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Symmetric presentations of finite groups

Joshua Anthony Roche

Symmetric Presentations of Finite Groups

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by

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Abstract

It is often the case that a progenitor, $P = m^{\star n} : N$, factored by a subgroup generated by one or more relators, $M = \langle \pi_1 w_1, \pi_2 w_2, \ldots, \pi_k w_k \rangle$, gives a finite group F, particularly, a classical group, simple group, or a sporadic group. In such instances, the presentation of the factor group, $G = P/M = \langle x, y, t \rangle$, is also a symmetric presentation of the finite group F. Symmetric presentations of groups allow us to represent, and manipulate, group elements in a manner that is typically more convenient than conventional techniques; in this sense, symmetric presentations are particularly useful in the study of large finite groups.

In this thesis, we first construct, by manual double coset enumeration, the groups A_5, S_5, S_6, S_7, and $S_7 \times 3$ as finite homomorphic images of the progenitors $2^{*3} : S_3$, $2^{*4} : A_4$, $2^{*5} : A_5$, $3^{*5} : S_5$, and $3^{*5} : S_5$, respectively. We also demonstrate that their respective symmetric presentations enable us to represent, and manipulate, their group elements in a convenient (symmetric) fashion as well as to obtain, in most cases, useful permutation representations for their group elements.

We devote the majority of our efforts to the construction, and manipulation, of $M_{12}\colon 2$, or $\mathrm{Aut}(M_{12})$, the outer automorphism group of the Mathieu group M_{12}. In particular, we construct, by the technique of manual double coset enumeration over S_4, the group $\mathrm{Aut}(M_{12})$ as a finite homomorphic image of the progenitor $3^{\star 4} : S_4$. By way of this construction, we show that $\mathrm{Aut}(M_{12})$ is isomorphic to $3^{\star 4} : S_4$ factored by two relations and we conclude that the symmetric presentation $\langle x, y, t \mid x^4 = y^2 = (yx)^3 = t^3 = [t, y] = [t^x, y] = (yxt)^{10} = ((x^2 y)^2 t)^5 = e \rangle$ defines the group $\mathrm{Aut}(M_{12})$. Finally, we demonstrate that this symmetric presentation enables us to express and manipulate every element of $\mathrm{Aut}(M_{12})$ either as a symmetric representation of the form πw, where π is a permutation of S_4 on 4 letters and w is a word of concatenated generators of length at most eight, or as a permutation representation on 7920 letters.

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Table of Contents

List of Figures

Chapter 1

Introduction

In this chapter, we introduce several important definitions, we introduce the concepts and techniques necessary for proving that a factor group $G = P/M$ is isomorphic to a finite group F, and we give the context and motivation for studying progenitors in general and M_{12} in particular.

1.1 Definitions

We make reference to the following definitions. (For a more detailed treatment of these definitions, see [Rot95].)

Definition 1.1: G-sets. Let G be a group, and let X be a nonempty set. Then X is a *G-set of degree* $|X|$ if there is a function (called an *action*), $\widehat{\phi}\colon G \times X \to X$, denoted by $\widehat{\phi}\colon (g, x) \mapsto gx$, such that:

(1) $ex = x$ for all $x \in X$, where e is the identity of G; and

(2) $(gh)x = g(hx)$ for all $g, h \in G$ and all $x \in X$.

Definition 1.2: Faithful. Let X be a G-set with action $\widehat{\phi}$. Then X is *faithful* if the homomorphism $\phi\colon G \to S_X$ is injective.

Definition 1.3: Transitive. A G-set is *transitive* if, for every $x, y \in X$, there exists a

$g \in G$ such that $y = gx$. Note also that a G-set is transitive if it has only one orbit.

Definition 1.4: k-Transitive. Let X be a G-set of degree n and let $k \leq n$ be a positive integer. Then X is *k-transitive* if, for every pair of k-tuples having distinct entries in X, say, $(x_1, x_1, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$, there is a $g \in G$ such that $gx_i = y_i$ for all $i \in \{1, 2, \ldots, k\}$.

Definition 1.5: Sharply k-Transitive. A k-transitive G-set X is *sharply k-transitive* if only the identity fixes k distinct elements of X.

Definition 1.6: Conjugate. If $H \leq G$ and $g \in G$, then the *conjugate* $g^{-1}Hg$ is $\{g^{-1}hg \mid h \in H\}$. The conjugate is often denoted by H^g.

Definition 1.7: Normalizer. If $H \leq G$, then the *normalizer* of H in G, denoted by $N_G(H)$, is $N_G(H) = \{a \in G \mid a^{-1}Ha = H\}$.

Definition 1.8: Centralizer. If $a \in G$, then the *centralizer* of a in G, denoted by $C_G(a)$, is the set of all $x \in G$ which commute with a.

Definition 1.9: Simple Group. A group G is *simple* if it has no normal subgroups other than the trivial subgroup $\{1\}$ and itself.

Definition 1.10: Semi-Direct Product. Let G be a group, and let $H, K \leq G$. If

1. $H \cap K = \langle 1 \rangle$,
2. $G = HK$, and
3. $K \lhd G$ and $H \leq G$,

then G is an *internal semi-direct product* of K by H.

1.2 The Progenitor m^{*n}: N

1.2.1 Free Products of n Copies of Cyclic Groups of Order m

Consider a group generated by two elements of order 2, say, $\langle t_1, t_2 \mid t_1^2 = t_2^2 = e\rangle$. Since the element t_1t_2 has infinite order, and since $\langle t_1t_2, t_1\rangle = \langle t_1, t_2\rangle$, we may refer to $\langle t_1, t_2 \mid t_1^2 = t_2^2 = e\rangle$ as an infinite dihedral group

$$\langle t_1, t_2 \mid t_1^2 = t_2^2 = e\rangle = \{e, t_1, t_2, t_1t_2, t_2t_1, t_1t_2t_1, \ldots\},$$

where elements of odd length in t_1 and t_2 are involutions (meaning they are of order 2), and elements of even length in t_1 and t_2 are of order infinity.

We denote $2^{*2} = \langle t_1, t_2 \mid t_1^2 = t_2^2 = e\rangle$. Since 2^{*2} is generated by two cyclic subgroups of order 2 with no relation between them, 2^{*2} is isomorphic to the free product of two copies of the cyclic group C_2 of order 2. That is,

$$2^{*2} = \langle t_1, t_2 \mid t_1^2 = t_2^2 = e\rangle = \{e, t_1, t_2, t_1t_2, t_2t_1, t_1t_2t_1, \ldots\} = \langle t_1\rangle * \langle t_2\rangle \cong C_2 * C_2$$

In fact, we can extend this notion to n generators and define a *free product of n copies of the cyclic group of order* 2 by:

$$2^{*n} = \langle t_1, t_2, \ldots, t_n \mid t_1^2 = t_2^2 = \cdots = t_n^2 = e\rangle = \langle t_1\rangle * \langle t_2\rangle * \cdots * \langle t_n\rangle \cong \underbrace{C_2 * C_2 * \cdots * C_2}_{n \text{ times}}$$

Even more generally, we can define m^{*n} to be a free product of n copies of the cyclic group C_m, where m is the order of the generators t_i. That is, we can define m^{*n} so that

$$m^{*n} = \langle t_1, t_2, \ldots, t_n \mid t_1^m = t_2^m = \cdots = t_n^m = e\rangle = \langle t_1\rangle * \langle t_2\rangle * \cdots * \langle t_n\rangle \cong \underbrace{C_m * C_m * \cdots * C_m}_{n \text{ times}}.$$

1.2.2 The Control Subgroup N

The *control subgroup* N is a subgroup of S_n which acts transitively on m^{*n} by permuting the generators of each cyclic group. In particular, the control subgroup N acts on the generators of m^{*n}, the *symmetric generators*, by *conjugation.* That is, for any element $\pi \in N$ and any symmetric generator t_i,

$$\pi^{-1}t_i\pi = t_i^\pi = t_{(i)\pi}.$$

Suppose, for example, that m^{*n}: N is a progenitor with control subgroup $N \cong S_3$ and symmetric generators $\{t_0, t_1, t_2\}$. Then $(0\ 2\ 1)^{-1}t_2(0\ 2\ 1) = t_2^{(0\ 2\ 1)} = t_{2(0\ 2\ 1)} = t_1$.

1.2.3 Definition of a Progenitor

Definition 1.11: Progenitor. A *progenitor* is an infinite semi-direct product of the form

$$m^{*n} : N,$$

where m^{*n} is a free product of n copies of the cyclic group of order m generated by elements t_i of order m in the set $T = \{t_1, t_2, \ldots, t_n\}$, and where N is a subgroup of S_n which acts transitively (and by conjugation) on $m^{*n}: N$ by permuting the generators of m^{*n}. As was mentioned above, we call N the *control subgroup* and we call the generators $t_1, t_2, \ldots, t_n$ of the free product m^{*n} the *symmetric generators.*

Multiplication of Elements in a Progenitor. Since N acts by conjugation as permutations of the n symmetric generators, the multiplication of any two elements $\pi_1 w_1, \pi_2 w_2 \in m^{*n} : N$, where $\pi_1, \pi_2 \in N$ and w_1 and w_2 are words in the free generators t_i, is given by:

$$(\pi_1 w_1)(\pi_2 w_2) = \pi_1(\pi_2\pi_2^{-1})w_1\pi_2 w_2$$
$$= (\pi_1\pi_2)(\pi_2^{-1}w_1\pi_2)w_2 = (\pi_1\pi_2)w_1^{\pi_2}w_2.$$

Inversion of Elements in a Progenitor. The inverse of any element $\pi t_{k_i}t_{k_j}\cdots t_{k_n}$ in $m^{*n}: N$ is given by

$$(\pi t_{k_i}t_{k_j}\cdots t_{k_n})^{-1} = t_{k_n}^{-1}t_{k_i}^{-1}\cdots t_{k_j}^{-1}\pi^{-1} = (\pi^{-1}\pi)t_{k_n}^{-1}t_{k_i}^{-1}\cdots t_{k_j}^{-1}\pi^{-1}$$
$$= \pi^{-1}(\pi t_{k_n}^{-1}t_{k_i}^{-1}\cdots t_{k_j}^{-1}\pi^{-1}) = \pi^{-1}(t_{k_n}^{-1}t_{k_i}^{-1}\cdots t_{k_j}^{-1})^{\pi^{-1}}.$$

Representation of Elements in a Progenitor. Since $(\pi_1 w_1)(\pi_2 w_2) = (\pi_1\pi_2)w_1^{\pi_2}w_2$ (see above), every element of N can be gathered on the left by way of conjugation. Therefore, every element of $m^{*n}: N$ can be represented as an element of the form πw, where π is a permutation of N and w is a *word* in the symmetric generators t_i for $1 \leq i \leq n$. This representation is unique provided that w is simplified so that adjacent symmetric generators are distinct.

Definition 1.12: Point Stabilizer. Let $m^{*n}: N$ be a progenitor and let w be a reduced word in the symmetric generators t_i. Then the *point stabilizer* of w in N is defined by:

$$N^w = \{\pi \in N \mid w^\pi = w\} = C_N(w).$$

For example, the point stabilizer of the word t_1 in N is given by $N^1 = \{\pi \in N \mid t_1^\pi = t_1\} = C_N(t_1)$. Likewise, the point stabilizer of the word t_1t_2 in N is given by $N^{12} = \{\pi \in N \mid (t_1t_2)^\pi = t_1t_2\} = C_N(\langle t_1, t_2\rangle)$.

Definition 1.13: Coset Stabilizer. Let Nw be a (single) right coset of N in the progenitor $m^{*n}\colon N$, where w is a reduced word in the symmetric generators t_i. Then

$$N^{(w)} = \{\pi \in N \mid Nw\pi = Nw\} = \{\pi \in N \mid Nw^\pi = Nw\}$$

is the *coset stabilizer* subgroup of Nw.

1.3 Homomorphic Images and Factor Groups

1.3.1 Identifying an Image of a Progenitor that is Homomorphic to a Finite Group F

Under certain conditions, a group F may be a homomorphic image of a progenitor $m^{*n}\colon N$. We provide these conditions in Lemma 1.1 below.

Lemma 1.1. *Let F be a group, let $T = \{t_0, t_1, \ldots, t_n\} \subseteq F$, and let $N \leq F$. Define $N \cong N_F(T) = \{g \in F \mid g^{-1}Tg = T\}$ to be the set normalizer in F of T. If $F = \langle T\rangle$ and if N permutes T transitively (but not necessarily faithfully), then F is a homomorphic image of the (infinite) progenitor $m^{*n}\colon N$. In this case, T is called a symmetric generating set for F.*

Example 1.1: Identifying an Image of a Progenitor that is Homomorphic to S_5. Suppose that $F = S_5 = \langle (1\ 2), (1\ 3), (1\ 4), (1\ 5)\rangle$. Let $t_1 = (1\ 2)$, $t_2 = (1\ 3)$, $t_3 = (1\ 4)$, and $t_4 = (1\ 5)$. Define $T = \{t_1, t_2, t_3, t_4\}$. Then $N = N_F(T) = S_4 = \langle (2\ 3\ 4\ 5), (2\ 3)\rangle$, and N is transitive on T. Therefore, by Lemma 1.1, F is a homomorphic image of $2^{*4}\colon S_4$; that is, F is a homomorphic image of $2^{*4}\colon N$. We denote $P = 2^{*4}\colon S_4$. Then there exists a homomorphism $\alpha\colon P \to F$, and $P/\ker\alpha \cong F$.

1.3.2 Identifying a Factor Group $G = P/M$ that is Isomorphic to a Finite Group F

Let F be a *finite* group, and let $P = m^{*n}\colon N$ be a progenitor. If finite group F and progenitor $P = m^{*n}\colon N$ satisfy the conditions established by Lemma 1.1, then we may identify a normal subgroup M with which to factor $P = m^{*n}\colon N$ so that

$$G = P/M \cong F.$$

If $P = m^{*n}\colon N$ satisfys Lemma 1.1, then there exists a homomorphism $\alpha\colon P \to F$. The group $M = \ker\alpha$ is the smallest normal subgroup of P, and the factor group $G = P/M$ is isomorphic to F. We call the elements $w_1\pi_1, w_2\pi_2, \ldots, w_k\pi_k \in \ker\alpha$ generating $\ker\alpha$ the *relators.* The relators are often expressed as *relations* equal to the identity e of P:

$$w_1\pi_1 = e, \quad w_2\pi_2 = e, \quad \ldots \quad w_k\pi_k = e$$

For this reason, the factor group G is expressed in terms of the progenitor $m^{*n}\colon N$ factored by the appropriate relators $w_1\pi_1, w_2\pi_2, \ldots, w_k\pi_k$. That is,

$$G = P/M = \frac{m^{*n}\colon N}{\pi_1 w_1, \pi_2 w_2, \ldots, w_k\pi_k}.$$

Identifying the Relators. Let F be a finite group, let $P = m^{*n}\colon N$ be a progenitor, and suppose that there exists a homomorphism $\alpha\colon P \to F$. The relators with which to factor P to construct a symmetric presentation of F are the generators of $\ker\alpha$. In general, however, it is quite difficult to find all elements of the kernel explicitly. Lemma 1.2 below is a useful tool for finding the relators (that is, the generators of the kernel).

Lemma 1.2.

$$N \cap \langle t_i, t_j\rangle \leq C_N(N^{ij}),$$

where N^{ij} denotes the stabilizer in N of the two points i and j.

Proof. Let $\pi \in \langle t_i, t_j\rangle \cap N$. Then $\pi = w(t_i, t_j)$; that is, π is some word w in t_i and t_j. Now let $g \in N^{ij}$. Then, since $g \in N^{ij}$, we have $\pi^g = w(t_i, t_j)^g = w(t_{(i)g}, t_{(j)g}) = w(t_i, t_j) = \pi$.

□

Example 1.2: Identifying a Factor Group G that is Isomorphic to S_5. Suppose that $F = S_5 = \langle(1\ 2),(1\ 3),(1\ 4),(1\ 5)\rangle$. Let $t_1 = (1\ 2)$, $t_2 = (1\ 3)$, $t_3 = (1\ 4)$, and $t_4 = (1\ 5)$. Define $T = \{t_1, t_2, t_3, t_4\}$. Then $N = N_F(T) = S_4 = \langle(2\ 3\ 4\ 5),(2\ 3)\rangle$, and N is transitive on T. Therefore, by Lemma 1.1, F is a homomorphic image of $2^{*4}\colon S_4$; that is, F is a homomorphic image of $2^{*4}\colon N$. We denote $P = 2^{*4}\colon S_4$. Then there exists a homomorphism $\alpha\colon P \to F$, and $P/\ker\alpha \cong F$.

We now use Lemma 1.2 to aid our search for the appropriate relators with which to factor $2^{*4}\colon S_4$. Let $N = S_4 = \langle x, y\rangle$, where $x \sim (1\ 2\ 3\ 4)$, and $y \sim (1\ 2)$. We consider N^{12}, the point stabilizer of t_1 and t_2 in N. Now, $N^{12} = \{\pi \in N \mid (t_1t_2)^\pi = t_1t_2\} = \langle(3\ 4)\rangle$. Therefore, $C_N(N^{12})$, the centralizer of N^{12} in N, is $C_N(N^{12}) = \{e, (1\ 2)\}$. By Lemma 1.2, $N \cap \langle t_1, t_2\rangle \leq C_N(N^{12}) = \{e, (1\ 2)\}$. That is, the appropriate relations with which to factor P are of the form $w_k = \pi_k$ or, simply, $w_k\pi_k = e$, where $\pi_k \in \{e, (1\ 2)\}$ and w_k is a word in t_1 and t_2.

Possible relations include, for example, $t_1t_2 = (1\ 2)$ or $t_2t_1t_2t_1 = e$ or $t_1t_2t_1t_2t_1t_2t_1 = (1\ 2)$. It turns out, in this case, that the kernel of this homomorphism is equal to the normal closure of the relation $t_2^{t_1} = (1\ 2)$, namely $\langle t_2^{t_1} = (1\ 2)\rangle^P$. Therefore $G = \frac{2^{*4}:S_4}{t_1t_2t_1=(1\ 2)} \cong S_5$ or, in terms of the relator, $G = \frac{2^{*4}:S_4}{(1\ 2)t_1t_2t_1} \cong S_5$. Moreover, a symmetric presentation of S_5 is given by
$\langle x, y, t | x^4 = y^2 = (xy)^3 = t^2 = [t, y^{x^2}] = [t, y^x y^{x^2}] = e, tt^x = y\rangle$.

1.4 Proving that Factor Group G is Isomorphic to a Finite Group F

If a finite group F and a progenitor $P = m^{\star n} : N$ satisy Lemma 1.1, and if an appropriate factor group $G = P/M = \langle t_0, t_1, \ldots, t_n\rangle$ is identified by computer or by hand, then it is possible, using several techniques, to prove by hand that G is isomorphic F.

For the remainder of this thesis, in fact, we will set out to prove that, for some particular finite group F, progenitor P, and factor group $G = P/M$, F is a homomorphic image of P and, moreover, $G = P/M$ is isompohic to F. To prove that F is a homomorphic image of P and $G \cong F$, we start by constructing $G = P/M$, piece by piece, by way of a technique called *manual double coset enumeration.* Before describing the technique of manual double coset enumeration, however, we first illustrate the concept of *double* *coset decomposition.*

1.4.1 Double Coset Decomposition

Consider a group G having two subgroups, H and K. We define a relation $\sim$ on G so that, for all $x, y \in G$, $x \sim y$ if and only if there exists an $h \in H$ and $k \in K$ such that $y = hxk$. This relation is an equivalence relation, and its equivalence classes are sets of the form

$$HxK = \{hxk \mid h \in H, k \in K\} = \bigcup_{k \in K} Hxk = \bigcup_{h \in H} hxK$$

This subset of G, which is both a union of the right cosets of G and a union of the left cosets of G, is called a *double coset* of H and K in G. In fact, if G acts by right multiplication on the right cosets of H in G, then double cosets of the form HxK correspond to the orbits of K in this action. The number of (single) right cosets of H in HxK, is given by Lemma 1.3 below.

Lemma 1.3. *If H and K are finite subgroups of a group G, and if x is an element of G, then $|HxK| = |H|\,|K|\,/\,|H^x \cap K|$.*

Proof. We proceed by counting the number of (single) right cosets of H in HxK. Now,

$$Hxk_1 \neq Hxk_2 \iff Hxk_1k_2^{-1}x^{-1} \neq H$$

$$\iff k_1k_2^{-1} \notin (x^{-1}Hx) \cup K = H^x \cap K$$

$$\iff (H^x \cap K)k_1 \neq (H^x \cap K)k_2.$$

Therefore, the number of single cosets of H in HxK is equal to the number of single cosets of $H^x \cap K$ in K, and so

$$|HxK| = |H|\,|K : (H^x \cap K)| = |H|\,|K|\,/\,|H^x \cap K|\,.$$

□

We now return to the progenitor $m^{*n}: N$ and we consider the double cosets of the form NxN in $m^{*n}: N$. Note first that, since every element x in the progenitor can be represented as πw for some $\pi \in N$ and some reduced word w in the symmetric generators t_i, the double coset NxN can be represented by the reduced word w as follows:

$$NxN = N\pi wN = NwN.$$

We denote the double coset NwN by $[w]$. The double coset NwN is equal to the union of the distict (single) right cosets of the form Nw^{π} for some $\pi \in N$:

$$NwN = \bigcup_{\pi \in N} Nw^{\pi},$$

and the progenitor, in turn, is equal to the disjoint union of its double cosets:

$$m^{*n} : N = NeN \cup Nw_1N \cup Nw_2N \cup Nw_3N \cup \cdots \cup Nw_kN \cup \cdots .$$

To determine the number of *distinct* single cosets in a double coset NwN, we refer to Lemma 1.4 below.

Lemma 1.4. *Let NwN be a double coset in the progenitor $m^{*n} : N$, where w is a reduced word in the symmetric generators t_i. The number of distinct (single) right cosets in the double coset NwN is given by $|N : N^{(w)}|$.*

Proof. We note first that

$$N^{(w)} = \{\pi \in N \mid Nw\pi = Nw\} = \{\pi \in N \mid Nw\pi w^{-1} = N\}$$

$$= \{\pi \in N \mid w\pi w^{-1} \in N\} = \{\pi \in N \mid \pi \in N^{w}\} = N \cap N^{w}.$$

By Lemma 1.3, $|NwN| = |N : N^{(w)}|$.

□

1.4.2 Manual Double Coset Enumeration of G over N

In order to construct a factor group G by hand, we use a process called *manual double coset enumeration.* Construction by manual double coset enumeration helps us to determine the index of N in G and, ultimately, the order of G (that is, the number of distinct right cosets of N in G). Manual double coset enumeration of a factor group $G = (m^{*n} : N)/M$ over N involves the several steps. Before describing these steps, we first define (1) the action of $g \in G$ on the right cosets of N in G and (2) the orbit of $N^{(w)}$ on T.

Action of a Generator on a Right Coset. Let X denote the set of single (right) cosets of N in the factor group $G = (m^{*n} \colon N)/M$, let $Nw \in X$, where w is a reduced word in the symmetric generators t_i, and let $g \in G$. We define an *action* $\widehat{\phi}$ of g on Nw by right multiplication of g on the right coset Nw. That is, we define an action $\widehat{\phi}$ of g on Nw with the mapping $\widehat{\phi} : G \times X \to X$ given by

$$\widehat{\phi} : (g, Nw) \mapsto Nwg.$$

Orbits of $N^{(w)}$ on T. Let Nw be a right coset of N in the progenitor $m^{\star n} : N$, where w is a reduced word in the symmetric generators t_i, and let $N^{(w)}$ be the coset stabilizer subgroup of Nw. Then

$$O(t_i) = \{(t_i)^n \mid n \in N^{(w)}\},$$

where $i \in \{0, 1, 2, \ldots, n\}$, are the *orbits* of $N^{(w)}$ on T. The orbits of $N^{(w)}$ on T are subsets of $T = \{t_0, t_1, t_2, \ldots, t_n\}$ on which the coset stabilizer $N^{(w)}$ is transitive.

Procedure for Manual Double Coset Emuneration of G over N. The procedure for manual double coset emuneration of G over N is as follows:

1. We first consider the double coset characterized by a reduced word $w_0 = e$ of length zero. This is the double coset NeN, which we denote [*]. By Lemma 1.4, the number of distinct right cosets of N in G in a double coset NwN is given by $|N : N^{(w)}|$. Since [*] is a double coset with a word of length zero, the number of distinct right cosets in [*] is $|N : N| = 1$.

2. We next determine the orbits of N on T. Since N is transitive on $T = \{t_0, t_1, t_2, \ldots, t_n\}$, we take a representative from the orbit of N on T, say t_i, and multiply it by the elements of N on the right to get the (right) coset Nt_i. The relations $\pi_1 w_1, \pi_2 w_2, \ldots, \pi_k w_k$ indicate whether or not the new double coset Nt_iN is distinct. If Nt_iN is indeed distinct, we proceed to step 3.

3. We next consider the double coset characterized by a reduced word $w_1 = t_i$ of length one. This is the double coset Nt_iN, which we denote $[i]$. Since $[i]$ is a double coset with a word of length one, the coset stabilizer is $N^{(i)} = \{\pi \in N \mid Nt_i^{\pi} = Nt_i\}$.

Therefoere, by Lemma 1.4, the number of distinct (single) right cosets in $[i]$ is $|N : N^{(i)}|$.

4. We then determine the orbits of $N^{(i)}$ on T. We take a representative from each of the orbits, say t_j, t_h, and so on, and we multiply each of these representatives by Nt_i on the right to get Nt_it_j, Nt_it_h, and so on. The relations $\pi_1 w_1, \pi_2 w_2, \ldots, \pi_k w_k$ again indicate whether or not the new double cosets Nt_it_jN, Nt_it_hN, and so on, are distinct. If one or more of the double cosets Nt_it_jN, Nt_it_hN, and so on, are indeed distinct, we proceed to step 5.

5. We next consider the distinct double cosets characterized by reduced words $w_2 = t_it_j$, $w_3 = t_it_h$, and so on, of length two, and we repeat steps 3 and 4. When the relations $\pi_1 w_1, \pi_2 w_2, \ldots, \pi_k w_k$ indicate that there are no new distinct double cosets, or when the coset stabilizer $N^{(w_f)}$ is transitive on the symmetric generators, we conclude that right multiplication on the right cosets of N in G is closed. This signifies that our manual double coset enumeration of G over N is complete.

In Chapters 2 through 7, the construction of $G = P/M$ by way of manual double coset enumeration will play an important role when we prove that G is isomorphic to a finite group F. In addition to constructing $G = P/M$ by way of manual double coset enumeration, we will also need to determine the permutation representations of the generators of G and, ultimately, show that G is isomorphic to a group G_1 generated by these permutation representations. To determine the permutation representations of the generators of a factor group G, we determine the action of the generators of G on the set of all right cosets of N in G.

1.4.3 Determining the Action of the Generators of G on the Right Cosets of N in G

Let X denote the set of distinct right cosets of N in G, that is, let $X = \{Nw_0, Nw_1, \ldots, Nw_q\}$, where $w_0, w_1, \ldots, w_q$ are reduced words of concatenated symmetric generators t_i. Recall that we had defined an *action* $\widehat{\phi}$ of $g \in G$ on $Nw \in X$ by the right multiplication of g on the right coset Nw. That is, we had defined an action $\widehat{\phi}$ of g on Nw with the mapping $\widehat{\phi} : G \times X \to X$ given by $\widehat{\phi} : (g, Nw) \mapsto Nwg$.

Action of a Generator on the Set of Right Cosets. We now define a mapping $\phi : G \longrightarrow S_X$ so that ϕ maps a generator $g \in G$ to its action on X. That is, we define ϕ so that $\phi(g) = \widehat{\phi}(g) : X \to X$. The action $\phi(g)$ of g on the set of right cosets of N in G is a permutation on $|X|$ letters. In this sense, the action $\phi(g)$ of g on the set of right cosets of N in G is equivalent to a permutation representation of g on $|X|$ letters.

Procedure for Determining the Action of a Generator on X. Suppose that $t_i \in G$ is a symmetric generator of a factor group G and suppose that $\pi \in N$ is a generator of its control subgroup N.

To calculate the action $\phi(t_i)$ of the symmetric generator t_i on the right cosets of N in G, we multiply every right coset Nw by t_i on the right. We consider first the identity coset N: we multiply the identity coset N by t_i, we then multiply the new coset Nt_i again by t_i, and we repeat this process until the product, by right multiplication, is again the identity coset N. We consider next another right coset, say Nt_j: we multiply the right coset Nt_j by t_i, we then multiply the new coset Nt_jt_i again by t_i, and we repeat this process until the product, by right multiplication, is again the right coset Nt_j. We repeat this action for every right coset of N in G. By way of this process, we find the actions $\phi(t_i)$ of the symmetric generators t_i on the right cosets of N in G and, equivalently, we find the permutation representations $p = \phi(t_i)$ of the symmetric generators t_i in their actions on the right cosets of N in G. Note that for symmetric generators t_i of order m, the actions $\phi(t_i)$ of the symmetric generators on the right cosets of N in G will be products of m-cycles.

Now, note that $\pi \in N \Rightarrow Nw\pi = N\pi^{-1}w\pi = Nw^{\pi}$. Therefore, to calculate the action $\phi(\pi)$ of a generator π on the right cosets of N in G, we conjugate every right coset Nw by π. Since $\pi \in N \Rightarrow N^{\pi} = N$, we consider first a non-trivial right coset, say Nt_i: we conjugate the right coset Nt_i by π, we then conjugate the new coset $Nt_i^{\pi} = Nt_j$ again by π, and we repeat this process until the conjugated product is again the right coset Nt_i. We repeat this action for every right coset of N in G. By way of this process, we find the actions $\phi(\pi)$ of the generators $\pi \in N$ on the right cosets of N in G and, equivalently, we find the permutation representations $p = \phi(\pi)$ of the generators π in their actions on the right cosets of N in G. Note that for generators π of order k, the actions $\phi(\pi)$ of the generators on the right cosets of N in G will be products of k-cycles.

1.4.4 Proving Factor Group G is Isomorphic to Finite Group F

Suppose that a finite group F and a progenitor $P = m^{\star n} : N$ satisy Lemma 1.1. Let G denote the group P factored by the relations $\pi_1 w_1 = e, \ldots, \pi_k w_k = e$, that is, let

$$G = \frac{m^{\star n} : N}{\langle \pi_1 w_1, \pi_2 w_2, \ldots, \pi_k w_k \rangle},$$

and suppose G has been identified, with the help of Lemma 1.2, to be an appropriate factor group. To prove that F is a finite homomorphic image of P and, ultimately, to prove that $F \cong G$, we use the following general strategy:

1. We first show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $G = \langle x, y, t \rangle$ and that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong G$. Now, by way of manual double coset enumeration of G over N, we determine that $[G : N] \leq |X|$ (where X is the set of distinct right cosets of N in G) and, therefore, that $|G| \leq |N| \cdot |X| = s$. We then consider a permutation group $G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle \leq S_X$, where $\phi(x), \phi(y), \phi(t)$ are the permutation representations of the generators x, y, t of G, and we show that G_1 is a homomorphic image of a G. To do this, we demonstrate that the generators $\phi(x)$ and $\phi(y)$ conjugate $\phi(t)$ in the same way that the generators x and y conjugate of the the symmetric generator t. We also demonstrate that, if relators $\pi_1 w_1, \pi_2 w_2, \ldots, \pi_k w_k$ hold true in G, then $\phi(\pi_1)\phi(w_1), \phi(\pi_2)\phi(w_2), \ldots, \phi(\pi_k)\phi(w_k)$ also hold true in G_1. After showing that G_1 is a homomorphic image of G, and knowing that $|G_1| = s$, we are able to conclude $G_1 \leq G$ and so $s = |G_1| \leq |G|$. Hence, $s \leq |G| \leq s \Rightarrow |G| = s$.Finally, after showing that $G_1 \leq G$ and $|G_1| = s = |G|$, we conclude $G \cong G_1$.

2. We next show $G_1 \cong F$ and, ultimately, $G \cong F$. With the help of the computer algebra system MAGMA (see [BCP97]), we find elements $a, b, c \in G_1$ that generate a known presentation of F. That is, we find elements $a, b, c \in G_1$ such that $F \cong \langle a, b, c \rangle$. After finding these elements, we are able to conclude $\langle a, b, c \rangle \leq G_1$ and, knowing that $|F| = |\langle a, b, c \rangle| = s = |G_1|$, we can then conclude $G_1 \cong \langle a, b, c \rangle \cong F$. Knowing that G_1 is a homomorphic image of a G and further that $G_1 \cong F$, we can conclude F is a homomorphic image of a G. (Note, moreover, that since $G = P/M$, where $M = \langle \pi_1 w_1, \pi_2 w_2, \ldots, \pi_k w_k \rangle$, we can also conclude F is a homomorphic image of a P.) Finally, after showing that $G_1 \cong F$ and $G \cong G_1$, we conclude $G \cong F$.

By proving that finite group F is isomorphic to G, we demonstrate that F can be defined in terms G. In other words, we demonstrate that the presentation of G, say $G = \langle x, y, t \rangle$, is a symmetric presentation of F. Establishing that $G = \langle x, y, t \rangle$ is a symmetric presentation of F, in turn, enables us to express and manipulate every element g of $F \cong G$ either as a symmetric representation of the form $g = \pi w$, where $\pi \in N$ is a permutation on 4 letters and w is a word of concatenated symmetric generators of G, or as a permutation representation $p = \phi(g)$ on $|X|$ letters.

Below, we describe two algorithms important to the manipulation of elements as both symmetric representations and permutation representations. The first algorithm describes how a permutation representation $p = \phi(g)$ on $|X|$ letters may be converted to its symmetric representation form $g = \pi w$, and the second algorithm describes the reverse conversion.

1.5 Algorithm for Converting an Element of G from its Permutation Representation to its Symmetric Representation

Let F be a finite group and let $P = m^{\star n} : N$ be a progenitor. Suppose that finite group F and progenitor $P = m^{\star n} : N$ satisy Lemma 1.1, and suppose that the factor group $G = P/M = \langle t_0, t_1, \ldots, t_n \rangle$ is isomorphic to F. Let X denote the set of distinct right cosets of N in G.

Let $g \in G$ and suppose that $p = \phi(g)$ be the permutation representation of g on $|X|$ letters. Now, $N^p = \{\sigma_1^p \mid \sigma_1 \in N\} = \{p^{-1}\sigma_1 p \mid \sigma_1 \in N\} = \{\sigma_2 p \mid \sigma_2 \in N\} = Np$, since $p^{-1}\sigma_1 \in N$. Moreover, by the action of right multiplication, $N^p = Nw$ for some right coset Nw, where w is a word in the symmetric generators $T = \{t_0, t_1, \ldots, t_n\}$ (and their inverses, $t_0^{-1}, t_1^{-1}, \ldots, t_n^{-1}$, if the order of each symmetric generator is greater than 2). In this way, we determine w, the word component of the symmetric representation of $p = \phi(g)$.

Now, since $p = \phi(g)$ is the permutation representation of an element in $G = \langle t_0, t_1, \ldots, t_n \rangle$, $N^p = Np$ and $N^p = Nw$ imply that $Np = Nw$. Moreover, $Np = Nw$ implies that $p \in Nw$ which implies that $p \sim \pi w$ for some $\pi \in N$ or, more precisely, $p = \phi(\pi)\phi(w)$ for some $\pi \in N$. Finally, $p = \phi(\pi)\phi(w)$ implies that $\phi(\pi) = p(\phi(w))^{-1} = p\phi(w^{-1})$. To determine $\pi \in N$, the permutation component of the symmetric representation of $p = \phi(g)$, we calculate the action of $\pi \sim \phi(\pi) = p\phi(w^{-1})$ on the set of symmetric generators $T = \{t_0, t_1, \ldots, t_n\}$.

1.6 Algorithm for Converting an Element of G from its Symmetric Representation to its Permutation Representation

Let F be a finite group and let $P = m^{\star n} : N$ be a progenitor. Suppose that finite group F and progenitor $P = m^{\star n} : N$ satisy Lemma 1.1, and suppose that the factor group $G = P/M = \langle t_0, t_1, \ldots, t_n \rangle$ is isomorphic to F. Let X denote the set of distinct right cosets of N in G.

Let $g \in G$ and suppose that g has the symmetric representation $g = \pi w$, such that $\pi \in N$ is a permutation on n letters and w is a word in the symmetric generators $T = \{t_0, t_1, \ldots, t_n\}$ (including their inverses, $t_0^{-1}, t_1^{-1}, \ldots, t_n^{-1}$, if the order of each symmetric generator is greater than 2).

To determine the permutation representation $p = \phi(g)$ of g, we first calculate the action $\phi(\pi)$ of π on the set of right cosets of N in G, and we then calculate the action $\phi(t_i)$ of the symmetric generators t_i on the set of right cosets of N in G. In so doing, we determine the permutation representation $\phi(\pi)$ of π on $|X|$ letters and the permutation representation $\phi(w)$ of the word w on $|X|$ letters. To determine the permutation representation $p = \phi(g)$ of g, we calculate the product of the permutation representation of π and the permutation representation the word w. That is, we calculate $p = \phi(g) = \phi(\pi)\phi(w)$.

1.7 Motivation for the Subject

1.7.1 The Mathieu Group on 12 Letters, M_{12}

The Mathieu group on 12 letters, which we denote M_{12}, is a sporadic group of order 95,040. This group is sharply 5-transitive and its usual action is on 12 cosets of M_{11} (note that $[M_{12} \colon M_{11}] = 12$). To properly define M_{12}, we refer to the Steiner system $S = S(5, 6, 12)$.

Steiner System. A *Steiner system* $S(l, m, n)$ is a collection of m-element subsets of an n-element set Λ such that no two of the m-element subsets of Λ have l or more in common. The number of these special m-element subsets in a Steiner system $S(l, m, n)$, if that Steiner system indeed exists, is given by $\binom{n}{l}/\binom{m}{l}$.

Thus a Steiner system $S = S(5, 6, 12)$ is a collection of 6-element subsets, or *hexads*, of a 12-element set such that no two have 5 or more in common. There are $\binom{12}{5}/\binom{6}{5} = 132$ hexads in this Steiner system. One such example of a Steiner system S of the form $S(5, 6, 12)$ is $S = \{\{1, 4, 5, 7, 9, 10\}, \{1, 4, 5, 7, 8, 12\}, \{1, 4, 5, 6, 7, 11\}, \{1, 4, 5, 6, 8, 9\}, \{1, 4, 6, 9, 11, 12\}, \ldots\}$.

The Mathieu Group M_{12}. The Mathieu group on 12 letters, M_{12}, is the automorphism group of a Steiner system $S = S(5, 6, 12)$; that is, $M_{12} = \{\sigma \in S_{12} \mid S^\sigma = S\}$.

1.7.2 Motivation for Curtis' Investigation of Progenitors

The motivation for Curtis' investigation of progenitors and symmetric presentations was his interest in the behavior of the Mathieu groups M_{12} and M_{24} [Cur07]. The conclusions of his initial work with M_{12} are summarized below.

M_{12} as a Finite Homomorphic Image of 3^{*5}: A_5. M_{12} is generated by 5 elements of order 3 which are normalised, as a subset, by a subgroup of M_{12}, which is also transitive on the set of five elements, isomorphic to A_5. Thus, by Lemma 1.2, $F = M_{12}$ is a homomorphic image of the progenitor $P = 3^{*5}: N$, where $N \cong A_5$. A presentation for the progenitor is $P = 3^{*5} : N = \langle x, y, t | x^5 = y^3 = (xy)^2 = t^3 = [t, y] = [t, yx^{-1}yx^{-2}] = e\rangle$, where $t \sim t_0$. Now, by the information above, there exists a homomorphism $\alpha\colon P \to M_{12}$, and therefore $P/\ker\alpha \cong M_{12}$. In order to factor P so that it is isomorphic to F, we must determine $\ker\alpha$. In particular, the question now is what element or elements of $3^{*5} : N$ are needed to factor the progenitor P in order to obtain F. Since every element of P is of the form πw, where $\pi \in N$ and w is a word in the five t_i's, we must determine the elements of N that can be written as a product of the symmetric generators t_0, t_1, t_2, t_3, and t_4.

Let $N = \langle x, y\rangle$, where $x \sim (0\ 1\ 2\ 3\ 4)$, and $y \sim (4\ 2\ 1)$. Now the point stabilizers of N are $N^0 \cong A_4 = \langle (1\ 4\ 2), (1\ 2)(3\ 4)\rangle$ and $N^{01} \cong A_3 = \langle (2\ 4\ 3)\rangle$, and the centralizer in

N of N^{01} is $\langle(2\ 4\ 3)\rangle$. Therefore, by Lemma 1.3, the elements of $\langle t_0, t_1\rangle$ may be written as $(2\ 4\ 3)$ or $(2\ 3\ 4)$ or the identity e. We find that the required relator is $(t_0^{-1}t_1)^2 = (2\ 3\ 4)$; that is, $(t^{-1}(t^x))^2 = yx^{-1}yx^{-2}$.

Therefore $F = \langle x, y, t | x^5 = y^3 = (xy)^2 = t^3 = [t, y] = [t, yx^{-1}yx^{-2}] = e, (t^{-1}(t^x))^2 = yx^{-1}yx^{-2}\rangle$ is a symmetric presentation of $C_3 \times M_{12}$. However, the center is $Z(C_3 \times M_{12}) = C_3 = \langle (xt)^8\rangle$. By factoring out the center, we obtain $M_{12} \cong F = \langle x, y, t | x^5 = y^3 = (xy)^2 = t^3 = [t, y] = [t, yx^{-1}yx^{-2}] = (xt)^8 = e, (t^{-1}(t^x))^2 = yx^{-1}yx^{-2}, \rangle$.

Let $M_{12} = \langle \alpha(x), \alpha(y), \alpha(t)\rangle$, so that $A_5 \cong \langle \alpha(x), \alpha(y) \mid [\alpha(x)]^5 = [\alpha(y)]^3 = [\alpha(x)\alpha(y)]^2 = \alpha(e)\rangle$. A homomorphism of the type described above can be given as $\alpha : 3^{*5} : N \longrightarrow M_{12}$, where $\alpha(x) = (1\ 3\ 9\ 5\ 4)(2\ 6\ 7\ 10\ 8)$, $\alpha(y) = (\infty\ 6\ 7)(0\ 3\ 9)(1\ 8\ 5)(2\ 4\ 10)$, and $\alpha(t) = (12\ 8\ 10)(11\ 3\ 9)(1\ 4\ 7)(2\ 6\ 5)$. Note that A_5 normalizes $\alpha(t)$, $\alpha(t)^{\alpha(x)}$, $\alpha(t)^{\alpha(x^2)}$, $\alpha(t)^{\alpha(x^3)}$, and $\alpha(t)^{\alpha(x^4)}$. Since $|\alpha(x)\alpha(y)| = 2$, the five elements of order 3 that generate M_{12} are the five conjugates $t_0 = \alpha(t) = (12\ 8\ 10)(11\ 3\ 9)(1\ 4\ 7)(2\ 6\ 5)$ under conjugation by $N = N_G(t_0, t_1, t_2, t_3, t_4) = \langle \alpha(x), \alpha(y)\rangle \cong A_5$. Moreover, the relation $[\alpha(t)^{-1}\alpha(t)^{\alpha(x)}]^2 = \alpha(y)\alpha(x)^{-1}\alpha(y)\alpha(x)^{-2}$ holds in M_{12}. Therefore, we can perform a manual double coset enumeration of M_{12} over A_5 to construct the group by hand.

1.7.3 Motivation for Studing Progenitors

Whereas every element of M_{12} is usually represented by a permutation on 12 letters, with the symmetric presentation discovered by Curtis,

$$\langle x, y, t | x^5 = y^3 = (xy)^2 = t^3 = [t, y] = [t, yx^{-1}yx^{-2}] = (xt)^8 = e, (t^{-1}(t^x))^2 = yx^{-1}yx^{-2}\rangle,$$

every element of M_{12} is represented by a permutation on 5 letters followed by a word generated by $\{t_1, t_2, t_3, t_4, t_5\}$.

In general, symmetric presentations offer a uniform and straight-forward way of constructing finite groups. The symmetric representation of elements of finite groups allows us to express each element in a convenient and shorter form, and the manipulation of elements written as symmetric representations is equivalent to the manipulation of permutations. For more examples of symmetric presentations and the manipulation of symmetrically-represented group elements, see [HK06], [HN05], [Con71], [Cur07], and [CH96].

Chapter 2

A_5 as a Homomorphic Image of the Progenitor $2^{*3} : S_3$

In this chapter, we investigate A_5 as a homomorphic image of the progenitor $2^{*3} : S_3$. The group A_5 is the alternating group on five letters having order $5!/2 = 60$. The progenitor $2^{*3} : S_3$ is a semi-direct product of 2^{*3}, a free product of three copies of the cyclic group of order 2, and S_3, the symmetric group on three letters which permutes the three symmetric generators, t_0, t_1, and t_2, by conjugation.

2.1 Introduction

Let $\bar{G}$ be a homomorphic image of the infinite semi-direct product, the *progenitor*, $2^{*3}: S_3$. A symmetric presentation of $2^{*3} : S_3$ is given by

$$\bar{G} = \langle x, y, t \mid x^3 = y^2 = (xy)^2 = e = t^2 = [t, x] \rangle,$$

where $[t, x] = txtx$ and e is the identity. In this case, $N \cong S_3 \cong \langle x, y \mid x^3 = y^2 = (xy)^2 = e \rangle$, and the action of N on the three symmetric generators is given by $x \sim (0\ 1\ 2)$, $y \sim (1\ 2)$, and $t \sim t_0$.

Let G denote the group $\bar{G}$ factored by the relations $(yxt)^5 = e$ and $(xt)^5 = e$. That is, let

$$G = \frac{\bar{G}}{(yxt)^5, (xt)^5}.$$

A symmetric presentation for G is given by

$$\langle x, y, t \mid x^3 = y^2 = (xy)^2 = e = t^2 = [t, x] = (yxt)^5 = (xt)^5 \rangle.$$

Now, we consider the following relations:

$$[(0\ 1\ 2)t_0]^5 = e$$

and

$$[(0\ 1)t_0]^5 = e.$$

According to a computer proof by [CHB96], the progenitor $2^{*3} : S_3$, factored by the relations $[(0\ 1\ 2)t_0]^5 = e$ and $[(0\ 1)t_0]^5 = e$, is isomorphic to A_5. We will construct A_5 by way of manual double coset enumeration of $G \cong \frac{2^{*3}:S_3}{[(0\ 1\ 2)t_0]^5, [(0\ 1)t_0]^5}$ over S_3. In so doing, we will show that A_5 is isomorphic to the symmetric presentation

$$G = \langle x, y, t \mid x^3 = y^2 = (xy)^2 = e = t^2 = [t, x] = (yxt)^5 = (xt)^5 \rangle.$$

2.2 Manual Double Coset Enumeration of G Over S_3

We first determine the order of the homomorphic image, G, of the progenitor. To determine the order of the homomorphic image G, we will determine the index of $N \cong S_3$ in G. We determine the index of $N \cong S_3$ in G first by expanding the relations $[(0\ 1\ 2)t_0]^5 = e$ and $[(0\ 1)t_0]^5 = e$, and next by performing manual double coset enumeration on G over $N \cong S_3$. To begin, we expand the relations that factor the progenitor $2^{*3} : S_3$:

$$[(0\ 1\ 2)t_0]^5 = e \tag{2.1}$$

$$[(0\ 1)t_0]^5 = e \tag{2.2}$$

We expand relations (2.1) and (2.2) in detail below:

1. Let $\pi = (0\ 1\ 2)$.

   Then $[(0\ 1\ 2)t_0]^5 = e \Rightarrow (\pi t_0)^5 = e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0$
   $= e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e \Rightarrow \pi t_0 \pi t_0 \pi^3 \pi^{-1} \pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e \Rightarrow$
   $\pi t_0 \pi^4 \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow \pi^5 \pi^{-1} \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0$
   $= e \Rightarrow \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow (0\ 1\ 2)^5 t_0^{(0\ 1\ 2)^4} t_0^{(0\ 1\ 2)^3} t_0^{(0\ 1\ 2)^2} t_0^{(0\ 1\ 2)} t_0 = e$
   $\Rightarrow (0\ 2\ 1) t_0^{(0\ 1\ 2)} t_0^{e} t_0^{(0\ 2\ 1)} t_0^{(0\ 1\ 2)} t_0 = e \Rightarrow (0\ 2\ 1) t_1 t_0 t_2 t_1 t_0 = e \Rightarrow (0\ 2\ 1) t_1 t_0 t_2 = t_0 t_1.$

Thus relation (2.1) implies that $(0\ 2\ 1)t_1t_0t_2 = t_0t_1$ or, equivalently, $Nt_1t_0t_2 = Nt_0t_1$. That is, using our short-hand notation, $102 \sim 01$.

2. Let $\pi = (0\ 1)$.

   Then $[(0\ 1)t_0]^5 = e \Rightarrow (\pi t_0)^5 = e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi^3 \pi^{-1} \pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e \Rightarrow \pi t_0 \pi^4 \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow \pi^5 \pi^{-1} \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow (0\ 1)^5 t_0^{(0\ 1)^4} t_0^{(0\ 1)^3} t_0^{(0\ 1)^2} t_0^{(0\ 1)} t_0 = e$
   $\Rightarrow (0\ 1) t_0^{e} t_0^{(0\ 1)} t_0^{3} t_0^{(0\ 1)} t_0 = e \Rightarrow (0\ 1)t_0t_1t_0t_1t_0 = e \Rightarrow (0\ 1)t_0t_1t_0 = t_0t_1$.

   Thus relation (2.2) implies that $(0\ 1)t_0t_1t_0 = t_0t_1$ or, equivalently, $Nt_0t_1t_0 = Nt_0t_1$. That is, using our short-hand notation, $010 \sim 01$.

We now perform manual double coset enumeration of G over S_3.

1. We first note that the double coset $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$. In this sense, we say that NeN is a double coset with a word in the t_i's of length zero.

   Let $[*]$ denote the double coset NeN.

   We first determine the order of the double coset $[*]$.

   The double coset $[*]$ has one distinct right coset: the identity right coset, $Ne = \{ne \mid n \in N\} = N$.

   We next determine the distinct double cosets of the form NwN, where w is a word of length one given by $w = t_i$, $i \in \{0, 1, 2\}$.

   Since $N \cong S_3$ is transitive, and since the orbit of N on T is $O(0) = \{g0 \mid g \in N\} = \{0, 1, 2\} = O(1) = O(2)$, N has one orbit on $T = \{t_0, t_1, t_2\}$: $\{0, 1, 2\}$.

   Therefore, there is one double coset of the form NwN, where w is a word of length one given by $w = t_i$, $i \in \{0, 1, 2\}$: Nt_0N.

2. We next consider the double coset Nt_0N.

   Let $[0]$ denote the double coset Nt_0N.

   Note that $Nt_0N = \{Nt_0n \mid n \in N\} = \{Nn^{-1}t_0n \mid n \in N\} = \{Nt_0^n \mid n \in N\}$
   $= \{Nt_0, Nt_1, Nt_2\}$.

We first determine the order of the double coset [0].

Note that the point stabilizer is $N^0 = \{n \in N \mid t_0^n = t_0\} = \langle(1\ 2)\rangle \cong S_2$, and note further that the coset stabilizer is $N^{(0)} \geq \{n \in N \mid Nt_0^n = Nt_0\} = N^0 = \langle(1\ 2)\rangle \cong S_2$. Thus $|N^{(0)}| \geq |S_2| = 2$, and, by Lemma 1.4, $|Nt_0N| = \frac{|N|}{|N^{(0)}|} = \frac{6}{2} = 3$.

That is, the double coset [0] has at most three distinct single cosets.

We next determine the distinct double cosets of the form NwN, where w is a word of length two given by $w = t_0t_i$, $i \in \{0,1,2\}$.

Since $O(0) = \{g0 \mid g \in N^{(0)}\} = \{0\}$ and since $O(1) = \{g1 \mid g \in N^{(0)}\} = \{1,2\} = O(2)$, $N^{(0)}$ has two orbits on $T = \{t_0, t_1, t_2\}$: $\{0\}$ and $\{1,2\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length two given by $w = t_0t_i$, $i \in \{0,1\}$: Nt_0t_0N and Nt_0t_1N.

But, since $Nt_0t_0N = Nt_0^2N = NeN = N$, we conclude that there is one distinct double coset of the form Nt_0t_iN, where $i \in \{0,1,2\}$: Nt_0t_1N.

3. We next consider the double coset Nt_0t_1N.

   Let [01] denote the double coset Nt_0t_1N.

   Note that $Nt_0t_1N = \{Nt_0t_1n \mid n \in N\} = \{Nn^{-1}t_0t_1n \mid n \in N\} = \{N(t_0t_1)^n \mid n \in N\} = \{Nt_it_j \mid i,j \in \{0,1,2\}, i \neq j\} = \{Nt_0t_1, Nt_0t_2, Nt_1t_0, Nt_1t_2, Nt_2t_0, Nt_2t_1\}$.

   We first determine the order of the double coset [01].

   Note that the point stabilizer is $N^{01} = \{n \in N \mid (t_0t_1)^n = t_0t_1\} = \{e\}$, and note further that the coset stabilizer is $N^{(01)} \geq \{n \in N \mid Nt_0^n = Nt_0\} = N^{01} = \{e\}$. Thus $|N^{(01)}| \geq |N^{01}| = |\{e\}| = 1$ and, by Lemma 1.4, $|Nt_0t_1| = \frac{|N|}{|N^{(0)}|} = \frac{6}{1} = 6$.

   That is, the double coset [01] has at most six distinct single cosets.

   We next determine the distinct double cosets of the form NwN, where w is a word of length one given by $w = t_0t_1t_i$, $i \in \{0,1,2\}$.

   Since $O(0) = \{g0 \mid g \in N^{(0)}\} = \{0\}$, since $O(1) = \{g1 \mid g \in N^{(0)}\} = \{1\}$, and since $O(2) = \{g2 \mid g \in N^{(0)}\} = \{2\}$, $N^{(01)}$ has three orbits on $T = \{t_0, t_1, t_2\}$: $\{0\}$ and $\{1\}$ and $\{2\}$.

   Therefore, there are at most three double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1t_i$, $i \in \{0,1,2\}$: $Nt_0t_1t_0N$, $Nt_0t_1t_1N$, and

$Nt_0t_1t_2N$.

But note that $Nt_0t_1t_1N = Nt_0t_1^2N = Nt_0eN = Nt_0N$.

Moreover, note that by relation (2.2), $(0\ 1)t_0t_1t_0 = t_0t_1$ implies that $Nt_0t_1t_0 = Nt_0t_1$ which implies that $Nt_0t_1t_0N = Nt_0t_1N$. That is, [01] = [010].

Further, by relation (2.1), $(0\ 2\ 1)t_1t_0t_2 = t_0t_1 \Rightarrow [(0\ 2\ 1)t_1t_0t_2]^{(0\ 1)} = [t_0t_1]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_0t_1t_2 = t_1t_0$ implies that $Nt_0t_1t_2 = Nt_1t_0$ which implies that $Nt_0t_1t_2N = Nt_0t_1N$. That is, [01] = [012].

Since $Nt_0t_1t_1N = Nt_0N$ and $Nt_0t_1t_0N = Nt_0t_1N$ and $Nt_0t_1t_2N = Nt_0t_1N$, we need not consider additional double cosets of the form $Nt_0t_1t_iN$, where $i \in \{0, 1, 2\}$.

In fact, since $N^{(01)}$ is transitive on the symmetric generators and since $Nt_0t_1t_1N = Nt_0t_1^2N = Nt_0eN = Nt_0N$ and $Nt_0t_1t_0N = Nt_0t_1N$ and $Nt_0t_1t_2N = Nt_0t_1N$ imply that the double cosets [011] = [0], [010] = [01], and [012] = [01], respectively, we have completed the double coset enumeration of G over S_3.

In total, therefore, there are at most 3 distinct double cosets of N in G and at most 10 distinct right (single) cosets of N in G. The double cosets of N in G are as follows: [*], [0], and [01].

2.3 Cayley Diagram of G Over S_3

The Cayley diagram of G over S_3 is illustrated in Figure 2.1. The *vertices* of the Cayley diagram indicate the set of right cosets of N in G, $\{Nw_i \mid w_i \text{ are words in } T\}$. The *nodes* represent the double cosets of N in G and each node is labeled with the number of distinct right (single) cosets of N in G within the double cosets. The *lines* between the nodes indicate relations between the images of the right cosets of one node and the right cosets of other nodes; the number of lines emanating from a particular node is determined by the number of orbits on the point stabilizer. The lines are labeled with integers indicating the number of pathways (or orbits) from the vertices (the right cosets) of one node (one double coset) to the vertices of another node. Put together, these pieces of the diagram illustrate the action of N on the right cosets of N in G by right multiplication.

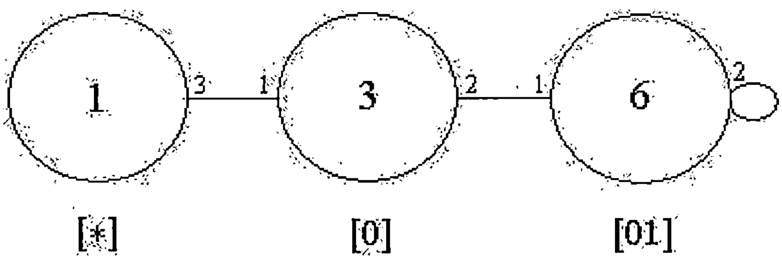

Figure 2.1: Cayley Diagram of G Over S_3

2.4 Action of the Symmetric Generators and the Generators of S_3 on the Right Cosets of G Over S_3

Let X denote the set of all (10) distinct right cosets of N in G, that is, let $X = \{N, Nt_0, Nt_1, Nt_2, Nt_0t_1, Nt_0t_2, Nt_1t_0, Nt_1t_2Nt_2t_0, Nt_2t_1\}$. We define a mapping $\phi\colon G \longrightarrow S_X$ so that ϕ maps a symmetric generator $g \in G$ to its action (by right multiplication) on X. That is, we define ϕ so that $\phi(g) = \widehat{\phi}(g) : X \to X$. Then the action $\phi(t) \sim \phi(t_0)$ of the symmetric generator $t \sim t_0$ on the set of right cosets of N in G may be expressed as

$$\phi(t) \sim \phi(t_0) = (*\ 0)(1\ 10)(2\ 20)(12\ 21),$$

and the action of the generator $x \sim (0\ 1\ 2)$ of S_3 on the set of right cosets of N in G may be expressed as

$$\phi(x) \sim \phi((0\ 1\ 2)) = (0\ 1\ 2)(01\ 12\ 20)(02\ 10\ 21)$$

and the action of the generator $y \sim (1\ 2)$ of S_3 on the set of right cosets of N in G may be expressed as

$$\phi(y) \sim \phi((1\ 2)) = (1\ 2)(01\ 02)(10\ 20)(12\ 21).$$

Since there are 10 distinct right cosets of N in G, these actions may be written as permutations on 10 letters. In fact, the actions of the generators on the set of right cosets of N in G are equivalent to the permutation representations of the generators in their action on the right cosets of N in G. To better manipulate the permutation representations of the symmetric generators t_i and the generators x and y, it is helpful to label the distinct single cosets of N in G as follows:

(10)	$*$	(5)	02
(1)	0	(6)	10
(2)	1	(7)	12
(3)	2	(8)	20
(4)	01	(9)	21

Having labeled each of the 10 distinct right cosets of N in G, we may express the permutation representation of the symmetric generators $t \sim t_0$, $t^x \sim t_1$, and $t^{x^2} \sim t_2$, and the generators $x \sim (0\ 1\ 2)$ and $y \sim (1\ 2)$, in their action on the right cosets of N in G as, respectively

$$\phi(t) \sim \phi(t_0) : (10\ 1)(2\ 6)(3\ 8)(7\ 9),$$

$$\phi(t^x) \sim \phi(t_1) : (10\ 2)(1\ 4)(3\ 9)(5\ 8),$$

$$\phi(t^{x^2}) \sim \phi(t_2) : (10\ 3)(1\ 5)(2\ 7)(4\ 6),$$

$$\phi(x) \sim \phi((0\ 1\ 2)) : (1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9),$$

$$\phi(y) \sim \phi((1\ 2)) : (2\ 3)(4\ 5)(6\ 8)(7\ 9)$$

2.5 Proof of Isomorphism between G and A_5

We now demonstrate that $G \cong A_5$.

Proof. To prove that $G \cong A_5$, we must first show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and that $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 60$ (from which we can conclude $G \cong \langle \phi(x), \phi(y), \phi(t) \rangle$), and we must next show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong A_5$ (from which we can conclude A_5 is a homomorphic image of G and $G \cong A_5$).

We first show $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 60$. From our construction of G using manual double coset enumeration of G over S_3, illustrated by the Cayley Diagram in Figure 2.1, we concluded that group G defined by the symmetric presentation must contain a homomorphic image of $N \cong S_3$ whose index $[G : N]$ is at most 10:

$$[G : N] = \frac{|N|}{|N^{(*)}|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} \leq \frac{6}{6} + \frac{6}{2} + \frac{6}{1} =$$

$$1 + 3 + 6 = 10$$

Since the index of N in G is at most 10, and since $|G| = \frac{|G|}{|N|} \cdot |N|$, the order of the homomorphic image group G is at most 60:

$$|G| = \frac{|G|}{|N|} \cdot |N| \leq 10 \cdot |N| = 10 \cdot 6 = 60 \Rightarrow |G| \leq 60$$

We now consider $\langle \phi(x), \phi(y), \phi(t) \rangle$. Note that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a group generated by the permutation representations of the generators x, y, and t and, as such, it is a subgroup of the symmetric group S_{10} acting on the ten right cosets of N in G. We now show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and, therefore, that $|G| \geq |\langle \phi(x), \phi(y), \phi(t) \rangle| = 60$. To show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we first demonstrate that $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$ is a homomorphic image of $\bar{G}$. Now, recall that $\bar{G} = \langle x, y, t \rangle$ is a homomorphic image of the progenitor $2^{*3} : S_3$, and its presentation is given by

$$\bar{G} = \langle x, y, t \mid x^3 = y^2 = (xy)^2 = e = t^2 = [t, x] \rangle,$$

where $x \sim (0\ 1\ 2)$, $y \sim (1\ 2)$, and $t \sim t_0$, and $N = \langle x, y \rangle \cong S_3$. Let $\alpha : \bar{G} \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ be a mapping from $\bar{G}$ to $\langle \phi(x), \phi(y), \phi(t) \rangle$. We note first that the mapping $\alpha : \bar{G} \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ is well defined. The generators $\phi(x), \phi(y)$, and $\phi(t)$ are the permutation representations of $x \sim (0\ 1\ 2)$, $y \sim (1\ 2)$, and $t \sim t_0$ on 10 letters. Since the order of $\phi(x)$ is 3, the order of $\phi(y)$ is 2, and the order of $\phi(x)\phi(y)$ is 2, we conclude $\langle \phi(x), \phi(y) \rangle \cong N \cong S_3$. Moreover, we can demonstrate that $\phi(t)$ has exactly three conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N \cong S_3$. Now, since $t \sim t_0$, we have that

$$\phi(t)^{\phi(x)} \sim \phi(t_0)^{\phi((0\ 1\ 2))} = [(10\ 1)(2\ 6)(3\ 8)(7\ 9)]^{(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)} =$$

$$[(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)][(10\ 1)(2\ 6)(3\ 8)(7\ 9)][(1\ 3\ 2)(4\ 8\ 7)(5\ 9\ 6)] =$$

$$(10\ 2)(1\ 4)(3\ 9)(5\ 8) = \phi(t_1) \sim \phi(t^x)$$

and further that

$$\phi(t)^{\phi(x)^2} \sim \phi(t_0)^{\phi((0\ 1\ 2))^2} = [(10\ 1)(2\ 6)(3\ 8)(7\ 9)]^{(1\ 3\ 2)(4\ 8\ 7)(5\ 9\ 6)} =$$

$$[(1\ 3\ 2)(4\ 8\ 7)(5\ 9\ 6)][(10\ 1)(2\ 6)(3\ 8)(7\ 9)][(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)] =$$

$$(10\ 3)(1\ 5)(2\ 7)(4\ 6) = \phi(t_2) \sim \phi(t^{x^2})$$

Therefore, $\phi(t)$ has exactly three conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N \cong S_3$; these conjugates are, namely, $\phi(t) \sim \phi(t_0)$, $\phi(t^x) \sim \phi(t_1)$, and $\phi(t^{x^2}) \sim \phi(t_2)$. Since $\langle \phi(x), \phi(y) \rangle \cong N \cong S_3$ and since $\phi(t)$ has exactly three conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N$, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $\bar{G} = \langle x, y, t \rangle$. That is, $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*3} : S_3$.

Next, to show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we must show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $\bar{G}$ factored by the relations $(yxt)^5 = e$ and $(xt)^5 = e$; that is, we must show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*3} : S_3$ factored by the relations $[(0\ 1\ 2)t_0]^5 = e$ and $[(0\ 1)t_0]^5 = e$. Let $\tilde{\alpha} : G \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ be a mapping from G to $\langle \phi(x), \phi(y), \phi(t) \rangle$. We note first that the mapping $\tilde{\alpha} : G \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ is well-defined, and we know already that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*3} : S_3$. Now, to show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we need only demonstrate that the relations $[(0\ 1\ 2)t_0]^5 = e$ and $[(0\ 1)t_0]^5 = e$, which hold true in G, also hold true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$.

To demonstrate that the relation $[(0\ 1\ 2)t_0]^5 = e$, or, equivalently, the relation $t_1 t_0 t_2 t_1 t_0 = (0\ 1\ 2)$, holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$, we show that $\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{10}$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation in the same way that $\phi((0\ 1\ 2)) \sim \phi(x)$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by $\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(10\ 1)(2\ 6)(3\ 8)(7\ 9)]^{(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)} = \phi(t_1),$$

$$\phi(t_1)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(10\ 2)(1\ 4)(3\ 9)(5\ 8)]^{(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)} = \phi(t_2),$$

$$\phi(t_2)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(10\ 3)(1\ 5)(2\ 7)(4\ 6)]^{(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)} = \phi(t_0)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by $\phi((0\ 1\ 2))$. This gives us

$$\phi(t_0)^{\phi((0\ 1\ 2))} = [(10\ 1)(2\ 6)(3\ 8)(7\ 9)]^{(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)} = \phi(t_1),$$

$$\phi(t_1)^{\phi((0\ 1\ 2))} = [(10\ 2)(1\ 4)(3\ 9)(5\ 8)]^{(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)} = \phi(t_2),$$

$$\phi(t_2)^{\phi((0\ 1\ 2))} = [(10\ 3)(1\ 5)(2\ 7)(4\ 6)]^{(1\ 2\ 3)(4\ 7\ 8)(5\ 6\ 9)} = \phi(t_0)$$

Since $\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{10}$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation in the same way that $\phi((0\ 1\ 2)) \sim \phi(x)$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation, we conclude that the relation $[(0\ 1\ 2)t_0]^5 = e$, which holds true in G, also holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$.

