Global Asymptotic Stability for Semilinear Equations via Thompson's Metric

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Abstract: In ordered Banach spaces we prove the global asymptotic stability of the unique strictly positive equilibrium of the semilinear equation $u' = Au + f(u)$, if A is the generator of a positive and exponentially stable C_0-semigroup and f is a contraction with respect to Thompson's metric. The given estimates show that convergence holds with a uniform exponential rate.

Key words: Ordered Banach spaces, Thompson metric, global stability, semilinear equations.

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1. Introduction

Let $(E, \| \cdot \|)$ be a real Banach space ordered by a normal cone K, that is K is a closed convex subset of E such that $\lambda K \subseteq K$ $(\lambda \geq 0)$, $K \cap (-K) = \{0\}$, inducing an ordering by $x \leq y :\iff y - x \in K$, and

$$\exists\, c \geq 1 : \qquad 0 \leq x \leq y \;\Rightarrow\; \|x\| \leq c\|y\| \,.$$

Moreover we assume that K is solid, that is K has nonempty interior K°, and we set $x \ll y :\iff y - x \in K^\circ$. In this situation K° endowed with the Thompson metric [11]

$$d(x, y) := \log \big(\min\{\alpha \geq 1 \,:\, x \leq \alpha y, \; y \leq \alpha x\} \big)$$

is a complete metric space. A function $f : K^\circ \to K^\circ$ is d-Lipschitz continuous with constant $l \geq 0$ if and only if, for all $x, y \in K^\circ$,

$$x \leq \alpha y \wedge y \leq \alpha x \quad\Rightarrow\quad f(x) \leq \alpha^l f(y) \wedge f(y) \leq \alpha^l f(x) \,.$$

We study in this paper the dynamical system

$$
\begin{aligned}
(1) \qquad\qquad u'(t) &= A\,u(t) + f(u(t)) \qquad (t \geq 0)\,, \\
u(0) &= x \;\in K^\circ\,,
\end{aligned}
$$

where A is the generator of a C_0-semigroup $(T(t))_{t \geq 0}$ of positive operators on E, i.e.,

$$x \geq 0 \quad \Rightarrow \quad T(t)x \geq 0 \ \text{ for all } \ t \geq 0 \,.$$

In particular, A is quasimonotone increasing (cf. Section 2). We are interested in the semiflow given by mild solutions of (1), i.e., by solutions of the integral equation

$$(2) \quad u(t) = T(t)x + \int_0^t T(t-s)f(u(s)) \, \mathrm{d}s \qquad \big(t \in [0, t_{\max}(x)), \ x \in K^\circ \big) \,.$$

If $l < 1$ and the semigroup is exponentially stable, then we obtain a unique $x_0 \in K^\circ \cap D(A)$ such that

$$Ax_0 + f(x_0) = 0$$

(cf. Proposition 6 below); i.e., the dynamical system given by (1) has a unique equilibrium x_0 in K°. In our main result Theorem 1 below we show that, under these assumptions, the equilibrium x_0 is globally asymptotically stable in K°, i.e., any mild solution $u(t,x)$ of (1) converges to x_0 as $t \to \infty$. Moreover, we establish an explicit estimate for $d(u(t,x), x_0)$ which shows that convergence holds with a uniform exponential rate. Under the additional assumption that x_0 is an eigenvalue of A we obtain in Theorem 2 an optimal estimate on $d(u(t,x), x_0)$.

As an introductory example consider $E = l^\infty(\mathbb{Z})$ endowed with the supremum norm $\|\cdot\|_\infty$ and ordered by the normal and solid cone

$$K := \big\{ x \in l^\infty(\mathbb{Z}) \ : \ x_n \geq 0 \ \text{ for all } n \in \mathbb{Z} \big\} \,.$$

Then

$$K^\circ = \Big\{ x \in l^\infty(\mathbb{Z}) \ : \ \inf_{n \in \mathbb{Z}} x_n > 0 \Big\},$$

and for $|\beta| < 2$ the function $f : K^\circ \to K^\circ$ defined by

$$f(x) = \big(2 + \beta \sin\big(\log(x_{n-1}x_{n+1})\big)\big)_{n \in \mathbb{Z}}$$

is d-Lipschitz continuous with constant $\dfrac{2|\beta|}{\sqrt{4-\beta^2}}$ (cf. Section 4). Thus our results yield that, for $|\beta| < 2/\sqrt{5}$, all solutions of the infinite system

$$u_n'(t) = u_{n+1}(t) - 4u_n(t) + u_{n-1}(t) + 2$$
$$+ \beta \sin\big(\log\big(u_{n-1}(t)u_{n+1}(t)\big)\big) \qquad (n \in \mathbb{Z}),$$

with initial values in K° are exponentially decaying to the constant sequence $(1)_{n\in\mathbb{Z}}$ in $l^\infty(\mathbb{Z})$ as $t \to \infty$, with respect to Thompson's metric.

Concerning our assumptions we remark that, for $l > 1$, global existence for solutions of (1) fails in general, as one can already see in the one-dimensional case $E = \mathbb{R}$ for $Ax = -x$ and $f(x) = x^l$. The case $l = 1$ is special in the sense that it includes linear functions. For $E = \mathbb{R}$, $Ax = -x$ and $f(x) = 2x$ in (1) existence of equilibria in K° fails, and all solutions in K° grow exponentially.

