Lorentzian Flat Lie Groups Admitting a Timelike Left-Invariant Killing Vector Field

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Abstract: We call a connected Lie group endowed with a left-invariant Lorentzian flat metric Lorentzian flat Lie group. In this Note, we determine all Lorentzian flat Lie groups admitting a timelike left-invariant Killing vector field. We show that these Lie groups are 2-solvable and unimodular and hence geodesically complete. Moreover, we show that a Lorentzian flat Lie group (G, μ) admits a timelike left-invariant Killing vector field if and only if G admits a left-invariant Riemannian metric which has the same Levi-Civita connection of μ. Finally, we give an useful characterization of left-invariant pseudo-Riemannian flat metrics on Lie groups G satisfying the property: for any couple of left invariant vector fields X and Y their Lie bracket $[X, Y]$ is a linear combination of X and Y.

Key words: Lie groups, Lie algebras, Flat Lorentzian metric, Killing vector field.

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1. Introduction and main results

A connected Lie group G together with a left-invariant pseudo-Riemannian metric is called *pseudo-Riemannian Lie group*. When the metric is definite positive or of signature $(-, +, \ldots, +)$ the group is called Riemannian or Lorentzian. Let (G, μ) be a pseudo-Riemannian Lie group and $\mathfrak{g}$ its Lie algebra endowed with the inner product $\langle \ , \ \rangle = \mu(e)$. The Levi-Civita connection of (G, μ) defines a product $(u, v) \mapsto uv$ on $\mathfrak{g}$ called *Levi-Civita product* given by:

$$2\langle uv, w \rangle = \langle [u, v], w \rangle - \langle [v, w], u \rangle + \langle [w, u], v \rangle. \tag{1}$$

For any $u \in \mathfrak{g}$, we denote by L_u and R_u respectively the left multiplication and the right multiplication on $\mathfrak{g}$ given by $\mathrm{L}_u(v) = uv$ and $\mathrm{R}_u(v) = vu$. For any $u \in \mathfrak{g}$, L_u is skew-symmetric with respect to $\langle \ , \ \rangle$, and $\mathrm{ad}_u = \mathrm{L}_u - \mathrm{R}_u$ where $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$ is given by $\mathrm{ad}_u(v) = [u, v]$.

The curvature of $(\mathfrak{g}, \langle \ , \ \rangle)$ is given by $K(u, v) = \mathrm{L}_{[u,v]} - [\mathrm{L}_u, \mathrm{L}_v]$. If K vanishes identically (G, μ) is called *pseudo-Riemannian flat Lie group* and

$(\mathfrak{g}, \langle\ ,\ \rangle)$ is called *pseudo-Riemannian flat Lie algebra*. This is equivalent to the fact that $\mathfrak{g}$ endowed with the Levi-Civita product is a left symmetric algebra, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\text{ass}(u, v, w) = \text{ass}(v, u, w) \,,$$

where $\text{ass}(u, v, w) = (uv)w - u(vw)$. This relation is equivalent to

$$\text{R}_{uv} - \text{R}_v \circ \text{R}_u = [\text{L}_u, \text{R}_v] \,, \tag{2}$$

for any $u, v \in \mathfrak{g}$.

Let $S(\mathfrak{g})$ be the Lie subalgebra of $\mathfrak{g}$ defined by

$$S(\mathfrak{g}) = \left\{ u \in \mathfrak{g} \,:\, \text{ad}_u + \text{ad}_u^* = 0 \right\} \,,$$

where ad_u^* is the adjoint of ad_u with respect to $\langle\ ,\ \rangle$. It is easy to see that a left invariant vector field X on G is a Killing vector field iff $X(e) \in S(\mathfrak{g})$.