To demonstrate that the relation $[(0\ 1)t_0]^5 = e$, or, equivalently, the relation $t_0t_1t_0t_1t_0 = (0\ 1)$, holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$, we show that $\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0) \sim \phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t) \in S_{10}$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation in the same way that the element $\phi((0\ 1)) \sim \phi(yx)$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by $\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(10\ 1)(2\ 6)(3\ 8)(7\ 9)]^{(1\ 2)(4\ 6)(5\ 7)(8\ 9)} = \phi(t_1),$$

$$\phi(t_1)^{\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(10\ 2)(1\ 4)(3\ 9)(5\ 8)]^{(1\ 2)(4\ 6)(5\ 7)(8\ 9)} = \phi(t_0)$$

$$\phi(t_2)^{\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(10\ 3)(1\ 5)(2\ 7)(4\ 6)]^{(1\ 2)(4\ 6)(5\ 7)(8\ 9)} = \phi(t_2)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by $\phi((0\ 1))$. This gives us

$$\phi(t_0)^{\phi((0\ 1))} = [(10\ 1)(2\ 6)(3\ 8)(7\ 9)]^{(1\ 2)(4\ 6)(5\ 7)(8\ 9)} = \phi(t_1),$$

$$\phi(t_1)^{\phi((0\ 1))} = [(10\ 2)(1\ 4)(3\ 9)(5\ 8)]^{(1\ 2)(4\ 6)(5\ 7)(8\ 9)} = \phi(t_0),$$

$$\phi(t_2)^{\phi((0\ 1))} = [(10\ 3)(1\ 5)(2\ 7)(4\ 6)]^{(1\ 2)(4\ 6)(5\ 7)(8\ 9)} = \phi(t_2)$$

Since $\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0) \sim \phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t) \in S_{10}$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation in the same way that the element $\phi((0\ 1)) \sim \phi(yx)$ acts on the three symmetric generators $\phi(t_0)$, $\phi(t_1)$, and $\phi(t_2)$ by conjugation, we conclude that the relation $[(0\ 1)t_0]^5 = e$, which holds true in G, also holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$.

Since $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*3} : S_3$, and since the relations $[(0\ 1\ 2)t_0]^5 = e$ and $[(0\ 1)t_0]^5 = e$ hold true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*3} : S_3$

factored by the relations $[(0\ 1\ 2)t_0]^5 = e$ and $[(0\ 1)t_0]^5 = e$; that is, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G.

More importantly, since $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we have that $\langle \phi(x), \phi(y), \phi(t) \rangle \leq G$. In fact, since $\langle \phi(x), \phi(y), \phi(t) \rangle \leq G$, we have that $|G| \geq |\langle \phi(x), \phi(y), \phi(t) \rangle|$. Since it is easily demonstrated, with MAGMA or by hand, that $|\langle \phi(x), \phi(y), \phi(t) \rangle| = 60$, we conclude finally that $|G| \geq |\langle \phi(x), \phi(y), \phi(t) \rangle| = 60$, that is, $|G| \geq 60$. Given $|G| \leq 60$ and $|G| \geq 60$, we conclude $|G| = 60$. Moreover, since $|\langle \phi(x), \phi(y), \phi(t) \rangle| = 60 = |G|$ and since $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we conclude

$$\langle \phi(x), \phi(y), \phi(t) \rangle \cong G.$$

We finally show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong A_5$. Let $G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle$. Now, with the help of MAGMA (see [BCP97]), we know that the elements $a = (1\ 5\ 6)(3\ 10\ 8)(4\ 9\ 7)$, $b = (3\ 4)(5\ 6)(7\ 10)(8\ 9)$, and $c = (1\ 2)(3\ 5)(4\ 6)(7\ 10)$ belong to G_1. (Note: The labels for the right cosets in this case were assigned by MAGMA and are different from the labels that we assigned earlier.) Therefore, $\langle a, b, c \rangle \leq G_1$, where $G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle$, a permutation group on 10 letters, is a permutation representation of G and, further, $|G_1| = 60$. But $|\langle a, b, c \rangle| = |G_1| = 60$. Therefore, $G_1 = \langle a, b, c \rangle$. However, $\langle a, b, c \rangle \cong A_5 \cong \langle a, b, c | a^3 = b^2 = c^2 = (ab)^3 = [c, a] = e \rangle$. Therefore, $G_1 \cong A_5$ and, since $G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle \cong G$, we conclude that $G \cong A_5$.

□

2.6 Converting an Element of G from its Permutation Representation to its Symmetric Representation

To illustrate how a permutation representation of an element of A_5 on 10 letters may be converted to its symmetric representation form, we consider the following example:

Example 2.1. Let $g \in G \cong A_5$ and let $p = \phi(g) = (10\ 7\ 5)(1\ 9\ 4)(2\ 6\ 8)$ be the permutation representation of g on 10 letters. Then $10^p = 7$ implies $N^p = Nt_1t_2$, since 10 and 7 are labels for the right cosets N and Nt_1t_2, respectively. Moreover, since $N^p = Np$ and $N^p = Nt_1t_2$, we have that $Np = Nt_1t_2$. Now, $Np = Nt_1t_2$ implies that $p \in Nt_1t_2$ which implies that $p \sim \pi t_1 t_2$ for some $\pi \in N \cong S_3$ or, more precisely, $p = \phi(\pi t_1 t_2) = \phi(\pi)\phi(t_1)\phi(t_2)$

for some $\pi \in N \cong S_3$. To determine $\pi \in N$, we note first that $p = \phi(\pi)\phi(t_1)\phi(t_2) \Rightarrow p(\phi(t_2))^{-1}(\phi(t_1))^{-1} = p\phi(t_2^{-1})\phi(t_1^{-1}) = p\phi(t_2)\phi(t_1) = \phi(\pi)$. We then calculate the action of $\pi \sim \phi(\pi) = p\phi(t_2)\phi(t_1)$ on the symmetric generators t_i, where $i \in \{0,1,2\}$. Now, $\phi(\pi) = p\phi(t_2)\phi(t_1) = [(10\ 7\ 5)(1\ 9\ 4)(2\ 6\ 8)][(10\ 3)(1\ 5)(2\ 7)(4\ 6)][(10\ 2)(1\ 4)(3\ 9)(5\ 8)] = (1\ 3\ 2)(4\ 8\ 7)(5\ 9\ 6)$. The element $\pi \sim \phi(\pi) = p\phi(t_2)\phi(t_1) = (1\ 3\ 2)(4\ 8\ 7)(5\ 9\ 6)$ acts on the right cosets Nt_0, Nt_1, and Nt_2 via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$Nt_0 = 1 \mapsto 1^p = 3 = Nt_2, \qquad Nt_2 = 3 \mapsto 3^p = 2 = Nt_1,$$

$$Nt_1 = 2 \mapsto 2^p = 1 = Nt_0$$

Therefore, the element $\phi(\pi)$ acts as $(0\ 2\ 1)$ on the right cosets Nt_0, Nt_1, and Nt_2, and so $\phi(\pi)$ is the permutation representation of $\pi = (0\ 2\ 1) \in S_3$ on 10 letters. Therefore, $\pi = (0\ 2\ 1)$ and $w = t_1t_2$, and so the symmetric representation of g is $(0\ 2\ 1)t_1t_2$.

2.7 Converting an Element of G from its Symmetric Representation to its Permutation Representation

To illustrate how an element of A_5 in symmetric representation form may be converted to its permutation representation on 10 letters, we consider the following example:

Example 2.2. Let $g \in G \cong A_5$ have the symmetric representation $(0\ 2\ 1)t_1t_2$. To determine the permutation representation $p = \phi(g)$ of g, we first calculate the action of $\pi = (0\ 2\ 1)$ on the right cosets of N in G. Now, the element $\pi = (0\ 2\ 1)$ acts on the right cosets N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$10 = N \mapsto N^{(0\ 2\ 1)} = N = 10$$

$$1 = Nt_0 \mapsto Nt_0^{(0\ 2\ 1)} = Nt_2 = 3$$

$$3 = Nt_2 \mapsto Nt_2^{(0\ 2\ 1)} = Nt_1 = 2$$

$$2 = Nt_1 \mapsto Nt_1^{(0\ 2\ 1)} = Nt_0 = 1$$

$$4 = Nt_0t_1 \mapsto N(t_0t_1)^{(0\ 2\ 1)} = Nt_2t_0 = 8$$

$$8 = Nt_2t_0 \mapsto N(t_2t_0)^{(0\ 2\ 1)} = Nt_1t_2 = 7$$

$$7 = Nt_1t_2 \mapsto N(t_1t_2)^{(0\ 2\ 1)} = Nt_0t_1 = 4$$

$$5 = Nt_0t_2 \mapsto N(t_0t_2)^{(0\ 2\ 1)} = Nt_2t_1 = 9$$

$$9 = Nt_2t_1 \mapsto N(t_2t_1)^{(0\ 2\ 1)} = Nt_1t_0 = 6$$

$$6 = Nt_1t_0 \mapsto N(t_1t_0)^{(0\ 2\ 1)} = Nt_0t_2 = 5$$

Therefore, the permutation representation of $\pi = (0\ 2\ 1)$ is $\phi(\pi) = (1\ 3\ 2)(4\ 8\ 7)(5\ 9\ 6)$. Similarly, we calculate the action of the symmetric generator t_1 on the right cosets of N in G. The symmetric generator t_1 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_1, Nw) = Nwt_1$. The mappings below illustrate this action:

$$10 = N \mapsto Nt_1 = 2$$

$$2 = Nt_1 \mapsto Nt_1t_1 = N = 10$$

$$1 = Nt_0 \mapsto Nt_0t_1 = 4$$

$$4 = Nt_0t_1 \mapsto Nt_0t_1t_2 = Nt_0 = 1$$

$$3 = Nt_2 \mapsto Nt_2t_1 = 9$$

$$9 = Nt_2t_1 \mapsto Nt_2t_1t_1 = Nt_2 = 3$$

$$5 = Nt_0t_2 \mapsto Nt_0t_2t_1 = Nt_2t_0 = 8$$

$$8 = Nt_2t_0 \mapsto Nt_2t_0t_1 = Nt_0t_2 = 5$$

Therefore, the permutation representation of t_1 is $\phi(t_1) = (10\ 2)(1\ 4)(3\ 9)(5\ 9)$. Finally, we calculate the action of the symmetric generator t_2 on the right cosets of N in G. The symmetric generator t_2 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_2, Nw) = Nwt_2$. The mappings below illustrate this action:

$$10 = N \mapsto Nt_2 = 3$$

$$3 = Nt_2 \mapsto Nt_2t_2 = N = 10$$

$$1 = Nt_0 \mapsto Nt_0t_2 = 5$$

$$5 = Nt_0t_2 \mapsto Nt_0t_2t_2 = Nt_0 = 1$$

$$2 = Nt_1 \mapsto Nt_1t_2 = 7$$

$$7 = Nt_1t_2 \mapsto Nt_1t_2t_2 = Nt_1 = 2$$

$$4 = Nt_0t_1 \mapsto Nt_0t_1t_2 = Nt_1t_0 = 6$$

$$6 = Nt_1t_0 \mapsto Nt_1t_0t_2 = Nt_0t_1 = 4$$

Therefore, the permutation representation of t_2 is $\phi(t_2) = (10\ 3)(1\ 5)(2\ 7)(4\ 6)$. Now, $g = (0\ 2\ 1)t_1t_2 \sim \phi(g) = \phi(\pi)\phi(t_1)\phi(t_2) = [(1\ 3\ 2)(4\ 8\ 7)(5\ 9\ 6)][(10\ 2)(1\ 4)(3\ 9)(5\ 9)]$ $[(10\ 3)(1\ 5)(2\ 7)(4\ 6)] = (10\ 7\ 5)(1\ 9\ 4)(2\ 6\ 8)$. Therefore, the permutation representation of g is $p = \phi(g) = (10\ 7\ 5)(1\ 9\ 4)(2\ 6\ 8)$.

Chapter 3

S_5 as a Homomorphic Image of the Progenitor $2^{*4} : A_4$

In this chapter, we investigate S_5 as a homomorphic image of the progenitor $2^{*4} : A_4$. The group S_5 is the symmetric group on five letters having order $5! = 120$. The progenitor $2^{*4} : A_4$ is a semi-direct product of 2^{*4}, a free product of four copies of the cyclic group of order 2, and A_4, the alternating group on four letters which permutes the four symmetric generators, t_0, t_1, t_2, and t_3, by conjugation.

3.1 Introduction

Let $\bar{G}$ be a homomorphic image of the infinite semi-direct product, the *progenitor*, $2^{*4}: A_4$. A symmetric presentation of $2^{*4} : A_4$ is given by

$$\bar{G} = \langle x, y, t \mid x^3 = y^3 = (xy)^2 = e = t^2 = [t, x] \rangle,$$

where $[t, x] = txtx$ and e is the identity. In this case, $N \cong A_4 \cong \langle x, y \mid x^3 = y^3 = (xy)^2 = e \rangle$, and the action of N on the four symmetric generators is given by $x \sim (1\ 2\ 3)$, $y \sim (0\ 1\ 2)$, and $t \sim t_0$.

Let G denote the group $\bar{G}$ factored by the relations $(yt)^4 = e$ and $(xyt)^6 = e$. That is, let

$$G = \frac{\bar{G}}{(yt)^4, (xyt)^6}.$$

A symmetric presentation for G is given by

$$\langle x, y, t \mid x^3 = y^3 = (xy)^2 = e = t^2 = [t, x] = (yt)^4 = (xyt)^6 \rangle.$$

Now, we consider the following relations:

$$[(0\ 1\ 2)t_0]^4 = e$$
and
$$[(0\ 1)(2\ 3)t_0]^6 = e.$$

According to a computer proof by [CHB96], the progenitor $2^{*4} : A_4$, factored by the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$, is isomorphic to S_5. In fact, factoring the progenitor $2^{*4} : A_4$ by the relation $[(0\ 1\ 2)t_0]^4 = e$ alone suffices. We will construct S_5 by way of manual double coset enumeration of $G \cong \frac{2^{*4}:A_4}{[(0\ 1\ 2)t_0]^4, [(0\ 1)(2\ 3)t_0]^6}$ over A_4. In so doing, we will show that S_5 is isomorphic to the symmetric presentation

$$G = \langle x, y, t \mid x^3 = y^3 = (xy)^2 = e = t^2 = [t, x] = (yt)^4 = (xyt)^6 \rangle.$$

3.2 Manual Double Coset Enumeration of G Over A_4

We first determine the order of the homomorphic image, G, of the progenitor. To determine the order of the homomorphic image G, we will determine the index of $N \cong A_4$ in G. We determine the index of $N \cong A_4$ in G first by expanding the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$, and next by performing manual double coset enumeration on G over $N \cong A_4$. To begin, we expand the relations that factor the progenitor $2^{*4} : A_4$:

$$[(0\ 1\ 2)t_0]^4 = e \tag{3.1}$$

$$[(0\ 1)(2\ 3)t_0]^6 = e \tag{3.2}$$

As mentioned above, relation (3.1), $[(0\ 1\ 2)t_0]^4 = e$, is required to determine the homomorphic image, G, of the progenitor, and the other relation, (3.2), can be derived from relation (3.1). We expand relations (3.1) and (3.2) in detail below:

1. Let $\pi = (0\ 1\ 2)$.

   Then $[(0\ 1\ 2)t_0]^4 = e \Rightarrow (\pi t_0)^4 = e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e \Rightarrow$
   $\pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e \Rightarrow \pi t_0 \pi^3 \pi^{-1} \pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e \Rightarrow$

$\pi^4\pi^{-1}\pi^{-1}\pi^{-1}t_0\pi^3t_0^{\pi^2}t_0^{\pi}t_0 = e \Rightarrow \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
$(0\ 1\ 2)^4 t_0^{(0\ 1\ 2)^3} t_0^{(0\ 1\ 2)^2} t_0^{(0\ 1\ 2)} t_0 = e \Rightarrow (0\ 1\ 2) t_0^e t_0^{(0\ 2\ 1)} t_0^{(0\ 1\ 2)} t_0 = e$
$\Rightarrow (0\ 1\ 2)t_0t_2t_1t_0 = e \Rightarrow (0\ 1\ 2)t_0t_2 = t_0t_1.$

Thus relation (3.1) implies that $(0\ 1\ 2)t_0t_2 = t_0t_1$ or, equivalently, $Nt_0t_2 = Nt_0t_1$. That is, using our short-hand notation, $02 \sim 01$.

2. Let $\pi = (0\ 1)(2\ 3)$.

   Then $[(0\ 1)(2\ 3)t_0]^6 = e \Rightarrow (\pi t_0)^6 = e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e \Rightarrow$
   $\pi t_0 \pi t_0 \pi t_0 \pi^3 \pi^{-1}\pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e \Rightarrow \pi t_0 \pi t_0 \pi^4 \pi^{-1}\pi^{-1}\pi^{-1} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow \pi t_0 \pi^5 \pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1} t_0 \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
   $\pi^6 \pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1} t_0 \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow \pi^6 t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
   $[(0\ 1)(2\ 3)]^6 t_0^{[(0\ 1)(2\ 3)]^5} t_0^{[(0\ 1)(2\ 3)]^4} t_0^{[(0\ 1)(2\ 3)]^3} t_0^{[(0\ 1)(2\ 3)]^2} t_0^{[(0\ 1)(2\ 3)]} t_0 = e$
   $\Rightarrow e t_0^{(0\ 1)(2\ 3)} t_0^e t_0^{(0\ 1)(2\ 3)} t_0^e t_0^{(0\ 1)(2\ 3)} t_0 = e \Rightarrow e t_1 t_0 t_1 t_0 t_1 t_0 = e \Rightarrow t_1 t_0 t_1 = t_0 t_1 t_0.$

   Thus relation (3.2) implies that $t_1t_0t_1 = t_0t_1t_0$ or, equivalently, $Nt_1t_0t_1 = Nt_0t_1t_0$. That is, using our short-hand notation, $101 \sim 010$.

We now perform manual double coset enumeration of G over A_4.

1. We first note that the double coset $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$.

   Let $[*]$ denote the double coset NeN.

   The double coset $[*]$ has one distinct right coset: the identity right coset, $Ne = \{ne \mid n \in N\} = N$.

   Moreover, since $N \cong A_4$ is transitive, and since $O(0) = \{0^g \mid g \in N\} = \{0, 1, 2, 3\} = O(1) = O(2) = O(3)$, N must have one orbit on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2, 3\}$.

   Therefore, there is one double coset of the form NwN, where w is a word of length one given by $w = t_i$, $i \in \{0, 1, 2, 3\}$: Nt_0N.

2. We next consider the double coset Nt_0N.

   Let $[0]$ denote the double coset Nt_0N.

   Note that $N^{(0)} \geq N^0 = \langle(1\ 2\ 3)\rangle \cong A_3$. Thus $|N^{(0)}| \geq |A_3| = 3$, and, by Lemma 1.4, $|Nt_0N| = \frac{|N|}{|N^{(0)}|} = \frac{12}{3} = 4$.

Therefore, the double coset [0] has at most four distinct single cosets.

Moreover, $N^{(0)}$ must have two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$ and $\{1, 2, 3\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length two given by $w = t_0 t_i$, $i \in \{0, 1\}$: Nt_0t_0N and Nt_0t_1N.

But, since $Nt_0t_0N = Nt_0^2N = NeN = N$, we need only consider one additional the double coset of the form Nt_0t_iN, where $i \in \{0, 1, 2, 3\}$: Nt_0t_1N.

3. We next consider the double coset Nt_0t_1N.

   Let [01] denote the double coset Nt_0t_1N.

   Now, by relation (3.1), $(0\ 1\ 2)t_0t_2 = t_0t_1$ and $[(0\ 1\ 2)t_0t_2]^{(1\ 2\ 3)} = (t_0t_1)^{(1\ 2\ 3)} \Rightarrow (0\ 2\ 3)t_0t_3 = t_0t_2$ imply that $t_0t_2 = (0\ 2\ 1)t_0t_1 = (0\ 2\ 3)t_0t_3$. Therefore, $t_0t_2 = (0\ 2\ 1)t_0t_1 = (0\ 2\ 3)t_0t_3$ implies that

$$01 \sim 02 \sim 03$$

   Similarly, by conjugation, we find that

$$10 \sim 12 \sim 13, \qquad 20 \sim 21 \sim 23, \qquad 30 \sim 31 \sim 32$$

   Since each of the twelve single cosets has three names, the double coset [01] has at most four distinct single cosets.

   An alternative approach for determining the order of the double coset is as follows: We note that $N^{(01)} \geq N^{01} = \langle e \rangle$. Now, by relation (3.1), $N(t_0t_1)^{(1\ 2\ 3)} = Nt_0t_2 = Nt_0t_1$ implies that $(1\ 2\ 3) \in N^{(01)}$. Therefore, $N^{(01)} \geq \langle (1\ 2\ 3) \rangle \cong A_3$, and so $|N^{(01)}| \geq |A_3| = 3$. Now, by Lemma 1.4, $|Nt_0t_1N| = \frac{|N|}{|N^{(01)}|} \leq \frac{12}{3} = 4$.

   Therefore, as we concluded earlier, the double coset [01] has at most four distinct single cosets.

   Now, $N^{(01)}$ must have two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$ and $\{1, 2, 3\}$.

   Therefore, there are at most two double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1t_i$, $i \in \{0, 1\}$: $Nt_0t_1t_0N$ and $Nt_0t_1t_1N$.

   But, since $Nt_0t_1t_1N = Nt_0t_1^2N = Nt_0eN = Nt_0N$, we need only consider one additional the double coset of the form $Nt_0t_1t_iN$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_0N$.

4. We next consider the double coset $Nt_0t_1t_0N$.

Let [010] denote the double coset $Nt_0t_1t_0N$.

Now, by relations (3.2), $t_1t_0t_1 = t_0t_1t_0$, and, by conjugation with elements of A_4, $(t_1t_0t_1)^{(0\ 1)(2\ 3))} = (t_0t_1t_0)^{(0\ 1)(2\ 3)} \Rightarrow t_0t_1t_0 = t_1t_0t_1$, and $(t_1t_0t_1)^{(1\ 3)(0\ 2)} = (t_0t_1t_0)^{(1\ 3)(0\ 2)} \Rightarrow t_3t_2t_3 = t_2t_3t_2$, and $(t_1t_0t_1)^{(1\ 2)(0\ 3)} = (t_0t_1t_0)^{(1\ 2)(0\ 3)} \Rightarrow t_2t_3t_2 = t_3t_2t_3$, and $(t_1t_0t_1)^{(0\ 1\ 2)} = (t_0t_1t_0)^{(0\ 1\ 2)} \Rightarrow t_2t_1t_2 = t_1t_2t_1$, and $(t_1t_0t_1)^{(0\ 2\ 1)} = (t_0t_1t_0)^{(0\ 2\ 1)} \Rightarrow t_0t_2t_0 = t_2t_0t_2$, and $(t_1t_0t_1)^{(0\ 1\ 3} = (t_0t_1t_0)^{(0\ 1\ 3)} \Rightarrow t_3t_1t_3 = t_1t_3t_1$, and $(t_1t_0t_1)^{(0\ 3\ 1)} = (t_0t_1t_0)^{(0\ 3\ 1)} \Rightarrow t_0t_3t_0 = t_3t_0t_3$, and $(t_1t_0t_1)^{(0\ 2\ 3)} = (t_0t_1t_0)^{(0\ 2\ 3)} \Rightarrow t_1t_2t_1 = t_2t_1t_2$, and $(t_1t_0t_1)^{(0\ 3\ 2)} = (t_0t_1t_0)^{(0\ 3\ 2)} \Rightarrow t_1t_3t_1 = t_3t_1t_3$, and $(t_1t_0t_1)^{(1\ 2\ 3)} = (t_0t_1t_0)^{(1\ 2\ 3)} \Rightarrow t_2t_0t_2 = t_0t_2t_0$, and $(t_1t_0t_1)^{(1\ 3\ 2)} = (t_0t_1t_0)^{(1\ 3\ 2)} \Rightarrow t_3t_0t_3 = t_0t_3t_0$. Furthermore, by relation (3.1),
$(0\ 1\ 2)t_0t_2 = t_0t_1 = (0\ 1\ 3)t_0t_3 \Rightarrow (0\ 1\ 2)t_0t_2t_0 = t_0t_1t_0 = (0\ 1\ 3)t_0t_3t_0$. Therefore, $(0\ 1\ 2)t_0t_2t_0 = t_0t_1t_0 = (0\ 1\ 3)t_0t_3t_0$ and, the above relations, imply that:

$$010 \sim 020 \sim 030 \sim 101 \sim 121 \sim 131 \sim 202 \sim 212 \sim 232 \sim 303 \sim 313 \sim 323$$

Since each of the twelve single cosets has twelve names, the double coset [010] has one distinct single coset.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(010)} \geq N^{010} = \langle e \rangle$. Now, by relations (3.1) and (3.2), $N(t_0t_1t_0)^{(0\ 1)(2\ 3)} = Nt_1t_0t_1 = Nt_0t_1t_0$ implies that $(0\ 1)(2\ 3) \in N^{(010)}$, and $N(t_0t_1t_0)^{(0\ 1\ 2)} = Nt_1t_2t_1 = Nt_0t_1t_0$ implies that $(0\ 1\ 2) \in N^{(010)}$. Therefore, $N^{(010)} \geq \langle (0\ 1)(2\ 3), (0\ 1\ 2) \rangle \cong A_4$. Therefore, $|N^{(010)}| \geq |A_4| = 12$. Now, by Lemma 1.4, $|Nt_0t_1t_0N| = \frac{|N|}{|N^{(010)}|} \leq \frac{12}{12} = 1$.

Therefore, as we concluded earlier, the double coset [010] has one distinct single coset.

Now, $N^{(010)}$ must have one orbit on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2, 3\}$.

Therefore, there is at most one double coset of the form NwN, where w is a word of length four given by $t_0t_1t_0t_i$, $i = 0$: $Nt_0t_1t_0t_0N$.

But, since $Nt_0t_1t_0t_0N = Nt_0t_1t_0^2N = Nt_0t_1eN = Nt_0t_1N$, we need not consider additional double cosets of the form $Nt_0t_1t_0t_iN$, where $i \in \{0, 1, 2, 3\}$.

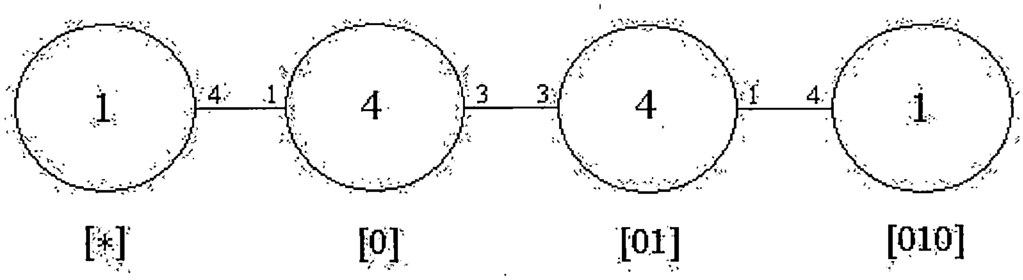

Figure 3.1: Cayley Diagram of G Over A_4

In fact, since $N^{(010)}$ is transitive on the symmetric generators and since $Nt_0t_1t_0t_0 = Nt_0t_1t_0^2 = Nt_0t_1e = Nt_0t_1$ implies that the double coset [0100] = [01], we must have completed the double coset enumeration of G over A_4.

In total, therefore, there are at most 4 distinct double cosets of N in G and at most 10 distinct right (single) cosets of N in G. The double cosets of N in G are as follows: [*], [0], [01], and [010].

3.3 Cayley Diagram of G Over A_4

The Cayley diagram of G over A_4 is illustrated in Figure 3.1. For a detailed explanation of the meaning of the component parts of a Cayley diagram, see Section 2.3.

3.4 Action of the Symmetric Generators and the Generators of A_4 on the Right Cosets of G Over A_4

Let X denote the set of all (10) distinct right cosets of N in G, that is, let $X = \{N, Nt_0, Nt_1, Nt_2, Nt_3, Nt_0t_1, Nt_1t_0, Nt_2t_0, Nt_3t_0, Nt_0t_1t_0\}$. We define a mapping $\phi\colon G \longrightarrow S_X$ so that ϕ maps a generator $g \in G$ to its action (by right multiplication) on X. That is, we define ϕ so that $\phi(g) = \widehat{\phi}(g) : X \to X$. Then the action $\phi(t) \sim \phi(t_0)$ of the symmetric generator $t \sim t_0$ on the right cosets of N in G may be expressed as

$$\phi(t) \sim \phi(t_0) = (*\ 0)(1\ 10)(2\ 20)(3\ 30)(01\ 010),$$

and the action $\phi(x) \sim \phi((1\ 2\ 3))$ of the generator $x \sim (1\ 2\ 3)$ of A_4 on the right cosets of N in G may be expressed as

$$\phi(x) \sim \phi((1\ 2\ 3)) = (1\ 2\ 3)(10\ 20\ 30),$$

and the action $\phi(y) \sim \phi((1\ 2))$ of the generator $y \sim (1\ 2)$ of S_3 on the right cosets of N in G may be expressed as

$$\phi(y) \sim \phi((1\ 2)) = (0\ 1\ 2)(01\ 10\ 20).$$

Since there are 10 distinct right cosets of N in G, these actions may be written as permutations on 10 letters. In fact, the actions of the generators on the set of right cosets of N in G are equivalent to the permutation representations of the generators in their action on the right cosets of N in G. To better manipulate the permutation representations of the symmetric generators t_i and the generators x and y, it is helpful to label the distinct single cosets of N in G as follows:

(10)	*	(5)	01
(1)	0	(6)	10
(2)	1	(7)	20
(3)	2	(8)	30
(4)	3	(9)	010

Having labeled each of the 10 distinct right cosets of N in G, we express the permutation representation of the symmetric generators $t \sim t_0$, $t^y \sim t_1$, $t^{y^2} \sim t_2$, and $t^{yx^2} \sim t_3$, and the generators $x \sim (1\ 2\ 3)$ and $y \sim (0\ 1\ 2)$ in their action on the right cosets of N in G as

$$\phi(t) \sim \phi(t_0) : (10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9),$$

$$\phi(t^y) \sim \phi(t_1) : (10\ 2)(1\ 5)(3\ 7)(4\ 8)(6\ 9),$$

$$\phi(t^{y^2}) \sim \phi(t_2) : (10\ 3)(1\ 5)(2\ 6)(4\ 8)(7\ 9),$$

$$\phi(t^{yx^2}) \sim \phi(t_3) : (10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9),$$

$$\phi(x) \sim \phi((1\ 2\ 3)) : (2\ 3\ 4)(6\ 7\ 8),$$

$$\phi(y) \sim \phi((0\ 1\ 2)) : (1\ 2\ 3)(5\ 6\ 7)$$

3.5 Proof of Isomorphism between G and S_5

We now demonstrate that $G \cong S_5$.

Proof. To prove that $G \cong S_5$, we must first show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and that $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 120$ (from which we can conclude $G \cong \langle \phi(x), \phi(y), \phi(t) \rangle$), and we must next show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong S_5$ (from which we can conclude S_5 is a homomorphic image of G and $G \cong S_5$).

We first show $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 120$. From our construction of G using manual double coset enumeration of G over A_4, illustrated by the Cayley Diagram in Figure 3.1, we concluded that group G defined by the symmetric presentation must contain a homomorphic image of $N \cong A_4$ whose index $[G : N]$ is at most 10:

$$[G:N] = \frac{|N|}{|N^{(*)}|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(010)}|} \leq \frac{12}{12} + \frac{12}{3} + \frac{12}{3} + \frac{12}{12} =$$

$$1 + 4 + 4 + 1 = 10$$

Since the index of N in G is at most 10, and since $|G| = \frac{|G|}{|N|} \cdot |N|$, the order of the homomorphic image group G is at most 120:

$$|G| = \frac{|G|}{|N|} \cdot |N| \leq 10 \cdot |N| = 10 \cdot 12 = 120 \Rightarrow |G| \leq 120$$

We now consider $\langle \phi(x), \phi(y), \phi(t) \rangle$. Note that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a group generated by the permutation representations of the generators x, y, and t and, as such, it is a subgroup of the symmetric group S_{10} acting on the ten right cosets of N in G. We now show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and, therefore, that $|G| \geq |\langle \phi(x), \phi(y), \phi(t) \rangle| = 120$. To show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we first demonstrate that $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$ is a homomorphic image of $\bar{G}$. Now, recall that $\bar{G} = \langle x, y, t \rangle$ is a homomorphic image of the progenitor $2^{*4} : A_4$, and its presentation is given by

$$\bar{G} = \langle x, y, t \mid x^3 = y^3 = (xy)^2 = e = t^2 = [t, x] \rangle,$$

where $x \sim (1\ 2\ 3)$, $y \sim (0\ 1\ 2)$, and $t \sim t_0$, and $N = \langle x, y \rangle \cong A_4$. Let $\alpha : \bar{G} \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ be a mapping from $\bar{G}$ to $\langle \phi(x), \phi(y), \phi(t) \rangle$. We note first that the mapping $\alpha : \bar{G} \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ is well defined. The generators $\phi(x), \phi(y)$, and $\phi(t)$ are the permutation representations of $x \sim (1\ 2\ 3)$, $y \sim (0\ 1\ 2)$, and $t \sim t_0$ on 10 letters. Since the order of $\phi(x)$ is 3, the order of $\phi(y)$ is 3, and the order of $\phi(x)\phi(y)$ is 2, we conclude $\langle \phi(x), \phi(y) \rangle \cong N \cong A_4$. Moreover, we can demonstrate that $\phi(t)$ has exactly four conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N \cong A_4$. Now, since $t \sim t_0$, we have that

$$\phi(t)^{\phi(y)} \sim \phi(t_0)^{\phi((0\ 1\ 2))} = [(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} =$$

$$[(1\ 2\ 3)(5\ 6\ 7)][(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)][(1\ 3\ 2)(5\ 7\ 6)] =$$

$$(10\ 2)(1\ 5)(3\ 7)(4\ 8)(6\ 9) = \phi(t_1) \sim \phi(t^y)$$

and further that

$$\phi(t)^{\phi(y^2)} \sim \phi(t_0)^{\phi((0\ 1\ 2)^2)} = [(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]^{(1\ 2\ 3)(5\ 6\ 7)} =$$

$$[(1\ 3\ 2)(5\ 7\ 6)][(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)][(1\ 2\ 3)(5\ 6\ 7)] =$$

$$(10\ 3)(1\ 5)(2\ 6)(4\ 8)(7\ 9) = \phi(t_2) \sim \phi(t^{y^2})$$

and further that

$$\phi(t)^{\phi(yx^2)} \sim \phi(t_0)^{\phi((0\ 1\ 2)(1\ 2\ 3)^2)} = [(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]^{(1\ 2\ 4)(5\ 6\ 8)} =$$

$$[(1\ 4\ 2)(5\ 8\ 6)][(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)][(1\ 2\ 4)(5\ 6\ 8)] =$$

$$(10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9) = \phi(t_3) \sim \phi(t^{yx^2})$$

Therefore, $\phi(t)$ has exactly four conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N \cong A_4$; these conjugates are, namely, $\phi(t) \sim \phi(t_0)$, $\phi(t^y) \sim \phi(t_1)$, $\phi(t^{y^2}) \sim \phi(t_2)$, and $\phi(t^{yx^2}) \sim \phi(t_3)$. Since $\langle \phi(x), \phi(y) \rangle \cong N \cong A_4$ and since $\phi(t)$ has exactly four conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N$, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $\bar{G} = \langle x, y, t \rangle$. That is, $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*4} : A_4$.

Next, to show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we must show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $\bar{G}$ factored by the relations $(yt)^4 = e$ and $(xyt)^6 = e$; that is, we must show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*4} : A_4$ factored by the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$.

Let $\tilde{\alpha} : G \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ be a mapping from G to $\langle \phi(x), \phi(y), \phi(t) \rangle$. We note first that the mapping $\tilde{\alpha} : G \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ is well-defined, and we know already that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*4} : A_4$. Now, to show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we need only demonstrate that the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$, which hold true in G, also hold true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$.

To demonstrate that the relation $[(0\ 1\ 2)t_0]^4 = e$, or, equivalently, the relation $t_0 t_2 t_1 t_0 = (0\ 2\ 1)$, holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$, we show that $\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t)\phi(t^{y^2})\phi(t^y)\phi(t) \in S_{10}$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation in the same way that $\phi((0\ 2\ 1)) \sim \phi(y^2)$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by $\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_2),$$

$$\phi(t_1)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(10\ 2)(1\ 5)(3\ 7)(4\ 8)(6\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_0),$$

$$\phi(t_2)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(10\ 3)(1\ 5)(2\ 6)(4\ 8)(7\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_1),$$

$$\phi(t_3)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_3)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by $\phi((0\ 2\ 1))$. This gives us

$$\phi(t_0)^{\phi((0\ 2\ 1))} = [(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_2),$$

$$\phi(t_1)^{\phi((0\ 2\ 1))} = [(10\ 2)(1\ 5)(3\ 7)(4\ 8)(6\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_0),$$

$$\phi(t_2)^{\phi((0\ 2\ 1))} = [(10\ 3)(1\ 5)(2\ 6)(4\ 8)(7\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_1),$$

$$\phi(t_3)^{\phi((0\ 2\ 1))} = [(10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9)]^{(1\ 3\ 2)(5\ 7\ 6)} = \phi(t_3)$$

Since $\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t)\phi(t^{y^2})\phi(t^y)\phi(t) \in S_{10}$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation in the same way that $\phi((0\ 2\ 1)) \sim \phi(y^2)$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation, we conclude that the relation $[(0\ 1\ 2)t_0]^4 = e$, which holds true in G, also holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$.

To demonstrate that the relation $[(0\ 1)(2\ 3)t_0]^6 = e$, or, equivalently, the relation $t_1 t_0 t_1 t_0 t_1 t_0 = e$, holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{10}$, we show that

$\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0) \sim \phi(t^y)\phi(t)\phi(t^y)\phi(t)\phi(t^y)\phi(t) \in S_{10}$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by $\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]^{\phi(e)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(10\ 2)(1\ 5)(3\ 7)(4\ 8)(6\ 9)]^{\phi(e)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(10\ 3)(1\ 5)(2\ 6)(4\ 8)(7\ 9)]^{\phi(e)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9)]^{\phi(e)} = \phi(t_3)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by the identity element $\phi(e)$. This gives us

$$\phi(t_0)^{\phi(e)} = [(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]^{\phi(e)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(e)} = [(10\ 2)(1\ 5)(3\ 7)(4\ 8)(6\ 9)]^{\phi(e)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(e)} = [(10\ 3)(1\ 5)(2\ 6)(4\ 8)(7\ 9)]^{\phi(e)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(e)} = [(10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9)]^{\phi(e)} = \phi(t_3)$$

Since $\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0) \sim \phi(t^y)\phi(t)\phi(t^y)\phi(t)\phi(t^y)\phi(t) \in S_{10}$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the four symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, and $\phi(t_3)$ by conjugation, we conclude that the relation $[(0\ 1)(2\ 3)t_0]^6 = e$, which holds true in G, also holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{10}$.

Since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $2^{*4} : A_4$, and since the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$ hold true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{10}$, we conclude that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $2^{*4} : A_4$ factored by the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$; that is, we conclude that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G.

More importantly, since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we have that $\langle\phi(x), \phi(y), \phi(t)\rangle \leq G$. In fact, since $\langle\phi(x), \phi(y), \phi(t)\rangle \leq G$, we have that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle|$. Since it is easily demonstrated, with MAGMA or by hand, that

$|\langle\phi(x), \phi(y), \phi(t)\rangle| = 120$, we conclude finally that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle| = 120$, that is, $|G| \geq 120$. Given $|G| \leq 120$ and $|G| \geq 120$, we conclude $|G| = 120$. Moreover, since $|\langle\phi(x), \phi(y), \phi(t)\rangle| = 120 = |G|$ and since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we conclude

$$\langle\phi(x), \phi(y), \phi(t)\rangle \cong G.$$

We finally show that $\langle\phi(x), \phi(y), \phi(t)\rangle \cong S_5$. Let $G_1 = \langle\phi(x), \phi(y), \phi(t)\rangle$. Now, with the help of MAGMA (see [BCP97]), we know that the elements $a = (1\ 9\ 5\ 8\ 7)$ $(2\ 3\ 6\ 4\ 10)$, $b = (1\ 2)(3\ 5)(4\ 7)(6\ 8)(9\ 10)$, and $c = (1\ 7)(3\ 6)(4\ 10)(5\ 8)$ belong to G_1. (Note: The labels for the right cosets in this case were assigned by MAGMA and are different from the labels that we assigned earlier.) Therefore, $\langle a, b, c\rangle \leq G_1$, where $G_1 = \langle\phi(x), \phi(y), \phi(t)\rangle$, a permutation group on 10 letters, is a permutation representation of G and, further, $|G_1| = 120$. But $|\langle a, b, c\rangle| = |G_1| = 120$. Therefore, $G_1 = \langle a, b, c\rangle$. However, $\langle a, b, c\rangle \cong S_5 \cong \langle a, b, c | a^5 = b^2 = c^2 = (ab)^4 = (a^{-2}(ab)^2)^3 = (a^{-2}ba^2b)^2 = [c, b] = e\rangle$. Therefore, $G_1 \cong S_5$ and, since $G_1 = \langle\phi(x), \phi(y), \phi(t)\rangle \cong G$, we conclude that $G \cong S_5$.

□

3.6 Converting an Element of G from its Permutation Representation to its Symmetric Representation

To illustrate how a permutation representation of an element of S_5 on 10 letters may be converted to its symmetric representation form, we consider the following example:

Example 3.1. Let $g \in G \cong S_5$ and let $p = \phi(g) = (10\ 8)(1\ 3)(4\ 9)(5\ 7)$ be the permutation representation of g on 10 letters. Then $10^p = 8$ implies $N^p = Nt_3t_0$, since 10 and 8 are labels for the right cosets N and Nt_3t_0, respectively. Moreover, since $N^p = Np$ and $N^p = Nt_3t_0$, we have that $Np = Nt_3t_0$. Now, $Np = Nt_3t_0$ implies that $p \in Nt_3t_0$ which implies that $p \sim \pi t_3t_0$ for some $\pi \in N \cong A_4$ or, more precisely, $p = \phi(\pi t_3 t_0) = \phi(\pi)\phi(t_3)\phi(t_0)$ for some $\pi \in N \cong A_4$. To determine π, we note first that $p = \phi(\pi)\phi(t_3)\phi(t_0) \Rightarrow p(\phi(t_0))^{-1}(\phi(t_3))^{-1} = p\phi(t_0^{-1})\phi(t_3^{-1}) = p\phi(t_0)\phi(t_3) = \phi(\pi)$. We then calculate the action of $\pi \sim \phi(\pi) = p\phi(t_0)\phi(t_3)$ on the symmetric generators t_i, where $i \in \{0, 1, 2, 3\}$. Now, $\pi \sim \phi(\pi) = p\phi(t_0)\phi(t_3) = [(10\ 8)(1\ 3)(4\ 9)(5\ 7)][(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)]$ $[(10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9)] = (1\ 3\ 4)(5\ 7\ 8)$. The element $\pi \sim \phi(\pi) =$

$p\phi(t_0)\phi(t_3)(1\ 3\ 4)(5\ 7\ 8)$ acts on the right cosets Nt_0, Nt_1, Nt_2, and Nt_3 via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$Nt_0 = 1 \mapsto 1^p = 3 = Nt_2, \qquad Nt_2 = 3 \mapsto 3^p = 4 = Nt_3,$$

$$Nt_3 = 4 \mapsto 4^p = 1 = Nt_0, \qquad Nt_1 = 2 \mapsto 2^p = 2 = Nt_1$$

Therefore, the element $\phi(\pi)$ acts as $(0\ 2\ 3)$ on the right cosets Nt_0, Nt_1, Nt_2, and Nt_3, and so $\phi(\pi)$ is the permutation representation of $\pi = (0\ 2\ 3) \in A_4$ on 10 letters. Therefore, $\pi = (0\ 2\ 3) \in A_4$ and $w = t_3t_0$, and so the symmetric representation of g is $(0\ 2\ 3)t_3t_0$.

3.7 Converting an Element of G from its Symmetric Representation to its Permutation Representation

To illustrate how an element of S_5 in symmetric representation form may be converted to its permutation representation on 10 letters, we consider the following example:

Example 3.2. Let $g \in G \cong S_5$ have the symmetric representation $(0\ 2\ 3)t_3t_0$. To determine the permutation representation $p = \phi(g)$ of g, we first calculate the action of $\pi = (0\ 2\ 3)$ on the right cosets of N in G. Now, the element $\pi = (0\ 2\ 3)$ acts on the right cosets N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$10 = N \mapsto N^{(0\ 2\ 3)} = N = 10$$

$$1 = Nt_0 \mapsto Nt_0^{(0\ 2\ 3)} = Nt_2 = 3$$

$$3 = Nt_2 \mapsto Nt_2^{(0\ 2\ 3)} = Nt_3 = 4$$

$$4 = Nt_3 \mapsto Nt_3^{(0\ 2\ 3)} = Nt_0 = 1$$

$$2 = Nt_1 \mapsto Nt_1^{(0\ 2\ 3)} = Nt_1 = 2$$

$$5 = Nt_0t_1 \mapsto N(t_0t_1)^{(0\ 2\ 3)} = Nt_2t_0 = 7$$

$$7 = Nt_2t_0 \mapsto N(t_2t_0)^{(0\ 2\ 3)} = Nt_3t_2 = Nt_3t_0 = 8$$

$$8 = Nt_3t_0 \mapsto N(t_3t_0)^{(0\ 2\ 3)} = Nt_0t_2 = Nt_0t_1 = 5$$

$$9 = Nt_0t_1t_0 \mapsto N(t_0t_1t_0)^{(0\ 2\ 3)} = Nt_2t_1t_2 = Nt_0t_1t_0 = 9$$

Therefore, the permutation representation of $\pi = (0\ 2\ 3)$ is $\phi(\pi) = (1\ 3\ 4)(5\ 7\ 8)$. Similarly, we calculate the action of the symmetric generator t_3 on the right cosets of N in G. The symmetric generator t_3 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_3, Nw) = Nwt_3$. The mappings below illustrate this action:

$$10 = N \mapsto Nt_3 = 4$$

$$4 = Nt_3 \mapsto Nt_3t_3 = N = 10$$

$$1 = Nt_0 \mapsto Nt_0t_3 = Nt_0t_1 = 5$$

$$5 = Nt_0t_1 \mapsto Nt_0t_1t_3 = Nt_0t_3t_3 = Nt_0 = 1$$

$$2 = Nt_1 \mapsto Nt_1t_3 = Nt_1t_0 = 6$$

$$6 = Nt_1t_0 \mapsto Nt_1t_0t_3 = Nt_1t_3t_3 = Nt_1 = 2$$

$$3 = Nt_2 \mapsto Nt_2t_3 = Nt_2t_0 = 7$$

$$7 = Nt_2t_0 \mapsto Nt_2t_0t_3 = Nt_2t_3t_3 = Nt_2 = 3$$

$$8 = Nt_3t_0 \mapsto Nt_3t_0t_3 = Nt_0t_1t_0 = 9$$

$$9 = Nt_0t_1t_0 \mapsto Nt_0t_1t_0t_3 = Nt_3t_0t_3t_3 = Nt_3t_0 = 8$$

Therefore, the permutation representation of t_3 is $\phi(t_3) = (10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9)$. Finally, we calculate the action of the symmetric generator t_0 on the right cosets of N in G. The symmetric generator t_0 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_0, Nw) = Nwt_0$. The mappings below illustrate this action:

$$10 = N \mapsto Nt_0 = 1$$

$$1 = Nt_0 \mapsto Nt_0t_0 = N = 10$$

$$2 = Nt_1 \mapsto Nt_1t_0 = 6$$

$$6 = Nt_1t_0 \mapsto Nt_1t_0t_0 = Nt_1 = 2$$

$$3 = Nt_2 \mapsto Nt_2t_0 = 7$$

$$7 = Nt_2t_0 \mapsto Nt_2t_0t_0 = Nt_2 = 3$$

$$4 = Nt_3 \mapsto Nt_3t_0 = 8$$

$$8 = Nt_3t_0 \mapsto Nt_3t_0t_0 = Nt_3 = 4$$

$$5 = Nt_0t_1 \mapsto Nt_0t_1t_0 = 9$$

$$9 = Nt_0t_1t_0 \mapsto Nt_0t_1t_0t_0 = Nt_0t_1 = 5$$

Therefore, the permutation representation of t_0 is $\phi(t_0) = (10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)$. Now, $g = (0\ 2\ 3)t_3t_0 \sim \phi(g) = \phi((0\ 2\ 3))\phi(t_3)\phi(t_0) = [(1\ 3\ 4)(5\ 7\ 8)][(10\ 4)(1\ 5)(2\ 6)(3\ 7)(8\ 9)]$ $[(10\ 1)(2\ 6)(3\ 7)(4\ 8)(5\ 9)] = (10\ 8)(1\ 3)(4\ 9)(5\ 7)$. Therefore, the permutation representation of g is $p = \phi(g) = (10\ 8)(1\ 3)(4\ 9)(5\ 7)$.

Chapter 4

S_6 as a Homomorphic Image of the Progenitor $2^{*5}: A_5$

In this chapter, we investigate S_6 as a homomorphic image of the progenitor $2^{*5} : A_5$. The group S_6 is the symmetric group on six letters having order $6! = 720$. The progenitor $2^{*5} : A_5$ is a semi-direct product of 2^{*5}, a free product of five copies of the cyclic group of order 2, and A_5, the alternating group on five letters which permutes the five symmetric generators, t_0, t_1, t_2, t_3, and t_4, by conjugation.

4.1 Introduction

Let $\bar{G}$ be a homomorphic image of the infinite semi-direct product, the *progenitor*, $2^{*5}: A_5$. A symmetric presentation of $2^{*5}: A_5$ is given by

$$\bar{G} = \langle x, y, t \mid x^5 = y^3 = (xy)^2 = e = [t, y] = [t, y^{x^2}]\rangle,$$

where $[t, y] = tyty$, $[t, y^{x^2}] = ty^{x^2}ty^{x^2}$, and e is the identity. In this case, $N \cong A_5 \cong \langle x, y \mid x^5 = y^3 = (xy)^2 = e\rangle$, and the action of N on the five symmetric generators is given by $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (4\ 2\ 1)$, and $t \sim t_0$.

Let G denote the group $\bar{G}$ factored by the relations $(xy^{-1}x^2y^{-1}t)^4 = e$ and $(x^2y^{-1}x^2t)^6 = e$. That is, let

$$G = \frac{\bar{G}}{(xy^{-1}x^2y^{-1}t)^4, (x^2y^{-1}x^2t)^6}.$$

A symmetric presentation for G is given by

$$\langle x, y, t \mid x^5 = y^3 = (xy)^2 = e = [t, y] = [t, y^{x^2}] = (xy^{-1}x^2y^{-1}t)^4 = (x^2y^{-1}x^2t)^6\rangle.$$

Now, we consider the following relations:

$$[(0\ 1\ 2)t_0]^4 = e$$
and
$$[(0\ 1)(2\ 3)t_0]^6 = e.$$

According to a computer proof by [CHB96], the progenitor $2^{*5}: A_5$, factored by the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$, is isomorphic to S_6. In fact, factoring the progenitor $2^{*5}: A_5$ by the relation $[(0\ 1\ 2)t_0]^4 = e$ alone suffices. We will construct S_6 by hand by way of manual double coset enumeration of $G \cong \frac{2^{*5}: A_5}{[(0\ 1\ 2)t_0]^4, [(0\ 1)(2\ 3)t_0]^6}$ over S_3. In so doing, we will show that S_6 is isomorphic to the symmetric presentation

$$\langle x, y, t \mid x^5 = y^3 = (xy)^2 = e = [t, y] = [t, y^{x^2}] = (xy^{-1}x^2y^{-1}t)^4 = (x^2y^{-1}x^2t)^6\rangle.$$

4.2 Manual Double Coset Enumeration of G Over A_5

We first determine the order of the homomorphic image, G, of the progenitor. To determine the order of the homomorphic image G, we will determine the index of $N \cong A_5$ in G. We determine the index of $N \cong A_5$ in G first by expanding the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$, and next by performing manual double coset enumeration on G over $N \cong A_5$. To begin, we expand the relations that factor the progenitor $2^{*5} : A_5$:

$$[(0\ 1\ 2)t_0]^4 = e \tag{4.1}$$

$$[(0\ 1)(2\ 3)t_0]^6 = e \tag{4.2}$$

As mentioned above, relation (4.1), $[(0\ 1\ 2)t_0]^4 = e$, is required to determine the homomorphic image, G, of the progenitor, and the other relation, (4.2), can be derived from relation (4.1). We expand relations (4.1) and (4.2) in detail below:

1. Let $\pi = (0\ 1\ 2)$.

   Then $[(0\ 1\ 2)t_0]^4 = e \Rightarrow (\pi t_0)^4 = e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e \Rightarrow$
   $\pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e \Rightarrow \pi t_0 \pi^3 \pi^{-1} \pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e \Rightarrow$

$\pi^4\pi^{-1}\pi^{-1}\pi^{-1}t_0\pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
$(0\ 1\ 2)^4 t_0^{(0\ 1\ 2)^3} t_0^{(0\ 1\ 2)^2} t_0^{(0\ 1\ 2)} t_0 = e \Rightarrow (0\ 1\ 2) t_0^e t_0^{(0\ 2\ 1)} t_0^{(0\ 1\ 2)} t_0 = e \Rightarrow$
$(0\ 1\ 2) t_0 t_2 t_1 t_0 = e \Rightarrow (0\ 1\ 2) t_0 t_2 = t_0 t_1.$

Thus relation (4.1) implies that $(0\ 1\ 2)t_0t_2 = t_0t_1$ or, equivalently, $Nt_0t_2 = Nt_0t_1$. That is, using our short-hand notation, $02 \sim 01$.

2. Let $\pi = (0\ 1)(2\ 3)$.

   Then $[(0\ 1)(2\ 3)t_0]^6 = e \Rightarrow (\pi t_0)^6 = e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e \Rightarrow$
   $\pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e \Rightarrow$
   $\pi t_0 \pi t_0 \pi t_0 \pi^3 \pi^{-1} \pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e \Rightarrow \pi t_0 \pi t_0 \pi^4 \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow \pi t_0 \pi^5 \pi^{-1} \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
   $\pi^6 \pi^{-1} \pi^{-1} \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow \pi^6 t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
   $[(0\ 1)(2\ 3)]^6 t_0^{[(0\ 1)(2\ 3)]^5} t_0^{[(0\ 1)(2\ 3)]^4} t_0^{[(0\ 1)(2\ 3)]^3} t_0^{[(0\ 1)(2\ 3)]^2} t_0^{[(0\ 1)(2\ 3)]} t_0 = e$
   $\Rightarrow e t_0^{[(0\ 1)(2\ 3)]} t_0^e t_0^{[(0\ 1)(2\ 3)]} t_0^e t_0^{[(0\ 1)(2\ 3)]} t_0 = e \Rightarrow e t_1 t_0 t_1 t_0 t_1 t_0 = e \Rightarrow$
   $t_1 t_0 t_1 = t_0 t_1 t_0.$

   Thus relation (4.2) implies that $t_1t_0t_1 = t_0t_1t_0$ or, equivalently, $Nt_1t_0t_1 = Nt_0t_1t_0$. That is, using our short-hand notation, $101 \sim 010$.

We now perform manual double coset enumeration of G over A_5.

1. We first note that the double coset $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$.

   Let $[*]$ denote the double coset NeN.

   The double coset $[*]$ has one distinct right coset: the identity right coset, $Ne = \{ne \mid n \in N\} = N$. Moreover, since $N \cong A_5$ is transitive, and since $O(0) = \{0^g \mid g \in N^{(*)}\} = \{0, 1, 2, 3, 4\} = O(1) = O(2) = O(3) = O(4)$, N must have one orbit on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$.

   Therefore, there is one double coset of the form NwN, where w is a word of length one given by $w = t_i$, $i \in \{0, 1, 2, 3, 4\}$: Nt_0N.

2. We next consider the double coset Nt_0N.

   Let $[0]$ denote the double coset Nt_0N.

   Now, note that $N^{(0)} \geq N^0 = \langle (1\ 2)(3\ 4), (1\ 2\ 3) \rangle \cong A_4$. Thus $|N^{(0)}| \geq |A_4| = 12$ and, by Lemma 1.4, $|Nt_0N| = \frac{|N|}{|N^{(0)}|} = \frac{60}{12} = 5$.

Therefore, the double coset [*] has at most five distinct single cosets.

Moreover, $N^{(0)}$ must have two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$ and $\{1, 2, 3, 4\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length two given by $w = t_0 t_i$, $i \in \{0, 1\}$: Nt_0t_0N and Nt_0t_1N.

But, since $Nt_0t_0N = Nt_0^2N = NeN = N$, we need only consider one additional the double coset of the form Nt_0t_iN, where $i \in \{0, 1, 2, 3, 4\}$: Nt_0t_1N.

3. We next consider the double coset Nt_0t_1N.

   Let [01] denote the double coset Nt_0t_1N.

   Now, by relation (4.1), $(0\ 1\ 2)t_0t_2 = t_0t_1$, and $[(0\ 1\ 2)t_0t_2]^{(2\ 4)(1\ 3)}$ $= (t_0t_1)^{(2\ 4)(1\ 3)} \Rightarrow (0\ 3\ 4)t_0t_4 = t_0t_3$, and $[(0\ 1\ 2)t_0t_2]^{(1\ 3\ 2)} = (t_0t_1)^{(1\ 3\ 2)}$ $\Rightarrow (0\ 3\ 1)t_0t_1 = t_0t_3$. Therefore, $(0\ 3\ 4)t_0t_4 = t_0t_3 = (0\ 3\ 1)t_0t_1 = (1\ 2\ 3)t_0t_2$ implies that

   $$01 \sim 02 \sim 03 \sim 04$$

   Similarly, by conjugation, we find that

   $$10 \sim 12 \sim 13 \sim 14, \qquad 20 \sim 21 \sim 23 \sim 24$$

   $$30 \sim 31 \sim 32 \sim 34, \qquad 40 \sim 41 \sim 42 \sim 43$$

   Since each of the twelve single cosets has three names, the double coset [01] must have at most four distinct single cosets.

   An alternative approach for determining the order of the double coset is as follows: We note that $N^{(01)} \geq N^{01} = \langle (2\ 3\ 4) \rangle \cong A_3$. This means that $(2\ 3\ 4) \in N^{(01)}$. Now, by relation (4.1), $N(t_0t_1)^{(1\ 2)(3\ 4)} = Nt_0t_2 = Nt_0t_1$ implies that $(1\ 2)(3\ 4) \in N^{(01)}$. Therefore, $(2\ 3\ 4), (1\ 2)(3\ 4) \in N^{(01)}$, and so $N^{(01)} \geq \langle (1\ 2)(3\ 4), (2\ 3\ 4) \rangle \cong A_4$. That is, $\left|N^{(01)}\right| \geq |A_4| = 12$. Now, by Lemma 1.4, $|Nt_0t_1N| = \frac{|N|}{\left|N^{(01)}\right|} \leq \frac{60}{12} = 5$. Therefore, as we concluded earlier, the double coset [01] has at most five distinct single cosets.

   Now, $N^{(01)}$ must have two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$ and $\{1, 2, 3, 4\}$.

   Therefore, there are at most two double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1t_i$, $i \in \{0, 1\}$: $Nt_0t_1t_0N$ and $Nt_0t_1t_1N$.

But, since $Nt_0t_1t_1N = Nt_0t_1^2N = Nt_0eN = Nt_0N$, we need only consider one additional the double coset of the form $Nt_0t_1t_iN$, where $i \in \{0,1,2,3,4\}$: $Nt_0t_1t_0N$.

4. We next consider the double coset $Nt_0t_1t_0N$.

   Let [010] denote the double coset $Nt_0t_1t_0N$.

   Now, by relation (4.2), $t_1t_0t_1 = t_0t_1t_0$, and, by conjugation with elements of A_5, $(t_1t_0t_1)^{(1\ 2)(3\ 4)} = (t_0t_1t_0)^{(1\ 2)(3\ 4)} \Rightarrow t_2t_0t_2 = t_0t_2t_0$,
   and $(t_1t_0t_1)^{(1\ 3)(2\ 4)} = (t_0t_1t_0)^{(1\ 3)(2\ 4)} \Rightarrow t_3t_0t_3 = t_0t_3t_0$,
   and $(t_1t_0t_1)^{(1\ 4)(2\ 3)} = (t_0t_1t_0)^{(1\ 4)(2\ 3)} \Rightarrow t_4t_0t_4 = t_0t_4t_0$,
   and $(t_1t_0t_1)^{(0\ 2)(3\ 4)} = (t_0t_1t_0)^{(0\ 2)(3\ 4)} \Rightarrow t_1t_2t_1 = t_2t_1t_2$,
   and $(t_1t_0t_1)^{(0\ 3)(2\ 4)} = (t_0t_1t_0)^{(0\ 3)(2\ 4)} \Rightarrow t_1t_3t_1 = t_3t_1t_3$,
   and $(t_1t_0t_1)^{(0\ 4)(2\ 3)} = (t_0t_1t_0)^{(0\ 4)(2\ 3)} \Rightarrow t_1t_4t_1 = t_4t_1t_4$,
   and $(t_1t_0t_1)^{(0\ 3)(1\ 2)} = (t_0t_1t_0)^{(0\ 3)(1\ 2)} \Rightarrow t_2t_3t_2 = t_3t_2t_3$,
   and $(t_1t_0t_1)^{(0\ 4)(1\ 2)} = (t_0t_1t_0)^{(0\ 4)(1\ 2)} \Rightarrow t_2t_4t_2 = t_4t_2t_4$,
   and $(t_1t_0t_1)^{(0\ 4)(1\ 3)} = (t_0t_1t_0)^{(0\ 4)(1\ 3)} \Rightarrow t_3t_4t_3 = t_4t_3t_4$.
   Furthermore, by relation (4.1), $(0\ 3\ 4)t_0t_4 = t_0t_3 = (0\ 3\ 1)t_0t_1 = (1\ 2\ 3)t_0t_2 \Rightarrow (0\ 3\ 4)t_0t_4t_0 = t_0t_3t_0 = (0\ 3\ 1)t_0t_1t_0 = (1\ 2\ 3)t_0t_2t_0$ Therefore, $(0\ 3\ 4)t_0t_4t_0 = t_0t_3t_0 = (0\ 3\ 1)t_0t_1t_0 = (1\ 2\ 3)t_0t_2t_0$ implies that

   $$010 \sim 020 \sim 030 \sim 040 \sim 101 \sim 121 \sim 131 \sim 141 \sim 202 \sim 212 \sim$$

   $$232 \sim 242 \sim 303 \sim 313 \sim 323 \sim 343 \sim 404 \sim 414 \sim 424 \sim 434$$

   Since each of the twenty single cosets has twenty names, the double coset [010] must have one distinct single coset.

   An alternative approach for determining the order of the double coset is as follows: We note that $N^{(010)} \geq N^{010} = \langle(2\ 3\ 4)\rangle$. This means that $(2\ 3\ 4) \in N^{(010)}$. Now, by relations (4.1) and (4.2), $N(t_0t_1t_0)^{(0\ 1)(2\ 3)} = Nt_1t_0t_1 = Nt_0t_1t_0$ implies that $(0\ 1)(2\ 3) \in N^{(010)}$, and $N(t_0t_1t_0)^{(0\ 1\ 2\ 3\ 4)} = Nt_1t_2t_1 = Nt_0t_1t_0$ implies that $(0\ 1\ 2\ 3\ 4) \in N^{(010)}$. Therefore, $(2\ 3\ 4), (1\ 2)(3\ 4), (0\ 1\ 2\ 3\ 4) \in N^{(010)}$, and so $N^{(010)} \geq \langle(1\ 2)(3\ 4), (2\ 3\ 4), (0\ 1\ 2\ 3\ 4)\rangle \cong A_5$. That is, $|N^{(010)}| \geq |A_5| = 60$. Now, by Lemma 1.4, $|Nt_0t_1t_0N| = \frac{|N|}{|N^{(010)}|} \leq \frac{60}{60} = 1$.

   Therefore, as we concluded earlier, the double coset [010] has one distinct single coset.

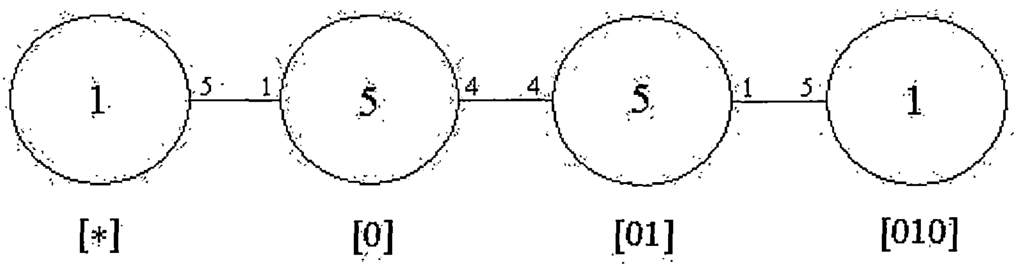

Figure 4.1: Cayley Diagram of G Over A_5

Now, $N^{(010)}$ must have one orbit on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$.

Therefore, there is at most one double coset of the form NwN, where w is a word of length four given by $w = t_0t_1t_0t_i$, $i = 0$: $Nt_0t_1t_0t_0N$.

But, since $Nt_0t_1t_0t_0N = Nt_0t_1t_0^2N = Nt_0t_1eN = Nt_0t_1N$, we need not consider additional double cosets of the form $Nt_0t_1t_0t_iN$, where $i \in \{0, 1, 2, 3\}$.

In fact, since $N^{(010)}$ is transitive on the symmetric generators and since $Nt_0t_1t_0t_0 = Nt_0t_1t_0^2 = Nt_0t_1e = Nt_0t_1$ implies that the double coset [0100] = [01], we have completed the double coset enumeration of G over A_5.

In total, therefore, there are at most 4 distinct double cosets of N in G and at most 12 distinct right (single) cosets of N in G. The double cosets of N in G are as follows: [*], [0], [01], and [010].

4.3 Cayley Diagram of G Over A_5

The Cayley diagram of G over A_5 is illustrated in Figure 4.1. For a detailed explanation of the meaning of the component parts of a Cayley diagram, see Section 2.3.

