R is a coalgebra via the Frobenius structure given by the quasibasis $1_{(1)} \otimes \Pi^R(1_2) \in R \otimes R$. Similarly, $A^L = \Pi^L(B)$ is a coalgebra via the Frobenius quasibasis $\Pi^L(1_{(1)}) \otimes 1_{(2)} \in R \otimes R$ and moreover, A^L and A^R are anti–isomorphic as coalgebras as well. The maps $\alpha : H \to C$ and $\beta : H \to C_{cop}$ are given by Π^R and the composition of Π^L with the coalgebra anti–isomorphism $A^L \xrightarrow{\sim} C_{cop}$.

Define the bicoalgebroid multiplication as the composition of the canonical injection of the $\square_C$–product and weak bialgebra multiplication,

$$\mu = m \circ \iota_{H,H} : H \,\square_C\, H \to H \otimes H \to H$$

which is a $C-C$–bicomodule map and is checked to satisfy 6.1.2. The unit map is just the inclusion $\eta : C \subseteq B$.

As shown in lemma 6.6.8, Frobenius co–separability of the coalgebra C follows immediately from the Frobenius separability of the algebra A^R.

As expected from a duality argument, the reverse implication hinges on the natural transformation $\omega_{M,N} : M \underset{k}{\otimes} N \to M \,\square_C\, N$ of 6.6.5. By the Frobenius co–separability of C, we have for the C–comodules B^C and ${}^C B$ a natural map $\omega_{B,B} : B \underset{k}{\otimes} B \to B \,\square_C\, B$. Define the WBA multiplication as the composite map

$$m := \mu \circ \omega_{B,B} : B \otimes B \to B \,\square_C\, B \to B, \qquad b \otimes b' \mapsto b_{[0]}\varepsilon \circ \omega(b_{[1]} \otimes b'_{[-1]})b'_{[0]}. \tag{6.6.12}$$

The WBA unit map, $i := \eta \circ t : k \to C \to B$, is the image of the co–Frobenius element under the unit map. It is checked that the algebra structure $\langle B, \mu, \eta\rangle$ and coalgebra structure $\langle B, \gamma, \pi\rangle$ are compatible in the sense of weak bialgebras, i.e. axioms 2.0.1, 2.0.2 are satisfied. □

CHAPTER 7

An application to Algebraic Quantum Field Theory

Algebraic Quantum Field Theory emerged out of axiomatic approaches to quantum field theory through the seminal work of R. Haag and D. Kastler. Its departure from earlier axiomatics (such as the Wightman framework) was the realization that the physical content of a quantum field theory resides in the abstract algebraic structure of its algebra of *observables* rather than in the unobservable fields (e.g. Dirac spinor valued fermionic fields in QED) or in the different representations of the field content (the most common being the Fock space representation), which are only of secondary nature. From this ambitious point of view, the complete field algebra and the (global) gauge group of a quantum field theory are to be reconstructed from the algebra of observables or more precisely, from a net of local subalgebras of observables.

This is closely connected to the problem of superselection sectors. To fix ideas, consider Quantum Electrodynamics, which exhibits charge superselection, meaning that the Hilbert space of the theory splits into a direct sum of subspaces indexed by electric charge, each carrying a distinct representation of the global gauge group $U(1)$. The 'behind the Moon' argument, due originally to Haag, is often used to illustrate that any which of these sectors contains all physical information on the theory. Following the account of [**43**], assume we have to calculate a scattering process (for example Mœller scattering) but only the charge–11 sector is at our disposal. This is not an obstacle since we may prepare an in–state with 2 e^{-}s and place 13 e^{+}s far enough (e.g. behind the moon) so as not to interfere with our scattering experiment. One can convince oneself that the same result will be calculated for this scattering process in sectors of all charges and more generally, that it is impossible to distinguish between different sectors by such local measurements.

The basic mathematical object of study in AQFT is the net of observable local subalgebras. The local algebras are modelled mathematically as abstract C^*–algebras (von Neumann algebras) and the most important physical property required of the net is (Einstein) locality, meaning that observables in spatially separated regions should commute. *Abstract* means that no representation of the observable algebra –as an algebra of bounded operators acting on a concrete Hilbert space– is assumed. Different representations of the observable algebra are seen to roughly correspond to different states of the same theory.

This philosophy underlies the AQFT interpretation of charge superselection sectors. Consider two distinct superselection sectors in the conventional (Wigner) sense, i.e. two Hilbert subspaces $\mathcal{H}$ and $\mathcal{H}'$ carrying different irreducible representations π_G and π'_G of the global gauge group G. It can then be proven that $\mathcal{H}$ and $\mathcal{H}'$ carry inequivalent irreducible representations of the observable algebra. This result suggests that it should be possible to define charge superselection sectors without reference either to charge carrying fields or a gauge group. Indeed, Doplicher, Haag and Roberts defined a special class of representations of the observable algebra (the DHR representations) which corresponds intuitively to localized charges. Charge superselection sectors in AQFT are then defined as the irreducible DHR representations.

This definition went a long way and achieved great successes, culminating in the Doplicher–Roberts reconstruction theorem. In mathematical terms, the set of DHR representations carry the structure of a braided and rigid monoidal category. In physical terms, the monoidal product corresponds to the composition of charges, the braiding corresponds to particle statistics and rigidity reflects the existence of antiparticles. The essence of the reconstruction theorem is that in more than three space–time dimensions, the DHR category is equivalent to the representation category of a compact group, which is interpreted as the global gauge group. Furthermore, there is a canonical embedding of the observable algebra into an extended field algebra carrying an action of the gauge group such that the observable algebra is the invariant subalgebra. Thus, two problems are solved simultaneously: both the field algebra and the gauge group are reconstructed from the observable net alone, at least for $d + 1 \geq 3$. Much work has been done to extend this result to $1 + 1$ space–time dimensions. It is known from model studies (e.g. the chiral Ising model) that physically relevant fusion rules exist which preclude group symmetry. Also, in low space–time dimension, the DHR category is not symmetric but only braided. Mathematically, the reconstruction problem is one of finding the appropriate algebraic structure that is to be reconstructed.

In this chapter, we first review the axioms of AQFT and the basic results of the DHR theory. We then consider the problem of field algebra reconstruction (i.e. half of the Doplicher–Roberts theorem's undertaking) from the perspective of quantum groupoid Galois theory. Note that we shall only be using a small part of the mathematical structure of AQFT. The physical essence is in the locality of the observable algebra, which has no place (yet) in the framework of Galois theory. We conclude with remarks on the full Doplicher–Roberts problem in arbitrary dimensions.

1. The Observable Net

The standard exposition of algebraic quantum field theory intends to capture the most salient properties of a local, relativistic quantum field theory defined on Minkowski space. The local subsystems to consider are the algebras $\mathfrak{A}(\mathcal{O})$ of 'observables measurable within a given causally complete spacetime region $\mathcal{O}$'. The coherent association of algebras to spacetime regions provides the observable algebra with a locality structure. The axioms of AQFT identify structural properties that hold in most models of relativistic quantum field theory.

(I) Net structure.

Let $\mathcal{K}$ be the poset of causally complete spacetime regions (these are usually taken to be 'diamonds', i.e. intersections of forward– and backward oriented light cones) in Minkowski space, ordered by inclusion. $\mathcal{K}$ satisfies the additional property that for any $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$, there exists an $\mathcal{O}_3 \in \mathcal{K}$, such that both $\mathcal{O}_1 \subseteq \mathcal{O}_3$ and $\mathcal{O}_2 \subseteq \mathcal{O}_3$, i.e. $\mathcal{K}$ is not only a poset, but a *net*. The local structure of the observable algebra is then encoded by a covariant functor $\iota : \mathcal{K} \to C^* - \mathsf{Alg}$ with

$$\begin{aligned}&\text{object map } \ \mathcal{O} \mapsto \iota(\mathcal{O}) = \mathfrak{A}(\mathcal{O}) \ \text{ and}\\ &\text{arrow map } \ \langle \mathcal{O}_1 \subseteq \mathcal{O}_2 \rangle \mapsto \iota_{\mathcal{O}_1,\mathcal{O}_2} = \langle \mathfrak{A}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}(\mathcal{O}_2)\rangle,\end{aligned}$$

where the category $C^*-\mathsf{Alg}$ is understood to have arrows the unital *–homomorphisms, and $\mathcal{K}$ is a poset category. The *isotony* property of the net is the functoriality of ι, i.e. for $\mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3$,

$$\iota_{\mathcal{O}_1 \subseteq \mathcal{O}_2} \circ \iota_{\mathcal{O}_2 \subseteq \mathcal{O}_3} = \iota_{\mathcal{O}_1 \subseteq \mathcal{O}_3}.$$

The net property implies that the mapping $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ constitutes an inductive system and the inductive limit algebra $\mathfrak{A}_{loc} = \bigcup_{\mathcal{O}\in\mathcal{K}} \mathfrak{A}(\mathcal{O})$ of strictly local observables exists[1]. $\mathfrak{A}_{loc}$ possesses a unique C^*–norm; the *quasi–local* algebra $\mathfrak{A} = \overline{\mathfrak{A}_{loc}}^{C^*} = \overline{\bigcup_{\mathcal{O}\in\mathcal{K}} \mathfrak{A}(\mathcal{O})}^{C^*}$ is the completion in this norm.

In non–simply connected spacetimes, the set $\mathcal{K}$ of double cones is not directed and the isotony property fails. It may still be possible to define the quasilocal algebra, e.g. for the case that spacetime is S^1. Then the non–trivial topology of spacetime is reflected in that the quasilocal algebra $\mathfrak{A}$ develops a non–trivial center.

(II) Locality.

The fact that we are considering relativistic field theories is represented by the axiom of locality, which is sometimes referred to as microcausality or Einstein causality. Denote $\mathcal{O}_1 \times \mathcal{O}_2$ that $\mathcal{O}_1$ and $\mathcal{O}_2$ are space–like separated; the corresponding subalgebras are then required to commute,

$$[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0, \quad \text{for } \mathcal{O}_1 \times \mathcal{O}_2.$$

Define $\mathfrak{A}(\mathcal{O}') = \overline{\bigcup_{\mathcal{O}_1 \times \mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)}^{C^*}$ the C^*–subalgebra generated by all $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \times \mathcal{O}$; then $[\mathfrak{A}(\mathcal{O}), \mathfrak{A}(\mathcal{O}')] = 0$.

(III) Spacetime (Poincaré–) Covariance.

The action of the Poincaré group on Minkowski space induces an action on the observable net. There is a group homomorphism $\alpha : \mathcal{P}_+^\uparrow \to \mathrm{Aut}(\mathfrak{A})$, such that for all $\mathcal{O} \in \mathcal{K}$ and all

[1]Note that this coincides with the categorical colimit of the functor ι. Although the local structure of the observable algebra is essential from a physical point of view, our algebraic constructions will always refer to the global algebra $\mathfrak{A}$. Nevertheless, the above formulation suggests a strategy for generalization, replacing algebras by algebra–valued functors.

$\Lambda \in \mathcal{P}_+^\uparrow$

$$\alpha_\Lambda(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\Lambda(\mathcal{O})) \tag{7.1.1}$$

(IV) Vacuum State.

We specify a vacuum state on $\mathfrak{A}$, which is equivalent to specifying –via the GNS-construction– a vacuum representation of the observable algebra. There is a pure state ω_0 on $\mathfrak{A}$, with GNS–construction $\langle \pi_0, \mathcal{H}_0, \Omega, U_0 \rangle$, such that π_0 is faithful and irreducible and the spectrum of U_0 lies in the future light cone,

$$\mathsf{Sp}U_0 \subseteq \bar{V}_+$$

This requirement is usually referred to as the *spectrum condition* and is a Lorentz covariant expression of the positivity of energy, or stability of the vacuum. This additional input is in fact enough to translate all pure (C^*–) algebraic constructions to concrete C^*–algebras realized as operators in Hilbert space, if we content ourselves with using only the vacuum Hilbert space $\mathcal{H}_0$.

(V) Haag duality.

This axiom is of topological nature. For each $\mathcal{O} \in \mathcal{K}$, we require that $\pi_0(\mathfrak{A}(\mathcal{O}))$ be equal to the commutant of its complement (i.e. as large as possible, since it is always contained in it):

$$\pi_0(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}'))'$$

More precisely, the above requirement should be called 'Haag duality in the vacuum representation'. One particular consequence is that all local algebras become von Neumann algebras, since they are equal to their double commutants: $\pi_0(\mathfrak{A}(\mathcal{O}))'' = \pi_0(\mathfrak{A}(\mathcal{O}))$.

(VI) Additivity.

If $\bigcup_\alpha \mathcal{O}_\alpha$ is a covering of the double cone $\mathcal{O} \in \mathcal{K}$, then $\mathfrak{A}(\mathcal{O})$ is contained in the von Neumann algebra generated by the $\{\mathfrak{A}(\mathcal{O}_\alpha)\}_\alpha$,

$$\mathfrak{A}(\mathcal{O}) \subseteq (\cup_\alpha \mathfrak{A}(\mathcal{O}_\alpha))''$$

This property may be interpreted to state the absence of a smallest length scale, in the sense that any observable can be generated with observables from arbitrarily small regions.

2. Representations of the Observable Net

In this section we recall the Doplicher–Haag–Roberts definition of superselection sectors and show how the structure of a rigid, monoidal, symmetric (resp. braided) category emerges.

As already noted, the different inequivalent representations of the C^*–algebra of observables correspond to different states of the system. For the purposes of particle physics, it seems reasonable to select from the plethora of representations those, which are localized in some sense. Such a representation is then said to describe a *localized charge*.

The category of representations of the (quasilocal) observable algebra will be denoted $\mathsf{Rep}_{\mathfrak{A}}$. It has objects $\langle \pi, \mathcal{H}_\pi \rangle \in \mathsf{Rep}_{\mathfrak{A}}$, with $\mathcal{H}_\pi \in \mathsf{Hilb}$ and $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ a C^*–morphism. An arrow $T : \langle \pi, \mathcal{H}_\pi \rangle \to \langle \pi', \mathcal{H}_{\pi'} \rangle$ is an interwiner, i.e. a unitary $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ satisfying $T\,\pi(A) = \pi'(A)\,T$ for all $A \in \mathfrak{A}$. The module category $\mathsf{Rep}_{\mathfrak{A}}$ comes equipped with the forgetful functor $U : \mathsf{Rep}_{\mathfrak{A}} \to \mathsf{Hilb}$, $\langle \pi, \mathcal{H}_\pi \rangle \mapsto \mathcal{H}\pi$ sending all representations to their underlying Hilbert space. The additional *– and topological structure of $\mathfrak{A}$ enriches the category of representations, making $\mathsf{Rep}_{\mathfrak{A}}$ a C^**–category* (see below).

2.1. The category of localized representations. Doplicher, Haag and Roberts proposed the following criterion to single out those representations of the observable algebra which describe localized charges. Essentially, these are the representations which are equivalent to the vacuum outside a bounded region.

Definition 7.2.1. (The DHR selection criterion) An irreducible representation π of $\mathfrak{A}$ is localized in a double cone $\mathcal{O} \in \mathcal{K}$, if the restriction of π to the complement of $\mathcal{O}$ is unitarily equivalent to the vacuum:

$$\exists V : \mathcal{H}_\pi \to \mathcal{H}_0, \text{ so that } V\pi(A) = \pi_0(A)V \text{ for all } A \in \mathfrak{A}(\mathcal{O}') \tag{7.2.2}$$

The subcategory of localized representations will be denoted $\mathsf{Rep}_{loc}(\mathfrak{A})$.

It should be noted that this selection criterion is quite restrictive in view of realistic models of particle physics. For example, an electrical charge in quantum electrodynamics cannot be described as a DHR charge due to Gauss' law. Indeed, we could measure the charge by measuring the flux of the electrical field strength through a sphere of arbitrary radius around the charge. This is just a manifestation of the long–range character of the electromagnetic interaction or, equivalently, the vanishing of the photon mass. This problem is to be expected in all gauge theories. Buchholz and Fredenhagen have shown that massive gauge theories may be accomodated by modifying the selection criterion to representations that are equivalent to the vacuum outside an infinitely extended *cone* around some space–like direction.

2.2. The category of localized endomorphisms. The pivotal trick in the DHR analysis is the following observation. Consider a localized representation $\pi \in \mathsf{Rep}_{loc}(\mathfrak{A})$ acting in a Hilbert space $\mathcal{H}_\pi$; we can replace π by an equivalent representation acting on the vacuum Hilbert space $\mathcal{H}_0$ by introducing the algebra endomorphism $\rho : \mathfrak{A} \to \mathfrak{A}$

$$\rho(A) = V^{-1}\pi_0(A)V, \quad A \in \mathfrak{A}, \tag{7.2.3}$$

where $V : \mathcal{H}_\pi \to \mathcal{H}_0$ is a unitary. In this way, all superselection sectors may be represented inside the vacuum Hilbert space $\mathcal{H}_0$. Furthermore, by the selection criterion 7.2.2 applied to π, the unitary $V : \mathcal{H}_\pi \to \mathcal{H}_0$ may be chosen such that it defines an equivalence between π and the vacuum representation π_0 on the complement $\mathcal{O}'$ of some double cone $\mathcal{O}$. The endomorphism ρ will then be localized in $\mathcal{O}$ in the sense that $\rho(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}')$.

PROPOSITION 7.2.4. *For a double cone $\mathcal{O} \in \mathcal{K}$, there is a one-to-one correspondence up to unitary equivalence between the following:*

- *representations $\pi \in \mathsf{Rep}_{loc}(\mathfrak{A})$, i.e. for which there is a unitary $V : \mathcal{H}_\pi \to \mathcal{H}_0$ such that*

$$\pi(B) = V^{-1}\pi_0(B)V, \ \ \forall B \in \mathfrak{A}(\mathcal{O}')$$

- *representations of $\mathfrak{A}$ which are of the form $\pi_0 \circ \rho$ where $\rho : \mathfrak{A} \to \mathfrak{A}$ is a localized endomorphism, i.e.*

$$\rho(B) = B, \ \ \forall B \in \mathfrak{A}(\mathcal{O}').$$

The correspondence is established by $\rho(A) = V^{-1}\pi_0(A)V$.