Motivated by the analysis of generalized Riccati equations and other non-linear matrix equations, the contraction rate of flows with respect to Thompson's metric has been investigated in [7], [10]. In particular, Gaubert and Qu [3] studied the time-dependent case for order-preserving flows. In our case the semiflow generated by (1) is not order-preserving in general, since f need not to be quasimonotone increasing in the sense of Volkmann [12].

The paper is organized as follows: In the preliminary Section 2 we state some properties of Thompson's metric in our situation and of positive semi-groups. Moreover, we show existence of global solutions to (1) and nonexpansiveness of the semiflow with respect to Thompson's metric under the assumption that $l < 1$ and that A generates a semigroup of positive operators. In Section 3 we state and prove our main results on global asymptotic stability. Section 4 contains some examples.

2. Preliminaries

The interrelation between the topologies generated by d and $\|\cdot\|$ on K° are characterized by the following Proposition 1, see [5]. Let $p \in K^\circ$ be fixed and let $\|\cdot\|_p$ denote the Minkowski functional

$$\|x\|_p = \min\left\{\alpha \geq 0 \,:\, -\alpha p \leq x \leq \alpha p\right\} \qquad (x \in E).$$

Then $\|\cdot\|_p$ is an equivalent norm on E [2, Prop.19.9], and we write

$$\operatorname{dist}_p(x, \partial K) := \inf\left\{\|x - y\|_p \,:\, y \in \partial K\right\}$$

for the distance to the boundary of K with respect to this norm.

PROPOSITION 1. *For all* $x, y \in K^\circ$:

(i) $\|x - y\|_p \leq \big(\exp(d(x,y)) - 1\big)\exp\big(\max\{d(x,p), d(y,p)\}\big)$;

(ii) $d(x,y) \leq \|x - y\|_p / \min\big\{\operatorname{dist}_p(x, \partial K), \operatorname{dist}_p(y, \partial K)\big\}$;

(iii) $-\log(\operatorname{dist}_p(x, \partial K)) \leq d(x, p)$.

In particular, if f is d-Lipschitz continuous then f is locally Lipschitz continuous in norm, and thus (2) has unique local solutions, i.e., initial value problems for (1) have unique mild solutions on maximal time intervals. In case that f is a d-contraction, we will see that (1) generates a nonexpansive semiflow on $[0, \infty)$.

We assume from now on that A is the generator of a C_0-semigroup $(T(t))_{t \geq 0}$ of positive operators, i.e., $T(t)(K) \subseteq K$ for each $t \geq 0$. Then A is quasimonotone increasing, i.e., for all $x \in D(A) \cap K$ and each

$$\phi \in K^* := \left\{\psi \in E^* \, : \, \psi \geq 0 \text{ on } K\right\}$$

such that $\phi(x) = 0$ we have $\phi(Ax) \geq 0$ (cf. [1], [6]). We recall the following result from [4], for which we give a new proof here. We also recall the directional derivative m_+ from [8], which we use for $\|\cdot\|_p$ here,

$$m_+[x, y] := \lim_{h \to 0+} \frac{\|x + hy\|_p - \|x\|_p}{h} \qquad (x, y \in E),$$

and the property

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_+ \|T(t)x\|_p = m_+[T(t)x, AT(t)x] \qquad (x \in D(A)),$$

where $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_+$ denotes the right side derivative.

PROPOSITION 2. If $p \in D(A) \cap K^\circ$ and $\omega := \min\{\alpha \in \mathbb{R} \, : \, Ap \leq \alpha p\}$ then $m_+[p, Ap] = \omega$ and $m_+[x, Ax] \leq \omega\|x\|_p$ for all $x \in D(A)$. Moreover, $\|T(t)x\|_p \leq \mathrm{e}^{\omega t}\|x\|_p$ for all $t \geq 0$ and $x \in E$, in particular $T(t)p \leq \mathrm{e}^{\omega t}p$ for all $t \geq 0$.

Proof. Replacing A by $A - \omega I$ and $T(t)$ by $\mathrm{e}^{-\omega t}T(t)$ we may assume $\omega = 0$. Since $p \in K^\circ$ we have $p + hAp \in K^\circ$ for small $h > 0$. For these h we have

$$\|p + hAp\|_p = \min\left\{\alpha \in \mathbb{R} \, : \, p + hAp \leq \alpha p\right\}$$

$$= \min\left\{\alpha \in \mathbb{R} \, : \, Ap \leq \frac{\alpha - 1}{h}p\right\} = 1,$$

which implies $m_+[p, Ap] = 0$. If $x \in D(A)$ and $\alpha = \|x\|_p$ then $-\alpha p \leq x \leq \alpha p$, which implies

$$-\alpha T(t)p \leq T(t)x \leq \alpha T(t)p \qquad \text{and} \qquad \|T(t)x\|_p \leq \alpha\|T(t)p\|_p.$$

But then

$$m_+[x, Ax] = \lim_{t \to 0+} \frac{\|T(t)x\|_p - \alpha}{t}$$

$$\leq \alpha \lim_{t \to 0+} \frac{\|T(t)p\|_p - \|p\|_p}{t} = \alpha m_+[p, Ap] = 0 \,.$$