4.4 Action of the Symmetric Generators and the Generators of A_5 on the Right Cosets of G Over A_5

Let X denote the set of all (12) distinct right cosets of N in G, that is, let $X = \{N, Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, Nt_0t_1, Nt_1t_0, Nt_2t_0, Nt_3t_0, Nt_4t_0, Nt_0t_1t_0\}$. We define a mapping $\phi\colon G \longrightarrow S_X$ so that ϕ maps a generator $g \in G$ to its action (by right multiplication) on X. That is, we define ϕ so that $\phi(g) = \widehat{\phi}(g) : X \to X$. Then the action $\phi(t) \sim \phi(t_0)$ of the symmetric generator $t \sim t_0$ on the right cosets of N in G may be expressed as

$$\phi(t) \sim \phi(t_0) = (*\ 0)(1\ 10)(2\ 20)(3\ 30)(4\ 40)(01\ 010),$$

and the action $\phi(x) \sim \phi((0\ 1\ 2\ 3\ 4))$ of the generator $x \sim (0\ 1\ 2\ 3\ 4)$ of A_5 on the right cosets of N in G may be expressed as

$$\phi(x) \sim \phi((0\ 1\ 2\ 3\ 4)) = (0\ 1\ 2\ 3\ 4)(01\ 12\ 23\ 34\ 40),$$

and the action $\phi(y) \sim \phi((4\ 2\ 1))$ of the generator $y \sim (4\ 2\ 1)$ of A_5 on the right cosets of N in G may be expressed as

$$\phi(y) \sim \phi((4\ 2\ 1)) = (1\ 4\ 2)(10\ 40\ 20).$$

Since there are 12 distinct right cosets of N in G, these actions may be written as permutations on 12 letters. In fact, the actions of the generators on the set of right cosets of N in G are equivalent to the permutation representations of the generators in their action on the right cosets of N in G. To better manipulate the permutation representations of the symmetric generators t_i and the generators x and y, it is helpful to label the distinct single cosets of N in G as follows:

(12)	*	(3)	2	(6)	01	(9)	30
(1)	0	(4)	3	(7)	10	(10)	40
(2)	1	(5)	4	(8)	20	(11)	010

Having labeled each of the 12 distinct right cosets of N in G, we express the permutation representation of the symmetric generator $t \sim t_0$, $t^x \sim t_1$, $t^{x^2} \sim t_2$, $t^{x^3} \sim t_3$, and $t^{x^4} \sim t_4$, and the generators $x \sim (0\ 1\ 2\ 3\ 4)$ and $y \sim (4\ 2\ 1)$, in their action on the right cosets of N in G as, resprectively,

$$\phi(t) \sim \phi(t_0) : (12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11),$$

$$\phi(t^x) \sim \phi(t_1) : (12\ 2)(1\ 6)(3\ 8)(4\ 9)(5\ 10)(7\ 11),$$

$$\phi(t^{x^2}) \sim \phi(t_2) : (12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11),$$

$$\phi(t^{x^3}) \sim \phi(t_3) : (12\ 4)(1\ 6)(2\ 7)(3\ 8)(5\ 10)(9\ 11),$$

$$\phi(t^{x^4}) \sim \phi(t_4) : (12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11),$$

$$\phi(x) \sim \phi((0\ 1\ 2\ 3\ 4)) : (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10),$$

$$\phi(y) \sim \phi((4\ 2\ 1)) : (2\ 5\ 3)(7\ 10\ 8)$$

4.5 Proof of Isomorphism between G and S_6

We now demonstrate that $G \cong S_6$.

Proof. To prove that $G \cong S_6$, we must first show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and that $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 720$ (from which we can conclude $G \cong \langle \phi(x), \phi(y), \phi(t) \rangle$), and we must next show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong S_6$ (from which we can conclude S_6 is a homomorphic image of G and $G \cong S_6$).

We first show $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 720$. From our construction of G using manual double coset enumeration of G over A_5, illustrated by the Cayley Diagram in Figure 4.1, we concluded that group G defined by the symmetric presentation must contain a homomorphic image of $N \cong A_5$ whose index $[G : N]$ is at most 12:

$$[G : N] = \frac{|N|}{|N^{(*)}|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(010)}|} \leq \frac{60}{60} + \frac{60}{12} + \frac{60}{12} + \frac{60}{60} =$$

$$1 + 5 + 5 + 1 = 12$$

Since the index of N in G is at most 12, and since $|G| = \frac{|G|}{|N|} \cdot |N|$, the order of the homomorphic image group G is at most 720:

$$|G| = \frac{|G|}{|N|} \cdot |N| \leq 12 \cdot |N| = 12 \cdot 60 = 720 \Rightarrow |G| \leq 720$$

We now consider $\langle \phi(x), \phi(y), \phi(t) \rangle$. Note that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a group generated by the permutation representations of the generators x, y, and t and, as such, it is a subgroup of the symmetric group S_{12} acting on the twelve right cosets of N in G. We now show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and, therefore, that $|G| \geq |\langle \phi(x), \phi(y), \phi(t) \rangle| = 720$. To show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we first demonstrate that $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{12}$ is a homomorphic image of $\bar{G}$.

Now, recall that $\bar{G} = \langle x, y, t\rangle$ is a homomorphic image of the progenitor $2^{*5} : A_5$, and its presentation is given by

$$\bar{G} = \langle x, y, t \mid x^5 = y^3 = (xy)^2 = e = [t, y] = [t, y^{x^2}]\rangle,$$

where $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (4\ 2\ 1)$, and $t \sim t_0$, and $N = \langle x, y\rangle \cong A_5$. Let $\alpha : \bar{G} \longrightarrow \langle \phi(x), \phi(y), \phi(t)\rangle$ be a mapping from $\bar{G}$ to $\langle \phi(x), \phi(y), \phi(t)\rangle$. We note first that the mapping $\alpha : \bar{G} \longrightarrow \langle \phi(x), \phi(y), \phi(t)\rangle$ is well defined. The generators $\phi(x), \phi(y)$, and $\phi(t)$ are the permutation representations of $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (4\ 2\ 1)$, and $t \sim t_0$ on 12 letters. Since the order of $\phi(x)$ is 5, the order of $\phi(y)$ is 3, and the order of $\phi(x)\phi(y)$ is 2, we conclude $\langle \phi(x), \phi(y)\rangle \cong N \cong A_5$. Moreover, we can demonstrate that $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle \phi(x), \phi(y)\rangle \cong N \cong A_5$. Now, since $t \sim t_0$, we have that

$$\phi(t)^{\phi(x)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4))} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{(1\ 5\ 4\ 3\ 2)(6\ 10\ 9\ 8\ 7)} =$$

$$[(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)][(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)][(1\ 5\ 4\ 3\ 2)(6\ 10\ 9\ 8\ 7)]$$

$$= (12\ 2)(1\ 6)(3\ 8)(4\ 9)(5\ 10)(7\ 11) = \phi(t_1) \sim \phi(t^x)$$

and further that

$$\phi(t)^{\phi(x^2)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^2)} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{(1\ 4\ 2\ 5\ 3)(6\ 9\ 7\ 10\ 8)} =$$

$$[(1\ 3\ 5\ 2\ 4)(6\ 8\ 10\ 7\ 9)][(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)][(1\ 4\ 2\ 5\ 3)(6\ 9\ 7\ 10\ 8)]$$

$$= (12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11) = \phi(t_2) \sim \phi(t^{x^2})$$

and further that

$$\phi(t)^{\phi(x^3)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^3)} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{(1\ 3\ 5\ 2\ 4)(6\ 8\ 10\ 7\ 9)} =$$

$$[(1\ 4\ 2\ 5\ 3)(6\ 9\ 7\ 10\ 8)][(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)][(1\ 3\ 5\ 2\ 4)(6\ 8\ 10\ 7\ 9)]$$

$$= (12\ 4)(1\ 6)(2\ 7)(3\ 8)(5\ 10)(9\ 11) = \phi(t_3) \sim \phi(t^{x^3})$$

and further that

$$\phi(t)^{\phi(x^4)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^4)} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)} =$$

$$[(1\ 5\ 4\ 3\ 2)(6\ 10\ 9\ 8\ 7)][(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)][(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)]$$

$$= (12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11) = \phi(t_4) \sim \phi(t^{x^4})$$

Therefore, $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N \cong A_5$; these conjugates are, namely, $\phi(t) \sim \phi(t_0)$, $\phi(t^x) \sim \phi(t_1)$, $\phi(t^{x^2}) \sim \phi(t_2)$, $\phi(t^{x^3}) \sim \phi(t_3)$, and $\phi(t^{x^4}) \sim \phi(t_4)$. Since $\langle \phi(x), \phi(y) \rangle \cong N \cong A_5$ and since $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N$, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $\bar{G} = \langle x, y, t \rangle$. That is, $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*5} : A_5$.

Next, to show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we must show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $\bar{G}$ factored by the relations $(xy^{-1}x^2y^{-1}t)^4 = e$ and $(x^2y^{-1}x^2t)^6 = e$; that is, we must show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*5} : A_5$ factored by the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$. Let $\tilde{\alpha} : G \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ be a mapping from G to $\langle \phi(x), \phi(y), \phi(t) \rangle$. We note first that the mapping $\tilde{\alpha} : G \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ is well-defined, and we know already that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*5} : A_5$. Now, to show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we need only demonstrate that the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$, which hold true in G, also hold true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{12}$.

To demonstrate that the relation $[(0\ 1\ 2)t_0]^4 = e$, or, equivalently, the relation $t_0t_2t_1t_0 = (0\ 2\ 1)$, holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{12}$, we show that $\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t)\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{12}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 2\ 1)) \sim \phi(xy^{-1}x^2y^{-1})$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_2),$$

$$\phi(t_1)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(12\ 2)(1\ 6)(3\ 8)(4\ 9)(5\ 10)(7\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_0),$$

$$\phi(t_2)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_1),$$

$$\phi(t_3)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(12\ 4)(1\ 6)(2\ 7)(3\ 8)(5\ 10)(9\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = [(12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_4)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi((0\ 2\ 1))$. This gives us

$$\phi(t_0)^{\phi((0\ 2\ 1))} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_2),$$

$$\phi(t_1)^{\phi((0\ 2\ 1))} = [(12\ 2)(1\ 6)(3\ 8)(4\ 9)(5\ 10)(7\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_0),$$

$$\phi(t_2)^{\phi((0\ 2\ 1))} = [(12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_1),$$

$$\phi(t_3)^{\phi((0\ 2\ 1))} = [(12\ 4)(1\ 6)(2\ 7)(3\ 8)(5\ 10)(9\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_3),$$

$$\phi(t_4)^{\phi((0\ 2\ 1))} = [(12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11)]^{(1\ 3\ 2)(6\ 8\ 7)} = \phi(t_4)$$

Since $\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t)\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{12}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 2\ 1)) \sim \phi(xy^{-1}x^2y^{-1})$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation, we conclude that the relation $[(0\ 1\ 2)t_0]^4 = e$, which holds true in G, also holds true in $\langle \phi(x), \phi(y), \phi(t)\rangle \leq S_{12}$.

To demonstrate that the relation $[(0\ 1)(2\ 3)t_0]^6 = e$, or, equivalently, the relation $t_1t_0t_1t_0t_1t_0 = e$, holds true in $\langle \phi(x), \phi(y), \phi(t)\rangle \leq S_{12}$, we show that $\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t) \in S_{12}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{\phi(e)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(12\ 2)(1\ 6)(3\ 8)(4\ 9)(5\ 10)(7\ 11)]^{\phi(e)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11)]^{\phi(e)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(12\ 4)(1\ 6)(2\ 7)(3\ 8)(5\ 10)(9\ 11)]^{\phi(e)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = [(12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11)]^{\phi(e)} = \phi(t_4)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by the identity element $\phi(e)$. This gives us

$$\phi(t_0)^{\phi(e)} = [(12\ 1)(2\ 7)(3\ 8)(4\ 9)(5\ 10)(6\ 11)]^{\phi(e)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(e)} = [(12\ 2)(1\ 6)(3\ 8)(4\ 9)(5\ 10)(7\ 11)]^{\phi(e)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(e)} = [(12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11)]^{\phi(e)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(e)} = [(12\ 4)(1\ 6)(2\ 7)(3\ 8)(5\ 10)(9\ 11)]^{\phi(e)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(e)} = [(12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11)]^{\phi(e)} = \phi(t_4)$$

Since $\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t) \in S_{12}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation, we conclude that the relation $[(0\ 1)(2\ 3)t_0]^6 = e$. which holds true in G, also holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{12}$.

Since $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*5} : A_5$, and since the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$ hold true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{12}$, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $2^{*5} : A_5$ factored by the relations $[(0\ 1\ 2)t_0]^4 = e$ and $[(0\ 1)(2\ 3)t_0]^6 = e$; that is, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G.

More importantly, since $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we have that $\langle \phi(x), \phi(y), \phi(t) \rangle \leq G$. In fact, since $\langle \phi(x), \phi(y), \phi(t) \rangle \leq G$, we have that $|G| \geq |\langle \phi(x), \phi(y), \phi(t) \rangle|$. Since it is easily demonstrated, with MAGMA or by hand, that $|\langle \phi(x), \phi(y), \phi(t) \rangle| = 720$, we conclude finally that $|G| \geq |\langle \phi(x), \phi(y), \phi(t) \rangle| = 720$, that is, $|G| \geq 720$. Given $|G| \leq 720$ and $|G| \geq 720$, we conclude $|G| = 720$. Moreover, since $|\langle \phi(x), \phi(y), \phi(t) \rangle| = 720 = |G|$ and since $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G, we conclude

$$\langle \phi(x), \phi(y), \phi(t) \rangle \cong G.$$

We finally show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong S_6$. Let $G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle$. Now, with the help of MAGMA (see [BCP97]), we know that the elements $a = (1\ 5\ 7\ 3\ 11\ 2)(4\ 6\ 8\ 10\ 12\ 9)$, $b = (1\ 2)(3\ 6)(4\ 7)(5\ 9)(8\ 11)(10\ 12)$, and $c = (1\ 12)(2\ 9)(3\ 6)(4\ 7)(5\ 10)(8\ 11)$ belong to G_1. (Note: The labels for the right cosets in this case were assigned by MAGMA and are different from the labels that we assigned earlier.) Therefore, $\langle a, b, c \rangle \leq G_1$, where $G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle$, a permutation group on 12 letters, is a permutation representation of G and, further, $|G_1| = 720$. But $|\langle a, b, c \rangle| = |G_1| = 720$. Therefore, $G_1 = \langle a, b, c \rangle$. Moreover, $\langle a, b, c \rangle \cong S_6 \cong \langle a, b, c | a^6 = b^2 = c^2 = (ab)^5 = (a^{-2}(ab)^2)^3 =$

$(a^{-2}ba^2b)^2 = [c,b] = e\rangle$. Therefore, $G_1 \cong S_6$ and, since $G_1 = \langle\phi(x), \phi(y), \phi(t)\rangle \cong G$, we conclude $G \cong S_6$.

□

4.6 Converting an Element of G from its Permutation Representation to its Symmetric Representation

To illustrate how a permutation representation of an element of S_6 on 12 letters may be converted to its symmetric representation form, we consider the following example:

Example 4.1. Let $g \in G \cong S_6$ and let $p = \phi(g) = (1\ 8\ 2\ 6\ 3\ 7)(4\ 10)(5\ 9)(11\ 12)$ be the permutation representation of g on 12 letters. Then $12^p = 11$ implies $N^p = Nt_0t_1t_0$, since 12 and 11 are labels for the right cosets N and $Nt_0t_1t_0$, respectively. Moreover, since $N^p = Np$ and $N^p = Nt_0t_1t_0$, we have that $Np = Nt_0t_1t_0$. Now, $Np = Nt_0t_1t_0$ implies that $p \in Nt_0t_1t_0$ which implies that $p \sim \pi t_0t_1t_0$ for some $\pi \in N \cong A_5$ or, more precisely, $p = \phi(\pi t_0t_1t_0) = \phi(\pi)\phi(t_0)\phi(t_1)\phi(t_0)$ for some $\pi \in N \cong S_3$. To determine $\pi \in N$, we note first that $p = \phi(\pi)\phi(t_0)\phi(t_1)\phi(t_0) \Rightarrow p(\phi(t_0))^{-1}(\phi(t_1))^{-1}(\phi(t_0))^{-1} = p\phi(t_0^{-1})\phi(t_1^{-1})\phi(t_0^{-1}) = p\phi(t_0)\phi(t_1)\phi(t_0) = \phi(\pi)$. We then calculate the action of $\pi \sim \phi(\pi) = \phi(t_0)\phi(t_1)\phi(t_0)$ on the symmetric generators t_i, where $i \in \{0,1,2,3,4\}$. Now, $\phi(\pi) = \phi(t_0)\phi(t_1)\phi(t_0) = [(1\,8\,2\,6\,3\,7)(4\,10)(5\,9)(11\,12)][(12\,1)(2\,7)(3\,8)(4\,9)(5\,10)(6\,11)]$ $[(12\,2)(1\,6)(3\,8)(4\,9)(5\,10)(7\,11)][(12\,1)(2\,7)(3\,8)(4\,9)(5\,10)(6\,11)] = (1\,3)(4\,5)(6\,8)(9\,10)$. The element $\phi(\pi) = \phi(t_0)\phi(t_1)\phi(t_0) = (1\ 3)(4\ 5)(6\ 8)(9\ 10)$ acts on the right cosets Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4 via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^\pi$. The mappings below illustrate this action:

$$Nt_0 = 1 \mapsto 1^p = 3 = Nt_2, \qquad Nt_2 = 3 \mapsto 3^p = 1 = Nt_0,$$

$$Nt_1 = 2 \mapsto 2^p = 2 = Nt_1, \qquad Nt_3 = 4 \mapsto 4^p = 5 = Nt_4,$$

$$Nt_4 = 5 \mapsto 5^p = 4 = Nt_3$$

Therefore, the element $\phi(\pi)$ acts as $(0\ 2)(3\ 4)$ on the right cosets Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4, and so $\phi(\pi)$ is the permutation representation of $\pi = (0\ 2)(3\ 4) \in A_5$ on 12 letters. Therefore, $\pi = (0\ 2)(3\ 4)$ and $w = t_0t_1t_0$, and so the symmetric representation of

g is $(0\ 2)(3\ 4)t_0t_1t_0$.

4.7 Converting an Element of G from its Symmetric Representation to its Permutation Representation

To illustrate how an element of S_6 in symmetric representation form may be converted to its permutation representation on 12 letters, we consider the following example:

Example 4.2. Let $g \in G \cong S_6$ have the symmetric representation $g = (0\ 4\ 3\ 2\ 1)t_2t_4t_2$. To determine the permutation representation $p = \phi(g)$ of g, we first calculate the action of $\pi = (0\ 3\ 2\ 1)$ on the right cosets of N in G. Now, the element $\pi = (0\ 4\ 3\ 2\ 1)$ acts on the right cosets N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$12 = N \mapsto N^{(0\ 4\ 3\ 2\ 1)} = N = 12$$

$$1 = Nt_0 \mapsto Nt_0^{(0\ 4\ 3\ 2\ 1)} = Nt_4 = 5$$

$$5 = Nt_4 \mapsto Nt_4^{(0\ 4\ 3\ 2\ 1)} = Nt_3 = 4$$

$$4 = Nt_3 \mapsto Nt_3^{(0\ 4\ 3\ 2\ 1)} = Nt_2 = 3$$

$$3 = Nt_2 \mapsto Nt_2^{(0\ 4\ 3\ 2\ 1)} = Nt_1 = 2$$

$$2 = Nt_1 \mapsto N(t_1)^{(0\ 4\ 3\ 2\ 1)} = Nt_0 = 1$$

$$6 = Nt_0t_1 \mapsto N(t_0t_1)^{(0\ 4\ 3\ 2\ 1)} = Nt_4t_0 = 10$$

$$10 = Nt_4t_0 \mapsto N(t_4t_0)^{(0\ 4\ 3\ 2\ 1)} = Nt_3t_4 = Nt_3t_0 = 9$$

$$9 = Nt_3t_0 \mapsto N(t_3t_0)^{(0\ 4\ 3\ 2\ 1)} = Nt_2t_4 = Nt_2t_0 = 8$$

$$8 = Nt_2t_0 \mapsto N(t_2t_0)^{(0\ 4\ 3\ 2\ 1)} = Nt_1t_4 = Nt_1t_0 = 7$$

$$7 = Nt_1t_0 \mapsto N(t_1t_0)^{(0\ 4\ 3\ 2\ 1)} = Nt_0t_4 = Nt_0t_1 = 6$$

$$11 = Nt_0t_1t_0 \mapsto N(t_0t_1t_0)^{(0\ 4\ 3\ 2\ 1)} = Nt_4t_0t_4 = Nt_0t_1t_0 = 11$$

Therefore, the permutation representation of $\pi = (0\,4\,3\,2\,1)$ is $\phi(\pi) = (1\,5\,4\,3\,2)(6\,10\,9\,8\,7)$. Similarly, we calculate the action of the symmetric generator t_2 on the right cosets of N

in G. The symmetric generator t_2 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_2, Nw) = Nwt_2$. The mappings below illustrate this action:

$$12 = N \mapsto Nt_2 = 3$$

$$3 = Nt_2 \mapsto Nt_2t_2 = N = 12$$

$$1 = Nt_0 \mapsto Nt_0t_2 = Nt_0t_1 = 6$$

$$6 = Nt_0t_1 \mapsto Nt_0t_1t_2 = Nt_0t_2t_2 = Nt_0 = 1$$

$$2 = Nt_1 \mapsto Nt_1t_2 = Nt_1t_0 = 7$$

$$7 = Nt_1t_0 \mapsto Nt_1t_0t_2 = Nt_1t_2t_2 = Nt_1 = 2$$

$$4 = Nt_3 \mapsto Nt_3t_2 = Nt_3t_0 = 9$$

$$9 = Nt_3t_0 \mapsto Nt_3t_0t_2 = Nt_3t_2t_2 = Nt_3 = 4$$

$$5 = Nt_4 \mapsto Nt_4t_2 = Nt_4t_0 = 10$$

$$10 = Nt_4t_0 \mapsto Nt_4t_0t_2 = Nt_4t_2t_2 = Nt_4 = 5$$

$$8 = Nt_2t_0 \mapsto Nt_2t_0t_2 = Nt_0t_1t_0 = 11$$

$$11 = Nt_0t_1t_0 \mapsto Nt_0t_1t_0t_2 = Nt_2t_0t_2t_2 = Nt_2t_0 = 8$$

Therefore, the permutation representation of t_2 is $\phi(t_2) = (12\,3)(1\,6)(2\,7)(4\,9)(5\,10)(8\,11)$. Finally, we calculate the action of the symmetric generator t_0 on the right cosets of N in G. The symmetric generator t_0 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_0, Nw) = Nwt_0$. The mappings below illustrate this action:

$$10 = N \mapsto Nt_0 = 1$$

$$1 = Nt_0 \mapsto Nt_0t_0 = N = 10$$

$$2 = Nt_1 \mapsto Nt_1t_0 = 6$$

$$6 = Nt_1t_0 \mapsto Nt_1t_0t_0 = Nt_1 = 2$$

$$3 = Nt_2 \mapsto Nt_2t_0 = 7$$

$$7 = Nt_2t_0 \mapsto Nt_2t_0t_0 = Nt_2 = 3$$

$$4 = Nt_3 \mapsto Nt_3t_0 = 8$$

$$8 = Nt_3t_0 \mapsto Nt_3t_0t_0 = Nt_3 = 4$$

$$5 = Nt_0t_1 \mapsto Nt_0t_1t_0 = 9$$

$$9 = Nt_0t_1t_0 \mapsto Nt_0t_1t_0t_0 = Nt_0t_1 = 5$$

Therefore, the permutation representation of t_4 is $\phi(t_4) = (12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11)$. Now, $g = (0\ 4\ 3\ 2\ 1)t_2t_4t_2 \sim \phi(g) = \phi((0\ 4\ 3\ 2\ 1))\phi(t_2)\phi(t_4)\phi(t_2) = [(1\ 5\ 4\ 3\ 2)(6\ 10\ 9\ 8\ 7)]$ $[(12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11)][(12\ 5)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(10\ 11)]$ $[(12\ 3)(1\ 6)(2\ 7)(4\ 9)(5\ 10)(8\ 11)] = (1\ 8\ 2\ 6\ 3\ 7)(4\ 10)(5\ 9)(11\ 12)$. Therefore, the permutation representation of g is $p = \phi(g) = (1\ 8\ 2\ 6\ 3\ 7)(4\ 10)(5\ 9)(11\ 12)$.

Chapter 5

S_7 as a Homomorphic Image of the Progenitor $3^{*5} : S_5$

In this chapter, we investigate S_7 as a homomorphic image of the progenitor $3^{*5} : S_5$. The group S_7 is the symmetric group on seven letters having order $7! = 5040$. The progenitor $3^{*5} : S_5$ is a semi-direct product of 3^{*5}, a free product of five copies of the cyclic group of order 3, and S_5, the symmetric group on five letters which permutes the five symmetric generators, t_0, t_1, t_2, t_3, and t_4, (and their inverses, $t_0^2 = t_0^{-1}$, $t_1^2 = t_1^{-1}$, $t_2^2 = t_2^{-1}$, $t_3^2 = t_3^{-1}$, and $t_4^2 = t_4^{-1}$) by conjugation.

5.1 Introduction

Let $\bar{G}$ be a homomorphic image of the infinite semi-direct product, the *progenitor*, $3^{*5} : S_5$. A symmetric presentation of $3^{*5} : S_5$ is given by

$$\bar{G} = \langle x, y, t \mid x^5 = y^2 = (yx)^4 = [x,y]^3 = t^3 = [t,y] = [t^x, y] = [t^{x^2}, y] = e \rangle,$$

where $[x,y]^3 = xyxyxy$, $[t,y] = tyty$, $[t^x, y] = t^x y t^x y$, $[t^{x^2}, y] = t^{x^2} y t^{x^2} y$, and e is the identity. In this case, $N \cong S_5 \cong \langle x, y \mid x^5 = y^2 = (yx)^4 = [x,y]^3 = e \rangle$, and the action of N on the five symmetric generators is given by $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (3\ 4)$, and $t \sim t_0$.

Let G denote the group $\bar{G}$ factored by the relations $(xyx^{-1}yxt)^5 = e$,

$(x^{-2}yx^2t)^4 = e$, $(t^{-1}t^x)^3 = e$, and $(xyx^{-1}yxt^{-1}t^x)^2 = e$. That is, let

$$G = \frac{\bar{G}}{(xyx^{-1}yxt)^5, (x^{-2}yx^2t)^4, (t^{-1}t^x)^3, (xyx^{-1}yxt^{-1}t^x)^2}.$$

A symmetric presentation for G is given by

$$\langle x, y, t \mid x^5, y^2, (yx)^4, [x, y]^3, t^3, [t, y], [t^x, y], [t^{x^2}, y], (xyx^{-1}yxt)^5,$$
$$(x^{-2}yx^2t)^4, (t^{-1}t^x)^3, (xyx^{-1}yxt^{-1}t^x)^2\rangle.$$

Now, we consider the following relations:

$$[(0\ 1\ 2)t_0]^5 = e,$$
$$[(0\ 1)t_0]^4 = e,$$
$$[t_0^{-1}t_1]^3 = e,$$
and
$$[(0\ 1\ 2)t_0^{-1}t_1]^2 = e.$$

According to a computer proof by [CHB96], the progenitor $3^{*5} : S_5$, factored by the relations $[(0\ 1\ 2)t_0]^5 = e$, $[(0\ 1)t_0]^4 = e$, $[t_0^{-1}t_1]^3 = e$, and $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, is isomorphic to S_7. In fact, factoring the progenitor $3^{*5} : S_5$ by the relation $[(0\ 1\ 2)t_0]^5 = e$ alone suffices. We will construct S_7 by hand by way of manual double coset enumeration of $G \cong \frac{3^{*5}:S_5}{[(0\ 1\ 2)t_0]^5, [(0\ 1)t_0]^4, [t_0^{-1}t_1]^3, [(0\ 1\ 2)t_0^{-1}t_1]^2}$ over S_5. In so doing, we will show that S_7 is isomorphic to the symmetric presentation

$$\langle x, y, t \mid x^5, y^2, (yx)^4, [x, y]^3, t^3, [t, y], [t^x, y], [t^{x^2}, y], (xyx^{-1}yxt)^5, (x^{-2}yx^2t)^4,$$
$$(t^{-1}t^x)^3, (xyx^{-1}yxt^{-1}t^x)^2\rangle.$$

5.2 Manual Double Coset Enumeration of G Over S_5

We first determine the order of the homomorphic image, G, of the progenitor. To determine the order of the homomorphic image G, we will determine the index of $N \cong S_5$ in G. We determine the index of $N \cong S_5$ in G first by expanding the relations $[(0\ 1\ 2)t_0]^5 = e$, $[(0\ 1)t_0]^4 = e$, $[t_0^{-1}t_1]^3 = e$, and $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and next by performing manual double coset enumeration on G over $N \cong S_5$. To begin, we expand the relations that factor the progenitor $3^{*5} : S_5$:

$$[(0\ 1\ 2)t_0]^5 = e \tag{5.1}$$

$$[(0\ 1)t_0]^4 = e \tag{5.2}$$

$$[t_0^{-1}t_1]^3 = e \tag{5.3}$$

$$[(0\ 1\ 2)t_0^{-1}t_1]^2 = e \tag{5.4}$$

As mentioned above, relation (5.1), $[(0\ 1\ 2)t_0]^5 = e$, is required to determine the homomorphic image, G, of the progenitor, and the other relations can be derived from relation (5.1). We expand relations (5.1), (5.2), (5.3), and (5.4) in detail below:

1. Let $\pi = (0\ 1\ 2)$.

   Then $[(0\ 1\ 2)t_0]^5 = e$
   $\Rightarrow (\pi t_0)^5 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi^3 \pi^{-1} \pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e$
   $\Rightarrow \pi t_0 \pi^4 \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow \pi^5 \pi^{-1} \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow (0\ 1\ 2)^5 t_0^{(0\ 1\ 2)^4} t_0^{(0\ 1\ 2)^3} t_0^{(0\ 1\ 2)^2} t_0^{(0\ 1\ 2)} t_0 = e$
   $\Rightarrow (0\ 2\ 1) t_0^{(0\ 1\ 2)} t_0^{e} t_0^{(0\ 2\ 1)} t_0^{(0\ 1\ 2)} t_0 = e$
   $\Rightarrow (0\ 2\ 1) t_1 t_0 t_2 t_1 t_0 = e$
   $\Rightarrow (0\ 2\ 1) t_1 t_0 t_2 = t_0^{-1} t_1^{-1}$.

   Thus relation (5.1) implies that $(0\ 2\ 1)t_1t_0t_2 = t_0^{-1}t_1^{-1}$ or, equivalently, $Nt_1t_0t_2 = Nt_0^{-1}t_1^{-1}$. That is, using our short-hand notation, $102 \sim \bar{0}\bar{1}$.

2. Let $\pi = (0\ 1)$.

   Then $[(0\ 1)t_0]^4 = e$
   $\Rightarrow (\pi t_0)^4 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e$
   $\Rightarrow \pi t_0 \pi^3 \pi^{-1} \pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e$
   $\Rightarrow \pi^4 \pi^{-1} \pi^{-1} \pi^{-1} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow (0\ 1)^4 t_0^{(0\ 1)^3} t_0^{(0\ 1)^2} t_0^{(0\ 1)} t_0 = e$

$\Rightarrow et_0^{(0\ 1)}t_0^e t_0^{(0\ 1)}t_0 = e$
$\Rightarrow et_1t_0t_1t_0 = e$
$\Rightarrow t_1t_0 = t_0^{-1}t_1^{-1}$.

Thus relation (5.2) implies that $t_1t_0 = t_0^{-1}t_1^{-1}$ or, equivalently, $Nt_1t_0 = Nt_0^{-1}t_1^{-1}$. That is, using our short-hand notation, $10 \sim \bar{0}\bar{1}$.

3. Now $[t_0^{-1}t_1]^3 = e$
$\Rightarrow [t_0^{-1}t_1][t_0^{-1}t_1][t_0^{-1}t_1] = e$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1t_0^{-1}t_1 = e$
$\Rightarrow t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}$.

Thus relation (5.3) implies that $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}$ or, equivalently, $Nt_0^{-1}t_1t_0^{-1} = Nt_1^{-1}t_0t_1^{-1}$. That is, using our short-hand notation, $\bar{0}1\bar{0} \sim \bar{1}0\bar{1}$.

4. Let $\pi = (0\ 1\ 2)$.

Then $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$
$\Rightarrow (\pi t_0^{-1}t_1)^2 = e$
$\Rightarrow \pi t_0^{-1}t_1\pi t_0^{-1}t_1 = e$
$\Rightarrow \pi^2\pi^{-1}t_0^{-1}t_1\pi t_0^{-1}t_1 = e$
$\Rightarrow \pi^2(t_0^{-1}t_1)^\pi t_0^{-1}t_1 = e$
$\Rightarrow \pi^2(t_0^{-1})^\pi t_1^\pi t_0^{-1}t_1 = e$
$\Rightarrow (0\ 1\ 2)^2(t_0^{-1})^{(0\ 1\ 2)}t_1^{(0\ 1\ 2)}t_0^{-1}t_1 = e$
$\Rightarrow (0\ 2\ 1)(t_1^{-1})t_2t_0^{-1}t_1 = e$
$\Rightarrow (0\ 2\ 1)(t_1^{-1})t_2 = t_1^{-1}t_0$.

Thus relation (5.4) implies that $(0\ 2\ 1)(t_1^{-1})t_2 = t_1^{-1}t_0$ or, equivalently, $Nt_1^{-1}t_2 = Nt_1^{-1}t_0$. That is, using our short-hand notation, $\bar{1}2 \sim \bar{1}0$.

We now perform manual double coset enumeration of G over S_5.

1. We first note that the double coset $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$.

Let $[*]$ denote the double coset NeN.

The double coset $[*]$ has one distinct right coset: the identity right coset, $Ne = \{ne \mid n \in N\} = N$.

Moreover, N has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are two double cosets of the form NwN, where w is a word of length one given by $w = t_i^{\pm 1}$, $i \in \{0, 1, 2, 3, 4\}$: Nt_0N and $Nt_0^{-1}N$.

2. We next consider the double coset Nt_0N.

   Let [0] denote the double coset Nt_0N.

   Note that $N^{(0)} \geq N^0 = \langle (1\ 2), (1\ 2\ 3\ 4) \rangle \cong S_4$. Thus $|N^{(0)}| \geq |S_4| = 24$ and, by Lemma 1.4, $|Nt_0N| = \frac{|N|}{|N^{(0)}|} \leq \frac{120}{24} = 5$.

   Therefore, the double coset [0] has at most five distinct single cosets.

   Moreover, $N^{(0)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

   Therefore, there are at most four double cosets of the form NwN, where w is a word of length two given by $w = t_0t_i^{\pm 1}$, $i \in \{0, 1\}$: Nt_0t_0N, Nt_0t_1N, $Nt_0t_0^{-1}N$, and $Nt_0t_1^{-1}N$.

   But, since $Nt_0t_0N = Nt_0^2N = Nt_0^{-1}N$, and since $Nt_0t_0^{-1}N = NeN = N$, we need only consider two addotional double cosets of the form $Nt_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$: Nt_0t_1N and $Nt_0t_1^{-1}N$.

3. We next consider the double coset $Nt_0^{-1}N$.

   Let $[\bar{0}]$ denote the double coset $Nt_0^{-1}N$.

   Note that $N^{(\bar{0})} \geq N^{\bar{0}} = \langle (1\ 2), (1\ 2\ 3\ 4) \rangle \cong S_4$. Thus $|N^{(0)}| \geq |S_4| = 24$ and, by Lemma 1.4, $|Nt_0^{-1}N| = \frac{|N|}{|N^{(\bar{0})}|} \leq \frac{120}{24} = 5$.

   Therefore, the double coset $[\bar{0}]$ has at most five distinct single cosets.

   Moreover, $N^{(\bar{0})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

   Therefore, there are at most four double cosets of the form NwN, where w is a word of length two given by $w = t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0^{-1}t_0N$, $Nt_0^{-1}t_1N$, $Nt_0^{-1}t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}N$.

   But note that $Nt_0^{-1}t_0^{-1}N = Nt_0^{-2}N = Nt_0N$ and $Nt_0^{-1}t_0N = NeN = N$.

   Moreover, by relation (5.2), since $t_1t_0 = t_0^{-1}t_1^{-1}$ implies that $Nt_0^{-1}t_1^{-1} = Nt_1t_0$, and since $Nt_0^{-1}t_1^{-1} = Nt_1t_0$ implies that $\{N(t_i^{-1}t_j^{-1}) \mid i, j \in \{0, 1, 2, 3, 4\}, i \neq j\} =$

$\{Nt_jt_i \mid i, j \in \{0,1,2,3,4\}, i \neq j\}$, we have that $Nt_0t_1N = Nt_0^{-1}t_1^{-1}N$. That is, $[01] = [\bar{0}\bar{1}]$.

Since $Nt_0^{-1}t_0^{-1}N = Nt_0^{-2}N = Nt_0N$ and $Nt_0^{-1}t_0N = NeN = N$, and since, by relation (5.2), $Nt_0t_1N = Nt_0^{-1}t_1^{-1}N$, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3,4\}$: $Nt_0^{-1}t_1N$.

4. We next consider the double coset $Nt_0t_1^{-1}N$.

   Let $[0\bar{1}]$ denote the double coset $Nt_0t_1^{-1}N$.

   Now, by relation (5.4), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0$, and by right multiplication, $(0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}$, and by conjugation, $(0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1}$.

   Therefore, $t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1}$. Moreover, by conjugation with (2 3) and (2 4), we have

   $[t_0t_1^{-1}]^{(2\ 3)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(2\ 3)} \Rightarrow t_0t_1^{-1} = (0\ 1\ 3)t_0t_3^{-1}$

   and $[t_0t_1^{-1}]^{(2\ 4)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(2\ 4)} \Rightarrow t_0t_1^{-1} = (0\ 1\ 4)t_0t_4^{-1}$.

   Therefore, we find that $t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} = (0\ 1\ 3)t_0t_3^{-1} = (0\ 1\ 4)t_0t_4^{-1}$.

   That is, using our short-hand notation, we have

   $$0\bar{1} \sim 0\bar{2} \sim 0\bar{3} \sim 0\bar{4}$$

   By conjugating the above relationships, we have also

   $$1\bar{0} \sim 1\bar{2} \sim 1\bar{3} \sim 1\bar{4}, \qquad 2\bar{0} \sim 2\bar{1} \sim 2\bar{3} \sim 2\bar{4},$$

   $$3\bar{0} \sim 3\bar{1} \sim 3\bar{2} \sim 3\bar{4}, \qquad 4\bar{0} \sim 4\bar{1} \sim 4\bar{2} \sim 4\bar{3}$$

   Since each of the twenty single cosets has four names, the double coset $[0\bar{1}]$ must have at most five distinct single cosets.

   An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1})} \geq N^{0\bar{1}} = \langle (2\ 3), (2\ 4) \rangle \cong S_3$. In fact, by relation (5.4),

$N(t_0t_{\bar{1}}^{-1})^{(1\ 2)} = Nt_0t_{\bar{2}}^{-1} = Nt_0t_{\bar{1}}^{-1}$ implies that $(1\ 2) \in N^{(0\bar{1})}$, and $N(t_0t_{\bar{1}}^{-1})^{(1\ 2\ 3\ 4)} = Nt_0t_{\bar{2}}^{-1} = Nt_0t_{\bar{1}}^{-1}$ implies that $(1\ 2\ 3\ 4) \in N^{(0\bar{1})}$. Therefore, $(1\ 2), (1\ 2\ 3\ 4) \in N^{(0\bar{1})}$, and so $N^{(0\bar{1})} \geq \langle(1\ 2), (1\ 2\ 3\ 4)\rangle \cong S_4$. Therefore, $\left|N^{(0\bar{1})}\right| \geq |S_4| = 24$. Now, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1})}\right|} \leq \frac{120}{24} = 5$.

Therefore, as we concluded earlier, the double coset $[0\bar{1}]$ has at most five distinct single cosets.

Now, $N^{0\bar{1}}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length three given by $w = t_0t_{\bar{1}}^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0t_{\bar{1}}^{-1}t_0N$, $Nt_0t_{\bar{1}}^{-1}t_1N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N$, and $Nt_0t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N$.

But note that $Nt_0t_{\bar{1}}^{-1}t_1N = Nt_0eN = Nt_0N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-2}N = Nt_0t_1N$.

Moreover, by relation (5.2), since $t_1t_0 = t_{\bar{0}}^{-1}t_{\bar{1}}^{-1} \Rightarrow [t_1t_0]^{(0\ 1)} = [t_{\bar{0}}^{-1}t_{\bar{1}}^{-1}]^{(0\ 1)} \Rightarrow t_0t_1 = t_{\bar{1}}^{-1}t_{\bar{0}}^{-1} \Rightarrow t_0t_0t_1 = t_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1} \Rightarrow t_{\bar{0}}^{-1}t_1 = t_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}$ we have that $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N = Nt_{\bar{0}}^{-1}t_1N$. That is, $[\bar{0}1] = [0\bar{1}\bar{0}]$.

Since $Nt_0t_{\bar{1}}^{-1}t_1N = Nt_0eN = Nt_0N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-2}N = Nt_0t_1N$, and since, by relation (5.2), $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N = Nt_{\bar{0}}^{-1}t_1N$, we conclude that there is one distinct double coset of the form $Nt_0t_{\bar{1}}^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$: $Nt_0t_{\bar{1}}^{-1}t_0N$.

5. We next consider the double coset Nt_0t_1N.

   Let $[01]$ denote the double coset Nt_0t_1N.

   Note that by relation (5.2), since $t_1t_0 = t_{\bar{0}}^{-1}t_{\bar{1}}^{-1}$ implies that $Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1} = Nt_1t_0$, and since $Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1} = Nt_1t_0$ implies that $\{Nt_{\bar{i}}^{-1}t_{\bar{j}}^{-1} \mid i, j \in \{0, 1, 2, 3, 4\}, i \neq j\} = \{Nt_jt_i \mid i, j \in \{0, 1, 2, 3, 4\}, i \neq j\}$, we have that $Nt_0t_1N = Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}N$. That is, $[01] = [\bar{0}\bar{1}]$.

   Therefore, note that $Nt_0t_1N = \{Nt_0t_1n \mid n \in N\} = \{Nn^{-1}t_0t_1n \mid n \in N\} = \{N(t_0t_1)^n \mid n \in N\} = \{Nt_it_j \mid i, j \in \{0, 1, 2, 3, 4\}, i \neq j\} = \{Nt_0t_1, Nt_0t_2, Nt_0t_3, Nt_0t_4, Nt_1t_0, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_2t_0, Nt_2t_1, Nt_2t_3, Nt_2t_4, Nt_3t_0, Nt_3t_1, Nt_3t_2, Nt_3t_4, Nt_4t_0, Nt_4t_1, Nt_4t_2, Nt_4t_3, Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}, Nt_{\bar{0}}^{-1}t_{\bar{2}}^{-1}, Nt_{\bar{0}}^{-1}t_{\bar{3}}^{-1}, Nt_{\bar{0}}^{-1}t_{\bar{4}}^{-1}, Nt_{\bar{1}}^{-1}t_{\bar{0}}^{-1}, Nt_{\bar{1}}^{-1}t_{\bar{2}}^{-1}, Nt_{\bar{1}}^{-1}t_{\bar{3}}^{-1}, Nt_{\bar{1}}^{-1}t_{\bar{4}}^{-1}, Nt_{\bar{2}}^{-1}t_{\bar{0}}^{-1}, Nt_{\bar{2}}^{-1}t_{\bar{1}}^{-1}, Nt_{\bar{2}}^{-1}t_{\bar{3}}^{-1}, Nt_{\bar{2}}^{-1}t_{\bar{4}}^{-1}, Nt_{\bar{3}}^{-1}t_{\bar{0}}^{-1},$

$Nt_3^{-1}t_1^{-1}, Nt_3^{-1}t_2^{-1}, Nt_3^{-1}t_4^{-1}, Nt_4^{-1}t_0^{-1}, Nt_4^{-1}t_1^{-1}, Nt_4^{-1}t_2^{-1}, Nt_4^{-1}t_3^{-1}\}$
$= \{Nt_j^{-1}t_i^{-1} \mid i, j \in \{0,1,2,3,4\}, i \neq j\} = \{N(t_1^{-1}t_0^{-1})^n \mid n \in N\}$
$= \{Nn^{-1}t_1^{-1}t_0^{-1}n \mid n \in N\} = \{Nt_1^{-1}t_0^{-1}n \mid n \in N\} = Nt_0^{-1}t_1^{-1}N.$

Now, by relation (5.2), $t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow [t_1t_0]^{(0\ 1)} = [t_0^{-1}t_1^{-1}]^{(0\ 1)} \Rightarrow t_0t_1 = t_1^{-1}t_0^{-1}$.

That is, using our short-hand notation, we have

$$01 \sim \bar{1}\bar{0}$$

Similarly, by conjugating the above relationship, we have

$$02 \sim \bar{2}\bar{0} \quad 03 \sim \bar{3}\bar{0} \quad 04 \sim \bar{4}\bar{0} \quad 10 \sim \bar{0}\bar{1} \quad 12 \sim \bar{2}\bar{1}$$

$$13 \sim \bar{3}\bar{1} \quad 14 \sim \bar{4}\bar{1} \quad 20 \sim \bar{0}\bar{2} \quad 21 \sim \bar{1}\bar{2} \quad 23 \sim \bar{3}\bar{2}$$

$$24 \sim \bar{4}\bar{2} \quad 30 \sim \bar{0}\bar{3} \quad 31 \sim \bar{1}\bar{3} \quad 32 \sim \bar{2}\bar{3} \quad 34 \sim \bar{4}\bar{3}$$

$$40 \sim \bar{0}\bar{4} \quad 41 \sim \bar{1}\bar{4} \quad 42 \sim \bar{2}\bar{4} \quad 43 \sim \bar{3}\bar{4}$$

Since each of the forty single cosets has two names, the double coset $[01] = [\bar{0}\bar{1}]$ must have at most twenty distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(01)} \geq N^{01} = \langle (2\ 3), (2\ 4) \rangle \cong S_3$. Therefore, $|N^{(01)}| \geq |S_3| = 6$. Now, by Lemma 1.4, $|Nt_0t_1N| = \frac{|N|}{|N^{(01)}|} \leq \frac{120}{6} = 20$.

Therefore, as we concluded earlier, the double coset $[01] = [\bar{0}\bar{1}]$ has at most twenty distinct single cosets.

Now, N^{01} has six orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1\}$, $\{2, 3, 4\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1t_i^{\pm 1}$, $i \in \{0, 1, 2\}$: $Nt_0t_1t_0N$, $Nt_0t_1t_1N$, $Nt_0t_1t_2N$, $Nt_0t_1t_0^{-1}N$, $Nt_0t_1t_1^{-1}N$, and $Nt_0t_1t_2^{-1}N$.

But note that $Nt_0t_1t_1^{-1}N = Nt_0eN = Nt_0N$ and $Nt_0t_1t_1N = Nt_0t_1^2N = Nt_0t_1^{-1}N$.

Moreover, by relation (5.2), since $t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_0t_1t_0 = t_0t_0^{-1}t_1^{-1} \Rightarrow t_0t_1t_0 = t_1^{-1}$ implies that $Nt_0t_1t_0 = Nt_1^{-1}$, we have that $Nt_0t_1t_0N = Nt_0^{-1}N$. That is, $[\bar{0}] = [010]$.

Similarly, by relation (5.2), since $t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow [t_1t_0]^{(0\ 1)}$
$= [t_0^{-1}t_1^{-1}]^{(0\ 1)} \Rightarrow t_0t_1 = t_1^{-1}t_0^{-1} \Rightarrow t_0t_1t_0^{-1} = t_1^{-1}t_0^{-1}t_0^{-1} \Rightarrow t_0t_1t_0^{-1} = t_1^{-1}t_0$ implies that $Nt_0t_1t_0^{-1} = Nt_1^{-1}t_0$, we have that $Nt_0t_1t_0^{-1}N = Nt_0^{-1}t_1N$. That is, $[\bar{0}1] = [01\bar{0}]$.

Likewise, by relation (5.1), since $(0\ 2\ 1)t_1t_0t_2 = t_0^{-1}t_1^{-1} \Rightarrow$
$[(0\ 2\ 1)t_1t_0t_2]^{(0\ 1)} = [t_0^{-1}t_1^{-1}]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_0t_1t_2 = t_1^{-1}t_0^{-1}$ implies that $Nt_0t_1t_2 = Nt_1^{-1}t_0^{-1}$ we have that $Nt_0t_1t_2N = Nt_0^{-1}t_1^{-1}N$. That is, $[\bar{0}\bar{1}] = [012]$.

Similarly, by relation (5.4), since $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2$
$= t_2t_1^{-1}t_0 \Rightarrow (0\,2\,1)(0\,1\,2)t_2(0\,2\,1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\,2\,1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow$
$(0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1}$
$= t_2t_1^{-1}e \Rightarrow (0\,2\,1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\,2\,1)t_2t_0^{-1}]^{(1\ 2)} = [t_2t_1^{-1}]^{(1\ 2)} \Rightarrow (0\,1\,2)t_1t_0^{-1} = t_1t_2^{-1} \Rightarrow t_0(0\ 1\ 2)t_1t_0^{-1} = t_0t_1t_2^{-1} \Rightarrow$
$(0\ 1\ 2)(0\ 2\ 1)t_0(0\ 1\ 2)t_1t_0^{-1} = t_0t_1t_2^{-1} \Rightarrow (0\ 1\ 2)t_0^{(0\ 1\ 2)}t_1t_0^{-1} = t_0t_1t_2^{-1} \Rightarrow$
$(0\ 1\ 2)t_1t_1t_0^{-1} = t_0t_1t_2^{-1} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1} = t_0t_1t_2^{-1}$ implies that $Nt_0t_1t_2^{-1}$
$= Nt_1^{-1}t_0^{-1}$, we have that $Nt_0t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}N$. That is, $[\bar{0}\bar{1}] = [01\bar{2}]$.

Since $Nt_0t_1t_1^{-1}N = Nt_0N$, $Nt_0t_1t_1N = Nt_0t_1^{-1}N$, $Nt_0t_1t_0N = Nt_0^{-1}N$, $Nt_0t_1t_0^{-1}N = Nt_0^{-1}t_1N$, $Nt_0t_1t_2N = Nt_0^{-1}t_1^{-1}N$, and $Nt_0t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}N$, we need not consider additional double cosets of the form $Nt_0t_1t_i^{\pm 1}N$, $i \in \{0, 1, 2, 3, 4\}$.

6. We next consider the double coset $Nt_0^{-1}t_1N$.

   Let $[\bar{0}1]$ denote the double coset $Nt_0^{-1}t_1N$.

   Now, by relation (5.4), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow [(0\ 2\ 1)t_1^{-1}t_2]^{(0\ 1)} = [t_1^{-1}t_0]^{(0\ 1)} \Rightarrow$ $(0\ 1\ 2)t_0^{-1}t_2 = t_0^{-1}t_1$ and, by conjugation with elements of $N \cong S_5$,
   $[(0\ 1\ 2)t_0^{-1}t_2]^{(2\ 3)} = [t_0^{-1}t_1]^{(2\ 3)} \Rightarrow (0\ 1\ 3)t_0^{-1}t_3 = t_0^{-1}t_1$ and $[(0\ 1\ 2)t_0^{-1}t_2]^{(2\ 4)} = [t_0^{-1}t_1]^{(2\ 4)} \Rightarrow (0\ 1\ 4)t_0^{-1}t_4 = t_0^{-1}t_1$.

   Therefore, by relation (5.4), $(0\ 1\ 2)t_0^{-1}t_2 = t_0^{-1}t_1 = (0\ 1\ 3)t_0^{-1}t_3 = (0\ 1\ 4)t_0^{-1}t_4$.

   That is, using our short-hand notation, we have

   $$\bar{0}1 \sim \bar{0}2 \sim \bar{0}3 \sim \bar{0}4$$

   By conjugating the above relationship, we have also that

   $$\bar{1}0 \sim \bar{1}2 \sim \bar{1}3 \sim \bar{1}4, \qquad \bar{2}0 \sim \bar{2}1 \sim \bar{2}3 \sim \bar{2}4,$$

$$\bar{3}0 \sim \bar{3}1 \sim \bar{3}2 \sim \bar{3}4, \qquad \bar{4}0 \sim \bar{4}1 \sim \bar{4}2 \sim \bar{4}3$$

Since each of the twenty single cosets has four names, the double coset $[\bar{0}1]$ must have at most five distinct single cosets.

Now, $N^{\bar{0}1}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1,2,3,4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length three given by $w = t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0,1\}$: $Nt_0^{-1}t_1t_0N$, $Nt_0^{-1}t_1t_1N$, $Nt_0^{-1}t_1t_0^{-1}N$, and $Nt_0^{-1}t_1t_1^{-1}N$.

But note that $Nt_0^{-1}t_1t_1N = Nt_0^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}eN = Nt_0^{-1}N$.

Moreover, by relation (5.2), since $t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1}$ $\Rightarrow t_0^{-1}t_1t_0 = t_0t_1^{-1}$ implies that $Nt_0^{-1}t_1t_0 = Nt_0t_1^{-1}$, we have that $Nt_0^{-1}t_1t_0N = Nt_0t_1^{-1}N$. That is, $[0\bar{1}] = [\bar{0}10]$.

Similarly, by relation (5.2), since $t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1}$ $\Rightarrow t_0^{-1}t_1t_0 = t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0t_0 = t_0t_1^{-1}t_0 \Rightarrow t_0^{-1}t_1t_0^{-1} = t_0t_1^{-1}t_0$ implies that $Nt_0^{-1}t_1t_0^{-1} = Nt_0t_1^{-1}t_0$ we have that $Nt_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_0N$. That is, $[0\bar{1}0] = [\bar{0}1\bar{0}]$.

Since $Nt_0^{-1}t_1t_1N = Nt_0^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}eN = Nt_0^{-1}N$, $Nt_0^{-1}t_1t_0N = Nt_0t_1^{-1}N$, and $Nt_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_0N$, we need not consider additional double cosets of the form $Nt_0^{-1}t_1t_i^{\pm 1}N$, $i \in \{0,1,2,3,4\}$.

7. We finally consider the double coset $Nt_0^{-1}t_1t_0^{-1}N$.

   Let $[\bar{0}1\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}N$.

   Note again that, by relation (5.2), since $t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}t_1t_0 = t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0t_0 = t_0t_1^{-1}t_0 \Rightarrow t_0^{-1}t_1t_0^{-1} = t_0t_1^{-1}t_0$ implies that $Nt_0^{-1}t_1t_0^{-1} = Nt_0t_1^{-1}t_0$ we have that $Nt_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_0N$. That is, $[0\bar{1}0] = [\bar{0}1\bar{0}]$.

   Note that $Nt_0^{-1}t_1t_0^{-1}N = \{Nt_0^{-1}t_1t_0^{-1}n \mid n \in N\} = \{Nn^{-1}t_0^{-1}t_1t_0^{-1}n \mid n \in N\} = \{N(t_0^{-1}t_1t_0^{-1})^n \mid n \in N\} = \{Nt_i^{-1}t_jt_i^{-1} \mid i,j \in \{0,1,2,3,4\}, i \neq j\} = \{Nt_0^{-1}t_1t_0^{-1}, Nt_0^{-1}t_2t_0^{-1}, Nt_0^{-1}t_3t_0^{-1}, Nt_0^{-1}t_4t_0^{-1}, Nt_1^{-1}t_0t_1^{-1}, Nt_1^{-1}t_2t_1^{-1}, Nt_1^{-1}t_3t_1^{-1}, Nt_1^{-1}t_4t_1^{-1}, Nt_2^{-1}t_0t_2^{-1}, Nt_2^{-1}t_1t_2^{-1}, Nt_2^{-1}t_3t_2^{-1}, Nt_2^{-1}t_4t_2^{-1}, Nt_3^{-1}t_0t_3^{-1}, Nt_3^{-1}t_1t_3^{-1}, Nt_3^{-1}t_2t_3^{-1}, Nt_3^{-1}t_4t_3^{-1}, Nt_4^{-1}t_0t_4^{-1}, Nt_4^{-1}t_1t_4^{-1}, Nt_4^{-1}t_2t_4^{-1}, Nt_4^{-1}t_3t_4^{-1}, Nt_0t_1^{-1}t_0,$

$Nt_0t_2^{-1}t_0, Nt_0t_3^{-1}t_0, Nt_0t_4^{-1}t_0, Nt_1t_0^{-1}t_1, Nt_1t_2^{-1}t_1, Nt_1t_3^{-1}t_1, Nt_1t_4^{-1}t_1, Nt_2t_0^{-1}t_2,$
$Nt_2t_1^{-1}t_2, Nt_2t_3^{-1}t_2, Nt_2t_4^{-1}t_2, Nt_3t_0^{-1}t_3, Nt_3t_1^{-1}t_3, Nt_3t_2^{-1}t_3, Nt_3t_4^{-1}t_3, Nt_4t_0^{-1}t_4,$
$Nt_4t_1^{-1}t_4, Nt_4t_2^{-1}t_4, Nt_4t_3^{-1}t_4\} = \{Nt_it_j^{-1}t_i \mid i,j \in \{0,1,2,3,4\}, i \neq j\}$
$= \{N(t_0t_1^{-1}t_0)^n \mid n \in N\} = \{Nn^{-1}t_0t_1^{-1}t_0n \mid n \in N\} = \{Nt_0t_1^{-1}t_0n \mid n \in N\}$
$= Nt_0t_1^{-1}t_0N.$

Now, by relation (5.4), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 = (0\ 3\ 1)t_1^{-1}t_3 = (0\ 4\ 1)t_1^{-1}t_4$ and $(0\ 2\ 1)t_1^{-1}t_2t_1^{-1} = t_1^{-1}t_0t_1^{-1} = (0\ 3\ 1)t_1^{-1}t_3t_1^{-1} = (0\ 4\ 1)t_1^{-1}t_4t_1^{-1}$. Similarly, by conjugation of these relations, $(0\ 1\ 2)t_0^{-1}t_2t_0^{-1} = t_0^{-1}t_1t_0^{-1} = (0\ 1\ 3)t_0^{-1}t_3t_0^{-1} = (0\ 1\ 4)t_0^{-1}t_4t_0^{-1}$ and $(0\ 1\ 2)t_2^{-1}t_1t_2^{-1} = t_2^{-1}t_0t_2^{-1} = (0\ 3\ 2)t_2^{-1}t_3t_2^{-1} = (0\ 4\ 2)t_2^{-1}t_4t_2^{-1}$ and $(0\ 2\ 3)t_3^{-1}t_2t_3^{-1} = t_3^{-1}t_0t_3^{-1} = (0\ 1\ 3)t_3^{-1}t_1t_3^{-1} = (0\ 4\ 3)t_3^{-1}t_4t_3^{-1}$ and
$(0\ 2\ 4)t_4^{-1}t_2t_4^{-1} = t_4^{-1}t_0t_4^{-1} = (0\ 3\ 4)t_4^{-1}t_3t_4^{-1} = (0\ 1\ 4)t_4^{-1}t_1t_4^{-1}$. Finally, by relation (5.3), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}$ and $[t_0^{-1}t_1t_0^{-1}]^{(0\ 2)} = [t_1^{-1}t_0t_1^{-1}]^{(0\ 2)} \Rightarrow t_2^{-1}t_1t_2^{-1} = t_1^{-1}t_2t_1^{-1}$ and $[t_0^{-1}t_1t_0^{-1}]^{(0\ 3)} = [t_1^{-1}t_0t_1^{-1}]^{(0\ 3)} \Rightarrow t_3^{-1}t_1t_3^{-1} = t_1^{-1}t_3t_1^{-1}$ and
$[t_0^{-1}t_1t_0^{-1}]^{(0\ 4)} = [t_1^{-1}t_0t_1^{-1}]^{(0\ 4)} \Rightarrow t_4^{-1}t_1t_4^{-1} = t_1^{-1}t_4t_1^{-1}$.

Therefore, the following single cosets, expressed in our short-hand notation, are equivalent:

$$0\bar{1}0 \sim 0\bar{2}0 \sim 0\bar{3}0 \sim 0\bar{4}0 \sim 1\bar{0}1 \sim 1\bar{2}1 \sim 1\bar{3}1 \sim 1\bar{4}1 \sim 2\bar{0}2 \sim 2\bar{1}2 \sim$$
$$2\bar{3}2 \sim 2\bar{4}2 \sim 3\bar{0}3 \sim 3\bar{1}3 \sim 3\bar{2}3 \sim 3\bar{4}3 \sim 4\bar{0}4 \sim 4\bar{1}4 \sim 4\bar{2}4 \sim 4\bar{3}4$$
$$\bar{0}1\bar{0} \sim \bar{0}2\bar{0} \sim \bar{0}3\bar{0} \sim \bar{0}4\bar{0} \sim \bar{1}0\bar{1} \sim \bar{1}2\bar{1} \sim \bar{1}3\bar{1} \sim \bar{1}4\bar{1} \sim \bar{2}0\bar{2} \sim \bar{2}1\bar{2} \sim$$
$$\bar{2}3\bar{2} \sim \bar{2}4\bar{2} \sim \bar{3}0\bar{3} \sim \bar{3}1\bar{3} \sim \bar{3}2\bar{3} \sim \bar{3}4\bar{3} \sim \bar{4}0\bar{4} \sim \bar{4}1\bar{4} \sim \bar{4}2\bar{4} \sim \bar{4}3\bar{4}$$

Since each of the forty singe cosets has forty names, the double coset $[\bar{0}1\bar{0}] = [0\bar{1}0]$ must have one distinct single coset.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(\bar{0}1\bar{0})} \geq N^{\bar{0}1\bar{0}} = \langle (2\ 3), (2\ 4) \rangle \cong S_3$. In fact, by relations (5.2), (5.3), and (5.4), $N(t_0^{-1}t_1t_0^{-1})^{(0\ 1)} = Nt_1^{-1}t_0t_1^{-1} = Nt_0^{-1}t_1t_0^{-1}$ implies that $(0\ 1) \in N^{(\bar{0}1\bar{0})}$, and $N(t_0^{-1}t_1t_0^{-1})^{(0\ 1\ 2\ 3\ 4)} = Nt_1^{-1}t_0t_1^{-1} = Nt_0^{-1}t_1t_0^{-1}$ implies that $(0\ 1\ 2\ 3\ 4) \in N^{(\bar{0}1\bar{0})}$. Therefore, $(0\ 1), (0\ 1\ 2\ 3\ 4) \in N^{(\bar{0}1\bar{0})}$, and so $N^{(\bar{0}1\bar{0})} \geq \langle (0\ 1), (0\ 1\ 2\ 3\ 4) \rangle \cong S_5$. Therefore, $\left|N^{(\bar{0}1\bar{0})}\right| \geq |S_5| = 120$. Now, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_0^{-1}\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{0})}\right|} \leq \frac{120}{120} = 1$.

Therefore, as we concluded earlier, the double coset $[\bar{0}1\bar{0}] = [0\bar{1}0]$ has at most one distinct single coset.

Now, $N^{\bar{0}1\bar{0}}$ has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i = 0$: $Nt_0^{-1}t_1t_0^{-1}t_0N$ and $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_0N = Nt_0^{-1}t_1eN = Nt_0^{-1}t_1N$.

Moreover, by relation (5.2), since $t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1}$ $\Rightarrow t_0^{-1}t_1t_0 = t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0^{-1}t_0^{-1} = t_0^{-1}t_1t_0 = t_0t_1^{-1}$ implies that $Nt_0^{-1}t_1t_0^{-1}t_0^{-1} = Nt_0^{-1}t_1t_0 = Nt_0t_1^{-1}$, we have that $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0N = Nt_0t_1^{-1}N$. That is, $[0\bar{1}] = [\bar{0}1\bar{0}\bar{0}]$.

Since $Nt_0^{-1}t_1t_0^{-1}t_0N = Nt_0^{-1}t_1N$ and $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}N$, we need not consider additional double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3, 4\}$.

In fact, since $N^{(\bar{0}1\bar{0})}$ is transitive on the symmetric generators and since $Nt_0^{-1}t_1t_0^{-1}t_0N = Nt_0^{-1}t_1tN$ and $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}N$ imply that the double coset $[\bar{0}1\bar{0}0] = [\bar{0}1]$ and the double coset $[\bar{0}1\bar{0}\bar{0}] = [0\bar{1}]$, we have completed the double coset enumeration of G over S_5.

In total, therefore, there are at most 7 distinct double cosets of N in G and at most 42 distinct right (single) cosets of N in G. The double cosets of N in G are as follows: $[*]$, $[0]$, $[\bar{0}]$, $[\bar{0}1]$, $[01] = [\bar{0}\bar{1}]$, $[0\bar{1}]$, and $[\bar{0}1\bar{0}] = [0\bar{1}0]$.

5.3 Cayley Diagram of G Over S_5

The Cayley diagram of G over S_5 is illustrated in Figure 5.1. For a detailed explanation of the meaning of the component parts of a Cayley diagram, see Section 2.3.

