TILTING MODULES AND THE SUBCATEGORIES C_i^M

María Inés Platzeck,[1] * and Nilda Isabel Pratti[2] †

[1]Instituto de Matemática, Universidad Nacional del Sur,
8000 Bahía Blanca, Argentina
[2]Dpto. de Matemática, F.C.E. y N., Universidad Nacional de Mar del Plata,
7600 Mar del Plata, Argentina

ABSTRACT

In this paper we further study the full subcategories C_i^M of the category of finitely generated modules over an artin algebra introduced in [PP], consisting of the modules having an add$M$ resolution of length i, which remains exact under the functor $\mathrm{Hom}_A(M,-)$. In particular, we characterize tilting modules in terms of these categories and determine when the transpose of a tilting module is a tilting module.

Introduction

Let A be an artin algebra and mod$A$ be the category of finitely generated right A-modules. Let M be an A-module and denote by add$M$ the full subcategory of mod$A$ consisting of the direct sums of direct summands of M. In [PP] we considered for an A-module M and for every $n \geq 0$ the full subcategories C_n^M of mod$A$ consisting of the modules X such that there is an exact sequence $M_n \to \cdots \to M_1 \to M_0 \to X \to 0$ with $M_i \in \mathrm{add}M$, and such that the induced sequence $\mathrm{Hom}_A(M, M_n) \to \cdots \to \mathrm{Hom}_A(M, M_1) \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$ is exact, generalizing work of M. Auslander in [A] about the subcategory C_1^M. The results in [PP] refer mainly to C_0^M and C_1^M, and the modules M with the property that $C_0^M = C_1^M$ are studied there. Examples of such modules are semisimple modules, tilting modules, *-modules (as defined en [C]) and the transpose of tilting modules.

In this paper we give some applications of the results in [PP]. On one side, we prove that the transpose TrM of a tilting module M is a *-module. Using then a result by D'Este and Happel about *-modules it follows that TrM is a tilting module over the algebra $\mathrm{End}(_BM)/P(_BM,_BM)$, where $B = \mathrm{End}(M_A)$ and $P(_BM,_BM)$ is the set of the endomorphisms of $_BM$ which factor through a projective module. As a consequence we obtain conditions for the transpose of a tilting module M to be a tilting module. This is the case, for example, when M is a splitting or a separating tilting module with no nonzero projective summands.

Tilting modules M satisfy $C_0^M = C_1^M$. The converse is not true, even if we assume that $DA \in C_0^M$. An example is provided by the module M direct sum of a complete set of representatives of the isomorphism classes of indecomposable modules over an algebra of finite representation type. However, we characterize tilting modules in terms of the categories C_i^M in the following way. Let $B =$

*E-mail platzeck@uns.edu.ar
†E-mail nilprat@mdp.edu.ar

End(M_A). The module M is a tilting module if and only if $DA \in C_0^{M_A}$, $C_0^{M_A} = C_1^{M_A}$ and $C_0^{{}_BM} = C_1^{{}_BM}$. We study the relation between the validity of some properties defining a tilting module, and the conditions $C_0^{M_A} = C_1^{M_A}$ and $C_0^{{}_BM} = C_1^{{}_BM}$.

Finally, we consider generalized tilting modules and prove that generalized tilting modules M of projective dimension n have the property that $DA \in C_{n-1}^{M_A}$, and the equalities $C_{n-1}^{M_A} = C_n^{M_A}$ and $C_{n-1}^{{}_BM} = C_n^{{}_BM}$ hold. However, the converse does not hold.

1. Preliminaries

Throughout this paper A denotes an artin algebra, mod$A$ the category of finitely generated right A-modules and A^{op} the opposite algebra of A. The word module means finitely generated module and we will write M_A or M to indicate that the A-module M is a right module, and ${}_AM$ to indicate that it is a left module. All subcategories considered are full. We will denote by D: mod$A \to$ modA^{op} the usual duality for artin algebras. Moreover, pd$M$ denotes the projective dimension and id$M$ the injective dimension of the module M. We denote by TrM the transpose of M and by Gen$M$ (respectively, Cogen$M$) the subcategory of mod$A$ generated (respectively, cogenerated) by M.

According to [HR, 3] we will say that the module M_A is a tilting module if it satisfies the following conditions:

(T1) pd $M_A \leq 1$.

(T2) $Ext_A^1(M_A, M_A) = 0$.

(T3) There exists an exact sequence $0 \to A \to M' \to M'' \to 0$ with M', M'' in addM_A.

It was shown in [BB] and [HR] that if M is a tilting module and $B = \text{End}(M)$ then: 1) ${}_BM$ is a tilting module and $A \simeq \text{End}({}_BM)$ and 2) the functors $\text{Hom}_A(M, -)$ and $- \otimes_B M$ induce mutually inverse equivalences between the full subcategories $\mathcal{T}(M) = \{X : \text{Ext}_A^1(M, X) = 0\}$ and $\mathcal{Y}(M) = \{Y : \text{Tor}_1^B(Y, M) = 0\}$ while the functors $\text{Ext}_A^1(M, -)$ and $\text{Tor}_1^B(-, M)$ induce mutually inverse equivalences between the full subcategories $\mathcal{F}(M) = \{X : \text{Hom}_A(M, X) = 0\}$ and $\mathcal{X}(M) = \{Y : Y \otimes_B M = 0\}$.