Hence

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_+ \|T(t)x\|_p = m_+[T(t)x, AT(t)x] \leq 0$$

for all $t \geq 0$ and $x \in D(A)$, which implies $\|T(t)x\|_p \leq \|x\|_p$ for all $t \geq 0$ and $x \in E$. ∎

Here we remark that if $p \in D(A) \cap K^\circ$ and $-Ap \in K^\circ$ then $\omega < 0$ and the semigroup is exponentially stable. On the other hand, exponential stability of the semigroup implies that $A : D(A) \to E$ is bijective and $-A^{-1}$ is positive as can be seen from

$$(\lambda I - A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t}T(t)x \,\mathrm{d}t \qquad (x \in E)$$

for $\lambda \geq 0$.

PROPOSITION 3. *For each $t \geq 0$ we have $T(t)(K^\circ) \subseteq K^\circ$. If the semigroup is exponentially stable then we have $-A^{-1}(K^\circ) \subseteq K^\circ$.*

Proof. Choose $p \gg 0$. By strong continuity, we find $\delta > 0$ such that $T(t)p \gg 0$ for $t \in [0, \delta]$. If $x \gg 0$ we find $\alpha > 0$ with $p \leq \alpha x$ which implies $T(t)p \leq \alpha T(t)x$ for any $t \geq 0$. In particular, $T(t)x \gg 0$ for all $t \in [0, \delta]$. We have shown $T(t)(K^\circ) \subseteq K^\circ$ for $t \in [0, \delta]$. By the semigroup property, this holds for all $t \geq 0$.

Let the semigroup be exponentially stable and $x \gg 0$. Since $-A^{-1}$ is positive, we have $y := -A^{-1}x \geq 0$, and if $y \notin K^\circ$ we find $\phi \in K^* \setminus \{0\}$ such that $\phi(y) = 0$. But then $\phi(Ay) \geq 0$ in contradiction to $Ay = -x \in -K^\circ$. ∎

For $T > 0$ let $C([0, T], E)$ be endowed with the maximum norm $\|\cdot\|_\infty$ and ordered by the normal and solid cone

$$K_T := \big\{u \in C([0, T], E) \,:\, u(t) \geq 0 \text{ for all } t \in [0, T]\big\}.$$

Note that (since $[0, T]$ is compact)

$$K_T^\circ = \big\{u \in C([0, T], E) \,:\, u(t) \gg 0 \text{ for all } t \in [0, T]\big\} \,.$$

The Thompson metric with respect to K_T is denoted by d_T. We will see that d-contractivity of f yields existence of global mild solutions for (1):

PROPOSITION 4. *Let $f : K^\circ \to K^\circ$ be d-Lipschitz continuous with constant $l < 1$ and let A be the generator of a positive semigroup. Then each initial value problem*

$$u'(t) = Au(t) + f(u(t)), \qquad u(0) = x \gg 0,$$

has a unique mild solution on $[0, \infty)$. In particular, $u(t) \gg 0$ for each $t \geq 0$.

Proof. Fix $T > 0$ and consider the integral operator $S : K_T^\circ \to K_T^\circ$ defined as

$$(Sv)(t) = T(t)x + \int_0^t T(t-s)f(v(s))\,\mathrm{d}s\,.$$

Let $v, w \in K_T^\circ$. We have $v \leq \exp(d_T(v,w))w$ and $w \leq \exp(d_T(v,w))v$, that is to say

$$v(t) \leq \exp(d_T(v,w))w(t) \quad \text{and} \quad w(t) \leq \exp(d_T(v,w))v(t) \quad \text{for all } t \in [0,T]\,.$$

Thus

$$(Sv)(t) \leq T(t)x + \int_0^t T(t-s)\exp(ld_T(v,w))f(w(s))\,\mathrm{d}s$$

$$\leq \exp(ld_T(v,w))S(w)(t) \qquad (t \in [0,T])\,,$$

and therefore $Sv \leq \exp(ld_T(v,w))Sw$; analogously $Sw \leq \exp(ld_T(v,w))Sv$. Thus

$$d_T(Sv, Sw) \leq ld_T(v,w) \qquad (v,w \in K_T^\circ)\,,$$

and therefore S has a unique fixed point $u \in K_T^\circ$ which is a mild solution of (1) on $[0,T]$. Since $T > 0$ was arbitrary, the assertion follows. ∎

In the sequel, let $u(\cdot, x) : [0, \infty) \to K^\circ$ denote the mild solution of (1) with initial value $u(0) = x \gg 0$.

PROPOSITION 5. *Under the assumptions of Proposition 4, for all $x, y \gg 0$*

$$t \mapsto d(u(t,x), u(t,y)) \text{ is monotone decreasing on } [0, \infty),$$

i.e., the semiflow of (1) is non-expansive with respect to Thompson's metric.