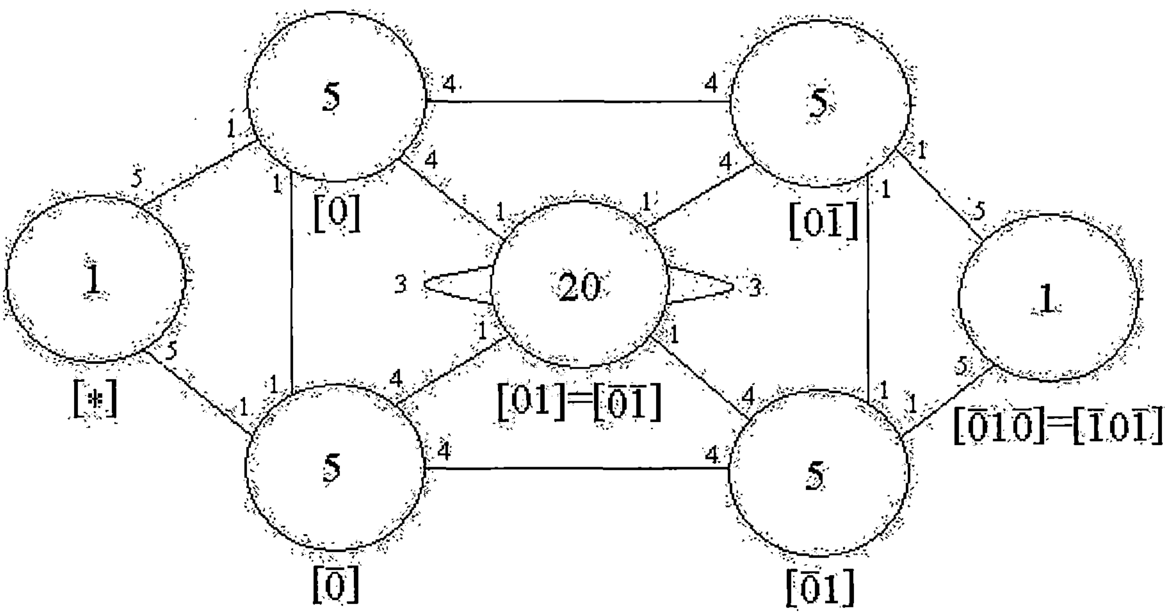

Figure 5.1: Cayley Diagram of G Over S_5

5.4 Action of the Symmetric Generators and the Generators of S_5 on the Right Cosets of G Over S_5

Let X denote the set of all (42) distinct right cosets of N in G, that is, let $X = \{N, Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, Nt_0^{-1}, Nt_1^{-1}, Nt_2^{-1}, Nt_3^{-1}, Nt_4^{-1}, Nt_0^{-1}t_1, Nt_1^{-1}t_0, Nt_2^{-1}t_0, Nt_3^{-1}t_0, Nt_4^{-1}t_0, Nt_0t_1^{-1}, Nt_1t_0^{-1}, Nt_2t_0^{-1}, Nt_3t_0^{-1}, Nt_4t_0^{-1}, Nt_0t_1, Nt_0t_2, Nt_0t_3, Nt_0t_4, Nt_1t_0, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_2t_0, Nt_2t_1, Nt_2t_3, Nt_2t_4, Nt_3t_0, Nt_3t_1, Nt_3t_2, Nt_3t_4, Nt_4t_0, Nt_4t_1, Nt_4t_2, Nt_4t_3, Nt_0t_1^{-1}t_0\}$. We define a mapping $\phi \colon G \longrightarrow S_X$ so that ϕ maps a generator $g \in G$ to its action (by right multiplication) on X. That is, we define ϕ so that $\phi(g) = \widehat{\phi}(g) : X \to X$. Then the action $\phi(t) \sim \phi(t_0)$ of the symmetric generator $t \sim t_0$ on the right cosets of N in G may be expressed as

$$\phi(t) \sim \phi(t_0) = (*\ 0\ \bar{0})(1\ 10\ 1\bar{0})(2\ 20\ 2\bar{0})(3\ 30\ 3\bar{0})(4\ 40\ 4\bar{0})(\bar{1}\ \bar{1}0\ 01)$$

$$(\bar{2}\ \bar{2}0\ 02)(\bar{3}\ \bar{3}0\ 03)(\bar{4}\ \bar{4}0\ 04)(0\bar{1}\ 0\bar{1}0\ \bar{0}1),$$

and the action $\phi(x) \sim \phi((0\ 1\ 2\ 3\ 4))$ of the generator $x \sim (0\ 1\ 2\ 3\ 4)$ of S_5 on the right cosets of N in G may be expressed as

$$\phi(x) \sim \phi((0\,1\,2\,3\,4)) = (0\,1\,2\,3\,4)(\bar{0}\,\bar{1}\,\bar{2}\,\bar{3}\,\bar{4})(\bar{0}1\ \bar{1}2\ \bar{2}3\ \bar{3}4\ \bar{4}0)(0\bar{1}\ 1\bar{2}\ 2\bar{3}\ 3\bar{4}\ 4\bar{0})(01\ 12\ 23\ 34\ 40)$$

$$(02\ 13\ 24\ 30\ 41)(03\ 14\ 20\ 31\ 42)(04\ 10\ 21\ 32\ 43),$$

and the action $\phi(y) \sim \phi((0\ 1\ 2\ 3\ 4))$ of the generator $y \sim (3\ 4)$ of S_5 on the right cosets of N in G may be expressed as

$$\phi(y) \sim \phi((0\ 1\ 2\ 3\ 4)) = (3\ 4)(\bar{3}\ \bar{4})(\bar{3}0\ \bar{4}0)(3\bar{0}\ 4\bar{0})(03\ 04)(13\ 14)(23\ 24)$$

$$(30\ 40)(31\ 41)(32\ 42)(34\ 43).$$

Since there are 42 distinct right cosets of N in G, these actions may be written as permutations on 42 letters. In fact, the actions of the generators on the set of right cosets of N in G are equivalent to the permutation representations of the generators in their action on the right cosets of N in G. To better manipulate the permutation representations of the symmetric generators t_i and the generators x and y, it is helpful to label the distinct single cosets of N in G as follows:

(42)	*	(7)	$\bar{1}$	(14)	$\bar{3}0$	(21)	01	(28)	14	(35)	32
(1)	0	(8)	$\bar{2}$	(15)	$\bar{4}0$	(22)	02	(29)	20	(36)	34
(2)	1	(9)	$\bar{3}$	(16)	$0\bar{1}$	(23)	03	(30)	21	(37)	40
(3)	2	(10)	$\bar{4}$	(17)	$1\bar{0}$	(24)	04	(31)	23	(38)	41
(4)	3	(11)	$\bar{0}1$	(18)	$2\bar{0}$	(25)	10	(32)	24	(39)	42
(5)	4	(12)	$\bar{1}0$	(19)	$3\bar{0}$	(26)	12	(33)	30	(40)	43
(6)	$\bar{0}$	(13)	$\bar{2}0$	(20)	$4\bar{0}$	(27)	13	(34)	31	(41)	$0\bar{1}0$

Having labeled each of the 42 distinct right cosets of N in G, we may express the permutation representation of the symmetric generators $t \sim t_0$, $t^x \sim t_1$, $t^{x^2} \sim t_2$, $t^{x^3} \sim t_3$, and $t^{x^4} \sim t_4$, and the generators $x \sim (0\ 1\ 2\ 3\ 4)$ and $y \sim (3\ 4)$, in their action on the right cosets of N in G as, respectively,

$$\phi(t) \sim \phi(t_0) : (42\ 1\ 6)(2\ 25\ 17)(3\ 29\ 18)(4\ 33\ 19)(5\ 37\ 20)(7\ 12\ 21)$$

$$(8\ 13\ 22)(9\ 14\ 23)(10\ 15\ 24)(16\ 41\ 11),$$

$$\phi(t^x) \sim \phi(t_1) : (42\ 2\ 7)(1\ 21\ 16)(3\ 30\ 18)(4\ 34\ 19)(5\ 38\ 20)(6\ 11\ 25)$$

$$(8\ 13\ 26)(9\ 14\ 27)(10\ 15\ 28)(12\ 17\ 41),$$

$$\phi(t^{x^2}) \sim \phi(t_2) : (42\ 3\ 8)(1\ 22\ 16)(2\ 26\ 17)(4\ 35\ 19)(5\ 39\ 20)(6\ 11\ 29)$$

$$(7\ 12\ 30)(9\ 14\ 31)(10\ 15\ 32)(13\ 18\ 41),$$

$$\phi(t^{x^3}) \sim \phi(t_3) : (42\ 4\ 9)(1\ 23\ 16)(2\ 27\ 17)(3\ 31\ 18)(5\ 40\ 20)(6\ 11\ 33)$$

$$(7\ 12\ 34)(8\ 13\ 35)(10\ 15\ 36)(14\ 19\ 41),$$

$$\phi(t^{x^4}) \sim \phi(t_4) : (42\ 5\ 10)(1\ 24\ 16)(2\ 28\ 17)(3\ 32\ 18)(4\ 36\ 19)(6\ 11\ 37)$$

$$(7\ 12\ 38)(8\ 13\ 39)(9\ 14\ 40)(15\ 20\ 41),$$

$$\phi(x) \sim \phi((0\ 1\ 2\ 3\ 4)) : (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15)(16\ 17\ 18\ 19\ 20)$$

$$(21\ 26\ 31\ 36\ 37)(22\ 27\ 32\ 33\ 38)(23\ 28\ 29\ 34\ 39)(24\ 25\ 30\ 35\ 40),$$

$$\phi(y) \sim \phi(3\ 4) : (4\ 5)(9\ 10)(14\ 15)(19\ 20)(23\ 24)(27\ 28)(31\ 32)$$

$$(33\ 37)(34\ 38)(35\ 39)(36\ 40)$$

5.5 Proof of Isomorphism between G and S_7

We now demonstrate that $G \cong S_7$.

Proof. To prove that $G \cong S_7$, we must first show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and that $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 5040$ (from which we can conclude $G \cong \langle \phi(x), \phi(y), \phi(t) \rangle$), and we must next show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong S_7$ (from which we can conclude S_7 is a homomorphic image of G and $G \cong S_7$).

We first show $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 5040$. From our construction of G using manual double coset enumeration of G over S_5, illustrated by the Cayley Diagram in Figure 5.1, we concluded that group G defined by the symmetric presentation must contain a homomorphic image of $N \cong S_5$ whose index $[G : N]$ is at most 42:

$$[G:N] = \frac{|N|}{|N^{(*)}|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(\bar{0})}|} + \frac{|N|}{|N^{(0\bar{1})}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(\bar{0}1)}|} + \frac{|N|}{|N^{(0\bar{1}0)}|} \leq$$

$$\frac{120}{120} + \frac{120}{24} + \frac{120}{24} + \frac{120}{24} + \frac{120}{6} + \frac{120}{24} + \frac{120}{120} = 1 + 5 + 5 + 5 + 20 + 5 + 1 = 42$$

Since the index of N in G is at most 42, and since $|G| = \frac{|G|}{|N|} \cdot |N|$, the order of the homomorphic image group G is at most 5040:

$$|G| = \frac{|G|}{|N|} \cdot |N| \leq 42 \cdot |N| = 42 \cdot 120 = 5040 \Rightarrow |G| \leq 5040$$

We now consider $\langle \phi(x), \phi(y), \phi(t) \rangle$. Note that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a group generated by the permutation representations of the generators x, y, and t and, as such, it is a subgroup of the symmetric group S_{42} acting on the forty-two right cosets of N in G. We now show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G and, therefore, that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle| = 5040$. To show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we first demonstrate that $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{42}$ is a homomorphic image of $\bar{G}$. Now, recall that $\bar{G} = \langle x, y, t\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$, and its presentation is given by

$$\bar{G} = \langle x, y, t \mid x^5 = y^2 = (yx)^4 = [x,y]^3 = t^3 = [t,y] = [t^x, y] = [t^{x^2}, y] = e\rangle,$$

where $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (3\ 4)$, and $t \sim t_0$, and $N = \langle x, y\rangle \cong S_5$. Let $\alpha : \bar{G} \longrightarrow \langle\phi(x), \phi(y), \phi(t)\rangle$ be a mapping from $\bar{G}$ to $\langle\phi(x), \phi(y), \phi(t)\rangle$. We note first that the mapping $\alpha : \bar{G} \longrightarrow \langle\phi(x), \phi(y), \phi(t)\rangle$ is well defined. The generators $\phi(x), \phi(y)$, and $\phi(t)$ are the permutation representations of $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (3\ 4)$, and $t \sim t_0$ on 42 letters. Since the order of $\phi(x)$ is 5, the order of $\phi(y)$ is 2, and the order of $\phi(x)\phi(y)$ is 4, we conclude $\langle\phi(x), \phi(y)\rangle \cong N \cong S_5$. Moreover, we can demonstrate that $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle\phi(x), \phi(y)\rangle \cong N \cong S_5$. Now, since $t \sim t_0$, we have that

$$\phi(t)^{\phi(x)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4))} = [(42\ 1\ 6)(2\ 25\ 17)(3\ 29\ 18)(4\ 33\ 19)(5\ 37\ 20)(7\ 12\ 21)$$

$$(8\ 13\ 22)(9\ 14\ 23)(10\ 15\ 24)(16\ 41\ 11)]^{\phi((0\ 1\ 2\ 3\ 4))} =$$

$$[(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15)(16\ 17\ 18\ 19\ 20)(21\ 26\ 31\ 36\ 37)$$

$$(22\ 27\ 32\ 33\ 38)(23\ 28\ 29\ 34\ 39)(24\ 25\ 30\ 35\ 40)][(42\ 1\ 6)(2\ 25\ 17)$$

$$(3\ 29\ 18)(4\ 33\ 19)(5\ 37\ 20)(7\ 12\ 21)(8\ 13\ 22)(9\ 14\ 23)(10\ 15\ 24)(16\ 41\ 11)]$$

$$[(1\ 5\ 4\ 3\ 2)(6\ 10\ 9\ 8\ 7)(11\ 15\ 14\ 13\ 12)(16\ 20\ 19\ 18\ 17)(21\ 37\ 36\ 31\ 26)$$

$$(22\ 38\ 33\ 32\ 27)(23\ 39\ 34\ 29\ 28)(24\ 40\ 35\ 30\ 25)] =$$

$$(42\ 2\ 7)(1\ 21\ 16)(3\ 30\ 18)(4\ 34\ 19)(5\ 38\ 20)(6\ 11\ 25)$$

$$(8\ 13\ 26)(9\ 14\ 27)(10\ 15\ 28)(12\ 17\ 41) = \phi(t_1) \sim \phi(t^x)$$

and similarly,

$$\phi(t)^{\phi(x^2)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^2)} = \phi(t_2) \sim \phi(t^{x^2})$$

$$\phi(t)^{\phi(x^3)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^3)} = \phi(t_3) \sim \phi(t^{x^3})$$

$$\phi(t)^{\phi(x^4)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^4)} = \phi(t_4) \sim \phi(t^{x^4})$$

Therefore, $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle\phi(x), \phi(y)\rangle \cong N \cong S_5$; these conjugates are, namely, $\phi(t) \sim \phi(t_0)$, $\phi(t^x) \sim \phi(t_1)$, $\phi(t^{x^2}) \sim \phi(t_2)$, $\phi(t^{x^3}) \sim \phi(t_3)$, and $\phi(t^{x^4}) \sim \phi(t_4)$. Since $\langle\phi(x), \phi(y)\rangle \cong N \cong S_5$ and since $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle\phi(x), \phi(y)\rangle \cong N$, we conclude that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of $\bar{G} = \langle x, y, t\rangle$. That is, $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$.

Next, to show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of $\bar{G}$ factored by the relations $(xyx^{-1}yxt)^5 = e$, $(x^{-2}yx^2t)^4 = e$, $(t^{-1}t^x)^3 = e$, and $(xyx^{-1}yxt^{-1}t^x)^2 = e$; that is, we we show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$ factored by the relations $[(0\ 1\ 2)t_0]^5 = e$, $[(0\ 1)t_0]^4 = e$, $[t_0^{-1}t_1]^3 = e$, and $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$. Let $\tilde{\alpha} : G \longrightarrow \langle\phi(x), \phi(y), \phi(t)\rangle$ be a mapping from G to $\langle\phi(x), \phi(y), \phi(t)\rangle$. We note first that the mapping $\tilde{\alpha} : G \longrightarrow \langle\phi(x), \phi(y), \phi(t)\rangle$ is well-defined, and we know already that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$. Now, to show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we need only demonstrate that the relations $[(0\ 1\ 2)t_0]^5 = e$, $[(0\ 1)t_0]^4 = e$, $[t_0^{-1}t_1]^3 = e$, and $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, which hold true in G, also hold true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{42}$.

To demonstrate that the relation $[(0\ 1\ 2)t_0]^5 = e$, or, equivalently, the relation $t_1t_0t_2t_1t_0 = (0\ 1\ 2)$, holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{42}$, we show that $\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{42}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 1\ 2)) \sim \phi(xyx^{-1}yx)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_1),$$

$$\phi(t_1)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_2),$$

$$\phi(t_2)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_0),$$

$$\phi(t_3)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_4)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi((0\ 1\ 2))$. This gives us

$$\phi(t_0)^{\phi((0\ 1\ 2))} = \phi(t_1),$$

$$\phi(t_1)^{\phi((0\ 1\ 2))} = \phi(t_2),$$

$$\phi(t_2)^{\phi((0\ 1\ 2))} = \phi(t_0),$$

$$\phi(t_3)^{\phi((0\ 1\ 2))} = \phi(t_3),$$

$$\phi(t_4)^{\phi((0\ 1\ 2))} = \phi(t_4)$$

Since $\phi(t_1)\phi(t_0)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{42}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 1\ 2)) \sim \phi(xyx^{-1}yx)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation, we conclude that the relation $[(0\ 1\ 2)t_0]^5 = e$, which holds true in G, also holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{42}$. By way of a similar process, we find that the relations $[(0\ 1)t_0]^4 = e$, $[t_0^{-1}t_1]^3 = e$, and $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$ also hold true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{42}$.

Since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$, and since the relations $[(0\ 1\ 2)t_0]^5 = e$, $[(0\ 1)t_0]^4 = e$, $[t_0^{-1}t_1]^3 = e$, and $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$ hold true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{42}$, we conclude that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$ factored by the relations $[(0\ 1\ 2)t_0]^5 = e$, $[(0\ 1)t_0]^4 = e$, $[t_0^{-1}t_1]^3 = e$, and $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$; that is, we conclude that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G.

More importantly, since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we have that $\langle\phi(x), \phi(y), \phi(t)\rangle \leq G$. In fact, since $\langle\phi(x), \phi(y), \phi(t)\rangle \leq G$, we have that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle|$. Since it is easily demonstrated, with MAGMA or by hand, that $|\langle\phi(x), \phi(y), \phi(t)\rangle| = 5040$, we conclude finally that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle| = 5040$, that is, $|G| \geq 5040$. Given $|G| \leq 5040$ and $|G| \geq 5040$, we conclude $|G| = 5040$. Moreover, since $|\langle\phi(x), \phi(y), \phi(t)\rangle| = 5040 = |G|$ and since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we conclude

$$\langle\phi(x), \phi(y), \phi(t)\rangle \cong G.$$

We finally show that $\langle\phi(x), \phi(y), \phi(t)\rangle \cong S_7$. Let $G_1 = \langle\phi(x), \phi(y), \phi(t)\rangle$. Now, with the help of MAGMA (see [BCP97]), we know that the elements $a = (1\ 4\ 22\ 36\ 33\ 19\ 3)$ $(2\ 6\ 27\ 41\ 39\ 24\ 13)(5\ 16\ 14\ 31\ 35\ 20\ 25)(7\ 29\ 10\ 8\ 34\ 42\ 32)(9\ 21\ 11\ 38\ 28\ 17\ 18)$

(12 23 37 15 30 40 26), $b = (5\ 11)(7\ 17)(12\ 24)(13\ 25)(18\ 31)(19\ 32)(26\ 33)(34\ 38)(35\ 41)(36\ 42)(39\ 40)$, and $c = (1\ 2\ 3)(4\ 9\ 10)(5\ 12\ 13)(6\ 15\ 16)(7\ 18\ 19)(8\ 20\ 21)(11\ 24\ 25)(14\ 27\ 28)(17\ 31\ 32)(23\ 37\ 29)$ belong to G_1. (Note: The labels for the right cosets in this case were assigned by MAGMA and are different from the labels that we assigned earlier.) Therefore, $\langle a, b, c\rangle \leq G_1$, a permutation group on 42 letters, is a permutation representation of G and, further, $|G_1| = 5040$. But $|\langle a, b, c\rangle| = 5040 = |G_1|$. Therefore, $G_1 = \langle a, b, c\rangle$. Moreover, $\langle a, b, c\rangle \cong S_7 \cong \langle a, b, c | a^7 = b^2 = (ab)^6 = (a^{-2}(ab)^2)^3 = (a^{-2}ba^2b)^2 = c^3 = [c, b] = [c^a, b] = [c^{a^2}, b] = e\rangle$. Therefore, $G_1 \cong S_7$ and, since $G_1 = \langle \phi(x), \phi(y), \phi(t)\rangle \cong G$, we conclude $G \cong S_7$.

□

5.6 Converting an Element of G from its Permutation Representation to its Symmetric Representation

To illustrate how a permutation representation of an element of S_7 on 42 letters may be converted to its symmetric representation form, we consider the following example:

Example 5.1. Let $g \in G \cong S_7$ and let $p = \phi(g) = (1\ 28\ 22\ 17\ 23\ 2\ 24\ 26\ 16\ 27)(3\ 10\ 35\ 42\ 40)(4\ 15\ 39\ 18\ 9)(5\ 32\ 8\ 19\ 14)(6\ 34\ 11\ 38\ 29\ 7\ 33\ 12\ 37\ 30)(13\ 20\ 31\ 41\ 36)(21\ 25)$ be the permutation representation of g on 42 letters. Then $42^p = 40$ implies $N^p = Nt_4t_3$, since 42 and 40 are labels for the right cosets N and Nt_4t_3, respectively. Moreover, since $N^p = Np$ and $N^p = Nt_4t_3$, we have that $Np = Nt_4t_3$. Now, $Np = Nt_4t_3$ implies that $p \in Nt_4t_3$ which implies that $p \sim \pi t_4t_3$ for some $\pi \in N \cong S_5$ or, more precisely, $p = \phi(\pi t_4 t_3) = \phi(\pi)\phi(t_4)\phi(t_3)$ for some $\pi \in N \cong S_5$. To determine $\pi \in N$, we note first that $p = \phi(\pi)\phi(t_4)\phi(t_3) \Rightarrow p(\phi(t_3))^{-1}(\phi(t_4))^{-1} = p\phi(t_3^{-1})\phi(t_4^{-1}) = \phi(\pi)$. We then calculate the action of $\pi \sim \phi(\pi) = p\phi(t_3^{-1})\phi(t_4^{-1})$ on the symmetric generators t_i, where $i \in \{0, 1, 2, 3, 4\}$. Now, $\phi(\pi) = p\phi(t_3^{-1})\phi(t_4^{-1}) =$

$$[(1\ 28\ 22\ 17\ 23\ 2\ 24\ 26\ 16\ 27)(3\ 10\ 35\ 42\ 40)(4\ 15\ 39\ 18\ 9)(5\ 32\ 8\ 19\ 14)$$

$$(6\ 34\ 11\ 38\ 29\ 7\ 33\ 12\ 37\ 30)(13\ 20\ 31\ 41\ 36)(21\ 25)][(42\ 4\ 9)(1\ 23\ 16)$$

$$(2\ 27\ 17)(3\ 31\ 18)(5\ 40\ 20)(6\ 11\ 33)(7\ 12\ 34)(8\ 13\ 35)(10\ 15\ 36)(14\ 19\ 41)]^{-1}$$

$$[(42\ 5\ 10)(1\ 24\ 16)(2\ 28\ 17)(3\ 32\ 18)(4\ 36\ 19)(6\ 11\ 37)(7\ 12\ 38)(8\ 13\ 39)$$
$$(9\ 14\ 40)(15\ 20\ 41)]^{-1}$$
$$= (1\ 2)(3\ 4\ 5)(6\ 7)(8\ 9\ 10)(11\ 12)(13\ 14\ 15)(16\ 17)(18\ 19\ 20)(21\ 25)$$
$$(22\ 27\ 24\ 26\ 23\ 28)(29\ 34\ 37\ 30\ 33\ 38)(31\ 36\ 39)(32\ 35\ 40).$$

The element $\pi \sim \phi(\pi) = p\phi(t_3^{-1})\phi(t_4^{-1}) = (1\ 2)(3\ 4\ 5)(6\ 7)(8\ 9\ 10)(11\ 12)(13\ 14\ 15)(16\ 17)(18\ 19\ 20)(21\ 25)(22\ 27\ 24\ 26\ 23\ 28)(29\ 34\ 37\ 30\ 33\ 38)(31\ 36\ 39)(32\ 35\ 40)$ acts on the right cosets Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4 via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$Nt_0 = 1 \mapsto 1^p = 2 = Nt_1, \qquad Nt_1 = 2 \mapsto 2^p = 1 = Nt_0,$$
$$Nt_2 = 3 \mapsto 3^p = 4 = Nt_3, \qquad Nt_3 = 4 \mapsto 4^p = 5 = Nt_4,$$
$$Nt_4 = 5 \mapsto 5^p = 3 = Nt_2$$

Therefore, the element $\phi(\pi)$ acts as $(0\ 1)(2\ 3\ 4)$ on the right cosets Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4, and so $\phi(\pi)$ is the permutation representation of $\pi = (0\ 1)(2\ 3\ 4) \in S_5$ on 42 letters. Therefore, $\pi = (0\ 1)(2\ 3\ 4)$ and $w = t_4t_3$, and so the symmetric representation of g is $(0\ 1)(2\ 3\ 4)t_4t_3$.

5.7 Converting an Element of G from its Symmetric Representation to its Permutation Representation

To illustrate how an element of S_7 in symmetric representation form may be converted to its permutation representation on 42 letters, we consider the following example:

Example 5.2. Let $g \in G \cong S_7$ have the symmetric representation $g = (0\ 1)(2\ 3\ 4)t_4t_3$. To determine the permutation representation $p = \phi(g)$ of g, we first calculate the action of $\pi = (0\ 1)(2\ 3\ 4)$ on the right cosets of N in G. Now, the element $\pi = (0\ 1)(2\ 3\ 4)$ acts on the right cosets N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$42 = N \mapsto N^{(0\ 1)(2\ 3\ 4)} = N = 42$$

$$1 = Nt_0 \mapsto Nt_0^{(0\ 1)(2\ 3\ 4)} = Nt_1 = 2$$

$$2 = Nt_1 \mapsto Nt_1^{(0\ 1)(2\ 3\ 4)} = Nt_0 = 1$$

$$3 = Nt_2 \mapsto Nt_2^{(0\ 1)(2\ 3\ 4)} = Nt_3 = 4$$

$$4 = Nt_3 \mapsto Nt_3^{(0\ 1)(2\ 3\ 4)} = Nt_4 = 5$$

$$5 = Nt_4 \mapsto Nt_4^{(0\ 1)(2\ 3\ 4)} = Nt_2 = 3$$

$$6 = Nt_0^{-1} \mapsto N(t_0^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_1^{-1} = 7$$

$$7 = Nt_1^{-1} \mapsto N(t_1^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_0^{-1} = 6$$

$$8 = Nt_2^{-1} \mapsto N(t_2^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_3^{-1} = 9$$

$$9 = Nt_3^{-1} \mapsto N(t_3^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_4^{-1} = 10$$

$$10 = Nt_4^{-1} \mapsto N(t_4^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_2^{-1} = 8$$

$$11 = Nt_0^{-1}t_1 \mapsto N(t_0^{-1}t_1)^{(0\ 1)(2\ 3\ 4)} = Nt_1^{-1}t_0 = 12$$

$$12 = Nt_1^{-1}t_0 \mapsto N(t_1^{-1}t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_0^{-1}t_1 = 11$$

$$13 = Nt_2^{-1}t_0 \mapsto N(t_2^{-1}t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_3^{-1}t_1 = Nt_3^{-1}t_0 = 14$$

$$14 = Nt_3^{-1}t_0 \mapsto N(t_3^{-1}t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_4^{-1}t_1 = Nt_4^{-1}t_0 = 15$$

$$15 = Nt_4^{-1}t_0 \mapsto N(t_4^{-1}t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_2^{-1}t_1 = Nt_2^{-1}t_0 = 13$$

$$16 = Nt_0t_1^{-1} \mapsto N(t_0t_1^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_1t_0^{-1} = 17$$

$$17 = Nt_1t_0^{-1} \mapsto N(t_1t_0^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_0t_1^{-1} = 16$$

$$18 = Nt_2t_0^{-1} \mapsto N(t_2t_0^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_3t_1^{-1} = Nt_3t_0^{-1} = 19$$

$$19 = Nt_3t_0^{-1} \mapsto N(t_3t_0^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_4t_1^{-1} = Nt_4t_0^{-1} = 20$$

$$20 = Nt_4t_0^{-1} \mapsto N(t_4t_0^{-1})^{(0\ 1)(2\ 3\ 4)} = Nt_2t_1^{-1} = Nt_2t_0^{-1} = 18$$

$$21 = Nt_0t_1 \mapsto N(t_0t_1)^{(0\ 1)(2\ 3\ 4)} = Nt_1t_0 = 25$$

$$25 = Nt_1t_0 \mapsto N(t_1t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_0t_1 = 21$$

$$22 = Nt_0t_2 \mapsto N(t_0t_2)^{(0\ 1)(2\ 3\ 4)} = Nt_1t_3 = 27$$

$$27 = Nt_1t_3 \mapsto N(t_1t_3)^{(0\ 1)(2\ 3\ 4)} = Nt_0t_4 = 24$$

$$24 = Nt_0t_4 \mapsto N(t_0t_4)^{(0\ 1)(2\ 3\ 4)} = Nt_1t_2 = 26$$

$$26 = Nt_1t_2 \mapsto N(t_1t_2)^{(0\ 1)(2\ 3\ 4)} = Nt_0t_3 = 23$$

$$23 = Nt_0t_3 \mapsto N(t_0t_3)^{(0\ 1)(2\ 3\ 4)} = Nt_1t_4 = 28$$

$$28 = Nt_1t_4 \mapsto N(t_1t_4)^{(0\ 1)(2\ 3\ 4)} = Nt_0t_2 = 22$$

$$29 = Nt_2t_0 \mapsto N(t_2t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_3t_1 = 34$$

$$34 = Nt_3t_1 \mapsto N(t_3t_1)^{(0\ 1)(2\ 3\ 4)} = Nt_4t_0 = 37$$

$$37 = Nt_4t_0 \mapsto N(t_4t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_2t_1 = 30$$

$$30 = Nt_2t_1 \mapsto N(t_2t_1)^{(0\ 1)(2\ 3\ 4)} = Nt_3t_0 = 33$$

$$33 = Nt_3t_0 \mapsto N(t_3t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_4t_1 = 38$$

$$38 = Nt_4t_1 \mapsto N(t_4t_1)^{(0\ 1)(2\ 3\ 4)} = Nt_2t_0 = 29$$

$$31 = Nt_2t_3 \mapsto N(t_2t_3)^{(0\ 1)(2\ 3\ 4)} = Nt_3t_4 = 36$$

$$36 = Nt_3t_4 \mapsto N(t_3t_4)^{(0\ 1)(2\ 3\ 4)} = Nt_4t_2 = 39$$

$$39 = Nt_4t_2 \mapsto N(t_4t_2)^{(0\ 1)(2\ 3\ 4)} = Nt_2t_3 = 31$$

$$32 = Nt_2t_4 \mapsto N(t_2t_4)^{(0\ 1)(2\ 3\ 4)} = Nt_3t_2 = 35$$

$$35 = Nt_3t_2 \mapsto N(t_3t_2)^{(0\ 1)(2\ 3\ 4)} = Nt_4t_3 = 40$$

$$40 = Nt_4t_3 \mapsto N(t_4t_3)^{(0\ 1)(2\ 3\ 4)} = Nt_2t_4 = 32$$

$$41 = Nt_0t_1^{-1}t_0 \mapsto N(t_0t_1^{-1}t_0)^{(0\ 1)(2\ 3\ 4)} = Nt_1t_0^{-1}t_1 = Nt_0t_1^{-1}t_0 = 41$$

Therefore, the permutation representation of $\pi = (0\ 1)(2\ 3\ 4)$ is
$\phi(\pi) = (1\ 2)(3\ 4\ 5)(6\ 7)(8\ 9\ 10)(11\ 12)(13\ 14\ 15)(16\ 17)(18\ 19\ 20)(21\ 25)$
$(22\ 27\ 24\ 26\ 23\ 28)(29\ 34\ 37\ 30\ 33\ 38)(31\ 36\ 39)(32\ 35\ 40)$.

Similarly, we calculate the action of the symmetric generator t_4 on the right cosets of N in G. The symmetric generator t_4 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_4, Nw) = Nwt_4$. By this mapping, the permutation representation of t_4 in its action on the right cosets of N in G is
$\phi(t_4) = (42\ 5\ 10)(1\ 24\ 16)(2\ 28\ 17)(3\ 32\ 18)(4\ 36\ 19)(6\ 11\ 37)(7\ 12\ 38)(8\ 13\ 39)$
$(9\ 14\ 40)(15\ 20\ 41)$.

Finally, we calculate the action of the symmetric generator t_3 on the right cosets of N in G. The symmetric generator t_3 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_3, Nw) = Nwt_3$. By this mapping, the permutation representation of t_3 in its action on the right cosets of N in G, therefore, is $\phi(t_3) =$ (42 4 9)(1 23 16)(2 27 17)(3 31 18)(5 40 20)(6 11 33)(7 12 34)(8 13 35)(10 15 36) (14 19 41). Now, $(0\ 1)(2\ 3\ 4)t_4t_3 \sim \phi((0\ 1)(2\ 3\ 4))\phi(t_4)\phi(t_3) =$

[(1 2)(3 4 5)(6 7)(8 9 10)(11 12)(13 14 15)(16 17)(18 19 20)(21 25)

(22 27 24 26 23 28)(29 34 37 30 33 38)(31 36 39)(32 35 40)][(42 5 10)

(1 24 16)(2 28 17)(3 32 18)(4 36 19)(6 11 37)(7 12 38)(8 13 39)(9 14 40)

(15 20 41)][(42 4 9)(1 23 16)(2 27 17)(3 31 18)(5 40 20)(6 11 33)

(7 12 34)(8 13 35)(10 15 36)(14 19 41)]

= (1 28 22 17 23 2 24 26 16 27)(3 10 35 42 40)(4 15 39 18 9)(5 32 8 19 14)

(6 34 11 38 29 7 33 12 37 30)(13 20 31 41 36)(21 25).

Therefore, the permutation representation of $g = (0\ 1)(2\ 3\ 4)t_4t_3$ is
$p = \phi(g) =$ (1 28 22 17 23 2 24 26 16 27)(3 10 35 42 40)(4 15 39 18 9)(5 32 8 19 14) (6 34 11 38 29 7 33 12 37 30)(13 20 31 41 36)(21 25).

Chapter 6

$S_7 \times 3$ as a Homomorphic Image of the Progenitor $3^{*5} : S_5$

In this chapter, we investigate $S_7 \times 3$ as a homomorphic image of the progenitor $3^{*5} : S_5$. The group $S_7 \times 3$ is the direct product of three copies of the symmetric group on seven letters; its order is $7! \times 3 = 15120$. The progenitor $3^{*5} : S_5$ is a semi-direct product of 3^{*5}, a free product of five copies of the cyclic group of order 3, and S_5, the symmetric group on five letters which permutes the five symmetric generators, t_0, t_1, t_2, t_3, and t_4, (and their inverses, $t_0^2 = t_0^{-1}$, $t_1^2 = t_1^{-1}$, $t_2^2 = t_2^{-1}$, $t_3^2 = t_3^{-1}$, and $t_4^2 = t_4^{-1}$) by conjugation.

6.1 Introduction

Let $\bar{G}$ be a homomorphic image of the infinite semi-direct product, the *progenitor*, 3^{*5}: S_5. A symmetric presentation of $3^{*5} : S_5$ is given by

$$\bar{G} = \langle x, y, t \mid x^5 = y^2 = (yx)^4 = [x, y]^3 = t^3 = [t, y] = [t^x, y] = [t^{x^2}, y] = e \rangle,$$

where $[x, y]^3 = xyxyxy$, $[t, y] = tyty$, $[t^x, y] = t^x y t^x y$, $[t^{x^2}, y] = t^{x^2} y t^{x^2} y$, and e is the identity. In this case, $N \cong S_5 \cong \langle x, y \mid x^5 = y^2 = (yx)^4 = [x, y]^3 = e \rangle$, and the action of N on the five symmetric generators is given by $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (3\ 4)$, and $t \sim t_0$.

Let G denote the group $\bar{G}$ factored by the relations $(yxt)^6 = e$, $(t^{-1}t^x)^3 = e$,

$(xyx^{-1}yxt^{-1}t^x)^2 = e$, and $(x^{-2}yx^2t)^{12} = e$. That is, let

$$G = \frac{\bar{G}}{(yxt)^6, (t^{-1}t^x)^3, (xyx^{-1}yxt^{-1}t^x)^2, (x^{-2}yx^2t)^{12}}.$$

A symmetric presentation for G is given by

$$\langle x, y, t \mid x^5, y^2, (yx)^4, [x,y]^3, t^3, [t,y], [t^x, y], [t^{x^2}, y], (yxt)^6,$$
$$(t^{-1}t^x)^3, (xyx^{-1}yxt^{-1}t^x)^2, (x^{-2}yx^2t)^{12}\rangle.$$

We now consider the following relations:

$$[(0\ 1\ 2\ 3)t_0]^6 = e,$$
$$[t_0^{-1}t_1]^3 = e,$$
$$[(0\ 1\ 2)t_0^{-1}t_1]^2 = e,$$
and
$$[(0\ 1)t_0]^{12} = e.$$

According to a computer proof by [CHB96], the progenitor $3^{*5} : S_5$, factored by the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$, is isomorphic to $S_7 \times 3$. In fact, factoring the progenitor $3^{*5} : S_5$ by the relation $[(0\ 1\ 2\ 3)t_0]^6 = e$ alone suffices. We will construct $S_7 \times 3$ by hand by way of manual double coset enumeration of $G \cong \frac{3^{*5}:S_5}{[(0\ 1\ 2\ 3)t_0]^6, [t_0^{-1}t_1]^3, [(0\ 1\ 2)t_0^{-1}t_1]^2, [(0\ 1)t_0]^{12}}$ over S_5. In so doing, we will show that $S_7 \times 3$ is isomorphic to the symmetric presentation

$$\langle x, y, t \mid x^5, y^2, (yx)^4, [x,y]^3, t^3, [t,y], [t^x, y], [t^{x^2}, y], (yxt)^6,$$
$$(t^{-1}t^x)^3, (xyx^{-1}yxt^{-1}t^x)^2, (x^{-2}yx^2t)^{12}\rangle.$$

6.2 Manual Double Coset Enumeration of G Over S_5

We first determine the order of the homomorphic image, G, of the progenitor. To determine the order of the homomorphic image G, we will determine the index of $N \cong S_5$ in G. We determine the index of $N \cong S_5$ in G first by expanding the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$, and next by performing manual double coset enumeration on G over $N \cong S_5$. To begin, we expand the relations that factor the progenitor $3^{*5} : S_5$:

$$[(0\ 1\ 2\ 3)t_0]^6 = e \qquad (6.1)$$

$$[t_0^{-1}t_1]^3 = e \tag{6.2}$$

$$[(0\ 1\ 2)t_0^{-1}t_1]^2 = e \tag{6.3}$$

$$[(0\ 1)t_0]^{12} = e \tag{6.4}$$

As mentioned above, relation (6.1), $[(0\ 1\ 2\ 3)t_0]^6 = e$, is required to determine the homomorphic image, G, of the progenitor, and the other relations can be derived from relation (6.1). We expand relations (6.1), (6.2), (6.3), and (6.4) in detail below:

1. Let $\pi = (0\ 1\ 2\ 3)$.

   Then $[(0\ 1\ 2\ 3)t_0]^6 = e \Rightarrow (\pi t_0)^6 = e \Rightarrow$
   $\pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e \Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e \Rightarrow$
   $\pi t_0 \pi t_0 \pi t_0 \pi^3 \pi^{-1}\pi^{-1} t_0 \pi^2 t_0^{\pi} t_0 = e \Rightarrow \pi t_0 \pi t_0 \pi^4 \pi^{-1}\pi^{-1}\pi^{-1} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$
   $\Rightarrow \pi t_0 \pi^5 \pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1} t_0 \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
   $\pi^6 \pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1} t_0 \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow \pi^6 t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
   $(0\ 1\ 2\ 3)^6 t_0^{(0\ 1\ 2\ 3)^5} t_0^{(0\ 1\ 2\ 3)^4} t_0^{(0\ 1\ 2\ 3)^3} t_0^{(0\ 1\ 2\ 3)^2} t_0^{(0\ 1\ 2\ 3)} t_0 = e$
   $\Rightarrow (0\ 2)(1\ 3) t_0^{(0\ 1\ 2\ 3)} t_0^{e} t_0^{(0\ 3\ 2\ 1)} t_0^{(0\ 2)(1\ 3)} t_0^{(0\ 1\ 2\ 3)} t_0 = e \Rightarrow$
   $(0\ 2)(1\ 3) t_1 t_0 t_3 t_2 t_1 t_0 = e \Rightarrow (0\ 2)(1\ 3) t_1 t_0 t_3 = t_0^{-1} t_1^{-1} t_2^{-1}$.

   Thus relation (6.1) implies that $(0\ 2)(1\ 3)t_1 t_0 t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$ or, equivalently, $Nt_1t_0t_3 = Nt_0^{-1}t_1^{-1}t_2^{-1}$. That is, using our short-hand notation, $103 \sim \bar{0}\bar{1}\bar{2}$.

2. Now $[t_0^{-1}t_1]^3 = e \Rightarrow [t_0^{-1}t_1][t_0^{-1}t_1][t_0^{-1}t_1] = e \Rightarrow t_0^{-1}t_1t_0^{-1}t_1t_0^{-1}t_1 = e \Rightarrow t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}$.

   Thus relation (6.2) implies that $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}$ or, equivalently, $Nt_0^{-1}t_1t_0^{-1} = Nt_1^{-1}t_0t_1^{-1}$. That is, using our short-hand notation, $\bar{0}1\bar{0} \sim \bar{1}0\bar{1}$.

3. Let $\pi = (0\ 1\ 2)$.

   Then $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e \Rightarrow (\pi t_0^{-1}t_1)^2 = e \Rightarrow \pi t_0^{-1}t_1\pi t_0^{-1}t_1 = e$
   $\Rightarrow \pi^2\pi^{-1}t_0^{-1}t_1\pi t_0^{-1}t_1 = e \Rightarrow \pi^2(t_0^{-1}t_1)^{\pi}t_0^{-1}t_1 = e$
   $\Rightarrow \pi^2(t_0^{-1})^{\pi}t_1^{\pi}t_0^{-1}t_1 = e \Rightarrow (0\ 1\ 2)^2(t_0^{-1})^{(0\ 1\ 2)}t_1^{(0\ 1\ 2)}t_0^{-1}t_1 = e$
   $\Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0^{-1}t_1 = e \Rightarrow (0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0$.

   Thus relation (6.3) implies that $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0$ or, equivalently, $Nt_1^{-1}t_2 = Nt_1^{-1}t_0$. That is, using our short-hand notation, $\bar{1}2 \sim \bar{1}0$.

4. Let $\pi = (0\ 1)$.

Then $[(0\ 1)t_0]^{12} = e \Rightarrow (\pi t_0)^{12} = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^2\pi^{-1}t_0\pi t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^3\pi^{-1}\pi^{-1}t_0\pi^2 t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^4\pi^{-1}\pi^{-1}\pi^{-1}t_0\pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^5\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}t_0\pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^6\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}t_0\pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^7\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}t_0\pi^6 t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi t_0\pi^8\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}\pi^{-1}t_0\pi^7 t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi t_0\pi^9\pi^{-8}t_0\pi^8 t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi t_0\pi^{10}\pi^{-9}t_0\pi^9 t_0^{\pi^8} t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0\pi^{11}\pi^{-10}t_0\pi^{10} t_0^{\pi^9} t_0^{\pi^8} t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi^{12}\pi^{-11}t_0\pi^{11} t_0^{\pi^{10}} t_0^{\pi^9} t_0^{\pi^8} t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi^{12} t_0^{\pi^{11}} t_0^{\pi^{10}} t_0^{\pi^9} t_0^{\pi^8} t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e \Rightarrow$
$(0\ 1)^{12} t_0^{(0\ 1)^{11}} t_0^{(0\ 1)^{10}} t_0^{(0\ 1)^{9}} t_0^{(0\ 1)^{8}} t_0^{(0\ 1)^{7}} t_0^{(0\ 1)^{6}} t_0^{(0\ 1)^{5}} t_0^{(0\ 1)^{4}} t_0^{(0\ 1)^{3}} t_0^{(0\ 1)^{2}} t_0^{(0\ 1)} t_0$
$= e \Rightarrow e t_0^{(0\ 1)} t_0^{e} t_0^{(0\ 1)} t_0^{e} t_0^{(0\ 1)} t_0^{e} t_0^{(0\ 1)} t_0^{e} t_0^{(0\ 1)} t_0^{e} t_0^{(0\ 1)} t_0 = e$

$\Rightarrow t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0 = e$

$\Rightarrow t_1t_0t_1t_0 = t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}$.

Thus relation (6.4) implies that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ or, equivalently, $Nt_1t_0t_1t_0t_1t_0t_1t_0 = Nt_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$. That is, using our short-hand notation, $10101010 \sim \bar{1}\bar{0}\bar{1}\bar{0}$.

We now perform manual double coset enumeration of G over S_5.

1. We first note that the double coset $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$.

   Let $[*]$ denote the double coset NeN.

   The double coset $[*]$ has one distinct right coset: the identity right coset, $Ne = \{ne \mid n \in N\} = N$.

   Moreover, since $N \cong S_5$ is transitive on $\{0, 1, 2, 3, 4\}$, and since $N \cong S_5$ is also transitive on the inverses $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, N has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are two double cosets of the form NwN, where w is a word of length one given by $w = t_i^{\pm 1}$, $i \in \{0, 1, 2, 3, 4\}$: Nt_0N and $Nt_0^{-1}N$.

2. We next consider the double coset Nt_0N.

   Let [0] denote the double coset Nt_0N.

   Note that $N^{(0)} \geq N^0 = \langle (1\ 2), (1\ 2\ 3\ 4) \rangle \cong S_4$. Thus $|N^{(0)}| \geq |S_4| = 24$ and, by Lemma 1.4, $|Nt_0N| = \frac{|N|}{|N^{(0)}|} \leq \frac{120}{24} = 5$.

   Therefore, the double coset [0] has at most five distinct single cosets.

   Moreover, $N^{(0)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

   Therefore, there are at most four double cosets of the form NwN, where w is a word of length two given by $w = t_0 t_i^{\pm 1}$, $i \in \{0, 1\}$: Nt_0t_0N, Nt_0t_1N, $Nt_0t_0^{-1}N$, and $Nt_0t_1^{-1}N$.

   But, since $Nt_0t_0N = Nt_0^2N = Nt_0^{-1}N$, and since $Nt_0t_0^{-1}N = NeN = N$, we conclude that there are two distinct double cosets of the form $Nt_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$: Nt_0t_1N and $Nt_0t_1^{-1}N$.

3. We next consider the double coset $Nt_0^{-1}N$.

   Let $[\bar{0}]$ denote the double coset $Nt_0^{-1}N$.

   Note that $N^{(\bar{0})} \geq N^{\bar{0}} \langle (1\ 2), (1\ 2\ 3\ 4) \rangle \cong S_4$. Thus $\left|N^{(\bar{0})}\right| \geq |S_4| = 24$ and, by Lemma 1.4, $\left|Nt_0^{-1}N\right| = \frac{|N|}{|N^{(\bar{0})}|} \leq \frac{120}{24} = 5$.

   Therefore, the double coset $[\bar{0}]$ has at most five distinct single cosets.

   Moreover, $N^{(\bar{0})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

   Therefore, there are at most four double cosets of the form NwN, where w is a word of length two given by $w = t_0^{-1} t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0^{-1}t_0N$, $Nt_0^{-1}t_1N$, $Nt_0^{-1}t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}N$. But, since $Nt_0^{-1}t_0^{-1}N = Nt_0^{-2}N = Nt_0N$, and since $Nt_0^{-1}t_0N = NeN = N$, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$: $Nt_0^{-1}t_1N$ and $Nt_0^{-1}t_1^{-1}N$.

4. We next consider the double coset $Nt_0t_1^{-1}N$.

Let $[0\bar{1}]$ denote the double coset $Nt_0t_1^{-1}N$.

Now, by relation (6.3) and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2$ $= t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0$ and, by right multiplication, $(0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} =$ $t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}$, and finally by conjugation, $(0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow$ $[(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1}$. Furthermore, $[t_0t_1^{-1}]^{(2\ 3)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(2\ 3)} \Rightarrow t_0t_1^{-1} = (0\ 1\ 3)t_0t_3^{-1}$ and $[t_0t_1^{-1}]^{(2\ 4)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(2\ 4)} \Rightarrow t_0t_1^{-1} = (0\ 1\ 4)t_0t_4^{-1}$.

Therefore, by relation (6.3), $t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} = (0\ 1\ 3)t_0t_3^{-1} = (0\ 1\ 4)t_0t_4^{-1}$.

That is, using our short-hand notation, we have

$$0\bar{1} \sim 0\bar{2} \sim 0\bar{3} \sim 0\bar{4}$$

By conjugating the above relationship, we also have that

$$1\bar{0} \sim 1\bar{2} \sim 1\bar{3} \sim 1\bar{4}, \qquad 2\bar{0} \sim 2\bar{1} \sim 2\bar{3} \sim 2\bar{4},$$

$$3\bar{0} \sim 3\bar{1} \sim 3\bar{2} \sim 3\bar{4}, \qquad 4\bar{0} \sim 4\bar{1} \sim 4\bar{2} \sim 4\bar{3}$$

Since each of the twenty single cosets has four names, the double coset $[0\bar{1}]$ must have at most five distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1})} \geq N^{0\bar{1}} = \langle(2\ 3)(3\ 4)\rangle \cong S_3$. In fact, by relation (6.3), $N(t_0t_1^{-1})^{(1\ 2)} = Nt_0t_2^{-1} = Nt_0t_1^{-1}$ implies that $(1\ 2) \in N^{(0\bar{1})}$, and $N(t_0t_1^{-1})^{(1\ 3)} =$ $Nt_0t_3^{-1} = Nt_0t_1^{-1}$ implies that $(1\ 3) \in N^{(0\bar{1})}$, and $N(t_0t_1^{-1})^{(1\ 4)} = Nt_0t_4^{-1} = Nt_0t_1^{-1}$ implies that $(1\ 4) \in N^{(0\bar{1})}$. Therefore, $(1\ 2), (1\ 3), (1\ 4) \in N^{(0\bar{1})}$, and so $N^{(0\bar{1})} \geq$ $\langle(1\ 2), (1\ 3), (1\ 4)\rangle \cong S_4$. That is, $\left|N^{(0\bar{1})}\right| \geq |S_4| = 24$. Now, by Lemma 1.4, $\left|Nt_0t_1^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1})}|} \leq \frac{120}{24} = 5$.

Therefore, as we concluded earlier, the double coset $[0\bar{1}]$ has at most five distinct single cosets.

Now, $N^{(0\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1\}$: $Nt_0t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_1N$, $Nt_0t_1^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_1^{-1}N$.

But, since $Nt_0t_1^{-1}t_1N = Nt_0eN = Nt_0N$, and since $Nt_0t_1^{-1}t_1^{-1}N = Nt_0t_1^{-2}N = Nt_0t_1N$, we conclude that there are two distinct double cosets of the form $Nt_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3,4\}$: $Nt_0t_1^{-1}t_0N$ and $Nt_0t_1^{-1}t_0^{-1}N$.

5. We next consider the double coset Nt_0t_1N.

   Let [01] denote the double coset Nt_0t_1N.

   Note that the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$ do not apply to the single cosets in the double coset Nt_0t_1N; therefore, each of the twenty single cosets has one name.

   Since each of the twenty single cosets has one name, the double coset [01] must have at most twenty distinct single cosets.

   An alternative approach for determining the order of the double coset is as follows: We note that $N^{(01)} \geq N^{01} \cong S_3 = \langle (2\ 3), (2\ 4) \rangle \cong S_3$. Therefore, $|N^{(01)}| \geq |S_3| = 6$. Now, by Lemma 1.4, $|Nt_0t_1N| = \frac{|N|}{|N^{(01)}|} \leq \frac{120}{6} = 20$.

   Therefore, as we concluded earlier, the double coset [01] has at most twenty distinct single cosets.

   Now, $N^{(01)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1\}$, $\{2,3,4\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}, \bar{4}\}$.

   Therefore, there are at most six double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1t_i^{\pm 1}$, $i \in \{0,1,2\}$: $Nt_0t_1t_0N$, $Nt_0t_1t_1N$, $Nt_0t_1t_2N$, $Nt_0t_1t_0^{-1}N$, $Nt_0t_1t_1^{-1}N$, and $Nt_0t_1t_2^{-1}N$.

   But note that $Nt_0t_1t_1^{-1}N = Nt_0eN = Nt_0N$ and $Nt_0t_1t_1N = Nt_0t_1^2N = Nt_0t_1^{-1}N$.

   Moreover, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0$
   $\Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0$
   $\Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0$
   $\Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e$
   $\Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(1\ 2)} = [t_2t_1^{-1}]^{(1\ 2)} \Rightarrow (0\ 1\ 2)t_1t_0^{-1} = t_1t_2^{-1}$
   $\Rightarrow t_0(0\ 1\ 2)t_1t_0^{-1} = t_0t_1t_2^{-1} \Rightarrow (0\ 1\ 2)(0\ 2\ 1)t_0(0\ 1\ 2)t_1t_0^{-1} = t_0t_1t_2^{-1}$

$\Rightarrow (0\ 1\ 2)t_0^{(0\ 1\ 2)}t_1t_0^{-1} = t_0t_1t_2^{-1} \Rightarrow (0\ 1\ 2)t_1t_1t_0^{-1} = t_0t_1t_2^{-1} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1} = t_0t_1t_2^{-1}$ implies that $Nt_0t_1t_2^{-1} = Nt_1^{-1}t_0^{-1}$, and therefore $Nt_0t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}N$. That is, $[\bar{0}\bar{1}] = [01\bar{2}]$.

Since $Nt_0t_1t_1^{-1}N = Nt_0N$, $Nt_0t_1t_1N = Nt_0t_1^{-1}N$, and $Nt_0t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}N$, we conclude that there are three distinct double cosets of the form $Nt_0t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$: $Nt_0t_1t_0N$, $Nt_0t_1t_0^{-1}N$, and $Nt_0t_1t_2N$.

6. We next consider the double coset $Nt_0^{-1}t_1N$.

   Let $[\bar{0}1]$ denote the double coset $Nt_0^{-1}t_1N$.

   Now, by relation (6.3), and by conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0$ $\Rightarrow [(0\ 2\ 1)t_1^{-1}t_2]^{(0\ 1)} = [t_1^{-1}t_0]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_0^{-1}t_2 = t_0^{-1}t_1$ and, by conjugation with elements of $N \cong S_5$, $[(0\ 1\ 2)t_0^{-1}t_2]^{(2\ 3)} = [t_0^{-1}t_1]^{(2\ 3)} \Rightarrow (0\ 1\ 3)t_0^{-1}t_3 = t_0^{-1}t_1$ and $[(0\ 1\ 2)t_0^{-1}t_2]^{(2\ 4)} = [t_0^{-1}t_1]^{(2\ 4)} \Rightarrow (0\ 1\ 4)t_0^{-1}t_4 = t_0^{-1}t_1$.

   Thus, by relation (6.3), $(0\ 1\ 2)t_0^{-1}t_2 = t_0^{-1}t_1 = (0\ 1\ 3)t_0^{-1}t_3 = (0\ 1\ 4)t_0^{-1}t_4$.

   That is, using our short-hand notation, we have

$$\bar{0}1 \sim \bar{0}2 \sim \bar{0}3 \sim \bar{0}4$$

   By conjugating the above relationship, we also have that

$$\bar{0}1 \sim \bar{0}2 \sim \bar{0}3 \sim \bar{0}4, \qquad \bar{1}0 \sim \bar{1}2 \sim \bar{1}3 \sim \bar{1}4,$$

$$\bar{2}0 \sim \bar{2}1 \sim \bar{2}3 \sim \bar{2}4, \qquad \bar{3}0 \sim \bar{3}1 \sim \bar{3}2 \sim \bar{3}4,$$

$$\bar{4}0 \sim \bar{4}1 \sim \bar{4}2 \sim \bar{4}3$$

   Since each of the twenty single cosets has four names, the double coset $[\bar{0}1]$ must have at most five distinct single cosets.

   Now, $N^{(\bar{0}1)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

   Therefore, there are at most four double cosets of the form NwN, where w is a word of length three given by $w = t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0^{-1}t_1t_0N$, $Nt_0^{-1}t_1t_1N$, $Nt_0^{-1}t_1t_0^{-1}N$, and $Nt_0^{-1}t_1t_1^{-1}N$.

But note that $Nt_0^{-1}t_1t_1N = Nt_0^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}eN = Nt_0^{-1}N$. Since $Nt_0^{-1}t_1t_1N = Nt_0^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}eN = Nt_0^{-1}N$, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3,4\}$: $Nt_0^{-1}t_1t_0N$ and $Nt_0^{-1}t_1t_0^{-1}N$.

7. We next consider the double coset $Nt_0^{-1}t_1^{-1}N$.

   Let $[\bar{0}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1^{-1}N$.

   Note that the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$ do not apply to the single cosets in the double coset $Nt_0^{-1}t_1^{-1}N$; therefore, each of the twenty single cosets has one name.

   Since each of the twenty single cosets has one name, the double coset $[\bar{0}\bar{1}]$ must have at most twenty distinct single cosets.

   An alternative approach for determining the order of the double coset is as follows: We note that $N^{(\bar{0}\bar{1})} \geq N^{\bar{0}\bar{1}} = \langle(2\ 3),(2\ 4)\rangle \cong S_3$. Therefore, $\left|N^{(\bar{0}\bar{1})}\right| \geq |S_3| = 6$. Now, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1})}|} \leq \frac{120}{6} = 20$.

   Therefore, as we concluded earlier, the double coset $[\bar{0}\bar{1}]$ has at most twenty distinct single cosets.

   Now, $N^{(\bar{0}\bar{1})}$ has six orbits on $T = \{t_0,t_1,t_2,t_3,t_4\}$: $\{0\}$, $\{1\}$, $\{2,3,4\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3},\bar{4}\}$.

   Therefore, there are at most six double cosets of the form NwN, where w is a word of length three given by $w = t_0^{-1}t_1^{-1}t_i^{\pm1}$, $i \in \{0,1,2\}$: $Nt_0^{-1}t_1^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

   But note that $Nt_0^{-1}t_1^{-1}t_1N = Nt_0^{-1}eN = Nt_0^{-1}N$ and
   $Nt_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-2}N = Nt_0t_1N$.

   Moreover, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_1^{-1}t_0$
   $\Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1{-}1(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_1^{-1}t_1^{-1}t_0$
   $\Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2 = t_1^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2 = t_1t_0$ implies that $Nt_0^{-1}t_1^{-1}t_2 = Nt_1t_0$, and therefore, $Nt_0^{-1}t_1^{-1}t_2N = Nt_0t_1N$. That is, $[01] = [\bar{0}\bar{1}2]$.

   Since $Nt_0^{-1}t_1^{-1}t_1N = Nt_0^{-1}eN = Nt_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-2}N = Nt_0^{-1}t_1N$, and $Nt_0^{-1}t_1^{-1}t_2N = Nt_1t_0N$, we conclude that there are three distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3,4\}$: $Nt_0^{-1}t_1^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

8. We next consider the double coset $Nt_0t_1^{-1}t_0N$.

   Let $[0\bar{1}0]$ denote the double coset $Nt_0t_1^{-1}t_0N$.

   Now, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 = (0\ 3\ 1)t_1^{-1}t_3 = (0\ 4\ 1)t_1^{-1}t_4$ and, by left multiplication, $t_0(0\ 2\ 1)t_1^{-1}t_2 = t_0t_1^{-1}t_0 = t_0(0\ 3\ 1)t_1^{-1}t_3 = t_0(0\ 4\ 1)t_1^{-1}t_4 \Rightarrow$ $(0\ 2\ 1)(0\ 1\ 2)t_0(0\ 2\ 1)t_1^{-1}t_2 = t_0t_1^{-1}t_0 = (0\ 3\ 1)(0\ 1\ 3)t_0(0\ 3\ 1)t_1^{-1}t_3$ $= (0\ 4\ 1)(0\ 1\ 4)t_0(0\ 4\ 1)t_1^{-1}t_4 \Rightarrow t_0^{(0\ 2\ 1)}t_1^{-1}t_2 = t_0t_1^{-1}t_0 = t_0^{(0\ 3\ 1)}t_1^{-1}t_3 = t_0^{(0\ 4\ 1)}t_1^{-1}t_4$ $\Rightarrow (0\ 2\ 1)t_2t_1^{-1}t_2 = t_0t_1^{-1}t_0 = (0\ 3\ 1)t_3t_1^{-1}t_3 = (0\ 4\ 1)t_4t_1^{-1}t_4$. Similarly, by conjugation of these relations, $(0\ 1\ 2)t_2t_0^{-1}t_2 = t_1t_0^{-1}t_1 = (0\ 1\ 3)t_3t_0^{-1}t_3 = (0\ 1\ 4)t_4t_0^{-1}t_4$ and $(0\ 1\ 2)t_1t_2^{-1}t_1 = t_0t_2^{-1}t_0 = (0\ 3\ 2)t_3t_2^{-1}t_3 = (0\ 4\ 2)t_4t_2^{-1}t_4$ and $(0\ 2\ 3)t_2t_3^{-1}t_2 = t_0t_3^{-1}t_0 = (0\ 1\ 3)t_1t_3^{-1}t_1 = (0\ 4\ 3)t_4t_3^{-1}t_4$ and $(0\ 2\ 4)t_2t_4^{-1}t_2 = t_0t_4^{-1}t_0 = (0\ 3\ 4)t_3t_4^{-1}t_3$ $= (0\ 1\ 4)t_1t_4^{-1}t_1$. Furthermore, by relation (6.2), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0$ and so $[t_0t_1^{-1}t_0]^{(0\ 2)} = [t_1t_0^{-1}t_1]^{(0\ 2)} \Rightarrow t_2t_1^{-1}t_2 = t_1t_2^{-1}t_1$ and $[t_0t_1^{-1}t_0]^{(0\ 3)} = [t_1t_0^{-1}t_1]^{(0\ 3)} \Rightarrow t_3t_1^{-1}t_3 = t_1t_3^{-1}t_1$ and $[t_0t_1^{-1}t_0]^{(0\ 4)} = [t_1t_0^{-1}t_1]^{(0\ 4)} \Rightarrow t_4t_1^{-1}t_4 = t_1t_4^{-1}t_1$.

   These relations imply that:

$$0\bar{1}0 \sim 0\bar{2}0 \sim 0\bar{3}0 \sim 0\bar{4}0 \sim 1\bar{0}1 \sim 1\bar{2}1 \sim 1\bar{3}1 \sim 1\bar{4}1 \sim 2\bar{0}2 \sim 2\bar{1}2 \sim$$

$$2\bar{3}2 \sim 2\bar{4}2 \sim 3\bar{0}3 \sim 3\bar{1}3 \sim 3\bar{2}3 \sim 3\bar{4}3 \sim 4\bar{0}4 \sim 4\bar{1}4 \sim 4\bar{2}4 \sim 4\bar{3}4$$

   Since each of the twenty single cosets has twenty names, the double coset $[0\bar{1}0]$ must have one distinct single coset.

   Now, $N^{(0\bar{1}0)}$ has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0,1,2,3,4\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$.

   Therefore, there are at most two double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1^{-1}t_0t_i^{\pm1}$, $i = 0$: $Nt_0t_1^{-1}t_0t_0N$ and $Nt_0t_1^{-1}t_0t_0^{-1}N$.

   But note that $Nt_0t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}eN = Nt_0t_1^{-1}N$ and note further that $Nt_0t_1^{-1}t_0t_0N = Nt_0t_1^{-1}t_0^2N = Nt_0t_1^{-1}t_0^{-1}N$.

   Since $Nt_0t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}eN = Nt_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_0t_0N = Nt_0t_1^{-1}t_0^2N = Nt_0t_1^{-1}t_0^{-1}N$, we need not consider additional double cosets of the form $Nt_0t_1^{-1}t_0t_i^{\pm1}N$, where $i \in \{0,1,2,3,4\}$.

9. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}N$.

Let $[0\bar{1}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}N$.

Recall that in step 4 of our manual double coset enumeration, we determined, by relation (6.3), that $t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} = (0\ 1\ 3)t_0t_3^{-1} = (0\ 1\ 4)t_0t_4^{-1}$. Now, by right multiplication, we find that $t_0t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_2^{-1}t_0^{-1} = (0\ 1\ 3)t_0t_3^{-1}t_0^{-1} = (0\ 1\ 4)t_0t_4^{-1}t_0^{-1}$.

These relations imply that:

$$0\bar{1}\bar{0} \sim 0\bar{2}\bar{0} \sim 0\bar{3}\bar{0} \sim 0\bar{4}\bar{0}$$

By conjugating the relationship above, we find also that

$$1\bar{0}\bar{1} \sim 1\bar{2}\bar{1} \sim 1\bar{3}\bar{1} \sim 1\bar{4}\bar{1}, \qquad 2\bar{1}\bar{2} \sim 2\bar{0}\bar{2} \sim 2\bar{3}\bar{2} \sim 2\bar{4}\bar{2},$$

$$3\bar{1}\bar{3} \sim 3\bar{2}\bar{3} \sim 3\bar{0}\bar{3} \sim 3\bar{4}\bar{3}, \qquad 4\bar{1}\bar{4} \sim 4\bar{2}\bar{4} \sim 4\bar{3}\bar{4} \sim 4\bar{0}\bar{4}$$

Since each of the twenty single cosets has four names, the double coset $[0\bar{1}\bar{0}]$ must have at most five distinct single cosets.

Now, $N^{(0\bar{1}\bar{0})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0t_1^{-1}t_0^{-1}t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_1N$, and $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}(t_0^{-1})^2N = Nt_0t_1^{-1}t_0N$ and note further that $Nt_0t_1^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}eN = Nt_0t_1^{-1}N$.

Moreover, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow [t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$. Further, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow$

$t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0$ $\Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$. Therefore, $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ $= (0\ 1\ 2)t_0t_1t_2$ and this implies, in turn, that $Nt_1t_0^{-1}t_1^{-1}t_0 = Nt_0t_1t_2$. Therefore, $Nt_0t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2N$. That is, $[012] = [0\bar{1}\bar{0}1]$.

Since $Nt_0t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_0N$ and $Nt_0t_1^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2N$, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3,4\}$: $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$.

10. We next consider the double coset $Nt_0t_1t_0N$.

Let $[010]$ denote the double coset $Nt_0t_1t_0N$.

By relation (6.3) and by conujugation and right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2$ $= t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2$ $\Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow [(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1}]^{(0\ 1)} = [t_1t_0t_2]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = t_0t_1t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_2^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2t_2^{-1} \Rightarrow t_1^{-1}t_0^{-1}t_2$ $= (0\ 2\ 1)t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_2t_0 = (0\ 2\ 1)t_0t_1t_0 \Rightarrow t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$. Also, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2 = t_1^{-1}t_0^{-1}t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = t_1^{-1}t_0^{-1}t_2t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1}$. Now, $t_1^{-1}t_0^{-1}t_2t_0t_1^{-1}$ $= (0\ 2\ 1)t_0t_1t_0t_1^{-1}$ and $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1}$ imply that $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$, and so $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1} \Rightarrow$ $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1}t_1 = (0\ 2\ 1)t_0t_1t_0t_1^{-1}t_1 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = (0\ 2\ 1)t_0t_1t_0 \Rightarrow$ $(0\ 1\ 2)(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = (0\ 1\ 2)(0\ 2\ 1)t_0t_1t_0 \Rightarrow t_1^{-1}t_2t_0t_2t_0 = t_0t_1t_0 \Rightarrow t_1t_1^{-1}t_2t_0t_2t_0$ $= t_1t_0t_1t_0 \Rightarrow t_2t_0t_2t_0 = t_1t_0t_1t_0 \Rightarrow t_2t_0t_2t_0t_0^{-1} = t_1t_0t_1t_0t_0^{-1} \Rightarrow t_2t_0t_2 = t_1t_0t_1$. Finally, by conjugation, $[t_2t_0t_2]^{(2\ 3)} = [t_1t_0t_1]^{(2\ 3)} \Rightarrow t_3t_0t_3 = t_1t_0t_1$ and, also by conjugation, $[t_2t_0t_2]^{(2\ 4)} = [t_1t_0t_1]^{(2\ 4)} \Rightarrow t_4t_0t_4 = t_1t_0t_1 \Rightarrow$ and so $t_2t_0t_2 = t_1t_0t_1$ and $t_3t_0t_3 = t_1t_0t_1$ and $t_4t_0t_4 = t_1t_0t_1$ imply that $t_2t_0t_2 = t_1t_0t_1 = t_3t_0t_3 = t_4t_0t_4$.

These relations imply that:

$$101 \sim 202 \sim 303 \sim 404$$

Similarly, by conjugating the above relationship, we find that:

$$010 \sim 212 \sim 313 \sim 414, \qquad 020 \sim 121 \sim 323 \sim 424,$$

$$030 \sim 131 \sim 232 \sim 434, \qquad 040 \sim 141 \sim 242 \sim 343$$

Since each of the twenty single cosets has four names, the double coset [010] must have at most five distinct single cosets.

Now, $N^{(010)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1t_0t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0t_1t_0t_0N$, $Nt_0t_1t_0t_0^{-1}N$, $Nt_0t_1t_0t_1N$, and $Nt_0t_1t_0t_1^{-1}N$.

But note that $Nt_0t_1t_0t_0N = Nt_0t_1t_0^2N = Nt_0t_1t_0^{-1}N$ and note further that $Nt_0t_1t_0t_0^{-1}N = Nt_0t_1eN = Nt_0t_1N$.

Since $Nt_0t_1t_0t_0N = Nt_0t_1t_0^{-1}N$ and $Nt_0t_1t_0t_0^{-1}N = Nt_0t_1N$, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$: $Nt_0t_1t_0t_1N$ and $Nt_0t_1t_0t_1^{-1}N$.

11. We next consider the double coset $Nt_0t_1t_0^{-1}N$.

    Let $[01\bar{0}]$ denote the double coset $Nt_0t_1t_0^{-1}N$.