A tilting module M is said to be a separating (respectively, splitting) tilting module if the torsion theory $(\mathcal{T}(M), \mathcal{F}(M))$ splits in mod$A$ (respectively, the torsion theory $(\mathcal{X}(M), \mathcal{Y}(M))$ splits in mod$B$).

For a general reference for tilting theory we refer the reader to [As], [R] and [HR].

We recall that the module M is a *-module, as defined in [C], when the functor $\text{Hom}_A(M, -)$ induces an equivalence of categories between Gen$M$ and CogenD_BM.

2. The transpose of a tilting module

In this section we use results of the subcategories C_i^M to prove that the transpose ${}_A(\text{Tr}M_A)$ of a tilting module M_A is a *-module and give also a necessary and sufficient condition for ${}_A(\text{Tr}M_A)$ to be a tilting module.

Furthermore, we apply this result to obtain that the transpose of a splitting or a separating tilting module without nonzero projective summands is a tilting module.

We start by stating a theorem of D'Este and Happel [DH] which motivated this section.

Theorem 2.1. *([DH]) Let M_A be an A-module. Then: M_A is a *-module if and only if $M_{\overline{A}}$ is a tilting $\overline{A}$-module, where $\overline{A} = A/\mathrm{ann}M_A$.*

It is well known that if M_A is a tilting module and $B = \mathrm{End}(M_A)$, then ${}_BM$ is also a tilting module and $\psi : A \to \mathrm{End}({}_BM)$ defined by $\psi(a)(t) = t.a$, $t \in {}_BM$, $a \in A$ is an isomorphism [HR, 2]. Moreover, we need the following result.

Lemma 2.2. *Let M_A be a tilting A-module and $B = \mathrm{End}(M_A)$. Then ψ induces an isomorphism $\mathrm{ann}(\mathrm{Tr}M_A) \simeq P({}_BM, {}_BM)$, where $P({}_BM, {}_BM)$ is the set of the endomorphisms of ${}_BM$ which factor through a projective module.*

Proof. If M_A is a tilting module then $\psi : A \to \mathrm{End}({}_BM)$ is an isomorphism. By [PP, 4.1] we know that ${}_A(\mathrm{Tr}M_A) \simeq_A (\mathrm{Tr}_BM)$. Then $a \in \mathrm{ann}_A(\mathrm{Tr}M_A)$ if and only if $a \in \mathrm{ann}_A(\mathrm{Tr}_BM)$. On the other hand, $a.x = \psi(a).x = \mathrm{Tr}\psi(a)(x) = 0$ for $x \in \mathrm{Tr}_BM$. Hence $a \in \mathrm{ann}_A(\mathrm{Tr}M_A)$ if and only if $\psi(a) \in \mathrm{P}({}_BM, {}_BM)$.

□

We prove next that the transpose of a tilting module is a *-module, and using D'Este and Happel's result stated in Theorem 2.1 we prove the following theorem.

Theorem 2.3. *Let M_A be a tilting A-module and $B = \mathrm{End}(M_A)$. Then:*

*a) ${}_A(\mathrm{Tr}M_A)$ is a *-module.*

b) ${}_{\underline{\mathrm{End}}({}_BM)}(\mathrm{Tr}M_A)$ is a tilting $\underline{\mathrm{End}}({}_BM)$-module.

c) ${}_A(\mathrm{Tr}M_A)$ is a tilting module if and only if $P({}_BM, {}_BM) = 0$.

Proof. a) It is proven in [PP, 3.8] that a module M_A is a *-module if and only if $C_0^{M_A} = C_1^{M_A}$ and the functor $\mathrm{Hom}_A(M, -)$ is exact on $C_0^{M_A}$.

On the other hand, using that $A \simeq \mathrm{End}({}_BM)$ and ${}_BM$ is a tilting module, we get that $C_0^{\mathrm{Tr}M_A} = C_1^{\mathrm{Tr}M_A}$ and the functor $\mathrm{Hom}_A(\mathrm{Tr}M_A, -)$ is exact on $C_0^{\mathrm{Tr}M_A}$, from [PP, 4.11] and [PP, 4.7] respectively. It follows that ${}_A(\mathrm{Tr}M_A)$ is a *-module.

b) By a) we know that ${}_A(\mathrm{Tr}M_A)$ is a *-module. Then from Theorem 2.1 and Lemma 2.2 we obtain that ${}_{\overline{A}}(\mathrm{Tr}M_A)$ is a tilting $\overline{A}$-module, where $\overline{A} = A/\mathrm{ann}(\mathrm{Tr}M_A) \simeq \mathrm{End}({}_BM)/P({}_BM, {}_BM)$.

c) Assume that $P({}_BM, {}_BM) \neq 0$. By Lemma 2.2, $\mathrm{ann}_A(\mathrm{Tr}M_A) \neq 0$. Therefore ${}_A(\mathrm{Tr}M_A)$ is not faithful and consequently ${}_A(\mathrm{Tr}M_A)$ is not a tilting module.

The converse follows directly from b). □

Lemma 2.4. *Let M_A be a tilting module. Then the number of isomorphism classes of indecomposable projective summands of M_A is equal to the number of isomorphism classes of indecomposable projective summands of ${}_BM$.*

Proof. This result follows from the Connecting Lemma [HR, 2]. In fact, if I, P are respectively the injective envelope and the projective cover of the simple module S, then $\mathrm{TrD}\,\mathrm{Hom}_A(M_A, I) \simeq \mathrm{Ext}^1_A(M_A, P)$. Moreover, $P \in \mathrm{add}M_A$ if and only if $\mathrm{D}\,\mathrm{Hom}_A(M_A, I)$ is a projective B^{op}-module. On the other hand, we know that $D_BM = \mathrm{Hom}_A(M_A, DA)$, therefore $\mathrm{D}\,\mathrm{Hom}(M_A, I) \in \mathrm{add}_BM$.