Put differently, we have defined an inclusion $\mathsf{Rep}_{loc}(\mathfrak{A}) \hookrightarrow \mathrm{End}(\mathfrak{A})$ of categories, where $\mathrm{End}(\mathfrak{A})$ is the category of endomorphisms of $\mathfrak{A}$. The category $\mathrm{End}(\mathfrak{A})$ has objects the endomorphisms of $\mathfrak{A}$ and arrows $T : \rho \to \sigma$ the intertwiners $T \in \mathfrak{A} : \ T\rho(A) = \sigma(A)T$. The crucial observation is that the category $\mathrm{End}(\mathfrak{A})$ has a much richer structure than $\mathsf{Rep}_{loc}(\mathfrak{A}) \subseteq \mathsf{Rep}(\mathfrak{A})$— in particular, it is monoidal with the monoidal product given by the composition of endomorphisms. This suggests that the superselection sector structure should be captured as an appropriate subcategory (the DHR category) of $\mathrm{End}(\mathfrak{A})$, rather than $\mathsf{Rep}(\mathfrak{A})$.

It is clear from Proposition 7.2.4 that we should restrict ourselves to the localized endomorphisms. A further restriction is *transportability.*

DEFINITION 7.2.5 (Transportable morphisms). A morphism is transportable if we can find morphisms equivalent to it in all translates of its region of localization: for $\rho \in \mathrm{End}(\mathcal{O})$, and any double–cone $\mathcal{O}_1 = \mathcal{O} + \mathbf{x}$ there is a unitary U and ρ_1 localized in $\mathcal{O}_1$ such that[2]

$$U\rho(A) = \rho_1(A)U.$$

We can now define the objects of the DHR category as elements of Δ:

DEFINITION 7.2.6. For any double cone $\mathcal{O} \in \mathcal{K}$, let $\Delta(\mathcal{O})$ denote the set of transportable morphisms localized in $\mathcal{O}$ and further denote $\Delta = \bigcup_{\mathcal{O}\in\mathcal{K}} \Delta(\mathcal{O})$.

The arrows of the DHR category are the arrows induced from $\mathrm{End}(\mathfrak{A})$ such that $\Delta \hookrightarrow \mathrm{End}(\mathfrak{A})$ is a full subcategory, i.e. for objects ρ, ρ' in Δ, the set $\mathrm{Hom}(\rho, \rho')$ of arrows between ρ and ρ' is defined as follows:

$$\mathrm{Hom}_\Delta(\rho_1, \rho_2) \equiv (\rho_2|\rho_1) = \{T \in \mathfrak{A} \,|\, \rho_2(A)T = T\rho_1(A), \ \forall A \in \mathfrak{A}\}.$$

Elements of the Hom–space $(\rho_2|\rho_1)$ are called intertwiners from ρ_1 to ρ_2. Spaces of intertwiners are linear subspaces of $\mathfrak{A}$, and respect the localization of the source– and target morphisms in the sense that for $\rho_i \in \Delta(\mathcal{O}_i)$ $(i = 1, 2)$, $(\rho_2 \,|\, \rho_1) \subseteq \mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2)$.

[2]From a physical point of view, it is reasonable to further restrict attention to Poincaré covariant morphisms $\Delta_\mathcal{P} \subseteq \Delta$ for which there is a unitary representation $V : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{H}_0)$ of the whole Poincaré group implementing Poincaré transformations on ρ, i.e. $\rho(\alpha_{(\Lambda,x)}(A)) = V_{(\Lambda,x)}\rho(A)V^{-1}_{(\Lambda,x)}$.

Composition of arrows is simply by multiplication in $\mathfrak{A}$: for $T : \rho \to \rho'$ and $T' : \rho' \to \rho''$, the product $T' \circ T \equiv T'T : \rho \to \rho''$ is an intertwiner from ρ to ρ''. The identity of $\mathfrak{A}$ is the identity arrow for all objects $\rho \in \mathsf{Ob}\,\Delta$, $\mathbf{1} = \mathrm{id}_{\rho}$.

Remark 7.2.7 (Charge quantum numbers). The set of objects $\mathsf{Ob}\,\Delta$ is much too large to be identified with the superselection sectors in the physical sense. Instead, we take unitary equivalence classes of morphisms. Call a unitary U on $\mathcal{H}_0$ localized in $\mathcal{O} \in \mathcal{K}$ if $U \in \mathfrak{A}(\mathcal{O})$; then $\mathsf{Ad}_U(A) = UAU^{-1}$ defines a localized automorphism. We consider two morphisms ρ and ρ' equivalent if $\rho' = \mathsf{Ad}_U \circ \rho$ for some localized unitary U. The set of equivalence classes

$$[\Delta] = \{[\rho] \,|\, \rho' \in [\rho] \Leftrightarrow \rho' = \mathsf{Ad}_U \circ \rho\} = \Delta/\sim \tag{7.2.8}$$

is then identified with physical superselection sectors, or 'charge quantum nubers'.

Remark 7.2.9 (Local structure of Δ). Although the local aspect of the observable algebra is ignored in the application to Galois theory, it is of utmost importance to AQFT.

The causal structure of the observable algebra $\mathfrak{A}$ carries over to the morphisms, in the sense that morphisms with space–like separated supports commute, i.e. for $\rho_1 \in \Delta(\mathcal{O}_1)$, $\rho_2 \in \Delta(\mathcal{O}_2)$

$$\text{for } \mathcal{O}_1 \times \mathcal{O}_2 : \quad \rho_1\,\rho_2 = \rho_2\,\rho_1 \tag{7.2.10}$$

To prove 7.2.10, it is sufficient to prove that $\rho_1\,\rho_2(A) = \rho_2\,\rho_1(A)$ holds for any localized observable A. However, since the ρ_i are transportable, we can choose for any region $\mathcal{O} \in \mathcal{K}$ and $A \in \mathfrak{A}(\mathcal{O})$ transported morphisms $\rho_1' \sim \rho_1$ and $\rho_2' \sim \rho_2$ which are localized space–like to ρ_1 and ρ_2, respectively and both are space–like to $\mathcal{O}$. Then clearly $\rho_1'\rho_2'(A) = \rho_2'\rho_1'(A)$ and this holds for the equivalent morphisms ρ_1 and ρ_2 as well.

The category of endomorphisms $\mathrm{End}\,\mathfrak{A}$ of an algebra $\mathfrak{A}$ is enriched by the extra structure that $\mathfrak{A}$ may possess. In particular, the quasilocal C^*–algebra structure of the observable algebra $\mathfrak{A}$ has the consequence that

(1) Δ is a monoidal $*$–category, and
(2) Δ is a C^*–category.

2.3. $\Delta(\mathfrak{A})$ is a C^*–category. The notion of C^*–category is the categorical analogue of a C^*–algebra. For

Definition 7.2.11. A *C^*–category* has Hom–spaces which are $\mathbf{C}$–linear, such that for all $X, Y, Z \in \mathsf{Ob}\,\mathcal{C}$ the following properties hold:

- composition is a bilinear map $\circ : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$,
- there is a positive *–operation $^* : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$ and
- a norm $\|\cdot\|_{X,Y}$ on $\mathrm{Hom}(X,Y)$ making it a Banach space, such tha

$$\|g \circ f\|_{X,Z} \le \|g\|_{Y,Z}\|f\|_{X,Y} \;\text{ and }\; \|f^* \circ f\|_{X,X} = \|f\|^2_{X,Y}$$

As promised, we have that

LEMMA 7.2.12. *The DHR category Δ is a C^*–category: for all $\rho, \sigma \in \mathsf{Ob}\,\Delta$,* $\mathrm{Hom}_\Delta(\rho,\sigma) = (\sigma\,|\,\rho)$ *is a complex vector space with positive $*$–operation and norm $\|\cdot\|_{\rho,\sigma}$ inherited from $\mathfrak{A}$.*

Two important properties which follow from the C^*–structure is that the DHR category Δ has *subobjects* and *direct sums*. $\mathsf{Rep}(\mathfrak{A})$ has subobjects and direct sums, inherited from Hilb via the forgetful functor $\mathsf{Rep}(\mathfrak{A}) \to \mathsf{Hilb}$. For an abstract category, however, establishing these properties is far from trivial.

DEFINITION 7.2.13. Let $\mathcal{C}$ be a *–category. For objects $X, Y \in \mathsf{Ob}\,\mathcal{C}$, X is a *subobject* of Y if there is an isometry $s \in \mathrm{Hom}(X,Y)$ i.e. X is isometrically embedded into Y. $\mathcal{C}$ *has subobjects* if for each $Y \in \mathsf{Ob}\,\mathcal{C}$ and each projection $p \in \mathrm{End}(Y)$ there is an object $X \in \mathsf{Ob}\,\mathcal{C}$ and an isometry $s : X \to Y$ such that $p = s \circ s^*$.

For objects $X, Y, Z \in \mathsf{Ob}\,\mathcal{C}$, $Z \cong X \oplus Y$ is a *direct sum* of X and Y, defined up to isomorphism, if there are isometries $f \in \mathrm{Hom}(X,Z)$, $g \in \mathrm{Hom}(Y,Z)$ such that $f \circ f^* + g \circ g^* = \mathrm{id}_Z$ and $f^* \circ g = 0$. $\mathcal{C}$ *has direct sums* if for any pair of objects $X, Y \in \mathsf{Ob}\,\mathcal{C}$, there is a direct sum $Z \cong X \oplus Y$.

PROPOSITION 7.2.14. *The category Δ has subobjects and direct sums*[3]*; for $\rho, \sigma \in \mathsf{Ob}\,\mathcal{C}$, the direct sum is denoted $\rho \oplus \sigma$.*

An object X in a $\mathbf{C}$–linear category $\mathcal{C}$ is said to be *simple*, or irreducible, if it is nonzero and $\mathrm{End}(X) = \mathbf{C}\,\mathrm{id}_X$. The category $\mathcal{C}$ is *semisimple* if every object is a finite direct sum of irreducible objects.

2.4. $\Delta(\mathfrak{A})$ is a monoidal $*$–category. The DHR category Δ inherits its monoidal structure from $\mathrm{End}(\mathfrak{A})$. As a first step, consider that $\Delta \subset \mathrm{End}(\mathfrak{A})$ is a submonoid, i.e. composition restricts properly to transportable, localized morphisms.

It is clear from the definitions that for $\rho \in \Delta(\mathcal{O}_1)$ and $\sigma \in \Delta(\mathcal{O}_2)$, the composition $\rho\sigma$ is again localized, in $\mathcal{O}_1 \cup \mathcal{O}_2$. To see that $\rho\sigma$ is also transportable, consider an arbitrary double cone $\mathcal{O}_3$. Since ρ and σ are transportable, there are unitaries $U : \rho \to \rho'$ and $V : \sigma \to \sigma'$ with $\rho', \sigma' \in \Delta(\mathcal{O}_3)$. But then $U\rho(V) : \rho\sigma \to \rho'\sigma'$ and $\rho'\sigma' \in \Delta(\mathcal{O}')$. Since $\mathcal{O}_3$ was arbitrary, $\rho\sigma$ is also transportable.

REMARK 7.2.15. By an entirely similar reasoning, we can check that composition of endomorphsims is well–defined on the unitary equivalence classes of $[\Delta]$,

$$[\rho_1\rho_2] = [\rho_1][\rho_2]. \tag{7.2.16}$$

It is in fact the monoid structure on $[\Delta]$ which we take to mean the 'charge composition algebra'.

[3]Both assertions rely on Borcher's 'Property B', which is satisfied iff for each $\mathcal{O}, \mathcal{O}_1 \in \mathcal{K}$ such that $\mathcal{O}^- \subseteq \mathcal{O}_1$, every projection E of $\mathfrak{A}(\mathcal{O})$ is equivalent to the unit in $\mathfrak{A}(\mathcal{O}_1)$, i.e. there is an isometry $W \in \mathfrak{A}(\mathcal{O}_1)$ such that $WW^* = E$ and $W^*W = \mathbf{1}$. This is a technical assumption, since it is fulfilled for any additive, causal net satisfying the spectral condition.

It remains to define the monoidal product on arrows. For intertwiners $S_1 : \rho_1 \to \rho_1'$, $S_2 : \rho_2 \to \rho_2'$, let

$$S_1 \times S_2 = S_1\rho_1(S_2) = \rho_1'(S_2)S_1 : \rho_1\rho_2 \to \rho_1'\rho_2',$$

where the two expression coincide because of the intertwiner property of S_1. The cross product is associative,

$$S_1 \times (S_2 \times S_3) = (S_1 \times S_2) \times S_3 \tag{7.2.17}$$

and is compatible with the composition of intertwiners in the sense of the *interchange law*:

$$(S_1' \circ S_1) \times (S_2' \circ S_2) = (S_1' \times S_2') \circ (S_1 \times S_2), \tag{7.2.18}$$

whenever both sides are well–defined, e.g. $S_i : \rho_i \to \rho_i'$ and $S_i' : \rho_i' \to \rho_i''$ are composable intertwiners. In other words, we have that

Proposition 7.2.19. $\langle \Delta, \otimes, \mathrm{id}_{\mathfrak{A}} \rangle$ *is a (strict) monoidal* **–category. The monoidal unit* $\mathrm{id}_{\mathfrak{A}}$ *is the identity endomorphism in* $\mathrm{End}\,\mathfrak{A}$ *and the monoidal product is given on objects* $\rho, \sigma \in \mathsf{Ob}\,\Delta$ *by the composition of endomorphisms* $\rho\otimes\sigma = \rho\sigma$ *and on arrows* $S \in \mathrm{Hom}(\rho, \rho'), T \in \mathrm{Hom}(\sigma, \sigma')$ *by* $S \otimes T = S \times T = S\rho(T)$.

Furthermore, since for all objects ρ, σ, $\mathrm{Hom}(\rho, \sigma)$ *is closed in the norm inherited from* $\mathfrak{A}$ *and for any pair of arrows* $\|S \times T\| \leq \|S\| \cdot \|T\|$, Δ *is a* C^*–monoidal category.

2.5. $\Delta(\mathfrak{A})$ is a braided and rigid monoidal category. The composition of charges is reflected mathematically in the monoidal structure on Δ. Two other fundamental features of relativistic quantum field theory is particle statistics (in more than 3 spacetime dimensions, the Bose/Fermi alternative) and the existence of antiparticles. AQFT captures these properties mathematically in the braiding and rigidity structure on the DHR category.

For two arbitrary morphisms $\rho_1 \in \Delta(\mathcal{O}_1)$ and $\rho_1 \in \Delta(\mathcal{O}_1)$, let $\tilde{\mathcal{O}}_{1,2}$ be space–like separated double cones and introduce unitaries $U_i \in \mathrm{Hom}(\rho_i, \tilde{\rho}_i)$ such that $\tilde{\rho}_i \in \Delta(\mathcal{O}_i)$. Then $U_1 \times U_2 \in \mathrm{Hom}(\rho_1\rho_2, \tilde{\rho}_1\tilde{\rho}_2)$ and $U_1^* \times U_2^* \in \mathrm{Hom}(\tilde{\rho}_2\tilde{\rho}_1, \rho_2\rho_1)$. Since $\mathcal{O}_1 \times \mathcal{O}_2$, the morphisms $\tilde{\rho}_1$ and $\tilde{\rho}_2$ commute. Consider the following unitary intertwiner in $\mathrm{Hom}(\rho_1\rho_2, \rho_2\rho_1)$:

$$\varepsilon_{\rho_1,\rho_2}(U_1, U_2) := (U_2 \times U_1)^* \circ (U_1 \times U_2) = \rho_2(U_1^*)U_2^*\, U_1\rho_1(U_2). \tag{7.2.20}$$

It is then proved that $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is independent of the choice of U_1 and U_2, except for their relative localization. The analysis boils down to two distinct cases, depending on the dimensionality of spacetime[4]. Note that for $D = d + 1 \leq 2$, there is a well–defined notion of left and right; we use $\mathcal{O} < \mathcal{O}'$ to denote that $\mathcal{O}$ is to the *left* of $\mathcal{O}'$.

Lemma 7.2.21. *In space–time dimension* $D = d + 1 \leq 2$, $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ *is definable in terms of* ρ_1 *and* ρ_2 *depends on* $\tilde{\mathcal{O}}_1$ *and* $\tilde{\mathcal{O}}_2$ *only through their relative spatial orientation; let* $U_i : \rho_i \to \tilde{\rho}_i$ *with* $\mathcal{O}_1 < \mathcal{O}_2$ *and let* $U_i' : \rho_i \to \tilde{\tilde{\rho}}_i$ *with* $\tilde{\tilde{\mathcal{O}}}_1 > \tilde{\tilde{\mathcal{O}}}_2$ *having the opposite orientation.*

[4]This is the first instance in the discussion of AQFT, where the dimension of the underlying spacetime enters.

Then

$$\varepsilon_{\rho_1,\rho_2}(U_1,U_2) = [\varepsilon_{\rho_2,\rho_1}(U_2',U_1')]^* .$$

For $D = d+1 \geq 3$, $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ is independant of the choice of $(\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2)$ as long as they are space–like separated and

$$\varepsilon_{\rho_1,\rho_2} = (\varepsilon_{\rho_2,\rho_1})^{-1}.$$

For $D \geq 3$, the braiding is defined with $\varepsilon_{\rho_1,\rho_2} = \varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ for U_1 and U_2 space–like separated; in the case that $D \leq 2$, there are two possible conventions to define $\varepsilon_{\rho_1,\rho_2}$. According to Lemma 7.2.21, one convention is the complex conjugate of the other.

PROPOSITION 7.2.22. *$\varepsilon_{\rho_1,\rho_2}$ is a braiding on the DHR category $\langle \Delta, \otimes, \mathrm{id}_{\mathfrak{A}} \rangle$. For spacetime dimension $D \leq 2$, $\varepsilon_{\rho_1,\rho_2}$ is the unique braiding such that $\varepsilon_{\rho_1,\rho_2} = \mathrm{id}$ for morphisms $\rho_i \in \Delta(\mathcal{O}_i)$ with $\mathcal{O}_2 < \mathcal{O}_1$. For $D \geq 3$, $\varepsilon_{\rho_1,\rho_2}$ is the unique braiding such that $\varepsilon_{\rho_1,\rho_2} = \mathrm{id}$ for morpshims $\rho_i \in \Delta(\mathcal{O}_i)$ with $\mathcal{O}_2$ and $\mathcal{O}_1$ space–like separated. In this case, $\varepsilon_{\rho_1,\rho_2}$ is in fact a symmetry.*

The existence of antiparticles is fundamental to relativistic quantum field theories. In AQFT, the conjugate $\bar{\rho}$ of a charge $\rho \in \Delta$ is identified with the conjugate object in the categorical sense.