It is a well-known result that a pseudo-Riemannian flat Lie group is geodesically complete if and only if it is unimodular [3]. In [5], J. Milnor characterized Riemannian flat Lie groups. In [1, 2] there is a more precise version of Milnor's Theorem: A Lie group is a Riemannian flat Lie group if and only if its Lie algebra $\mathfrak{g}$ splits as an orthogonal direct sum: $\mathfrak{g} = S(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, where $S(\mathfrak{g})$ and $[\mathfrak{g}, \mathfrak{g}]$ are abelian. Moreover, in this case the dimension of $[\mathfrak{g}, \mathfrak{g}]$ is even and the Levi-Civita product is given by:

$$\text{L}_a = \begin{cases} \text{ad}_a & \text{if } a \in S(\mathfrak{g}) \,, \\ 0 & \text{if } a \in [\mathfrak{g}, \mathfrak{g}] \,. \end{cases} \tag{3}$$

In the Lorentzian case, the only known Lorentzian flat Lie groups are those nilpotent [3] and those with degenerate center [1]. Their Lie algebras are determined by the double extension process, and the metric in both cases is complete.

Recall that a vector field X on a Lorentzian manifold (M, g) is called timelike (resp. spacelike, null) if $g_p(X_p, X_p) < 0$ (resp. $g_p(X_p, X_p) > 0$, $g_p(X_p, X_p) = 0$) for any $p \in M$. Many authors in mathematics and physics are interested in the existence of timelike Killing vector field in a Lorentzian manifold. This condition implies interesting information on the structure of the manifold. In General Relativity, if this condition is satisfied, then the Lorentzian manifold is called stationary, and it is known that any compact stationary manifold is geodesically complete [7].

Our first and main purpose in this paper is to determine Lorentzian flat Lie groups admitting a timelike left-invariant killing vector field. More precisely, we will prove the following result.

THEOREM 1.1. *Let* (G, μ) *be a Lorentzian Lie group. Then it is flat and carries a timelike left invariant killing vector field if and only if its Lie algebra* $\mathfrak{g}$ *splits as an orthogonal direct sum* $\mathfrak{g} = S(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$*, where* $[\mathfrak{g}, \mathfrak{g}]$ *is abelian, and* $S(\mathfrak{g})$ *is abelian and contains a timelike vector. Moreover, in this case the dimension of* $[\mathfrak{g}, \mathfrak{g}]$ *is even and the Levi-Civita product is given by* (3).

From this Theorem, we deduce the following corollaries:

COROLLARY 1.2. *A Lorentzian flat Lie group which admits a timelike left-invariant killing vector field is 2-solvable and unimodular and hence geodesically complete.*

COROLLARY 1.3. *Let* (G, μ) *be a Lorentzian flat Lie group. The existence of timelike left-invariant killing vector field on* G *is equivalent to the existence of left-invariant Riemannian metric on* G *with the same Levi-Civita connection of* μ.

On the other hand, we define the class $\mathcal{C}$ consisting of those Lie groups G which are non commutative and their Lie algebras $\mathfrak{g}$ satisfy: for all x and y in $\mathfrak{g}$, $[x, y]$ is a linear combination of x and y. This class $\mathcal{C}$ has been studied by several authors. J. Milnor in [5] showed that every left-invariant Riemannian metric on $G \in \mathcal{C}$ has constant negative sectional curvature. K. Nomizu in [6] showed that every left invariant Lorentzian metric on $G \in \mathcal{C}$ has constant sectional curvature which can be positive, negative or null.

The following theorem gives an useful characterization of flat pseudo-Riemannian left invariant metric on a Lie group belonging to $\mathcal{C}$.

THEOREM 1.4. *Let* G *be a non commutative Lie group that belongs to* $\mathcal{C}$*, and* μ *a left-invariant pseudo-Riemannian metric on* G*. Then* μ *is flat if and only if the restriction of* $\mu(e)$ *to* $[\mathfrak{g}, \mathfrak{g}]$ *is degenerate.*

We deduce the following corollary.

