EXTRACTA MATHEMATICAE Vol. **29**, Núm. 1 - 2, 65 – 71 (2014)

Duality of Almost Limited Operators

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Presented by Manuel González Received July 7, 2014

Abstract: We establish necessary and sufficient conditions under which positive almost limited operators between Banach lattices have almost limited adjoints and positive operators with almost limited adjoints are themselves almost limited. Also, we give some consequences.

Key words: Almost limited operator, almost limited set, order continuous norm, dual positive Schur property.

AMS *Subject Class.* (2010): 46B42, 47B60, 47B65.

1. Introduction

Recall from [5] that an operator T from a Banach space X into Banach lattice E is called almost limited if $T(B_X)$ is an almost limited set in E, equivalently, $\|T'(f_n)\| \longrightarrow 0$ for every disjoint weak* null sequence (f_n) in E'. We denote by $L_{a-lim}(X, E)$ the classe of almost limited operators.

In this paper, we investigate the direct and reciprocal duality property of almost limited operators. We will prove that if E and F are Banach lattices such that F is Dedekind $\sigma-$complete then, each positive almost limited operator $T : E \longrightarrow F$ admits an almost limited adjoint T' if, and only if, E' has the dual positive Schur property or the norm of F is order continuous (Theorem 3.5). Next, we will establish that if E and F are Banach lattices such that F has the property (d) then, each positive operator $T : E \longrightarrow F$ is almost limited whenever its adjoint T' is almost limited if, and only if, the norm of E' is order continuous or F has the dual positive Schur property (Theorem 3.7).

The article is organized as follows, after the introduction section, we give in preliminaries section all notations and definitions of Banach lattice theory that we will need in this paper. In the main results section, we study in the first subsection the direct duality property for the class of almost limited operators and in the second subsection we study the reciprocal duality property of that class of operators.

2. Preliminary

Let us recall from [2] that a norm bounded subset A of a Banach lattice E is said to be almost limited, if every disjoint weak* null sequence (f_n) in E' converges uniformly to zero on A. Clearly that all relatively compact sets and all limited sets in a Banach lattice are almost limited. The converse does not hold in general. For example, the closed unit ball of the Banach lattice B_{ℓ^∞} is almost limited, but it is neither relatively compact nor limited.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. Also, a vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. We will use the term operator $T : E \longrightarrow F$ to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ is positive then, its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

A Banach lattice E has:

- the *property* (d) if, $|f_n| \wedge |f_m| = 0$ in E' and $f_n \longrightarrow 0$ in the $\sigma(E', E)$-topology of E' implies $|f_n| \longrightarrow 0$ in the $\sigma(E', E)$-topology of E'. It should be noted, by [7, Proposition 1.4], that if E is Dedekind σ-complete then, E has property (d). But the converse is false. In fact, the Banach lattice ℓ^∞/c_0 has the property (d) but is not Dedekind σ-complete [7, Remark 1.5].

- the *dual positive Schur property* if, $\|f_n\| \to 0$ for every weak* null sequence $(f_n) \subset (E')^+$, equivalently, $\|f_n\| \to 0$ for every weak* null sequence $(f_n) \subset (E')^+$ consisting of pairwise disjoint terms [7, Proposition 2.3].

- the *dual Schur property* if, $\|f_n\| \to 0$ for every weak* null sequence $(f_n) \subset E'$ consisting of pairwise disjoint terms [5, Definition 3.2].

For terminologies concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

3. Mains results

Firstly, we give the following definitions,

Definition 3.1. Let E and F be two Banach spaces and $\mathcal{U}(E,F)$ a class of operators from E to F.

- We shall say that the class $\mathcal{U}(E,F)$ satisfies the *direct duality property*, if we have $T \in \mathcal{U}(E,F) \Longrightarrow T' \in \mathcal{U}(F',E')$.
- By duality, we say that the class $\mathcal{U}(E,F)$ satisfies the *reciprocal duality property*, if we have $T' \in \mathcal{U}(F',E') \Longrightarrow T \in \mathcal{U}(E,F)$.

3.1. Direct duality property for almost limited operators.

Note that there exists an almost limited operator whose adjoint is not almost limited. Indeed, the identity operator of the Banach lattice ℓ^∞ is almost limited (because ℓ^∞ has the dual Schur property), but its adjoint, which is the identity operator of the Banach lattice $(\ell^\infty)'$, is not almost limited (because $(\ell^\infty)'$ does not have the dual Schur property [7, Proposition 2.1]).