DEFINITION 7.2.23. Let $\mathcal{C}$ be a monoidal *-category. The *conjugate* of an object $X \in \mathsf{Ob}\,\mathcal{C}$ is a triple $(\bar{X}, r, \bar{r})$, where $\bar{X} \in \mathsf{Ob}\,\mathcal{C}$ and the arrows $R : \mathbf{1} \to \bar{X} \otimes X$ and $\bar{R} : \mathbf{1} \to X \otimes \bar{X}$ satisfy

(7.2.24) $$(\bar{R}^* \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes R) = \mathrm{id}_X$$

(7.2.25) $$(R^* \otimes \mathrm{id}_{\bar{X}}) \circ (\mathrm{id}_{\bar{X}} \otimes \bar{R}) = \mathrm{id}_{\bar{X}}.$$

The category $\mathcal{C}$ *has conjugates* if there is a conjugate for every object in $\mathcal{C}$; in this case, $\mathcal{C}$ is said to be *rigid*.

In AQFT, the existence of a conjugate $(\bar{\rho}, R, \bar{R})$ for an object ρ means that there is a subobject of $\rho\,\bar{\rho}$ equivalent to the monoidal unit. This is consistent with the intuition that conjugate charges annihilate each other.

The notion of categorical (or quantum–) dimension is closely connected to the theory of conjugates. For an object $X \in \mathsf{Ob}\,\mathcal{C}$ with standard conjugate $(\bar{X}, r, \bar{r})$, the dimension $d(X)$ of X is defined by

(7.2.26) $$d(X)\,\mathrm{id}_{\mathbf{1}} = R^* \circ R.$$

If no conjugate exists, the dimension is formally set to be $d(X) = +\infty$. The dimension function satisfies properties one would expect, in particular $d(X) \geq 0$ for all $X \in \mathsf{Ob}\,\mathcal{C}$, and the following relation hold:

(7.2.27) $$d(\bar{X}) = d(X), \quad d(X \otimes Y) = d(X) \cdot d(Y), \quad d(X \oplus Y) = d(X) + d(Y),$$

and the dimension of the monoidal unit is $d(\iota) = 1$. In a symmetric C^*–category, the quantum dimensions are restricted to the integers. For the representation category of a compact group in particular, the quantum dimensions coincide with the linear dimensions of the representation spaces. A strong result holds for the restriction of the DHR category Δ to the full subcategory of finite–dimensional objects Δ_f with objects

$$\mathsf{Ob}\,\Delta_f = \{\rho \in \mathsf{Ob}\,\Delta \,|\, d(\rho) < +\infty\},$$

i.e. the full subcategory of objects with conjugates. It can be proved that $d : \mathsf{Ob}\,\Delta_f \to \mathbf{R}^+$ takes values only in $[1,\infty)$ and in the interval $[1,2]$ only the discrete values $2\cos(\pi/k)$ $(k \geq 3)$ can appear[5]

Remark 7.2.28 (Non–integer quantum dimensions in low spacetime dimension). Another salient feature of low–dimensional QFT is the appearance of non–integer quantum dimensions. Recall that for $D = d+1 \geq 3$ the DHR category is symmetric, and this forces quantum dimensions to be integers; for $D \leq 2$, this restriction does not apply. In the Lee–Yang model, for example, there is a self–dual sector ρ with the fusion rule $\rho \otimes \rho = \rho \oplus \mathbf{1}$. Thus, the quantum dimension must satisfy $d(\rho)^2 = d(\rho) + 1$ which has the irrational solution $d(\rho) = \frac{1+\sqrt{5}}{2}$.

The Doplicher–Roberts reconstruction theorem identifies the DHR category Δ with the representation category of the gauge group. Clearly, in a theory exhibiting non–integer quantum dimensions, the reconstructed 'gauge group' cannot possibly be a group. In fact, it cannnot even be a Hopf algebra: it can be proved that the quantum dimensions in $\mathsf{Rep}(H)$ with H a semi–simple Hopf algebra are also necessarily integers. This observation is one of the main motivations for studying quantum groupoids, as a candidate symmetry in low–dimensional QFT models.

3. The Doplicher–Roberts reconstruction theorem

Formulating QFT in terms of the observable net leads to an elegant interpretation of superselection sectors and particle statistics, but to make contact with conventional QFT, two major ingredients are missing: so far, we are ignorant of the charge carrying fields and the gauge group. The DR theorem solves both problems in one stroke, by simultaneously reconstructing both the field algebra –the extension of the observable algebra by charge–carrying fields– and the gauge group from the DHR category Δ.

The approach of conventional QFT is exactly opposite. There one starts with a field algebra $\mathcal{F} \subseteq \mathcal{B}(\mathcal{H}^u)$ acting irreducibly as bounded operators on the state space $\mathcal{H}^u$ (the superscript refers to 'universal') of the theory and gauge group G, together with a unitary representation $\pi^u : G \to \mathcal{U}(\mathcal{H}^u)$ on $\mathcal{H}^u$. There is a privileged direct sum decomposition of $\mathcal{H}^u$ into orthogonal subspaces $\mathcal{H}^u = \bigoplus_\xi \mathcal{H}^u_\xi$, where the subspaces $\mathcal{H}^u_\xi$ are labelled by the characters (unitary

[5]This is an instance of the connection of AQFT to the theory of subfactors. Assuming that every local von Neumann algebra $\mathfrak{A}(\mathcal{O})$ is a factor, i.e. has trivial center, an endomorphism $\rho \in \Delta(\mathcal{O})$ gives rise to an inclusion of factors $\rho(\mathfrak{A}(\mathcal{O})) \subseteq \mathfrak{A}(\mathcal{O})$. The *index* of this inclusion is related to the categorical dimension through $[\mathfrak{A}(\mathcal{O}) : \rho(\mathfrak{A}(\mathcal{O}))] = d(\rho)^2$. The index of subfactors has been classified in the work of V.F.R. Jones.

equivalence classes of irreducible representations) of G, such that $\mathcal{H}^u_\xi$ carries the representation $D_\xi \in \pi^u(G)$. This is, in the conventional sense, the decomposition of the state space into superselection sectors. The observable algebra is *defined* as the algebra of gauge invariant elements of $\mathcal{F}$, i.e. $\pi^u(\mathfrak{A}) = \mathcal{F} \cap \pi^u(G)'$. The representation $\pi^u : \mathfrak{A} \to \mathcal{B}(\mathcal{H}^u)$ is reducible, and is in fact reduced by the subspaces $\mathcal{H}^u_\xi$, i.e. there is a direct sum decomposition $\pi^u = \bigoplus \pi_\xi$, into representations π_ξ on subspaces $\mathcal{H}^u_\xi$, respectively.

AQFT captures this setting in the notion of *field system*.

Definition 7.3.1. Denoting $(\mathcal{H}_0, \pi_0)$ the vacuum representation on $\mathfrak{A}$, a field system is a triple $\{\pi^u, G, \mathcal{F}\}$ where $(\mathcal{H}^u, \pi^u)$ is a (reducible) representation of the observable algebra, G is a strongly compact group of unitaries on $\mathcal{H}^u$ and $\mathcal{F}$ is a local net of field algebras acting irreducibly on $\mathcal{H}^u$ such that the following properties are satisfied:

(1) $(\mathcal{H}_0, \pi_0)$ is a subrepresentation of $(\mathcal{H}^u, \pi^u)$: there is an isometry $V : \mathcal{H}_0 \to \mathcal{H}^u$, $V\pi_0 = \pi^u V$;
(2) V maps $\mathcal{H}_0$ into the G–invariant subspace of $\mathcal{H}^u$;
(3) the automorphisms $\alpha_g : G \to \mathrm{Aut}\,\mathcal{F}$ induced by G leave the local field algebras $\mathcal{F}(\mathcal{O})$ globally fixed and $(\pi^u(\mathfrak{A}(\mathcal{O})))'' \subseteq \mathcal{F}(\mathcal{O})$ is the invariant subalgebra under the G–action;
(4) for any $\mathcal{O} \in \mathcal{K}$, $\mathcal{F}(\mathcal{O})V(\mathcal{H}_0)$ spans a dense subspace of $\mathcal{H}^u$ (i.e. $V(\mathcal{H}_0)$ is *cyclic* for any local field algebra);
(5) fields are local relative to observables: whenever $\mathcal{O}_1 \times \mathcal{O}_2$, $\mathcal{F}(\mathcal{O}_1)$ commutes with $\pi^u(\mathfrak{A}(\mathcal{O}_2))$.

The field system is called *normal* if the field algebra $\mathcal{F}$ satisfies $\mathbf{Z}_2$–graded commutativity, i.e. odd (fermionic) elements anticommute at spacelike separations while even (bosonic) elements commute. The field system is called *complete* if every superselection sector appears as a subrepresentation of π^u. Lastly, two field systems $(\pi^u_i, G_i, \mathcal{F}_i)$ $(i = 1, 2)$ are equivalent if there is a unitary $W : \mathcal{H}^u_1 \to \mathcal{H}^u_2$ intertwining the representations π^u_1 and π^u_2, the two gauge groups and the two field nets.

Remark 7.3.2 (Recovering DHR representations from the field system). In this rigorous formulation of the field system, it is possible to show that the subrepresentations appearing in $(\mathcal{H}^u, \pi^u)$ are precisely the DHR representations. In other words, localized representations (in the DHR sense) are precisely the ones that can be reached from the vacuum by the application of localized fields, see e.g. [**59**] Section 9.

Stated briefly, the DR reconstruction theorem says that with the standing assumptions of AQFT and the observable net there is a complete, normal field system $(\pi^u, G, \mathcal{F})$ which is unique up to equivalence. In regard to the reconstruction of the gauge group, the following Theorem tells us what to expect.

THEOREM 7.3.3. *A strict symmetric rigid monoidal category*[6] $\langle \mathfrak{T}, \otimes, \iota \rangle$ *with subobjects, direct sums and one–dimensional monoidal unit* $\mathrm{End}\,\iota = \mathbf{C}\,1$ *is the abstract dual of a unique compact group* G*, i.e. there is a concrete representation category* $\mathsf{Rep}(G)$ *and a functor* $\Delta \to \mathsf{Rep}(G)$ *which is an isomorphism of symmetric monoidal categories.*

3.1. Tannaka–Krein theory. Note that $\langle \mathfrak{T}, \otimes, \iota \rangle$ is an abstract monoidal category; hence, 7.3.3 is a much stronger result than the classical Tannaka–Krein theorem, which reconstructs a compact group from a concrete monoidal category, i.e. a monoidal category equipped with a *fiber functor*. A fiber functor for $\mathfrak{T}$ is a faithful and exact strong monoidal functor $E : \mathfrak{T} \to \mathsf{Vec}_{\mathbf{C}}$. If it exists, it can be proven to be unique up to unitary natural isomorphism. A fiber functor from a symmetric monoidal category $\langle \mathfrak{T}, \otimes, \iota; c_{X,Y} \rangle$ is *symmetric* if it maps the symmetry of $\mathfrak{T}$ to the canonical symmetry of $\mathsf{Vec}_{\mathbf{C}}$, $E(c_{X,Y}) = \sum_{E(X),E(Y)}$. Thus, 7.3.3 characterizes compact group duals within abstract categories, wheras Tannaka–Krein theory characterizes compact group duals within categories of finite dimensional Hilbert spaces.

Assuming $\mathfrak{T}$ is equipped with a $*$–preserving symmetric fiber functor E, denote G_E the set of unitary monoidal natural transformations from E to itself. G_E has the structure of a compact topological group with the identical natural transformation as unit. By definition, G_E acts on spaces $E(X)$ for all $X \in \mathfrak{T}$. Taking an element $g \in G_E$, the representation $\pi_X(g)$ on $E(X)$ is the X–component of the natural transformation g, $\pi_X(g) = g_X : E(X) \to E(X)$. Tannaka–Krein reconstruction is essentially the following

PROPOSITION 7.3.4. *Let* $\mathfrak{T}$ *be a symmetric monoidal* $*$*–category with* $\mathrm{End}\,\iota = \mathbf{C}\,1$*, equipped with a symmetric* $*$*–preserving fiber functor* $E : \mathfrak{T} \to \mathsf{Vec}_{\mathbf{C}}$*. With* G_E *defined as above,* $F : \mathfrak{T} \to \mathsf{Rep}_f(G_E)$*,* $X \mapsto (E(X), \pi_X)$ *is a faithful symmetric* $*$*–functor to the finite–dimensional representations of* G_E*, such that* $U \circ F = E$ *where* $U : \mathsf{Rep}_f(G_E) \to \mathsf{Vec}_{\mathbf{C}}$ *is the canonical forgetful functor* $(V, \pi) \mapsto V$*; moreover,* F *is an equivalence of symmetric monoidal categories.*

The uniqueness of fiber functors implies the uniqueness of the reconstructed group: if $E_1, E_2 : \mathfrak{T} \to \mathsf{Vec}_{\mathbf{C}}$ are two symmetric $*$–preserving fiber functors then $E_1 \cong E_2$ and hence $G_{E_1} \cong G_{E_2}$. The Tannaka–Krein theorem can be generalized to Hopf algebras and quantum groupoids [**88**], [**42**], [**71**].

3.2. The cross product construction. The reconstruction of field system in AQFT is a more difficult problem in that no fiber functor is available *a priori*. Interestingly, it can nevertheless be solved, and the fiber functor arises as a by–product of the field algebra construction. However, since the gauge group and the field algebra are constructed simultaneously, the classical Tannaka–Krein theory is circumvented altogether.

The key idea is to costruct the field algebra as a cross product C^*–algebra $\mathcal{F} = \mathfrak{A} \rtimes \mathfrak{T}$ of the observable algebra and the DHR category. In the case that $\mathfrak{T}$ maybe identified as a compact

[6]In the discussion of DR duality, an abstract category satisfying the requirements of 7.3.3 shall be denoted $\mathfrak{T}$ in accordance with the literature. In relation to AQFT, it should be thought of as the DHR category Δ.

group dual $\Delta \cong \mathsf{Rep}G$, and $\mathcal{F}$ will carry an action of the gauge group such that the invariant subalgebra is just $\mathfrak{A}$. The main result is the following:

THEOREM 7.3.5. *Let A be a C^*–algebra with trivial center and $\mathfrak{T} \subseteq \mathrm{End}\,\mathfrak{A}$ a full monoidal subcategory with subobjects, direct sums, conjugates and $(\mathfrak{T}, \varepsilon)$ symmetric. Then there is a unique C^*–algebra $\mathcal{F}$ and a group $G \subseteq \mathrm{Aut}(\mathcal{F})$ such that*

(1) *$\mathfrak{A} = \mathcal{F}^G$ is the G–invariant subalgebra;*
(2) *the extension $\mathfrak{A} \subseteq \mathcal{F}$ is irreducible, i.e. $\mathfrak{A}' \cap \mathcal{F} = \mathbf{C}\,1$;*
(3) *$\mathcal{F}$ is generated as a C^*–algebra by $\mathfrak{A}$ and the $\mathcal{H}_\rho$, where*

$$\mathcal{H}_\rho := \{\psi \in \mathcal{F} \mid \psi A = \rho(A)\psi,\ \forall A \in \mathfrak{A}\}$$

are finite dimensional Hilbert spaces in $\mathcal{F}$ for al objects $\rho \in \mathfrak{T}$;
(4) *$\varepsilon(\rho, \sigma) = \theta_{H_\rho, H_\sigma}$ for each pair of objects $\rho, \sigma \in \mathfrak{T}$, where $\theta_{H,K}$ is the unique unitary in $\mathcal{F}$ such that $\theta_{H,K}\psi\psi' = \psi'\psi$ for all $\psi \in H$, $\psi' \in K$.*

With the conditions (1) through (4) above, $G = \mathsf{Fix}(\mathfrak{A}) \subseteq \mathrm{Aut}(\mathcal{F})$ is compact in the strong topology. Moreover, the system $(\mathcal{F}, G)$ is unique up to $*$–isomorphism.

Assuming the field algebra structure promised by the above theorem, one can see at this point how the missing fiber functor emerges. The Hilbert spaces $\mathcal{H}_\rho$ are naturally interpreted as multiplets of field operators creating charge ρ; the endomorphisms of $\mathfrak{A}$ become *inner* in the extended field algebra $\mathfrak{A} \rtimes \mathfrak{T}$. Restricting the action of G to $\mathcal{H}_\rho \subseteq \mathcal{F}$, one obtains a representation $(\mathcal{H}_\rho, D_\rho)$ of G. Intertwiners $(D_\rho, \mathcal{H}_\rho) \to (D_{\rho'}, \mathcal{H}_{\rho'})$ are in one–to–one correspondance with arrows $\rho \to \rho'$; the mapping between Hom–spaces is given by

$$\mathrm{Hom}_{\mathfrak{T}}(\rho, \rho') \to \mathrm{Hom}_{\mathsf{Rep}(G)}(D_\rho, D_{\rho'}) \tag{7.3.6}$$

$$(\rho \xrightarrow{T} \rho') \mapsto \{T \cdot \_ : \mathcal{H}_\rho \ni \psi \to T\psi \in \mathcal{H}_{\rho'}\}. \tag{7.3.7}$$

Hence, there is an isomorphism of $\mathfrak{T}$ with a concrete representation category $\mathsf{Rep}(G)$.

The proof and detailed analysis of Theorem 7.3.5 can be found in the classic papers [**32**], [**33**] and [**34**], we only sketch the main points. First, it suffices to construct the cross product of the observable algebra with a single endomorphism $\rho \in \mathfrak{T}$, in the sense that every category fulfilling the requirements of 7.3.3 is shown to be an inductive limit $\mathfrak{T} = \varinjlim \mathfrak{T}_\rho$ of full subcategories each of which is generated by a single object ρ. Denoting $\mathfrak{T}_\rho \overset{full}{\subseteq} \mathfrak{T}$ the full subcategory on objects $\{\iota, \rho, \rho^2, \rho^3, ...\}$, the cross product $\mathcal{F}^\rho = \mathfrak{A} \rtimes \mathfrak{T}_\rho$ incorporates a single multiplet of field operators creating charge ρ. The full gauge group is then obtained from the $G_\rho = \mathrm{Aut}_{\mathfrak{A}}\mathfrak{A} \rtimes \mathfrak{T}_\rho$ as the projective limit $G = \varprojlim G\rho$.