    Recall that in step 4 of our manual double coset enumeration, we determined, by relation (6.3), that $t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} = (0\ 1\ 3)t_0t_3^{-1} = (0\ 1\ 4)t_0t_4^{-1}$. Now, by conjugating this relationship, we find that $[t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} = [(0\ 1\ 3)t_0t_3^{-1}]^{(0\ 1)} = [(0\ 1\ 4)t_0t_4^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} = (0\ 3\ 1)t_1t_3^{-1} = (0\ 4\ 1)t_1t_4^{-1}$ and, by left multiplication, $t_0t_1t_0^{-1} = t_0(0\ 2\ 1)t_1t_2^{-1} = t_0(0\ 3\ 1)t_1t_3^{-1} = t_0(0\ 4\ 1)t_1t_4^{-1} \Rightarrow t_0t_1t_0^{-1} = (0\ 2\ 1)(0\ 1\ 2)t_0(0\ 2\ 1)t_1t_2^{-1} = (0\ 3\ 1)(0\ 1\ 3)t_0(0\ 3\ 1)t_1t_3^{-1} = (0\ 4\ 1)(0\ 1\ 4)t_0(0\ 4\ 1)t_1t_4^{-1} \Rightarrow t_0t_1t_0^{-1} = (0\ 2\ 1)t_0^{(0\ 2\ 1)}t_1t_2^{-1} = (0\ 3\ 1)t_0^{(0\ 3\ 1)}t_1t_3^{-1} = (0\ 4\ 1)t_0^{(0\ 4\ 1)}t_1t_4^{-1} \Rightarrow t_0t_1t_0^{-1} = (0\ 2\ 1)t_2t_1t_2^{-1} = (0\ 3\ 1)t_3t_1t_3^{-1} = (0\ 4\ 1)t_4t_1t_4^{-1}$. These relations imply that:

$$01\bar{0} \sim 21\bar{2} \sim 31\bar{3} \sim 41\bar{4}$$

Similarly, by conjugating the above relationship, we find that

$$10\bar{1} \sim 20\bar{2} \sim 30\bar{3} \sim 40\bar{4}, \qquad 02\bar{0} \sim 12\bar{1} \sim 32\bar{3} \sim 42\bar{4},$$

$$03\bar{0} \sim 13\bar{1} \sim 23\bar{2} \sim 43\bar{4}, \qquad 04\bar{0} \sim 14\bar{1} \sim 24\bar{2} \sim 34\bar{3}$$

Since each of the twenty single cosets has four names, the double coset $[01\bar{0}]$ must have at most five distinct single cosets.

Now, $N^{(01\bar{0})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1,2,3,4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1\}$: $Nt_0t_1t_0^{-1}t_0N$, $Nt_0t_1t_0^{-1}t_0^{-1}N$, $Nt_0t_1t_0^{-1}t_1N$, and $Nt_0t_1t_0^{-1}t_1^{-1}N$.

But note that $Nt_0t_1t_0^{-1}t_0^{-1}N = Nt_0t_1(t_0^{-1})^2N = Nt_0t_1t_0N$ and note further that $Nt_0t_1t_0^{-1}t_0N = Nt_0t_1eN = Nt_0t_1N$.

Moreover, by relation (6.2), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0^{-1}t_0^{-1}t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0t_1^{-1} \Rightarrow t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0t_1^{-1} \Rightarrow t_0t_1t_0^{-1}t_1 = t_0^{-1}t_1^{-1}t_0t_1^{-1}t_1 \Rightarrow t_0t_1t_0^{-1}t_1 = t_0^{-1}t_1^{-1}t_0$ implies that $Nt_0t_1t_0^{-1}t_1 = Nt_0^{-1}t_1^{-1}t_0$. Therefore, $Nt_0t_1t_0^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_0N$. That is, $[\bar{0}\bar{1}0] = [01\bar{0}1]$.

Since $Nt_0t_1t_0^{-1}t_0^{-1}N = Nt_0t_1t_0N$ and $Nt_0t_1t_0^{-1}t_0N = Nt_0t_1N$ and $Nt_0t_1t_0^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_0N$, we conclude that there is one distinct double coset of the form $Nt_0t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3,4\}$: $Nt_0t_1t_0^{-1}t_1^{-1}N$.

12. We next consider the double coset $Nt_0t_1t_2N$.

Let $[012]$ denote the double coset $Nt_0t_1t_2N$.

Note that, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2$
$\Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2$
$\Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2$ and, by conjugation, $[(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1}]^{(0\ 1)}$
$= [t_1t_0t_2]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = t_0t_1t_2$. This implies that $Nt_1^{-1}t_0^{-1}t_2^{-1} = Nt_0t_1t_2$ and, therefore, $Nt_0^{-1}t_1^{-1}t_2^{-1}N = Nt_0t_1t_2N$ Thus, $Nt_0t_1t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}N$. That is, $[012] = [\bar{0}\bar{1}\bar{2}]$.

Thus, note that $Nt_0t_1t_2N = \{Nt_0t_1t_2n \mid n \in N\} = \{Nn^{-1}t_0t_1t_2n \mid n \in N\} = \{N(t_0t_1t_2)^n \mid n \in N\} = \{Nt_it_jt_k \mid i,j,k \in \{0,1,2,3,4\}, i \neq j \neq k\} = \{Nt_i^{-1}t_j^{-1}t_k^{-1} \mid i,j,k \in \{0,1,2,3,4\}, i \neq j \neq k\} = \{N(t_0^{-1}t_1^{-1}t_2^{-1})^n \mid n \in N\} = \{Nn^{-1}t_0^{-1}t_1^{-1}t_2^{-1}n \mid n \in N\} = \{Nt_0^{-1}t_1^{-1}t_2^{-1}n \mid n \in N\} = Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

Now, by relation (6.3) and by right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0$ $\Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2$ $\Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$. Moreover, by relation (6.1), $(0\ 2)(1\ 3)t_1t_0t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$, and so $t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$ and $(0\ 2)(1\ 3)t_1t_0t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$ imply that $(0\ 2)(1\ 3)t_1t_0t_3 = (0\ 1\ 2)t_1t_0t_2$. Therefore, by conjugation, $[(0\ 2)(1\ 3)t_1t_0t_3]^{(0\ 1)} = [(0\ 1\ 2)t_1t_0t_2]^{(0\ 1)} \Rightarrow (1\ 2)(0\ 3)t_0t_1t_3 = (0\ 2\ 1)t_0t_1t_2$. By further conjugation, $[(1\ 2)(0\ 3)t_0t_1t_3]^{(3\ 4)} = [(0\ 2\ 1)t_0t_1t_2]^{(3\ 4)} \Rightarrow (1\ 2)(0\ 4)t_0t_1t_4 = (0\ 2\ 1)t_0t_1t_2$. Therefore, by relations (6.1) and (6.3),
$(1\ 2)(0\ 3)t_0t_1t_3 = (0\ 2\ 1)t_0t_1t_2 = (1\ 2)(0\ 4)t_0t_1t_4$.

Thererfore, in terms of our short-hand notation, these relations imply that:

$$012 \sim 013 \sim 014$$

Similarly, by conjugation, we have

$021 \sim 023 \sim 024,$	$031 \sim 032 \sim 034,$	$041 \sim 042 \sim 043,$
$102 \sim 103 \sim 104,$	$120 \sim 123 \sim 124,$	$130 \sim 132 \sim 134,$
$140 \sim 142 \sim 143,$	$201 \sim 203 \sim 204,$	$210 \sim 213 \sim 214,$
$230 \sim 231 \sim 234,$	$240 \sim 241 \sim 243,$	$301 \sim 302 \sim 304,$
$310 \sim 312 \sim 314,$	$320 \sim 321 \sim 324,$	$340 \sim 341 \sim 342,$
$401 \sim 402 \sim 403,$	$410 \sim 412 \sim 413,$	$420 \sim 421 \sim 423,$

$430 \sim 431 \sim 432, \qquad 430 \sim 431 \sim 432$

Since each of the sixty single cosets has three names, the double coset $[012] = [\bar{0}\bar{1}\bar{2}]$ must have at most twenty distinct single cosets.

Now, $N^{(012)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1\}$, $\{2, 3, 4\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1t_2t_i^{\pm1}$, $i \in \{0, 1, 2\}$: $Nt_0t_1t_2t_0N$, $Nt_0t_1t_2t_0^{-1}N$, $Nt_0t_1t_2t_1N$, $Nt_0t_1t_2t_1^{-1}N$, $Nt_0t_1t_2t_2N$, and $Nt_0t_1t_2t_2^{-1}N$.

But note that $Nt_0t_1t_2t_2^{-1}N = Nt_0t_1eN = Nt_0t_1N$ and, by relation (6.3), $Nt_0t_1t_2t_2N = Nt_0t_1t_2^2N = Nt_0t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}N$.

Moreover, by relation (6.3) and by left and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0$ $\Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0^{-1} = t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0^{-1} = t_1^{-1} \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2t_0^{-1} = t_2^{-1}t_1^{-1} \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2t_0^{-1} = t_2^{-1}t_1^{-1} \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2t_0^{-1} = t_2^{-1}t_1^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_1^{-1}t_2t_0^{-1} = t_2^{-1}t_1^{-1} \Rightarrow (0\ 2\ 1)t_1t_2t_0^{-1} = t_2^{-1}t_1^{-1} \Rightarrow t_1(0\ 2\ 1)t_1t_2t_0^{-1} = t_1t_2^{-1}t_1^{-1} \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2t_0^{-1} = t_1t_2^{-1}t_1^{-1} \Rightarrow (0\ 2\ 1)(t_1)^{(0\ 2\ 1)}t_1t_2t_0^{-1} = t_1t_2^{-1}t_1^{-1} \Rightarrow (0\ 2\ 1)t_0t_1t_2t_0^{-1} = t_1t_2^{-1}t_1^{-1}$. This implies that $Nt_0t_1t_2t_0^{-1} = Nt_1t_2^{-1}t_1^{-1}$ and, therefore, $Nt_0t_1t_2t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}N$. That is, $[0\bar{1}\bar{0}] = [012\bar{0}]$.

Similarly, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow [t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$, and also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$. Now, $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$. This implies that $Nt_1t_0^{-1}t_1^{-1}t_0^{-1} = Nt_0t_1t_2t_0$ and, therefore, $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0N$. That is, $[0\bar{1}\bar{0}\bar{1}] = [0120]$.

Similarly, by relation (6.3) and by left and right multiplication and conjugation, $t_1t_0^{-1}t_1^{-1}t_0 = (0\,2\,1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\,2\,1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 =$

$(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0t_1^{-1} = (0\ 1\ 2)t_0t_1t_2t_1^{-1}$. Now, by relation (6.2), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0t_1^{-1} \Rightarrow t_1t_0t_1t_0^{-1} = t_1t_0^{-1}t_1^{-1}t_0t_1^{-1}$. Therefore, $t_1t_0^{-1}t_1^{-1}t_0t_1^{-1} = (0\ 1\ 2)t_0t_1t_2t_1^{-1}$ and $t_1t_0t_1t_0^{-1} = t_1t_0^{-1}t_1^{-1}t_0t_1^{-1}$ imply that $t_1t_0t_1t_0^{-1} = t_1t_0^{-1}t_1^{-1}t_0t_1^{-1} = (0\ 1\ 2)t_0t_1t_2t_1^{-1}$ and this implies, in turn, that $Nt_1t_0t_1t_0^{-1} = Nt_0t_1t_2t_1^{-1}$. Therefore, $Nt_0t_1t_0t_1^{-1}N = Nt_0t_1t_2t_1^{-1}N$. That is, $[010\bar{1}] = [012\bar{1}]$.

Finally, by relation (6.3) and by conjugation and right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow [(0\ 2\ 1)t_1^{-1}t_2]^{(0\ 1)} = [t_1^{-1}t_0 rbrack^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_0^{-1}t_2 = t_0^{-1}t_1 \Rightarrow t_0^{-1}(0\ 1\ 2)t_0^{-1}t_2 = t_0^{-1}t_0^{-1}t_1 \Rightarrow (0\ 1\ 2)(0\ 2\ 1)t_0^{-1}(0\ 1\ 2)t_0^{-1}t_2 = t_0t_1$ $\Rightarrow (0\ 1\ 2)(t_0^{-1})^{(0\ 1\ 2)}t_0^{-1}t_2 = t_0t_1 \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2 = t_0t_1 \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2t_2 = t_0t_1t_2 \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = t_0t_1t_2 \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}t_1 = t_0t_1t_2t_1$, and, also by relation (6.3) and conjugation and left and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)}$ $\Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_2t_0t_1^{-1} = t_2(0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_2t_0t_1^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_2(0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_2t_0t_1^{-1} = (0\ 1\ 2)t_2^{(0\ 1\ 2)}t_0t_2^{-1} \Rightarrow t_2t_0t_1^{-1} = (0\ 1\ 2)t_0t_0t_2^{-1} \Rightarrow t_2t_0t_1^{-1} = (0\ 1\ 2)t_0^{-1}t_2^{-1} \Rightarrow t_2t_0t_1^{-1}t_1 = (0\ 1\ 2)t_0^{-1}t_2^{-1}t_1 \Rightarrow t_2t_0 = (0\ 1\ 2)t_0^{-1}t_2^{-1}t_1 \Rightarrow t_0^{-1}t_2t_0 = t_0^{-1}(0\ 1\ 2)t_0^{-1}t_2^{-1}t_1 \Rightarrow t_0^{-1}t_2t_0$ $= (0\ 1\ 2)(0\ 2\ 1)t_0^{-1}(0\ 1\ 2)t_0^{-1}t_2^{-1}t_1 \Rightarrow t_0^{-1}t_2t_0 = (0\ 1\ 2)(t_0^{-1})^{(0\ 1\ 2)}t_0^{-1}t_2^{-1}t_1 \Rightarrow t_0^{-1}t_2t_0 = (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}t_1$. Now, $(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}t_1 = t_0t_1t_2t_1$ and $t_0^{-1}t_2t_0 = (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}t_1$ imply that $t_0^{-1}t_2t_0 = (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}t_1 = t_0t_1t_2t_1 \Rightarrow t_0^{-1}t_2t_0 = t_0t_1t_2t_1$ which implies that $Nt_0^{-1}t_2t_0 = Nt_0t_1t_2t_1$. Therefore, $Nt_0t_1t_2t_1N$ $= Nt_0^{-1}t_1t_0N$. That is, $[\bar{0}10] = [0121]$.

Since $Nt_0t_1t_2t_2N = Nt_0t_1t_2^2N = Nt_0t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}N$ and $Nt_0t_1t_2t_2^{-1}N = Nt_0t_1N$ and $Nt_0t_1t_2t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}N$ and $Nt_0t_1t_2t_0N = Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$ and $Nt_0t_1t_2t_1N = Nt_0^{-1}t_1t_0N$ and $Nt_0t_1t_2t_1^{-1}N = Nt_0t_1t_0t_1^{-1}N$, we need not consider additional double cosets of the form $Nt_0t_1t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

13. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_0N$.

    Let $[\bar{0}\bar{1}0]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_0N$.

Now, by relation (6.3), and by left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow$ $t_0^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_0^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)(t_0^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{-1}t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0$. Note further that $[(0\ 2\ 1)t_2^{-1}t_1^{-1}t_2]^{(2\ 3)} = [t_0^{-1}t_1^{-1}t_0]^{(2\ 3)} \Rightarrow (0\ 3\ 1)t_3^{-1}t_1^{-1}t_3 = t_1^{-1}t_1^{-1}t_0$ and $[(0\ 2\ 1)t_2^{-1}t_1^{-1}t_2]^{(2\ 4)} = [t_0^{-1}t_1^{-1}t_0]^{(2\ 4)} \Rightarrow (0\ 4\ 1)t_4^{-1}t_1^{-1}t_4 = t_1^{-1}t_1^{-1}t_0$. Thus, by relation (6.3), $(0\ 2\ 1)t_2^{-1}t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 = (0\ 3\ 1)t_3^{-1}t_1^{-1}t_3 = (0\ 4\ 1)t_4^{-1}t_1^{-1}t_4$. These relations imply that:

$$\bar{0}\bar{1}0 \sim \bar{2}\bar{1}2 \sim \bar{3}\bar{1}3 \sim \bar{4}\bar{1}4$$

Similarly, by conjugation, we find that

$$\bar{1}\bar{0}1 \sim \bar{2}\bar{0}2 \sim \bar{3}\bar{0}3 \sim \bar{4}\bar{0}4, \qquad \bar{0}\bar{2}0 \sim \bar{1}\bar{2}1 \sim \bar{3}\bar{2}3 \sim \bar{4}\bar{2}4,$$

$$\bar{0}\bar{3}0 \sim \bar{1}\bar{3}1 \sim \bar{2}\bar{3}2 \sim \bar{4}\bar{3}4, \qquad \bar{0}\bar{4}0 \sim \bar{1}\bar{4}1 \sim \bar{2}\bar{4}2 \sim \bar{3}\bar{4}3$$

Since each of the twenty single cosets has four names, the double coset $[\bar{0}\bar{1}0]$ must have at most five distinct single cosets.

Now, $N^{(\bar{0}\bar{1}0)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0^{-1}t_1^{-1}t_0t_0N$, $Nt_0^{-1}t_1^{-1}t_0t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_0t_1N$, and $Nt_0^{-1}t_1^{-1}t_0t_1^{-1}N$.

But note that $Nt_0^{-1}t_1^{-1}t_0t_0N = Nt_0^{-1}t_1^{-1}t_0^2N = Nt_0^{-1}t_1^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}eN = Nt_0^{-1}t_1^{-1}N$.

Moreover, by relation (6.2), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0t_1^{-1}$ implies that $Nt_0^{-1}t_1^{-1}t_0t_1^{-1} = Nt_0t_1t_0^{-1}$. Therefore, $Nt_0^{-1}t_1^{-1}t_0t_1^{-1}N = Nt_0t_1t_0^{-1}N$. That is, $[01\bar{0}] = [\bar{0}\bar{1}0\bar{1}]$.

Similarly, by relation (6.2), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0t_1^{-1}$ $\Rightarrow t_0t_1t_0^{-1}t_1^{-1} = t_0^{-1}t_1^{-1}t_0t_1$ implies that $Nt_0^{-1}t_1^{-1}t_0t_1 = Nt_0t_1t_0^{-1}t_1^{-1}$. Therefore, $Nt_0^{-1}t_1^{-1}t_0t_1N = Nt_0t_1t_0^{-1}t_1^{-1}N$. That is, $[01\bar{0}\bar{1}] = [\bar{0}\bar{1}01]$.

Since $Nt_0^{-1}t_1^{-1}t_0t_0N = Nt_0^{-1}t_1^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_0t_1^{-1}N = Nt_0t_1t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_0t_1N = Nt_0t_1t_0^{-1}t_1^{-1}N$, we need not consider additional double cosets of the form $Nt_0^{-1}t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

14. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_0^{-1}N$.

Let $[\bar{0}\bar{1}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_0^{-1}N$.

Now, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0, 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow [t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$. Also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$. Now, $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2$. Moreover, by relation (6.3), $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0t_0 = (0\ 1\ 2)t_0t_1t_2t_0 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$, and $t_1t_1t_0^{-1}t_1^{-1}t_0 = t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_2t_0t_1t_2 \Rightarrow [t_1^{-1}t_0^{-1}t_1^{-1}t_0]^{(0\ 1\ 2)} = [(0\ 1\ 2)t_2t_0t_1t_2]^{(0\ 1\ 2)} \Rightarrow t_2^{-1}t_1^{-1}t_2^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0$. Therefore, $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_2^{-1}t_1^{-1}t_2^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1$. Now, by conjugation, we have that $[t_1t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 3)} = [t_2^{-1}t_1^{-1}t_2^{-1}t_1]^{(0\ 3)} \Rightarrow t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ and $[t_1t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 4)} = [t_2^{-1}t_1^{-1}t_2^{-1}t_1]^{(0\ 4)} \Rightarrow t_1t_4^{-1}t_1^{-1}t_4^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ and so $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$. Finally, $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$ and $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ imply that $t_1t_3^{-1}t_1^{-1}t_3^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and so, by conjugation, $[t_1t_3^{-1}t_1^{-1}t_3^{-1}]^{(2\ 3)} = [(0\ 1\ 2)t_0t_1t_2t_0]^{(2\ 3)} \Rightarrow t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 3)t_0t_1t_3t_0 \Rightarrow (0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 1)(0\ 1\ 3)t_0t_1t_3t_0 \Rightarrow (0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = t_0t_1t_3t_0 \Rightarrow (0\ 3\ 2)(0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0 \Rightarrow (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0$. Now, by relation (6.3) and by right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow$

$(0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2$ $\Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2$ $\Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$. By relation (6.1), $(0\ 2)(1\ 3)t_1t_0t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$, and so $t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$ and $(0\ 2)(1\ 3)t_1t_0t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$ imply that $(0\ 2)(1\ 3)t_1t_0t_3 = (0\ 1\ 2)t_1t_0t_2$. Therefore, by conjugation and right multiplication, $[(0\ 2)(1\ 3)t_1t_0t_3]^{(0\ 1)} = [(0\ 1\ 2)t_1t_0t_2]^{(0\ 1)} \Rightarrow (1\ 2)(0\ 3)t_0t_1t_3 = (0\ 2\ 1)t_0t_1t_2 \Rightarrow (1\ 2)(0\ 3)t_0t_1t_3t_0 = (0\ 2\ 1)t_0t_1t_2t_0 \Rightarrow (0\ 2\ 1)(1\ 2)(0\ 3)t_0t_1t_3t_0 = (0\ 2\ 1)(0\ 2\ 1)t_0t_1t_2t_0 \Rightarrow (0\ 3\ 2)t_0t_1t_3t_0 = (0\ 1\ 2)t_0t_1t_2t_0$. Therefore,
$(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0$ and $(0\ 3\ 2)t_0t_1t_3t_0 = (0\ 1\ 2)t_0t_1t_2t_0$ imply that $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$, and $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$ imply that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$. Therefore, $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = t_1t_3^{-1}t_1^{-1}t_3^{-1}t_1^{-1}$ $= t_1t_4^{-1}t_1^{-1}t_4^{-1}t_1^{-1}$. Now, $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$ and so, by conjugation, $[t_2^{-1}t_1^{-1}t_2^{-1}]^{(0\ 2)} = [t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2)} \Rightarrow t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$.
Since $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$, we have that $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}$ $= (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$. Similarly, by conjugation, $[t_2^{-1}t_1^{-1}t_2^{-1}]^{(2\ 3)} = [t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}]^{(2\ 3)} \Rightarrow t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}$. Since $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$, we have that $t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$. Finally, by conjugation, $[t_2^{-1}t_1^{-1}t_2^{-1}]^{(2\ 4)} = [t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}]^{(2\ 4)} \Rightarrow t_4^{-1}t_1^{-1}t_4^{-1}$ $= t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}$. Since $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$, we have that $t_4^{-1}t_1^{-1}t_4^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$. By further conjugation, $[(1\ 2)(0\ 3)t_0t_1t_3]^{(3\ 4)} = [(0\ 2\ 1)t_0t_1t_2]^{(3\ 4)} \Rightarrow (1\ 2)(0\ 4)t_0t_1t_4 = (0\ 2\ 1)t_0t_1t_2$. Therefore, by relations (6.1) and (6.3), $(1\ 2)(0\ 3)t_0t_1t_3 = (0\ 2\ 1)t_0t_1t_2$ That is, by relations (6.1) and (6.3), $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1} = t_3^{-1}t_1^{-1}t_3^{-1} = t_4^{-1}t_1^{-1}t_4^{-1}$. These relations imply that:

$$\bar{0}\bar{1}\bar{0} \sim \bar{2}\bar{1}\bar{2} \sim \bar{3}\bar{1}\bar{3} \sim \bar{4}\bar{1}\bar{4}$$

Similarly, by conjugation, we find that

$$\bar{1}\bar{0}\bar{1} \sim \bar{2}\bar{0}\bar{2} \sim \bar{3}\bar{0}\bar{3} \sim \bar{4}\bar{0}\bar{4}, \qquad \bar{0}\bar{2}\bar{0} \sim \bar{1}\bar{2}\bar{1} \sim \bar{3}\bar{2}\bar{3} \sim \bar{4}\bar{2}\bar{4},$$

$$\bar{0}\bar{3}\bar{0} \sim \bar{1}\bar{3}\bar{1} \sim \bar{2}\bar{3}\bar{2} \sim \bar{4}\bar{3}\bar{4}, \qquad \bar{0}\bar{4}\bar{0} \sim \bar{1}\bar{4}\bar{1} \sim \bar{2}\bar{4}\bar{2} \sim \bar{3}\bar{4}\bar{3}$$

Since each of the twenty single cosets has four names, the double coset $[\bar{0}\bar{1}\bar{0}]$ must have at most five distinct single cosets.

Now, $N^{(\bar{0}\bar{1}\bar{0})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1,2,3,4\}$, $\{\bar{0}\}$, and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1\}$: $Nt_0^{-1}t_1^{-1}t_0^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_0^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N$, and $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}N$.

But note that $Nt_0^{-1}t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}(t_0^{-1})^2N = Nt_0^{-1}t_1^{-1}t_0N$ and note further that $Nt_0^{-1}t_1^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}eN = Nt_0^{-1}t_1^{-1}N$.

Moreover, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow$ $[(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow$ $[t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}$ $\Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$. Also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow$ $t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$. Now, $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_1t_0^{-1}t_1^{-1}t_0 = t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_2t_0t_1t_2$ and this implies, in turn, that $Nt_1^{-1}t_0^{-1}t_1^{-1}t_0 = Nt_2t_0t_1t_2$. Therefore, $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2t_0N$; therefore, since $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0N$ and since $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2t_0N$, we conclude that $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$. That is, $[0\bar{1}\bar{0}\bar{1}] = [\bar{0}\bar{1}\bar{0}1]$.

Since $Nt_0^{-1}t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3,4\}$: $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}N$.

15. We next consider the double coset $Nt_0^{-1}t_1t_0N$.

Let $[\bar{0}10]$ denote the double coset $Nt_0^{-1}t_1t_0N$.

Now, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 = (0\ 3\ 1)t_1^{-1}t_3 = (0\ 4\ 1)t_1^{-1}t_4$ and, by right multiplication, $(0\ 2\ 1)t_1^{-1}t_2t_1 = t_1^{-1}t_0t_1 = (0\ 3\ 1)t_1^{-1}t_3t_1 = (0\ 4\ 1)t_1^{-1}t_4t_1$. Similarly, by conjugation of these relations, $(0\ 1\ 2)t_0^{-1}t_2t_0 = t_0^{-1}t_1t_0 = (0\ 1\ 3)t_0^{-1}t_3t_0 = (0\ 1\ 4)t_0^{-1}t_4t_0$ and $(0\ 1\ 2)t_2^{-1}t_1t_2 = t_2^{-1}t_0t_2 = (0\ 3\ 2)t_2^{-1}t_3t_2 = (0\ 4\ 2)t_2^{-1}t_4t_2$ and $(0\ 2\ 3)t_3^{-1}t_2t_3 = t_3^{-1}t_0t_3 = (0\ 1\ 3)t_3^{-1}t_1t_3 = (0\ 4\ 3)t_3^{-1}t_4t_3$ and $(0\ 2\ 4)t_4^{-1}t_2t_4 = t_4^{-1}t_0t_4 = (0\ 3\ 4)t_4^{-1}t_3t_4 = (0\ 1\ 4)t_4^{-1}t_1t_4$. These relations imply that:

$$\bar{0}10 \sim \bar{0}20 \sim \bar{0}30 \sim \bar{0}40$$

Similarly, by further conjugation, we find that

$$\bar{1}01 \sim \bar{1}21 \sim \bar{1}31 \sim \bar{1}41, \qquad \bar{2}02 \sim \bar{2}12 \sim \bar{2}32 \sim \bar{2}42,$$

$$\bar{3}03 \sim \bar{3}13 \sim \bar{3}23 \sim \bar{3}43, \qquad \bar{4}04 \sim \bar{4}14 \sim \bar{4}24 \sim \bar{4}34$$

Since each of the twenty single cosets has four names, the double coset $[\bar{0}10]$ must have at most five distinct single cosets.

Now, $N^{(\bar{0}10)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1t_0t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0^{-1}t_1t_0t_0N$, $Nt_0^{-1}t_1t_0t_0^{-1}N$, $Nt_0^{-1}t_1t_0t_1N$, and $Nt_0^{-1}t_1t_0t_1^{-1}N$.

But note that $Nt_0^{-1}t_1t_0t_0N = Nt_0^{-1}t_1t_0^2N = Nt_0^{-1}t_1t_0^{-1}N$ and note further that $Nt_0^{-1}t_1t_0t_0^{-1}N = Nt_0^{-1}t_1eN = Nt_0^{-1}t_1N$.

Moreover, by relation (6.3) and by conjugation and left and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = t_1(0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0^{-1}t_1t_0t_1^{-1} = t_0^{-1}(0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0^{-1}t_1t_0t_1^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_0^{-1}(0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0^{-1}t_1t_0t_1^{-1} = (0\ 1\ 2)(t_0^{-1})^{(0\ 1\ 2)}t_2t_0t_2^{-1}$

$\Rightarrow t_0^{-1}t_1t_0t_1^{-1} = (0\ 1\ 2)t_1^{-1}t_2t_0t_2^{-1} \Rightarrow (0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 1\ 2)(0\ 1\ 2)t_1^{-1}t_2t_0t_2^{-1} \Rightarrow$ $(0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1}$ and, also by relation (6.3) and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0^{-1} \Rightarrow$ $(0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1} = t_1^{-1}t_0^{-1}t_2^{-1}$ and, also by relation (6.3) and by conjugation and left and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow$ $(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2 \Rightarrow$ $(0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow$ $(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow [(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1}]^{(0\ 1)} = [t_1t_0t_2]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}$ $= t_0t_1t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2$ and, therefore, $(0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1}$ and $(0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1} = t_1^{-1}t_0^{-1}t_2^{-1}$ and $t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2$ imply that $(0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1} =$ $t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow (0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_2$ which implies that $Nt_0^{-1}t_1t_0t_1^{-1} = Nt_0t_1t_2$ and which implies, in turn, that the double cosets $Nt_0^{-1}t_1t_0t_1^{-1}N = Nt_0t_1t_2N$. That is, $[012] = [\bar{0}10\bar{1}]$.

Similarly, by relation (6.4), $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and, by relation (6.2) and by left and right multiplication, $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0t_1^{-1}t_1$ $\Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0 \Rightarrow t_0t_0^{-1}t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_1t_0^{-1}t_1 =$ $t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1 = t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1t_1 = t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1} =$ $t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$. Now, $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and $t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ imply that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} =$ $t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ which implies that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow$ $t_1^{-1}t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_1t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0t_1t_0 = t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow$ $t_0^{-1}t_0t_1t_0t_1t_0t_1t_0 = t_0^{-1}t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1^{-1}t_1t_0t_1t_0t_1t_0 =$ $t_1^{-1}t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0 = t_1t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0t_0^{-1} = t_1t_0t_1t_0^{-1}t_0^{-1} \Rightarrow t_0t_1t_0t_1 =$ $t_1t_0t_1t_0 \Rightarrow t_0t_1t_0t_1t_1^{-1} = t_1t_0t_1t_0t_1^{-1} \Rightarrow t_0t_1t_0 = t_1t_0t_1t_0t_1^{-1} \Rightarrow t_1^{-1}t_0t_1t_0 =$ $t_1^{-1}t_1t_0t_1t_0t_1^{-1} \Rightarrow t_1^{-1}t_0t_1t_0 = t_0t_1t_0t_1^{-1}$, and so this implies that $Nt_1^{-1}t_0t_1t_0 =$ $Nt_0t_1t_0t_1^{-1}$. Therefore, $Nt_1^{-1}t_0t_1t_0N = Nt_0t_1t_0t_1^{-1}N$. That is, $[010\bar{1}] = [\bar{0}101]$.

Since $Nt_0^{-1}t_1t_0t_0N = Nt_0^{-1}t_1t_0^2N = Nt_0^{-1}t_1t_0^{-1}N$ and $Nt_0^{-1}t_1t_0t_0^{-1}N = Nt_0^{-1}t_1eN =$ $Nt_0^{-1}t_1N$ and $Nt_0^{-1}t_1t_0t_1^{-1}N = Nt_0t_1t_2N$ and $Nt_0^{-1}t_1t_0t_1N = Nt_0t_1t_0t_1^{-1}N$, we need not consider additional double cosets of the form $Nt_0^{-1}t_1t_0t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

16. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}N$.

Let $[\bar{0}1\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}N$.

Recall that in step 6 of our manual double coset enumeration, we determined, by relation (6.3), that $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 = (0\ 3\ 1)t_1^{-1}t_3 = (0\ 4\ 1)t_1^{-1}t_4$. Now, by right multiplication, we have $(0\ 2\ 1)t_1^{-1}t_2t_1^{-1} = t_1^{-1}t_0t_1^{-1} = (0\ 3\ 1)t_1^{-1}t_3t_1^{-1} = (0\ 4\ 1)t_1^{-1}t_4t_1^{-1}$. Similarly, by conjugation of these relations, $(0\ 1\ 2)t_0^{-1}t_2t_0^{-1} = t_0^{-1}t_1t_0^{-1} = (0\ 1\ 3)t_0^{-1}t_3t_0^{-1} = (0\ 1\ 4)t_0^{-1}t_4t_0^{-1}$ and $(0\ 1\ 2)t_2^{-1}t_1t_2^{-1} = t_2^{-1}t_0t_2^{-1} = (0\ 3\ 2)t_2^{-1}t_3t_2^{-1} = (0\ 4\ 2)t_2^{-1}t_4t_2^{-1}$ and $(0\ 2\ 3)t_3^{-1}t_2t_3^{-1} = t_3^{-1}t_0t_3^{-1} = (0\ 1\ 3)t_3^{-1}t_1t_3^{-1} = (0\ 4\ 3)t_3^{-1}t_4t_3^{-1}$ and $(0\ 2\ 4)t_4^{-1}t_2t_4^{-1} = t_4^{-1}t_0t_4^{-1} = (0\ 3\ 4)t_4^{-1}t_3t_4^{-1} = (0\ 1\ 4)t_4^{-1}t_1t_4^{-1}$. Finally, by relation (6.2), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}$ and $[t_0^{-1}t_1t_0^{-1}]^{(0\ 2)} = [t_1^{-1}t_0t_1^{-1}]^{(0\ 2)} \Rightarrow t_2^{-1}t_1t_2^{-1} = t_1^{-1}t_2t_1^{-1}$ and $[t_0^{-1}t_1t_0^{-1}]^{(0\ 3)} = [t_1^{-1}t_0t_1^{-1}]^{(0\ 3)} \Rightarrow t_3^{-1}t_1t_3^{-1} = t_1^{-1}t_3t_1^{-1}$ and $[t_0^{-1}t_1t_0^{-1}]^{(0\ 4)} = [t_1^{-1}t_0t_1^{-1}]^{(0\ 4)} \Rightarrow t_4^{-1}t_1t_4^{-1} = t_1^{-1}t_4t_1^{-1}$. These relations imply that:

$$\bar{0}1\bar{0} \sim \bar{0}2\bar{0} \sim \bar{0}3\bar{0} \sim \bar{0}4\bar{0} \sim \bar{1}0\bar{1} \sim \bar{1}2\bar{1} \sim \bar{1}3\bar{1} \sim \bar{1}4\bar{1} \sim \bar{2}0\bar{2} \sim \bar{2}1\bar{2} \sim$$

$$\bar{2}3\bar{2} \sim \bar{2}4\bar{2} \sim \bar{3}0\bar{3} \sim \bar{3}1\bar{3} \sim \bar{3}2\bar{3} \sim \bar{3}4\bar{3} \sim \bar{4}0\bar{4} \sim \bar{4}1\bar{4} \sim \bar{4}2\bar{4} \sim \bar{4}3\bar{4}$$

Since each of the twenty single cosets has twenty names, the double coset $[\bar{0}1\bar{0}]$ must have one distinct single coset.

Now, $N^{(\bar{0}1\bar{0})}$ has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i = 0$: $Nt_0^{-1}t_1t_0^{-1}t_0N$ and $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_0N = Nt_0^{-1}t_1eN = Nt_0^{-1}t_1N$ and note further that $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0N$. Since $Nt_0^{-1}t_1t_0^{-1}t_0N = Nt_0^{-1}t_1eN = Nt_0^{-1}t_1N$ and $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0N$, we need not consider additional double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

17. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{0}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$.

Note first that, by relation (6.3) and by left and right multiplication and conjugation, $(0\,2\,1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\,2\,1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\,2\,1)(0\,1\,2)t_2(0\,2\,1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow$

$[(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow [t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$, and also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$, and so $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and this implies, in turn, that $Nt_1t_0^{-1}t_1^{-1}t_0^{-1} = Nt_0t_1t_2t_0$ and thus $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0N$.

Similarly, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow$
$[(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow [t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$, and also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$, and so $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_1t_0^{-1}t_1^{-1}t_0 = t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_2t_0t_1t_2$ and this implies, in turn, that $Nt_1^{-1}t_0^{-1}t_1^{-1}t_0 = Nt_2t_0t_1t_2$ and thus $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2t_0N$.

Therefore, since $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0N$ and since $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2t_0N$, we have that $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$. We conclude therefore that $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2t_0N$. That is, $[0\bar{1}\bar{0}\bar{1}] = [\bar{0}\bar{1}\bar{0}1] = [0120]$.

Hence note that $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N = \{Nt_0t_1^{-1}t_0^{-1}t_1^{-1}n \mid n \in N\} = \{Nn^{-1}t_0t_1^{-1}t_0^{-1}t_1^{-1}n \mid n \in N\} = \{N(t_0t_1^{-1}t_0^{-1}t_1^{-1})^n \mid n \in N\} = \{Nt_it_j^{-1}t_i^{-1}t_j^{-1} \mid i,j \in \{0,1,2,3,4\}, i \neq$

$j\} = \{Nt_i^{-1}t_j^{-1}t_i^{-1}t_j \mid i, j \in \{0,1,2,3,4\}, i \neq j\} = \{N(t_0^{-1}t_1^{-1}t_0^{-1}t_1)^n \mid n \in N\} = \{Nn^{-1}t_0^{-1}t_1^{-1}t_0^{-1}t_1 n \mid n \in N\} = \{Nt_0^{-1}t_1^{-1}t_0^{-1}t_1 n \mid n \in N\} = Nt_0^{-1}t_1^{-1}t_0^{-1}t_1 N = \{Nt_it_jt_kt_i \mid i, j, k \in \{0,1,2,3,4\}, i \neq j \neq k\} = \{N(t_0t_1t_2t_0)^n \mid n \in N\} = \{Nn^{-1}t_0t_1t_2t_0 n \mid n \in N\} = \{Nt_0t_1t_2t_0 n \mid n \in N\} = Nt_0t_1t_2t_0N.$

Now, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow$ $[(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow [t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$, and also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$, and so $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2$. Moreover, by relation (6.3), $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0t_0 = (0\ 1\ 2)t_0t_1t_2t_0 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_1t_1t_0^{-1}t_1^{-1}t_0 = t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_2t_0t_1t_2 \Rightarrow [t_1^{-1}t_0^{-1}t_1^{-1}t_0]^{(0\ 1\ 2)} = [(0\ 1\ 2)t_2t_0t_1t_2]^{(0\ 1\ 2)} \Rightarrow t_2^{-1}t_1^{-1}t_2^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0$. Now, $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_2^{-1}t_1^{-1}t_2^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1$. By conjugation, we see that $[t_1t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 3)} = [t_2^{-1}t_1^{-1}t_2^{-1}t_1]^{(0\ 3)} \Rightarrow t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ and $[t_1t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 4)} = [t_2^{-1}t_1^{-1}t_2^{-1}t_1]^{(0\ 4)} \Rightarrow t_1t_4^{-1}t_1^{-1}t_4^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ and so $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$. Finally, $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$ and $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ imply that $t_1t_3^{-1}t_1^{-1}t_3^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$. Therefore, by conjugation, $[t_1t_3^{-1}t_1^{-1}t_3^{-1}]^{(2\ 3)} = [(0\ 1\ 2)t_0t_1t_2t_0]^{(2\ 3)} \Rightarrow t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 3)t_0t_1t_3t_0 \Rightarrow (0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 1)(0\ 1\ 3)t_0t_1t_3t_0 \Rightarrow (0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = t_0t_1t_3t_0 \Rightarrow (0\ 3\ 2)(0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0 \Rightarrow (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0$. Now, by relation (6.3) and by right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow$

$(0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2$ $\Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow$ $(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$ and, by relation (6.1), $(0\ 2)(1\ 3)t_1t_0t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$, and so $t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$ and $(0\ 2)(1\ 3)t_1t_0t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$ imply that $(0\ 2)(1\ 3)t_1t_0t_3 = (0\ 1\ 2)t_1t_0t_2$, and so, by conjugation and right multiplication, $[(0\ 2)(1\ 3)t_1t_0t_3]^{(0\ 1)} = [(0\ 1\ 2)t_1t_0t_2]^{(0\ 1)} \Rightarrow (1\ 2)(0\ 3)t_0t_1t_3 =$ $(0\ 2\ 1)t_0t_1t_2 \Rightarrow (1\ 2)(0\ 3)t_0t_1t_3t_0 = (0\ 2\ 1)t_0t_1t_2t_0 \Rightarrow (0\ 2\ 1)(1\ 2)(0\ 3)t_0t_1t_3t_0 =$ $(0\ 2\ 1)(0\ 2\ 1)t_0t_1t_2t_0 \Rightarrow (0\ 3\ 2)t_0t_1t_3t_0 = (0\ 1\ 2)t_0t_1t_2t_0$. Therefore, $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0$ and $(0\ 3\ 2)t_0t_1t_3t_0 = (0\ 1\ 2)t_0t_1t_2t_0$ imply that $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$, and $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} =$ $(0\ 1\ 2)t_0t_1t_2t_0$ and $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} =$ $t_1t_4^{-1}t_1^{-1}t_4^{-1}$ imply that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = t_1t_3^{-1}t_1^{-1}t_3^{-1}$ $= t_1t_4^{-1}t_1^{-1}t_4^{-1}$. These relations imply that:

$$1\bar{0}\bar{1}\bar{0} \sim 1\bar{2}\bar{1}\bar{2} \sim 1\bar{3}\bar{1}\bar{3} \sim 1\bar{4}\bar{1}\bar{4}$$

Similiarly, by conjugation, we find that

$$0\bar{1}\bar{0}\bar{1} \sim 0\bar{2}\bar{0}\bar{2} \sim 0\bar{3}\bar{0}\bar{3} \sim 0\bar{4}\bar{0}\bar{4}, \qquad 2\bar{1}\bar{2}\bar{1} \sim 2\bar{0}\bar{2}\bar{0} \sim 2\bar{3}\bar{2}\bar{3} \sim 2\bar{4}\bar{2}\bar{4},$$

$$3\bar{1}\bar{3}\bar{1} \sim 3\bar{2}\bar{3}\bar{2} \sim 3\bar{0}\bar{3}\bar{0} \sim 3\bar{4}\bar{3}\bar{4}, \qquad 4\bar{1}\bar{4}\bar{1} \sim 4\bar{2}\bar{4}\bar{2} \sim 4\bar{3}\bar{4}\bar{3} \sim 4\bar{0}\bar{4}\bar{0}$$

Since each of the twenty single cosets has four names, the double coset $[0\bar{1}\bar{0}\bar{1}] =$ $[\bar{0}\bar{1}\bar{0}1] = [0120]$ must have at most five distinct single cosets.

Now, $N^{(0\bar{1}\bar{0}\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_0^{-1}t_1^{-1}t_i^{\pm1}$, $i \in \{0, 1\}$: $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_1N$, and $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_1^{-1}N$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_0^{-1}N$, and note further that, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 =$ $t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 =$ $t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow$

$(0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow$ $[(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow$ $[t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}$ $\Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$, and also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$, and so $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2$. Therefore, $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_1N = Nt_0t_1t_2N$. That is, $[012] = [0\bar{1}\bar{0}\bar{1}\bar{1}] = [0\bar{1}\bar{0}1]$.

Moreover, by relation (6.3), $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0t_0 =$ $(0\ 1\ 2)t_0t_1t_2t_0 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_1t_1t_0^{-1}t_1^{-1}t_0 = t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow$ $t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_1t_2 \Rightarrow$ $t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_2t_0t_1t_2 \Rightarrow [t_1^{-1}t_0^{-1}t_1^{-1}t_0]^{(0\ 1\ 2)} = [(0\ 1\ 2)t_2t_0t_1t_2]^{(0\ 1\ 2)} \Rightarrow$ $t_2^{-1}t_1^{-1}t_2^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0$, and so $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_2^{-1}t_1^{-1}t_2^{-1}t_1 =$ $(0\ 1\ 2)t_0t_1t_2t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ which implies, in turn, that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1 =$ $(0\ 1\ 2)t_0t_1t_2t_0t_1 = t_2^{-1}t_1^{-1}t_2^{-1}t_1t_1 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0t_1 = t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$. Therefore, $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1 = t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$ implies that $Nt_1t_0^{-1}t_1^{-1}t_0^{-1}t_1 =$ $Nt_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$. Therefore, $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}N$. That is, $[\bar{0}\bar{1}\bar{0}\bar{1}] = [0\bar{1}\bar{0}\bar{1}0]$.

Similarly, by relation (6.3), $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow$ $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = (0\ 1\ 2)t_0t_1t_2t_0t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} =$ $(0\ 1\ 2)t_0t_1t_2t_0t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$. Therefore, $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$ implies that $Nt_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = Nt_2^{-1}t_1^{-1}t_2^{-1}$. Therefore, $Nt_0t_1^{-1}t_0^{-1}t_1^{1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0^{-1}N$. That is, $[\bar{0}\bar{1}\bar{0}] = [0\bar{1}\bar{0}\bar{1}\bar{0}]$.

Since $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_1N =$ $Nt_0t_1t_2N$, we need not consider additional double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

18. We next consider the double coset $Nt_0t_1t_0t_1N$.

Let [0101] denote the double coset $Nt_0t_1t_0t_1N$.

By relation (6.3) and by conujugation and right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow [(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1}]^{(0\ 1)} = [t_1t_0t_2]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = t_0t_1t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2t_2^{-1} \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2 = (0\ 2\ 1)t_0t_1 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2t_0 = (0\ 2\ 1)t_0t_1t_0 \Rightarrow t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$. Also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2 = t_1^{-1}t_0^{-1}t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = t_1^{-1}t_0^{-1}t_2t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1}$. Now, $t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$ and $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1}$ imply that $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$, and so $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1}t_1 = (0\ 2\ 1)t_0t_1t_0t_1^{-1}t_1 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = (0\ 2\ 1)t_0t_1t_0 \Rightarrow (0\ 1\ 2)(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = (0\ 1\ 2)(0\ 2\ 1)t_0t_1t_0 \Rightarrow t_1^{-1}t_2t_0t_2t_0 = t_0t_1t_0 \Rightarrow t_1t_1^{-1}t_2t_0t_2t_0 = t_1t_0t_1t_0 \Rightarrow t_2t_0t_2t_0 = t_1t_0t_1t_0$. Moreover, by conjugation, $[t_2t_0t_2t_0]^{(2\ 3)} = [t_1t_0t_1t_0]^{(2\ 3)} \Rightarrow t_3t_0t_3t_0 = t_1t_0t_1t_0$ and, also by conjugation, $[t_2t_0t_2t_0]^{(2\ 4)} = [t_1t_0t_1t_0]^{(2\ 4)} \Rightarrow t_4t_0t_4t_0 = t_1t_0t_1t_0$. Therefore, $t_2t_0t_2t_0 = t_1t_0t_1t_0$ and $t_3t_0t_3t_0 = t_1t_0t_1t_0$ and $t_4t_0t_4t_0 = t_1t_0t_1t_0$ imply that $t_2t_0t_2t_0 = t_1t_0t_1t_0 = t_3t_0t_3t_0 = t_4t_0t_4t_0$. Therefore, by conjugation, we see that $[t_2t_0t_2t_0]^{(0\ 1)} = [t_1t_0t_1t_0]^{(0\ 1)} = [t_3t_0t_3t_0]^{(0\ 1)} = [t_4t_0t_4t_0]^{(0\ 1)} \Rightarrow t_2t_1t_2t_1 = t_0t_1t_0t_1 = t_3t_1t_3t_1 = t_4t_1t_4t_1$ and $[t_2t_0t_2t_0]^{(0\ 2)} = [t_1t_0t_1t_0]^{(0\ 2)} = [t_3t_0t_3t_0]^{(0\ 2)} = [t_4t_0t_4t_0]^{(0\ 2)} \Rightarrow t_0t_2t_0t_2 = t_1t_2t_1t_2 = t_3t_2t_3t_2 = t_4t_2t_4t_2$ and $[t_2t_0t_2t_0]^{(0\ 3)} = [t_1t_0t_1t_0]^{(0\ 3)} = [t_3t_0t_3t_0]^{(0\ 3)} = [t_4t_0t_4t_0]^{(0\ 3)} \Rightarrow t_2t_3t_2t_3 = t_1t_3t_1t_3 = t_0t_3t_0t_3 = t_4t_3t_4t_3$ and $[t_2t_0t_2t_0]^{(0\ 4)} = [t_1t_0t_1t_0]^{(0\ 4)} = [t_3t_0t_3t_0]^{(0\ 4)} = [t_4t_0t_4t_0]^{(0\ 4)} \Rightarrow t_2t_4t_2t_4 = t_1t_4t_1t_4 = t_3t_4t_3t_4 = t_0t_4t_0t_4$. Finally, by relation (6.4), $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and, by relation (6.2) and by left and right multiplication, $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0t_1^{-1}t_1 \Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0 \Rightarrow t_0t_0^{-1}t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_1t_0^{-1}t_1 = t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1 = t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1t_1 = t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$. Now, $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and $t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} =$

$t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ imply that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ which implies that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1^{-1}t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ $\Rightarrow t_0t_1t_0t_1t_0t_1t_0 = t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0^{-1}t_0t_1t_0t_1t_0t_1t_0 = t_0^{-1}t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1t_0t_1t_0t_1t_0$ $= t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1^{-1}t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0 = t_1t_0t_1t_0^{-1} \Rightarrow$ $t_0t_1t_0t_1t_0t_0^{-1} = t_1t_0t_1t_0^{-1}t_0^{-1} \Rightarrow t_0t_1t_0t_1 = t_1t_0t_1t_0$. Therefore, by conjugation, we see that $[t_0t_1t_0t_1]^{(1\ 2)} = [t_1t_0t_1t_0]^{(1\ 2)} \Rightarrow t_0t_2t_0t_2 = t_2t_0t_2t_0$ and $[t_0t_1t_0t_1]^{(1\ 3)} =$ $[t_1t_0t_1t_0]^{(1\ 3)} \Rightarrow t_0t_3t_0t_3 = t_3t_0t_3t_0$ and $[t_0t_1t_0t_1]^{(1\ 4)} = [t_1t_0t_1t_0]^{(1\ 4)} \Rightarrow t_0t_4t_0t_4 =$ $t_4t_0t_4t_0$. Therefore, the relations $t_2t_0t_2t_0 = t_1t_0t_1t_0 = t_3t_0t_3t_0 = t_4t_0t_4t_0$ and $t_2t_1t_2t_1 = t_0t_1t_0t_1 = t_3t_1t_3t_1 = t_4t_1t_4t_1$ and $t_0t_2t_0t_2 = t_1t_2t_1t_2 = t_3t_2t_3t_2 = t_4t_2t_4t_2$ and $t_2t_3t_2t_3 = t_1t_3t_1t_3 = t_0t_3t_0t_3 = t_4t_3t_4t_3$ and $t_2t_4t_2t_4 = t_1t_4t_1t_4 = t_3t_4t_3t_4 =$ $t_0t_4t_0t_4$, and $t_0t_1t_0t_1 = t_1t_0t_1t_0$ and $t_0t_2t_0t_2 = t_2t_0t_2t_0$ and $t_0t_3t_0t_3 = t_3t_0t_3t_0$ and $t_0t_4t_0t_4 = t_4t_0t_4t_0$ imply, in terms of our short-hand notation, that:

$$1010 \sim 2020 \sim 3030 \sim 4040 \sim 0101 \sim 2121 \sim 3131 \sim 4141 \sim 1212 \sim 0202 \sim$$

$$3232 \sim 4242 \sim 1313 \sim 2323 \sim 0303 \sim 4343 \sim 1414 \sim 2424 \sim 3434 \sim 0404$$

Since each of the twenty single cosets has twenty names, the double coset [0101] must have at most one distinct single coset.

Now, $N^{(0101)}$ has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0,1,2,3,4\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_0t_1t_i^{\pm 1}$, $i = 0$: $Nt_0t_1t_0t_1t_0N$ and $Nt_0t_1t_0t_1t_0^{-1}N$.

But note that, by relations (6.2) and (6.4), $t_0t_1t_0t_1 = t_1t_0t_1t_0 \Rightarrow t_0t_1t_0t_1t_0 =$ $t_1t_0t_1t_0t_0 \Rightarrow t_0t_1t_0t_1t_0 = t_1t_0t_1t_0^{-1}$ implies that $Nt_0t_1t_0t_1t_0 = Nt_1t_0t_1t_0^{-1}$. Therefore, $Nt_0t_1t_0t_1t_0N = Nt_0t_1t_0t_1^{-1}N$. That is, $[010\bar{1}] = [01010]$.

Similarly, by relations (6.2) and (6.4), $t_0t_1t_0t_1 = t_1t_0t_1t_0 \Rightarrow t_0t_1t_0t_1t_0^{-1} = t_1t_0t_1t_0t_0^{-1}$ $\Rightarrow t_0t_1t_0t_1t_0^{-1} = t_1t_0t_1$ implies that $Nt_0t_1t_0t_1t_0^{-1} = Nt_1t_0t_1$. Therefore, $Nt_0t_1t_0t_1t_0^{-1}N = Nt_0t_1t_0N$. That is, $[010] = [0101\bar{0}]$.

Since $Nt_0t_1t_0t_1t_0N = Nt_0t_1t_0t_1^{-1}N$ and $Nt_0t_1t_0t_1t_0^{-1}N = Nt_0t_1t_0N$, we need not consider additional double cosets of the form $Nt_0t_1t_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3,4\}$.

19. We next consider the double coset $Nt_0t_1t_0^{-1}t_1^{-1}N$.

Let $[01\bar{0}\bar{1}]$ denote the double coset $Nt_0t_1t_0^{-1}t_1^{-1}N$.

Note that by relation (6.2), $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1^{-1}t_0t_1 \Rightarrow t_0t_1t_0^{-1}t_1^{-1} = t_0^{-1}t_1^{-1}t_0t_1$ implies that $Nt_0t_1t_0^{-1}t_1^{-1} = Nt_0^{-1}t_1^{-1}t_0t_1$ which implies that $Nt_0t_1t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_0t_1N$. Therefore, $Nt_0t_1t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_0t_1N$. That is, $[01\bar{0}\bar{1}] = [\bar{0}\bar{1}01]$.

Hence note that $Nt_0t_1t_0^{-1}t_1^{-1}N = \{Nt_0t_1t_0^{-1}t_1^{-1}n \mid n \in N\} = \{Nn^{-1}t_0t_1t_0^{-1}t_1^{-1}n \mid n \in N\} = \{N(t_0t_1t_0^{-1}t_1^{-1})^n \mid n \in N\} = \{Nt_it_jt_i^{-1}t_j^{-1} \mid i,j \in \{0,1,2,3,4\}, i \neq j\} = \{Nt_i^{-1}t_j^{-1}t_it_j \mid i,j \in \{0,1,2,3,4\}, i \neq j\} = \{N(t_0^{-1}t_1^{-1}t_0t_1)^n \mid n \in N\} = \{Nn^{-1}t_0^{-1}t_1^{-1}t_0t_1n \mid n \in N\} = \{Nt_0^{-1}t_1^{-1}t_0t_1n \mid n \in N\} = Nt_0^{-1}t_1^{-1}t_0t_1N$.

Recall that in step 4 of our manual double coset enumeration, we determined, by relation (6.3), that $t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} = (0\ 1\ 3)t_0t_3^{-1} = (0\ 1\ 4)t_0t_4^{-1}$ and so, by conjugation, $[t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} = [(0\ 1\ 3)t_0t_3^{-1}]^{(0\ 1)} = [(0\ 1\ 4)t_0t_4^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} = (0\ 3\ 1)t_1t_3^{-1} = (0\ 4\ 1)t_1t_4^{-1}$ and finally, by left multiplication, $t_0t_1t_0^{-1} = t_0(0\ 2\ 1)t_1t_2^{-1} = t_0(0\ 3\ 1)t_1t_3^{-1} = t_0(0\ 4\ 1)t_1t_4^{-1} \Rightarrow t_0t_1t_0^{-1} = (0\,2\,1)(0\,1\,2)t_0(0\,2\,1)t_1t_2^{-1} = (0\,3\,1)(0\,1\,3)t_0(0\,3\,1)t_1t_3^{-1} = (0\,4\,1)(0\,1\,4)t_0(0\,4\,1)t_1t_4^{-1} \Rightarrow t_0t_1t_0^{-1} = (0\ 2\ 1)t_0^{(0\ 2\ 1)}t_1t_2^{-1} = (0\ 3\ 1)t_0^{(0\ 3\ 1)}t_1t_3^{-1} = (0\ 4\ 1)t_0^{(0\ 4\ 1)}t_1t_4^{-1} \Rightarrow t_0t_1t_0^{-1} = (0\ 2\ 1)t_2t_1t_2^{-1} = (0\ 3\ 1)t_3t_1t_3^{-1} = (0\ 4\ 1)t_4t_1t_4^{-1}$ and, by right multiplication, $t_0t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_2t_1t_2^{-1}t_1^{-1} = (0\ 3\ 1)t_3t_1t_3^{-1}t_1^{-1} = (0\ 4\ 1)t_4t_1t_4^{-1}t_1^{-1}$. By conjugation, we find that $[t_0t_1t_0^{-1}t_1^{-1}]^{(0\ 1)} = [(0\ 2\ 1)t_2t_1t_2^{-1}t_1^{-1}]^{(0\ 1)} =$
$[(0\ 3\ 1)t_3t_1t_3^{-1}t_1^{-1}]^{(0\ 1)} = [(0\ 4\ 1)t_4t_1t_4^{-1}t_1^{-1}]^{(0\ 1)} \Rightarrow t_1t_0t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_2t_0t_2^{-1}t_0^{-1} = (0\,1\,3)t_3t_0t_3^{-1}t_0^{-1} = (0\,1\,4)t_4t_0t_4^{-1}t_0^{-1}$ and $[t_0t_1t_0^{-1}t_1^{-1}]^{(1\ 2)} = [(0\,2\,1)t_2t_1t_2^{-1}t_1^{-1}]^{(1\ 2)} = [(0\ 3\ 1)t_3t_1t_3^{-1}t_1^{-1}]^{(1\ 2)} = [(0\ 4\ 1)t_4t_1t_4^{-1}t_1^{-1}]^{(1\ 2)} \Rightarrow t_0t_2t_0^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_2t_1^{-1}t_2^{-1} = (0\,3\,2)t_3t_2t_3^{-1}t_2^{-1} = (0\,4\,2)t_4t_2t_4^{-1}t_2^{-1}$ and $[t_0t_1t_0^{-1}t_1^{-1}]^{(1\ 3)} = [(0\,2\,1)t_2t_1t_2^{-1}t_1^{-1}]^{(1\ 3)} = [(0\ 3\ 1)t_3t_1t_3^{-1}t_1^{-1}]^{(1\ 3)} = [(0\ 4\ 1)t_4t_1t_4^{-1}t_1^{-1}]^{(1\ 3)} \Rightarrow t_0t_3t_0^{-1}t_3^{-1} = (0\ 2\ 3)t_2t_3t_2^{-1}t_3^{-1} = (0\,1\,3)t_1t_3t_1^{-1}t_3^{-1} = (0\,4\,3)t_4t_3t_4^{-1}t_3^{-1}$ and $[t_0t_1t_0^{-1}t_1^{-1}]^{(1\ 4)} = [(0\,2\,1)t_2t_1t_2^{-1}t_1^{-1}]^{(1\ 4)} = [(0\ 3\ 1)t_3t_1t_3^{-1}t_1^{-1}]^{(1\ 4)} = [(0\ 4\ 1)t_4t_1t_4^{-1}t_1^{-1}]^{(1\ 4)} \Rightarrow t_0t_4t_0^{-1}t_4^{-1} = (0\ 2\ 4)t_2t_4t_2^{-1}t_4^{-1} = (0\ 3\ 4)t_3t_4t_3^{-1}t_4^{-1} = (0\ 1\ 4)t_1t_4t_1^{-1}t_4^{-1}$. Now, by relation (6.2) and by right and left multiplication, $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0t_0^{-1}t_1t_0^{-1} = t_0t_1^{-1}t_0t_1^{-1} \Rightarrow t_1t_0^{-1} = t_0t_1^{-1}t_0t_1^{-1} \Rightarrow t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0t_1^{-1}t_1 \Rightarrow t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_0^{-1}t_1t_0 = t_0t_1^{-1}t_0t_0 \Rightarrow t_1t_0^{-1}t_1t_0 = t_0t_1^{-1}t_0^{-1} \Rightarrow t_1t_1t_0^{-1}t_1t_0 = t_1t_0t_1^{-1}t_0^{-1} \Rightarrow t_1^{-1}t_0^{-1}t_1t_0 = t_1t_0t_1^{-1}t_0^{-1}$ and, by relation (6.4), $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and, finally, by relation (6.2) and by left and right multiplication, $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0t_1^{-1}t_1 \Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0 \Rightarrow t_0t_0^{-1}t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_1t_0^{-1}t_1 =$

$t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1 = t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1t_1 = t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$. Now $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and $t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ imply that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ which implies that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow$ $t_1^{-1}t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_1t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0t_1t_0 = t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow$ $t_0^{-1}t_0t_1t_0t_1t_0t_1t_0 = t_0^{-1}t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1^{-1}t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0 = t_1t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0t_0^{-1} = t_1t_0t_1t_0^{-1}t_0^{-1} \Rightarrow t_0t_1t_0t_1 = t_1t_0t_1t_0 \Rightarrow t_0t_1t_0t_1t_1^{-1} = t_1t_0t_1t_0t_1^{-1} \Rightarrow t_0t_1t_0 = t_1t_0t_1t_0t_1^{-1} \Rightarrow t_0t_1t_0t_0^{-1} = t_1t_0t_1t_0t_1^{-1}t_0^{-1} \Rightarrow t_0t_1 = t_1t_0t_1t_0t_1^{-1}t_0^{-1} \Rightarrow t_1^{-1}t_0t_1 = t_1^{-1}t_1t_0t_1t_0t_1^{-1}t_0^{-1} \Rightarrow t_1^{-1}t_0t_1 = t_0t_1t_0t_1^{-1}t_0^{-1} \Rightarrow t_0^{-1}t_1^{-1}t_0t_1 = t_0^{-1}t_0t_1t_0t_1^{-1}t_0^{-1} \Rightarrow t_0^{-1}t_1^{-1}t_0t_1 = t_1t_0t_1^{-1}t_0^{-1} \Rightarrow$ $[t_0^{-1}t_1^{-1}t_0t_1]^{(0\ 1)} = [t_1t_0t_1^{-1}t_0^{-1}]^{(0\ 1)} \Rightarrow t_1^{-1}t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1}$; therefore, $t_1^{-1}t_0^{-1}t_1t_0 = t_1t_0t_1^{-1}t_0^{-1}$ and $t_1^{-1}t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1}$ imply that $t_1t_0t_1^{-1}t_0^{-1} = t_1^{-1}t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1} \Rightarrow t_1t_0t_1^{-1}t_0^{-1} = t_0t_1t_0^{-1}t_1^{-1}$. By conjugation, we find that $[t_1t_0t_1^{-1}t_0^{-1}]^{(1\ 2)} = [t_0t_1t_0^{-1}t_1^{-1}]^{(1\ 2)} \Rightarrow t_2t_0t_2^{-1}t_0^{-1} = t_0t_2t_0^{-1}t_2^{-1}$ and $[t_1t_0t_1^{-1}t_0^{-1}]^{(1\ 3)} = [t_0t_1t_0^{-1}t_1^{-1}]^{(1\ 3)} \Rightarrow t_3t_0t_3^{-1}t_0^{-1} = t_0t_3t_0^{-1}t_3^{-1}$ and $[t_1t_0t_1^{-1}t_0^{-1}]^{(1\ 4)} = [t_0t_1t_0^{-1}t_1^{-1}]^{(1\ 4)} \Rightarrow t_4t_0t_4^{-1}t_0^{-1} = t_0t_4t_0^{-1}t_4^{-1}$. These relations imply that:

$$01\bar{0}\bar{1} \sim 21\bar{2}\bar{1} \sim 31\bar{3}\bar{1} \sim 41\bar{4}\bar{1} \sim 10\bar{1}\bar{0} \sim 20\bar{2}\bar{0} \sim 30\bar{3}\bar{0} \sim 40\bar{4}\bar{0} \sim 02\bar{0}\bar{2} \sim 12\bar{1}\bar{2} \sim$$

$$32\bar{3}\bar{2} \sim 42\bar{4}\bar{2} \sim 03\bar{0}\bar{3} \sim 13\bar{1}\bar{3} \sim 23\bar{2}\bar{3} \sim 43\bar{4}\bar{3} \sim 04\bar{0}\bar{4} \sim 14\bar{1}\bar{4} \sim 24\bar{2}\bar{4} \sim 34\bar{3}\bar{4}$$

Since each of the twenty single cosets has twenty names, the double coset $[01\bar{0}\bar{1}] = [\bar{0}\bar{1}01]$ must have at most one distinct single coset.

Now, $N^{(01\bar{0}\bar{1})}$ has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i = 0$: $Nt_0t_1t_0^{-1}t_1^{-1}t_0N$ and $Nt_0t_1t_0^{-1}t_1^{-1}t_0^{-1}N$.

But note that, by relations (6.2), (6.3), and (6.4), $t_0t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0^{-1} \Rightarrow$ $t_0t_1t_0^{-1}t_1^{-1}t_0 = t_1t_0t_1^{-1}t_0^{-1}t_0 \Rightarrow t_0t_1t_0^{-1}t_1^{-1}t_0 = t_1t_0t_1^{-1}$ and this implies that $Nt_0t_1t_0^{-1}t_1^{-1}t_0 = Nt_1t_0t_1^{-1}$. Therefore, $Nt_0t_1t_0^{-1}t_1^{-1}t_0N = Nt_0t_1t_0^{-1}N$. That is, $[01\bar{0}] = [01\bar{0}\bar{1}0]$.

Similarly, by relations (6.2) and (6.4), $t_1^{-1}t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1} \Rightarrow t_1^{-1}t_0^{-1}t_1t_0t_0^{-1} = t_0t_1t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow t_1^{-1}t_0^{-1}t_1 = t_0t_1t_0^{-1}t_1^{-1}t_0^{-1}$ and this implies that $Nt_0t_1t_0^{-1}t_1^{-1}t_0^{-1} = Nt_1^{-1}t_0^{-1}t_1$. Therefore, $Nt_0t_1t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0N$. That is, $[\bar{0}\bar{1}0] = [01\bar{0}\bar{1}\bar{0}]$.

Since $Nt_0t_1t_0^{-1}t_1^{-1}t_0N = Nt_0t_1t_0^{-1}N$ and $Nt_0t_1t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0N$, we need not consider additional double cosets of the form $Nt_0t_1t_0^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

20. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}N$.

Let $[\bar{0}\bar{1}\bar{0}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}N$.