So to each indecomposable projective summand P of M_A we associate the indecomposable projective summand $\theta(P) = \mathrm{D}\,\mathrm{Hom}_A(M_A, I)$ of ${}_BM$. Since $DA \in \mathrm{Gen}M_A$ and the functor $\mathrm{Hom}_A(M_A, -)|_{GenM_A}$ is faithful, the correspondence θ is injective. By [HR, 2][As, 2.3] we know that each indecomposable projective module ${}_BP$ is of the form $\mathrm{D}\,\mathrm{Hom}_A(M_A, I_A)$, with I_A injective, so θ is also surjective. □

Next, we give equivalent conditions for the transpose of a tilting module M to be a tilting module. Let $(M_A)^* = \mathrm{Hom}_A(M_A, A)$.

Proposition 2.5. *Let M_A be a tilting module and $B=\mathrm{End}(M_A)$. Then the following conditions are equivalent:*
a) ${}_A(\mathrm{Tr}M_A)$ is a tilting module
b) $(M_A)^* = 0$
c) $P(M_A, M_A) = 0$
d) $(\mathrm{Tr}_B M)_B$ is a tilting module
e) $({}_BM)^* = 0$
f) $P({}_BM, {}_BM) = 0$

Proof. To prove that a) implies b) we assume ${}_A(\mathrm{Tr}M_A)$ is a tilting module. Then the module M_A does not have projective summands and $\mathrm{id}(\mathrm{DTr}M_A) \leq 1$. That is to say, $\mathrm{TrTr}M_A \simeq M_A$ and $\mathrm{Hom}_A(\mathrm{TrTr}M_A, A) = 0$. Then $(M_A)^* = 0$.

Clearly b) implies c), and it follows directly from Theorem 2.3 that c) implies d). The remaining implications follow from the previous ones using that ${}_BM$ is a tilting module. □

As a consequence we obtain that the transpose of a separating (or splitting) tilting module with no projective summands is a tilting module. This result can also be obtained from the work of Hoshino [Ho,2].

Corollary 2.6. *Let M_A be a splitting (or separating) tilting module without nonzero projective summands and $B=\mathrm{End}(M_A)$. Then ${}_A(\mathrm{Tr}M_A)$ is a tilting A-module.*

Proof. By the above proposition we only need to prove that $({}_BM)^* = 0$. So we assume that this is not the case and consider a nonzero morphism $f{:}\,{}_BM \to B$. Then $Df : DB \to D_BM$ is also nonzero. Therefore there exist an indecomposable injective module I_B and a nonzero morphism $h : I_B \to D_BM$, and since $D_BM \in \mathcal{Y}(M)$ we conclude that $I_B \notin \mathcal{X}(M)$.

We assume first that the tilting module M_A is splitting. Then $I_B \in \mathcal{Y}(M)$. By [As, 2.3] we conclude that there exists an injective module I_A in mod$A$ such that $I_B \simeq \mathrm{Hom}_A(M, I_A)$, and from the Connecting Lemma we obtain $0 = \mathrm{TrD}I_B \simeq \mathrm{TrD}\ \mathrm{Hom}(M, I_A) \simeq \mathrm{Ext}^1_A(M, P_0(I_A/rI_A))$. So the module P_0 belongs to $\mathcal{T}(M)$ and is a projective. Thus $P_0 \in \mathrm{add}M_A$, contradicting our hypothesis.

The proof when M_A is separating is similar. □

3. Modules M_A such that $C_0^{M_A} = C_1^{M_A}$ and tilting modules

In this section we consider an A-module M_A, $B = \mathrm{End}(M_A)$ and we show that M_A is a tilting module if and only if $DA \in C_0^{M_A}$, $C_0^{M_A} = C_1^{M_A}$ and $C_0^{{}_BM} = C_1^{{}_BM}$.

We start by studying the relation between the validity of some of the properties (T1), (T2) and (T3) defining a tilting module, and the conditions $C_0^{M_A} = C_1^{M_A}$ and $C_0^{{}_BM} = C_1^{{}_BM}$.

It is well known that $C_0^{M_A} = \mathrm{Gen} M_A$ for any module M_A. By [PP, 3.12], when M_A is a tilting module it satisfies the condition $C_0^{M_A} = C_1^{M_A}$. We will prove that the validity of (T2) and (T3) implies that $C_0^{M_A} = C_1^{M_A}$, and exhibit examples showing that no other combination of two of the properties (T1), (T2) and (T3) implies $C_0^{M_A} = C_1^{M_A}$.

We introduce now the following notation. For a module $M = M_A$, let $M^{\perp_n} = \cap_{i=1}^{n} \mathrm{Ker}(\mathrm{Ext}_A^i(M,-))$ and $M^{\perp} = \cap_{i>0} \mathrm{Ker}(\mathrm{Ext}_A^i(M,-))$.

It is well known for a tilting module M that $M^{\perp_1} = \mathrm{Gen} M$. In the next lemma we prove that it is enough to assume that M satisfies (T3) for the inclusion $M^{\perp_1} \subseteq \mathrm{Gen} M$ to hold.