The key idea in the construction of the cross product $\mathcal{F}^\rho = \mathfrak{A} \rtimes \mathfrak{T}_\rho$ with a single object is to embed $\mathfrak{T}_\rho$ into an appropriate C^*–algebra, the *Cuntz algebra* $\mathcal{O}_\rho$ and to construct $\mathfrak{A} \rtimes \mathcal{O}_\rho$. The details of this procedure lie outside our scope (see [**33**]), but we sketch how the Cuntz algebra structure can be identified within $\mathcal{F}^\rho$.

Consider a d–dimensional Hilbert space of isometries $\mathcal{H}_\rho$ corresponding to the sector ρ. The elements $\{\psi_k\}_{k=1\ldots d}$ spanning $\mathcal{H}_\rho$ may then be chosen to satisfy the relations

$$\psi_i^*\psi_k = \delta_{ik}\,\mathbf{1} \;\; (i,k=1,...,d) \;\text{ and } \sum_{k=1}^{d}\psi_k\psi_k^* = \mathbf{1}. \tag{7.3.8}$$

There is a unique norm on the $*$–algebra generated by $\{\mathbf{1},\psi_k,\psi_k^*\}$; the Cuntz algebra is the C^*–algebra obtained by completion in this norm and is denoted $\mathcal{O}_d$. The scalar product is defined, by 7.3.8 as

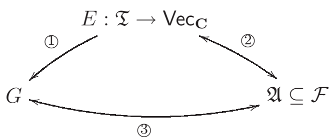
$$\psi'^*\psi = \langle\psi',\psi\rangle\,\mathbf{1}$$

Furthermore, every Cuntz algebra $\mathcal{O}_d$ comes equipped with a canonical endomorphism $\sigma_{\mathcal{O}_d}$ which, in the case of a field multiplet of charge ρ, reproduces the endomorphism ρ. Taking an element $\psi\in\mathcal{H}_\rho$ and an arbitrary algebra element $C\in\mathcal{O}_d$, we find

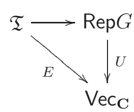
$$\psi\, C = \sigma_{\mathcal{O}_d}(C)\,\psi, \tag{7.3.9}$$

where $\sigma_{\mathcal{O}_d}\in \mathrm{End}\,\mathcal{O}_d$ is the inner endomorphism $\sigma_{\mathcal{O}_d}(C)=\sum_{k=1}^{d}\psi_k C\psi_k^*$. The canonical endomorphism is independent of the choice of basis $\{\psi_k\}$ and clearly, $\sigma_{\mathcal{O}_d}=\rho$.

It seems useful to separate the problems raised –and solved– by the DR theorem according to the following schematic[7].



① represents the setting of Tannaka Reconstruction: determining from a category $\mathfrak{T}$ and fiber functor $E:\mathfrak{T}\to\mathsf{Vec}_{\mathbf{C}}$ a group (resp. Hopf algebra, quantum groupoid) such that E factorizes through an equivalence of categories $\mathfrak{T}\xrightarrow{\sim}\mathsf{Rep}G$ and U is the canonical forgetful functor $U:\mathsf{Rep}G\to\mathsf{Vec}_{\mathbf{C}}$, $(V_a,D_a)\mapsto V_a$.



② represents a two–way relation between the field algebra extension and the fiber functor. Given a field algebra, identifying the subspaces $\mathcal{H}_\rho\subseteq\mathcal{F}$ of 'operators of charge ρ', as explained

[7]The interrelations represented by the arrows should be read in an informal sense. They do not necessarily refer to mathematically rigorous results, but should be seen as plausible conjectures

earlier, defines a fiber functor

$$\operatorname{End}\mathfrak{A} \supset \mathfrak{T} \to \mathsf{Vec}_{\mathbf{C}}$$
$$\rho \mapsto \mathcal{H}_\rho$$

In the reverse direction, given a fiber functor, the field algebra arises as a cross product $\mathcal{F} = \mathfrak{A} \rtimes E(P)$, where $P \in \mathfrak{T}$ is a distinguished (generating) object. This connection shall be examined in the next chapter

③ represents a two–way direction between the gauge group G and the field algebra extension $\mathfrak{A} \subseteq \mathcal{F}$: this is the setting of (noncommutative) Galois theory.

4. Field algebra from a fiber functor

To investigate the connection between fiber functors on the DHR category and Galois extensions of the observable algebra, we develop the example of a fiber functor which arises from a generating universal object in the category. This is in fact the setting of the Doplicher–Roberts theory. Recall, that if one assumes finitely many simple objects, the existence of such a (simple) generating object is automatic.

We make the following assumptions.

Assumption 7.4.1. *Let A be a noncommutative k–algebra. We shall consider a DHR category $\mathcal{C} \hookrightarrow \operatorname{End} A$ such that*

(1) *$\mathcal{C}$ is abelian such that every epi in $\mathcal{C}$ is split*
(2) *there is a generator $\gamma \in \mathcal{C}$ with right dual $(\bar\gamma, R, \bar R)$*

A definition and brief discussion of abelian categories may be found in [**59**], Appendix A.5 or [**53**]. Recall that in an abelian category, finite sums and finite direct products coincide. For objects $\alpha, \beta \in \mathsf{Ob}\,\mathcal{C}$ in an abelian category, the binary product or coproduct $a \prod b \cong a \coprod b \cong a \otimes b$ is called the biproduct or also the direct sum[8]. Hence, with the additional assumption that all epimorphisms should be split, we reproduce Doplicher and Roberts' requirements that the DHR category should have direct sums and subobjects.

Also, in abelian categories, generators and finite cogenerators coincide. With the assumptions of 7.4.1, an object $\gamma \in \mathsf{Ob}\,\mathcal{C}$ is a generator if every object $\alpha \in \mathsf{Ob}\,\mathcal{C}$ is contained in a finite direct sum of γ's as a direct summand. This is a strong requirement; one particular conseqence that we shall often rely on is that there exist dual bases $\{f_i, g_i\}_{i=1\ldots N}$

$$\gamma\bar\gamma\gamma\ldots\gamma\bar\gamma \xrightarrow{f_i} \gamma \xrightarrow{g_i} \gamma\bar\gamma\gamma\ldots\gamma\bar\gamma$$

such that $\sum_{i=1}^N g_i f_i = \mathrm{id}_{\gamma\bar\gamma\ldots\bar\gamma\gamma}$ for all finite products of γ's and $\bar\gamma$'s. The Hom spaces of arrows between such objects are finite direct sums of $\operatorname{Hom}(\gamma,\gamma) = \operatorname{End}\gamma$. For example, take dual bases

[8]To take a familiar example of an abelian category, the module category M_R over a ring R, the biproduct is in fact the direct sum of R–modules.

$p_k \in \mathrm{Hom}(\gamma\gamma,\gamma)$ and $q_k \in \mathrm{Hom}(\gamma,\gamma\gamma)$, satisfying $\sum_k q_k p_k = \mathrm{id}_{\gamma^2}$. Then for $f \in \mathrm{Hom}(\gamma\gamma,\gamma\gamma)$ and all q_k and p_l, the composite

$$\gamma \xrightarrow{q_k} \gamma\gamma \xrightarrow{f} \gamma\gamma \xrightarrow{p_l} \gamma$$

defines a map in $f_{k,l} \in \mathrm{Hom}(\gamma,\gamma)$. By the dual basis property of the q_k and p_k, the map f is written in components as $f = \sum_{m,n} q_n\, f_{m,n}\, p_m$, hence $\mathrm{Hom}(\gamma\gamma,\gamma\gamma) \cong \bigoplus_{m,n} \mathrm{Hom}(\gamma,\gamma)$.

We assume our category to be embedded in End A, hence every object is an endomorphism of A and the monoidal product is the composition of endomorphisms. We do not, however, assume a local structure on A which is the most important departure from the setting of AQFT. Treating the general case of a quasilocal observable algebra would require a quantum groupoid Galois theory of nets of local algebras, which is not yet available.

We shall construct the field algebra as a cross product of A and the fiber functor image of the generator $\gamma \in \mathsf{Ob}\,\mathcal{C}$. The resulting field algebra extension will then be shown to be right depth 2, hence bialgebroid Galois, by explicit construction of the depth 2 quasibases. Restricting to the special case of a semi–simple category, we also present the field algebra on generators and relations and show that it is essentially equivalent to the non–braided generalization of reduced field bundle ('exchange algebra') construction of Fredenhagen, Rehren and Schroer.

The fiber functor we consider is motivated by the following result (see [**69**]).

THEOREM 7.4.2. *For a cocomplete, closed monoidal category* $\langle \mathcal{C}, \otimes, I\rangle$ *with progenerator* P, *the functor* $\mathrm{Hom}_{\mathcal{C}}(P, P \otimes \_)$ *is a right exact monoidal embedding into* ${}_R\mathsf{M}_R$, *where* $R = \mathrm{End}\,P$.

Our aim is to construct the field algebra from the strong monoidal fiber functor $U = \mathrm{Hom}(\gamma, \_ \circ \gamma) : \Delta \to {}_R\mathsf{M}_R$, denoting $\mathrm{End}\,\gamma = R$. Besides the dual bases $p_k \in \mathrm{Hom}(\gamma\gamma,\gamma)$, $q_k \in \mathrm{Hom}(\gamma,\gamma\gamma)$, we shall need

$$\mathbf{1} \xrightarrow{u_i} \gamma \xrightarrow{e_i} \mathbf{1}$$

satisfying $\sum_i e_i u_i = \mathrm{id}_{\mathbf{1}}$. We define the cross product field algebra as follows.

THEOREM & DEFINITION 7.4.3. *Denoting* ${}_RV = U(\gamma) = \mathrm{Hom}(\gamma, {}_R\gamma\gamma)$, *define the field algebra with* $F = A \underset{R}{\otimes} V$. *The multiplication*

$$a \otimes v,\, b \otimes w \in A \underset{R}{\otimes} V : \quad (a \otimes v)(b \otimes w) = a\gamma(b)q_k \otimes p_k\gamma(w)v \tag{7.4.4}$$

and unit

$$1_F = e_i \otimes u_i \tag{7.4.5}$$

define an associative unital algebra structure on F, *and*

$$\iota : A \hookrightarrow A \underset{R}{\otimes} V, \quad a \mapsto ae_i \otimes u_i$$

is a unital algebra inclusion[9] *of the observable algebra into the field algebra, making ${}_AF_A$ an A–ring.*

PROOF. To see associativity we compute, on the one hand:

$$((a \otimes u)(b \otimes v))(c \otimes w) = \sum_i \sum_j a\gamma(b)q_j\gamma(c)q_i \underset{R}{\otimes} p_i\gamma(w)p_j\gamma(v)u =$$
$$= \sum_i \sum_j a\gamma(b)\gamma^2(c)q_jq_i \underset{R}{\otimes} p_ip_j\gamma^2(w)\gamma(v)u,$$

and on the other hand:

$$(a \otimes u)((b \otimes v)(c \otimes w)) = \sum_k \sum_l a\gamma(b\gamma(c)q_l)q_k \underset{R}{\otimes} p_k\gamma(p_l\gamma(w)v)u =$$
$$= \sum_k \sum_l a\gamma(b)\gamma^2(c)\gamma(q_l)q_k \underset{R}{\otimes} p_k\gamma(p_l)\gamma^2(w)\gamma(v)u.$$

Associativity holds if and only if

$$\sum_i \sum_j q_jq_i \underset{R}{\otimes} p_ip_j = \sum_k \sum_l \gamma(q_l)q_k \underset{R}{\otimes} p_k\gamma(q_l). \tag{7.4.6}$$

This is proved using that $q_jq_i = \sum_k \sum_l \gamma(q_l)q_kC^{ji}_{kl}$, where $C^{ji}_{kl} = p_k\gamma(p_l)q_jq_i$. C^{ji}_{kl} lies in R, and using the definition of the amalgamated tensor product over R yields 7.4.6. The unit property is an easy calculation using the definition of the multiplication,

$$(a \otimes v)(e_i \otimes u_i) = \sum_j \sum_i a \underset{R}{\otimes} \gamma(e_i)q_jp_j\gamma(u_i)v = a \underset{R}{\otimes} v$$
$$(e_l \otimes u_l)(b \otimes w) = \sum_m \sum_l be_jq_m \underset{R}{\otimes} p_m\gamma(w)u_j = b \underset{R}{\otimes} e_j\gamma(w)u_j = b \underset{R}{\otimes} w$$

It is easily checked that ι is a unital algebra morphism:

$$\iota(a)\iota(b) = (ae_i \otimes u_i)(be_j \otimes u_j) = ae_i\gamma(be_j)q_k \otimes p_k\gamma(u_j)u_i =$$
$$= ae_i\gamma(b)\gamma(e_j)q_kp_k\gamma(u_j) \underset{R}{\otimes} u_i = abe_i\gamma(e_j)\gamma(u_j) \underset{R}{\otimes} u_i = abe_i \underset{R}{\otimes} u_i = \iota(ab),$$

where in the second equality we used that $p_k\gamma(u_j) \in R$, and the unit of A is mapped onto the unit of F, $1_F = \iota(1_A) = 1_A\, e_i \underset{R}{\otimes} u_i$.

F is an A–ring with the inclusion $\iota : A \to F$. The corresponding A–bimodule structure is:

$$a' \cdot (a \underset{R}{\otimes} v) = (a'e_i \otimes u_i)(a \otimes v) = a'e_i\gamma(a)q_k \underset{R}{\otimes} p_k\gamma(v)u_i = \tag{7.4.7}$$
$$= a'a \underset{R}{\otimes} (e_iq_kp_ku_i)v = a'a \underset{R}{\otimes} v$$

$$(a \otimes v) \cdot a'' = (a \otimes v)(a''e_j \otimes u_j) = a\gamma(a''e_j)q_k \underset{R}{\otimes} p_k\gamma(u_j)v = \tag{7.4.8}$$
$$= a\gamma(a'')\gamma(e_j)q_kp_k\gamma(u_j) \underset{R}{\otimes} v = a\gamma(a'') \underset{R}{\otimes} v.$$

[9] Note that $u_i \in \mathrm{Hom}(1, \gamma) \subseteq \mathrm{Hom}(\gamma, \gamma^2)$

We only check that multiplication is a right A–module map:

$$(a \underset{R}{\otimes} v)((b \underset{R}{\otimes} w) \cdot a') = (a \underset{R}{\otimes} v)(b\gamma(a') \underset{R}{\otimes} w) = a\gamma(b)\gamma^2(a')q_k \underset{R}{\otimes} p_k\gamma(w)v$$
$$= a\gamma(b)q_k\gamma(a') \underset{R}{\otimes} p_k\gamma(w)v = ((a \underset{R}{\otimes} v)(b \underset{R}{\otimes} w)) \cdot a'.$$

□

Knowing the A–actions on F explicitly, we can identify the subspaces F^ρ of charge creation operators within F. The $\mathcal{F}^\rho$ are the analogues of the internal Hilbert spaces $\mathcal{H}_\rho$ in the setting of Doplicher–Roberts. An element $a \otimes v \in A \underset{R}{\otimes} V$ creates charge $\alpha \in \mathcal{C}$ if $(a \otimes v) \cdot b = \alpha(b) \cdot (a \otimes v)$. Substituting the A–actions above:

(7.4.9) $$(a \otimes v) \in F^\rho \quad \Leftrightarrow \quad a\gamma(b) \otimes v = \alpha(b)a \otimes v \quad \text{for all } \; b \in A$$

If ${}_RV$ is finitely generated projective over R, with dual basis $\varphi^i \otimes v_i \in V^* \underset{R}{\otimes} V$, then any element $v \in V$ can be written $v = \sum_i \varphi^i(v)v_i$, hence $a \otimes v = \sum_i a\varphi^i(v) \underset{R}{\otimes} v_i$. Applying φ^i to 7.4.9, we find

$$a\varphi^i(v)\gamma(b) = \alpha(b)a\varphi^i(v),$$

meaning that $a\varphi^i(v) \in \mathrm{Hom}(\gamma, \alpha)$ for all $(a \otimes v) \in F^\alpha$, and

(7.4.10) $$F^\alpha \simeq \mathrm{Hom}(\gamma, \alpha) \underset{R}{\otimes} V$$

Similarly, we can introduce the charge annihilation operators. An element $a \otimes v \in A \underset{R}{\otimes} V$ annihilates charge $\beta \in \mathcal{C}$ if $b \cdot (a \otimes v) = (a \otimes v) \cdot \beta(b)$ (alternatively, it creates charge β from the right). Using 7.4.7, we have

(7.4.11) $$(a \otimes v) \in \bar{F}^\beta \quad \Leftrightarrow \quad ca \otimes v = a\gamma\beta(c) \otimes v \quad \text{for all } \; c \in A$$

Appealing again to the finitely generated projectivity of ${}_RV$, we have

(7.4.12) $$\bar{F}^\beta \simeq \mathrm{Hom}(\gamma\beta, \mathbf{1}) \underset{R}{\otimes} V \simeq \mathrm{Hom}(\gamma, \bar{\beta}) \underset{R}{\otimes} V,$$

where $\bar{\beta}$ is the *right* dual of β.

We can now establish the inverse relation between the fiber functor and the field algebra. As proposed earlier, the fiber functor should associate to any sector $\rho \in \mathsf{Ob}\,\mathcal{C}$ the subalgebra $\mathcal{F}^\rho \subset \mathcal{F}$ of field operators creating charge ρ.