COROLLARY 1.5. *Let* G *in* $\mathcal{C}$*. Then* G *is non unimodular and hence any left-invariant pseudo-Riemannian metric* μ *on* G *such that the restriction of* $\mu(e)$ *to* $[\mathfrak{g}, \mathfrak{g}]$ *is degenerate is geodesically incomplete.*

2. Proofs of theorems

This section is devoted to the proofs of the results of the last section. The following proposition appeared first in [1, Lemma 3.1] will be useful later.

Proposition 2.1. *Let* (G, μ) *be a Riemannian or Lorentzian flat Lie group, then:*

$$S(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^{\perp} = \left\{ u \in \mathfrak{g} \, : \, R_u = 0 \right\}.$$

In particular $S(\mathfrak{g})$ *is abelian.*

Proof. Let (G, μ) be a pseudo-Riemannian flat Lie group and $(\mathfrak{g}, \langle \, , \, \rangle)$ its pseudo-Riemannian flat Lie algebra. We have obviously

$$(\mathfrak{g}\mathfrak{g})^{\perp} = \left\{ u \in \mathfrak{g} \, : \, R_u = 0 \right\}.$$

Let $u \in S(\mathfrak{g})$. By using (1) one can see easily that $u.u = 0$, and deduce from (2) that $[R_u, L_u] = R_u^2$. Since R_u and L_u are skew-symmetric then $[R_u, L_u]$ is skew-symmetric. But R_u^2 is symmetric, then $R_u^2 = 0$.

If the metric is Riemannian then $R_u = 0$. If the metric is Lorentzian, then $Im(R_u)$ is a totally isotropic subspace and hence there exists an isotropic vector $e \in \mathfrak{g}$ and a covector $\alpha \in \mathfrak{g}^*$ such that $R_u(v) = \alpha(v)e$ for any $v \in \mathfrak{g}$. Choose a basis $\{e, \bar{e}, f_1, \ldots, f_{n-2}\}$ of $\mathfrak{g}$ such that $\mathrm{span}\{e, \bar{e}\}$ and $\mathrm{span}\{f_1, \ldots, f_{n-2}\}$ are orthogonal, $\{f_1, \ldots, f_{n-2}\}$ is orthonormal, $\bar{e}$ is isotropic and $\langle e, \bar{e} \rangle = 1$. We have, for any $i = 1, \ldots, n-2$,

$$\langle R_u(e), \bar{e} \rangle = \alpha(e) = -\langle e, R_u(\bar{e}) \rangle,$$

$$\langle R_u(\bar{e}), \bar{e} \rangle = 0 = \alpha(\bar{e}),$$

$$\langle R_u(f_i), \bar{e} \rangle = \alpha(f_i) = -\langle f_i, R_u(\bar{e}) \rangle,$$

hence $\alpha = 0$ and then $R_u = 0$. ∎

Proof of Theorem 1.1.   Let $(\mathfrak{g}, \langle \, , \, \rangle)$ be a flat Lorentzian Lie algebra. Suppose that there exists $u \in S(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^{\perp}$ such that $\langle u, u \rangle < 0$. Then the restriction of $\langle \, , \, \rangle$ to $u^{\perp}$ is definite positive and so for $S(\mathfrak{g})^{\perp} = \mathfrak{g}\mathfrak{g} \subset u^{\perp}$. Then $\mathfrak{g} = S(\mathfrak{g}) \oplus \mathfrak{g}\mathfrak{g}$ and the restriction of $\langle \, , \, \rangle$ to $S(\mathfrak{g})$ is nondegenerate Lorentzian. Let us show that $\mathfrak{h} = \mathfrak{g}\mathfrak{g}$ is abelian and $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$. It is clear that $\mathfrak{h}$ is a Riemannian flat Lie algebra and hence