Lemma 3.1. *If M is an A-module and satisfies (T3) then $M^{\perp_1} \subseteq \mathrm{Gen} M$.*

Proof. Since M satisfies (T3) then there is an exact sequence $0 \to A \xrightarrow{f} M' \to M'' \to 0$ with M', M'' in $\mathrm{add} M$.

Assume now that $X \in M^{\perp_1}$. Let $f_1, \cdots, f_n$ be generators of the $Z(A)$-module $\mathrm{Hom}_A(M', X)$ and $\varphi = (f_1, \cdots, f_n)^t : (M')^n \to X$, where ($Z(A)$ denotes the center of A). We will prove that φ is an epimorphism, so that $X \in \mathrm{Gen}\ M$.

By applying the functor $\mathrm{Hom}_A(-, X)$ to the above sequence we get the exact sequence:

$$0 \to \mathrm{Hom}_A(M'', X) \to \mathrm{Hom}_A(M', X) \overset{\mathrm{Hom}(f,X)}{\longrightarrow} \mathrm{Hom}_A(A, X) \to \mathrm{Ext}_A^1(M'', X) = 0\ ,$$

and the commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
Hom_A(A, M'^n) & \xrightarrow[\mathrm{Hom}(A,\varphi)]{} & \mathrm{Hom}_A(A, X) & & \\
\mathrm{Hom}(f, M'^n) \uparrow & & \uparrow \mathrm{Hom}(f,X) & & \\
Hom_A(M', M'^n) & \xrightarrow[\mathrm{Hom}(M',\varphi)]{} & \mathrm{Hom}_A(M', X) & \to & 0
\end{array}$$

shows that $\mathrm{Hom}(A, \varphi)$ is an epimorphism. Terefore so is φ, and $X \in \mathrm{Gen} M$. $\square$

We will prove that if M satisfies the conditions (T2) and (T3) then $C_0^M = C_1^M$. The converse is not true. In fact, any simple module M satisfies $C_0^M = C_1^M$, and (T3) does not hold for M. Moreover, such a module M can be chosen so that neither (T1) nor (T2) hold, as it is the case when M is the simple module over the algebra $K[X]/(X^2)$.

Proposition 3.2. *Let M be an A-module. If M satisfies (T2) and (T3) then $C_0^M = C_1^M$.*

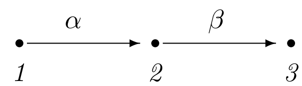
Proof. Let X in C_0^M and consider an exact sequence $0 \to K \to M' \to X \to 0$ with $M' \in \mathrm{add} M$ such that the induced sequence $0 \to \mathrm{Hom}_A(M, K) \to \mathrm{Hom}_A(M, M') \to \mathrm{Hom}_A(M, X) \to 0$ is exact. In order to prove that $C_0^M = C_1^M$ we only need to show that $K \in C_0^M$, by [PP, 3.5].

From the long exact sequence associated to $\mathrm{Hom}_A(M, -)$ and $0 \to K \to M' \to X \to 0$, and using the fact that $\mathrm{Ext}_A^1(M, M') = 0$ because M satisfies (T2), we get that $\mathrm{Ext}_A^1(M, K) = 0$, so $K \in M^{\perp_1}$. On the other hand, we know from

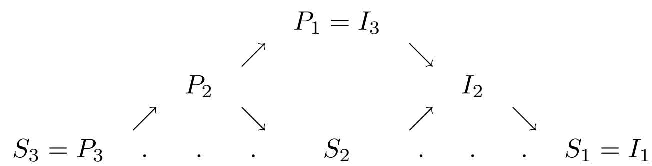
Lemma 3.1 that $M^{\perp_1} \subseteq GenM = C_0^M$, because M satisfies (T3). We conclude that $K \in C_0^M$. □

The next examples show that neither (T1) and (T2), nor (T1) and (T3) imply $C_0^M = C_1^M$.

Example 3.3. *Let A be the hereditary K-algebra given by the quiver:*



The Auslander-Reiten quiver Γ_A is:



We consider the projective module $M = P_2 \oplus P_1$. Clearly M satisfies (T1) and (T2). We observe that $S_2 \in C_0^M$ and $S_2 \notin C_1^M$, so $C_0^M \neq C_1^M$.

Example 3.4. *The module $M = P_1 \oplus P_2 \oplus I_2$ over the hereditary algebra of Example 3.3 has projective dimension 1, satisfies (T3) and $C_0^M \neq C_1^M$.*

Though, as we observed above, there are modules M_A such that $C_0^M = C_1^M$ which do not satisfy any of the three properties of the definition of a tilting module, we will prove that the condition $C_0^M = C_1^M$ guarantees that the module ${}_BM$ satisfies (T2), where $B = \mathrm{End}(M)$. To prove this result we need the following lemma.

Lemma 3.5. *Let M be an A-module such that $C_0^M = C_1^M$. Then* $\mathrm{Im}(\mathrm{Hom}_A(M, -)) \subseteq \mathrm{Ker}(\mathrm{Tor}_1^B(-,_B M))$.