PROPOSITION 7.4.13. *Let* $U_F : \mathcal{C} \to {}_R\mathsf{M}_R$ *be the functor with object–map* $\alpha \mapsto F^\alpha$ *and arrow–map* $\langle \alpha \xrightarrow{T} \beta \rangle \mapsto T \cdot \_ : F^\alpha \to F^\beta$, *where* $R = A' \cap F$ *is the centralizer of* A *in* F. *Then* $\langle U_F, U_{\alpha,\beta}, U_0 \rangle$ *is strong monoidal with the structure maps*

(7.4.14) $$U_{\alpha,\beta} : U(\alpha) \underset{R}{\otimes} U(\beta) \to U(\alpha\beta), \quad F^\alpha \underset{R}{\otimes} F^\beta \mapsto F^\alpha F^\beta$$

and

(7.4.15) $$U_0 : R \to U_F(\mathbf{1}), \quad R \simeq V(\gamma, \mathbf{1}) \underset{R}{\otimes} V.$$

Furthermore, $R = \operatorname{End}\gamma$ *and* $U_F \cong U = \operatorname{Hom}(\gamma, \_ \circ \gamma)$.

Proof. For all $\alpha \in \mathcal{C}$, F^α is obviously in ${}_R\mathsf{M}_R$. For any $T \in \operatorname{Hom}(\alpha, \beta)$, $U_F(T) : \operatorname{Hom}(\gamma, \alpha) \underset{R}{\otimes} V \to \operatorname{Hom}(\gamma, \beta) \underset{R}{\otimes} V$, $a \otimes v \mapsto Ta \otimes V$ is an ${}_R\mathsf{M}_R$–map. It is a left R–module map precisely because of the centralizer property. The composition of arrows corresponds to the multiplication of intertwiners in A.

As for the monoidal structure, U_0 is simply the isomorphism between R and $V(\gamma, \mathbf{1}) \underset{R}{\otimes} V$. Indeed, $a \otimes v \in A' \cap (A \underset{R}{\otimes} V)$ means $(a \otimes v) \cdot b = b \cdot (a \otimes v)$ for all $b \in A$. Using that ${}_RV$ if finitely generated projective,

$$ba\varphi^i(v) \otimes v_i = a\varphi^i(v)\gamma(b) \otimes v_i$$

hence, by the familiar argument,

$$A' \cap (A \underset{R}{\otimes} V) \simeq \operatorname{Hom}(\gamma, \mathbf{1}) \underset{R}{\otimes} V = U_F(\mathbf{1})$$

Take $(a^\alpha \otimes v) \in F^\alpha$, $(a^\beta \otimes w) \in F^\beta$ with $a^\alpha \in \operatorname{Hom}(\gamma, \alpha)$ and $a^\beta \in \operatorname{Hom}(\gamma, \beta)$. Using 7.4.4, we find

$$(a^\alpha \otimes v)(a^\beta \otimes w) = a^\alpha\gamma(a^\beta)q_k \underset{R}{\otimes} p_k\gamma(w)v \in \operatorname{Hom}(\gamma, \alpha\beta) \underset{R}{\otimes} V,$$

i.e. $U_{\alpha,\beta}(F^\alpha \underset{R}{\otimes} F^\beta) = F^\alpha F^\beta \subset F^{\alpha\beta} = F^{\alpha\beta}$. Introducing bases $\{a_i^\alpha\}_i$ in $\operatorname{Hom}(\gamma, \alpha)$ and $\{a_j^\beta\}_j$ in $\operatorname{Hom}(\gamma, \beta)$, respectively, calculate

$$U_{\alpha,\beta}((a_i^\alpha \otimes v)(a_j^\beta \otimes w)) = a_i^\alpha\gamma(a_j^\beta)q_k \underset{R}{\otimes} p_k\gamma(w)v$$

Observe that $\gamma \xrightarrow{q_k} \gamma^2 \xrightarrow{\gamma(a_j^\beta)} \gamma\beta \xrightarrow{a_i^\alpha} \alpha\beta$ is a basis in $\operatorname{Hom}(\gamma, \alpha\beta)$, hence

(7.4.16) $$F^\alpha F^\beta = F^{\alpha\beta},$$

meaning that $U_{\alpha,\beta}$ is an isomorphism, i.e. U_F is strong monoidal. The coherence diagrams required of a monoidal functor are equivalent to the associativity and unitalness of F.

The natural isomorphism $\tau : \operatorname{Hom}(\gamma, \gamma\gamma) \xrightarrow{\sim} U_F$,

$$\begin{array}{ccc} \operatorname{Hom}(\gamma, \alpha\gamma) & \xrightarrow{\tau_\alpha} & \operatorname{Hom}(\gamma, \alpha) \otimes_R \operatorname{Hom}(\gamma, \gamma\gamma) \\ {\scriptstyle U(f)}\downarrow & & \downarrow{\scriptstyle U_F(f)} \\ \operatorname{Hom}(\gamma, \beta\gamma) & \xrightarrow[\tau_\beta]{} & \operatorname{Hom}(\gamma, \beta) \otimes_R \operatorname{Hom}(\gamma, \gamma\gamma) \end{array}$$

is given by

$$\tau_\alpha : \operatorname{Hom}(\gamma, \alpha) \underset{R}{\otimes} \operatorname{Hom}(\gamma, \gamma\gamma) \to \operatorname{Hom}(\gamma, \alpha\gamma), \quad f \underset{R}{\otimes} v \mapsto fv$$

with inverse

$$\tau_\alpha^{-1} : \operatorname{Hom}(\gamma, \alpha\gamma) \to \operatorname{Hom}(\gamma, \alpha) \underset{R}{\otimes} \operatorname{Hom}(\gamma, \gamma\gamma), \quad g \mapsto q_i^\alpha \underset{R}{\otimes} p_i^\alpha g$$

where $\{q_i^\alpha, p_i^\alpha\}$ is the pair of dual bases $\alpha \xrightarrow{p_i^\alpha} \gamma \xrightarrow{q_i^\alpha} \alpha$, satisfying $\sum_i q_i^\alpha p_i^\alpha = \mathrm{id}_\alpha$. They are clearly inverses,

$$\tau_\alpha^{-1} \circ \tau_\alpha(f \underset{R}{\otimes} v) = q_i^\alpha \underset{R}{\otimes} (p_i^\alpha f)v = q_i^\alpha p_i^\alpha f \underset{R}{\otimes} v = f \underset{R}{\otimes} v,$$

and

$$\tau_\alpha \circ \tau_\alpha^{-1}(g) = q_i^\alpha p_i^\alpha g = g.$$

□

Next, we look at the extension $A \subset F$ and prove that it is depth 2 in the sense of [**47**]. From 4.4.3, it will follow that the field algebra extension is bialgebroid Galois. We shall construct the depth 2 quasibases explicitly as elements in $\mathrm{End}\,_A F_A$ and $(F \underset{A}{\otimes} F)^A$, respectively. First, a useful

LEMMA 7.4.17. *As k–modules,*

$$\mathrm{End}\,_A F_A \simeq \mathrm{End}\,_R V \quad \textit{and} \tag{7.4.18}$$

$$(F \underset{A}{\otimes} F)^A \simeq (\mathrm{Hom}(\gamma^2, \mathbf{1}) \underset{R}{\otimes} V) \underset{R}{\otimes} V \tag{7.4.19}$$

PROOF. Denote $\lambda_R : R \underset{R}{\otimes} M \xrightarrow{\sim} M$ the natural isomorphism $\{r \otimes m \mapsto r \cdot m\}$ in ${}_R\mathsf{M}_R$, with inverse $\{m \mapsto 1 \otimes m\}$. To prove 7.4.18, define the isomorphism

$$\rho : \mathrm{End}\,_R V \xrightarrow{\sim} \mathrm{End}_{A-A}(A \underset{R}{\otimes} V), \;\; \beta \mapsto \{a \otimes v \mapsto a \otimes \beta(v)\} \tag{7.4.20}$$

Obviously, $\rho(\beta)$ is an A–bimodule map. It is easily seen that

$$\rho^{-1} : \mathrm{End}_{A-A}(A \underset{R}{\otimes} V) \xrightarrow{\sim} \mathrm{End}\,_R V, \;\; f \mapsto \{v \mapsto \lambda_R \circ f(1_A \underset{R}{\otimes} v)\} \tag{7.4.21}$$

is an inverse for ρ. From the A–bimodule property of f, it follows that $f(r \otimes v) \in R \underset{R}{\otimes} V$ for any $r \in R$ and $v \in V$, hence also $f(1_A \otimes v) \in R \underset{R}{\otimes} V$ so that the definition makes sense. Moreover, $\rho^{-1}(f)$ is left R–linear, $\rho^{-1}(f)(r \cdot v) = \lambda_R \circ f(1_A \underset{R}{\otimes} r \cdot v) = \lambda_R(r \cdot f(1_A \underset{R}{\otimes} v)) = r \cdot \rho^{-1}(f)(v)$.

We check that ρ and ρ^{-1} are indeed inverses, i.e.

$$(\rho^{-1} \circ \rho)(\beta) = \{v \mapsto \lambda_R \circ (1_A \underset{R}{\otimes} \beta(v)) = \beta(v)\}$$

and

$$(\rho \circ \rho^{-1})(f) = \{a \otimes v \mapsto a \underset{R}{\otimes} \lambda_R \circ f(1_A \underset{R}{\otimes} v) = a \cdot f(1_A \underset{R}{\otimes} v) = f(a \underset{R}{\otimes} v)\}.$$

To prove 7.4.19, we first observe that $F \underset{A}{\otimes} F \cong (A \underset{R}{\otimes} V) \underset{R}{\otimes} V$ in ${}_R\mathsf{M}_R$, by the isomorphism

$$\rho' : (A \underset{R}{\otimes} V) \underset{A}{\otimes} (A \underset{R}{\otimes} V) \to (A \underset{R}{\otimes} V) \underset{R}{\otimes} V \tag{7.4.22}$$

$$(a \underset{R}{\otimes} v) \underset{A}{\otimes} (b \underset{R}{\otimes} w) = (a\gamma(b) \underset{R}{\otimes} v) \underset{R}{\otimes} w$$

with inverse $(\rho')^{-1}((a \underset{R}{\otimes} v) \underset{R}{\otimes} w) = (a \underset{R}{\otimes} v) \underset{A}{\otimes} (1_A \underset{R}{\otimes} w)$. The placement of parentheses is important: $(A \underset{R}{\otimes} V) \underset{R}{\otimes} V$ is the kernel of maps $((a \cdot r \underset{R}{\otimes} v) \underset{R}{\otimes} w - (a \underset{R}{\otimes} r \cdot v) \underset{R}{\otimes} w)$ and $((a \underset{R}{\otimes} v) \underset{R}{\otimes} r \cdot w - (a \cdot \gamma(r) \underset{R}{\otimes} v) \underset{R}{\otimes} w)$. With this in mind, it is easy to check that ρ' and $(\rho')^{-1}$ are well–defined R–bimodule maps.

$(A \underset{R}{\otimes} V) \underset{R}{\otimes} V$ is an A–bimodule with the actions $b \cdot ((a \underset{R}{\otimes} v) \underset{R}{\otimes} w) \cdot c = (ba\gamma^2(c) \underset{R}{\otimes} v) \underset{R}{\otimes} w$. Assuming ${}_R(V \underset{R}{\otimes} V)$ is finitely generated projective with dual bases $\{\Phi^{ij}, v_i \otimes v_j\}$, we may write elements of $(A \underset{R}{\otimes} V) \underset{R}{\otimes} V$ as $(a \otimes v) \otimes w = (a\Phi^{ij}(v \underset{R}{\otimes} w) \underset{R}{\otimes} v_i) \underset{R}{\otimes} w_j$. Then, $(a \underset{R}{\otimes} v) \underset{R}{\otimes} w \in (F \underset{A}{\otimes} F)^A$ precisely if

$$b(a\, \Phi^{ij}(v \underset{R}{\otimes} w)) \underset{R}{\otimes} v_i \underset{R}{\otimes} w_j = ((a\, \Phi^{ij}(v \underset{R}{\otimes} w))\gamma^2(b) \underset{R}{\otimes} v_i) \underset{R}{\otimes} w_j, \tag{7.4.23}$$

i.e. for all $\Phi \in (V \underset{R}{\otimes} V)^*$, $a\Phi(v \underset{R}{\otimes} w) \in \mathrm{Hom}(\gamma^2, \mathbf{1})$, proving 7.4.19. □

Recall that $\iota : A \hookrightarrow F = A \underset{R}{\otimes} V$ is a *right* depth 2 extension if there exists a pair of dual D2 quasibases $\{\gamma^i\}_{i=1}^m \in \mathrm{End}\,{}_AF_A$, $\{c_i^1 \otimes c_i^2\}_{i=1}^m \in (F \underset{A}{\otimes} F)^A$, such that

$$\forall f \in F: \quad \gamma^i(f)\, c_i^1 \underset{A}{\otimes} c_i^2 = \mathbf{1}_F \underset{A}{\otimes} f, \tag{7.4.24}$$

and it is a *left* depth 2 extension if there exists a pair of dual D2 quasibases $\{\beta^i\}_{i=1}^m \in \mathrm{End}\,{}_AF_A$, $\{b_i^1 \otimes b_i^2\}_{i=1}^m \in (F \underset{A}{\otimes} F)^A$, such that

$$\forall f \in F: \quad b_1^i \underset{A}{\otimes} b_2^i \beta^i(f) = f \underset{A}{\otimes} \mathbf{1}_F \tag{7.4.25}$$

Denote $\{(a_i \underset{R}{\otimes} r_i^1) \underset{R}{\otimes} r_i^2\}_{i=1\ldots m} \in (A \underset{R}{\otimes} V) \underset{R}{\otimes} V$ and $\{\rho^i\}_{i=1\ldots m} \in \mathrm{End}\, V$ the images of the right depth 2 quasibases under the isomorphsims $(\rho')^{-1} : (F \underset{A}{\otimes} F)^A \xrightarrow{\sim} (A \underset{R}{\otimes} V) \underset{R}{\otimes} V$ and $\rho^{-1} : \mathrm{End}\,{}_AF_A \xrightarrow{\sim} \mathrm{End}\, V$. Then, using 7.4.4 and 7.4.22, the depth 2 condition 7.4.24 takes the following form.

$$\forall (a \underset{R}{\otimes} v) \in A \underset{R}{\otimes} V: \quad (a\gamma(a_i)q_k \underset{R}{\otimes} p_k\gamma(r_i^1)\, \rho^i(v)) \underset{R}{\otimes} r_i^2 = (ae_i \underset{R}{\otimes} u_i) \underset{R}{\otimes} v \tag{7.4.26}$$

We shall make use of two further pairs of dual bases,

$$(1) \qquad \gamma\bar\gamma \xrightarrow{\bar p_k} \gamma \xrightarrow{\bar q_k} \gamma\bar\gamma, \quad \textstyle\sum_k \bar q_k \bar p_k = \mathrm{id}_{\gamma\bar\gamma}$$

and

$$(2) \qquad \bar\gamma\gamma\bar\gamma \xrightarrow{s_l} \bar\gamma \xrightarrow{t_l} \bar\gamma\gamma\bar\gamma, \quad \textstyle\sum_l t_l s_l = \mathrm{id}_{\bar\gamma\gamma\bar\gamma}.$$

The second condition in particular means that that $\gamma \in \mathcal{C} \subset \mathrm{End}\, A$ is a depth 2 arrow in the sense of [**82**] and [**12**].

We shall obtain the depth 2 quasibases $\{\rho^i\}_{i=1\dots m}$ and $\{(a_i \underset{R}{\otimes} r_i^1) \underset{R}{\otimes} r_i^2\}_{i=1\dots m}$ as the images of the abstract depth 2 quasibases $\{t_l\}_{i=1\dots m}$ and $\{s_l\}_{i=1\dots m}$ under isomorphisms $\mathrm{Hom}(\bar\gamma, \bar\gamma\gamma\bar\gamma) \to \mathrm{End}\,_R V$ and $\mathrm{Hom}(\bar\gamma\gamma\bar\gamma, \bar\gamma) \to (\mathrm{Hom}(\gamma^2, \mathbf{1}) \underset{R}{\otimes} V) \underset{R}{\otimes} V$, respectively.

Proposition 7.4.27. *The maps*

$$\begin{aligned}\kappa : \mathrm{Hom}(\bar\gamma, \bar\gamma\gamma\bar\gamma) &\xrightarrow{\sim} \mathrm{End}\,_R V \\ t &\mapsto \{\beta : v \mapsto \gamma(R) v R \gamma(t) \gamma(\bar R)\}\end{aligned} \tag{7.4.28}$$

$$\begin{aligned}\kappa' : \mathrm{Hom}(\bar\gamma\gamma\bar\gamma, \bar\gamma) &\xrightarrow{\sim} (\mathrm{Hom}(\gamma^2, \mathbf{1}) \underset{R}{\otimes} V) \underset{R}{\otimes} V \\ s &\mapsto \left(R\gamma(s)\bar q_k \gamma(\bar q_l) \underset{R}{\otimes} \bar p_k \gamma(\bar R) \right) \underset{R}{\otimes} \bar p_l \gamma(\bar R)\end{aligned} \tag{7.4.29}$$

are isomorphisms.

Proof. Fig. 1 shows the pictorial representation of the maps κ and κ', in the standard notation for tensor categories (see e.g. [**59**], Appendix): vertical lines represent objects, juxtaposition of lines denotes monoidal product and intertwiners are represented as boxes with in– and outgoing lines the source and target, respectively (to be read from top to bottom).