Proof. For $x, y \in K^\circ$ and $v := u(\cdot, x)$, $w := u(\cdot, y)$ we have

$$v(t) = T(t)x + \int_0^t T(t-s)f(v(s))\,\mathrm{d}s \qquad (t \geq 0)\,,$$

$$w(t) = T(t)y + \int_0^t T(t-s)f(w(s))\,\mathrm{d}s \qquad (t \geq 0)\,,$$

thus

$$v(\tau) \leq \max\big\{\exp(d(x,y)), \exp(ld_t(v,w))\big\}w(\tau) \qquad (\tau \in [0,t])\,,$$

$$w(\tau) \leq \max\big\{\exp(d(x,y)), \exp(ld_t(v,w))\big\}v(\tau) \qquad (\tau \in [0,t])\,,$$

which implies

$$d_t(v,w) \leq \max\big\{d(x,y), ld_t(v,w)\big\} \qquad (t \geq 0)\,.$$

Since $l < 1$ and $d(v(t), w(t)) \leq d_t(v, w)$ $(t \geq 0)$ we get

$$d(v(t), w(t)) \leq d(x, y) \qquad (t \geq 0)\,,$$

and the assertion follows by translation of v and w. ∎

3. GLOBAL ASYMPTOTIC STABILITY

PROPOSITION 6. *Let $f : K^\circ \to K^\circ$ be d-Lipschitz continuous with constant $l < 1$ and let A be the generator of a positive semigroup which is exponentially stable. Then there is a unique solution $x_0 \in K^\circ \cap D(A)$ of the equation $Ax_0 + f(x_0) = 0$.*

Remark. Observe that existence of $x_0 \in K^\circ \cap D(A)$ with $Ax_0 + f(x_0) = 0$ implies via $-Ax_0 \in K^\circ$ exponential stability of the semigroup (cf. Section 2), which thus is a necessary assumption.

Proof. Proposition 3 yields $-A^{-1}(f(K^\circ)) \subseteq K^\circ$. Since $-A^{-1} : E \to E$ is linear and order-preserving, the function $-A^{-1} \circ f : K^\circ \to K^\circ$ is again d-Lipschitz continuous with constant l. Thus $-A^{-1} \circ f$ has a unique fixed point x_0 in K° by Banach's Fixed Point Theorem. Since $A^{-1}(E) = D(A)$, we have $x_0 \in D(A)$, and the assertion follows. ∎

The following is our main result on global asymptotic stability.

THEOREM 1. *Let* $f : K^\circ \to K^\circ$ *be d-Lipschitz continuous with constant* $l < 1$ *and let* A *be the generator of a positive semigroup which is exponentially stable. Let* $x_0 \in K^\circ \cap D(A)$ *be the unique solution of the equation* $Ax_0 + f(x_0) = 0$. *Then for every* $x \in K^\circ$ *we have* $u(t, x) \to x_0$ *as* $t \to \infty$ *and*

$$d(u(t, x), x_0) \leq \frac{\mathrm{e}^{d(x, x_0)} - 1}{\mu}\, \mathrm{e}^{-\gamma t} \qquad (t \geq 0) \,,$$

where

$$\delta := -\min\big\{\omega \in \mathbb{R} \,:\, Ax_0 \leq \omega x_0\big\} \,>\, 0 \,,$$

$$\mu := l\big(1 - \mathrm{e}^{-\delta/2}\big) + \mathrm{e}^{-\delta/2} \,\in (0, 1) \,,$$

$$\gamma := -\log \mu \,>\, 0 \,.$$

Remark. We recall from Proposition 2 that, with $p := x_0$, we have in Theorem 1

$$-\delta = m_+[x_0, Ax_0] = \lim_{h \to 0+} \frac{\|x_0 + hAx_0\|_p - \|x_0\|_p}{h} \,.$$

Proof. Let $x \in K^\circ$ and $u(t) := u(t, x)$, $h(t) := \mathrm{e}^{d(u(t), x_0)}$ $(t \geq 0)$. Recall that h is decreasing by Proposition 5 and ≥ 1, and observe $-Ax_0 = f(x_0) \in K^\circ$. By Proposition 2 we have $T(t)x_0 \leq \mathrm{e}^{-\delta t}x_0$ where $\delta > 0$ here. Hence

$$T(t + s)x_0 = T(t)T(s)x_0 \leq T(t)\left(\mathrm{e}^{-\delta s}x_0\right) \leq T(t)x_0 \qquad (s, t \geq 0) \,,$$

which means that $t \mapsto T(t)x_0$ is decreasing on $[0, \infty)$. From

$$u(t) = T(t)x + \int_0^t T(t - s)f(u(s))\,\mathrm{d}s \qquad (t \geq 0)$$

we obtain by splitting the integral as $\int_0^{t/2} + \int_{t/2}^t$:

$$u(t) \leq h(0)T(t)x_0 + \int_0^t h(s)^l T(t-s)(-Ax_0)\,\mathrm{d}s$$

$$\leq h(0)T(t)x_0 + h(0)^l \int_0^{t/2} T(t-s)(-Ax_0)\,\mathrm{d}s$$

$$+ h(t/2)^l \int_{t/2}^t T(t-s)(-Ax_0)\,\mathrm{d}s$$

$$= h(0)T(t)x_0 + h(0)^l\Big(T(t/2)x_0 - T(t)x_0\Big) + h(t/2)^l\Big(x_0 - T(t/2)x_0\Big)$$