Proof. It is proven in [PP, 3.7] that $\mathrm{Im}(\mathrm{Hom}_A(M, -)) = \mathrm{Ker}(\mathrm{Tor}_1^B(-,_B M))$ when $C_0^M = C_1^M$ under the additional hypothesis that the functor $\mathrm{Hom}_A(M, -)$ is exact in C_0^M. It is not difficult to see that without this additional hypothesis the argument used there applies to prove the desired inclusion. □

Proposition 3.6. *Let M be an A-module and $B =$*End(M)*. If $C_0^M = C_1^M$ then ${}_BM$ satisfies (T2).*

Proof. We consider the following natural isomorphisms:

$$\mathrm{Ext}_B^1({}_BM,_B M) \simeq \mathrm{DTor}_1^B(D_BM,_B M) \simeq \mathrm{DTor}_1^B(\mathrm{Hom}_A(M, DA),_B M).$$

By Lemma 3.5, $\mathrm{Tor}_1^B(\mathrm{Hom}_A(M, DA),_B M) = 0$. Then $\mathrm{Ext}_B^1({}_BM,_B M) = 0$. □

Let ${}_BM_A$ be a B-A-bimodule. We denote by $F = \mathrm{Hom}_A(M, -)$: $\mathrm{mod}A \to \mathrm{mod}B$ and $G = - \otimes_B M$: $\mathrm{mod}B \to \mathrm{mod}A$ the pair of adjoint functors determined by M. Let ϵ_X denote the counit and μ_Y the unit of the adjunction, for $X \in \mathrm{mod}A$ and $Y \in \mathrm{mod}B$. We recall that $F(\epsilon_X)\mu_{FX} = id_{FX}$ and $\epsilon_{GY}G(\mu_Y) = id_{GY}$.

In [PP, 2.2] we proved that if $\epsilon_M : GFM \to M$ is an isomorphism then $C_1^M \subset \mathrm{Im}G \subset C_0^M$. In the next lemma we show that if $B = \mathrm{End}(M)$ and $\mu_{DM} : D_BM \to FGD_BM$ is an isomorphism then $DC_1^{{}_BM} \subset \mathrm{Im}F \subset DC_0^{{}_BM}$.

Lemma 3.7. *Let M be an A-module and $B = \mathrm{End}(M)$. Then:*

i) $\mathrm{Im}F \subset DC_0^{_BM} = \mathrm{Cogen}D_BM$

ii) If $\mu_{D_BM} : D_BM \to FGD_BM$ *is an isomorphism then* $DC_1^{_BM} \subset \mathrm{Im}F$

iii) If μ_{D_BM} *is an isomorphism,* $C_0^M = C_1^M$ *and* $C_0^{_BM} = C_1^{_BM}$ *then M is a *-module*

Proof. i) Let Y_B in $\mathrm{Im}F$ and X' be in $\mathrm{mod}A$ such that $FX' = Y_B$. Since any module X can be immersed in an injective module, there is an exact sequence $0 \to X' \to (DA)^n$, with $n \in N$. Applying the functor F we get the exact sequence: $0 \to FX' \to (FDA)^n$. Since $D_BM = FDA$ we obtain an exact sequence $0 \to Y_B \to (DM)^n$, proving that $FX' = Y_B \in \mathrm{Cogen}D_BM$.

ii) Let ${}_BX \in C_1^{_BM}$, and let ${}_BM_1 \to_B M_0 \to_B X \to 0$ be an exact sequence such that $\mathrm{Hom}_B({}_BM,{}_BM_1) \to \mathrm{Hom}_B({}_BM,{}_BM_0) \to \mathrm{Hom}_B({}_BM,{}_BX) \to 0$ is exact.

By applying the duality we get exact sequences $0 \to D_BX \to D_BM_0 \to D_BM_1$ and $0 \to \mathrm{D}\,\mathrm{Hom}_B({}_BM,{}_BX) \to \mathrm{D}\,\mathrm{Hom}_B({}_BM,{}_BM_0) \to \mathrm{D}\,\mathrm{Hom}_B({}_BM,{}_BM_1)$.
Since there is an isomorphism $\mathrm{D}\,\mathrm{Hom}_B({}_BM,{}_BX) \simeq D_BX \otimes_B M = GD_BX$, natural in ${}_BX$, we get that the sequence

$$0 \to GD_BX \to GD_BM_0 \to GD_BM_1$$

is exact. We apply the functor F and obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & FGD_BX & \to & FGD_BM_0 & \to & FGD_BM_1 \\ & & \downarrow^{\mu_{D_BX}} & & \downarrow^{\mu_{D_BM_0}} & & \downarrow^{\mu_{D_BM_1}} \\ 0 & \to & D_BX & \to & D_BM_0 & \to & D_BM_1 \end{array}$$

where $\mu_{D_BM_0}$ and $\mu_{D_BM_1}$ are isomorphisms. So μ_{D_BX} is an isomorphism and therefore $D_BX \in \mathrm{Im}F$.

iii) Since $C_0^M = C_1^M$, the functor F induces an equivalence of categories between $\mathrm{Gen}M$ and $\mathrm{Im}F$, as follows from [PP, 3.1]. By hypothesis, i) and ii) we know that $DC_1^{_BM} = \mathrm{Im}F = DC_0^{_BM} = \mathrm{Cogen}D_BM$. Then $\mathrm{Im}F = \mathrm{Cogen}D_BM$, so M is a *-module □

Any tilting module M satisfies that $C_0^M = C_1^M$ and it is well known that $DA \in \mathrm{Gen}M = C_0^M$ [HR, 2]. The converse is not true. We observe then that if M is a tilting module and $B = \mathrm{End}(M)$, then ${}_BM$ is also a tilting module, so that also $C_0^{_BM} = C_1^{_BM}$. Now we prove the main result of this section.