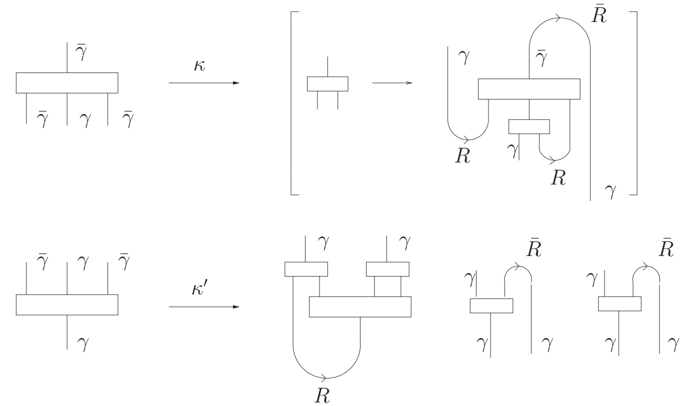


Figure 1. Graphical representation of the maps κ and κ'

The map

$$\begin{aligned}(\kappa')^{-1} : (\mathrm{Hom}(\gamma^2, \mathbf{1}) \underset{R}{\otimes} V) \underset{R}{\otimes} V &\to \mathrm{Hom}(\bar\gamma\gamma\bar\gamma, \bar\gamma) \\ (a \underset{R}{\otimes} v) \underset{R}{\otimes} w &\mapsto \bar\gamma(a)\bar\gamma\gamma^2(R)\bar\gamma\gamma(w)\bar\gamma\gamma(R)\bar\gamma(v)\bar R\end{aligned} \tag{7.4.30}$$

is an inverse for κ. Indeed:

$$\begin{aligned}\left(\kappa' \circ (\kappa')^{-1}\right)((a \underset{R}{\otimes} v) \underset{R}{\otimes} w) &= \kappa' \left(\bar\gamma(a)\bar\gamma\gamma^2(R)\bar\gamma\gamma(w)\bar\gamma\gamma(R)\bar\gamma(v)\bar R\right)\\ &= R\ \left[\gamma\bar\gamma(a)\gamma\bar\gamma\gamma^2(R)\gamma\bar\gamma\gamma(w)\gamma\bar\gamma\gamma(R)\gamma\bar\gamma(v)\right]\ \gamma(\bar R)\bar q_k\gamma(\bar q_l) \underset{R}{\otimes} \bar p_k\gamma(\bar R) \underset{R}{\otimes} \bar p_l\gamma(\bar R)\\ &= a\gamma^2(R)\gamma(w)\gamma(R)v\ [R\gamma(\bar R)]\ \bar q_k\gamma(\bar q_l) \underset{R}{\otimes} \bar p_k\gamma(\bar R) \underset{R}{\otimes} \bar p_l\gamma(\bar R)\\ &= a\gamma^2(R)\gamma(w)\ [\gamma(R)v\bar q_k]\ \gamma(\bar q_l) \underset{R}{\otimes} \bar p_k\gamma(\bar R) \underset{R}{\otimes} \bar p_l\gamma(\bar R)\\ &= a\gamma^2(R)\gamma(w)\gamma(\bar q_l) \underset{R}{\otimes} \gamma(R)v\ [\bar q_k\bar p_k]\ \gamma(\bar R) \underset{R}{\otimes} \bar p_l\gamma(\bar R)\\ &= a\gamma(\gamma(R)w\bar q_l) \underset{R}{\otimes} \gamma(R)v\gamma(\bar R) \underset{R}{\otimes} \bar p_l\gamma(\bar R)\\ &= a \underset{R}{\otimes} \gamma(R)v\gamma(\bar R) \underset{R}{\otimes} \gamma(R)w\ [\bar q_l\bar p_l]\ \gamma(\bar R) = (a \underset{R}{\otimes} v) \underset{R}{\otimes} w.\end{aligned}$$

In the fifth equality, we used that $\gamma(R)v\bar q_k \in \operatorname{End}\gamma = R$ and the definition of the amalgamated product $\underset{R}{\otimes}$; in the sixth equality, we used that $\sum \bar q_k\bar p_k = \mathbf{1}_{\gamma\bar\gamma}$; in the seventh, the A–bimodule structure of F 7.4.7, and the dual basis property again in the eighth. On the other hand,

$$\begin{aligned}\left((\kappa')^{-1} \circ \kappa'\right)(s) &= (\kappa')^{-1}\left[(R\gamma(s)\bar q_k\gamma(\bar q_l) \otimes \bar p_k\gamma(\bar R)) \otimes \bar p_l\gamma(\bar R)\right]\\ &= \bar\gamma\left(R\gamma(s)\bar q_k\gamma(\bar q_l)\right)\ \bar\gamma\gamma^2(R)\bar\gamma\gamma(\bar p_l\gamma(\bar R))\ \bar\gamma\gamma(R)\bar\gamma(\bar p_k\gamma(\bar R))\ \bar R\\ &= \bar\gamma(R\gamma(s)\bar q_k\gamma(\bar q_l))\bar\gamma\gamma(\bar p_l)\bar\gamma(\bar p_k)\bar R = \bar\gamma(R)\bar\gamma\gamma(s)\bar\gamma(\bar q_k)\bar\gamma\gamma(\bar q_l\bar p_l)\bar\gamma(\bar p_k)\bar R\\ &= \bar\gamma(R)\bar\gamma\gamma(s)\bar R = \bar\gamma(R)\bar R s = s.\end{aligned}$$

In the third equality, we used that $\bar p_k, \bar p_l$ are $\gamma\bar\gamma \to \gamma$ intertwiners and the rigidity relation; in the fifth the dual basis properties of $\{\bar q_k, \bar p_k\}$ and $\{\bar q_l, \bar p_l\}$ and finally, the rigidity relation once more.

The map

(7.4.31)
$$\begin{aligned}\kappa^{-1} : \operatorname{End}{}_R V &\to \operatorname{Hom}(\bar\gamma, \bar\gamma\gamma\bar\gamma)\\ \beta &\mapsto \bar\gamma(\bar q_k)\bar\gamma\gamma(R)\bar\gamma(\alpha(\bar p_k\gamma(\bar R)))\bar R\end{aligned}$$

is an inverse for κ.

$$\begin{aligned}(\kappa \circ \kappa^{-1})(\beta) &= \kappa(\bar\gamma(\bar q_k)\bar\gamma\gamma(R)\bar\gamma(\alpha(\bar p_k\gamma(\bar R)))\bar R)\\ &= \left\{v \mapsto \gamma(R)vR\ \left[\gamma\left(\bar\gamma(\bar q_k)\bar\gamma\gamma(R)\bar\gamma(\alpha(\bar p_k\gamma(\bar R)))\,\bar R\right)\right]\ \gamma(\bar R)\right\}\\ &= \left\{v \mapsto \gamma(R)vR\ \gamma\bar\gamma\left(\bar q_k\gamma(R)\,\alpha(\bar p_k\gamma(\bar R))\right)\ \gamma(\bar R)\gamma(\bar R)\right\}\\ &= \left\{v \mapsto \gamma(R)v\bar q_k\,\gamma(R)\alpha(\bar p_k\gamma(\bar R))\,[R\gamma(\bar R)]\,\gamma(\bar R)\right\}\\ &= \left\{v \mapsto \gamma(R)\,\alpha\left(\gamma(R)v\,[\bar q_k\bar p_k]\,\gamma(\bar R)\right)\ \gamma(\bar R)\right\}\\ &= \left\{v \mapsto \gamma(R)\,\alpha\left(\,[\gamma(R)\gamma^2(\bar R)]\,v\right)\ \gamma(\bar R)\right\} = [\gamma(R)\gamma^2(\bar R)]\,\alpha(v) = \alpha(v).\end{aligned}$$

In the fourth equality, we use that R is an $\gamma\bar\gamma \to \mathbf{1}$ intertwiner; in the fifth, we note that $\gamma(R)v\bar q_k \in \operatorname{End}\gamma = R$ and the R–linearity of $\alpha \in \operatorname{End}_{R-} V$. In the sixth equality, we use the dual basis property of $\{\bar q_k, \bar p_k\}$ and the rigidity axiom $R\gamma(\bar R) = \mathbf{1}_\gamma$ in the last two steps. We also have

$$\begin{aligned}(\kappa^{-1}\circ\kappa)(t) &= \kappa^{-1}\left(\{v\mapsto\gamma(R)vR\gamma(t)\gamma(\bar R)\}\right) = \bar\gamma(\bar q_k)\bar\gamma\gamma(R)\;\bar\gamma\left(\gamma(R)\bar p_k\gamma(\bar R)R\gamma(t)\gamma(\bar R)\right)\;\bar R\\ &= \bar\gamma(\bar q_k)\bar\gamma\gamma(R)\bar\gamma\gamma(R)\bar\gamma(\bar p_k\gamma(\bar R))\,\bar\gamma(R)\,\bar\gamma\gamma(t\bar R)\bar R = \bar\gamma(\bar q_k)\bar\gamma\gamma(R)\bar\gamma\gamma(R)\bar\gamma(\bar p_k\gamma(\bar R))\,[\bar\gamma(R)\bar R]\,t\bar R\\ &= \bar\gamma(\bar q_k)\bar\gamma\gamma(R)\;\bar\gamma\gamma(R)\bar\gamma(\bar p_k\gamma(\bar R))\,t\bar R = \bar\gamma(\bar q_k)\bar\gamma\gamma(R)\;\bar\gamma(\bar p_k\,[\gamma\bar\gamma(R)\gamma(\bar R)])\,t\bar R\\ &= \bar\gamma(\gamma\bar\gamma(R)\bar q_k)\bar\gamma(\bar p_k)t\bar R = \bar\gamma(\gamma\bar\gamma(R)\,[\bar q_k\bar p_k])t\bar R = t\bar\gamma(R)\bar R = t\end{aligned}$$

In the fourth equality, we use the intertwiner property of $\bar R : \mathbf{1}\to\bar R R$; we use rigidity in the fifth, seventh and last equalities; in the sixth, the intertwiner property of $\bar p_k : \gamma\bar\gamma\gamma$ and in the eighth, the dual basis property of $\{\bar q_k, \bar p_k\}$. □

Substituting the abstract depth 2 bases $\{t_i, s_i\}$ into 7.4.28 and κ', we obtain

(7.4.32) $$(a_i \underset{R}{\otimes} r_i^1)\underset{R}{\otimes} r_i^2 = \kappa'(s_i) = (R\gamma(s_i)\bar q_k\gamma(\bar q_l)\underset{R}{\otimes}\bar p_k\gamma(\bar R))\underset{R}{\otimes}\bar p_l\gamma(\bar R)$$

(7.4.33) $$\rho^i = \kappa(t_i) = \{v\mapsto\gamma(R)vR\gamma(t_i)\gamma(\bar R)\}$$

It remains to prove 7.4.26. Inserting the identity $\sum_i e_i u_i = \mathrm{id}$, and using $\sum_k q_k p_k = \mathrm{id}_{\gamma^2}$ in the second equality, we find

$$\begin{aligned}\left(a\gamma(a_i)q_k\underset{R}{\otimes}p_k\gamma(r_i^1)\,\rho^i(v)\right)\underset{R}{\otimes} r_i^2 &= \left(ae_i\underset{R}{\otimes}u_i\gamma(a_i)\,[q_k p_k]\,\gamma(r_i^1)\,\rho^i(v)\right)\underset{R}{\otimes} r_i^2\\ &= ae_i\underset{R}{\otimes}u_i\underset{R}{\otimes}\gamma(a_i)\gamma(r_i^1)\,\rho^i(v)r_i^2.\end{aligned}$$

Substituting 7.4.33 and 7.4.32, the rightmost tensorand evaluates to

$$\begin{aligned}\gamma(a_i)\gamma(r_i^1)\,\rho^i(v)r_i^2 &= \gamma(R\gamma(s_i)\bar q_k\gamma(\bar q_l))\;\gamma(\bar p_k\gamma(\bar R))\;\gamma(R)vR\gamma(t_i)\gamma(\bar R)\;\bar p_l\gamma(\bar R)\\ &= \gamma(R\gamma(s_i)\gamma\bar\gamma(\bar q_l))\,[\bar q_k\bar p_k]\,\gamma^2(\bar R)\gamma(R)vR\gamma(t_i)\gamma(\bar R)\bar p_l\gamma(\bar R)\\ &= \gamma(R)\;\gamma^2(s_i\bar\gamma(\bar q_l)\bar R)\;\gamma(R)vR\gamma(t_i)\gamma(\bar R)\;\bar p_l\gamma(\bar R)\\ &= \gamma(R)\;\gamma(R)vR\gamma(t_i)\gamma(\bar R)\;\gamma(s_i)\gamma\bar\gamma(\bar q_l)\gamma(\bar R))\;\bar p_l\gamma(\bar R)\\ &= \gamma(R)vR\gamma(t_i)\;[\gamma(\bar R\bar\gamma(R))]\;\gamma(s_i)\gamma\bar\gamma(\bar q_l)\gamma(\bar R))\;\bar p_l\gamma(\bar R)\\ &= \gamma(R)vR\;[\gamma(t_i s_i)]\;\gamma\bar\gamma(\bar q_l)\gamma(\bar R))\bar p_l\gamma(\bar R)\\ &= \gamma(R)v\bar q_l\;[R\gamma(\bar R)]\;\bar p_l\gamma(\bar R) = \gamma(R)v\;[\bar q_l\bar p_l]\;\gamma(\bar R)\\ &= \gamma(R)v\gamma(\bar R) = \gamma(R\gamma(\bar R))v = v,\end{aligned}$$

proving 7.3.5. In the fourth equality, we used the intertwining property of $\gamma(R)vR\gamma(t_i)\gamma(\bar R) : \gamma\to\gamma\gamma$, and in the fifth, the intertwining property of $vR\gamma(t_i) : \gamma\to\gamma\bar\gamma\gamma\bar\gamma$.

4.1. The reduced field bundle. We add the assumption that the category $\mathcal{C}$ is semi-simple, with finitely many simple objects. A skeleton of the subcategory of simple morphisms will be denoted $\mathcal{C}^0$, i.e. $\mathcal{C}^0$ contains one representative of each equivalence class of simple objects.

We take a more concrete approach, and let the universal object be a direct sum

$$\gamma = \bigoplus_{\alpha \in \mathcal{C}^0} n_\alpha \alpha, \tag{7.4.34}$$

containing all simple objects as direct summands with multiplicities n_α. The universal object then defines the functor

$$U : \Delta \to {}_R\mathsf{M}_R, \;\; U(\rho) = \mathrm{Hom}(\gamma, \gamma\rho).$$

Clearly, $R = \mathrm{End}\,\gamma = \bigoplus_\alpha \mathsf{Mat}_{n_\alpha}(\mathbf{C})$. If we choose $n_\alpha = 1$ for all $\alpha \in \mathcal{C}^0$, then R would be a commutative ring generated by the idempotent projections on the simple direct summands. We shall keep the multiplicities at our disposal, and work over the bimodule category over the noncommutative ring R.

The monoidal structure of U is the triple $\langle U, U_{\rho,\rho'}, U_0 \rangle$, where

$$\begin{aligned} & U_{\rho,\rho'} : U(\rho) \underset{R}{\otimes} U(\rho') \to U(\rho\rho') \\ & U_{\rho,\rho'}(T \underset{R}{\otimes} S) = T \circ S = TS : P \to P\rho' \to P\rho\rho' \end{aligned} \tag{7.4.35}$$

is a natural map, and

$$U_0 : R \to U(\mathbf{1}), \;\; R \simeq \mathrm{Hom}(\gamma, \gamma)$$

is an isomorphism (in fact, the definition of R). The semi–simplicity of $\mathcal{C}$ allows us to coordinatize functors and natural transformations in the distinguished basis of simple objects.

Considering first a simple object $\delta \in \mathcal{C}^0$, $U(\rho)$ may be decomposesed as

$$U(\delta) = \mathrm{Hom}(\gamma, \gamma\delta) = \bigoplus_{\alpha,\beta} n_\alpha n_\beta \,\mathrm{Hom}(\alpha, \beta\delta) \tag{7.4.36}$$

In order to obtain explicit formulas and also to make contact with the AQFT literature, we introduce bases $\{(T^\alpha_{\beta\delta})^i\}_{i=1}^{N^\alpha_{\beta\delta}}$ for the Hom–spaces $\mathrm{Hom}(\alpha, \beta\delta)$, where $N^\alpha_{\beta\delta}$ is the fusion coefficient:

$$\mathsf{dim}(\beta)\,\mathsf{dim}(\delta) = \sum_{\alpha \in \mathcal{C}^0} N^\alpha_{\beta\delta}\,\mathsf{dim}(\alpha) \tag{7.4.37}$$

The $(T)^i$ are thus intertwiners $(T^\alpha_{\beta\delta})^i : \alpha \to \beta\delta$. Taking into account the multiplicities n_α, n_β, we can label the basis elements of the $U(\delta)$ as

$$\{(T^\alpha_{\beta\delta})^{i,ab}\}\,;\, i = 1...N^\alpha_{\beta\delta}\,,\; a = 1...n_\alpha\,,\; b = 1...n_\beta \tag{7.4.38}$$

The index–pair a, b carries the R–bimodule structure of $U(\rho)$. More precisely, $U(\rho)_{\alpha,\beta}$ is an $\mathsf{Mat}_{n_\alpha}(\mathbf{C})$-$\mathsf{Mat}_{n_\beta}(\mathbf{C})$–bimodule.

Since any object is equivalent to an appropriate direct sum of copies of simple objects, we can express the functor U in the basis introduced above. For $\rho = \bigoplus_{\delta \in \mathcal{C}^0} k_\delta \gamma$,

$$U(\rho) = \bigoplus_{\delta \in \mathcal{C}^0} k_\delta \bigoplus_{\alpha,\beta \in \mathcal{C}^0} n_\alpha n_\beta \,\mathrm{Hom}(\alpha, \beta\delta) \tag{7.4.39}$$

Next, we obtain a similar decomposition in irreducibles of the natural transformation $U_{\rho,\rho'}$. To this end, we choose bases for the intertwiner spaces $\mathrm{Hom}(\alpha, \beta\,(\rho_1\rho_2...\rho_m))$, where the $\rho_i \in \mathcal{C}^0$ are irreducibles. To write down the general formula for $\mathrm{Hom}(\alpha, \beta(\rho_1...\rho_n))$, we employ a convenient graphical representation[10].