To give our first main result we will need the following Lemmas,

Lemma 3.2. [4, Proposition 2.1] *An operator $T : X \longrightarrow E$ from a Banach space X into a Banach lattice E with the property (d) is almost limited if, and only if, $\|T'(f_n)\| \longrightarrow 0$ for every weak* null sequence (f_n) in E' consisting of positive pairwise disjoint elements.*

Lemma 3.3. [4, Remark 2.4] *Let E and F be two Banach lattice and let X be a Banach space. If E has the dual positive Schur property (for example, $E = \ell^\infty$), F has the property (d) and S is positive then, $T = S \circ R : X \longrightarrow E \longrightarrow F$ is an almost limited operator.*

Lemma 3.4. *A Banach lattice E does not have the dual positive Schur property if, and only if, there exists a positive disjoint weak* null sequence (f_n) of E', $(y_n) \subset B_E^+$ and some $\varepsilon > 0$ such that $f_n(y_n) \geq \varepsilon$ for all n.*

Proof. If E does not have the dual positive Schur property, we know that there exists a positive disjoint weak* null sequence (f_n) of E' which is not norm convergent to 0. By passing to a subsequence if necessarily, we may assume that there exists some $\varepsilon > 0$ with $\|f_n\| > \varepsilon$ for all n. As $\|f_n\| = \sup\{f_n(y) : y \in B_F^+\}$, for each n there exist $y_n \in B_F^+$ such that $f_n(y_n) \geq \varepsilon$. The converse is easy. ∎

In the following theorem we give necessary and sufficient conditions of Banach lattices under which the adjoint of each positive almost limited operator $T : E \longrightarrow F$ is also almost limited.

THEOREM 3.5. *Let E and F be two Banach lattices such that F is Dedekind $\sigma-$complete. The following conditions are equivalent:*

1. *For each positive operator $T \in L_{a-lim}(E, F)$ we have $T' \in L_{a-lim}(F', E')$;*
2. *At least one of the following conditions is valid:*
   (a) *E' has the dual positive Schur property;*
   (b) *The norm of F is order continuous.*

Proof. (1) $\Longrightarrow$ (2) Assume by way of contradiction that E' does not have the dual positive Schur property and the norm of F is not order continuous. We have to construct a positive operator $T \in L_{a-lim}(E, F)$ with an adjoint $T' \notin L_{a-lim}(F', E')$. Indeed, since the norm of F is not order continuous then, by [6, Corollary 2.4.3] we may assume that ℓ^∞ is a closed sublattice of F. As E' does not have the dual positive Schur property, Lemma 3.4 implies that there exists a positive disjoint weak* null sequence (f_n) of E'', $(g_n) \subset B_{E'}^+$ and some $\varepsilon > 0$ such that $f_n(g_n) \geq \varepsilon$ for all n. Consider the positive operator $T : E \longrightarrow \ell^\infty \subset F$ defined by

$$T(x) = (g_n(x))_{n=1}^\infty$$

for all $x \in E$. By Lemma 3.3, T is an almost limited operator. But T' is not almost limited. In fact, note that

$$\begin{aligned} T' : (\ell^\infty)' &\longrightarrow E', \\ (\lambda_n)_{n=1}^\infty &\mapsto \sum_{n=1}^\infty \lambda_n g_n \end{aligned}$$

and for every n, we have

$$\begin{aligned}\|T''(f_n)\| &= \sup_{x\in B_{(\ell^\infty)'}} |T''(f_n)(x)| = \sup_{x\in B_{(\ell^\infty)'}} |f_n(T'(x))| \\ &\geq f_n(T'(e_n)) = f_n(g_n) \geq \varepsilon,\end{aligned}$$

(where $(e_n)_{n=1}^{\infty}$ is the canonical basis of $\ell^1 \subset (\ell^\infty)'$). As (f_n) is a disjoint weak* null sequence of E'', we conclude that T' is not almost limited.

(2.a)$\Longrightarrow$(1) Obvious.

(2.b)$\Longrightarrow$(1) Let $T : E \longrightarrow F$ be a positive almost limited operator. Since E' has the property (d) (because E' is Dedekind σ-complete) then, by Lemma 3.2 it suffices to show that $\|T''(f_n)\| \longrightarrow 0$ for each positive disjoint weak* null sequence $(f_n) \subset E''$. It is clear that $0 \leq T''(f_n) \longrightarrow 0$ for $\sigma(F'', F')$. By [3, Corollary 2.7], it suffices to show that $T''(f_n)(g_n) \longrightarrow 0$ for each disjoint norm bounded sequence $(g_n) \subset (F')^+$. As the norm of F is order continuous, it follows from [6, Corollary 2.4.3] that $g_n \longrightarrow 0$ for $\sigma(F', F)$. Now, since T is almost limited, we have $\|T'(g_n)\| \longrightarrow 0$. As $f_n \longrightarrow 0$ for $\sigma(E'', E')$, (f_n) is norm bounded. Hence $T''(f_n)(g_n) = f_n(T'(g_n)) \leq \|f_n\|\|T'(g_n)\| \longrightarrow 0$. This complete the proof. ∎

COROLLARY 3.6. *Let E be a Dedekind $\sigma-$complete Banach lattice. Then, the following statements are equivalent:*

1. *For each positive operator $T \in L_{a-lim}(E, E)$ we have $T' \in L_{a-lim}(E', E')$;*
2. *The norm of E is order continuous.*

3.2. RECIPROCAL DUALITY PROPERTY FOR ALMOST LIMITED OPERATORS. In this section, we characterize Banach lattice under which each positive operator is almost limited whenever it's adjoint is almost limited. Note that there exist an operator which is not almost limited while its adjoint is almost limited. Indeed, the identity operator of the Banach lattice ℓ^1, is not almost limited (because ℓ^1 does not have the dual Schur property), however its adjoint, which is the identity operator of the Banach lattice ℓ^∞, is almost limited.