$$\mathfrak{h} = S(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}],$$

where $S(\mathfrak{h})$ and $[\mathfrak{h}, \mathfrak{h}]$ are abelian. Moreover, there exist $\lambda_1, \ldots, \lambda_p \in S(\mathfrak{h})^* \backslash \{0\}$ and orthonormal basis $(e_1, \ldots, e_{2p})$ of $[\mathfrak{h}, \mathfrak{h}]$ such that for all $i = 1, \ldots, p$ and for all $s \in S(\mathfrak{h})$:

$$\mathrm{ad}_s(e_{2i-1}) = \lambda_i(s)e_{2i} \,,$$

$$\mathrm{ad}_s(e_{2i}) = -\lambda_i(s)e_{2i-1} \,.$$

On the other hand, for any $s \in S(\mathfrak{g})$, ad_s which is skew-symmetric leaves $\mathfrak{h}$ invariant and so it leaves $[\mathfrak{h}, \mathfrak{h}]$ also invariant and hence $S(\mathfrak{h})$. Moreover, if $x \in S(\mathfrak{h})$ satisfies $[x, s] = 0$ for any $s \in S(\mathfrak{g})$, then ad_x is skew-symmetric and then $x \in S(\mathfrak{g})$ and hence $x = 0$. So there exists also $\mu_1, \ldots, \mu_q \in S(\mathfrak{g})^* \backslash \{0\}$ and orthonormal basis $(f_1, \ldots, f_{2q})$ of $S(\mathfrak{h})$ such that for all $i = 1, \ldots, q$ and for all $s \in S(\mathfrak{g})$

$$\mathrm{ad}_s(f_{2i-1}) = \mu_i(s)f_{2i} \,,$$

$$\mathrm{ad}_s(f_{2i}) = -\mu_i(s)f_{2i-1} \,.$$

Suppose that $\mathfrak{h}$ is non abelian. Without loss of generality we can suppose that $[f_1, e_1] = \lambda_1(f_1)e_2 \neq 0$. We have also $[f_1, e_2] = -\lambda_1(f_1)e_1 \neq 0$. Moreover, there exists $s \in S(\mathfrak{g})$ such that $[s, f_1] = \mu_1(s)f_2 \neq 0$. We have necessarily $[f_2, e_1] = \lambda_1(f_2)e_2 \neq 0$. Otherwise, if $[f_2, e_1] = 0$ then $[f_2, e_2] = 0$ and hence

$$\mathrm{ad}_{[s,f_2]}(e_1) = -\mu_1(s)\lambda_1(f_1)e_2 = [\mathrm{ad}_s, \mathrm{ad}_{f_2}](e_1)$$

$$= [[s, e_1], f_2] = \sum_{i=3}^{2p} \alpha_i e_i \,.$$

This is impossible since $\mu_1(s)\lambda_1(f_1) \neq 0$. On the other hand, we have

$$0 = [e_1, [f_1, s]] + [f_1, [s, e_1]] + [s, [e_1, f_1]]$$

$$= -\mu_1(s)[e_1, f_2] + [f_1, \sum_{i=2}^{2p} \beta_i e_i] - \lambda_1(f_1)[s, e_2]$$

$$= \mu_1(s)\lambda_1(f_2)e_2 + \gamma_1 e_1 + \sum_{i=3}^{2p} \gamma_i e_i \,.$$

This is impossible and hence $\mathfrak{h}$ is abelian.

Now it is obvious that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \subset [\mathfrak{g}, \mathfrak{g}]$. On the other hand, for any $i = 1, \ldots, 2q$, there exists $s \in S(\mathfrak{g})$ such that

$$[s, f_{2i-1}] = \mu_i(s) f_{2i} \neq 0\,,$$

$$[s, f_{2i}] = -\mu_i(s) f_{2i-1} \neq 0\,.$$

This shows that $f_{2i-1}, f_{2i} \in [\mathfrak{g}, \mathfrak{g}]$. Conversely, if the Lie algebra $\mathfrak{g}$ of the Lorentzian Lie group (G, μ) splits as an orthogonal direct sum $\mathfrak{g} = S(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ where $S(\mathfrak{g})$ and $[\mathfrak{g}, \mathfrak{g}]$ are abelian, and there exists a timelike vector in $S(\mathfrak{g})$ then the Levi-Civita product is given by (3) which implies that (G, μ) is a Lorentzian flat Lie group which admits a timelike left-invariant Killing vector field. ∎