$T^{\alpha,i}_{\beta\rho}$

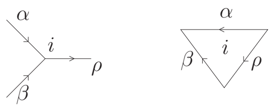


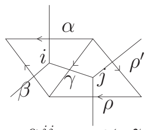
FIGURE 2. Representations of intertwiners in $\mathrm{Hom}(\alpha, \beta\rho)$

From semi–simplicity, it follows that a basis for $\mathrm{Hom}(\alpha, \beta\,(\rho_1\rho_2))$ is afforded by

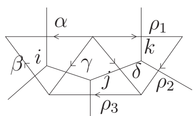
$$(T^{\alpha,ij}_{\beta(\rho_1\rho_2)}) := \beta(T^{\gamma,i}_{\rho_1\rho_2})\,T^{\alpha,j}_{\beta\gamma},\;\; i = 1...N^{\gamma}_{\rho_1\rho_2}\,,\; j = 1...N^{\alpha}_{\beta\gamma} \tag{7.4.40}$$

A basis for $\mathrm{Hom}(\alpha, \beta\,(\rho_1\rho_2\rho_3))$ may be built up in a similar way:

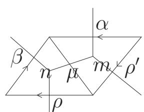
$$(T^{\alpha,ijk}_{\beta(\rho_1\rho_2\rho_3)}) := \beta(\rho_1(T^{\varepsilon,k}_{\rho_2\rho_3}))\,\beta(T^{\gamma,j}_{\rho_1\varepsilon})\,T^{\alpha,k}_{\beta\gamma},\;\; i = 1...N^{\varepsilon}_{\rho_2\rho_3}, j = 1...N^{\gamma}_{\rho_1\varepsilon}, k = 1...N^{\alpha}_{\beta\gamma}$$



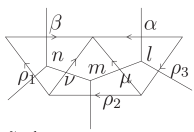
$T^{\alpha,ij}_{\beta(\rho\rho')} = \beta(T^{\gamma,i}_{\rho\rho'})\,T^{\alpha,j}_{\beta\gamma}$



$T^{\alpha,ijk}_{\beta(\rho_1\rho_2\rho_3)} = \beta(\rho_1(T^{\delta,i}_{\rho_2\rho_3}))\,\beta(T^{\gamma,j}_{\rho_1\delta})\,T^{\alpha,k}_{\beta\gamma}$



$\tilde{T}^{\alpha,mn}_{\beta(\rho\rho')} = T^{\mu,m}_{\beta\rho}\,T^{\alpha,m}_{\mu\rho'}$



$\tilde{T}^{\alpha,lmn}_{\beta(\rho_1\rho_2\rho_3)} = T^{\nu,n}_{\beta\rho_1}\,T^{\mu,m}_{\nu\rho_2}\,T^{\alpha,l}_{\mu\rho 3}$

FIGURE 3. 'Right ordered' and 'left ordered' bases

For irreducibles ρ and ρ', choose $T^{\alpha,i}_{\gamma,\rho} \underset{R}{\otimes} T^{\gamma,j}_{\beta\rho'} \in U(\rho) \underset{R}{\otimes} U(\rho')$, then

$$U_{\rho,\rho'}(T^{\alpha,i}_{\gamma,\rho} \underset{R}{\otimes} T^{\gamma,j}_{\beta\rho'}) = T^{\alpha,i}_{\gamma\rho}T^{\gamma,j}_{\beta\rho'} \in \mathrm{Hom}(\alpha, \beta\rho\rho'), \tag{7.4.41}$$

is the product in A of the two interwiners. It may be re–expressed in the canonical basis 7.4.40 via the 6j–symbol. By the associativity of the monoidal product, a three–fold product of simple objects can be decomposed into simple objects in two equivalent ways; the 6j–symbol connects the two decompositions. More precisely, taking irreducible morphisms β, ρ and ρ' for concreteness we have $(\beta\rho)\rho' \simeq \beta(\rho\rho')$, so for all irreducibles α, $\mathrm{Hom}(\alpha, (\beta\rho)\rho') \simeq \mathrm{Hom}(\alpha, \beta(\rho\rho'))$.

[10]We shall stick to the 'triangle representation', which is geometrically dual to the more conventional 'star' notation for intertwiners.

Decompositions corresponding to the first Hom–space are given by intertwiners of the type $\alpha \to \beta\gamma \to \beta(\rho\rho')$. A basis is given by

$$\{\beta(T^{\gamma,i}_{\rho_1\rho_2})\, T^{\alpha,j}_{\beta\gamma}\}_{i,j} \tag{7.4.42}$$

The decomposition corresponding to the second Hom–space are given by interwiners of the type $\alpha \to \delta\rho' \to (\beta\rho)\rho'$, which are spanned by

$$\{T^{\delta,k}_{\beta\rho}\, T^{\alpha,l}_{\delta\rho'}\}_{k,l} \tag{7.4.43}$$

The 6j–symbol is the unitary relating $\mathrm{Hom}(\alpha, (\beta\rho)\rho')$ and $\mathrm{Hom}(\alpha, \beta(\rho\rho'))$, expressed in the bases of 7.4.42 and 7.4.43,

$$T^{\delta,k}_{\beta\rho}\, T^{\alpha,}_{\delta\rho'} = D\big(\begin{smallmatrix}\alpha\\ \beta\rho\rho'\end{smallmatrix}\big)\left[\begin{smallmatrix}jk\\ \gamma\delta\\ il\end{smallmatrix}\right] \beta(T^{\gamma,i}_{\rho_1\rho_2})\, T^{\alpha,j}_{\beta\gamma} \tag{7.4.44}$$

The tensor notation employed in the previous equation lies close to physicist practice, but it is somewhat redundant and misleading. The set of 10 indices are not independent of each other, e.g. the range of the indices i, j, k and l depend on the remaining indices through the fusion coefficients. The 6j–symbol is best visualized in the geometric formalism, which also makes explicit the summation convention implied in 7.4.44. Comparing with 7.4.40 and 7.4.41,

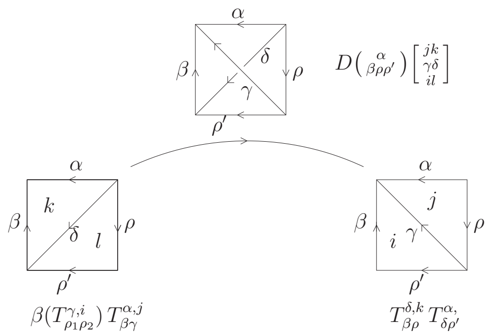


FIGURE 4. Definition of the 6j–symbol

we see that expressing $U_{\rho,\rho'}$ in the basis 7.4.40 requires precisely the 6j–symbol. An element $\Phi \underset{R}{\otimes} \Psi \in U(\rho) \underset{R}{\otimes} U(\rho')$ may be expanded as $\Phi \underset{R}{\otimes} \Psi = (\Phi^{\beta}_{\delta,k}\Psi^{\delta}_{\alpha,l})\, T^{\delta,k}_{\beta\rho} \underset{R}{\otimes} T^{\alpha,l}_{\delta\rho'}$, and an element $\Omega \in U(\rho\rho')$ takes the form $\Omega = \Omega^{\beta}_{\alpha,ij}\ \beta(T^{\gamma,i}_{\rho_1\rho_2})\, T^{\alpha,j}_{\beta\gamma}$. The desired explicit formula is then

$$U_{\rho,\rho'}(\Phi \underset{R}{\otimes} \Psi)^{\beta}_{\alpha,ij} = \sum_{k,l} \Phi^{\beta}_{\delta,k}\Psi^{\delta}_{\alpha,l}\ D\big(\begin{smallmatrix}\alpha\\ \beta\rho\rho'\end{smallmatrix}\big)\left[\begin{smallmatrix}jk\\ \gamma\delta\\ il\end{smallmatrix}\right] \tag{7.4.45}$$

Each equivalent basis in $\mathrm{Hom}(\alpha, \beta(\rho_1 ... \rho_m))$ corresponds to a different decompositions into irreducibles of the m–fold composite charge $(\rho_1 ... \rho_m)$, which in turn correspond to different orderings of an m–fold product of $T^{\alpha,i}_{\beta\rho}$'s. Considering $\mathrm{Hom}(\alpha, \beta(\rho_1\rho_2\rho_3))$, the left– and right–ordered bases of Figure 3 are two of the possible decompositions. Figure 5 shows two ways of obtaining the transformation connecting the bases $\{\tilde{T}^{\alpha,lmn}_{\beta(\rho_1\rho_2\rho_3)} = T^{\nu,n}_{\beta\rho_1}\, T^{\mu,m}_{\nu\rho_2}\, T^{\alpha,l}_{\mu\rho3}\}_{l,m,n}$ and $\{T^{\alpha,ijk}_{\beta(\rho_1\rho_2\rho_3)} = \beta(\rho_1(T^{\delta,i}_{\rho_2\rho_3}))\, \beta(T^{\gamma,j}_{\rho_1\delta})\, T^{\alpha,k}_{\beta\gamma}\}_{i,j,k}$ through successive applications of the $6j$–symbol.

The two composite maps must coincide, yielding a coherence condition for the $6j$–symbol, known as the Pentagon Equation. If we represent the $6j$–symbol as the tetrahedron connecting its Northern and Southern hemispheres, then Figure 5 has the following geometric interpretation. Identify the boundaries of the two pentagons of Figure 3, obtaining a solid with 6 triangular faces. The two sides of the Pentagon Equation are then represented by the decomposition of this solid into two– and three tetrahedra, respectively. We shall write down the explicit form of the Pentagon Equation momentarily, showing at the same time that is equivalent to the strong monoidality of the functor $U = \mathrm{Hom}(\gamma, \gamma \circ \_)$.

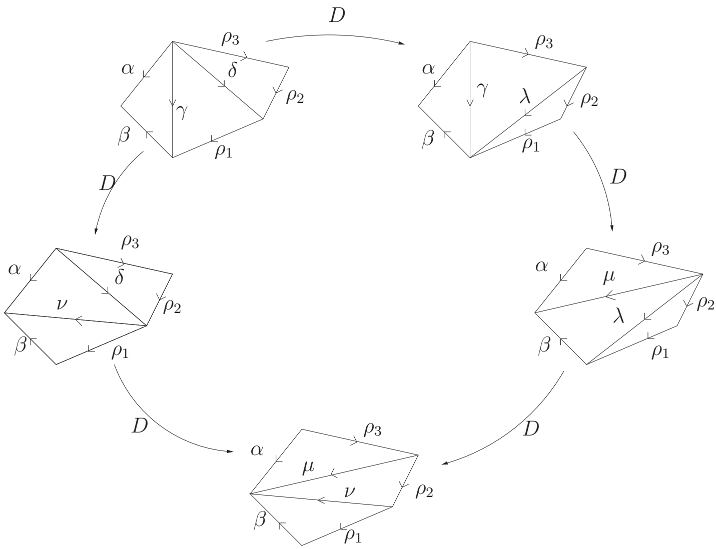


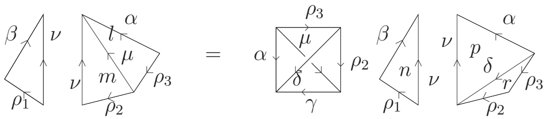
Figure 5. The Pentagon Equation

Omitting associators (i.e. assuming that our monoidal categories are strict), the hexagon expressing the monoidality of $\langle U, U_{\rho_1,\rho_2}, U_0\rangle$ reduces to the rectangle

(7.4.46) $$U_{\rho_1\rho_2,\rho_3} \circ (U_{\rho_1,\rho_2} \underset{R}{\otimes} U) = U_{\rho_1,\rho_2\rho_3} \circ (U \underset{R}{\otimes} U_{\rho_2,\rho_3})$$

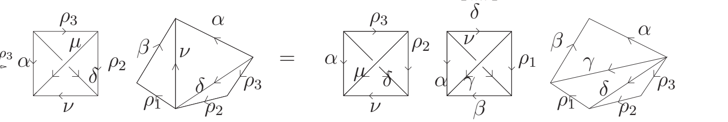
We can now write down 7.4.46 in our canonical (i.e. right ordered) basis. We present the calculation in the geometric formalism, which shows the precise meanings of contractions of indices.

$(U \underset{R}{\otimes} U_{\rho_2,\rho_3})(T^{nu,n}_{\beta\rho_1} \otimes T^{\mu,m}_{\nu\rho_2} \otimes T^{\alpha,l}_{\mu\rho_3}) =$



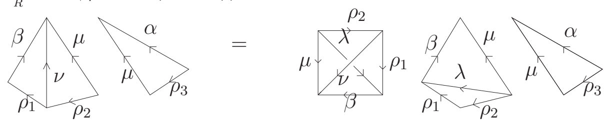
$T^{nu,n}_{\beta\rho_1} \otimes T^{\mu,m}_{\nu\rho_2}\, T^{\alpha,l}_{\mu\rho_3}$ $\quad$ $T^{\nu,n}_{\beta\rho_1} \otimes D\big({}^{\alpha}_{\nu\rho_2\rho_3}\big)\left[\begin{smallmatrix}pr\\ \mu\delta\\ ml\end{smallmatrix}\right] \nu(T^{\delta,r}_{\rho_2\rho_3})\, T^{\alpha,p}_{\nu\delta}$

$\xrightarrow{U_{\rho_1,\rho_2\rho_3}}$



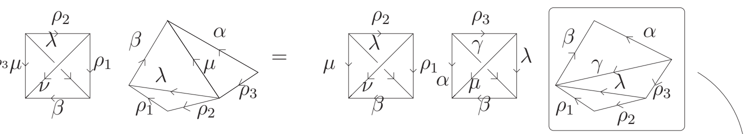
$D\big({}^{\alpha}_{\nu\rho_2\rho_3}\big)\left[\begin{smallmatrix}pr\\ \mu\delta\\ ml\end{smallmatrix}\right] \beta\rho_1(T^{\delta,r}_{\rho_2\rho_3})\, T^{\nu,n}_{\beta\rho_1}\, T^{\alpha,p}_{\nu\delta}$ $\quad$ $D\big({}^{\alpha}_{\nu\rho_2\rho_3}\big)\left[\begin{smallmatrix}pr\\ \mu\delta\\ ml\end{smallmatrix}\right] D\big({}^{\alpha}_{\beta\rho_1\delta}\big)\left[\begin{smallmatrix}sq\\ \gamma\mu\\ np\end{smallmatrix}\right] \beta\rho_1(T^{\delta,r}_{\rho_2\rho_3})\, \beta(T^{\gamma,q}_{\rho_1\delta})\, T^{\alpha,s}_{\beta\gamma}$

$(U_{\rho_1,\rho_2} \underset{R}{\otimes} U)(T^{\nu,a}_{\beta\rho_1} \otimes T^{\mu,b}_{\nu\rho_2} \otimes T^{\alpha,c}_{\mu\rho_3}) =$

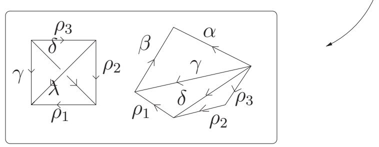


$T^{\nu,a}_{\beta\rho_1}\, T^{\mu,b}_{\nu\rho_2} \underset{R}{\otimes} T^{\alpha,c}_{\mu\rho_3}$ $\quad$ $D\big({}^{\mu}_{\beta\rho_1\rho_2}\big)\left[\begin{smallmatrix}de\\ \nu\lambda\\ ab\end{smallmatrix}\right] \beta(T^{\lambda,e}_{\rho_1\rho_2})\, T^{\mu,d}_{\beta\lambda} \underset{R}{\otimes} T^{\alpha,c}_{\mu\rho_3}$

$\xrightarrow{U_{\rho_1\rho_2,\rho_3}}$



$D\big({}^{\mu}_{\beta\rho_1\rho_2}\big)\left[\begin{smallmatrix}de\\ \nu\lambda\\ ab\end{smallmatrix}\right] \beta(T^{\lambda,e}_{\rho_1\rho_2})\, T^{\mu,d}_{\beta\lambda}\, T^{\alpha,c}_{\mu\rho_3}$ $\quad$ $D\big({}^{\mu}_{\beta\rho_1\rho_2}\big)\left[\begin{smallmatrix}de\\ \nu\lambda\\ ab\end{smallmatrix}\right] D\big({}^{\alpha}_{\beta\lambda\rho_3}\big)\left[\begin{smallmatrix}gh\\ \mu\gamma\\ de\end{smallmatrix}\right] \beta(T^{\lambda,e}_{\rho_1\rho_2})\, \beta(T^{\gamma,h}_{\lambda\rho_3})\, T^{\alpha,g}_{\beta\gamma}$



$$\beta(T^{\lambda,e}_{\rho_1\rho_2}\, T^{\gamma,h}_{\lambda\rho_3})\, T^{\alpha,g}_{\beta\gamma} = D\big({}^{\gamma}_{\rho_1\rho_2\rho_3}\big)\left[\begin{smallmatrix}jk\\ \lambda\delta\\ he\end{smallmatrix}\right] \beta\rho_1(T^{\delta,r}_{\rho_2\rho_3})\, \beta(T^{\gamma,q}_{\rho_1\delta})\, T^{\alpha,s}_{\beta\gamma}$$

Comparing the left/lower and upper/right sides of the rectangle, the Pentagon Equation reads:

$$D\big({}^{\alpha}_{\nu\rho_2\rho_3}\big)\left[\begin{smallmatrix}pr\\ \mu\delta\\ ml\end{smallmatrix}\right] D\big({}^{\alpha}_{\beta\rho_1\delta}\big)\left[\begin{smallmatrix}sq\\ \gamma\mu\\ np\end{smallmatrix}\right] = D\big({}^{\mu}_{\beta\rho_1\rho_2}\big)\left[\begin{smallmatrix}de\\ \nu\lambda\\ ab\end{smallmatrix}\right] D\big({}^{\alpha}_{\beta\lambda\rho_3}\big)\left[\begin{smallmatrix}gh\\ \mu\gamma\\ de\end{smallmatrix}\right] D\big({}^{\gamma}_{\rho_1\rho_2\rho_3}\big)\left[\begin{smallmatrix}jk\\ \lambda\delta\\ he\end{smallmatrix}\right]$$

Recall that in DHR–theory, there is an equivalence of categories $\mathsf{DHR}(\mathfrak{A}) \cong \mathsf{Rep}_{loc}(\mathfrak{A})$. The object map of the equivalence functor given by $\rho \mapsto \langle \pi_\rho, \mathcal{H}_\rho \rangle$, where π_ρ is a DHR–representation with respect to the vacuum representation $\langle\pi_0, \mathcal{H}_0\rangle$. There exists an isometry $V_\rho : \mathcal{H}_\rho \to \mathcal{H}_0$ such that $V_\rho \pi_\rho = (\pi_0 \circ \rho)V_\rho$. The image under the equivalence of the object $P = \bigoplus_{\alpha\in\Delta^0} n_\alpha \alpha$ is $\pi^\gamma = \bigoplus_{\alpha\in\Delta^0} n_\alpha \pi_\alpha$, a (completely) reducible representation of $\mathfrak{A}$ on $\mathcal{H} = \oplus\mathcal{H}_\alpha$.