THEOREM 3.7. *Let E and F be two Banach lattices such that F has the property (d). Then, the following assertions are equivalent:*

1. *For each positive operator $T : E \longrightarrow F$ such that $T' \in L_{a-lim}(F', E')$ we have $T \in L_{a-lim}(E, F)$;*

2. *One of the following is valid:*
   (a) *The norm of E' is order continuous;*
   (b) *F has the dual positive Schur property.*

Proof. Assume by way of contradiction that the norm of E' is not order continuous and F does not have the dual positive Schur property. We have to construct a positive operator $T \notin L_{a-lim}(E,F)$ but its adjoint $T' \in L_{a-lim}(F',E')$.

Indeed, since the the norm of E' is not order continuous, it follows from Theorem 2.4.14 and Proposition 2.3.11 in [6] that E contains a sub-lattice isomorphic to ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$. On the other hand, since F does not have the dual positive Schur property, Lemma 3.4 implies that there exists a positive disjoint weak* null sequence (f_n) of F', $(y_n) \subset B_F^+$ and some $\varepsilon > 0$ such that $f_n(y_n) \geq \varepsilon$ for all n. Consider the positive operator $S : \ell^1 \longrightarrow F$ defined by $S((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i y_i$ for each $(\alpha_i) \in \ell^1$.

Now, we consider the positive operator $T = S \circ P : E \longrightarrow \ell^1 \longrightarrow F$. By Lemma 3.4, $T' : F' \longrightarrow \ell^\infty \longrightarrow E'$ is almost limited. But T is not almost limited. Indeed, the sequence (f_n) is a positive disjoint weak* null sequence in F'. As the operator $P : E \longrightarrow \ell^1$ is surjective, there exists $\delta > 0$ such that $\delta B_{\ell^1} \subset P(B_E)$. Hence

$$\begin{aligned} \|T'(f_n)\| &= \sup_{x \in B_E} |T'(f_n)(x)| = \sup_{x \in B_E} |f_n(T(x))| \\ &= \sup_{x \in B_E} |f_n \circ S(P(x))| \geq \delta\, |f_n \circ S(e_n)| \geq \delta\, |f_n(y_n)| \geq \delta\, \varepsilon \end{aligned}$$

(where $(e_n)_{n=1}^{\infty}$ is the canonical basis of ℓ^1). Then $\|T'(f_n)\| > \delta\,\varepsilon$ for all n. On the other hand, since (f_n) is a positive disjoint weak* null sequence in F', we conclude that T is not almost limited.

(2.b)$\Longrightarrow$(1) Obvious.

(2.a)$\Longrightarrow$(1) Let $T : E \longrightarrow F$ be a positive operator such that its adjoint T' is almost limited. Since F has the property (d) then, by Lemma 3.2 it suffices to show that $\|T'(f_n)\| \longrightarrow 0$ for each positive disjoint weak* null sequence $(f_n) \subset F'$.

It is clear that $0 \leq T'(f_n) \longrightarrow 0$ for $\sigma(E',E)$. By [3, Corollary 2.7], it suffices to show that $T'(f_n)(x_n) \longrightarrow 0$ for each disjoint norm bounded sequence $(x_n) \subset E^+$. As the norm of E' is order continuous, it follows from [6, Theorem 2.4.14] that $x_n \longrightarrow 0$ for $\sigma(E,E')$. As the canonical embedding $\tau : E \longrightarrow E''$ is a lattice homomorphism then, $(\tau(x_n))$ is a positive disjoint weak* null sequence in E''. Now, since T' is almost limited, we obtain $\|T''(\tau(x_n))\| \longrightarrow 0$. As $f_n \longrightarrow 0$ for $\sigma(F', F)$, (f_n) is norm bounded and $T'(f_n)(x_n) = \tau(x_n)(T'(f_n)) = T''(\tau(x_n))(f_n) \leq \|T''(\tau(x_n))\|\|f_n\| \longrightarrow 0$ and this proves that T is almost limited. ∎

Corollary 3.8. *Let E be a Banach lattice with the property (d). Then the following statements are equivalent:*

1. *For each positive operator $T : E \longrightarrow E$ such that $T' \in L_{a-lim}(E', E')$ we have $T \in L_{a-lim}(E, E)$;*
2. *The norm of E' is order continuous.*

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