Now, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow [t_0t_1^{-1}]^{(0\ 1)} = [(0\ 1\ 2)t_0t_2^{-1}]^{(0\ 1)} \Rightarrow t_1t_0^{-1} = (0\ 2\ 1)t_1t_2^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1} = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$, and also by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_2^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_2 = t_2^{-1}t_1^{-1}t_0 \Rightarrow t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1(0\ 2\ 1)t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{(0\ 2\ 1)}t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$, and so $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 2\ 1)t_1t_2^{-1}t_1^{-1}t_0$ and $(0\ 2\ 1)t_0t_1t_2 = t_1t_2^{-1}t_1^{-1}t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2$. Moreover, by relation (6.3), $t_1t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0t_0 = (0\ 1\ 2)t_0t_1t_2t_0 \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_1t_1t_0^{-1}t_1^{-1}t_0 = t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (0\ 1\ 2)t_2t_0t_1t_2 \Rightarrow [t_1^{-1}t_0^{-1}t_1^{-1}t_0]^{(0\ 1\ 2)} = [(0\ 1\ 2)t_2t_0t_1t_2]^{(0\ 1\ 2)} \Rightarrow t_2^{-1}t_1^{-1}t_2^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0$, and thus $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and $t_2^{-1}t_1^{-1}t_2^{-1}t_1 = (0\ 1\ 2)t_0t_1t_2t_0$ imply that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1$. Now, by conjugation, we see that $[t_1t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 3)} = [t_2^{-1}t_1^{-1}t_2^{-1}t_1]^{(0\ 3)} \Rightarrow t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ and $[t_1t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 4)} = [t_2^{-1}t_1^{-1}t_2^{-1}t_1]^{(0\ 4)} \Rightarrow t_1t_4^{-1}t_1^{-1}t_4^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ and so $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$. Finally, $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}$ and $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1$ imply that $t_1t_3^{-1}t_1^{-1}t_3^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$ and so, by conjugation, $[t_1t_3^{-1}t_1^{-1}t_3^{-1}]^{(2\ 3)} = [(0\ 1\ 2)t_0t_1t_2t_0]^{(2\ 3)} \Rightarrow t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 3)t_0t_1t_3t_0 \Rightarrow (0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 1)(0\ 1\ 3)t_0t_1t_3t_0 \Rightarrow (0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = t_0t_1t_3t_0 \Rightarrow (0\ 3\ 2)(0\ 3\ 1)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0 \Rightarrow (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0$.

Now, by relation (6.3) and by right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2$ $\Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow$ $(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$ and, by relation (6.1), $(0\ 2)(1\ 3)t_1t_0t_3 = t_0^{-1}t_1^{-1}t_2^{-1}$, and so $t_0^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_1t_0t_2$ and $(0\ 2)(1\ 3)t_1t_0t_3 =$ $t_0^{-1}t_1^{-1}t_2^{-1}$ imply that $(0\ 2)(1\ 3)t_1t_0t_3 = (0\ 1\ 2)t_1t_0t_2$, and so, by conjugation and right multiplication, $[(0\ 2)(1\ 3)t_1t_0t_3]^{(0\ 1)} = [(0\ 1\ 2)t_1t_0t_2]^{(0\ 1)} \Rightarrow (1\ 2)(0\ 3)t_0t_1t_3 =$ $(0\ 2\ 1)t_0t_1t_2 \Rightarrow (1\ 2)(0\ 3)t_0t_1t_3t_0 = (0\ 2\ 1)t_0t_1t_2t_0 \Rightarrow (0\ 2\ 1)(1\ 2)(0\ 3)t_0t_1t_3t_0 =$ $(0\ 2\ 1)(0\ 2\ 1)t_0t_1t_2t_0 \Rightarrow (0\ 3\ 2)t_0t_1t_3t_0 = (0\ 1\ 2)t_0t_1t_2t_0$. Therefore, $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 3\ 2)t_0t_1t_3t_0$ and $(0\ 3\ 2)t_0t_1t_3t_0 = (0\ 1\ 2)t_0t_1t_2t_0$ imply that $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = (0\ 1\ 2)t_0t_1t_2t_0$, and $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} =$ $(0\ 1\ 2)t_0t_1t_2t_0$ and $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_1t_2t_0 = t_2^{-1}t_1^{-1}t_2^{-1}t_1 = t_1t_3^{-1}t_1^{-1}t_3^{-1} =$ $t_1t_4^{-1}t_1^{-1}t_4^{-1}$ imply that $t_1t_0^{-1}t_1^{-1}t_0^{-1} = (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1} = t_1t_3^{-1}t_1^{-1}t_3^{-1} =$ $t_1t_4^{-1}t_1^{-1}t_4^{-1}$ which implies that $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} =$ $t_1t_3^{-1}t_1^{-1}t_3^{-1}t_1^{-1} = t_1t_4^{-1}t_1^{-1}t_4^{-1}t_1^{-1}$. Now $t_1t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow$ $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1t_1^{-1} \Rightarrow t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$ and so, by conjugation, $[t_2^{-1}t_1^{-1}t_2^{-1}]^{(0\ 2)} = [t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2)} \Rightarrow t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$, and since $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = (0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$, we see that $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1} =$ $(0\ 2)(1\ 3)t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$. Similarly, by conjugation, $[t_2^{-1}t_1^{-1}t_2^{-1}]^{(2\ 3)} = [t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}]^{(2\ 3)} \Rightarrow t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}$ and since $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$, we see that $t_3^{-1}t_1^{-1}t_3^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$; and, finally, by conjugation, $[t_2^{-1}t_1^{-1}t_2^{-1}]^{(2\ 4)} = [t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}]^{(2\ 4)} \Rightarrow t_4^{-1}t_1^{-1}t_4^{-1} =$ $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}$ and since $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$, we see that $t_4^{-1}t_1^{-1}t_4^{-1} =$ $t_1t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}$. And, by further conjugation, $[(1\ 2)(0\ 3)t_0t_1t_3]^{(3\ 4)} =$ $[(0\ 2\ 1)t_0t_1t_2]^{(3\ 4)} \Rightarrow (1\ 2)(0\ 4)t_0t_1t_4 = (0\ 2\ 1)t_0t_1t_2$. Thus, by relations (6.1) and (6.3), $(1\ 2)(0\ 3)t_0t_1t_3 = (0\ 2\ 1)t_0t_1t_2$. Thus, by relations (6.1) and (6.3), $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1} = t_3^{-1}t_1^{-1}t_3^{-1} = t_4^{-1}t_1^{-1}t_4^{-1}$ and, by right multiplication, $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = t_3^{-1}t_1^{-1}t_3^{-1}t_1^{-1} = t_4^{-1}t_1^{-1}t_4^{-1}t_1^{-1}$. Note that, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_0^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)(0\ 1\ 2)t_0^{-1}(0\ 2\ 1)t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(t_0^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)t_2^{-1}t_1^{-1}t_2 = t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{-1}t_1^{-1}t_2^{-1} = t_0^{-1}t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} =$ $t_0^{-1}t_1^{-1}t_0t_2t_1^{-1} \Rightarrow (0\ 1\ 2)(0\ 2\ 1)t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = (0\ 1\ 2)t_0^{-1}t_1^{-1}t_0t_2t_1^{-1} \Rightarrow t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} =$

$(0\ 1\ 2)t_0^{-1}t_1^{-1}t_0t_2t_1^{-1}$. Also, by relation (6.3) and by left and right multiplication and conjugation, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow$ $(0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow$ $(0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 1\ 2)(0\ 2\ 1)t_2t_0^{-1} = (0\ 1\ 2)t_2t_1^{-1} \Rightarrow t_2t_0^{-1} = (0\ 1\ 2)t_2t_1^{-1} \Rightarrow$ $t_2t_2t_0^{-1} = t_2(0\ 1\ 2)t_2t_1^{-1} \Rightarrow t_2^{-1}t_0^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_2(0\ 1\ 2)t_2t_1^{-1} \Rightarrow t_2^{-1}t_0^{-1} =$ $(0\ 1\ 2)t_2^{(0\ 1\ 2)}t_2t_1^{-1} \Rightarrow t_2^{-1}t_0^{-1} = (0\ 1\ 2)t_0t_2t_1^{-1} \Rightarrow t_0^{-1}t_2^{-1}t_0^{-1} = t_0^{-1}(0\ 1\ 2)t_0t_2t_1^{-1} \Rightarrow$ $t_0^{-1}t_2^{-1}t_0^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_0^{-1}(0\ 1\ 2)t_0t_2t_1^{-1} \Rightarrow t_0^{-1}t_2^{-1}t_0^{-1} = (0\ 1\ 2)(t_0^{-1})^{(0\ 1\ 2)}t_0t_2t_1^{-1}$ $\Rightarrow t_0^{-1}t_2^{-1}t_0^{-1} = (0\ 1\ 2)t_1^{-1}t_0t_2t_1^{-1} \Rightarrow t_2^{-1}t_0^{-1}t_2^{-1}t_0^{-1} = t_2^{-1}(0\ 1\ 2)t_1^{-1}t_0t_2t_1^{-1} \Rightarrow$ $t_2^{-1}t_0^{-1}t_2^{-1}t_0^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_2^{-1}(0\ 1\ 2)t_1^{-1}t_0t_2t_1^{-1} \Rightarrow t_2^{-1}t_0^{-1}t_2^{-1}t_0^{-1} =$ $(0\ 1\ 2)(t_2^{-1})^{(0\ 1\ 2)}t_1^{-1}t_0t_2t_1^{-1} \Rightarrow t_2^{-1}t_0^{-1}t_2^{-1}t_0^{-1} = (0\ 1\ 2)t_0^{-1}t_1^{-1}t_0t_2t_1^{-1}$. Now $t_2^{-1}t_0^{-1}t_2^{-1}t_0^{-1} = (0\ 1\ 2)t_0^{-1}t_1^{-1}t_0t_2t_1^{-1}$ and $t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = (0\ 1\ 2)t_0^{-1}t_1^{-1}t_0t_2t_1^{-1}$ imply that $t_2^{-1}t_0^{-1}t_2^{-1}t_0^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$. Therefore, $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} =$ $t_3^{-1}t_1^{-1}t_3^{-1}t_1^{-1} = t_4^{-1}t_1^{-1}t_4^{-1}t_1^{-1} = t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = t_2^{-1}t_0^{-1}t_2^{-1}t_0^{-1} =$ $(1\ 2)(0\ 3)t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_3^{-1}t_0^{-1}t_3^{-1}t_0^{-1} = t_4^{-1}t_0^{-1}t_4^{-1}t_0^{-1} = t_0^{-1}t_2{-}1t_0^{-1}t_2^{-1} =$ $(0\ 1)(2\ 3)t_1^{-1}t_2^{-1}t_1^{-1}t_2^{-1} = t_3^{-1}t_2^{-1}t_3^{-1}t_2^{-1} = t_4^{-1}t_2^{-1}t_4^{-1}t_2^{-1} = t_0^{-1}t_3{-}1t_0^{-1}t_3^{-1} =$ $(0\ 2\ 1)t_1^{-1}t_3^{-1}t_1^{-1}t_3^{-1} = t_2^{-1}t_3^{-1}t_2^{-1}t_3^{-1} = t_4^{-1}t_3^{-1}t_4^{-1}t_3^{-1} = t_0^{-1}t_4{-}1t_0^{-1}t_4^{-1} =$ $(0\ 1)(3\ 4))t_1^{-1}t_4^{-1}t_1^{-1}t_4^{-1} = t_2^{-1}t_4^{-1}t_2^{-1}t_4^{-1} = t_3^{-1}t_4^{-1}t_3^{-1}t_4^{-1}$. These relations imply that:

$$\bar{0}\bar{1}\bar{0}\bar{1} \sim \bar{2}\bar{1}\bar{2}\bar{1} \sim \bar{3}\bar{1}\bar{3}\bar{1} \sim \bar{4}\bar{1}\bar{4}\bar{1} \sim \bar{1}\bar{0}\bar{1}\bar{0} \sim \bar{2}\bar{0}\bar{2}\bar{0} \sim \bar{3}\bar{0}\bar{3}\bar{0} \sim \bar{4}\bar{0}\bar{4}\bar{0} \sim \bar{0}\bar{2}\bar{0}\bar{2} \sim \bar{1}\bar{2}\bar{1}\bar{2} \sim$$

$$\bar{3}\bar{2}\bar{3}\bar{2} \sim \bar{4}\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{3}\bar{0}\bar{3} \sim \bar{1}\bar{3}\bar{1}\bar{3} \sim \bar{2}\bar{3}\bar{2}\bar{3} \sim \bar{4}\bar{3}\bar{4}\bar{3} \sim \bar{0}\bar{4}\bar{0}\bar{4} \sim \bar{1}\bar{4}\bar{1}\bar{4} \sim \bar{2}\bar{4}\bar{2}\bar{4} \sim \bar{3}\bar{4}\bar{3}\bar{4}$$

Since each of the twenty single cosets has twenty names, the double coset $[\bar{0}\bar{1}\bar{0}\bar{1}]$ must have at most one distinct single coset.

Now, $N^{(\bar{0}\bar{1}\bar{0}\bar{1})}$ has two orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0, 1, 2, 3, 4\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

Thus there are at most two double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i = 0$: $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N$.

But note that, by relations (6.1) and (6.3), $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} =$ $(1\ 2)(0\ 3)t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and so, by right multiplication, $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0 =$ $(1\ 2)(0\ 3)t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}t_0 \Rightarrow (0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0 = (1\ 2)(0\ 3)t_1^{-1}t_0^{-1}t_1^{-1}$ which implies that $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0 = Nt_1^{-1}t_0^{-1}t_1^{-1}$. Therefore, $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_0^{-1}N$. That is, $[\bar{0}\bar{1}\bar{0}] = [\bar{0}\bar{1}\bar{0}\bar{1}0]$.

Similarly, by relations (6.1) and (6.3), $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1} = (1\ 2)(0\ 3)t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and so, by right multiplication, $(0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = (1\ 2)(0\ 3)t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}t_0^{-1} \Rightarrow (0\ 2)(1\ 3)t_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = (1\ 2)(0\ 3)t_1^{-1}t_0^{-1}t_1^{-1}t_0$ which implies that $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = Nt_1^{-1}t_0^{-1}t_1^{-1}t_0$. Therefore, $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$. That is, $[0\bar{1}\bar{0}\bar{1}] = [\bar{0}\bar{1}\bar{0}1] = [\bar{0}\bar{1}\bar{0}\bar{1}\bar{0}]$.

Since $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$, we need not consider additional double cosets of the form $Nt_0^{-1}t_1^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

21. We next consider the double coset $Nt_0t_1t_0t_1^{-1}N$.

Let $[010\bar{1}]$ denote the double coset $Nt_0t_1t_0t_1^{-1}N$.

Now, by relation (6.4), $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and, by relation (6.2) and by left and right multiplication, $t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1} \Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0t_1^{-1}t_1 \Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = t_1^{-1}t_0 \Rightarrow t_0t_0^{-1}t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_0^{-1}t_1 = t_0t_1^{-1}t_0 \Rightarrow t_1t_1t_0^{-1}t_1 = t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1 = t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0^{-1}t_1t_1 = t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0t_1 \Rightarrow t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$. Now $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$ and $t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ imply that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = t_1t_0t_1^{-1}t_0t_1t_0^{-1}$ which implies that $t_1t_0t_1t_0t_1t_0t_1t_0 = t_1t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1^{-1}t_1t_0t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_1t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0t_1t_0 = t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0^{-1}t_0t_1t_0t_1t_0t_1t_0 = t_0^{-1}t_0t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_1^{-1}t_1t_0t_1t_0t_1t_0 = t_1^{-1}t_1^{-1}t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0 = t_1t_0t_1t_0^{-1} \Rightarrow t_0t_1t_0t_1t_0t_0^{-1} = t_1t_0t_1t_0^{-1}t_0^{-1} \Rightarrow t_0t_1t_0t_1 = t_1t_0t_1t_0 \Rightarrow t_0t_1t_0t_1t_1^{-1} = t_1t_0t_1t_0t_1^{-1} \Rightarrow t_0t_1t_0 = t_1t_0t_1t_0t_1^{-1} \Rightarrow t_1^{-1}t_0t_1t_0 = t_1^{-1}t_1t_0t_1t_0t_1^{-1} \Rightarrow t_1^{-1}t_0t_1t_0 = t_0t_1t_0t_1^{-1}$, and this implies that $Nt_0t_1t_0t_1^{-1} = Nt_0^{-1}t_1t_0t_1$ which implies that $Nt_0t_1t_0t_1^{-1}N = Nt_0^{-1}t_1t_0t_1N$. Therefore, $Nt_0t_1t_0t_1^{-1}N = Nt_0^{-1}t_1t_0t_1N$ and so $[010\bar{1}] = [\bar{0}101]$. Also, by relation (6.3) and by conjugation and left and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} =$

$t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = t_1(0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0t_1t_0t_1^{-1} = t_0(0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0t_1t_0t_1^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_0(0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0t_1t_0t_1^{-1} = (0\ 1\ 2)t_0^{(0\ 1\ 2)}t_2t_0t_2^{-1} \Rightarrow t_0t_1t_0t_1^{-1} = (0\ 1\ 2)t_1t_2t_0t_2^{-1}$, and this implies that $Nt_0t_1t_0t_1^{-1} = Nt_1t_2t_0t_2^{-1}$. Therefore, $Nt_0t_1t_0t_1^{-1}N = Nt_0t_1t_2t_1^{-1}N$; that is, $Nt_0t_1t_0t_1^{-1}N = Nt_0t_1t_2t_1^{-1}N$ and so $[010\bar{1}] = [012\bar{1}]$. Therefore, we conclude that $Nt_0t_1t_0t_1^{-1}N = Nt_0^{-1}t_1t_0t_1N = Nt_0t_1t_2t_1^{-1}N$. That is, $[010\bar{1}] = [\bar{0}101] = [012\bar{1}]$.

Hence note that $Nt_0t_1t_0t_1^{-1}N = Nt_0^{-1}t_1t_0t_1N = Nt_0t_1t_2t_1^{-1}N$.

Now, by relation (6.3) and by conjugation and right and left multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow [(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1}]^{(0\ 1)} = [t_1t_0t_2]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = t_0t_1t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2t_2^{-1} \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2 = (0\ 2\ 1)t_0t_1 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2t_0 = (0\ 2\ 1)t_0t_1t_0 \Rightarrow t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$. Also, by relation (6.3), $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2 = t_1^{-1}t_0^{-1}t_2 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = t_1^{-1}t_0^{-1}t_2t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1}$. Now, $t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$ and $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1}$ imply that $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = t_1^{-1}t_0^{-1}t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1}$, and so $(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_0t_1^{-1} \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0t_1^{-1}t_1 = (0\ 2\ 1)t_0t_1t_0t_1^{-1}t_1 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = (0\ 2\ 1)t_0t_1t_0 \Rightarrow (0\ 1\ 2)(0\ 2\ 1)t_1^{-1}t_2t_0t_2t_0 = (0\ 1\ 2)(0\ 2\ 1)t_0t_1t_0 \Rightarrow t_1^{-1}t_2t_0t_2t_0 = t_0t_1t_0 \Rightarrow t_1t_1^{-1}t_2t_0t_2t_0 = t_1t_0t_1t_0 \Rightarrow t_2t_0t_2t_0 = t_1t_0t_1t_0 \Rightarrow t_2t_0t_2t_0t_0 = t_1t_0t_1t_0t_0 \Rightarrow t_2t_0t_2t_0^{-1} = t_1t_0t_1t_0^{-1}$. Moreover, by conjugation, $[t_2t_0t_2t_0^{-1}]^{(2\ 3)} = [t_1t_0t_1t_0^{-1}]^{(2\ 3)} \Rightarrow t_3t_0t_3t_0^{-1} = t_1t_0t_1t_0^{-1}$ and, also by conjugation, $[t_2t_0t_2t_0^{-1}]^{(2\ 4)} = [t_1t_0t_1t_0^{-1}]^{(2\ 4)} \Rightarrow t_4t_0t_4t_0^{-1} = t_1t_0t_1t_0^{-1}$, and so $t_2t_0t_2t_0^{-1} = t_1t_0t_1t_0^{-1}$ and $t_3t_0t_3t_0^{-1} = t_1t_0t_1t_0^{-1}$ and $t_4t_0t_4t_0^{-1} = t_1t_0t_1t_0^{-1}$ imply that $t_2t_0t_2t_0^{-1} = t_1t_0t_1t_0^{-1} = t_3t_0t_3t_0^{-1} = t_4t_0t_4t_0^{-1}$. Therefore, by conjugation, we see that $[t_2t_0t_2t_0^{-1}]^{(0\ 1)} = [t_1t_0t_1t_0^{-1}]^{(0\ 1)} = [t_3t_0t_3t_0^{-1}]^{(0\ 1)} = [t_4t_0t_4t_0^{-1}]^{(0\ 1)} \Rightarrow t_2t_1t_2t_1^{-1} = t_0t_1t_0t_1^{-1} = t_3t_1t_3t_1^{-1} = t_4t_1t_4t_1^{-1}$ and $[t_2t_0t_2t_0^{-1}]^{(0\ 2)} = [t_1t_0t_1t_0^{-1}]^{(0\ 2)} =$

$[t_3t_0t_3t_0^{-1}]^{(0\ 2)} = [t_4t_0t_4t_0^{-1}]^{(0\ 2)} \Rightarrow t_0t_2t_0t_2^{-1} = t_1t_2t_1t_2^{-1} = t_3t_2t_3t_2^{-1} = t_4t_2t_4t_2^{-1}$ and $[t_2t_0t_2t_0^{-1}]^{(0\ 3)} = [t_1t_0t_1t_0^{-1}]^{(0\ 3)} = [t_3t_0t_3t_0^{-1}]^{(0\ 3)} = [t_4t_0t_4t_0^{-1}]^{(0\ 3)} \Rightarrow t_2t_3t_2t_3^{-1} = t_1t_3t_1t_3^{-1} = t_0t_3t_0t_3^{-1} = t_4t_3t_4t_3^{-1}$ and $[t_2t_0t_2t_0^{-1}]^{(0\ 4)} = [t_1t_0t_1t_0^{-1}]^{(0\ 4)} = [t_3t_0t_3t_0^{-1}]^{(0\ 4)} = [t_4t_0t_4t_0^{-1}]^{(0\ 4)} \Rightarrow t_2t_4t_2t_4^{-1} = t_1t_4t_1t_4^{-1} = t_3t_4t_3t_4^{-1} = t_0t_4t_0t_4^{-1}$.

Therefore, in terms of our short-hand notation, these relations imply that:

$$101\bar{0} \sim 202\bar{0} \sim 303\bar{0} \sim 404\bar{0}$$

Similiarly, by conjugation, we find that

$$010\bar{1} \sim 212\bar{1} \sim 313\bar{1} \sim 414\bar{1}, \qquad 020\bar{2} \sim 121\bar{2} \sim 323\bar{2} \sim 424\bar{2},$$

$$030\bar{3} \sim 131\bar{3} \sim 232\bar{3} \sim 434\bar{3}, \qquad 040\bar{4} \sim 141\bar{4} \sim 242\bar{4} \sim 343\bar{4}$$

Since each of the twenty single cosets has four names, the double coset $[010\bar{1}] = [\bar{0}101] = [012\bar{1}]$ must have at most five distinct single cosets.

Now, $N^{(010\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3, t_4\}$: $\{0\}$, $\{1,2,3,4\}$, $\{\bar{0}\}$, and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1\}$: $Nt_0t_1t_0t_1^{-1}t_0N$, $Nt_0t_1t_0t_1^{-1}t_0^{-1}N$, $Nt_0t_1t_0t_1^{-1}t_1N$, and $Nt_0t_1t_0t_1^{-1}t_1^{-1}N$.

But note that $Nt_0t_1t_0t_1^{-1}t_1^{-1}N = Nt_0t_1t_0(t_1^{-1})^2N = Nt_0t_1t_0t_1N$ and note further that $Nt_0t_1t_0t_1^{-1}t_1N = Nt_0t_1t_0eN = Nt_0t_1t_0N$.

Moreover, by relations (6.2) and (6.4), $t_1^{-1}t_0t_1t_0 = t_0t_1t_0t_1^{-1} \Rightarrow t_1^{-1}t_0t_1t_0t_0^{-1} = t_0t_1t_0t_1^{-1}t_0^{-1} \Rightarrow t_1^{-1}t_0t_1 = t_0t_1t_0t_1^{-1}t_0^{-1}$, and this implies that $Nt_1^{-1}t_0t_1 = Nt_0t_1t_0t_1^{-1}t_0^{-1}$. Therefore, $Nt_0t_1t_0t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0N$. That is, $[\bar{0}10] = [010\bar{1}\bar{0}]$.

Similarly, by relations (6.2) and (6.4), $t_1^{-1}t_0t_1t_0 = t_0t_1t_0t_1^{-1} \Rightarrow t_1^{-1}t_0t_1t_0t_0 = t_0t_1t_0t_1^{-1}t_0 \Rightarrow t_1^{-1}t_0t_1t_0^{-1} = t_0t_1t_0t_1^{-1}t_0$ and, by relation (6.3) and by conjugation and left and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_2(0\ 2\ 1)t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2^{(0\ 2\ 1)}t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1t_1^{-1}t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)et_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2 = t_2t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}t_0t_0^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1}e \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow (0\ 2\ 1)t_2t_0^{-1} = t_2t_1^{-1} \Rightarrow [(0\ 2\ 1)t_2t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (0\ 2\ 1)t_0t_1^{-1} = t_0t_2^{-1} \Rightarrow t_0t_1^{-1} = (0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = t_1(0\ 1\ 2)t_0t_2^{-1} \Rightarrow$

$t_1t_0t_1^{-1} = (0\ 1\ 2)(0\ 2\ 1)t_1(0\ 1\ 2)t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)t_1^{(0\ 1\ 2)}t_0t_2^{-1} \Rightarrow t_1t_0t_1^{-1} = (0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0^{-1}t_1t_0t_1^{-1} = t_0^{-1}(0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0^{-1}t_1t_0t_1^{-1} =$ $(0\ 1\ 2)(0\ 2\ 1)t_0^{-1}(0\ 1\ 2)t_2t_0t_2^{-1} \Rightarrow t_0^{-1}t_1t_0t_1^{-1} = (0\ 1\ 2)(t_0^{-1})^{(0\ 1\ 2)}t_2t_0t_2^{-1} \Rightarrow t_0^{-1}t_1t_0t_1^{-1}$ $= (0\ 1\ 2)t_1^{-1}t_2t_0t_2^{-1} \Rightarrow (0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 1\ 2)(0\ 1\ 2)t_1^{-1}t_2t_0t_2^{-1} \Rightarrow$ $(0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1}$ and, also by relation (6.3) and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_0 = t_1^{-1}t_0^{-1} \Rightarrow$ $(0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1} = t_1^{-1}t_0^{-1}t_2^{-1}$ and, also by relation (6.3) and by conjugation and left and right multiplication, $(0\ 2\ 1)t_1^{-1}t_2 = t_1^{-1}t_0 \Rightarrow (0\ 2\ 1)t_1^{-1}t_2t_2 = t_1^{-1}t_0t_2 \Rightarrow$ $(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_0t_2 \Rightarrow t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1^{-1}t_1^{-1}t_0t_2 \Rightarrow$ $(0\ 2\ 1)(0\ 1\ 2)t_1^{-1}(0\ 2\ 1)t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow (t_1^{-1})^{(0\ 2\ 1)}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow$ $(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1} = t_1t_0t_2 \Rightarrow [(0\ 2\ 1)t_0^{-1}t_1^{-1}t_2^{-1}]^{(0\ 1)} = [t_1t_0t_2]^{(0\ 1)} \Rightarrow (0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1}$ $= t_0t_1t_2 \Rightarrow (0\ 2\ 1)(0\ 1\ 2)t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2$ and, therefore, $(0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1}$ and $(0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1} = t_1^{-1}t_0^{-1}t_2^{-1}$ and $t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2$ imply that $(0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_1^{-1}t_2t_0t_2^{-1} =$ $t_1^{-1}t_0^{-1}t_2^{-1} = (0\ 2\ 1)t_0t_1t_2 \Rightarrow (0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_2$; therefore, $t_1^{-1}t_0t_1t_0^{-1} =$ $t_0t_1t_0t_1^{-1}t_0$ and $(0\ 1\ 2)t_0^{-1}t_1t_0t_1^{-1} = (0\ 2\ 1)t_0t_1t_2$ imply that $(0\ 1\ 2)t_0t_1t_0t_1^{-1}t_0 =$ $(0\ 2\ 1)t_0t_1t_2$ and this implies that $Nt_0t_1t_0t_1^{-1}t_0 = Nt_0t_1t_2$. Therefore, $Nt_0t_1t_0t_1^{-1}t_0N = Nt_0t_1t_2N$. That is, $[012] = [010\bar{1}0]$.

Since $Nt_0t_1t_0t_1^{-1}t_1^{-1}N = Nt_0t_1t_0t_1N$ and $Nt_0t_1t_0t_1^{-1}t_1N = Nt_0t_1t_0N$ and $Nt_0t_1t_0t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0N$ and $Nt_0t_1t_0t_1^{-1}t_0N = Nt_0t_1t_2N$, we need not consider additional double cosets of the form $Nt_0t_1t_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3, 4\}$.

In fact, since $N^{(010\bar{1})}$ is transitive on the symmetric generators and since $Nt_0t_1t_0t_1^{-1}t_1^{-1}N = Nt_0t_1t_0t_1N$ and $Nt_0t_1t_0t_1^{-1}t_1N = Nt_0t_1t_0N$ and $Nt_0t_1t_0t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0N$ and $Nt_0t_1t_0t_1^{-1}t_0N = Nt_0t_1t_2N$ imply that the double coset $[010\bar{1}\bar{1}] = [0101]$ and the double coset $[010\bar{1}1] = [010]$ and the double coset $[010\bar{1}\bar{0}] = [\bar{0}10]$ and the double coset $[010\bar{1}0] = [012]$, we have effectively completed the double coset enumeration of G over S_5.

In total, therefore, there are at most 21 distinct double cosets of N in G and at most 126 distinct right (single) cosets of N in G. The double cosets of N in G are as follows: $[*]$, $[0]$, $[\bar{0}]$, $[\bar{0}1]$, $[01]$, $[\bar{0}\bar{1}]$, $[0\bar{1}]$, $[0\bar{1}0]$, $[0\bar{1}\bar{0}]$, $[010]$, $[01\bar{0}]$, $[012] = [\bar{0}\bar{1}\bar{2}]$, $[\bar{0}\bar{1}0]$, $[\bar{0}1\bar{0}]$, $[\bar{0}10]$, $[\bar{0}1\bar{0}]$, $[0\bar{1}0\bar{1}] = [\bar{0}\bar{1}01] = [0120]$, $[0101]$, $[010\bar{1}] = [\bar{0}\bar{1}01]$, $[\bar{0}\bar{1}0\bar{1}]$, and $[010\bar{1}] = [\bar{0}101] = [012\bar{1}]$.

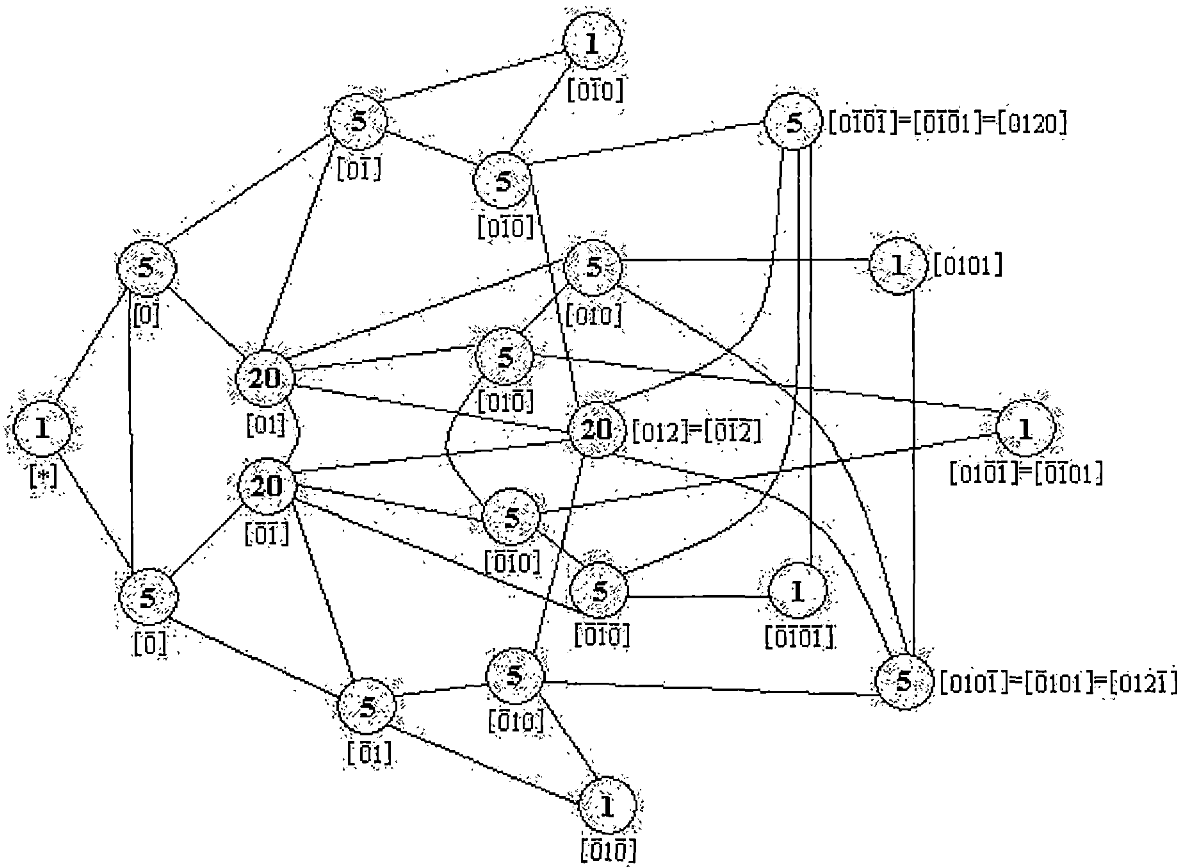

Figure 6.1: Cayley Diagram of G Over S_5

6.3 Cayley Diagram of G Over S_5

The Cayley diagram of G over S_5 is illustrated in Figure 6.1. For a detailed explanation of the meaning of the component parts of a Cayley diagram, see Section 2.3.

6.4 Action of the Symmetric Generators and the Generators of S_5 on the Right Cosets of G Over S_5

Let X denote the set of all (126) distinct right cosets of N in G. We define a mapping $\phi\colon G \longrightarrow S_X$ so that ϕ maps a generator $g \in G$ to its action (by right multiplication) on X. That is, we define ϕ so that $\phi(g) = \widehat{\phi}(g) : X \to X$. Then the action $\phi(t) \sim \phi(t_0)$ of the symmetric generator $t \sim t_0$ on the right cosets of N in G may be expressed as

$$\phi(t) \sim \phi(t_0) = (\ast\ 0\ \bar{0})(1\ 10\ 1\bar{0})(2\ 20\ 2\bar{0})(3\ 30\ 3\bar{0})(4\ 40\ 4\bar{0})(\bar{1}\ \bar{1}0\ \bar{1}\bar{0})(\bar{2}\ \bar{2}0\ \bar{2}\bar{0})(\bar{3}\ \bar{3}0\ \bar{3}\bar{0})$$

$$(\bar{4}\ \bar{4}0\ \bar{4}\bar{0})(0\bar{1}\ 0\bar{1}0\ 0\bar{1}\bar{0})(\bar{0}1\ \bar{0}10\ \bar{0}1\bar{0})(01\ 010\ 01\bar{0})(02\ 020\ 02\bar{0})(03\ 030\ 03\bar{0})(04\ 040\ 04\bar{0})$$

$$(12\ 120\ \bar{2}\bar{1})(13\ 130\ \bar{3}\bar{1})(14\ 140\ \bar{4}\bar{1})(21\ 210\ \bar{1}\bar{2})(23\ 230\ \bar{3}\bar{2})(24\ 240\ \bar{4}\bar{2})(31\ 310\ \bar{1}\bar{3})(32\ 320\ \bar{2}\bar{3})$$

$$(34\ 340\ \bar{4}\bar{3})(41\ 410\ \bar{1}\bar{4})(42\ 420\ \bar{2}\bar{4})(43\ 430\ \bar{3}\bar{4})(\bar{0}\bar{1}\ \bar{0}\bar{1}0\ \bar{0}\bar{1}\bar{0})(\bar{0}\bar{2}\ \bar{0}\bar{2}0\ \bar{0}\bar{2}\bar{0})(\bar{0}\bar{3}\ \bar{0}\bar{3}0\ \bar{0}\bar{3}\bar{0})$$

$$(\bar{0}\bar{4}\ \bar{0}\bar{4}0\ \bar{0}\bar{4}\bar{0})(1\bar{0}\bar{1}\ 012\ 1\bar{0}\bar{1}\bar{0})(2\bar{0}\bar{2}\ 021\ 2\bar{0}\bar{2}\bar{0})(3\bar{0}\bar{3}\ 031\ 3\bar{0}\bar{3}\bar{0})(4\bar{0}\bar{4}\ 041\ 4\bar{0}\bar{4}\bar{0})(101\ 0101\ 101\bar{0})$$

$$(10\bar{1}\ \bar{1}01\ 01\bar{0}\bar{1})(102\ \bar{1}01\ 010\bar{1})(201\ \bar{2}02\ 020\bar{2})(301\ \bar{3}03\ 030\bar{3})(401\ \bar{4}04\ 040\bar{4})(\bar{1}0\bar{1}\ 0\bar{1}0\bar{1}\ \bar{0}\bar{1}0\bar{1}),$$

and the action $\phi(x) \sim \phi((0\ 1\ 2\ 3\ 4))$ of the generator $x \sim (0\ 1\ 2\ 3\ 4)$ of S_5 on the right cosets of N in G may be expressed as

$$\phi(x) \sim \phi((0\,1\,2\,3\,4)) = (0\,1\,2\,3\,4)(\bar{0}\,\bar{1}\,\bar{2}\,\bar{3}\,\bar{4})(\bar{0}1\ \bar{1}0\ \bar{2}0\ \bar{3}0\ \bar{4}0)(0\bar{1}\ 1\bar{0}\ 2\bar{0}\ 3\bar{0}\ 4\bar{0})(01\ 12\ 23\ 34\ 40)$$

$$(02\ 13\ 24\ 30\ 41)(03\ 14\ 20\ 31\ 42)(04\ 10\ 21\ 32\ 43)(\bar{0}\bar{1}\ \bar{1}\bar{2}\ \bar{2}\bar{3}\ \bar{3}\bar{4}\ \bar{4}\bar{0})(\bar{0}\bar{2}\ \bar{1}\bar{3}\ \bar{2}\bar{4}\ \bar{3}\bar{0}\ \bar{4}\bar{1})$$

$$(\bar{0}\bar{3}\ \bar{1}\bar{4}\ \bar{2}\bar{0}\ \bar{3}\bar{1}\ \bar{4}\bar{2})(\bar{0}\bar{4}\ \bar{1}\bar{0}\ \bar{2}\bar{1}\ \bar{3}\bar{2}\ \bar{4}\bar{3})(0\bar{1}\bar{0}\ 1\bar{0}\bar{1}\ 2\bar{0}\bar{2}\ 3\bar{0}\bar{3}\ 4\bar{0}\bar{4})(010\ 020\ 030\ 040\ 101)$$

$$(01\bar{0}\ 02\bar{0}\ 03\bar{0}\ 04\bar{0}\ 10\bar{1})(\bar{0}\bar{1}0\ \bar{0}\bar{2}0\ \bar{0}\bar{3}0\ \bar{0}\bar{4}0\ \bar{1}\bar{0}1)(012\ 120\ 230\ 340\ 401)(021\ 130\ 240\ 301\ 410)$$

$$(031\ 140\ 201\ 310\ 420)(041\ 102\ 210\ 320\ 430)(\bar{0}\bar{1}\bar{0}\ \bar{0}\bar{2}\bar{0}\ \bar{0}\bar{3}\bar{0}\ \bar{0}\bar{4}\bar{0}\ \bar{1}\bar{0}\bar{1})$$

$$(\bar{0}10\ \bar{1}01\ \bar{2}02\ \bar{3}03\ \bar{4}04)(0\bar{1}0\bar{1}\ 1\bar{0}1\bar{0}\ 2\bar{0}2\bar{0}\ 3\bar{0}3\bar{0}\ 4\bar{0}4\bar{0})(010\bar{1}\ 020\bar{2}\ 030\bar{3}\ 040\bar{4}\ 101\bar{0}),$$

and the action $\phi(y) \sim \phi((3\ 4))$ of the generator $y \sim (3\ 4)$ of S_5 on the right cosets of N in G may be expressed as

$$\phi(y) \sim \phi((3\ 4)) = (3\ 4)(\bar{3}\ \bar{4})(\bar{3}0\ \bar{4}0)(3\bar{0}\ 4\bar{0})(03\ 04)(13\ 14)(23\ 24)(30\ 40)(31\ 41)(32\ 42)$$

$$(34\ 43)(\bar{0}\bar{3}\ \bar{0}\bar{4})(\bar{1}\bar{3}\ \bar{1}\bar{4})(\bar{2}\bar{3}\ \bar{2}\bar{4})(\bar{3}\bar{0}\ \bar{4}\bar{0})(\bar{3}\bar{1}\ \bar{4}\bar{1})(\bar{3}\bar{2}\ \bar{4}\bar{2})(\bar{3}\bar{4}\ \bar{4}\bar{3})(3\bar{0}\bar{3}\ 4\bar{0}\bar{4})(030\ 040)(03\bar{0}\ 04\bar{0})$$

$$(\bar{0}\bar{3}0\ \bar{0}\bar{4}0)(031\ 041)(130\ 140)(230\ 240)(301\ 401)(310\ 410)(320\ 420)(340\ 430)(\bar{0}\bar{3}\bar{0}\ \bar{0}\bar{4}\bar{0})$$

$$(\bar{3}03\ \bar{4}04)(3\bar{0}3\bar{0}\ 4\bar{0}4\bar{0})(030\bar{3}\ 040\bar{4}).$$

Since there are 126 distinct right cosets of N in G, these actions may be written as permutations on 126 letters. In fact, the actions of the generators on the set of right cosets of N in G are equivalent to the permutation representations of the generators in their action on the right cosets of N in G. To better manipulate the permutation representations of the symmetric generators t_i and the generators x and y, it is helpful to label the distinct single cosets of N in G as follows:

(126)	$*$	(21)	01	(42)	$\bar{0}\bar{2}$	(63)	$1\bar{0}\bar{1}$	(84)	140	(105)	$\bar{0}\bar{3}\bar{0}$
(1)	0	(22)	02	(43)	$\bar{0}\bar{3}$	(64)	$2\bar{0}\bar{2}$	(85)	201	(106)	$\bar{0}\bar{4}\bar{0}$
(2)	1	(23)	03	(44)	$\bar{0}\bar{4}$	(65)	$3\bar{0}\bar{3}$	(86)	210	(107)	$\bar{0}10$
(3)	2	(24)	04	(45)	$\bar{1}\bar{0}$	(66)	$4\bar{0}\bar{4}$	(87)	230	(108)	$\bar{1}01$
(4)	3	(25)	10	(46)	$\bar{1}\bar{2}$	(67)	010	(88)	240	(109)	$\bar{2}02$
(5)	4	(26)	12	(47)	$\bar{1}\bar{3}$	(68)	020	(89)	301	(110)	$\bar{3}03$
(6)	$\bar{0}$	(27)	13	(48)	$\bar{1}\bar{4}$	(69)	030	(90)	310	(111)	$\bar{4}04$
(7)	$\bar{1}$	(28)	14	(49)	$\bar{2}\bar{0}$	(70)	040	(91)	320	(112)	$\bar{0}1\bar{0}$
(8)	$\bar{2}$	(29)	20	(50)	$\bar{2}\bar{1}$	(71)	101	(92)	340	(113)	$0\bar{1}\bar{0}\bar{1}$
(9)	$\bar{3}$	(30)	21	(51)	$\bar{2}\bar{3}$	(72)	$01\bar{0}$	(93)	401	(114)	$1\bar{0}\bar{1}\bar{0}$
(10)	$\bar{4}$	(31)	23	(52)	$\bar{2}\bar{4}$	(73)	$02\bar{0}$	(94)	410	(115)	$2\bar{0}\bar{2}\bar{0}$
(11)	$0\bar{1}$	(32)	24	(53)	$\bar{3}\bar{0}$	(74)	$03\bar{0}$	(95)	420	(116)	$3\bar{0}\bar{3}\bar{0}$
(12)	$1\bar{0}$	(33)	30	(54)	$\bar{3}\bar{1}$	(75)	$04\bar{0}$	(96)	430	(117)	$4\bar{0}\bar{4}\bar{0}$
(13)	$2\bar{0}$	(34)	31	(55)	$\bar{3}\bar{2}$	(76)	$10\bar{1}$	(97)	$\bar{0}\bar{1}0$	(118)	0101
(14)	$3\bar{0}$	(35)	32	(56)	$\bar{3}\bar{4}$	(77)	012	(98)	$\bar{0}\bar{2}0$	(119)	$01\bar{0}\bar{1}$
(15)	$4\bar{0}$	(36)	34	(57)	$\bar{4}\bar{0}$	(78)	021	(99)	$\bar{0}\bar{3}0$	(120)	$\bar{0}\bar{1}\bar{0}\bar{1}$
(16)	$\bar{0}1$	(37)	40	(58)	$\bar{4}\bar{1}$	(79)	031	(100)	$\bar{0}\bar{4}0$	(121)	$010\bar{1}$
(17)	$\bar{1}0$	(38)	41	(59)	$\bar{4}\bar{2}$	(80)	041	(101)	$\bar{1}\bar{0}1$	(122)	$101\bar{0}$
(18)	$\bar{2}0$	(39)	42	(60)	$\bar{4}\bar{3}$	(81)	102	(102)	$\bar{0}\bar{1}\bar{0}$	(123)	$020\bar{2}$
(19)	$\bar{3}0$	(40)	43	(61)	$0\bar{1}0$	(82)	120	(103)	$\bar{1}\bar{0}\bar{1}$	(124)	$030\bar{3}$
(20)	$\bar{4}0$	(41)	$\bar{0}\bar{1}$	(62)	$0\bar{1}\bar{0}$	(83)	130	(104)	$\bar{0}\bar{2}\bar{0}$	(125)	$040\bar{4}$

Having labeled each of the 126 distinct right cosets of N in G, we may express the permutation representation of the symmetric generator $t \sim t_0$ in its action on the right cosets of N in G as

$\phi(t) \sim \phi(t_0)$: (126 1 6)(2 25 12)(3 29 13)(4 33 14)(5 37 15)(7 17 45)(8 18 49)(9 19 53)

(10 20 57)(11 61 62)(16 107 112)(21 67 72)(22 68 73)(23 69 74)(24 70 75)(26 82 50)

(27 83 54)(28 84 58)(30 86 46)(31 87 55)(32 88 59)(34 90 47)(35 91 51)(36 92 60)

(38 94 48)(39 95 52)(40 96 56)(41 97 102)(42 98 104)(43 99 105)(44 100 106)(63 77 114)

(64 78 115)(65 79 116)(66 80 117)(71 118 122)(76 101 119)(81 108 121)(85 109 123)

(89 110 124)(93 111 125)(103 113 120),

we may express the permutation representation of the symmetric generator $t^x \sim t_1$ in its action on the right cosets of N in G as

$\phi(t)^{\phi(x)} \sim \phi(t_1)$: (126 2 7)(1 21 11)(3 30 13)(4 34 14)(5 38 15)(6 16 41)(8 18 50)(9 19 54)

(10 20 58)(12 61 63)(17 108 112)(25 71 76)(26 68 73)(27 69 74)(28 70 75)(22 78 49)

(23 79 53)(24 80 57)(29 85 42)(31 87 55)(32 88 59)(33 89 43)(35 91 51)(36 92 60)

(37 93 44)(39 95 52)(40 96 56)(45 101 103)(46 98 104)(47 99 105)(48 100 106)(62 81 113)

(64 82 115)(65 83 116)(66 84 117)(67 118 121)(72 97 119)(77 107 122)(86 109 123)

(90 110 124)(94 111 125)(102 114 120),

we may express the permutation representation of the symmetric generator $t^{x^2} \sim t_2$ in its action on the right cosets of N in G as

$\phi(t)^{\phi(x)^2} \sim \phi(t_2)$: (126 3 8)(1 22 11)(2 26 12)(4 35 14)(5 39 15)(6 16 42)(7 17 46)(9 19 55)

(10 20 59)(13 61 64)(18 109 112)(29 71 76)(30 67 72)(31 69 74)(32 70 75)(21 77 45)

(23 79 53)(24 80 57)(25 81 41)(27 83 54)(28 84 58)(33 89 43)(34 90 47)(36 92 60)

(37 93 44)(38 94 48)(40 96 56)(49 101 103)(50 97 102)(51 99 105)(52 100 106)(62 85 113)

(63 86 114)(65 87 116)(66 88 117)(68 118 123)(73 98 119)(78 107 122)(82 108 121)

(91 110 124)(95 111 125)(104 115 120),

we may express the permutation representation of the symmetric generator $t^{x^3} \sim t_3$ in its action on the right cosets of N in G as

$\phi(t)^{\phi(x)^3} \sim \phi(t_3)$: (126 4 9)(1 23 11)(2 27 12)(3 31 13)(5 40 15)(6 16 43)(7 17 47)(8 18 51)

(10 20 60)(14 61 65)(19 110 112)(33 71 76)(34 67 72)(35 68 73)(36 70 75)(21 77 45)

(22 78 49)(24 80 57)(25 81 41)(26 82 50)(28 84 58)(29 85 42)(30 86 46)(32 88 59)

(37 93 44)(38 94 48)(39 95 52)(53 101 103)(54 97 102)(55 98 104)(56 100 106)

(62 89 113)(63 90 114)(64 91 115)(66 92 117)(69 118 124)(74 99 119)(79 107 122)

(83 108 121)(87 109 123)(96 111 125)(105 116 120),

we may express the permutation representation of the symmetric generator $t^{x^4} \sim t_4$ in its action on the right cosets of N in G as

$\phi(t)^{\phi(x)^4} \sim \phi(t_4)$: (126 5 10)(1 24 11)(2 28 12)(3 32 13)(4 36 14)(6 16 44)(7 17 48)(8 18 52)

(9 19 56)(15 61 66)(20 111 112)(37 71 76)(38 67 72)(39 68 73)(40 69 74)(21 77 45)

(22 78 49)(23 79 53)(25 81 41)(26 82 50)(27 83 54)(29 85 42)(30 86 46)(31 87 55)

(33 89 43)(34 90 47)(35 91 51)(57 101 103)(58 97 102)(59 98 104)(60 99 105)(62 93 113)

(63 94 114)(64 95 115)(65 96 116)(70 118 125)(75 100 119)(80 107 122)(84 108 121)

(88 109 123)(92 110 124)(106 117 120),

we may express the permutation representation of the generator $x \sim (0\ 1\ 2\ 3\ 4)$ in its action on the right cosets of N in G as

$\phi(x) \sim \phi((0\,1\,2\,3\,4))$: (1 2 3 4 5)(6 7 8 9 10)(11 12 13 14 15)(16 17 18 19 20)(21 26 31 36 37)

(22 27 32 33 38)(23 28 29 34 39)(24 25 30 35 40)(41 46 51 56 57)(42 47 52 53 58)

(43 48 49 54 59)(44 45 50 55 60)(62 63 64 65 66)(67 68 69 70 71)(72 73 74 75 76)

(97 98 99 100 101)(77 82 87 92 93)(78 83 88 89 94)(79 84 85 90 95)(80 81 86 91 96)

(102 104 105 106 103)(107 108 109 110 111)(113 114 115 116 117)(121 123 124 125 122),

and we may express the permutation representation of the generator $y \sim (3\ 4)$ in its action on the right cosets of N in G as

$\phi(y) \sim \phi((3\ 4))$: (4 5)(9 10)(19 20)(14 15)(23 24)(27 28)(31 32)(33 37)(34 38)(35 39)

(36 40)(43 44)(47 48)(51 52)(53 57)(54 58)(55 59)(56 60)(65 66)(69 70)(74 75)(99 100)

(79 80)(83 84)(87 88)(89 93)(90 94)(91 95)(92 96)(105 106)(110 111)(116 117)(124 125).

6.5 Proof of Isomorphism between G and $S_7 \times 3$

We now demonstrate that $G \cong S_7 \times 3$.

Proof. To prove that $G \cong S_7 \times 3$, we must first show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and that $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 15120$ (from which we can conclude $G \cong \langle \phi(x), \phi(y), \phi(t) \rangle$), and we must next show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong S_7 \times 3$ (from which we can conclude $S_7 \times 3$ is a homomorphic image of G and $G \cong S_7 \times 3$).

We first show $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G and $|G| = |\langle\phi(x), \phi(y), \phi(t)\rangle| = 15120$. From our construction of G using manual double coset enumeration of G over S_5, illustrated by the Cayley Diagram in Figure 6.1, we concluded that group G defined by the symmetric presentation must contain a homomorphic image of $N \cong S_5$ whose index $[G : N]$ is at most 126:

$$[G:N] = \frac{|N|}{|N^{(*)}|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(\bar{0})}|} + \frac{|N|}{|N^{(0\bar{1})}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(\bar{0}\bar{1})}|} + \frac{|N|}{|N^{(\bar{0}1)}|}$$
$$+\frac{|N|}{|N^{(0\bar{1}0)}|} + \frac{|N|}{|N^{(0\bar{1}\bar{0})}|} + \frac{|N|}{|N^{(010)}|} + \frac{|N|}{|N^{(01\bar{0})}|} + \frac{|N|}{|N^{(012)}|} + \frac{|N|}{|N^{(\bar{0}\bar{1}0)}|} + \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{0})}|}$$
$$+\frac{|N|}{|N^{(\bar{0}10)}|} + \frac{|N|}{|N^{(\bar{0}1\bar{0})}|} + \frac{|N|}{|N^{(0\bar{1}0\bar{1})}|} + \frac{|N|}{|N^{(0101)}|} + \frac{|N|}{|N^{(01\bar{0}\bar{1})}|} + \frac{|N|}{|N^{(\bar{0}\bar{1}0\bar{1})}|}$$
$$+\frac{|N|}{|N^{(010\bar{1})}|} \leq \frac{120}{120} + \frac{120}{24} + \frac{120}{24} + \frac{120}{24} + \frac{120}{6} + \frac{120}{6} + \frac{120}{24} + \frac{120}{120} + \frac{120}{24} + \frac{120}{24}$$
$$+\frac{120}{24} + \frac{120}{6} + \frac{120}{24} + \frac{120}{24} + \frac{120}{24} + \frac{120}{120} + \frac{120}{24} + \frac{120}{120} + \frac{120}{120} + \frac{120}{120} + \frac{120}{24}$$
$$= 1+5+5+5+20+20+5+1+5+5+5+20+5+5+5+1+5+1+1+1+5 = 126$$

Since the index of N in G is at most 126, and since $|G| = \frac{|G|}{|N|} \cdot |N|$, the order of the homomorphic image group G is at most 15120:

$$|G| = \frac{|G|}{|N|} \cdot |N| \leq 126 \cdot |N| = 126 \cdot 120 = 15120 \Rightarrow |G| \leq 15120$$

We now consider $\langle\phi(x), \phi(y), \phi(t)\rangle$. Note that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a group generated by the permutation representations of the generators x, y, and t and, as such, it is a subgroup of the symmetric group S_{126} acting on the one hundred twenty-six right cosets of N in G. We now show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G and, therefore, that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle| = 15120$. To show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we first demonstrate that $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{126}$ is a homomorphic image of $\bar{G}$. Now, recall that $\bar{G} = \langle x, y, t\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$, and its presentation is given by

$$\bar{G} = \langle x, y, t \mid x^5 = y^2 = (yx)^4 = [x, y]^3 = t^3 = [t, y] = [t^x, y] = [t^{x^2}, y] = e\rangle,$$

where $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (3\ 4)$, and $t \sim t_0$, and $N = \langle x, y\rangle \cong S_5$. Let $\alpha : \bar{G} \longrightarrow \langle\phi(x), \phi(y), \phi(t)\rangle$ be a mapping from $\bar{G}$ to $\langle\phi(x), \phi(y), \phi(t)\rangle$. We note first that the mapping $\alpha : \bar{G} \longrightarrow \langle \phi(x), \phi(y), \phi(t) \rangle$ is well defined. The generators $\phi(x), \phi(y)$, and $\phi(t)$ are the permutation representations of $x \sim (0\ 1\ 2\ 3\ 4)$, $y \sim (3\ 4)$, and $t \sim t_0$ on 126 letters. Since the order of $\phi(x)$ is 5, the order of $\phi(y)$ is 2, and the order of $\phi(x)\phi(y)$ is 4, we conclude $\langle \phi(x), \phi(y) \rangle \cong N \cong S_5$. Moreover, we demonstrate below that $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N \cong S_5$. Now, given $t \sim t_0$, we see that

$\phi(t)^{\phi(x)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4))} =$ [(126 1 6)(2 25 12)(3 29 13)(4 33 14)(5 37 15)(7 17 45)

(8 18 49)(9 19 53)(10 20 57)(11 61 62)(16 107 112)(21 67 72)(22 68 73)(23 69 74)(24 70 75)

(26 82 50)(27 83 54)(28 84 58)(30 86 46)(31 87 55)(32 88 59)(34 90 47)(35 91 51)

(36 92 60)(38 94 48)(39 95 52)(40 96 56)(41 97 102)(42 98 104)(43 99 105)(44 100 106)

(63 77 114)(64 78 115)(65 79 116)(66 80 117)(71 118 122)(76 101 119)(81 108 121)

(85 109 123)(89 110 124)(93 111 125)(103 113 120)]$^{\phi((0\ 1\ 2\ 3\ 4))}$

= [(1 2 3 4 5)(6 7 8 9 10)(11 12 13 14 15)(16 17 18 19 20)(21 26 31 36 37)

(22 27 32 33 38)(23 28 29 34 39)(24 25 30 35 40)(41 46 51 56 57)(42 47 52 53 58)

(43 48 49 54 59)(44 45 50 55 60)(62 63 64 65 66)(67 68 69 70 71)(72 73 74 75 76)

(97 98 99 100 101)(77 82 87 92 93)(78 83 88 89 94)(79 84 85 90 95)(80 81 86 91 96)

(102 104 105 106 103)(107 108 109 110 111)(113 114 115 116 117)(121 123 124 125 122)]

[(126 1 6)(2 25 12)(3 29 13)(4 33 14)(5 37 15)(7 17 45)(8 18 49)(9 19 53)(10 20 57)(11 61 62)

(16 107 112)(21 67 72)(22 68 73)(23 69 74)(24 70 75)(26 82 50)(27 83 54)(28 84 58)

(30 86 46)(31 87 55)(32 88 59)(34 90 47)(35 91 51)(36 92 60)(38 94 48)(39 95 52)

(40 96 56)(41 97 102)(42 98 104)(43 99 105)(44 100 106)(63 77 114)(64 78 115)(65 79 116)

(66 80 117)(71 118 122)(76 101 119)(81 108 121)(85 109 123)(89 110 124)(93 111 125)

(103 113 120)][(1 5 4 3 2)(6 10 9 8 7)(11 15 14 13 12)(16 20 19 18 17)(21 37 36 31 26)

(22 38 33 32 27)(23 39 34 29 28)(24 40 35 30 25)(41 57 56 51 46)(42 58 53 52 47)

(43 59 54 49 48)(44 60 55 50 45)(62 66 65 64 63)(67 71 70 69 68)(72 76 75 74 73)

(97 101 100 99 98)(77 93 92 87 82)(78 94 89 88 83)(79 95 90 85 84)(80 96 91 86 81)

(102 103 106 105 104)(107 111 110 109 108)(113 117 116 115 114)(121 122 125 124 123)]

= (126 2 7)(1 21 11)(3 30 13)(4 34 14)(5 38 15)(6 16 41)(8 18 50)(9 19 54)(10 20 58)

(12 61 63)(17 108 112)(25 71 76)(26 68 73)(27 69 74)(28 70 75)(22 78 49)(23 79 53)

(24 80 57)(29 85 42)(31 87 55)(32 88 59)(33 89 43)(35 91 51)(36 92 60)(37 93 44)

(39 95 52)(40 96 56)(45 101 103)(46 98 104)(47 99 105)(48 100 106)(62 81 113)(64 82 115)

(65 83 116)(66 84 117)(67 118 121)(72 97 119)(77 107 122)(86 109 123)(90 110 124)

(94 111 125)(102 114 120) $= \phi(t_1) \sim \phi(t^x)$,

and further that

$\phi(t)^{\phi(x^2)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^2)}$ = [(126 1 6)(2 25 12)(3 29 13)(4 33 14)(5 37 15)(7 17 45)

(8 18 49)(9 19 53)(10 20 57)(11 61 62)(16 107 112)(21 67 72)(22 68 73)(23 69 74)

(24 70 75)(26 82 50)(27 83 54)(28 84 58)(30 86 46)(31 87 55)(32 88 59)(34 90 47)

(35 91 51)(36 92 60)(38 94 48)(39 95 52)(40 96 56)(41 97 102)(42 98 104)(43 99 105)

(44 100 106)(63 77 114)(64 78 115)(65 79 116)(66 80 117)(71 118 122)(76 101 119)

(81 108 121)(85 109 123)(89 110 124)(93 111 125)(103 113 120)]$^{\phi((0\ 1\ 2\ 3\ 4)^2)}$

= [(1 3 5 2 4)(6 8 10 7 9)(11 13 15 12 14)(16 18 20 17 19)(21 31 37 26 36)(22 32 38 27 33)

(23 29 39 28 34)(24 30 40 25 35)(41 51 57 46 56)(42 52 58 47 53)(43 49 59 48 54)

(44 50 60 45 55)(62 64 66 63 65)(67 69 71 68 70)(72 74 76 73 75)(97 99 101 98 100)

(77 87 93 82 92)(78 88 94 83 89)(79 85 95 84 90)(80 86 96 81 91)(102 105 103 104 106)

(107 109 111 108 110)(113 115 117 114 116)(121 124 122 123 125)][(126 1 6)(2 25 12)

(3 29 13)(4 33 14)(5 37 15)(7 17 45)(8 18 49)(9 19 53)(10 20 57)(11 61 62)(16 107 112)

(21 67 72)(22 68 73)(23 69 74)(24 70 75)(26 82 50)(27 83 54)(28 84 58)(30 86 46)

(31 87 55)(32 88 59)(34 90 47)(35 91 51)(36 92 60)(38 94 48)(39 95 52)(40 96 56)

(41 97 102)(42 98 104)(43 99 105)(44 100 106)(63 77 114)(64 78 115)(65 79 116)(66 80 117)

$$(71\ 118\ 122)(76\ 101\ 119)(81\ 108\ 121)(85\ 109\ 123)(89\ 110\ 124)(93\ 111\ 125)(103\ 113\ 120)]$$

$$[(1\ 4\ 2\ 5\ 3)(6\ 9\ 7\ 10\ 8)(11\ 14\ 12\ 15\ 13)(16\ 19\ 17\ 20\ 18)(21\ 36\ 26\ 37\ 31)$$

$$(22\ 33\ 27\ 38\ 32)(23\ 34\ 28\ 39\ 29)(24\ 35\ 25\ 40\ 30)(41\ 56\ 46\ 57\ 51)(42\ 53\ 47\ 58\ 52)$$

$$(43\ 54\ 48\ 59\ 49)(44\ 55\ 45\ 60\ 50)(62\ 65\ 63\ 66\ 64)(67\ 70\ 68\ 71\ 69)(72\ 75\ 73\ 76\ 74)$$

$$(97\ 100\ 98\ 101\ 99)(77\ 92\ 82\ 93\ 87)(78\ 89\ 83\ 94\ 88)(79\ 90\ 84\ 95\ 85)(80\ 91\ 81\ 96\ 86)$$

$$(102\ 106\ 104\ 103\ 105)(107\ 110\ 108\ 111\ 109)(113\ 116\ 114\ 117\ 115)(121\ 125\ 123\ 122\ 124)]$$

$$= (126\ 3\ 8)(1\ 22\ 11)(2\ 26\ 12)(4\ 35\ 14)(5\ 39\ 15)(6\ 16\ 42)(7\ 17\ 46)(9\ 19\ 55)(10\ 20\ 59)$$

$$(13\ 61\ 64)(18\ 109\ 112)(29\ 71\ 76)(30\ 67\ 72)(31\ 69\ 74)(32\ 70\ 75)(21\ 77\ 45)(23\ 79\ 53)$$

$$(24\ 80\ 57)(25\ 81\ 41)(27\ 83\ 54)(28\ 84\ 58)(33\ 89\ 43)(34\ 90\ 47)(36\ 92\ 60)(37\ 93\ 44)$$

$$(38\ 94\ 48)(40\ 96\ 56)(49\ 101\ 103)(50\ 97\ 102)(51\ 99\ 105)(52\ 100\ 106)(62\ 85\ 113)$$

$$(63\ 86\ 114)(65\ 87\ 116)(66\ 88\ 117)(68\ 118\ 123)(73\ 98\ 119)(78\ 107\ 122)(82\ 108\ 121)$$

$$(91\ 110\ 124)(95\ 111\ 125)(104\ 115\ 120) = \phi(t_2) \sim \phi(t^{x^2}),$$

and further that

$$\phi(t)^{\phi(x^3)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^3)} = [(126\ 1\ 6)(2\ 25\ 12)(3\ 29\ 13)(4\ 33\ 14)(5\ 37\ 15)(7\ 17\ 45)$$

$$(8\ 18\ 49)(9\ 19\ 53)(10\ 20\ 57)(11\ 61\ 62)(16\ 107\ 112)(21\ 67\ 72)(22\ 68\ 73)(23\ 69\ 74)$$

$$(24\ 70\ 75)(26\ 82\ 50)(27\ 83\ 54)(28\ 84\ 58)(30\ 86\ 46)(31\ 87\ 55)(32\ 88\ 59)(34\ 90\ 47)$$

$$(35\ 91\ 51)(36\ 92\ 60)(38\ 94\ 48)(39\ 95\ 52)(40\ 96\ 56)(41\ 97\ 102)(42\ 98\ 104)(43\ 99\ 105)$$

$$(44\ 100\ 106)(63\ 77\ 114)(64\ 78\ 115)(65\ 79\ 116)(66\ 80\ 117)(71\ 118\ 122)(76\ 101\ 119)$$

$$(81\ 108\ 121)(85\ 109\ 123)(89\ 110\ 124)(93\ 111\ 125)(103\ 113\ 120)]^{\phi((0\ 1\ 2\ 3\ 4)^3)}$$

$$= [(1\ 4\ 2\ 5\ 3)(6\ 9\ 7\ 10\ 8)(11\ 14\ 12\ 15\ 13)(16\ 19\ 17\ 20\ 18)(21\ 36\ 26\ 37\ 31)$$

$$(22\ 33\ 27\ 38\ 32)(23\ 34\ 28\ 39\ 29)(24\ 35\ 25\ 40\ 30)(41\ 56\ 46\ 57\ 51)(42\ 53\ 47\ 58\ 52)$$

$$(43\ 54\ 48\ 59\ 49)(44\ 55\ 45\ 60\ 50)(62\ 65\ 63\ 66\ 64)(67\ 70\ 68\ 71\ 69)(72\ 75\ 73\ 76\ 74)$$

$$(97\ 100\ 98\ 101\ 99)(77\ 92\ 82\ 93\ 87)(78\ 89\ 83\ 94\ 88)(79\ 90\ 84\ 95\ 85)(80\ 91\ 81\ 96\ 86)$$

$$(102\ 106\ 104\ 103\ 105)(107\ 110\ 108\ 111\ 109)(113\ 116\ 114\ 117\ 115)(121\ 125\ 123\ 122\ 124)]$$

$$[(126\ 1\ 6)(2\ 25\ 12)(3\ 29\ 13)(4\ 33\ 14)(5\ 37\ 15)(7\ 17\ 45)(8\ 18\ 49)(9\ 19\ 53)(10\ 20\ 57)$$
$$(11\ 61\ 62)(16\ 107\ 112)(21\ 67\ 72)(22\ 68\ 73)(23\ 69\ 74)(24\ 70\ 75)(26\ 82\ 50)(27\ 83\ 54)$$
$$(28\ 84\ 58)(30\ 86\ 46)(31\ 87\ 55)(32\ 88\ 59)(34\ 90\ 47)(35\ 91\ 51)(36\ 92\ 60)(38\ 94\ 48)$$
$$(39\ 95\ 52)(40\ 96\ 56)(41\ 97\ 102)(42\ 98\ 104)(43\ 99\ 105)(44\ 100\ 106)(63\ 77\ 114)(64\ 78\ 115)$$
$$(65\ 79\ 116)(66\ 80\ 117)(71\ 118\ 122)(76\ 101\ 119)(81\ 108\ 121)(85\ 109\ 123)(89\ 110\ 124)$$
$$(93\ 111\ 125)(103\ 113\ 120)][(1\ 3\ 5\ 2\ 4)(6\ 8\ 10\ 7\ 9)(11\ 13\ 15\ 12\ 14)(16\ 18\ 20\ 17\ 19)$$
$$(21\ 31\ 37\ 26\ 36)(22\ 32\ 38\ 27\ 33)(23\ 29\ 39\ 28\ 34)(24\ 30\ 40\ 25\ 35)(41\ 51\ 57\ 46\ 56)$$
$$(42\ 52\ 58\ 47\ 53)(43\ 49\ 59\ 48\ 54)(44\ 50\ 60\ 45\ 55)(62\ 64\ 66\ 63\ 65)(67\ 69\ 71\ 68\ 70)$$
$$(72\ 74\ 76\ 73\ 75)(97\ 99\ 101\ 98\ 100)(77\ 87\ 93\ 82\ 92)(78\ 88\ 94\ 83\ 89)(79\ 85\ 95\ 84\ 90)$$
$$(80\ 86\ 96\ 81\ 91)(102\ 105\ 103\ 104\ 106)(107\ 109\ 111\ 108\ 110)(113\ 115\ 117\ 114\ 116)$$
$$(121\ 124\ 122\ 123\ 125)] = (126\ 4\ 9)(1\ 23\ 11)(2\ 27\ 12)(3\ 31\ 13)(5\ 40\ 15)(6\ 16\ 43)$$
$$(7\ 17\ 47)(8\ 18\ 51)(10\ 20\ 60)(14\ 61\ 65)(19\ 110\ 112)(33\ 71\ 76)(34\ 67\ 72)(35\ 68\ 73)(36\ 70\ 75)$$
$$(21\ 77\ 45)(22\ 78\ 49)(24\ 80\ 57)(25\ 81\ 41)(26\ 82\ 50)(28\ 84\ 58)(29\ 85\ 42)(30\ 86\ 46)$$
$$(32\ 88\ 59)(37\ 93\ 44)(38\ 94\ 48)(39\ 95\ 52)(53\ 101\ 103)(54\ 97\ 102)(55\ 98\ 104)(56\ 100\ 106)$$
$$(62\ 89\ 113)(63\ 90\ 114)(64\ 91\ 115)(66\ 92\ 117)(69\ 118\ 124)(74\ 99\ 119)(79\ 107\ 122)$$
$$(83\ 108\ 121)(87\ 109\ 123)(96\ 111\ 125)(105\ 116\ 120) = \phi(t_3) \sim \phi(t^{x^3}),$$

and further that

$$\phi(t)^{\phi(x^4)} \sim \phi(t_0)^{\phi((0\ 1\ 2\ 3\ 4)^4)} = [(126\ 1\ 6)(2\ 25\ 12)(3\ 29\ 13)(4\ 33\ 14)(5\ 37\ 15)(7\ 17\ 45)$$
$$(8\ 18\ 49)(9\ 19\ 53)(10\ 20\ 57)(11\ 61\ 62)(16\ 107\ 112)(21\ 67\ 72)(22\ 68\ 73)(23\ 69\ 74)(24\ 70\ 75)$$
$$(26\ 82\ 50)(27\ 83\ 54)(28\ 84\ 58)(30\ 86\ 46)(31\ 87\ 55)(32\ 88\ 59)(34\ 90\ 47)(35\ 91\ 51)$$
$$(36\ 92\ 60)(38\ 94\ 48)(39\ 95\ 52)(40\ 96\ 56)(41\ 97\ 102)(42\ 98\ 104)(43\ 99\ 105)(44\ 100\ 106)$$
$$(63\ 77\ 114)(64\ 78\ 115)(65\ 79\ 116)(66\ 80\ 117)(71\ 118\ 122)(76\ 101\ 119)(81\ 108\ 121)$$
$$(85\ 109\ 123)(89\ 110\ 124)(93\ 111\ 125)(103\ 113\ 120)]^{\phi((0\ 1\ 2\ 3\ 4)^4)}$$
$$= [(1\ 5\ 4\ 3\ 2)(6\ 10\ 9\ 8\ 7)(11\ 15\ 14\ 13\ 12)(16\ 20\ 19\ 18\ 17)(21\ 37\ 36\ 31\ 26)(22\ 38\ 33\ 32\ 27)$$

(23 39 34 29 28)(24 40 35 30 25)(41 57 56 51 46)(42 58 53 52 47)(43 59 54 49 48)

(44 60 55 50 45)(62 66 65 64 63)(67 71 70 69 68)(72 76 75 74 73)(97 101 100 99 98)

(77 93 92 87 82)(78 94 89 88 83)(79 95 90 85 84)(80 96 91 86 81)(102 103 106 105 104)

(107 111 110 109 108)(113 117 116 115 114)(121 122 125 124 123)][(126 1 6)(2 25 12)

(3 29 13)(4 33 14)(5 37 15)(7 17 45)(8 18 49)(9 19 53)(10 20 57)(11 61 62)(16 107 112)

(21 67 72)(22 68 73)(23 69 74)(24 70 75)(26 82 50)(27 83 54)(28 84 58)(30 86 46)

(31 87 55)(32 88 59)(34 90 47)(35 91 51)(36 92 60)(38 94 48)(39 95 52)(40 96 56)

(41 97 102)(42 98 104)(43 99 105)(44 100 106)(63 77 114)(64 78 115)(65 79 116)(66 80 117)

(71 118 122)(76 101 119)(81 108 121)(85 109 123)(89 110 124)(93 111 125)(103 113 120)]

[(1 2 3 4 5)(6 7 8 9 10)(11 12 13 14 15)(16 17 18 19 20)(21 26 31 36 37)

(22 27 32 33 38)(23 28 29 34 39)(24 25 30 35 40)(41 46 51 56 57)(42 47 52 53 58)

(43 48 49 54 59)(44 45 50 55 60)(62 63 64 65 66)(67 68 69 70 71)(72 73 74 75 76)

(97 98 99 100 101)(77 82 87 92 93)(78 83 88 89 94)(79 84 85 90 95)(80 81 86 91 96)

(102 104 105 106 103)(107 108 109 110 111)(113 114 115 116 117)(121 123 124 125 122)]

= (126 5 10)(1 24 11)(2 28 12)(3 32 13)(4 36 14)(6 16 44)(7 17 48)(8 18 52)(9 19 56)

(15 61 66)(20 111 112)(37 71 76)(38 67 72)(39 68 73)(40 69 74)(21 77 45)(22 78 49)

(23 79 53)(25 81 41)(26 82 50)(27 83 54)(29 85 42)(30 86 46)(31 87 55)(33 89 43)

(34 90 47)(35 91 51)(57 101 103)(58 97 102)(59 98 104)(60 99 105)(62 93 113)

(63 94 114)(64 95 115)(65 96 116)(70 118 125)(75 100 119)(80 107 122)(84 108 121)

(88 109 123)(92 110 124)(106 117 120) $= \phi(t_4) \sim \phi(t^{x^4})$.