We shall define the Field Algebra through the functor

$$U' = \mathrm{Hom}_{DHR}(\pi^\gamma, \pi^\gamma \circ \_) : \mathcal{C} \to {}_R\mathsf{M}_R \tag{7.4.47}$$

For $\rho \in \Delta^0$, $U'(\rho)$ decomposes as $\mathrm{Hom}(\pi^\gamma, \pi^\gamma \circ \rho) \simeq \bigoplus_{\alpha,\beta} n_\alpha n_\beta \,\mathrm{Hom}(\pi_\alpha, \pi_\beta \circ \rho)$. Furthermore, intertwiners $\varphi \in \mathrm{Hom}(\pi_\alpha, \pi_\beta \circ \rho)$ are in one–to–one correspondence with interwiners $V_\beta^*\varphi V_\alpha \in \mathrm{Hom}(\pi_0 \circ \alpha, \pi_0 \circ \beta\rho)$, or in other words

$$V_\beta^*(\_)V_\alpha : \mathrm{Hom}(\pi_\alpha, \pi_\beta \circ \rho) \to \mathrm{Hom}(\pi_0 \circ \alpha, \pi_0 \circ \beta)$$

is an isometry. Using our previously developed notation, we immediately find that the

$$(F^{\rho,i})_{\alpha,a;\beta,b} = (V_\beta^b)^* \pi_0(T^{\alpha;i,ab}_{\beta,\rho})V_\alpha^a\,;\ i = 1...N^\alpha_{\beta\rho}, a = 1...n_\alpha, b = 1...n_\beta \tag{7.4.48}$$

are a basis for $\mathrm{Hom}(\pi_\alpha, \pi_\beta \circ \rho)$. Denote $F^{\rho,i} = \bigoplus_{\alpha,a;\beta,b} (F^{\rho,i})_{\alpha,a;\beta,b}$. The F^ρ shall be called the *charge creation operators* for the sector ρ. The terminology is justified by the following property

Lemma 7.4.49. *The $F^{\rho,i}$, defined as above, create charge ρ in the representation π^γ in the sense*

$$F^{\rho,i}\pi^\gamma(A) = \pi^\gamma \circ \rho(A)F^{\rho,i}$$

Proof. We omit the multiplicity indices to un–clutter our formulas.

$$\begin{aligned}(V_\beta{}^*\pi_0(T^{\alpha,i}_{\beta,\rho})V_\alpha)\pi_\alpha(A) &= V_\beta{}^*T^{\alpha,i}_{\beta,\rho}\pi_0 \circ \alpha(A)V_\alpha = V_\beta{}^*\pi_0 \circ \beta\rho(A)T^{\alpha,i}_{\beta,\rho}V_\alpha = \\ &= \pi_\beta \circ \rho(A)(V_\beta{}^*\pi_0(T^{\alpha,i}_{\beta,\rho})V_\alpha)\end{aligned}$$

Extending the unitaries $V_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_0$ to isometries on $\mathcal{H}$, the statement follows. □

We proceed to calculate the relations satisied by the charge creation operators among themselves. Substituting the definitions,

$$\begin{aligned}F^{\rho',j}_{\beta\gamma}F^{\rho,i}_{\gamma\alpha} &= (V_\beta{}^*\pi_0(T^{\gamma,j}_{\beta\rho'})V_\gamma)(V_\gamma{}^*\pi_0(T^{\alpha,i}_{\gamma\rho})V_\alpha) = V_\beta{}^*\pi_0(T^{\gamma,j}_{\beta\rho'}T^{\alpha,i}_{\gamma\rho})V_\alpha = \\ &= V_\beta{}^*D\left(\begin{smallmatrix}\alpha\\ \beta\rho'\rho\end{smallmatrix}\right)\left[\begin{smallmatrix}ki\\ \delta\gamma\\ lj\end{smallmatrix}\right]\beta(T^{\delta,l}_{\rho'\rho})T^{\alpha,k}_{\beta\delta}V_\alpha = D\left(\begin{smallmatrix}\alpha\\ \beta\rho'\rho\end{smallmatrix}\right)\left[\begin{smallmatrix}ki\\ \alpha\gamma\\ lj\end{smallmatrix}\right]\pi_\beta(T^{\alpha,l}_{\rho'\rho})\,F^{\delta,k}_{\beta\alpha}\end{aligned} \tag{7.4.50}$$

Up to now, we have only worked with charge creation operators for irreducible morphisms (i.e. sectors). However, the definition is easily extendable to arbitrary composite charges, using our bases for the intertwiner spaces $\mathrm{Hom}(\alpha, \beta\,(\rho_1...\rho_n))$. In particular, charge creation operators for the composite charge $(\rho'\rho)$ will be labelled $F^{\rho\rho',kl}_{\alpha\beta} = V_\beta{}^*T^{\alpha,kl}_{\beta(\rho\rho')}V_\alpha = V_\beta{}^*\pi_\beta(T^{\delta,l}_{\rho'\rho})\,T^{\alpha,k}_{\beta\delta}V_\alpha$. Using our earlier, condensed notation, the matrix elements of charge creation operators for composite charges $(\rho_1...\rho_m)$ read

$$F^{(\rho_1..\rho_m),i_1..1_m}_{\alpha\beta} = V_\beta{}^*T^{\alpha,i_1..i_m}_{\beta(\rho_1..\rho_m)}V_\alpha \tag{7.4.51}$$

We can now cast 7.4.50 in the more transparent form

$$F^{\rho',j}_{\beta\gamma} F^{\rho,i}_{\gamma\alpha} = D\left(\begin{smallmatrix} \alpha \\ \beta\rho'\rho \end{smallmatrix}\right) \left[\begin{smallmatrix} ki \\ \alpha\gamma \\ lj \end{smallmatrix}\right] F^{(\rho'\rho),kl}_{\beta\alpha} \tag{7.4.52}$$

We are now ready to define the Field Algebra, as a 'cross–product' of the observable algebra and the charge creation operators.

Definition 7.4.53. The Field Algebra is generated by the $\{\pi^\gamma(\mathfrak{A}) \underset{R}{\otimes} F^\sigma\}_{\sigma\in\Delta^0}$, subject to the following relations:

$$(\pi^\gamma(A) \otimes F^{\rho,i})\,(\pi^\gamma(A') \otimes F^{\rho',j}) = (\pi^\gamma(A\rho(A')) \otimes F^{\rho,i}F^{\rho',j})$$

This is essentially Fredenhagen, Rehren and Schroer's reduced field bundle construction (see [**37**]). In their terminology, for irreducible α, β and a morphism ρ, triples $e = (\alpha, \rho, \beta) = (s(e), c(e), r(e))$ are called 'fusion channels' or 'coloured edges' with range β, source α and charge ρ. Our bases of intertwiners $T^\beta_{\alpha\rho}$ are denoted simply T_e. The reduced field bundle is generated by elements $F(e, A)$, which in turn are defined by their action on the Hilbert space $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$. Denoting (α, Ψ), $\Psi \in \mathcal{H}_0$ the elements of the Hilbert space $\mathcal{H}_\alpha$,

$$F(e, A)(\alpha, \Psi) = (\beta, (T_e^*\alpha(A))\Psi).$$

Thus, apart from the fact that our semidirect product field algebra is based on the *left* action of the charge creation operators on the observable algebra, the $F(e, A)$ correspond *to matrix elements* $(\pi^\beta \otimes F^{\rho,i}_{\alpha\beta})$. The reduced field bundle is said to have a 'path algebra' structure. A path η is a composition of edges $\eta = e_n \circ ... \circ e_i$ such that $s(e_{i+1}) = r(e_i)$. Field bundle elements $F(e_2, A_2)$, $F(e_1, A_1)$ can be multiplied only if e_2 and e_1 are compatible, i.e. $s(e_2) = r(e_1)$ must hold, but the charges c_1 and c_2 may be arbitrary. This is an intuitive formulation of the algebra of field operator matrix elements. It must be remarked that the reduced field bundle of [**37**] also possesses additional structure which does not arise in our construction. Namely, the statistics operator $\varepsilon : \rho_1\rho_2 \to \rho_2\rho_1$ endows the reduced field bundle with an additional 'exchange algebra' structure. Since we hadn't considered a braiding on the category $\mathcal{C}$, this piece of structure is of course missing.

Words of Thanks

I feel privileged to have been the student of Kornél Szlachányi. He introduced me to this field, as well as to any other that interested him, with enthusiasm and patience. His help and advice over the years have been invaluable. I am grateful to the colleagues and friends who always gave generously of their time, for discussions scientific or other. I have learned a lot, and most of what I know about algebraic quantum field theory in particular, from the clear and illuminating explanations of Péter Vecsernyés. Gabi Böhm is my older sister in the scientific genealogy, and indeed she took this responsibility seriously. This thesis especially benefited from her interest, enthusiasm and, not least, her careful proofreading. The many conversations with László Szabados were as entertaining as they were enlightening. My fellow PhD students, Viktor Czinner, Árpád Lukács, Péter Major, Gábor Purcsel and Zoltán Zimborás were not only helpful as colleagues but also helpful as friends. I am thankful for the support of the Theoretical Department of the Research Institute for Particle– and Nuclear Physics. Finally, I would like to thank my family for help and encouragement I could always trust.

Summary

The novel results presented in this Thesis are based on the publications [**3**] (by K. Szlachányi and the Author) and [**4**] (by the Author). We summarize the main points, as they appear in the Thesis.

In chapter 3, we introduce basic definitions and results leading up to, and including quantum groupoids. No new results are presented here, the emphasis is on presenting successive generalizations of the Hopf algebra structure together with the the respective module and comodule categories. We describe the Distributive Double Algebra (DDA) approach to Frobenius Hopf algebroids, introduced in [**79**], which we use throughout Chapters 4 and 5.

Following a brief overview of classical and Hopf Galois theory, Chapter 4 gives an account of quantum groupoid Galois theory, based on the paper [**3**]. We understand the term quantum groupoid to mean Frobenius Hopf algebroid or Distributive Double Algebra. The starting point is Theorem&Definition 4.3.8, which shows that a Galois extension over a DDA may be defined equivalently as a Galois extension over the vertical or over the horizontal constituent Hopf algebroids. A key result is Theorem 4.4.5, which establishes the equivalence between Galois extensions over a Frobenius Hopf algebroid and depth 2 balanced Frobenius extensions. This theorem characterizes quantum groupoid Galois extensions without explicitly assuming a quantum groupoid (co–) action. It is a generalization of the analogous result stated for bialgebroids in [**47**].

Chapter 5 introduces a notion of scalar extension for quantum groupoids based on the paper [**3**]. After collecting results on Yetter–Drinfel'd modules and braided commutative algebras (BCAs), the basic construction appears in Thm. 5.3.1. It associates to a bialgebroid H over R and a braided commutative algebra Q over H the scalar extension, a bialgebroid $H\#Q$ with base algebra extended to Q. It generalizes both Brzeziński and Militaru's construction for Hopf algebras [**21**] and the coring extensions of [**14**]. In Proposition 5.3.28, we extend the scalar extension construction to Hopf algebroids by defining an antipode for the extended bialgebroid. We prove in Proposition 5.3.13 that scalar extension is transitive in the sense that the composition of scalar extensions is again a scalar extension. An important application of the construction comes from Galois theory: in Propositions 5.5.1 and 5.5.8, we prove that the centralizer of a bialgebroid Galois extension is a braided commutative algebra and moreover, that scalar extension by the centralizer gives precisely the canonical endomorphism bialgebroid associated to the extension. This gives us partial control over the ambiguity in the 'Galois

group' for quantum groupoid Galois extension; notably, no similar result is available in Hopf Galois theory.

Chapter 6 is about bicoalgebroids and is based on the paper [**4**]. We define module and comodule categories over bicoalgebroids and prove, in Theorem 6.3.2 that the comodule category is monoidal with a strict monoidal forgetful functor to the bicomodule category over the base coalgebra. We prove a comonadic Schauenburg–type theorem for bicoalgebroids in 6.3.27. We consider the notions of cocenter and cocentralizer for bicomodules and coalgebra extensions, respectively, and prove the equivalence of different definitions in Section 4 of Chapter 5. We define the Yetter–Drinfel'd category and braided cocommutative coalgebras (BCCs) over bicoalgebroids and define a scalar extension for bicoalgebroids in Theorem 6.5.23.

Chapter 7 is devoted to an application from Algebraic Quantum Field Theory (AQFT). We recall the axioms of AQFT, the field algebra construction and the Doplicher–Roberts Reconstruction Theorem. With mild assumptions on the DR category we give a purely algebraic construction for the field algebra similar in spirit to Doplicher and Roberts' and prove that it defines a depth 2 extension of the observable algebra. This promises a quantum groupoid symmetry by our earlier results from quantum groupoid Galois theory. In the special case that the DR category is semi–simple, our field algebra construction reduces to a generalization of Fredenhagen, Rehren and Schroer's reduced field bundle.

Összefoglalás

Az dolgozatban talalható új eredmények a [**3**] (Szlachányi Kornél és a Szerző munkája), és a [**4**] (a Szerző munkája) publikaciókon alapulnak. A főbb pontokat a az értekezésbeli tárgyalásuk sorrendjében foglaljuk össze.

A harmadik fejezetben a kvantum gruppoidok elméletét megalapozó alapvető definíciókat és tételeket tekintjük át. Új eredményeket nem mutatunk be, a hangsúlyt a Hopf algebrát általánosító szimmetria struktúrák valamint a megfelelő modulus– és komodulus kategóriáik együttes bemutatására helyeztük. Ismertetjük a Frobenius Hopf algebroidokat leíró Disztributív Dupla Algebra (DDA) formalizmust (ld. [**79**]), amit a negyedik és ötödik fejezetben alkalmazunk.

A klasszikus– és Hopf Galois elmélet tömör áttekintését követően a negyedik fejezetben mutatjuk be a kvantum gruppoidok Galois elméletét a [**3**] cikk alapján. A kvantum gruppoid kifejezés alatt Frobenius Hopf algebroidot értünk, melynek leŕására a DDA formalizmust használjuk. Kiindulópontunk a 4.3.8 tétel&definíció, mely szerint egy DDA fölötti Galois kiterjesztés ekvivalens módon definiálható vertikális, illetve horizontális Hopf algebroidja fölötti Galois kiterjesztésként. Fontos eredmény továbbá a 4.4.5 tétel, amely megállapítja a Frobenius Hopf algebroid fölötti Galois kiterjesztések és a 2–es mélységű balanszírozott Frobenius kiterjesztések ekvivalenciáját. Ez a tétel a szimmetria struktúra explicit említése nélkül jellemzi a kvantum gruppoid–Galois kiterjesztéseket, és a [**47**]–ben található, bialgebroidokra vonatkozó eredmény általánosítása.

Az ötödik fejezet tárgyalja a kvantumgruppoidok skalár–kiterjesztését a [**3**] cikk alapján. A Yetter–Drinfel'd modulusok és fonott kommutatív algebrák (az angol kifejezés rövidítés alapján BCA–k) elmélete alapvető eredményeinek áttekintése után az alapvető konstrukciót a 5.3.1 tétel tartalmazza. A konstrukció egy R fölötti H bialgebroidhoz és egy H fölötti Q BCA–hoz egy $H\#Q$, Q fölötti bialgebroidot rendel és egyszerre általánosítja Brzeziński és Militaru Hopf algebrákra vonatkozó konstrukcióját valamint a [**14**] értelmében vett kogyűrű kiterjesztéseket. A skalár–kiterjesztést általánosítjuk Hopf algebroidokra is (5.3.28 állítás), azáltal, hogy kiterjesztett bialgebroidhoz antipódot konstruálunk. A 5.3.13 állítás szerint a skalár–kiterjesztés tranzitív abban az értelemben, hogy két egymást követő skalár–kiterjesztés kompozíciója szintén skalár–kiterjesztést eredményez. A skalár–kiterjesztés fontos szerephez jut a Galois elméletben: az 5.5.1 és 5.5.8 állításokban bebizonyítjuk, hogy egy bialgebroid–Galois kiterjesztés centralizátora mindig fonott kommutatív és továbbá, hogy a cenztralizátorral való skalár–kiterjesztés éppen a Galois bővítéshez tartozó kanonikus endomorfizmus bialgebroidot adja. Ezzel részben le tudjuk írni a kvantum gruppoid szimmetria többértelműségét – a Hopf Galois elméletben ismeretesen nem áll rendelkezésre hasonló eredmény.

A hatodik fejezetet a bialgebroid struktúrát dualizáló bikoalgebroidoknak szenteljük, és a [**4**] cikken alapul. Definiáljuk a bikoalgebroid fölötti modulus– és komodulus kategóriát. A 6.3.2 tételben bebizonyítjuk, hogy egy bikoalgebroid komodulus kategórián létezik monoidális struktúra valamint egy szigorúan monoidális felejtőfunktor a bázis koalgebra feletti bikomodulus kategóriába. A 6.3.27 tétel egy bikoalgebroidokra vonatkozó Schauenburg–típusú tétel komonádikus változata. A hatodik fejezet negyedik szakaszában áttekintjük a bikomodulus kocentrumának, illetve koalgebra kiterjesztés kocentralizátorának fogalmát és bizonyítjuk néhány különböző definíció ekvivalenciáját. Definiáljuk a bikoalgebroid fölötti Yetter–Drinfel'd modulusok és fonott kokommutatív koalgebrák (az angol kifejezés rövidítés alapján BCC–k) fogalmát és a 6.5.23 tételben bevezetjük a skalár–kiterjesztés fogalmát bikoalgebroidokra.

A hetedik fejezetben egy Algebrai Térelméletbeli alkalmazást mutatunk be. Vázoljuk az Algebrai Térelmélet axiómarendszerét, a téralgebra konstrukciót és a Doplicher–Roberts–féle rekonstrukciós tételt. Enyhe feltevésekkel élve a szuperszelekciós szektorok Doplicher–Roberts (DR) kategóriájáról, adunk egy Doplicher és Robertséhez hasonló ám tisztán algebrai téralgebra konstrukciót, és bebizonyítjuk, hogy az a megfigyelheő algebra 2–es mélysgű kiterjesztését definiálja. Ez az eredmény a korábbi, Galois–elméletbeli eredményeink alapján kvantum gruppoid szimmetriát ígér. Abban a speciális esetben amikor a Doplicher–Roberts kategória féligegyszerű, téralgebránk a Fredenhagen-Rehren-Schroer–féle 'reduced field bundle' konstrukció általánosítását adja vissza.

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