$$\leq h(0)T(t/2)x_0 + h(t/2)^l\Big(x_0 - T(t/2)x_0\Big)$$

$$= h(t/2)^l x_0 + \Big(h(0) - h(t/2)^l\Big)T(t/2)x_0$$

$$\leq h(t/2)^l x_0 + \Big(h(0) - h(t/2)^l\Big)\mathrm{e}^{-\delta t/2}x_0$$

$$= \Big[h(t/2)^l\Big(1 - \mathrm{e}^{-\delta t/2}\Big) + h(0)\mathrm{e}^{-\delta t/2}\Big]x_0$$

and

$$u(t) \geq h(0)^{-1}T(t)x_0 + \int_0^t h(s)^{-l} T(t-s)(-Ax_0)\,\mathrm{d}s$$

$$\geq h(0)^{-1}T(t)x_0 + h(0)^{-l} \int_0^{t/2} T(t-s)(-Ax_0)\,\mathrm{d}s$$

$$+ h(t/2)^{-l} \int_{t/2}^t T(t-s)(-Ax_0)\,\mathrm{d}s$$

$$= \frac{1}{h(0)}T(t)x_0 + \frac{1}{h(0)^l}\Big(T(t/2)x_0 - T(t)x_0\Big) + \frac{1}{h(t/2)^l}\Big(x_0 - T(t/2)x_0\Big)$$

$$\geq h(0)^{-1}T(t/2)x_0 + h(t/2)^{-l}\Big(x_0 - T(t/2)x_0\Big)$$

$$= h(t/2)^{-l} x_0 + \Big(h(0)^{-1} - h(t/2)^{-l}\Big)T(t/2)x_0$$

$$\geq h(t/2)^{-l} x_0 + \Big(h(0)^{-1} - h(t/2)^{-l}\Big)\mathrm{e}^{-\delta t/2}x_0$$

$$= \Big[h(t/2)^{-l}\Big(1 - \mathrm{e}^{-\delta t/2}\Big) + h(0)^{-1}\mathrm{e}^{-\delta t/2}\Big]x_0\,.$$

We thus have shown

$$h(t) \le \max\left\{ \left(\frac{1 - \mathrm{e}^{-\delta t/2}}{h(t/2)^l} + \frac{\mathrm{e}^{-\delta t/2}}{h(0)}\right)^{-1}, \; h(t/2)^l(1 - \mathrm{e}^{-\delta t/2}) + h(0)\mathrm{e}^{-\delta t/2}\right\}.$$

Now let $\alpha = h(0)$, $\beta = h(t/2)^l \le \alpha$, $\lambda = \mathrm{e}^{-\delta t/2} \in (0,1)$ and observe

$$\begin{aligned} 1 &= (1-\lambda)^2 + 2\lambda(1-\lambda) + \lambda^2 \\ &\le (1-\lambda)^2 + \lambda(1-\lambda)\big(\alpha/\beta + \beta/\alpha\big) + \lambda^2 \\ &= \big((1-\lambda)\beta + \lambda\alpha\big)\big((1-\lambda)\beta^{-1} + \lambda\alpha^{-1}\big), \end{aligned}$$

which means $\big((1-\lambda)\beta^{-1} + \lambda\alpha^{-1}\big)^{-1} \le \big((1-\lambda)\beta + \lambda\alpha\big)$. This yields

$$h(t) \le h(t/2)^l\big(1 - \mathrm{e}^{-\delta t/2}\big) + h(0)\mathrm{e}^{-\delta t/2} \qquad (t \ge 0).$$

Since the problem is autonomous we also have

$$h(s + t) \le h(s + t/2)^l\big(1 - \mathrm{e}^{-\delta t/2}\big) + h(s)\mathrm{e}^{-\delta t/2} \qquad (s, t \ge 0).$$

We know that h is decreasing, so $h(t)$ tends to some $c \ge 1$ as $t \to \infty$. The first inequality gives us $c \le c^l$, and $l < 1$ implies $c = 1$.

Rate of convergence: We let $t = 1$, $s = n \in \mathbb{N}_0$ in the second inequality. Then we have

$$h(n + 1) \le h(n + 1/2)^l\big(1 - \mathrm{e}^{-\delta/2}\big) + h(n)\mathrm{e}^{-\delta/2} \qquad (n \in \mathbb{N}_0),$$

and

$$h(n + 1) - 1 \le \big(h(n + 1/2)^l - 1\big)\big(1 - \mathrm{e}^{-\delta/2}\big) + (h(n) - 1)\mathrm{e}^{-\delta/2} \qquad (n \in \mathbb{N}_0).$$

By the mean value theorem and monotonicity of h we have

$$h(n + 1/2)^l - 1 \le l\big(h(n + 1/2) - 1\big) \le l(h(n) - 1),$$

and thus

$$h(n + 1) - 1 \le \mu(h(n) - 1) \qquad (n \in \mathbb{N}_0),$$

where $\mu = l(1 - \mathrm{e}^{-\delta/2}) + \mathrm{e}^{-\delta/2} < 1$. This yields

$$h(n) - 1 \le \mu^n(h(0) - 1) \qquad (n \in \mathbb{N}_0).$$

For $t > 0$, we choose $n \in \mathbb{N}$ such that $n - 1 < t \le n$ and obtain

$$h(t) - 1 \le h(n - 1) - 1 \le \mu^{n-1}(h(0) - 1) \le \mu^{t-1}(h(0) - 1),$$

and

$$d(u(t), x_0) = \log h(t) = \log(1 + h(t) - 1) \le h(t) - 1 \le \mu^{t-1}(h(0) - 1),$$

which is the claim. ∎

Observe that we have in the proof $Ax_0 \le -\delta x_0$. In case of equality $Ax_0 = -\delta x_0$, i.e., if x_0 is an eigenvalue of A, we give another estimate which turns out to be sharp.