Proof of Corollary 1.3.   Let (G, μ) be a Lorentzian flat Lie group and $(\mathfrak{g}, \langle\ ,\ \rangle)$ its Lorentzian flat Lie algebra. If (G, μ) admits a timelike left-invariant Killing vector field, then according to Theorem 1.1, $\mathfrak{g} = S(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ where $S(\mathfrak{g})$ is nondegenerate Lorentzian and $[\mathfrak{g}, \mathfrak{g}]$ is nondegenerate Euclidean. Choose any definite positive product $\langle\ ,\ \rangle_s$ on $S(\mathfrak{g})$ and define $\langle\ ,\ \rangle'$ on $\mathfrak{g}$ as the orthogonal sum of $\langle\ ,\ \rangle_s$ and the restriction of $\langle\ ,\ \rangle$ to $[\mathfrak{g}, \mathfrak{g}]$. It is easy to check that the left invariant Riemannian metric on G associated to $\langle\ ,\ \rangle'$ has the same Levi-Civita connection as μ.

Conversely, suppose that G possesses a left-invariant Riemannian metric μ' with the same Levi-Civita connection of μ. Put $\langle\ ,\ \rangle = \mu(e)$ and $\langle\ ,\ \rangle' = \mu'(e)$. Since μ and μ' have the same Levi-Civita connection then, according to Proposition 2.1 μ and μ' have also the same space of left invariant Killing vector fields identified to $S(\mathfrak{g})$. Then, by applying both [1, Theorem 3.1] and Theorem 1.1, we get $\mathfrak{g} = S(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ and this splitting is orthogonal with respect to $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$. If the restriction of $\langle\ ,\ \rangle$ to $S(\mathfrak{g})$ is Lorentzian then μ admits obviously a left invariant timelike Killing vector field. Suppose now that the restriction of $\langle\ ,\ \rangle$ to $[\mathfrak{g}, \mathfrak{g}]$ is Lorentzian. Then there exists a basis $\{e_1, \ldots, e_p\}$ of $[\mathfrak{g}, \mathfrak{g}]$ which is orthogonal with respect to $\langle\ ,\ \rangle$ and orthonormal with respect to $\langle\ ,\ \rangle'$ and such that, $\langle e_i, e_i \rangle > 0$ for $i = 1, \ldots, p - 1$ and $\langle e_p, e_p \rangle < 0$. Let $s \in S(\mathfrak{g})$. We have $\langle [s, e_p], e_p \rangle = 0$, and

$$\begin{aligned}
\langle [s, e_p], e_i \rangle &= \langle [s, e_p], e_i \rangle' \langle e_i, e_i \rangle \\
&= -\langle [s, e_i], e_p \rangle = -\langle [s, e_i], e_p \rangle' \langle e_p, e_p \rangle \\
&= \langle [s, e_p], e_i \rangle' \langle e_p, e_p \rangle
\end{aligned}$$

for any $i = 1, \ldots, p - 1$. Since $\langle e_p, e_p \rangle < 0$ and $\langle e_i, e_i \rangle > 0$ then $\langle [s, e_p], e_i \rangle = 0$

and hence $[s, e_p] = 0$ for any $s \in S(\mathfrak{g})$ and hence e_p lies in the center of $\mathfrak{g}$ and thus $e_p \in S(\mathfrak{g})$ which is impossible. This completes the proof of the corollary. ∎

Proof of Theorem 1.4. Let (G, μ) be a pseudo-Riemannian Lie group and $(\mathfrak{g}, \langle \, , \, \rangle)$ its pseudo-Riemannian Lie algebra.