Therefore, $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N \cong S_5$; these conjugates are, namely, $\phi(t) \sim \phi(t_0)$, $\phi(t^x) \sim \phi(t_1)$, $\phi(t^{x^2}) \sim \phi(t_2)$, $\phi(t^{x^3}) \sim \phi(t_3)$, and $\phi(t^{x^4}) \sim \phi(t_4)$. Since $\langle \phi(x), \phi(y) \rangle \cong N \cong S_5$ and since $\phi(t)$ has exactly five conjugates under conjugation by the elements of $\langle \phi(x), \phi(y) \rangle \cong N$, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $\bar{G} = \langle x, y, t \rangle$. That is, $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $3^{\star 5} : S_5$.

Next, to show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we must show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of $\bar{G}$ factored by the relations $(yxt)^6 = e$, $(t^{-1}t^x)^3 = e$, $(xyx^{-1}yxt^{-1}t^x)^2 = e$, and $(x^{-2}yx^2t)^{12} = e$; that is, we must show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$ factored by the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$. Let $\tilde{\alpha} : G \longrightarrow \langle\phi(x), \phi(y), \phi(t)\rangle$ be a mapping from G to $\langle\phi(x), \phi(y), \phi(t)\rangle$. We note that the mapping $\tilde{\alpha} : G \longrightarrow \langle\phi(x), \phi(y), \phi(t)\rangle$ is well-defined, and we know already that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$. Now, to show that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we need only demonstrate that the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$, which hold true in G, also hold true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{126}$.

To demonstrate that the relation $[(0\ 1\ 2\ 3)t_0]^6 = e$, or, equivalently, the relation $t_1t_0t_3t_2t_1t_0 = (0\ 2)(1\ 3)$, holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{126}$, we show that $\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^{x^3})\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 2)(1\ 3)) \sim \phi((yx)^2)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1), \phi(t_2), \phi(t_3)$, and $\phi(t_4)$ by $\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_2),$$

$$\phi(t_1)^{\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_3),$$

$$\phi(t_2)^{\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_0),$$

$$\phi(t_3)^{\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_1),$$

$$\phi(t_4)^{\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0)} = \phi(t_4)$$

We next conjugate the symmetric generators $\phi(t_0), \phi(t_1), \phi(t_2), \phi(t_3)$, and $\phi(t_4)$ by $\phi((0\ 2)(1\ 3))$. This gives us

$$\phi(t_0)^{\phi((0\ 2)(1\ 3))} = \phi(t_2),$$

$$\phi(t_1)^{\phi((0\ 2)(1\ 3))} = \phi(t_3),$$

$$\phi(t_2)^{\phi((0\ 2)(1\ 3))} = \phi(t_0),$$

$$\phi(t_3)^{\phi((0\ 2)(1\ 3))} = \phi(t_1),$$

$$\phi(t_4)^{\phi((0\ 2)(1\ 3))} = \phi(t_4)$$

Since $\phi(t_1)\phi(t_0)\phi(t_3)\phi(t_2)\phi(t_1)\phi(t_0) \sim \phi(t^x)\phi(t)\phi(t^{x^3})\phi(t^{x^2})\phi(t^x)\phi(t) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 2)(1\ 3)) \sim \phi((yx)^2)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation, we conclude that the relation $[(0\ 1\ 2\ 3)t_0]^6 = e$, which holds true in G, also holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{126}$.

To demonstrate that the relation $[t_0^{-1}t_1]^3 = e$, or, equivalently, the relation $t_0^{-1}t_1t_0^{-1}t_1t_0^{-1}t_1 = e$, holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{126}$, we must show that $\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1) \sim \phi(t^{-1})\phi(t^x)\phi(t^{-1})\phi(t^x)\phi(t^{-1})\phi(t^x) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0), \phi(t_1), \phi(t_2), \phi(t_3)$, and $\phi(t_4)$ by $\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)$. This gives us

$$\phi(t_0)^{\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)} = \phi(t_4)$$

We next conjugate the symmetric generators $\phi(t_0), \phi(t_1), \phi(t_2), \phi(t_3)$, and $\phi(t_4)$ by $\phi(e)$. This gives us

$$\phi(t_0)^{\phi(e)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(e)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(e)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(e)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(e)} = \phi(t_4)$$

Since $\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})\phi(t_1) \sim \phi(t^{-1})\phi(t^x)\phi(t^{-1})\phi(t^x)\phi(t^{-1})\phi(t^x) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the five symmetric generators $\phi(t_0)$,

$\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation, we conclude that the relation $[t_0^{-1}t_1]^3 = e$, which holds true in G, also holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{126}$.

To demonstrate that the relation $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, or, equivalently, the relation $t_1^{-1}t_2t_0^{-1}t_1 = (0\ 1\ 2)$, holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{126}$, we show that $\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1) \sim \phi((t^x)^{-1})\phi(t^{x^2})\phi(t^{-1})\phi(t^x) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 1\ 2)) \sim \phi(xyx^{-1}yxt^{-1})$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1)$. This gives us

$$\phi(t_0)^{\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1)} = \phi(t_1),$$

$$\phi(t_1)^{\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1)} = \phi(t_2),$$

$$\phi(t_2)^{\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1)} = \phi(t_0),$$

$$\phi(t_3)^{\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1)} = \phi(t_4)$$

We next conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi((0\ 1\ 2))$. This gives us

$$\phi(t_0)^{\phi((0\ 1\ 2))} = \phi(t_1),$$

$$\phi(t_1)^{\phi((0\ 1\ 2))} = \phi(t_2),$$

$$\phi(t_2)^{\phi((0\ 1\ 2))} = \phi(t_0),$$

$$\phi(t_3)^{\phi((0\ 1\ 2))} = \phi(t_3),$$

$$\phi(t_4)^{\phi((0\ 1\ 2))} = \phi(t_4)$$

Since $\phi(t_1^{-1})\phi(t_2)\phi(t_0^{-1})\phi(t_1) \sim \phi((t^x)^{-1})\phi(t^{x^2})\phi(t^{-1})\phi(t^x) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that $\phi((0\ 1\ 2)) \sim \phi(xyx^{-1}yxt^{-1})$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation, we conclude that the relation $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, which holds true in G, also holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{126}$.

To demonstrate that the relation $[(0\ 1)t_0]^{12} = e$, or, equivalently, the relation $t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0 = e$, holds true in $\langle\phi(x), \phi(y), \phi(t)\rangle \leq S_{126}$, we show that

$\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)$
$\sim \phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation. We first conjugate the symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by $\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)$. This gives us

$$\phi(t_0)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)} = \phi(t_4)$$

We next conjugate the symmetric generators $\phi(t_0), \phi(t_1), \phi(t_2), \phi(t_3)$, and $\phi(t_4)$ by $\phi(e)$. This gives us

$$\phi(t_0)^{\phi(e)} = \phi(t_0),$$

$$\phi(t_1)^{\phi(e)} = \phi(t_1),$$

$$\phi(t_2)^{\phi(e)} = \phi(t_2),$$

$$\phi(t_3)^{\phi(e)} = \phi(t_3),$$

$$\phi(t_4)^{\phi(e)} = \phi(t_4)$$

Since $\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0)\phi(t_1)\phi(t_0) \sim$
$\phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t)\phi(t^x)\phi(t) \in S_{126}$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation in the same way that the identity element $\phi(e)$ acts on the five symmetric generators $\phi(t_0)$, $\phi(t_1)$, $\phi(t_2)$, $\phi(t_3)$, and $\phi(t_4)$ by conjugation, we conclude that the relation $[(0\ 1)t_0]^{12} = e$, which holds true in G, also holds true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{126}$.

Since $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$, and since the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$ hold true in $\langle \phi(x), \phi(y), \phi(t) \rangle \leq S_{126}$, we conclude that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$ factored by the relations $[(0\ 1\ 2\ 3)t_0]^6 = e$, $[t_0^{-1}t_1]^3 = e$, $[(0\ 1\ 2)t_0^{-1}t_1]^2 = e$, and $[(0\ 1)t_0]^{12} = e$; that is, we conclude that $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G.

More importantly, since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we have that $\langle\phi(x), \phi(y), \phi(t)\rangle \leq G$. In fact, since $\langle\phi(x), \phi(y), \phi(t)\rangle \leq G$, we have that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle|$. Since it is easily demonstrated, with MAGMA or by hand, that $|\langle\phi(x), \phi(y), \phi(t)\rangle| = 15120$, we conclude finally that $|G| \geq |\langle\phi(x), \phi(y), \phi(t)\rangle| = 15120$, that is, $|G| \geq 15120$. Given $|G| \leq 15120$ and $|G| \geq 15120$, we conclude $|G| = 15120$. Moreover, since $|\langle\phi(x), \phi(y), \phi(t)\rangle| = 15120 = |G|$ and since $\langle\phi(x), \phi(y), \phi(t)\rangle$ is a homomorphic image of G, we conclude

$$\langle\phi(x), \phi(y), \phi(t)\rangle \cong G.$$

We finally show that $\langle\phi(x), \phi(y), \phi(t)\rangle \cong S_7 \times 3$. Let $G_1 = \langle\phi(x), \phi(y), \phi(t)\rangle$. Now, with the help of MAGMA (see [BCP97]), we know that the elements

$a =$ (1 78 87 98 52 116 71)(2 58 90 57 119 63 115)(3 23 118 83 51 55 97)
(4 42 70 124 59 33 46)(5 53 32 61 120 121 19)(6 85 45 122 9 50 126)
(7 110 49 112 75 103 25)(8 74 125 35 95 82 66)(10 106 54 111 96 102 34)
(11 113 30 37 109 36 88)(12 84 108 22 40 17 99)(13 31 76 104 73 89 107)
(14 43 64 48 39 123 38)(15 62 92 26 81 105 65)(16 67 80 94 101 56 47)
(18 69 41 77 79 24 91)(20 28 60 27 72 114 68)(21 117 44 100 86 29 93),

$b =$ (1 13)(2 19)(4 66)(5 46)(7 60)(8 88)(11 56)(12 48)(15 86)(18 65)(23 110)(28 126)
(29 102)(31 91)(33 82)(34 71)(37 80)(39 108)(44 111)(45 114)(47 115)(49 83)(50 123)
(53 58)(54 98)(64 122)(67 90)(69 76)(72 112)(84 118)(87 104)(95 109)(100 105), and

$c =$ (1 109 122)(2 110 100)(3 81 121)(4 108 69)(5 118 65)(6 52 113)(7 44 115)
(8 123 104)(9 78 36)(10 94 68)(11 126 98)(12 91 33)(13 95 64)(14 89 125)(15 53 83)
(16 27 96)(17 79 124)(18 46 84)(19 23 105)(20 106 101)(21 119 103)(22 41 42)(24 59 99)
(25 117 63)(26 120 97)(28 54 56)(29 90 112)(30 85 116)(31 82 48)(32 51 62)(34 80 114)
(35 43 107)(37 45 71)(38 73 74)(39 76 66)(40 77 70)(47 60 111)(49 86 58)(50 87 88)
(55 92 61)(57 75 93)(67 72 102)

belong to G_1. (Note: The labels for the right cosets in this case were assigned by MAGMA and are different from the labels that we assigned earlier.) Therefore, $\langle a,b,c\rangle \leq G_1$, a permutation group on 15120 letters, is a permutation representation of G and, further, $|G_1| = 15120$. But $|\langle a,b,c\rangle| = 15120 = |G_1|$. Therefore, $G_1 = \langle a,b,c\rangle$. Moreover, $\langle a,b,c\rangle \cong S_7 \times 3 \cong \langle a,b,c|a^7 = b^2 = (ab)^6 = (a^{-2}(ab)^2)^3 = (a^{-2}ba^2b)^2 = c^3 = [c,b] = [c^a,b] = [c^{a^2},b] = e\rangle$. Therefore, $G_1 \cong S_7 \times 3$ and, since $G_1 = \langle \phi(x), \phi(y), \phi(t)\rangle \cong G$, we conclude $G \cong S_7 \times 3$.

□

6.6 Converting an Element of G from its Permutation Representation to its Symmetric Representation

To illustrate how a permutation representation of an element of $S_7 \times 3$ on 126 letters may be converted to its symmetric representation form, we consider the following example:

Example 6.1. Let $g \in G \cong S_7 \times 3$ and let $p = \phi(g) =$

$$(1\ 11\ 122\ 107\ 103\ 101)(2\ 14\ 123\ 111\ 102\ 99\ 3\ 15\ 121\ 110\ 104\ 100)$$
$$(4\ 13\ 125\ 108\ 105\ 98\ 5\ 12\ 124\ 109\ 106\ 97)(6\ 76\ 71\ 62\ 113\ 16)$$
$$(7\ 74\ 68\ 66\ 114\ 19\ 8\ 75\ 67\ 65\ 115\ 20)(9\ 73\ 70\ 63\ 116\ 18\ 10\ 72\ 69\ 64\ 117\ 17)$$
$$(21\ 53\ 78\ 24\ 45\ 79\ 22\ 57\ 77\ 23\ 49\ 80)(25\ 43\ 85\ 37\ 41\ 89\ 29\ 44\ 81\ 33\ 42\ 93)$$
$$(26\ 60\ 86\ 40\ 50\ 92\ 30\ 56\ 82\ 36\ 46\ 96)(27\ 51\ 88\ 38\ 54\ 91\ 32\ 48\ 83\ 35\ 59\ 94)$$
$$(28\ 47\ 87\ 39\ 58\ 90\ 31\ 52\ 84\ 34\ 55\ 95)(126\ 112\ 118\ 119\ 120\ 61)$$

be the permutation representation of g on 126 letters. Then $126^p = 112$ implies $N^p = Nt_0^{-1}t_1t_0^{-1}$, since 126 and 112 are labels for the right cosets N and $Nt_0^{-1}t_1t_0^{-1}$, respectively. Moreover, since $N^p = Np$ and $N^p = Nt_0^{-1}t_1t_0^{-1}$, we have that $Np = Nt_0^{-1}t_1t_0^{-1}$. Now, $Np = Nt_0^{-1}t_1t_0^{-1}$ implies that $p \in Nt_0^{-1}t_1t_0^{-1}$ which implies that $p \sim \pi t_0^{-1}t_1t_0^{-1}$ for some $\pi \in N$ or, more precisely, $p = \phi(\pi)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1})$ for some $\pi \in N$. To determine $\pi \in N \cong S_5$, we note first that $p = \phi(\pi)\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1}) \Rightarrow p(\phi(t_0^{-1}))^{-1}(\phi(t_1))^{-1}(\phi(t_0^{-1}))^{-1} = p\phi((t_0^{-1})^{-1})\phi(t_1^{-1})\phi((t_0^{-1})^{-1}) = p\phi(t_0)\phi(t_1^{-1})\phi(t_0) =$

$\phi(\pi)$. We then calculate the action of $\phi(\pi) = p\phi(t_0)\phi(t_1^{-1})\phi(t_0)$ on the symmetric generators t_i, where $i \in \{0, 1, 2, 3, 4\}$. Now, $\phi(\pi) = p\phi(t_0)\phi(t_1^{-1})\phi(t_0) =$

[(1 11 122 107 103 101)(2 14 123 111 102 99 3 15 121 110 104 100)

(4 13 125 108 105 98 5 12 124 109 106 97)(6 76 71 62 113 16)

(7 74 68 66 114 19 8 75 67 65 115 20)(9 73 70 63 116 18 10 72 69 64 117 17)

(21 53 78 24 45 79 22 57 77 23 49 80)(25 43 85 37 41 89 29 44 81 33 42 93)

(26 60 86 40 50 92 30 56 82 36 46 96)(27 51 88 38 54 91 32 48 83 35 59 94)

(28 47 87 39 58 90 31 52 84 34 55 95)(126 112 118 119 120 61)]

[(126 1 6)(2 25 12)(3 29 13)(4 33 14)(5 37 15)(7 17 45)(8 18 49)(9 19 53)

(10 20 57)(11 61 62)(16 107 112)(21 67 72)(22 68 73)(23 69 74)(24 70 75)(26 82 50)

(27 83 54)(28 84 58)(30 86 46)(31 87 55)(32 88 59)(34 90 47)(35 91 51)(36 92 60)

(38 94 48)(39 95 52)(40 96 56)(41 97 102)(42 98 104)(43 99 105)(44 100 106)(63 77 114)

(64 78 115)(65 79 116)(66 80 117)(71 118 122)(76 101 119)(81 108 121)(85 109 123)

(89 110 124)(93 111 125)(103 113 120)][(126 7 2)(1 11 21)(3 13 30)(4 14 34)(5 15 38)

(6 41 16)(8 50 18)(9 54 19)(10 58 20)(12 63 61)(17 112 108)(25 76 71)(26 73 68)

(27 74 69)(28 75 70)(22 49 78)(23 53 79)(24 57 80)(29 42 85)(31 55 87)(32 59 88)

(33 43 89)(35 51 91)(36 60 92)(37 44 93)(39 52 95)(40 56 96)(45 103 101)(46 104 98)

(47 105 99)(48 106 100)(62 113 81)(64 115 82)(65 116 83)(66 117 84)(67 121 118)(72 119 97)

(77 122 107)(86 123 109)(90 124 110)(94 125 111)(102 120 114)]

[(126 1 6)(2 25 12)(3 29 13)(4 33 14)(5 37 15)(7 17 45)(8 18 49)(9 19 53)

(10 20 57)(11 61 62)(16 107 112)(21 67 72)(22 68 73)(23 69 74)(24 70 75)(26 82 50)

(27 83 54)(28 84 58)(30 86 46)(31 87 55)(32 88 59)(34 90 47)(35 91 51)(36 92 60)

(38 94 48)(39 95 52)(40 96 56)(41 97 102)(42 98 104)(43 99 105)(44 100 106)(63 77 114)

(64 78 115)(65 79 116)(66 80 117)(71 118 122)(76 101 119)(81 108 121)(85 109 123)

(89 110 124)(93 111 125)(103 113 120)]

= (1 2 3 4 5)(6 7 8 9 10)(11 12 13 14 15)(16 17 18 19 20)(21 26 31 36 37)

(22 27 32 33 38)(23 28 29 34 39)(24 25 30 35 40)(41 46 51 56 57)(42 47 52 53 58)

(43 48 49 54 59)(44 45 50 55 60)(62 63 64 65 66)(67 68 69 70 71)(72 73 74 75 76)

(97 98 99 100 101)(77 82 87 92 93)(78 83 88 89 94)(79 84 85 90 95)(80 81 86 91 96)

(102 104 105 106 103)(107 108 109 110 111)(113 114 115 116 117)(121 123 124 125 122).

The element $\pi \sim \phi(\pi) = p\phi(t_0)\phi(t_1^{-1})\phi(t_0) =$

(1 2 3 4 5)(6 7 8 9 10)(11 12 13 14 15)(16 17 18 19 20)(21 26 31 36 37)

(22 27 32 33 38)(23 28 29 34 39)(24 25 30 35 40)(41 46 51 56 57)(42 47 52 53 58)

(43 48 49 54 59)(44 45 50 55 60)(62 63 64 65 66)(67 68 69 70 71)(72 73 74 75 76)

(97 98 99 100 101)(77 82 87 92 93)(78 83 88 89 94)(79 84 85 90 95)(80 81 86 91 96)

(102 104 105 106 103)(107 108 109 110 111)(113 114 115 116 117)(121 123 124 125 122)

acts on the right cosets Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4 via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. The mappings below illustrate this action:

$$Nt_0 = 1 \mapsto 1^p = 2 = Nt_1, \qquad Nt_1 = 2 \mapsto 2^p = 4 = Nt_3,$$

$$Nt_3 = 4 \mapsto 4^p = 3 = Nt_2, \qquad Nt_2 = 3 \mapsto 3^p = 5 = Nt_4,$$

$$Nt_4 = 5 \mapsto 5^p = 1 = Nt_0$$

Therefore, the element $\phi(\pi)$ acts as (0 1 3 2 4) on the right cosets Nt_0, Nt_1, Nt_2, Nt_3, and Nt_4, and so $\phi(\pi)$ is the permutation representation of $\pi = (0\ 1\ 3\ 2\ 4) \in S_5$ on 126 letters. Therefore, $\pi = (0\ 1\ 3\ 2\ 4)$ and $w = t_0^{-1}t_1t_0^{-1}$, and so the symmetric representation of g is $(0\ 1\ 3\ 2\ 4)t_0^{-1}t_1t_0^{-1}$.

6.7 Converting an Element of G from its Symmetric Representation to its Permutation Representation

To illustrate how an element of $S_7 \times 3$ in symmetric representation form may be converted to its permutation representation on 126 letters, we consider the following example:

Example 6.2. Let $g \in G \cong S_7 \times 3$ have the symmetric representation $g = (0\ 1\ 3\ 2\ 4)t_0^{-1}t_1t_0^{-1}$. To determine the permutation representation $p = \phi(g)$ of g, we first calculate the action of $\pi = (0\ 1\ 3\ 2\ 4)$ on the right cosets of N in G. Now, the element $\pi = (0\ 1\ 3\ 2\ 4)$ acts on the right cosets N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. To illustrate this action, we provide several examples below:

$$126 = N \mapsto N^{(0\ 1\ 3\ 2\ 4)} = N = 126$$

$$1 = Nt_0 \mapsto Nt_0^{(0\ 1\ 3\ 2\ 4)} = Nt_1 = 2$$

$$2 = Nt_1 \mapsto Nt_1^{(0\ 1\ 3\ 2\ 4)} = Nt_3 = 4$$

$$4 = Nt_3 \mapsto Nt_3^{(0\ 1\ 3\ 2\ 4)} = Nt_2 = 3$$

$$3 = Nt_2 \mapsto Nt_2^{(0\ 1\ 3\ 2\ 4)} = Nt_4 = 5$$

$$5 = Nt_4 \mapsto Nt_4^{(0\ 1\ 3\ 2\ 4)} = Nt_0 = 1$$

$$6 = Nt_0^{-1} \mapsto N(t_0^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_1^{-1} = 7$$

$$7 = Nt_1^{-1} \mapsto N(t_1^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_3^{-1} = 9$$

$$9 = Nt_3^{-1} \mapsto N(t_3^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_2^{-1} = 8$$

$$8 = Nt_2^{-1} \mapsto N(t_2^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_4^{-1} = 10$$

$$10 = Nt_4^{-1} \mapsto N(t_4^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_0^{-1} = 6$$

$$\vdots$$

$$121 = Nt_0t_1t_0t_1^{-1} \mapsto N(t_0t_1t_0t_1^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_1t_3t_1t_3^{-1} = Nt_0t_3t_0t_3^{-1} = 124$$

$$124 = Nt_0t_3t_0t_3^{-1} \mapsto N(t_0t_3t_0t_3^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_1t_2t_1t_2^{-1} = Nt_0t_2t_0t_2^{-1} = 123$$

$$123 = Nt_0t_2t_0t_2^{-1} \mapsto N(t_0t_2t_0t_2^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_1t_4t_1t_4^{-1} = Nt_0t_4t_0t_4^{-1} = 125$$

$$125 = Nt_0t_4t_0t_4^{-1} \mapsto N(t_0t_4t_0t_4^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_1t_0t_1t_0^{-1} = 122$$

$$122 = Nt_1t_0t_1t_0^{-1} \mapsto N(t_1t_0t_1t_0^{-1})^{(0\ 1\ 3\ 2\ 4)} = Nt_3t_1t_3t_1^{-1} = Nt_0t_1t_0t_1^{-1} = 121$$

Therefore, the permutation representation of $\pi = (0\ 1\ 3\ 2\ 4)$ is

$\phi(\pi) = $ (1 2 3 4 5)(6 7 8 9 10)(11 12 13 14 15)(16 17 18 19 20)(21 26 31 36 37)

(22 27 32 33 38)(23 28 29 34 39)(24 25 30 35 40)(41 46 51 56 57)(42 47 52 53 58)

(43 48 49 54 59)(44 45 50 55 60)(62 63 64 65 66)(67 68 69 70 71)(72 73 74 75 76)

(97 98 99 100 101)(77 82 87 92 93)(78 83 88 89 94)(79 84 85 90 95)(80 81 86 91 96)

(102 104 105 106 103)(107 108 109 110 111)(113 114 115 116 117)(121 123 124 125 122).

Similarly, we calculate the action of the symmetric generator t_0^{-1} on the right cosets of N in G. The symmetric generator t_0 acts on the right cosets of N in G via the mapping ϕ: $G \longrightarrow S_X$ defined by $\phi(t_0, Nw) = Nwt_0$. By this mapping, the permutation representation of t_0 in its action on the right cosets of N in G is

$\phi(t_0) = $ (126 1 6)(2 25 12)(3 29 13)(4 33 14)(5 37 15)(7 17 45)(8 18 49)(9 19 53)

(10 20 57)(11 61 62)(16 107 112)(21 67 72)(22 68 73)(23 69 74)(24 70 75)(26 82 50)

(27 83 54)(28 84 58)(30 86 46)(31 87 55)(32 88 59)(34 90 47)(35 91 51)(36 92 60)

(38 94 48)(39 95 52)(40 96 56)(41 97 102)(42 98 104)(43 99 105)(44 100 106)(63 77 114)

(64 78 115)(65 79 116)(66 80 117)(71 118 122)(76 101 119)(81 108 121)(85 109 123)

(89 110 124)(93 111 125)(103 113 120).

Now, since ϕ: $G \longrightarrow S_X$ is a group homomorphism, $(\phi(t_0))^{-1} = \phi(t_0^{-1})$. Therefore, the permutation representation of t_0^{-1} in its action on the right cosets of N in G is

$\phi(t_0^{-1}) = (\phi(t_0))^{-1} = $ (126 6 1)(2 12 25)(3 13 29)(4 14 33)(5 15 37)(7 45 17)(8 49 18)(9 53 19)

(10 57 20)(11 62 61)(16 112 107)(21 72 67)(22 73 68)(23 74 69)(24 75 70)(26 50 82)

(27 54 83)(28 58 84)(30 46 86)(31 55 87)(32 59 88)(34 47 90)(35 51 91)(36 60 92)

(38 48 94)(39 52 95)(40 56 96)(41 102 97)(42 104 98)(43 105 99)(44 106 100)(63 114 77)

(64 115 78)(65 116 79)(66 117 80)(71 122 118)(76 119 101)(81 121 108)(85 123 109)

(89 124 110)(93 125 111)(103 120 113).

Finally, we calculate the action of the symmetric generator t_1 on the right cosets of N in G. The symmetric generator t_1 acts on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_1, Nw) = Nwt_1$. By this mapping, the permutation representation of t_1 in its action on the right cosets of N in G is

$\phi(t_1) =$ (126 2 7)(1 21 11)(3 30 13)(4 34 14)(5 38 15)(6 16 41)(8 18 50)(9 19 54)

(10 20 58)(12 61 63)(17 108 112)(25 71 76)(26 68 73)(27 69 74)(28 70 75)(22 78 49)

(23 79 53)(24 80 57)(29 85 42)(31 87 55)(32 88 59)(33 89 43)(35 91 51)(36 92 60)

(37 93 44)(39 95 52)(40 96 56)(45 101 103)(46 98 104)(47 99 105)(48 100 106)(62 81 113)

(64 82 115)(65 83 116)(66 84 117)(67 118 121)(72 97 119)(77 107 122)(86 109 123)

(90 110 124)(94 111 125)(102 114 120).

Now, $(0\ 1\ 3\ 2\ 4)t_0^{-1}t_1t_0^{-1} \sim \phi((0\ 1\ 3\ 2\ 4))\phi(t_0^{-1})\phi(t_1)\phi(t_0^{-1}) =$

[(1 2 4 3 5)(6 7 9 8 10)(11 12 14 13 15)(16 17 19 18 20)(21 27 35 32 37)

(22 28 33 30 40)(23 26 36 29 38)(24 25 34 31 39)(41 47 55 52 57)(42 48 53 50 60)

(43 46 56 49 58)(44 45 54 51 59)(62 63 65 64 66)(67 69 68 70 71)(72 74 73 75 76)

(77 83 91 88 93)(78 84 89 86 96)(79 82 92 85 94)(80 81 90 87 95)(97 99 98 100 101)

(102 105 104 106 103)(107 108 110 109 111)(113 114 116 115 117)(121 124 123 125 122)]

[(126 6 1)(2 12 25)(3 13 29)(4 14 33)(5 15 37)(7 45 17)(8 49 18)(9 53 19)(10 57 20)

(11 62 61)(16 112 107)(21 72 67)(22 73 68)(23 74 69)(24 75 70)(26 50 82)(27 54 83)

(28 58 84)(30 46 86)(31 55 87)(32 59 88)(34 47 90)(35 51 91)(36 60 92)(38 48 94)

(39 52 95)(40 56 96)(41 102 97)(42 104 98)(43 105 99)(44 106 100)(63 114 77)(64 115 78)

(65 116 79)(66 117 80)(71 122 118)(76 119 101)(81 121 108)(85 123 109)(89 124 110)

(93 125 111)(103 120 113)][(126 2 7)(1 21 11)(3 30 13)(4 34 14)(5 38 15)(6 16 41)

(8 18 50)(9 19 54)(10 20 58)(12 61 63)(17 108 112)(25 71 76)(26 68 73)(27 69 74)

(28 70 75)(22 78 49)(23 79 53)(24 80 57)(29 85 42)(31 87 55)(32 88 59)(33 89 43)

(35 91 51)(36 92 60)(37 93 44)(39 95 52)(40 96 56)(45 101 103)(46 98 104)(47 99 105)

(48 100 106)(62 81 113)(64 82 115)(65 83 116)(66 84 117)(67 118 121)(72 97 119)
(77 107 122)(86 109 123)(90 110 124)(94 111 125)(102 114 120)][(126 6 1)(2 12 25)(3 13 29)
(4 14 33)(5 15 37)(7 45 17)(8 49 18)(9 53 19)(10 57 20)(11 62 61)(16 112 107)
(21 72 67)(22 73 68)(23 74 69)(24 75 70)(26 50 82)(27 54 83)(28 58 84)(30 46 86)
(31 55 87)(32 59 88)(34 47 90)(35 51 91)(36 60 92)(38 48 94)(39 52 95)(40 56 96)
(41 102 97)(42 104 98)(43 105 99)(44 106 100)(63 114 77)(64 115 78)(65 116 79)(66 117 80)
(71 122 118)(76 119 101)(81 121 108)(85 123 109)(89 124 110)(93 125 111)(103 120 113)]
= (1 11 122 107 103 101)(2 14 123 111 102 99 3 15 121 110 104 100)
(4 13 125 108 105 98 5 12 124 109 106 97)(6 76 71 62 113 16)
(7 74 68 66 114 19 8 75 67 65 115 20)(9 73 70 63 116 18 10 72 69 64 117 17)
(21 53 78 24 45 79 22 57 77 23 49 80)(25 43 85 37 41 89 29 44 81 33 42 93)
(26 60 86 40 50 92 30 56 82 36 46 96)(27 51 88 38 54 91 32 48 83 35 59 94)
(28 47 87 39 58 90 31 52 84 34 55 95)(126 112 118 119 120 61).

Therefore, the permutation representation of $g = (0\ 1\ 3\ 2\ 4)t_0^{-1}t_1t_0^{-1}$ is $p = \phi(g) =$

(1 11 122 107 103 101)(2 14 123 111 102 99 3 15 121 110 104 100)
(4 13 125 108 105 98 5 12 124 109 106 97)(6 76 71 62 113 16)
(7 74 68 66 114 19 8 75 67 65 115 20)(9 73 70 63 116 18 10 72 69 64 117 17)
(21 53 78 24 45 79 22 57 77 23 49 80)(25 43 85 37 41 89 29 44 81 33 42 93)
(26 60 86 40 50 92 30 56 82 36 46 96)(27 51 88 38 54 91 32 48 83 35 59 94)
(28 47 87 39 58 90 31 52 84 34 55 95)(126 112 118 119 120 61).

Chapter 7

Aut(M_{12}) as a Homomorphic Image of the Progenitor $3^{\star 4}: S_4$

In our final chapter, we investigate Aut(M_{12}) as a homomorphic image of the progenitor $3^{\star 4}: S_4$. Aut(M_{12}), or $M_{12}: 2$, is an automorphism group of M_{12} having order $2 \times 95,040 = 190,080$. The progenitor $3^{\star 4}: S_4$ is a semi-direct product of $3^{\star 4}$, a free product of four copies of the cyclic group of order 3, and S_4, the symmetric group on four letters which permutes the four symmetric generators, t_0, t_1, t_2, and t_3, (and their inverses, $t_0^2 = t_0^{-1}$, $t_1^2 = t_1^{-1}$, $t_2^2 = t_2^{-1}$, and $t_3^2 = t_3^{-1}$) by conjugation.

7.1 Introduction

Let $\bar{G}$ be a homomorphic image of the infinite semi-direct product, the *progenitor*, $3^{\star 4}: S_4$. A symmetric presentation of $3^{\star 4}: S_4$ is given by

$$\bar{G} = \langle x, y, t \mid x^4 = y^2 = (yx)^3 = t^3 = [t, y] = [t^x, y] = e \rangle$$

where $[t, y] = tyty$, $[t^x, y] = t^x y t^x y$, and e is the identity. In this case, $N \cong S_4 \cong \langle x, y \mid x^4 = y^2 = (yx)^3 = e \rangle$, and the action of N on the four symmetric generators is given by $x \sim (0\ 1\ 2\ 3)$, $y \sim (2\ 3)$, and $t \sim t_0$.

Let G denote the group $\bar{G}$ factored by the relations $(yxt)^{10} = e$ and $[(x^2y)^2t]^5 = e$. That is, let

$$G = \frac{\bar{G}}{(yxt)^{10}, [(x^2y)^2t]^5}.$$

A symmetric presentation for G is given by

$$\langle x, y, t \mid x^4 = y^2 = (yx)^3 = t^3 = [t, y] = [t^x, y] = (yxt)^{10} = [(x^2y)^2t]^5 = e\rangle.$$

Now, we consider the following relations:

$$[(0\ 1\ 2)t_0]^{10} = e$$

and

$$[(0\ 1)(2\ 3)t_0]^5 = e.$$

According to a computer proof by [CHB96], the progenitor $3^{*4} : S_4$, factored by the relations $[(0\ 1\ 2)t_0]^{10} = e$ and $[(0\ 1)(2\ 3)t_0]^5 = e$, is isomorphic to $\mathrm{Aut}(M_{12})$. We will construct $\mathrm{Aut}(M_{12})$ by hand by way of manual double coset enumeration of $G \cong \frac{3^{*4}:S_4}{[(0\ 1\ 2)t_0]^{10},[(0\ 1)(2\ 3)t_0]^5}$ over S_4. In so doing, we will show that $\mathrm{Aut}(M_{12})$ is isomorphic to the symmetric presentation

$$\langle x, y, t \mid x^4 = y^2 = (yx)^3 = t^3 = [t, y] = [t^x, y] = (yxt)^{10} = [(x^2y)^2t]^5 = e\rangle$$

7.2 Manual Double Coset Enumeration of G Over S_4

We first determine the order of the homomorphic image, G, of the progenitor. To determine the order of the homomorphic image G, we must determine the index of $N \cong S_4$ in G. We determine the index of $N \cong S_4$ in G first by expanding the relations $[(0\ 1\ 2)t_0]^{10} = e$ and $[(0\ 1)(2\ 3)t_0]^5 = e$, and next by performing manual double coset enumeration on G over $N \cong S_4$. To begin, we expand the relations that factor the progenitor $3^{*4} : S_4$:

$$[(0\ 1\ 2)t_0]^{10} = e \tag{7.1}$$

$$[(0\ 1)(2\ 3)t_0]^5 = e \tag{7.2}$$

We expand relations (7.1) and (7.2) in detail below:

1. Let $\pi = (0\ 1\ 2)$.
   Then $[(0\ 1\ 2)t_0]^{10} = e$
   $\Rightarrow (\pi t_0)^{10} = e$
   $\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$
   $\Rightarrow \pi^{10} t_0^{\pi^9} t_0^{\pi^8} t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow (0\ 1\ 2)t_0^e t_0^{(0\ 2\ 1)} t_0^{(0\ 1\ 2)} t_0^e t_0^{(0\ 2\ 1)} t_0^{(0\ 1\ 2)} t_0^e t_0^{(0\ 2\ 1)} t_0^{(0\ 1\ 2)} t_0 = e$

$\Rightarrow (0\ 1\ 2)t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = e$

$\Rightarrow (0\ 1\ 2)t_0t_2t_1t_0t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$.

Thus relation (7.1) implies that $(0\ 1\ 2)t_0t_2t_1t_0t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$ or, equivalently, $Nt_0t_2t_1t_0t_2 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$. That is, using our short-hand notation, $02102 \sim \bar{0}\bar{1}\bar{2}\bar{0}\bar{1}$.

2. Let $\pi = (0\ 1)(2\ 3)$.

Then $[(0\ 1)(2\ 3)t_0]^5 = e$

$\Rightarrow (\pi t_0)^5 = e$

$\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$

$\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi^2 \pi^{-1} t_0 \pi t_0 = e$

$\Rightarrow \pi t_0 \pi t_0 \pi^3 \pi^{-2} t_0 \pi^2 t_0^{\pi} t_0 = e$

$\Rightarrow \pi t_0 \pi^4 \pi^{-3} t_0 \pi^3 t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi^5 \pi^{-4} t_0 \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = e$

$\Rightarrow [(0\ 1)(2\ 3)]^5 t_0^{[(0\ 1)(2\ 3)]^4} t_0^{[(0\ 1)(2\ 3)]^3} t_0^{[(0\ 1)(2\ 3)]^2} t_0^{(0\ 1)(2\ 3)} t_0 = e$

$\Rightarrow (0\ 1)(2\ 3)t_0^e t_0^{(0\ 1)(2\ 3)} t_0^e t_0^{(0\ 1)(2\ 3)} t_0 = e$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1t_0 = e$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$.

Thus relation (7.2) implies that $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$ or, equivalently, $Nt_0t_1t_0 = Nt_0^{-1}t_1^{-1}$. That is, using our short-hand notation, $010 \sim \bar{0}\bar{1}$.

We now perform manual double coset enumeration of G over S_4.

1. We first note that the double coset $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$.

   Let $[*]$ denote the double coset NeN.

   The double coset $[*]$ has one distinct right coset: the identity right coset, $Ne = \{ne \mid n \in N\} = N$.

   Moreover, since $N \cong S_4$ is transitive on $\{0, 1, 2, 3\}$ and also transitive on the inverses $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$, N has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.

   Therefore, we conclude that there are two distinct double cosets of the form NwN, where w is a word of length one given by $w = t_i^{\pm 1}$, $i = 0$: Nt_0N and $Nt_0^{-1}N$.

2. We next consider the double coset Nt_0N.

   Let [0] denote the double coset Nt_0N.

   Now, note that $N^{(0)} \geq N^0 = \langle (1\ 2), (1\ 3) \rangle \cong S_3$. Thus $\left|N^{(0)}\right| \geq |S_3| = 6$ and so, by Lemma 1.4, $|Nt_0N| = \frac{|N|}{|N^{(0)}|} \leq \frac{24}{6} = 4$.

   Therefore, the double coset [0] has at most four distinct single cosets.

   Moreover, $N^{(0)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2, 3\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}\}$.

   Therefore, there are at most four double cosets of the form NwN, where w is a word of length two given by $w = t_0t_i^{\pm 1}$, $i \in \{0, 1\}$: Nt_0t_0N, Nt_0t_1N, $Nt_0t_0^{-1}N$, and $Nt_0t_1^{-1}N$. But, since $Nt_0t_0N = Nt_0^2N = Nt_0^{-1}N$, and since $Nt_0t_0^{-1}N = NeN = N$, we conclude that there are two distinct double cosets of the form $Nt_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: Nt_0t_1N and $Nt_0t_1^{-1}N$.

3. We next consider the double coset $Nt_0^{-1}N$.

   Let $[\bar{0}]$ denote the double coset $Nt_0^{-1}N$.

   Now, note that $N^{(\bar{0})} \geq N^{\bar{0}} = \langle (1\ 2), (1\ 3) \rangle \cong S_3$. Thus $\left|N^{(\bar{0})}\right| \geq |S_3| = 6$ and so, by Lemma 1.4, $\left|Nt_0^{-1}N\right| = \frac{|N|}{|N^{(\bar{0})}|} \leq \frac{24}{6} = 4$.

   Therefore, the double coset $[\bar{0}]$ has at most four distinct single cosets.

   Moreover, $N^{(\bar{0})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2, 3\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}\}$.

   Therefore, there are at most four double cosets of the form NwN, where w is a word of length two given by $w = t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$: $Nt_0^{-1}t_0N$, $Nt_0^{-1}t_1N$, $Nt_0^{-1}t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}N$.

   But, since $Nt_0^{-1}t_0^{-1}N = Nt_0^{-2}N = Nt_0N$ and $Nt_0^{-1}t_0N = NeN = N$, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1N$ and $Nt_0^{-1}t_1^{-1}N$.

4. We next consider the double coset Nt_0t_1N.

   Let [01] denote the double coset Nt_0t_1N.

   Note that $N^{(01)} \geq N^{01} = \langle (2\ 3) \rangle \cong S_2$. Thus $\left|N^{(01)}\right| \geq |S_2| = 2$ and so, by Lemma 1.4, $|Nt_0t_1N| = \frac{|N|}{|N^{(01)}|} \leq \frac{24}{1} = 12$.

   Therefore, the double coset [01] has at most twelve distinct single cosets.

Now, $N^{(01)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1t_i^{\pm 1}$, $i \in \{0,1,2\}$: $Nt_0t_1t_0N$, $Nt_0t_1t_1N$, $Nt_0t_1t_2N$, $Nt_0t_1t_0^{-1}N$, $Nt_0t_1t_1^{-1}N$, and $Nt_0t_1t_2^{-1}N$.

But note that $Nt_0t_1t_1^{-1}N = Nt_0eN = Nt_0N$ and $Nt_0t_1t_1N = Nt_0t_1^2N = Nt_0t_1^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$ implies that $Nt_0t_1t_0 = Nt_0^{-1}t_1^{-1}$ which implies that $Nt_0t_1t_0N = Nt_0^{-1}t_1^{-1}N$. That is, $[010] = [\bar{0}\bar{1}]$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2N$ and $Nt_0t_1t_0^{-1}N$ and $Nt_0t_1t_2^{-1}N$.

5. We next consider the double coset $Nt_0t_1^{-1}N$.

   Let $[0\bar{1}]$ denote the double coset $Nt_0t_1^{-1}N$.

   Note that $N^{(0\bar{1})} \geq N^{0\bar{1}} = \langle(2\ 3)\rangle \cong S_2$. Therefore, $\left|N^{(0\bar{1})}\right| \geq |S_2| = 2$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1})}|} \leq \frac{24}{1} = 12$.

   Therefore, the double coset $[0\bar{1}]$ has at most twelve distinct single cosets.

   Now, $N^{(0\bar{1})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

   Therefore, there are at most six double cosets of the form NwN, where w is a word of length three given by $w = t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2\}$: $Nt_0t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_1N$, $Nt_0t_1^{-1}t_2N$, $Nt_0t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}N$.

   But note that $Nt_0t_1^{-1}t_1N = Nt_0eN = Nt_0N$ and $Nt_0t_1^{-1}t_1^{-1}N = Nt_0t_1^{-2}N = Nt_0t_1N$.

   Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_0N$ and $Nt_0t_1^{-1}t_2N$ and $Nt_0t_1^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}N$.

6. We next consider the double coset $Nt_0^{-1}t_1N$.

   Let $[\bar{0}1]$ denote the double coset $Nt_0^{-1}t_1N$.

   Note that $N^{(\bar{0}1)} \geq N^{\bar{0}1} = \{e, (2\ 3)\} \cong S_2$. Therefore, $\left|N^{(\bar{0}1)}\right| \geq |S_2| = 2$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1N\right| = \frac{|N|}{|N^{(\bar{0}1)}|} \leq \frac{24}{1} = 12$.

Therefore, the double coset $[\bar{0}1]$ has at most twelve distinct single cosets.

Now, $N^{(\bar{0}1)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length three given by $w = t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0,1,2\}$: $Nt_0^{-1}t_1t_0N$, $Nt_0^{-1}t_1t_1N$, $Nt_0^{-1}t_1t_2N$, $Nt_0^{-1}t_1t_0^{-1}N$, $Nt_0^{-1}t_1t_1^{-1}N$, and $Nt_0^{-1}t_1t_2^{-1}N$.

But note that $Nt_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}eN = Nt_0^{-1}N$ and $Nt_0^{-1}t_1t_1N = Nt_0^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$
$(0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$
$(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = (0\ 1)(2\ 3)t_1t_0^{-1}t_1^{-1} \Rightarrow$
$t_0^{-1}t_1t_0 = (0\ 1)(2\ 3)t_1t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}t_1t_0 = (0\ 1)(2\ 3)[t_0t_1^{-1}t_0^{-1}]^{(0\ 1)} \Rightarrow t_0^{-1}t_1t_0 = (0\ 1)(2\ 3)(0\ 1)[t_0t_1^{-1}t_0^{-1}](0\ 1) \Rightarrow t_0^{-1}t_1t_0 = (2\ 3)t_0t_1^{-1}t_0^{-1}(0\ 1)$. Therefore, $Nt_0^{-1}t_1t_0N = Nt_0t_1^{-1}t_0^{-1}N$. That is, $[\bar{0}10] = [0\bar{1}\bar{0}]$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2N$ and $Nt_0^{-1}t_1t_0^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}N$.

7. We next consider the double coset $Nt_0^{-1}t_1^{-1}N$.

   Let $[\bar{0}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1^{-1}N$.

   Note that $N^{(\bar{0}\bar{1})} \geq N^{\bar{0}\bar{1}} = \langle(2\ 3)\rangle \cong S_2$. Therefore, $\left|N^{(\bar{0}\bar{1})}\right| \geq |S_2| = 2$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1})}\right|} \leq \frac{24}{1} = 12$.

   Therefore, the double coset $[\bar{0}\bar{1}]$ has at most twelve distinct single cosets.

   Now, $N^{(\bar{0}\bar{1})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

   Therefore, there are at most six double cosets of the form NwN, where w is a word of length three given by $w = t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2\}$: $Nt_0^{-1}t_1^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

   But note that $Nt_0^{-1}t_1^{-1}t_1N = Nt_0^{-1}eN = Nt_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-2}N = Nt_0^{-1}t_1N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$, which implies that $Nt_0t_1t_0^{-1} = Nt_0^{-1}t_1^{-1}t_0$, and which implies that $Nt_0t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_0N$. Therefore, $Nt_0^{-1}t_1^{-1}t_0N = Nt_0t_1t_0^{-1}N$. That is, $[\bar{0}\bar{1}0] = [01\bar{0}]$.

Likewise, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0^{-1} = t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1 = t_0^{-1}t_1^{-1}t_0^{-1}$, which implies that $Nt_0t_1 = Nt_0^{-1}t_1^{-1}t_0^{-1}$, and which implies that $Nt_0t_1N = Nt_0^{-1}t_1^{-1}t_0^{-1}N$. Therefore, $Nt_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1N$. That is, $[\bar{0}\bar{1}\bar{0}] = [01]$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

8. We next consider the double coset $Nt_0t_1t_0^{-1}N$.

   Let $[01\bar{0}]$ denote the double coset $Nt_0t_1t_0^{-1}N$.

   Note that $N^{(01\bar{0})} \geq N^{01\bar{0}} = \langle(2\ 3)\rangle \cong S_2$. Thus $|N^{(01)}| \geq |S_2| = 2$ and so, by Lemma 1.4, $|Nt_0t_1t_0^{-1}N| = \frac{|N|}{|N^{(01\bar{0})}|} \leq \frac{24}{1} = 12$.

   Therefore, the double coset $[01\bar{0}]$ has at most twelve distinct single cosets.

   Now, $N^{(01\bar{0})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

   Therefore, there are at most six double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1t_0^{-1}t_i^{\pm1}$, $i \in \{0,1,2\}$: $Nt_0t_1t_0^{-1}t_0N$, $Nt_0t_1t_0^{-1}t_1N$, $Nt_0t_1t_0^{-1}t_2N$, $Nt_0t_1t_0^{-1}t_0^{-1}N$, $Nt_0t_1t_0^{-1}t_1^{-1}N$, and $Nt_0t_1t_0^{-1}t_2^{-1}N$.

   But note that $Nt_0t_1t_0^{-1}t_0N = Nt_0t_1eN = Nt_0t_1N$ and, by relation (7.2), $Nt_0t_1t_0^{-1}t_0^{-1}N = Nt_0t_1t_0^{-2}N = Nt_0t_1t_0N = Nt_0^{-1}t_1^{-1}N$.

   Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$
   $(0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$
   $(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow t_0(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1} \Rightarrow$
   $(0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_1t_0^{-1}t_1t_0 = t_0t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)[t_0t_1^{-1}t_0t_1]^{(0\ 1)}$
   $= t_0t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)[t_0t_1^{-1}t_0t_1](0\ 1) = t_0t_1t_0^{-1}t_1^{-1} \Rightarrow (2\ 3)t_0t_1^{-1}t_0t_1(0\ 1) =$

$t_0t_1t_0^{-1}t_1^{-1}$, which implies that $Nt_0t_1^{-1}t_0t_1N = Nt_0t_1t_0^{-1}t_1^{-1}N$. Therefore, $Nt_0t_1t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0t_1N$. That is, $[01\bar{0}\bar{1}] = [0\bar{1}01]$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_0^{-1}t_1N$ and $Nt_0t_1t_0^{-1}t_2N$ and $Nt_0t_1t_0^{-1}t_2^{-1}N$.

9. We next consider the double coset $Nt_0t_1t_2N$.

   Let [012] denote the double coset $Nt_0t_1t_2N$.

   Note that $N^{(012)} \geq N^{012} = \langle e \rangle$. Thus $|N^{(012)}| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2N| = \frac{|N|}{|N^{(012)}|} \leq \frac{24}{1} = 24$.

   Therefore, the double coset [012] has at most twenty-four distinct single cosets.

   Now, $N^{(012)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

   Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_0N$, $Nt_0t_1t_2t_1N$, $Nt_0t_1t_2t_2N$, $Nt_0t_1t_2t_3N$, $Nt_0t_1t_2t_0^{-1}N$, $Nt_0t_1t_2t_1^{-1}N$, $Nt_0t_1t_2t_2^{-1}N$, and $Nt_0t_1t_2t_3^{-1}N$.

   But note that $Nt_0t_1t_2t_2^{-1}N = Nt_0t_1eN = Nt_0t_1N$ and $Nt_0t_1t_2t_2N = Nt_0t_1t_2^2N = Nt_0t_1t_2^{-1}N$.

   Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0 \Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0 \Rightarrow [(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1}]^{(0\ 1\ 2)} = [t_2t_0^{-1}t_1^{-1}t_0]^{(0\ 1\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3t_1t_2t_1^{-1} = t_0t_1^{-1}t_2^{-1}t_1 \Rightarrow (1\ 2)(0\ 3)[t_0t_1t_2t_1^{-1}]^{(0\ 3)} = t_0t_1^{-1}t_2^{-1}t_1 \Rightarrow (1\ 2)(0\ 3)(0\ 3)[t_0t_1t_2t_1^{-1}](0\ 3) = t_0t_1^{-1}t_2^{-1}t_1 \Rightarrow (1\ 2)t_0t_1t_2t_1^{-1}(0\ 3) = t_0t_1^{-1}t_2^{-1}t_1$, which implies that $Nt_0t_1t_2t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1N$. Therefore, $Nt_0t_1t_2t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1N$. That is, $[012\bar{1}] = [0\bar{1}\bar{2}1]$.

   Likewise, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow$
   $(0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow$
   $[(0\ 1)(2\ 3)t_3t_0t_1t_0]^{(0\ 1\ 2)} = [t_2t_0^{-1}t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3t_1t_2t_1 = t_0t_1^{-1}t_2^{-1} \Rightarrow$

$(1\ 2)(0\ 3)[t_0t_1t_2t_1]^{(0\ 3)} = t_0t_1^{-1}t_2^{-1} \Rightarrow (1\ 2)(0\ 3)(0\ 3)[t_0t_1t_2t_1](0\ 3) = t_0t_1^{-1}t_2^{-1} \Rightarrow$ $(1\ 2)t_0t_1t_2t_1(0\ 3) = t_0t_1^{-1}t_2^{-1}$, which implies that $Nt_0t_1t_2t_1N = Nt_0t_1^{-1}t_2^{-1}N$. Therefore, $Nt_0t_1t_2t_1N = Nt_0t_1^{-1}t_2^{-1}N$. That is, $[0121] = [0\bar{1}\bar{2}]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_0N$ and $Nt_0t_1t_2t_0^{-1}N$ and $Nt_0t_1t_2t_3N$ and $Nt_0t_1t_2t_3^{-1}N$.

10. We next consider the double coset $Nt_0t_1t_2^{-1}N$.

Let $[01\bar{2}]$ denote the double coset $Nt_0t_1t_2^{-1}N$.

Note that $N^{(01\bar{2})} \geq N^{01\bar{2}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_0N$, $Nt_0t_1t_2^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_2N$, $Nt_0t_1t_2^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_2^{-1}N$, and $Nt_0t_1t_2^{-1}t_3^{-1}N$.

But note that $Nt_0t_1t_2^{-1}t_2N = Nt_0t_1eN = Nt_0t_1N$ and $Nt_0t_1t_2^{-1}t_2^{-1}N = Nt_0t_1t_2^{-2}N$ $= Nt_0t_1t_2N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow t_2(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1} \Rightarrow [(0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0]^{(0\ 2)} =$ $[t_2t_1t_0^{-1}t_1^{-1}]^{(0\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3t_2^{-1}t_1t_2 = t_0t_1t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)(0\ 3)[t_0t_1^{-1}t_2t_1]^{(1\ 2)(0\ 3)} =$ $t_0t_1t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)(0\ 3)(1\ 2)(0\ 3)[t_0t_1^{-1}t_2t_1](1\ 2)(0\ 3) = t_0t_1t_2^{-1}t_1^{-1} \Rightarrow$ $et_0t_1^{-1}t_2t_1(1\ 2)(0\ 3) = t_0t_1t_2^{-1}t_1^{-1}$, which implies that $Nt_0t_1^{-1}t_2t_1N = Nt_0t_1t_2^{-1}t_1^{-1}N$. Therefore, $Nt_0t_1t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_1N$. That is, $[01\bar{2}\bar{1}] = [0\bar{1}21]$.

Therefore, we conclude that there are five distinct double cosets of the form

$Nt_0t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_0N$, $Nt_0t_1t_2^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_0^{-1}N$, and $Nt_0t_1t_2^{-1}t_3^{-1}N$.

11. We next consider the double coset $Nt_0t_1^{-1}t_0N$.

Let $[0\bar{1}0]$ denote the double coset $Nt_0t_1^{-1}t_0N$.

Note that $N^{(0\bar{1}0)} \geq N^{0\bar{1}0} = \langle(2\ 3)\rangle \cong S_2$. Thus $|N^{(01)}| \geq |S_2| = 2$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_0N| = \frac{|N|}{|N^{(0\bar{1}0)}|} \leq \frac{24}{1} = 12$.

Therefore, the double coset $[0\bar{1}0]$ has at most twelve distinct single cosets.

Moreover, $N^{(0\bar{1}0)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,2\}$: $Nt_0t_1^{-1}t_0t_0N$, $Nt_0t_1^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_0t_2N$, $Nt_0t_1^{-1}t_0t_0^{-1}N$, $Nt_0t_1^{-1}t_0t_1^{-1}N$, and $Nt_0t_1^{-1}t_0t_2^{-1}N$.

But note that $Nt_0t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}eN = Nt_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_0t_0N = Nt_0t_1^{-1}t_0^2N = Nt_0t_1^{-1}t_0^{-1}N$.

Moreover, with the help of the computer algebra system MAGMA (see [BCP97]), we know that $(0\ 2)(1\ 3)t_3t_1^{-1}t_2t_1^{-1} = t_0t_1^{-1}t_0t_2^{-1}$. Now, $(0\ 2)(1\ 3)t_3t_1^{-1}t_2t_1^{-1} = t_0t_1^{-1}t_0t_2^{-1} \Rightarrow (0\,2)(1\,3)[t_0t_1^{-1}t_2t_1^{-1}]^{(0\ 3)} = t_0t_1^{-1}t_0t_2^{-1} \Rightarrow (0\,2)(1\,3)(0\,3)[t_0t_1^{-1}t_2t_1^{-1}](0\,3) =$

$t_0t_1^{-1}t_0t_2^{-1} \Rightarrow (0\ 1\ 3\ 2)t_0t_1^{-1}t_2t_1^{-1}(0\ 3) = t_0t_1^{-1}t_0t_2^{-1}$, which implies that $Nt_0t_1^{-1}t_2t_1^{-1}N = Nt_0t_1^{-1}t_0t_2^{-1}N$. That is, $[0\bar{1}0\bar{2}] = [0\bar{1}2\bar{1}]$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_0t_2N$.

12. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}N$.

Let $[0\bar{1}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{0})} \geq N^{0\bar{1}\bar{0}} = \langle(2\ 3)\rangle \cong S_2$. Thus $|N^{(01)}| \geq |S_2| = 2$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_0^{-1}N| = \frac{|N|}{|N^{(0\bar{1}\bar{0})}|} \leq \frac{24}{1} = 12$.

Therefore, the double coset $[0\bar{1}\bar{0}]$ has at most twelve distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2\}$: $Nt_0t_1^{-1}t_0^{-1}t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}eN = Nt_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_0N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0t_0 = t_1t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0^{-1}$ $= t_1t_0^{-1}t_1^{-1}t_0 \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}t_1t_0^{-1} = (0\ 1)(2\ 3)t_1t_0^{-1}t_1^{-1}t_0 \Rightarrow t_0^{-1}t_1t_0^{-1} =$ $(0\ 1)(2\ 3)t_1t_0^{-1}t_1^{-1}t_0 \Rightarrow t_0^{-1}t_1t_0^{-1} = (0\ 1)(2\ 3)[t_0t_1^{-1}t_0^{-1}t_1]^{(0\ 1)} \Rightarrow t_0^{-1}t_1t_0^{-1} =$ $(0\ 1)(2\ 3)(0\ 1)[t_0t_1^{-1}t_0^{-1}t_1](0\ 1) \Rightarrow t_0^{-1}t_1t_0^{-1} = (2\ 3)t_0t_1^{-1}t_0^{-1}t_1(0\ 1)$, which implies that $Nt_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_1N$. Therefore, $Nt_0t_1^{-1}t_0^{-1}t_1N = Nt_0^{-1}t_1t_0^{-1}N$. That is, $[0\bar{1}\bar{0}1] = [\bar{0}1\bar{0}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0t_0^{-1} = t_1t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1$ $= t_1t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}t_1 = (0\ 1)(2\ 3)t_1t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow t_0^{-1}t_1 =$ $(0\ 1)(2\ 3)t_1t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow t_0^{-1}t_1 = (0\ 1)(2\ 3)[t_0t_1^{-1}t_0^{-1}t_1^{-1}]^{(0\ 1)} \Rightarrow t_0^{-1}t_1 =$ $(0\ 1)(2\ 3)(0\ 1)[t_0t_1^{-1}t_0^{-1}t_1^{-1}](0\ 1) \Rightarrow t_0^{-1}t_1 = (2\ 3)t_0t_1^{-1}t_0^{-1}t_1^{-1}(0\ 1)$, which implies that $Nt_0^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N$. Therefore, $Nt_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1N$. That is, $[0\bar{1}\bar{0}\bar{1}] = [\bar{0}1]$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_0^{-1}t_2N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$.