THEOREM 2. *Let $f : K° \to K°$ be d-Lipschitz continuous with constant $l < 1$ and let A be the generator of a positive semigroup which is exponentially stable. Let $x_0 \in K° \cap D(A)$ be the unique solution of the equation $Ax_0 + f(x_0) = 0$ and assume $Ax_0 = -\delta x_0$. Then, for all $x \in K°$ and $t \ge 0$*

$$d(u(t, x), x_0) \le \frac{1}{1 - l} \log\left(1 + \left(e^{(1-l)d(x,x_0)} - 1\right)e^{-\delta(1-l)t}\right).$$

Remark. For each $a > 0$ the function $t \mapsto \log\left(1 + ae^{-\delta(1-l)t}\right)$ is strictly decreasing to 0 as $t \to \infty$ and

$$\log\left(1 + ae^{-\delta(1-l)t}\right) \le ae^{-\delta(1-l)t} \qquad (t \ge 0).$$

Thus, also Theorem 2 gives an exponential decay estimate with respect to d for global asymptotic stability of the equilibrium x_0 of (1). We shall see in the proof that the estimate is optimal.

Proof of Theorem 2. Let again $u := u(\cdot, x)$ and $h(t) := \exp(d(u(t), x_0))$ $(t \ge 0)$. We start the estimates as in the proof of Theorem 1. For $t \ge 0$ we have

$$u(t) \le h(0)T(t)x_0 + \int_0^t h(s)^l T(t - s)(-Ax_0)\,ds$$

$$= \left(h(0)e^{-\delta t} + \delta \int_0^t e^{-\delta(t-s)}h(s)^l\,ds\right)x_0$$

and

$$u(t) \geq h(0)^{-1}T(t)x_0 + \int_0^t h(s)^{-l}T(t-s)(-Ax_0)\,\mathrm{d}s$$

$$= \left(h(0)^{-1}\mathrm{e}^{-\delta t} + \delta \int_0^t \mathrm{e}^{-\delta(t-s)}(h(s))^{-l}\,\mathrm{d}s \right)x_0\,.$$

Thus

$$h(t) \leq \max \left\{ \left(\mathrm{e}^{-\delta t}\frac{1}{h(0)} + \int_0^t \mathrm{e}^{-\delta(t-s)}\frac{1}{(h(s))^l}\,\mathrm{d}s \right)^{-1}, \right.$$

$$\left. \mathrm{e}^{-\delta t}h(0) + \int_0^t \mathrm{e}^{-\delta(t-s)}(h(s))^l\,\mathrm{d}s \right\}.$$

We next prove for any $t \geq 0$:

$$\left(\mathrm{e}^{-\delta t}\frac{1}{h(0)} + \int_0^t \mathrm{e}^{-\delta(t-s)}\frac{1}{(h(s))^l}\,\mathrm{d}s \right)^{-1} \leq \mathrm{e}^{-\delta t}h(0) + \int_0^t \mathrm{e}^{-\delta(t-s)}(h(s))^l\,\mathrm{d}s\,.$$

This inequality is equivalent to

$$\left(h(0) + \delta \int_0^t \mathrm{e}^{\delta s}(h(s))^l\,\mathrm{d}s \right)\left(\frac{1}{h(0)} + \delta \int_0^t \mathrm{e}^{\delta s}\frac{1}{(h(s))^l}\,\mathrm{d}s \right) \geq \mathrm{e}^{2\delta t}.$$

According to the Cauchy-Schwarz inequality we have

$$\mathrm{e}^{\delta t} - 1 = \delta \int_0^t \mathrm{e}^{\delta s}\,\mathrm{d}s = \left(\int_0^t \sqrt{\delta}\mathrm{e}^{\delta s/2}(h(s))^{l/2}\sqrt{\delta}\mathrm{e}^{\delta s/2}\frac{1}{(h(s))^{l/2}}\,\mathrm{d}s \right)$$

$$\leq \left(\delta \int_0^t \mathrm{e}^{\delta s}(h(s))^l\,\mathrm{d}s \right)^{1/2}\left(\delta \int_0^t \mathrm{e}^{\delta s}\frac{1}{(h(s))^l}\,\mathrm{d}s \right)^{1/2}.$$

Thus, from $\sqrt{ab} + 1 \leq \sqrt{a+c}\sqrt{b+1/c}$ $(a,b,c > 0)$ we obtain

$$\mathrm{e}^{\delta t} \leq \left(\delta \int_0^t \mathrm{e}^{\delta s}(h(s))^l\,\mathrm{d}s \right)^{1/2}\left(\delta \int_0^t \mathrm{e}^{\delta s}\frac{1}{(h(s))^l}\,\mathrm{d}s \right)^{1/2} + 1$$

$$\leq \left(h(0) + \delta \int_0^t \mathrm{e}^{\delta s}(h(s))^l\,\mathrm{d}s \right)^{1/2}\left(\frac{1}{h(0)} + \delta \int_0^t \mathrm{e}^{\delta s}\frac{1}{(h(s))^l}\,\mathrm{d}s \right)^{1/2}.$$