Milnor in [5] has shown that $G \in \mathcal{C}$ if and only if $\mathfrak{g}$ contains an abelian ideal $\mathcal{U}$ of codimension 1 and an element $b \notin \mathcal{U}$ such that $[b, x] = x$ for every $x \in \mathcal{U}$. Note that $\mathcal{U} = [\mathfrak{g}, \mathfrak{g}]$.

If μ is flat, then since G is non unimodular, thus according to [1, Proposition 3.1], we deduce that the restriction of $\langle \, , \, \rangle$ to $[\mathfrak{g}, \mathfrak{g}]$ is degenerate.

Conversely, if the restriction of $\langle \, , \, \rangle$ to $[\mathfrak{g}, \mathfrak{g}]$ is degenerate, then there exists $e \in [\mathfrak{g}, \mathfrak{g}]$ such that $\langle e, x \rangle = 0$ for any $x \in [\mathfrak{g}, \mathfrak{g}]$. Let $y \in \mathfrak{g}$ such that $\langle e, y \rangle \neq 0$. We put:
$$d = \frac{y}{\langle y, e \rangle} - \frac{1}{2} \frac{\langle y, y \rangle}{\langle y, e \rangle^2} e \,,$$
then $\langle d, e \rangle = 1$ and $\langle d, d \rangle = 0$.

The restriction of $\langle \, , \, \rangle$ to $\operatorname{span}\{e, d\}$ is non degenerate, thus if we put $\mathcal{B} = \operatorname{span}\{e, d\}^\perp$ then we have
$$\mathfrak{g} = \mathbb{R}e \oplus \mathcal{B} \oplus \mathbb{R}d \,,$$
where $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}e \oplus \mathcal{B}$.

We have $d = \alpha b + u_0$ where $\alpha \in \mathbb{R} \setminus \{0\}$ and $u_0 \in [\mathfrak{g}, \mathfrak{g}]$. Then the only non-vanishing brackets are $[d, e] = \alpha e$, $[d, u] = \alpha u$ for any $u \in [\mathfrak{g}, \mathfrak{g}]$. By using (1), we deduce that the Levi-Civita product is given by:
$$L_e = 0 \,, \qquad de = \alpha e \,, \qquad dd = -\alpha d \,,$$
$$du = ue = 0 \,, \qquad ud = -\alpha u \,, \qquad uu' = \alpha \langle u, u' \rangle e$$
for any $u, u' \in \mathcal{B}$. From these relations we obtain
$$K(e, d) = K(e, u) = K(d, u) = K(u, u') = 0 \qquad \text{for any } u, u' \in \mathcal{B} \,,$$
which implies that the curvature K vanishes identically. ∎

References

[1] M. AIT BEN HADDOU, M. BOUCETTA, H. LEBZIOUI, Left-invariant Lorentzian flat metrics on Lie groups, *J. Lie Theory* **22** (2012), 269 – 289.

[2] A. BAHAYOU, M. BOUCETTA, Metacurvature of Riemannian Poisson-Lie groups, *J. Lie Theory* **19** (2009), 439 – 462.

[3] A. AUBERT, A. MEDINA, Groupes de Lie pseudo-riemanniens plats, *Tohoku Math. J. (2)* **55** (2003), 487 – 506.

[4] A. AUBERT, "Structures Affines et Pseudo-métriques Invariantes à Gauche sur des Groupes de Lie", Thesis, Univesité Montpellier II, 1996.

[5] J. MILNOR, Curvature of left invariant metrics on Lie Groups, *Advances in Math.* **21** (1976), 293 – 329.

[6] K. NOMIZU, Left-invariant lorentz metrics on Lie groups, *Osaka J. Math.* **16** (1979), 143 – 150.

[7] M. SÁNCHEZ, Lorentzian manifolds admitting a Killing vector field, *Nonlinear Anal.* **30** (1997), 643 – 654.

[8] B. O'NEILL, "Semi-Riemannian Geometry", Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.