Theorem 3.8. *Let M be an A-module and $B = \mathrm{End}(M)$. Then the following conditions are equivalent:*

a) M is a tilting module.

b) $C_0^M = C_1^M$, $C_0^{_BM} = C_1^{_BM}$ *and* $DA \in C_0^M$.

Proof. We just observed that a) implies b). So we prove that b) implies a). We use the following characterization of tilting modules, given in [PP, 3.12]: the module M is a tilting module if and only if $C_0^M = C_1^M$, the functor F is exact in C_0^M and

$DA \in C_0^M$. We also recall from [PP, 3.8] that the first two properties characterize *-modules.

Thus to prove that b) implies a) we only need to show that a module M satisfying b) is a *-module. This amounts to prove that $\mu_{D_B M}$ is an isomorphism, by Lemma 3.7, since we are assuming that $C_0^M = C_1^M$ and $C_0^{_BM} = C_1^{_BM}$. With this purpose we observe first that ϵ_{DA} is an isomorphism, because $DA \in C_1^M$ [PP, 2.2]. From the equality $1_{FDA} = F(\varepsilon_{DA}).\mu_{FDA}$ we get that μ_{FDA} is also an isomorphism. This proves that $\mu_{D_B M}$ is an isomorphism, since $FDA \simeq D_B M$, ending the proof of the theorem. □

4. Generalized tilting modules and the subcategory C_n^M

In this section we study which of the results proven in section 3 for tilting modules can be extended to generalized tilting modules, as defined in [M] and [H, 3]. In particular, we will prove that generalized tilting modules M_A of projective dimension n satisfy that $C_{n-1}^{M_A} = C_n^{M_A}$, $C_{n-1}^{_BM} = C_n^{_BM}$ and D$A \in C_n^{M_A}$, for $B = \mathrm{End}(M_A)$. However, the converse does not hold, so the characterization of tilting modules given in Theorem 3.8 can not be extended to generalized tilting modules.

We recall that a module $M \in \mathrm{mod}A$ is a generalized tilting module if it satisfies the following conditions:

(TG1) $\mathrm{pd}M \leq n$

(TG2) $\mathrm{Ext}_A^i(M, M) = 0$ for all $i \geq 1$

(TG3) There exists an exact sequence $0 \to A \to M_0 \to M_1 \to \cdots \to M_n \to 0$ with $M_i \in \mathrm{add}M$.

By [M, 1.16] and [H, 3] we know that if M is a generalized tilting A-module and $B = \mathrm{End}(M_A)$ then ${}_BM$ is a generalized tilting B^{op}-module. We prove some relations between the validity of some of the properties defining a generalized tilting module, the subcategories C_i^M and $M^{\perp_i}$. We denote $F = \mathrm{Hom}_A(M, -)$.

Proposition 4.1. *Let M be an A-module and n, $s \in \mathbb{N}$. Then:*
a) If M satisfies (TG2), $X \in C_{s-1}^M$ and $0 \to K_{s-1} \to M_{s-1} \to ... \to M_0 \to X \to 0$ is an exact sequence with $M_i \in addM$ and such that the induced sequence $0 \to FK_{s-1} \to FM_{s-1} \to ... \to FM_0 \to FX \to 0$ is exact then $K_{s-1} \in M^{\perp_s} = \cap_{i=1}^s \mathrm{KerExt}_A^i(M, -)$.
b) If M satisfies (TG3) for n then $M^{\perp_n} \subset C_0^M$.
c) If M satisfies (TG2) and (TG3) for n then $C_{n-1}^M = C_n^M$.
d) If M satisfies (TG2) and (TG3) for n then $M^\perp \subset \bigcap_{i\geq 0} C_i^M$.
e) If M satisfies (TG1) and (TG2), then $C_{n-1}^M \subset M^\perp$, where $\mathrm{pd}M = n$
f) If M is a generalized tilting A-module, and $\mathrm{pd}M = n$ then $M^\perp = C_{n-1}^M = C_n^M$.

Proof. We start by proving b). Assume M satisfies (TG3) for n. Then there exists an exact sequence $0 \to A \to M_0 \to M_1 \to \cdots \to M_n \to 0$ with $M_i \in \mathrm{add}M$.

We denote $K_j = \mathrm{Ker}(M_j \to M_{j+1})$ for $1 \leq j \leq n-1$. We get the exact sequences:

$$0 \to A \to M_0 \to K_1 \to 0,$$

$$0 \to K_j \to M_j \to K_{j+1} \to 0, \ for\ 1 \leq j \leq n-2$$

$$0 \to K_{n-1} \to M_{n-1} \to M_n \to 0.$$

Let $X \in M^{\perp_n}$. By applying the functor $\mathrm{Hom}_A(-,X)$ to these sequences we obtain the exact sequence:

$0 \to \mathrm{Hom}_A(K_1,X) \to \mathrm{Hom}_A(M_0,X) \to \mathrm{Hom}_A(A,X) \to \mathrm{Ext}^1(K_1,X) \to 0$
and the isomorphisms $\mathrm{Ext}^i_A(K_j,X) \simeq \mathrm{Ext}^{i+1}_A(K_{j+1},X)$ for $1 \le j \le n-2$ and $\mathrm{Ext}^i_A(K_{n-1},X) = 0$ for $1 \le i \le n-1$.