Summing up, we now have

$$\mathrm{e}^{d(u(t),x_0)} = h(t) \le \mathrm{e}^{-\delta t}h(0) + \delta \int_0^t \mathrm{e}^{-\delta(t-s)}(h(s))^l \,\mathrm{d}s \qquad (t \ge 0)\,.$$

For $T > 0$ consider $C([0,T], \mathbb{R})$ ordered by the cone

$$K_r := \left\{ \xi \in C([0,T], \mathbb{R}) \,:\, \xi(t) \ge 0 \text{ for all } t \in [0,T] \right\},$$

with corresponding Thompson metric d_r, and $S : K_r^\circ \to K_r^\circ$ defined by

$$(S\xi)(t) = \mathrm{e}^{-\delta t}h(0) + \delta \int_0^t \mathrm{e}^{-\delta(t-s)}(\xi(s))^l \,\mathrm{d}s\,.$$

As in the proof of Proposition 4 we have $d_r(S\xi, S\eta) \le l d_r(\xi, \eta)$ $(\xi, \eta \in K_r^\circ)$, and moreover S is monotone increasing. Thus, there is a unique fixed point ξ_0 of S, and $h(t) \le (Sh)(t)$ $(t \in [0,T])$ implies $h(t) \le \xi_0(t)$ $(t \in [0,T])$. Since $T > 0$ was arbitrary, this proves that

$$h(t) \le \xi_0(t) \qquad (t \ge 0)\,,$$

with $\xi_0 : [0, \infty) \to (0, \infty)$ the solution of the initial value problem

$$(3) \qquad \xi_0'(t) = \delta(\xi_0(t))^l - \delta\xi_0(t)\,, \qquad \xi_0(0) = h(0).$$

A simple calculation shows

$$\xi_0(t) = \left(1 + \mathrm{e}^{-\delta(1-l)t}\big(h(0)^{1-l} - 1\big)\right)^{1/(1-l)} \qquad (t \ge 0)\,.$$

Thus

$$d(u(t), x_0) = \log(h(t)) \le \log(\xi_0(t))$$

$$= \frac{1}{1-l} \log\left(1 + \mathrm{e}^{-\delta(1-l)t}\big(\mathrm{e}^{(1-l)d(x,x_0)} - 1\big)\right) \qquad (t \ge 0)\,.$$

∎

Remark. As can be seen from the proof, the inequality in Theorem 2 is best possible since the equation (3) is of the form (1) where $Ax = -\delta x$ and $f(x) = \delta x^l$ is a d-contraction with constant l.

4. Examples

The following lemma gives a simple sufficient condition for a function $f : (0, \infty) \to (0, \infty)$ to be a d-contraction.

LEMMA 1. *Let $f : (0, \infty) \to (0, \infty)$ be a C^1-function such that*

$$l := \sup_{x > 0} \frac{x|f'(x)|}{f(x)} < \infty \,.$$

Then f is d-Lipschitz with constant l.

Proof. First note that $d(x, y) = |\log(y/x)| = |\log y - \log x|$ for $x, y > 0$. We thus have to show

$$|\log f(y) - \log f(x)| \le l \, |\log y - \log x| \qquad (x, y > 0) \,.$$

Writing $\log f(z) = \log\left(f(e^{\log z})\right)$ this follows from the mean value theorem applied to $g(t) := \log\left(f(e^t)\right)$, since $g'(t) = e^t f'(e^t)/f(e^t)$. ∎

As the example $f(x) = x^l$ shows, the estimate in Lemma 1 is sharp.

EXAMPLES 1. (i) Consider $f(x) = \log(\beta + x)$ with $\beta > 1$. Then we have for $x > 0$ by the mean value theorem

$$\frac{x f'(x)}{f(x)} = \frac{x}{(\beta + x) \log(\beta + x)}$$

$$\le \frac{x}{(\beta + x) \log(\beta + x) - \beta \log \beta} \le \frac{1}{1 + \log \beta} \,.$$

By Lemma 1, f is a d-contraction on $(0, \infty)$ with constant $l = \frac{1}{1 + \log \beta} < 1$.

(ii) Consider $f(x) = \arctan(\beta + x)$ where $\beta > 0$. Then we have for $x > 0$ by the mean value theorem

$$\frac{x f'(x)}{f(x)} = \frac{x}{(1 + (\beta + x)^2) \arctan(\beta + x)}$$

$$\le \frac{x}{(1 + (\beta + x)^2) \arctan(\beta + x) - (1 + \beta^2) \arctan \beta}$$

$$\le \frac{1}{1 + 2\beta \arctan \beta} \,.$$

By Lemma 1, f is a d-contraction on $(0, \infty)$ with constant $l = \frac{1}{1+2\beta \arctan \beta} < 1$.

(iii)   Consider $f(x) = 2 + \beta \sin(\log x)$ with $|\beta| < \sqrt{2}$. Then we have for $x > 0$

$$\frac{x|f'(x)|}{f(x)} = \frac{|\beta \cos(\log x)|}{2 + \beta \sin(\log x)} = \frac{|\beta|\sqrt{1 - y^2}}{2 + \beta y},$$

where $y = \sin(\log x) \in [-1, 1]$.   A simple calculation shows that this is $\leq \frac{|\beta|}{\sqrt{4-\beta^2}}$ with equality for $y = -\frac{\beta}{2}$.   By Lemma 1, f is a d-contraction on $(0, \infty)$ with constant $l = \frac{|\beta|}{\sqrt{4-\beta^2}} < 1$.

We give an application of Theorem 1.

EXAMPLE 2. Consider the problem

$$\partial_t u(t, s) = \partial_s^2 u(t, s) + 2s\partial_s u(t, s) + \left(\frac{u(t, s)}{u(t, -s)}\right)^{1/4},$$

$$u(0, s) = u_0(s),$$

$$\partial_s u(t, 1) = -u(t, 1),$$

$$\partial_s u(t, -1) = u(t, -1),$$

with $s \in [-1, 1]$ and $t \geq 0$. We take $E := C([-1, 1], \mathbb{R})$ ordered by the normal and solid cone

$$K := \left\{x \in E \ : \ x(s) \geq 0 \ \text{for all } s \in [-1, 1]\right\}.$$

Clearly, $x \mapsto f(x)(s) := \left(\frac{x(s)}{x(-s)}\right)^{1/4}$ $(s \in [-1, 1])$ is d-Lipschitz with constant $l = 1/2$. We define

$$D(A) := \left\{x \in C^2([-1, 1], \mathbb{R}) \ : \ x'(1) = -x(1), \ x'(-1) = x(-1)\right\}$$

and $Ax := x'' + 2sx'$ for $x \in D(A)$. Then A is a second order elliptic operator with real coefficients and Robin boundary conditions and thus generates a positive C_0-semigroup in E (cf., e.g., [9]). Letting

$$w(s) := -e^{-s^2} \int_0^s e^{\sigma^2} \, d\sigma \qquad (s \in [-1, 1])$$

we have

$$w(-s) = -w(s) \qquad \text{and} \qquad w'(s) + 2sw(s) + 1 = 0 \qquad \text{on } [-1, 1],$$

and $c := w(-1) = \int_0^1 e^{\sigma^2 - 1}\, d\sigma \in [e^{-1}, 1]$. Thus

$$x_0(s) := w(-1) + \int_{-1}^s w(\sigma)\, d\sigma \qquad (s \in [-1, 1])$$

defines a function $x_0 \in K^\circ \cap D(A)$, satisfying

$$x_0''(s) + 2s x_0'(s) + 1 = 0 \qquad \text{and} \qquad x_0(-s) = x_0(s) \qquad \text{on } [-1, 1].$$

We conclude that $x_0 \in K^\circ \cap D(A)$ satisfies $A x_0 + f(x_0) = 0$. By the remark following Proposition 6 the semigroup generated by A is exponentially stable. Thus Theorem 1 is applicable. By

$$A x_0 = -1 = -\frac{1}{c} \cdot c \leq -\frac{1}{c} x_0, \qquad \min_{|s| \leq 1} x_0(s) = c,$$

we obtain

$$d(u(t, x), x_0) \leq \frac{e^{d(x, x_0)} - 1}{\mu}\, e^{-\gamma t} \qquad (t \geq 0,\ x \gg 0),$$

$\delta = c^{-1} \in [1, e]$, $\mu = (1 - e^{-1/(2c)})/2 + e^{-1/(2c)}$. The numerical values are $\delta \approx 1.85846$, $\mu \approx 0.69743$, $\gamma \approx 0.36036$. We also see that Theorem 2 cannot be applied.

For an application of Theorem 2 we recall the example from the introduction.

EXAMPLE 3. Consider again $E = l^\infty(\mathbb{Z})$ endowed with the supremum norm $\|\cdot\|_\infty$ and ordered by the normal and solid cone

$$K := \{x \in l^\infty(\mathbb{Z}) : x_n \geq 0 \text{ for all } n \in \mathbb{Z}\}.$$

For $|\beta| < \frac{2}{\sqrt{5}}$, let $f : K^\circ \to K^\circ$ be defined by

$$f(x) = \left(2 + \beta \sin\left(\log(x_{n-1} x_{n+1})\right)\right)_{n \in \mathbb{Z}}$$

and $A : E \to E$ be defined by $A(x_n) := (x_{n+1} - 4x_n + x_{n-1})_{n \in \mathbb{Z}}$. Then A is quasimonotone increasing and generates a semigroup of positive operators. Letting $p := (1)_{n \in \mathbb{Z}}$ we have $Ap = -2p = -f(p)$, and $\delta = -2$. For $|\beta| = \frac{2}{\sqrt{17}}$, the function f is a d-contraction with constant $l = 1/2$. By Theorem 2 we have the estimate

$$d(u(t, x), p) \leq 2 \log\left(1 + \left(e^{d(x, p)/2} - 1\right) e^{-t}\right)$$

$$\leq 2\left(e^{d(x, p)/2} - 1\right) e^{-t} \qquad (t \geq 0, x \gg 0).$$

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