In particular $\mathrm{Ext}^{n-1}_A(K_{n-1},X) = 0$. Then $\mathrm{Ext}^1_A(K_1,X) \simeq \mathrm{Ext}^2_A(K_2,X) \simeq \cdots \simeq \mathrm{Ext}^{n-1}_A(K_{n-1},X) = 0$. Therefore $0 \to \mathrm{Hom}_A(K_1,X) \to \mathrm{Hom}_A(M_0,X) \to \mathrm{Hom}_A(A,X) \to 0$ is exact. We deduce as in the proof of Lemma 3.1 that there exists an epimorphism $M' \to X \to 0$. Then $X \in \mathrm{Gen}M = C^M_0$, proving b)

In order to prove the remaining items we consider M satisfying (TG2), $X \in C^M_{s-1}$ and an exact sequence $0 \to K_{s-1} \to M_{s-1} \to ... \to M_0 \to X \to 0$, whith $M_i \in \mathrm{add}M$ and such that the induced sequence $0 \to FK_{s-1} \to FM_{s-1} \to ... \to FM_0 \to FX \to 0$ is exact. Then

i) $\mathrm{Ext}^{j+s}_A(M,K_{s-1}) \simeq \mathrm{Ext}^j_A(M,X)$ for all $j \ge 1$ and
ii) $\mathrm{Ext}^j_A(M,K_{s-1}) = 0$, for $1 \le j \le s$.

In fact, let $0 \to K_0 \to M_0 \to X \to 0$ be exact. Then $0 \to FK_0 \to FM_0 \to FX \to 0$ is also exact, and $K_0 \in C^M_{s-2}$. The long exact sequence associated to $0 \to K_0 \to M_0 \to X \to 0$ yields $\mathrm{Ext}^1_A(M,K_0) = 0$ and $\mathrm{Ext}^{j+1}_A(M,K_0) \simeq \mathrm{Ext}^j_A(M,X)$ for all $j \ge 1$, and i) and ii) follow then by induction on s.

Now, the equalities in ii) mean precisely that that $K_{s-1} \in M^{\perp_s}$, proving a). If we also assume that (TG3) holds for n, then from a) and b) we get that $K_{n-1} \in M^{\perp_n} \subseteq C^M_0$. Thus $M \in C^M_n$, proving c).

To prove d) we assume, moreover, that $X \in M^{\perp}$. Then from i) we obtain that $\mathrm{Ext}^{j+s}_A(M,K_{s-1}) = 0$ for all $j \ge 1$. Using that $K_{s-1} \in M^{\perp_s}$ it follows that $K_{s-1} \in M^{\perp}$. On the other hand, since $M^{\perp} \subseteq C^M_0$, by b), we get that $K_{s-1} \in C^M_0$ and therefore $X \in C^M_s$, ending the proof of d).

e) Let $X \in C_{n-1}$, and assume that the above considered module M satisfying (TG2) satisfies also (TG1) and has projective dimension n. Then $\mathrm{Ext}^{j+n}_A(M,K_{n-1}) = 0$ for all $j \ge 1$ and from i) we get that X is in $M^{\perp}$, as desired.

Finally, we observe that f) follows directly from c), d) and e). □

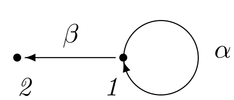
Theorem 4.2. *Let M be a generalized tilting A-module, with* $\mathrm{pd}M = n$ *and* $B = \mathrm{End}(M_A)$. *Then* $C^{M_A}_{n-1} = C^{M_A}_n$, $C^{{}_BM}_{n-1} = C^{{}_BM}_n$ *and* $DA \in C^{M_A}_n$

Proof. From Proposition 4.1 f) and the fact that both M_A and ${}_BM$ are generalized tilting modules we have that $C^{M_A}_{n-1} = C^{M_A}_n$ and $C^{{}_BM}_{n-1} = C^{{}_BM}_n$.

Since DA is an injective module we know that $\mathrm{Ext}^i_A(M,DA) = 0$, for all $i \ge 1$. Then $DA \in M^{\perp}_A$. By f), $C^{M_A}_n = M^{\perp}_A$, so $DA \in C^{M_A}_n$. □

The converse of the theorem is not true as we illustrate in the following example.

Example 4.3. *Let A be the K-algebra given by the quiver*



with $\alpha^2 = 0$, *and* $\beta\alpha = 0$

The Auslander-Reiten quiver Γ_A is

$$\begin{array}{ccccccc} S_1 & & & & I_2 & & \\ & \searrow & & \nearrow & & \searrow & \\ & & P_1 & & & & S_1 \\ & \nearrow & & \searrow & & \nearrow & \\ S_2 = P_2 & & & & I_1 & & \end{array}$$

Let $M = I_1 \oplus I_2$. Then $\mathrm{pd}M = \infty$, $DA \in C_1^M = C_2^M$ and $C_1^{{}_BM} = C_2^{{}_BM}$.

In the previous example we saw that the conditions $C_{n-1}^{M_A} = C_n^{M_A}$, $C_{n-1}^{{}_BM} = C_n^{{}_BM}$ and $DA \in C_n^{M_A}$ do not guarantee that M is a generalized tilting module. We will prove that when $DA \in C_n^{M_A}$ and $C_{n-1}^{M_A} = C_n^{M_A}$ the module ${}_BM$ satisfies (TG2).

We start with the following proposition.

Proposition 4.4. *Let M be an A-module and $n \geq 2$. If $X \in C_n^{M_A}$ then $\mathrm{Tor}_i^B(FX, M) = 0$, for $i = 1, ..., n-1$.*

Proof. This result is proven in [PP, 3.7] under the additional hypothesis that $C_0^M = C_1^M$, and the proof there can be easily addapted to the present situation, as we show next.

If $X \in C_n^{M_A}$ then there exists an exact sequence $M_n \to \cdots \to M_0 \to X \to 0$ with $M_i \in \mathrm{add}M$, and such that the induced sequence $FM_n \to \cdots \to FM_0 \to FX \to 0$ is exact. Denote $K_j = \mathrm{Ker}(M_j \to M_{j-1})$ and $K_0 = \mathrm{Ker}(M_0 \to X)$. We get the exact sequences

(1) $0 \to FK_0 \to FM_0 \to FX \to 0$

(2) $0 \to FK_j \to FM_j \to FK_{j-1} \to 0$, for $1 \leq j \leq n$

We apply the functor $G = - \otimes_B M$ to the sequence (1) and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \mathrm{Tor}_1^B(FM_0, M) & \to & \mathrm{Tor}_1^B(FX, M) & \to & GFK_0 & \to & GFM_0 & \to & GFX & \to & 0 \\ & & & & \downarrow^{\varepsilon_{K_0}} & & \downarrow^{\varepsilon_{M_0}} & & \downarrow^{\varepsilon_X} & & \\ & & 0 & \longrightarrow & K_0 & \longrightarrow & M_0 & \longrightarrow & X & \to & 0 \end{array}$$

where ε_{K_0}, ε_{M_0} and ε_X are isomorphisms by [PP, 2.2] ($K_0 \in C_1^M$ because $n \geq 2$) . Since FM_0 is B-projective, then $\mathrm{Tor}_i^B(FM_0, M) = 0$. We obtain $\mathrm{Tor}_1^B(FX, M) = 0$ and $\mathrm{Tor}_i^B(FK_0, M) \simeq \mathrm{Tor}_{i+1}^B(FX, M) = 0$ for $i \geq 1$. The result follows then by induction, using that $K_0 \in C_{n-1}^{M_A}$.

□

The previous result holds for all $i \geq 1$ under the additional hypothesis that $C_{n-1}^{M_A} = C_n^{M_A}$, $n \geq 1$, as we state next.

Corollary 4.5. *Let M be an A-module and $n \geq 1$. If $X \in C_n^{M_A}$ and $C_{n-1}^{M_A} = C_n^{M_A}$ then $\mathrm{Tor}_i^B(FX, {}_BM) = 0$ for all $i \geq 1$.*

Proof. If $C_{n-1}^{M_A} = C_n^{M_A}$, then $C_{n-1}^{M_A} = C_m^{M_A}$ for all $m \geq n \geq 1$. The result follows then from Proposition 4.4.

□

Corollary 4.6. *Let M be an A-module and $n \geq 1$. If $DA \in C_n^{M_A}$ and $C_{n-1}^{M_A} = C_n^{M_A}$ then $\mathrm{Ext}_B^i({}_BM, {}_BM) = 0$ for all $i \geq 1$ (${}_BM$ satisfies (TG2)).*

Proof. Since $DA \in C_n^{M_A}$ and $C_{n-1}^{M_A} = C_n^{M_A}$, we know by the above corollary that $\mathrm{Tor}_i^B(FDA, M) = 0$ for all $i \geq 1$. Then $\mathrm{Tor}_i^B(DM,_B M) = 0$ because $FDA = DM$. The corollary follows from the isomorphism $\mathrm{Ext}_B^i(M, M) \simeq \mathrm{Tor}_i^B(DM, M)$.

□

ACKNOWLEDGEMENTS

The authors gratefully acknowledge partial support from Universidad Nacional del Sur and from CONICET, of Argentina. The first author is a researcher from CONICET

References

[As] I. Assem. *Tilting theory. An introduction.* Topics in Algebra, Banach Centre Publications, vol. 26, PWN, Warsaw, 127-180, (1990).

[A] M. Auslander *Notes on representation theory.* Brandeis Univ., (1973).

[ARS] M. Auslander, I. Reiten, and S. O. Smal∅. *Representation theory of artin algebras*, vol. 36 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, (1995).

[BB] S. Brenner and M. C. R. Butler. *Generalization of the Bernstein-Gelfand-Ponomarev reflection functors.* Lecture Notes in Math., 832, Springer-Verlag, Berlin-Heidelberg, New York, 103-169, (1980).

[C] R. Colpi. *Some remarks on equivalences between categories of modules.* Comm. Algebra 18(6), 1935-1951, (1990).

[DH] G. D'Este and D. Happel. *Representable equivalences are represented by tilting modules.* Rend. Sem. Mat. Univ. Padova, Vol. 83. (1990).

[H] D. Happel. *Triangulated categories in the representation theory of finite dimensional algebras.* London Math. Soc. Lecture Note Series. 119, (1988).

[HR] D. Happel and C.M. Ringel. *Tilted algebras.* Trans. Amer. Math. Soc., 284, 399-443, (1982).

[Ho] M. Hoshino. *Splitting torsion theories induced by tilting modules.* Comm. Algebra 11, N° 4, 427-441, (1983).

[M] Y. Miyashita *Tilting modules of finite projective dimension.* Math. Z., 193, 113-146, (1986).

[PP] M. I. Platzeck and N. I. Pratti. *On a theorem of Auslander and applications.* Comm. in Algebra, 28(6), 2817-2835, (2000)

[R] C. M. Ringel. *Tame algebras and integral quadratic forms*, Lecture Notes in Math., 1099, Springer-Verlag, Berlin-Heidelberg, (1984).