13. We next consider the double coset $Nt_0t_1^{-1}t_2N$.

    Let $[0\bar{1}2]$ denote the double coset $Nt_0t_1^{-1}t_2N$.

Note that $N^{(0\bar{1}2)} \geq N^{0\bar{1}2} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_2N| = \frac{|N|}{|N^{(0\bar{1}2)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1^{-1}t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2t_0N$, $Nt_0t_1^{-1}t_2t_1N$, $Nt_0t_1^{-1}t_2t_2N$, $Nt_0t_1^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_3^{-1}N$.

But note that $Nt_0t_1^{-1}t_2t_2^{-1}N = Nt_0t_1^{-1}eN = Nt_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_2N = Nt_0t_1^{-1}t_2^2N = Nt_0t_1^{-1}t_2^{-1}N$.

Therefore, we conclude that there are six distinct double cosets of the form $Nt_0t_1^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2t_0N$, $Nt_0t_1^{-1}t_2t_1N$, $Nt_0t_1^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1^{-1}N$, and $Nt_0t_1^{-1}t_2t_3^{-1}N$.

14. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}N$.

    Let $[0\bar{1}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}N$.

    Note that $N^{(0\bar{1}\bar{2})} \geq N^{0\bar{1}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_2^{-1}N| = \frac{|N|}{|N^{(0\bar{1}\bar{2})}|} \leq \frac{24}{1} = 24$.

    Therefore, the double coset $[0\bar{1}\bar{2}]$ has at most twenty-four distinct single cosets.

    Moreover, $N^{(0\bar{1}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

    Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0t_1^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_2^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$.

    But note that $Nt_0t_1^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}eN = Nt_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2N$.

    Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0^{-1} = t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1 = t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow [(0\ 1)(2\ 3)t_0t_1]^{(0\ 1\ 2)} = [t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 1\ 2)}$

$\Rightarrow (1\ 2)(0\ 3)t_1t_2 = t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow t_0(1\ 2)(0\ 3)t_1t_2 = t_0t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow$ $(1\ 2)(0\ 3)(1\ 2)(0\ 3)t_0(1\ 2)(0\ 3)t_1t_2 = t_0t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)(0\ 3)t_0^{(1\ 2)(0\ 3)}t_1t_2 = t_0t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)(0\ 3)t_3t_1t_2 = t_0t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)(0\ 3)[t_0t_1t_2]^{(0\ 3)} = t_0t_1^{-1}t_2^{-1}t_1^{-1}$ $\Rightarrow (1\ 2)(0\ 3)(0\ 3)[t_0t_1t_2](0\ 3) = t_0t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)t_0t_1t_2(0\ 3) = t_0t_1^{-1}t_2^{-1}t_1^{-1}$, which implies that $Nt_0t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_1^{-1}N$. Therefore, $Nt_0t_1^{-1}t_2^{-1}t_1^{-1}N = Nt_0t_1t_2N$. That is, $[0\bar{1}\bar{2}\bar{1}] = [012]$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$.

15. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}N$.

Let $[\bar{0}1\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}N$.

Note that $N^{(\bar{0}1\bar{0})} \geq N^{\bar{0}1\bar{0}} = \langle(2\ 3)\rangle \cong S_2$. Therefore, $|N^{(01)}| \geq |S_2| = 2$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1t_0^{-1}N| = \frac{|N|}{|N^{(\bar{0}1\bar{0})}|} \leq \frac{24}{1} = 12$.

Therefore, the double coset $[\bar{0}1\bar{0}]$ has at most twelve distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1t_0^{-1}t_i^{\pm1}$, $i \in \{0,1,2\}$: $Nt_0^{-1}t_1t_0^{-1}t_0N$, $Nt_0^{-1}t_1t_0^{-1}t_1N$, $Nt_0^{-1}t_1t_0^{-1}t_2N$, $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_0N = Nt_0^{-1}t_1eN = Nt_0^{-1}t_1N$ and, by relation (7.2), $Nt_0^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0^{-2}N = Nt_0^{-1}t_1t_0N = Nt_0t_1^{-1}t_0^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$. Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1^{-1}t^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1 \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_0^{-1}t_1t_0 \Rightarrow$

$t_1^{-1}(0\,1)(2\,3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0 \Rightarrow (0\,1)(2\,3)(0\,1)(2\,3)t_1^{-1}(0\,1)(2\,3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0 \Rightarrow (0\ 1)(2\ 3)(t_1^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0 \Rightarrow$ $(0\ 1)(2\ 3)t_0^{-1}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0 \Rightarrow (0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0 \Rightarrow$ $(0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 \Rightarrow t_1t_0t_1^{-1}t_0 =$ $(0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$. Since $(0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$ and $t_1t_0t_1^{-1}t_0 =$ $(0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$, we conclude that $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$. Now, $t_0^{-1}t_1t_0^{-1}t_1^{-1} =$ $t_1t_0t_1^{-1}t_0 \Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1} = [t_0t_1t_0^{-1}t_1]^{(0\ 1)} \Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1(0\ 1)$, which implies that $Nt_0^{-1}t_1t_0^{-1}t_1^{-1}N = Nt_0t_1t_0^{-1}t_1N$. Therefore, $Nt_0^{-1}t_1t_0^{-1}t_1^{-1}N =$ $Nt_0t_1t_0^{-1}t_1N$. That is, $[\bar{0}1\bar{0}\bar{1}] = [01\bar{0}1]$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_0^{-1}t_1N$, $Nt_0^{-1}t_1t_0^{-1}t_2N$, and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$.

16. We next consider the double coset $Nt_0^{-1}t_1t_2N$.

Let $[\bar{0}12]$ denote the double coset $Nt_0^{-1}t_1t_2N$.

Note that $N^{(\bar{0}12)} \geq N^{\bar{0}12} = \langle e\rangle$. Therefore, $\left|N^{(\bar{0}12)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2N\right| = \frac{|N|}{|N^{(\bar{0}12)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}12]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}12)}$ has eight orbits on $T = \{t_0,t_1,t_2,t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2t_0N$, $Nt_0^{-1}t_1t_2t_1N$, $Nt_0^{-1}t_1t_2t_2N$, $Nt_0^{-1}t_1t_2t_3N$, $Nt_0^{-1}t_1t_2t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_1^{-1}N$, $Nt_0^{-1}t_1t_2t_2^{-1}N$, and $Nt_0^{-1}t_1t_2t_3^{-1}N$.

But note that $Nt_0^{-1}t_1t_2t_2^{-1}N = Nt_0^{-1}t_1eN = Nt_0^{-1}t_1N$ and $Nt_0^{-1}t_1t_2t_2N =$ $Nt_0^{-1}t_1t_2^2N = Nt_0^{-1}t_1t_2^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 =$ $t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $[(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0]^{(0\ 1\ 2)} = [t_2^{-1}t_0^{-1}t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3^{-1}t_1t_2t_1 = t_0^{-1}t_1^{-1}t_2^{-1} \Rightarrow$ $(1\,2)(0\,3)[t_0^{-1}t_1t_2t_1]^{(0\ 3)} = t_0^{-1}t_1^{-1}t_2^{-1} \Rightarrow (1\,2)(0\,3)(0\,3)[t_0^{-1}t_1t_2t_1](0\,3) = t_0^{-1}t_1^{-1}t_2^{-1} \Rightarrow$

$(1\ 2)t_0^{-1}t_1t_2t_1(0\ 3) = t_0^{-1}t_1^{-1}t_2^{-1}$, which implies that $Nt_0^{-1}t_1t_2t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}N$. Therefore, $Nt_0^{-1}t_1t_2t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}N$. That is, $[\bar{0}121] = [\bar{0}\bar{1}\bar{2}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $[(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0]^{(0\ 1\ 2)} = [t_2^{-1}t_0^{-1}t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3^{-1}t_1t_2t_1 = t_0^{-1}t_1^{-1}t_2^{-1} \Rightarrow$ $(1\ 2)(0\ 3)t_3^{-1}t_1t_2t_1t_1 = t_0^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow (1\ 2)(0\ 3)t_3^{-1}t_1t_2t_1^{-1} = t_0^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow$ $(1\ 2)(0\ 3)[t_0^{-1}t_1t_2t_1^{-1}]^{(0\ 3)} = t_0^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow (1\ 2)(0\ 3)(0\ 3)[t_0^{-1}t_1t_2t_1^{-1}](0\ 3) = t_0^{-1}t_1^{-1}t_2^{-1}t_1 \Rightarrow (1\ 2)t_0^{-1}t_1t_2t_1^{-1}(0\ 3) = t_0^{-1}t_1^{-1}t_2^{-1}t_1$, which implies that $Nt_0^{-1}t_1t_2t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$. Therefore, $Nt_0^{-1}t_1t_2t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$. That is, $[\bar{0}12\bar{1}] = [\bar{0}\bar{1}\bar{2}1]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2t_0N$, $Nt_0^{-1}t_1t_2t_3N$, $Nt_0^{-1}t_1t_2t_0^{-1}N$, and $Nt_0^{-1}t_1t_2t_3^{-1}N$.

17. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}N$.

Let $[\bar{0}1\bar{2}]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}N$.

Note that $N^{(\bar{0}1\bar{2})} \geq N^{\bar{0}1\bar{2}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}1\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2^{-1}t_0N$, $Nt_0^{-1}t_1t_2^{-1}t_1N$, $Nt_0^{-1}t_1t_2^{-1}t_2N$, $Nt_0^{-1}t_1t_2^{-1}t_3N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_2^{-1}N$, and $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_2N = Nt_0^{-1}t_1eN = Nt_0^{-1}t_1N$ and $Nt_0^{-1}t_1t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2^{-2}N = Nt_0^{-1}t_1t_2N$.

Moreover, with the help of MAGMA, we know that $(0\ 1)(2\ 3)t_3^{-1}t_1t_3^{-1}t_2 = t_0^{-1}t_1t_2^{-1}t_1$. Now, $(0\ 1)(2\ 3)t_3^{-1}t_1t_3^{-1}t_2 = t_0^{-1}t_1t_2^{-1}t_1 \Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1t_0^{-1}t_2]^{(0\ 3)} = t_0^{-1}t_1t_2^{-1}t_1 \Rightarrow$

$(0\,1)(2\,3)(0\,3)[t_0^{-1}t_1t_0^{-1}t_2](0\,3) = t_0^{-1}t_1t_2^{-1}t_1 \Rightarrow (0\,2\,3\,1)t_0^{-1}t_1t_0^{-1}t_2(0\,3) = t_0^{-1}t_1t_2^{-1}t_1$, which implies that $Nt_0^{-1}t_1t_0^{-1}t_2N = Nt_0^{-1}t_1t_2^{-1}t_1N$. That is, $[\bar{0}1\bar{2}1] = [\bar{0}1\bar{0}2]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$
$(0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow$
$(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1} \Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1} \Rightarrow$
$(0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1} \Rightarrow$
$(0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0]^{(0\ 2)} = [t_2^{-1}t_1t_0^{-1}t_1^{-1}]^{(0\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3^{-1}t_2^{-1}t_1t_2 =$
$t_0^{-1}t_1t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)(0\ 3)[t_0^{-1}t_1^{-1}t_2t_1]^{(0\ 3)(1\ 2)} = t_0^{-1}t_1t_2^{-1}t_1^{-1} \Rightarrow$
$(1\ 2)(0\ 3)(1\ 2)(0\ 3)[t_0^{-1}t_1^{-1}t_2t_1](1\ 2)(0\ 3) = t_0^{-1}t_1t_2^{-1}t_1^{-1} \Rightarrow et_0^{-1}t_1^{-1}t_2t_1(1\ 2)(0\ 3) = t_0^{-1}t_1t_2^{-1}t_1^{-1}$, which implies that $Nt_0^{-1}t_1^{-1}t_2t_1N = Nt_0^{-1}t_1t_2^{-1}t_1^{-1}N$. Therefore, $Nt_0^{-1}t_1t_2^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1N$. That is, $[\bar{0}1\bar{2}\bar{1}] = [\bar{0}\bar{1}21]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2^{-1}t_0N$, $Nt_0^{-1}t_1t_2^{-1}t_3N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$, and $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$.

18. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2N$.

Let $[\bar{0}\bar{1}2]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2N$.

Note that $N^{(\bar{0}\bar{1}2)} \geq N^{\bar{0}\bar{1}2} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}2)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1^{-1}t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_2^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_2^{-1}N = Nt_0^{-1}t_1^{-1}eN = Nt_0^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_2N = Nt_0^{-1}t_1^{-1}t_2^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_2 =$

$t_0^{-1}t_1^{-1}t_2 \Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_2t_0 = t_0^{-1}t_1^{-1}t_2t_0 \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1t_0t_2t_0 = (0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0 \Rightarrow t_0t_1t_0t_2t_0 = (0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0$. Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1} \Rightarrow t_1(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_1t_2t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_1t_2t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_3t_0t_1t_0 = t_1t_2t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0t_3t_0t_1t_0 = t_1t_2t_0^{-1}t_1^{-1} \Rightarrow [(0\ 1)(2\ 3)t_0t_3t_0t_1t_0]^{(1\ 2\ 3)} = [t_1t_2t_0^{-1}t_1^{-1}]^{(1\ 2\ 3)} \Rightarrow (0\ 2)(1\ 3)t_0t_1t_0t_2t_0 = t_2t_3t_0^{-1}t_2^{-1} \Rightarrow$ $(0\ 2)(1\ 3)(0\ 2)(1\ 3)t_0t_1t_0t_2t_0 = (0\ 2)(1\ 3)t_2t_3t_0^{-1}t_2^{-1} \Rightarrow t_0t_1t_0t_2t_0 =$ $(0\ 2)(1\ 3)t_2t_3t_0^{-1}t_2^{-1}$. Since $t_0t_1t_0t_2t_0 = (0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0$ and $t_0t_1t_0t_2t_0 = (0\ 2)(1\ 3)t_2t_3t_0^{-1}t_2^{-1}$, we conclude that $(0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0 = t_0t_1t_0t_2t_0 =$ $(0\ 2)(1\ 3)t_2t_3t_0^{-1}t_2^{-1}$; that is, $(0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0 = (0\ 2)(1\ 3)t_2t_3t_0^{-1}t_2^{-1}$. Now, $(0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0 = (0\ 2)(1\ 3)t_2t_3t_0^{-1}t_2^{-1} \Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0 =$ $(0\ 2)(1\ 3)[t_0t_1t_2^{-1}t_0^{-1}]^{(0\ 2)(1\ 3)} \Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0 =$ $(0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1t_2^{-1}t_0^{-1}](0\ 2)(1\ 3) \Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1^{-1}t_2t_0 =$ $et_0t_1t_2^{-1}t_0^{-1}(0\ 2)(1\ 3)$, which implies that $Nt_0^{-1}t_1^{-1}t_2t_0N = Nt_0t_1t_2^{-1}t_0^{-1}N$. Therefore, $Nt_0^{-1}t_1^{-1}t_2t_0N = Nt_0t_1t_2^{-1}t_0^{-1}N$. That is, $[\bar{0}\bar{1}20] = [01\bar{2}\bar{0}]$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

19. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2})} \geq N^{\bar{0}\bar{1}\bar{2}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length four given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_2^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_2N = Nt_0^{-1}t_1^{-1}eN = Nt_0^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-2}N = Nt_0^{-1}t_1^{-1}t_2N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0t_0^{-1} = t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1 = t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1} \Rightarrow$ $[(0\ 1)(2\ 3)t_3^{-1}t_0t_1]^{(0\ 1\ 2)} = [t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 1\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3^{-1}t_1t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$ $\Rightarrow (1\ 2)(0\ 3)[t_0^{-1}t_1t_2]^{(0\ 3)} = t_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)(0\ 3)(0\ 3)[t_0^{-1}t_1t_2](0\ 3) = t_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1} \Rightarrow (1\ 2)t_0^{-1}t_1t_2(0\ 3) = t_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1}$, which implies that $Nt_0^{-1}t_1t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1}N$. Therefore, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2N$. That is, $[\bar{0}\bar{1}\bar{2}\bar{1}] = [\bar{0}12]$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$.

20. We next consider the double coset $Nt_0t_1^{-1}t_2t_0N$.

Let $[0\bar{1}20]$ denote the double coset $Nt_0t_1^{-1}t_2t_0N$.

Note that $N^{(0\bar{1}20)} \geq N^{0\bar{1}20} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0N\right| = \frac{|N|}{|N^{(0\bar{1}20)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_0^{-1}N = Nt_0t_1^{-1}t_2eN = Nt_0t_1^{-1}t_2N$ and $Nt_0t_1^{-1}t_2t_0t_0N = Nt_0t_1^{-1}t_2t_0^2N = Nt_0t_1^{-1}t_2t_0^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow$ $t_0(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_0t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_0t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_0^{(0\ 1)(2\ 3)}t_3^{-1}t_0t_1t_0 = t_0t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_1t_3^{-1}t_0t_1t_0 =$

$t_0t_2^{-1}t_0^{-1}t_1^{-1} \Rightarrow [(0\,1)(2\,3)t_1t_3^{-1}t_0t_1t_0]^{(1\ 2)} = [t_0t_2^{-1}t_0^{-1}t_1^{-1}]^{(1\ 2)} \Rightarrow (0\,2)(1\,3)t_2t_3^{-1}t_0t_2t_0$ $= t_0t_1^{-1}t_0^{-1}t_2^{-1} \Rightarrow (0\ 2)(1\ 3)[t_0t_1^{-1}t_2t_0t_2]^{(0\ 2)(1\ 3)} = t_0t_1^{-1}t_0^{-1}t_2^{-1} \Rightarrow$ $(0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1^{-1}t_2t_0t_2](0\ 2)(1\ 3) = t_0t_1^{-1}t_0^{-1}t_2^{-1} \Rightarrow et_0t_1^{-1}t_2t_0t_2(0\ 2)(1\ 3) =$ $t_0t_1^{-1}t_0^{-1}t_2^{-1}$, which implies that $Nt_0t_1^{-1}t_2t_0t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$. Therefore, $Nt_0t_1^{-1}t_2t_0t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$. That is, $[0\bar{1}202] = [0\bar{1}\bar{0}\bar{2}]$.

Similarly, with the help of MAGMA, we know that $(0\ 2)(1\ 3)t_3t_1^{-1}t_2^{-1}t_3 =$ $t_0t_1^{-1}t_2t_0t_2^{-1}$. Now, $(0\ 2)(1\ 3)t_3t_1^{-1}t_2^{-1}t_3 = t_0t_1^{-1}t_2t_0t_2^{-1} \Rightarrow$ $(0\ 2)(1\ 3)[t_0t_1^{-1}t_2^{-1}t_0]^{(0\ 3)} = t_0t_1^{-1}t_2t_0t_2^{-1} \Rightarrow (0\ 2)(1\ 3)(0\ 3)t_0t_1^{-1}t_2^{-1}t_0(0\ 3) =$ $t_0t_1^{-1}t_2t_0t_2^{-1} \Rightarrow (0\ 1\ 3\ 2)t_0t_1^{-1}t_2^{-1}t_0(0\ 3) = t_0t_1^{-1}t_2t_0t_2^{-1}$, which implies that $Nt_0t_1^{-1}t_2^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_2^{-1}N$. That is, $[0\bar{1}20\bar{2}] = [0\bar{1}\bar{2}0]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_0t_1N$, $Nt_0t_1^{-1}t_2t_0t_3N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}N$.

21. We next consider the double coset $Nt_0t_1^{-1}t_2t_0^{-1}N$.

Let $[0\bar{1}2\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0^{-1}N$.

Now, with the help of MAGMA, we know that the following right cosets, or *single* cosets, are equivalent: $Nt_0t_1^{-1}t_2t_0^{-1} = Nt_1t_2^{-1}t_3t_1^{-1} = Nt_2t_3^{-1}t_0t_2^{-1} = Nt_3t_0^{-1}t_1t_3^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}2\bar{0} \sim 1\bar{2}3\bar{1} \sim 2\bar{3}0\bar{2} \sim 3\bar{0}1\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}2\bar{0}]$:

$$0\bar{1}2\bar{0} \sim 1\bar{2}3\bar{1} \sim 2\bar{3}0\bar{2} \sim 3\bar{0}1\bar{3}, \qquad 1\bar{0}2\bar{1} \sim 0\bar{2}3\bar{0} \sim 2\bar{3}1\bar{2} \sim 3\bar{1}0\bar{3},$$

$$2\bar{1}0\bar{2} \sim 1\bar{0}3\bar{1} \sim 0\bar{3}2\bar{0} \sim 3\bar{2}1\bar{3}, \qquad 3\bar{1}2\bar{3} \sim 1\bar{2}0\bar{1} \sim 2\bar{0}3\bar{2} \sim 0\bar{3}1\bar{0},$$

$$0\bar{2}1\bar{0} \sim 2\bar{1}3\bar{2} \sim 1\bar{3}0\bar{1} \sim 3\bar{0}2\bar{3}, \qquad 0\bar{1}3\bar{0} \sim 1\bar{3}2\bar{1} \sim 3\bar{2}0\bar{3} \sim 2\bar{0}1\bar{2}$$

Since each of the twenty-four single cosets has four names, the double coset $[0\bar{1}2\bar{0}]$ must have at most six distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1}2\bar{0})} \geq N^{0\bar{1}2\bar{0}} = \langle e \rangle$. In fact, by with the help of MAGMA, we know that $N(t_0t_1^{-1}t_2t_0^{-1})^{(0\ 1\ 2\ 3)} = Nt_1t_2^{-1}t_3t_1^{-1} = Nt_0t_1^{-1}t_2t_0^{-1}$ implies that $(0\ 1\ 2\ 3) \in N^{(0\bar{1}2\bar{0})}$. Therefore, $(0\ 1\ 2\ 3) \in N^{(0\bar{1}2\bar{0})}$, and so $N^{(0\bar{1}2\bar{0})} \geq \langle (0\ 1\ 2\ 3) \rangle$. That is, $\left|N^{(0\bar{1}2\bar{0})}\right| \geq |\langle (0\ 1\ 2\ 3) \rangle| = 4$ and so, by Lemma 1,4, $\left|Nt_0t_1^{-1}t_2t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}2\bar{0})}\right|} \leq \frac{24}{4} = 6$.

Therefore, as we concluded earlier, the double coset $[0\bar{1}2\bar{0}]$, as we noted earlier, must have at most six distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{0})}$ has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2t_0^{-1}t_i^{\pm 1}$, $i = 0$: $Nt_0t_1^{-1}t_2t_0^{-1}t_0N$ and $Nt_0t_1^{-1}t_2t_0^{-1}t_0^{-1}N$.

But note that $Nt_0t_1^{-1}t_2t_0^{-1}t_0N = Nt_0t_1^{-1}t_2eN = Nt_0t_1^{-1}t_2N$ and $Nt_0t_1^{-1}t_2t_0^{-1}t_0^{-1}N$ $= Nt_0t_1^{-1}t_2t_0^{-2}N = Nt_0t_1^{-1}t_2t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

22. We next consider the double coset $Nt_0t_1^{-1}t_2t_1N$.

    Let $[0\bar{1}21]$ denote the double coset $Nt_0t_1^{-1}t_2t_1N$.

    Note that $N^{(0\bar{1}21)} \geq N^{0\bar{1}21} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}21)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}21)}\right|} \leq \frac{24}{1} = 24$.

    Therefore, the double coset $[0\bar{1}21]$ has at most twenty-four distinct single cosets.

    Moreover, $N^{(0\bar{1}21)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

    Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

    But note that $Nt_0t_1^{-1}t_2t_1t_1^{-1}N = Nt_0t_1^{-1}t_2eN = Nt_0t_1^{-1}t_2N$ and $Nt_0t_1^{-1}t_2t_1t_1N = Nt_0t_1^{-1}t_2t_1^2N = Nt_0t_1^{-1}t_2t_1^{-1}N$.

    Moreover, with the help of MAGMA, we know that $(0\ 2\ 3)t_0t_3^{-1}t_2t_0t_1^{-1} = t_0t_1^{-1}t_2^{-1}t_1$. Now, $(0\ 2\ 3)t_0t_3^{-1}t_2t_0t_1^{-1} = t_0t_1^{-1}t_2^{-1}t_1 \Rightarrow (0\ 2\ 3)[t_0t_1^{-1}t_2t_0t_3^{-1}]^{(1\ 3)} = t_0t_1^{-1}t_2^{-1}t_1 \Rightarrow$ $(0\ 2\ 3)(1\ 3)t_0t_1^{-1}t_2t_0t_3^{-1}(1\ 3) = t_0t_1^{-1}t_2^{-1}t_1 \Rightarrow (0\ 2\ 3\ 1)t_0t_1^{-1}t_2t_0t_3^{-1}(1\ 3) = t_0t_1^{-1}t_2^{-1}t_1$, which implies that $Nt_0t_1^{-1}t_2t_0t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_0N$. That is, $[0\bar{1}210] = [0\bar{1}20\bar{3}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_0t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_0t_1^{-1} \Rightarrow$ $t_2(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow (0\,1)(2\,3)t_2^{(0\ 1)(2\ 3)}t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow (0\,1)(2\,3)t_3t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow$ $[(0\ 1)(2\ 3)t_3t_1^{-1}t_0t_1t_0]^{(0\ 1\ 2)} = [t_2t_0t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3t_2^{-1}t_1t_2t_1 = t_0t_1t_2^{-1} \Rightarrow$ $(1\,2)(0\,3)[t_0t_1^{-1}t_2t_1t_2]^{(1\ 2)(0\ 3)} = t_0t_1t_2^{-1} \Rightarrow (1\,2)(0\,3)(1\,2)(0\,3)[t_0t_1^{-1}t_2t_1t_2](1\,2)(0\,3)$ $= t_0t_1t_2^{-1} \Rightarrow et_0t_1^{-1}t_2t_1t_2(1\ 2)(0\ 3) = t_0t_1t_2^{-1}$, which implies that $Nt_0t_1^{-1}t_2t_1t_2N = Nt_0t_1t_2^{-1}N$. Therefore, $Nt_0t_1^{-1}t_2t_1t_2N = Nt_0t_1t_2^{-1}N$. That is, $[0\bar{1}212] = [01\bar{2}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1} \Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0t_1^{-1} \Rightarrow$ $(0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_0t_1^{-1} \Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_0t_1^{-1} \Rightarrow$ $t_2(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow (0\,1)(2\,3)t_2^{(0\ 1)(2\ 3)}t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow (0\,1)(2\,3)t_3t_1^{-1}t_0t_1t_0 = t_2t_0t_1^{-1} \Rightarrow$ $[(0\ 1)(2\ 3)t_3t_1^{-1}t_0t_1t_0]^{(0\ 1\ 2)} = [t_2t_0t_1^{-1}]^{(0\ 1\ 2)} \Rightarrow (1\ 2)(0\ 3)t_3t_2^{-1}t_1t_2t_1 = t_0t_1t_2^{-1} \Rightarrow$ $(1\,2)(0\,3)[t_0t_1^{-1}t_2t_1t_2]^{(1\ 2)(0\ 3)} = t_0t_1t_2^{-1} \Rightarrow (1\,2)(0\,3)(1\,2)(0\,3)[t_0t_1^{-1}t_2t_1t_2](1\,2)(0\,3)$ $= t_0t_1t_2^{-1} \Rightarrow et_0t_1^{-1}t_2t_1t_2(1\,2)(0\,3) = t_0t_1t_2^{-1} \Rightarrow t_0t_1^{-1}t_2t_1t_2(1\,2)(0\,3)t_1 = t_0t_1t_2^{-1}t_1 \Rightarrow$ $t_0t_1^{-1}t_2t_1t_2(1\ 2)(0\ 3)t_1(1\ 2)(0\ 3)(1\ 2)(0\ 3) = t_0t_1t_2^{-1}t_1 \Rightarrow$ $t_0t_1^{-1}t_2t_1t_2t_1^{(1\ 2)(0\ 3)}(1\ 2)(0\ 3) = t_0t_1t_2^{-1}t_1 \Rightarrow t_0t_1^{-1}t_2t_1t_2t_2(1\ 2)(0\ 3) = t_0t_1t_2^{-1}t_1 \Rightarrow$ $et_0t_1^{-1}t_2t_1t_2^{-1}(1\ 2)(0\ 3) = t_0t_1t_2^{-1}t_1$, which implies that $Nt_0t_1^{-1}t_2t_1t_2^{-1}N = Nt_0t_1t_2^{-1}t_1N$. Therefore, $Nt_0t_1^{-1}t_2t_1t_2^{-1}N = Nt_0t_1t_2^{-1}t_1N$. That is, $[0\bar{1}21\bar{2}] = [01\bar{2}1]$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3N$, and $Nt_0t_1^{-1}t_2t_1t_3^{-1}N$.

23. We next consider the double coset $Nt_0t_1^{-1}t_2t_1^{-1}N$.

Let $[0\bar{1}2\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2t_1^{-1}N$.

Note that $N^{(0\bar{1}2\bar{1})} \geq N^{0\bar{1}2\bar{1}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}2\bar{1})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_2t_1^{-1}N| = \frac{|N|}{|N^{(0\bar{1}2\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$,

$\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_1N = Nt_0t_{\bar{1}}^{-1}t_2eN = Nt_0t_{\bar{1}}^{-1}t_2N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N$ $= Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_2t_1N$.

Moreover, with the help of MAGMA, we know that $(0\ 2)(1\ 3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1} = t_0t_{\bar{1}}^{-1}t_0t_{\bar{2}}^{-1}$. Now, $(0\ 2)(1\ 3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1} = t_0t_{\bar{1}}^{-1}t_0t_{\bar{2}}^{-1} \Rightarrow (0\ 2)(1\ 3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2 = t_0t_{\bar{1}}^{-1}t_0t_{\bar{2}}^{-1}t_2 \Rightarrow$ $(0\ 2)(1\ 3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2 = t_0t_{\bar{1}}^{-1}t_0 \Rightarrow (0\ 2)(1\ 3)[t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2]^{(0\ 3)} = t_0t_{\bar{1}}^{-1}t_0 \Rightarrow$ $(0\ 2)(1\ 3)(0\ 3)[t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2](0\ 3) = t_0t_{\bar{1}}^{-1}t_0 \Rightarrow (0\ 1\ 3\ 2)t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2(0\ 3) = t_0t_{\bar{1}}^{-1}t_0$, which implies that $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2N = Nt_0t_{\bar{1}}^{-1}t_0N$. That is, $[0\bar{1}2\bar{1}2] = [0\bar{1}0]$.

Similarly, with the help of MAGMA, we know that $(0\,2)(1\,3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1} = t_0t_{\bar{1}}^{-1}t_0t_{\bar{2}}^{-1}$. Now, $(0\ 2)(1\ 3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1} = t_0t_{\bar{1}}^{-1}t_0t_{\bar{2}}^{-1} \Rightarrow (0\ 2)(1\ 3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2 = t_0t_{\bar{1}}^{-1}t_0t_{\bar{2}}^{-1}t_2 \Rightarrow$ $(0\ 2)(1\ 3)t_3t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2 = t_0t_{\bar{1}}^{-1}t_0 \Rightarrow (0\ 2)(1\ 3)[t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2]^{(0\ 3)} = t_0t_{\bar{1}}^{-1}t_0 \Rightarrow$ $(0\,2)(1\,3)(0\,3)[t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2](0\,3) = t_0t_{\bar{1}}^{-1}t_0 \Rightarrow (0\,1\,3\,2)t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2(0\,3) = t_0t_{\bar{1}}^{-1}t_0 \Rightarrow$ $(0\ 1\ 3\ 2)t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2(0\ 3)t_2 = t_0t_{\bar{1}}^{-1}t_0t_2 \Rightarrow (0\ 1\ 3\ 2)t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2(0\ 3)t_2(0\ 3)(0\ 3) =$ $t_0t_{\bar{1}}^{-1}t_0t_2 \Rightarrow (0\,1\,3\,2)t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2t_2^{(0\ 3)}(0\,3) = t_0t_{\bar{1}}^{-1}t_0t_2 \Rightarrow (0\,1\,3\,2)t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_2t_2(0\,3)$ $= t_0t_{\bar{1}}^{-1}t_0t_2 \Rightarrow (0\ 1\ 3\ 2)t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}(0\ 3) = t_0t_{\bar{1}}^{-1}t_0t_2$, which implies that $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_0t_2N$. That is, $[0\bar{1}2\bar{1}\bar{2}] = [0\bar{1}02]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0N$, $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_3N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{3}}^{-1}N$.

24. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_3N$.

    Let $[0\bar{1}23]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_3N$.

    Note that $N^{(0\bar{1}23)} \geq N^{0\bar{1}23} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}23)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_3N\right| = \frac{|N|}{|N^{(0\bar{1}23)}|} \leq \frac{24}{1} = 24$.

    Therefore, the double coset $[0\bar{1}23]$ has at most twenty-four distinct single cosets.

    Moreover, $N^{(0\bar{1}23)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

    Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_{\bar{1}}^{-1}t_2t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_3^{-1}N = Nt_0t_1^{-1}t_2eN = Nt_0t_1^{-1}t_2N$ and $Nt_0t_1^{-1}t_2t_3t_3N = Nt_0t_1^{-1}t_2t_3^2N = Nt_0t_1^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_0N = Nt_0t_1^{-1}t_2t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3N$, and $Nt_0t_1^{-1}t_2t_3t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3N$.

Finally, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow t_3(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_3^{-1}t_0t_1t_0 = t_3t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2t_3^{-1}t_0t_1t_0 = t_3t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_2t_3^{-1}t_0t_1t_0]^{(0\ 2\ 1\ 3)} = [t_3t_2^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0^{-1}t_2t_3t_2 = t_0t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0t_1^{-1}t_2t_3t_2]^{(0\ 1)} = t_0t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)[t_0t_1^{-1}t_2t_3t_2](0\ 1) = t_0t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0t_1^{-1}t_2t_3t_2(0\ 1) = t_0t_1^{-1}t_2^{-1}t_3^{-1}$, which implies that
$Nt_0t_1^{-1}t_2t_3t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$. That is, $[0\bar{1}232] = [0\bar{1}\bar{2}\bar{3}]$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_3t_1N$ and $Nt_0t_1^{-1}t_2t_3t_2^{-1}N$.

25. We next consider the double coset $Nt_0t_1^{-1}t_2t_3^{-1}N$.

Let $[0\bar{1}2\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_3^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2t_3^{-1} = Nt_1t_3^{-1}t_2t_0^{-1} = Nt_3t_0^{-1}t_2t_1^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}2\bar{3} \sim 1\bar{3}2\bar{0} \sim 3\bar{0}2\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}2\bar{3}]$:

$$0\bar{1}2\bar{3} \sim 1\bar{3}2\bar{0} \sim 3\bar{0}2\bar{1}, \qquad 1\bar{0}2\bar{3} \sim 0\bar{3}2\bar{1} \sim 3\bar{1}2\bar{0}, \qquad 2\bar{1}0\bar{3} \sim 1\bar{3}0\bar{2} \sim 3\bar{2}0\bar{1},$$

$$0\bar{1}3\bar{2} \sim 1\bar{2}3\bar{0} \sim 2\bar{0}3\bar{1}, \qquad 0\bar{2}1\bar{3} \sim 2\bar{3}1\bar{0} \sim 3\bar{0}1\bar{2}, \qquad 1\bar{2}0\bar{3} \sim 2\bar{3}0\bar{1} \sim 3\bar{1}0\bar{2},$$

$$2\bar{0}1\bar{3} \sim 0\bar{3}1\bar{2} \sim 3\bar{2}1\bar{0}, \qquad 2\bar{1}3\bar{0} \sim 1\bar{0}3\bar{2} \sim 0\bar{2}3\bar{1}$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}2\bar{3}]$ must have at most eight distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1}2\bar{3})} \geq N^{0\bar{1}2\bar{3}} = \langle e \rangle$. In fact, with the help of MAGMA, we know that, $N(t_0t_1^{-1}t_2t_3^{-1})^{(0\ 1\ 3)} = Nt_1t_3^{-1}t_2t_0^{-1} = Nt_0t_1^{-1}t_2t_3^{-1}$ implies that $(0\ 1\ 3) \in N^{(0\bar{1}2\bar{3})}$. Therefore, $(0\ 1\ 3) \in N^{(0\bar{1}2\bar{3})}$, and so $N^{(0\bar{1}2\bar{3})} \geq \langle(0\ 1\ 3)\rangle$. Thus $\left|N^{(0\bar{1}2\bar{3})}\right| \geq |\langle(0\ 1\ 3)\rangle| = 3$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}2\bar{3})}|} \leq \frac{24}{3} = 8$.

Therefore, as we concluded earlier, the double coset $[0\bar{1}2\bar{3}]$ has at most eight distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{3})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 3\}$, $\{2\}$, $\{\bar{0}, \bar{1}, \bar{3}\}$, and $\{\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2t_3^{-1}t_i^{\pm 1}$, $i \in \{2, 3\}$: $Nt_0t_1^{-1}t_2t_3^{-1}t_2N$, $Nt_0t_1^{-1}t_2t_3^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_3^{-1}t_3^{-1}N$.

But note that $Nt_0t_1^{-1}t_2t_3^{-1}t_3N = Nt_0t_1^{-1}t_2eN = Nt_0t_1^{-1}t_2N$ and $Nt_0t_1^{-1}t_2t_3^{-1}t_3^{-1}N$ $= Nt_0t_1^{-1}t_2t_3^{-2}N = Nt_0t_1^{-1}t_2t_3N$.

Moreover, with the help of MAGMA, we know that $(0\ 1)(2\ 3)t_3^{-1}t_1t_3^{-1}t_2 = t_0^{-1}t_1t_2^{-1}t_1$.
Now, $(0\ 1)(2\ 3)t_3^{-1}t_1t_3^{-1}t_2 = t_0^{-1}t_1t_2^{-1}t_1$
$\Rightarrow t_3(0\ 1)(2\ 3)t_3^{-1}t_1t_3^{-1}t_2 = t_3t_0^{-1}t_1t_2^{-1}t_1$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_3^{-1}t_1t_3^{-1}t_2 = t_3t_0^{-1}t_1t_2^{-1}t_1$
$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_3^{-1}t_1t_3^{-1}t_2 = t_3t_0^{-1}t_1t_2^{-1}t_1$
$\Rightarrow (0\ 1)(2\ 3)t_2t_3^{-1}t_1t_3^{-1}t_2 = t_3t_0^{-1}t_1t_2^{-1}t_1$
$\Rightarrow [(0\ 1)(2\ 3)t_2t_3^{-1}t_1t_3^{-1}t_2]^{(0\ 3\ 1\ 2)} = [t_3t_0^{-1}t_1t_2^{-1}t_1]^{(0\ 3\ 1\ 2)}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_0 = t_1t_3^{-1}t_2t_0^{-1}t_2$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1t_3^{-1}t_2t_0^{-1}t_2$

$\Rightarrow t_0t_1^{-1}t_2t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1t_3^{-1}t_2t_0^{-1}t_2$
$\Rightarrow t_0t_1^{-1}t_2t_1^{-1}t_0 = (0\ 1)(2\ 3)[t_0t_1^{-1}t_2t_3^{-1}t_2]^{(0\ 1\ 3)}$
$\Rightarrow t_0t_1^{-1}t_2t_1^{-1}t_0 = (0\ 1)(2\ 3)(0\ 3\ 1)t_0t_1^{-1}t_2t_3^{-1}t_2(0\ 1\ 3)$
$\Rightarrow t_0t_1^{-1}t_2t_1^{-1}t_0 = (0\ 2\ 3)t_0t_1^{-1}t_2t_3^{-1}t_2(0\ 1\ 3)$, which implies that $Nt_0t_1^{-1}t_2t_1^{-1}t_0N = Nt_0t_1^{-1}t_2t_3^{-1}t_2N$. That is, $[0\bar{1}2\bar{3}2] = [0\bar{1}2\bar{1}0]$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}N$.

26. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0N$.

Let $[0\bar{1}\bar{2}0]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0N$.

Note that $N^{(0\bar{1}\bar{2}0)} \geq N^{0\bar{1}\bar{2}0} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}0)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}0)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that, $Nt_0t_1^{-1}t_2^{-1}t_0t_2N = Nt_0t_1^{-1}t_2t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

27. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0})} \geq N^{0\bar{1}\bar{2}\bar{0}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0^{-1} = t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1 = t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1 = t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1 = t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1 = t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1 = t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow t_1(0\ 1)(2\ 3)t_3^{-1}t_0t_1 = t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_3^{-1}t_0t_1 = t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_3^{-1}t_0t_1 = t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_3^{-1}t_0t_1 = t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}$

$\Rightarrow [(0\ 1)(2\ 3)t_0t_3^{-1}t_0t_1]^{(0\ 2\ 1)} = [t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}]^{(0\ 2\ 1)}$

$\Rightarrow (0\ 2)(1\ 3)t_2t_3^{-1}t_2t_0 = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}$

$\Rightarrow (0\ 2)(1\ 3)[t_0t_1^{-1}t_0t_2]^{(0\ 2)(1\ 3)} = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}$

$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1^{-1}t_0t_2](0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}$

$\Rightarrow et_0t_1^{-1}t_0t_2(0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}$, which implies that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0t_2N$. That is, $[0\bar{1}\bar{2}\bar{0}\bar{2}] = [0\bar{1}02]$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$,
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

28. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1N$.

Let $[0\bar{1}\bar{2}1]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1N$.

Note that $N^{(0\bar{1}\bar{2}1)} \geq N^{0\bar{1}\bar{2}1} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}1)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_2^{-1}t_1N| = \frac{|N|}{|N^{(0\bar{1}\bar{2}1)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_2N = Nt_0t_1^{-1}t_0t_2N$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$.

29. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3N$.

Let $[0\bar{1}\bar{2}3]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3N$.

Note that $N^{(0\bar{1}\bar{2}3)} \geq N^{0\bar{1}\bar{2}3} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}3)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_3^{-1}t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_3^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_3^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_0^{-1}t_1t_0 = t_3^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_2(0\ 1)(2\ 3)t_2^{-1}t_0^{-1}t_1t_0 = t_2t_3^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_2^{-1}t_0^{-1}t_1t_0 = t_2t_3^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_2^{-1}t_0^{-1}t_1t_0 = t_2t_3^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3t_2^{-1}t_0^{-1}t_1t_0 = t_2t_3^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_3t_2^{-1}t_0^{-1}t_1t_0]^{(0\ 3\ 1\ 2)} = [t_2t_3^{-1}t_1t_0^{-1}t_1^{-1}]^{(0\ 3\ 1\ 2)}$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0^{-1}t_3^{-1}t_2t_3 = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0t_1^{-1}t_2^{-1}t_3t_2]^{(0\ 1)(2\ 3)} = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)[t_0t_1^{-1}t_2^{-1}t_3t_2](0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$, which implies that
$Nt_0t_1^{-1}t_2^{-1}t_3t_2N = Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}N$. That is, $[0\bar{1}\bar{2}32] = [0\bar{1}2\bar{3}\bar{2}]$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

30. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3})} \geq N^{0\bar{1}\bar{2}\bar{3}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_2(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_3t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow [(0\ 1)(2\ 3)t_3t_2^{-1}t_0t_1t_0^{-1}]^{(0\ 2\ 1\ 3)} = [t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_3t_2^{-1} = t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)[t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2]^{(0\ 1)}$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)(0\ 1)[t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2](0\ 1)$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (2\ 3)t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2(0\ 1)$, which implies that
$Nt_0t_1^{-1}t_2t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_2N$. That is, $[0\bar{1}23\bar{2}] = [0\bar{1}\bar{2}\bar{3}2]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0^{-1} = t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_2(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_3t_2^{-1}t_0t_1t_0^{-1} = t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0$
$\Rightarrow [(0\ 1)(2\ 3)t_3t_2^{-1}t_0t_1t_0^{-1}]^{(0\ 2\ 1\ 3)} = [t_2t_3^{-1}t_0^{-1}t_1^{-1}t_0]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_3t_2^{-1} = t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)[t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2]^{(0\ 1)}$

$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (0\ 1)(2\ 3)(0\ 1)[t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2](0\ 1)$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1} = (2\ 3)t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2(0\ 1)$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1}t_2 = (2\ 3)t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2(0\ 1)t_2$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1}t_2 = (2\ 3)t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2(0\ 1)t_2(0\ 1)(0\ 1)$
$\Rightarrow t_0t_1^{-1}t_2t_3t_2^{-1}t_2 = (2\ 3)t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2t_2^{(0\ 1)}(0\ 1)$
$\Rightarrow t_0t_1^{-1}t_2t_3 = (2\ 3)t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2t_2(0\ 1)$
$\Rightarrow t_0t_1^{-1}t_2t_3 = (2\ 3)t_0t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}(0\ 1)$, which implies that $Nt_0t_1^{-1}t_2t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}N$. That is, $[0\bar{1}23] = [0\bar{1}\bar{2}\bar{3}\bar{2}]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

31. We next consider the double coset $Nt_0t_1^{-1}t_0t_1N$.

Let $[0\bar{1}01]$ denote the double coset $Nt_0t_1^{-1}t_0t_1N$.

Note that $N^{(0\bar{1}01)} \geq N^{0\bar{1}01} = \langle(2\ 3)\rangle \cong S_2$. Thus $|N^{(01)}| \geq |S_2| = 2$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_0t_1N| = \frac{|N|}{|N^{(0\bar{1}01)}|} \leq \frac{24}{1} = 12$.

Therefore, the double coset $[0\bar{1}01]$ has at most twelve distinct single cosets.

Moreover, $N^{(0\bar{1}01)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_0t_1t_i^{\pm1}$, $i \in \{0,1,2\}$.

But note that $Nt_0t_1^{-1}t_0t_1t_1^{-1}N = Nt_0t_1^{-1}t_0eN = Nt_0t_1^{-1}t_0N$ and $Nt_0t_1^{-1}t_0t_1t_1N = Nt_0t_1^{-1}t_0t_1^2N = Nt_0t_1^{-1}t_0t_1^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)t_1t_0t_1^{-1}$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)t_1t_0t_1^{-1}$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)[t_0t_1t_0^{-1}]^{(0\ 1)}$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)(0\ 1)[t_0t_1t_0^{-1}](0\ 1)$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (2\ 3)t_0t_1t_0^{-1}(0\ 1)$, which implies that
$Nt_0t_1^{-1}t_0t_1t_0N = Nt_0t_1t_0^{-1}N$. That is, $[0\bar{1}010] = [01\bar{0}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_0t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1^{-1}t_0t_1t_0 = t_1t_0t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)t_1t_0t_1^{-1}$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)t_1t_0t_1^{-1}$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)[t_0t_1t_0^{-1}]^{(0\ 1)}$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (0\ 1)(2\ 3)(0\ 1)[t_0t_1t_0^{-1}](0\ 1)$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0 = (2\ 3)t_0t_1t_0^{-1}(0\ 1)$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0t_0 = (2\ 3)t_0t_1t_0^{-1}(0\ 1)t_0$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0^{-1} = (2\ 3)t_0t_1t_0^{-1}(0\ 1)t_0$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0^{-1} = (2\ 3)t_0t_1t_0^{-1}(0\ 1)t_0(0\ 1)(0\ 1)$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0^{-1} = (2\ 3)t_0t_1t_0^{-1}t_0^{(0\ 1)}(0\ 1)$
$\Rightarrow t_0t_1^{-1}t_0t_1t_0^{-1} = (2\ 3)t_0t_1t_0^{-1}t_1(0\ 1)$, which implies that
$Nt_0t_1^{-1}t_0t_1t_0^{-1}N = Nt_0t_1t_0^{-1}t_1N$. That is, $[0\bar{1}01\bar{0}] = [01\bar{0}1]$.

Similarly, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0t_1t_2N = Nt_0t_1t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_0t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_0t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

32. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}N$.

Let $[0\bar{1}0\bar{1}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1} = Nt_0t_{\bar{2}}^{-1}t_0t_{\bar{2}}^{-1} = Nt_0t_{\bar{3}}^{-1}t_0t_{\bar{3}}^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}0\bar{1} \sim 0\bar{2}0\bar{2} \sim 0\bar{3}0\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}0\bar{1}]$:

$$0\bar{1}0\bar{1} \sim 0\bar{2}0\bar{2} \sim 0\bar{3}0\bar{3}, \qquad 1\bar{0}1\bar{0} \sim 1\bar{2}1\bar{2} \sim 1\bar{3}1\bar{3},$$

$$2\bar{1}2\bar{1} \sim 2\bar{0}2\bar{0} \sim 2\bar{3}2\bar{3}, \qquad 3\bar{1}3\bar{1} \sim 3\bar{2}3\bar{2} \sim 3\bar{0}3\bar{0}$$

Since each of the twelve single cosets has three names, the double coset $[0\bar{1}0\bar{1}]$ must have at most four distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1}0\bar{1})} \geq N^{0\bar{1}0\bar{1}} = \langle (2\ 3) \rangle$. In fact, with the help of MAGMA, we know that $N(t_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1})^{(1\ 2)} = Nt_0t_{\bar{2}}^{-1}t_0t_{\bar{2}}^{-1} = Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}$ implies that $(1\ 2) \in N^{(0\bar{1}0\bar{1})}$, and $N(t_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1})^{(1\ 3)} = Nt_0t_{\bar{3}}^{-1}t_0t_{\bar{3}}^{-1} = Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}$ implies that $(1\ 3) \in N^{(0\bar{1}0\bar{1})}$. Therefore, $(1\ 2), (1\ 3) \in N^{(0\bar{1}0\bar{1})}$, and so $N^{(0\bar{1}0\bar{1})} \geq \langle (1\ 2), (1\ 3) \rangle \cong S_3$. Thus $\left|N^{(0\bar{1}0\bar{1})}\right| \geq |S_3| = 6$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}0\bar{1})}\right|} \leq \frac{24}{6} = 4$. Therefore, as we concluded earlier, the double coset $[0\bar{1}0\bar{1}]$ has at most four distinct single cosets.

Moreover, $N^{(0\bar{1}0\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2, 3\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length five given by $w = t_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}t_1N = Nt_0t_{\bar{1}}^{-1}t_0eN = Nt_0t_{\bar{1}}^{-1}t_0N$ and $Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N$ $= Nt_0t_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0t_1^{-1}t_0^{-1}N =$ $Nt_0^{-1}t_1t_0^{-1}t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_0t_1^{-1}t_0N$.

33. We next consider the double coset $Nt_0t_1^{-1}t_0t_2N$.

Let $[0\bar{1}02]$ denote the double coset $Nt_0t_1^{-1}t_0t_2N$.

Note that $N^{(0\bar{1}02)} \geq N^{0\bar{1}02} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}02)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}02)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}02]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}02)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_0t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0t_2t_2^{-1}N = Nt_0t_1^{-1}t_0eN = Nt_0t_1^{-1}t_0N$ and $Nt_0t_1^{-1}t_0t_2t_2N =$ $Nt_0t_1^{-1}t_0t_2^2N = Nt_0t_1^{-1}t_0t_2^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow t_1(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow [(0\ 1)(2\ 3)t_0t_2^{-1}t_0t_1t_0]^{(0\ 2\ 3\ 1)} = [t_1t_3^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2\ 3\ 1)}$

$\Rightarrow (0\ 2)(1\ 3)t_2t_3^{-1}t_2t_0t_2 = t_0t_1^{-1}t_2^{-1}t_0^{-1}$

$\Rightarrow (0\ 2)(1\ 3)[t_0t_1^{-1}t_0t_2t_0]^{(0\ 2)(1\ 3)} = t_0t_1^{-1}t_2^{-1}t_0^{-1}$

$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1^{-1}t_0t_2t_0](0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}$

$\Rightarrow et_0t_1^{-1}t_0t_2t_0(0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}$, which implies that $Nt_0t_1^{-1}t_0t_2t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$. That is, $[0\bar{1}020] = [0\bar{1}\bar{2}\bar{0}]$.

Similarly, relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0 = t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_2^{-1}t_0t_1t_0 = t_1t_3^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_0t_2^{-1}t_0t_1t_0]^{(0\ 2\ 3\ 1)} = [t_1t_3^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2\ 3\ 1)}$
$\Rightarrow (0\ 2)(1\ 3)t_2t_3^{-1}t_2t_0t_2 = t_0t_1^{-1}t_2^{-1}t_0^{-1}$
$\Rightarrow (0\ 2)(1\ 3)[t_0t_1^{-1}t_0t_2t_0]^{(0\ 2)(1\ 3)} = t_0t_1^{-1}t_2^{-1}t_0^{-1}$
$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1^{-1}t_0t_2t_0](0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}$
$\Rightarrow et_0t_1^{-1}t_0t_2t_0(0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}$
$\Rightarrow et_0t_1^{-1}t_0t_2t_0(0\ 2)(1\ 3)t_2 = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2$
$\Rightarrow et_0t_1^{-1}t_0t_2t_0(0\ 2)(1\ 3)t_2(0\ 2)(1\ 3)(0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2$
$\Rightarrow et_0t_1^{-1}t_0t_2t_0t_2^{(0\ 2)(1\ 3)}(0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2$
$\Rightarrow et_0t_1^{-1}t_0t_2t_0t_0(0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2$
$\Rightarrow et_0t_1^{-1}t_0t_2t_0^{-1}(0\ 2)(1\ 3) = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2$, which implies that
$Nt_0t_1^{-1}t_0t_2t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$. That is, $[0\bar{1}02\bar{0}] = [0\bar{1}\bar{2}\bar{0}2]$.

Similarly, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0t_2t_1N = Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_0t_2t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1N$, and $Nt_0t_1^{-1}t_0t_2t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_0t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_0t_2t_3N$.

34. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2N$.

    Let $[0\bar{1}\bar{0}2]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2N$.

    Note that $N^{(0\bar{1}\bar{0}2)} \geq N^{0\bar{1}\bar{0}2} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_0^{-1}t_2N| = \frac{|N|}{|N^{(0\bar{1}\bar{0}2)}|} \leq \frac{24}{1} = 24$.

    Therefore, the double coset $[0\bar{1}\bar{0}2]$ has at most twenty-four distinct single cosets.

    Moreover, $N^{(0\bar{1}\bar{0}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$,

$\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_0^{-1}t_2t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_0^{-1}N$ and
$Nt_0t_1^{-1}t_0^{-1}t_2t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2t_0N =$
$Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_1N =$
$Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_1^{-1}N = Nt_0^{-1}t_1t_2t_0^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form
$Nt_0t_1^{-1}t_0^{-1}t_2t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_0^{-1}t_2t_3N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$.

35. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$.

Let $[0\bar{1}0\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2})} \geq N^{0\bar{1}\bar{0}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4,
$\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_0^{-1}N$ and
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_0N =$
$Nt_0t_1^{-1}t_2^{-1}t_0N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_0N$.

Therefore, we conclude that there are four distinct double cosets of the form
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}N$,
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$.

36. We next consider the double coset $Nt_0t_1t_2t_0N$.

Let $[0120]$ denote the double coset $Nt_0t_1t_2t_0N$.

Note that $N^{(0120)} \geq N^{0120} = \langle e \rangle$. Thus $\left|N^{(0120)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_0N| = \frac{|N|}{\left|N^{(0120)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [0120] has at most twenty-four distinct single cosets.

Moreover, $N^{(0120)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_0^{-1}N = Nt_0t_1t_2eN = Nt_0t_1t_2N$ and $Nt_0t_1t_2t_0t_0N = Nt_0t_1t_2t_0^2N = Nt_0t_1t_2t_0^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow t_0(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0^{(0\ 1)(2\ 3)}t_3t_0t_1t_0 = t_0t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1t_3t_0t_1t_0 = t_0t_2t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_1t_3t_0t_1t_0]^{(1\ 2)} = [t_0t_2t_0^{-1}t_1^{-1}]^{(1\ 2)}$
$\Rightarrow (0\ 2)(1\ 3)t_2t_3t_0t_2t_0 = t_0t_1t_0^{-1}t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)[t_0t_1t_2t_0t_2]^{(0\ 2)(1\ 3)} = t_0t_1t_0^{-1}t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1t_2t_0t_2](0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}$
$\Rightarrow et_0t_1t_2t_0t_2(0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}$, which implies that $Nt_0t_1t_2t_0t_2N = Nt_0t_1t_0^{-1}t_2^{-1}N$. That is, $[01202] = [01\bar{0}\bar{2}]$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1t_2t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_0t_1N$, $Nt_0t_1t_2t_0t_1^{-1}N$, $Nt_0t_1t_2t_0t_2^{-1}N$, $Nt_0t_1t_2t_0t_3N$, and $Nt_0t_1t_2t_0t_3^{-1}N$.

37. We next consider the double coset $Nt_0t_1t_2t_0^{-1}N$.

Let $[012\bar{0}]$ denote the double coset $Nt_0t_1t_2t_0^{-1}N$.

Note that $N^{(012\bar{0})} \geq N^{012\bar{0}} = \langle e \rangle$. Thus $\left|N^{(012\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4,

$|Nt_0t_1t_2t_0^{-1}N| = \frac{|N|}{|N^{(012\bar{0})}|} \leq \frac{24}{1} = 24.$

Therefore, the double coset [$012\bar{0}$] has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_0N = Nt_0t_1t_2eN = Nt_0t_1t_2N$ and $Nt_0t_1t_2t_0^{-1}t_0^{-1}N = Nt_0t_1t_2t_0^{-2}N = Nt_0t_1t_2t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1N$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_0^{-1}t_1N$, $Nt_0t_1t_2t_0^{-1}t_2N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3N$, and $Nt_0t_1t_2t_0^{-1}t_3^{-1}N$.

38. We next consider the double coset $Nt_0t_1t_2t_3N$.

Let [0123] denote the double coset $Nt_0t_1t_2t_3N$.

Note that $N^{(0123)} \geq N^{0123} = \langle e \rangle$. Thus $|N^{(0123)}| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_3N| = \frac{|N|}{|N^{(0123)}|} \leq \frac{24}{1} = 24.$

Therefore, the double coset [0123] has at most twenty-four distinct single cosets.

Moreover, $N^{(0123)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_3t_0N$, $Nt_0t_1t_2t_3t_1N$, $Nt_0t_1t_2t_3t_2N$, $Nt_0t_1t_2t_3t_3N$, $Nt_0t_1t_2t_3t_0^{-1}N$, $Nt_0t_1t_2t_3t_1^{-1}N$, $Nt_0t_1t_2t_3t_2^{-1}N$, and $Nt_0t_1t_2t_3t_3^{-1}N$.

But note that $Nt_0t_1t_2t_3t_3^{-1}N = Nt_0t_1t_2eN = Nt_0t_1t_2N$ and $Nt_0t_1t_2t_3t_3N = Nt_0t_1t_2t_3^2N = Nt_0t_1t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

And, similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow t_3(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_3t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_3t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_3t_0t_1t_0 = t_3t_2t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_2t_3t_0t_1t_0]^{(0\ 2\ 1\ 3)} = [t_3t_2t_0^{-1}t_1^{-1}]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0t_2t_3t_2 = t_0t_1t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0t_1t_2t_3t_2]^{(0\ 1)} = t_0t_1t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)[t_0t_1t_2t_3t_2](0\ 1) = t_0t_1t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0t_1t_2t_3t_2(0\ 1) = t_0t_1t_2^{-1}t_3^{-1}$, which implies that
$Nt_0t_1t_2t_3t_2N = Nt_0t_1t_2^{-1}t_3^{-1}N$. That is, $[01232] = [01\bar{2}\bar{3}]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1t_2t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_3t_0N$, $Nt_0t_1t_2t_3t_0^{-1}N$, $Nt_0t_1t_2t_3t_1N$, and $Nt_0t_1t_2t_3t_2^{-1}N$.

39. We next consider the double coset $Nt_0t_1t_2t_3^{-1}N$.

Let $[012\bar{3}]$ denote the double coset $Nt_0t_1t_2t_3^{-1}N$.

Note that $N^{(012\bar{3})} \geq N^{012\bar{3}} = \langle e \rangle$. Thus $\left|N^{(012\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_3^{-1}N| = \frac{|N|}{|N^{(012\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_3N = Nt_0t_1t_2eN = Nt_0t_1t_2N$ and $Nt_0t_1t_2t_3^{-1}t_3^{-1}N = Nt_0t_1t_2t_3^{-2}N = Nt_0t_1t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1N$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1t_2t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_3^{-1}t_0N$, $Nt_0t_1t_2t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_1N$, $Nt_0t_1t_2t_3^{-1}t_2N$, and $Nt_0t_1t_2t_3^{-1}t_2^{-1}N$.

40. We next consider the double coset $Nt_0t_1t_2^{-1}t_0N$.

Let $[01\bar{2}0]$ denote the double coset $Nt_0t_1t_2^{-1}t_0N$.

Note that $N^{(01\bar{2}0)} \geq N^{01\bar{2}0} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0N\right| = \frac{|N|}{\left|N^{(01\bar{2}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_0t_0^{-1}N = Nt_0t_1t_2^{-1}eN = Nt_0t_1t_2^{-1}N$ and $Nt_0t_1t_2^{-1}t_0t_0N = Nt_0t_1t_2^{-1}t_0^2N = Nt_0t_1t_2^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$.

And, similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow t_2(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow t_0(0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0 = t_0t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0(0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0 = t_0t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0^{(0\ 1)(2\ 3)}t_3t_0^{-1}t_1t_0 = t_0t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1t_3t_0^{-1}t_1t_0 = t_0t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow [(0\ 1)(2\ 3)t_1t_3t_0^{-1}t_1t_0]^{(1\ 2)} = [t_0t_2t_1t_0^{-1}t_1^{-1}]^{(1\ 2)}$
$\Rightarrow (0\ 2)(1\ 3)t_2t_3t_0^{-1}t_2t_0 = t_0t_1t_2t_0^{-1}t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)[t_0t_1t_2^{-1}t_0t_2]^{(0\ 2)(1\ 3)} = t_0t_1t_2t_0^{-1}t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1t_2^{-1}t_0t_2](0\ 2)(1\ 3) = t_0t_1t_2t_0^{-1}t_2^{-1}$
$\Rightarrow et_0t_1t_2^{-1}t_0t_2(0\ 2)(1\ 3) = t_0t_1t_2t_0^{-1}t_2^{-1}$, which implies that
$Nt_0t_1t_2^{-1}t_0t_2N = Nt_0t_1t_2t_0^{-1}t_2^{-1}N$. That is, $[01\bar{2}02] = [012\bar{0}\bar{2}]$.

Finally, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0t_2^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}N$ and $Nt_0t_1t_2^{-1}t_0t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_0t_1N$ and $Nt_0t_1t_2^{-1}t_0t_3N$.

41. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}N$.

Let $[01\bar{2}\bar{0}]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}N$.

Note that $N^{(01\bar{2}\bar{0})} \geq N^{01\bar{2}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}N\right| = \frac{|N|}{|N^{(01\bar{2}\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_0N = Nt_0t_1t_2^{-1}eN = Nt_0t_1t_2^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_0^{-1}N$ $= Nt_0t_1t_2^{-1}t_0^{-2}N = Nt_0t_1t_2^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_2N =$ $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3N$, and $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

42. We next consider the double coset $Nt_0t_1t_2^{-1}t_1N$.

Let $[01\bar{2}1]$ denote the double coset $Nt_0t_1t_2^{-1}t_1N$.

Note that $N^{(01\bar{2}1)} \geq N^{01\bar{2}1} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_1N\right| = \frac{|N|}{\left|N^{(01\bar{2}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [01$\bar{2}$1] has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_1t_1^{-1}N = Nt_0t_1t_2^{-1}eN = Nt_0t_1t_2^{-1}N$ and $Nt_0t_1t_2^{-1}t_1t_1N = Nt_0t_1t_2^{-1}t_1^2N = Nt_0t_1t_2^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$ and $Nt_0t_1t_2^{-1}t_1t_2^{-1}N = Nt_0t_1^{-1}t_0t_1N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1t_2^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2^{-1}t_1t_0N$, $Nt_0t_1t_2^{-1}t_1t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3N$, and $Nt_0t_1t_2^{-1}t_1t_3^{-1}N$.

43. We next consider the double coset $Nt_0t_1t_2^{-1}t_3N$.

Let [01$\bar{2}$3] denote the double coset $Nt_0t_1t_2^{-1}t_3N$.

Note that $N^{(01\bar{2}3)} \geq N^{01\bar{2}3} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3N\right| = \frac{|N|}{\left|N^{(01\bar{2}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [01$\bar{2}$3] has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_3t_3^{-1}N = Nt_0t_1t_2^{-1}eN = Nt_0t_1t_2^{-1}N$ and $Nt_0t_1t_2^{-1}t_3t_3N = Nt_0t_1t_2^{-1}t_3^2N = Nt_0t_1t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_3t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_3t_2N = Nt_0t_1t_2t_3^{-1}t_2^{-1}N$, and $Nt_0t_1t_2^{-1}t_3t_2^{-1}N = Nt_0t_1t_2^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_3t_0N$ and $Nt_0t_1t_2^{-1}t_3t_1N$.

44. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}N$.

Let $[01\bar{2}\bar{3}]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}N$.

Note that $N^{(01\bar{2}\bar{3})} \geq N^{01\bar{2}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(01\bar{2}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_2^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_3N = Nt_0t_1t_2^{-1}eN = Nt_0t_1t_2^{-1}N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_3^{-1}N$ $= Nt_0t_1t_2^{-1}t_3^{-2}N = Nt_0t_1t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_0^{-1}N =$ $Nt_0t_1t_2t_3^{-1}t_0^{-1}N$.

And, similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow t_3(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_2t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow [(0\ 1)(2\ 3)t_2t_3t_0t_1t_0^{-1}]^{(0\ 2\ 1\ 3)} = [t_3t_2t_0^{-1}t_1^{-1}t_0]^{(0\ 2\ 1\ 3)}$

$\Rightarrow (0\ 1)(2\ 3)t_1t_0t_2t_3t_2^{-1} = t_0t_1t_2^{-1}t_3^{-1}t_2$

$\Rightarrow (0\ 1)(2\ 3)[t_0t_1t_2t_3t_2^{-1}]^{(0\ 1)} = t_0t_1t_2^{-1}t_3^{-1}t_2$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)[t_0t_1t_2t_3t_2^{-1}](0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2$

$\Rightarrow (2\ 3)t_0t_1t_2t_3t_2^{-1}(0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2$, which implies that $Nt_0t_1t_2^{-1}t_3^{-1}t_2N = Nt_0t_1t_2t_3t_2^{-1}N$. That is, $[01\bar{2}\bar{3}2] = [0123\bar{2}]$.

Finally, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow t_3(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_2t_3t_0t_1t_0^{-1} = t_3t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow [(0\ 1)(2\ 3)t_2t_3t_0t_1t_0^{-1}]^{(0\ 2\ 1\ 3)} = [t_3t_2t_0^{-1}t_1^{-1}t_0]^{(0\ 2\ 1\ 3)}$

$\Rightarrow (0\ 1)(2\ 3)t_1t_0t_2t_3t_2^{-1} = t_0t_1t_2^{-1}t_3^{-1}t_2$

$\Rightarrow (0\ 1)(2\ 3)[t_0t_1t_2t_3t_2^{-1}]^{(0\ 1)} = t_0t_1t_2^{-1}t_3^{-1}t_2$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)[t_0t_1t_2t_3t_2^{-1}](0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2$

$\Rightarrow (2\ 3)t_0t_1t_2t_3t_2^{-1}(0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2$

$\Rightarrow (2\ 3)t_0t_1t_2t_3t_2^{-1}(0\ 1)t_2 = t_0t_1t_2^{-1}t_3^{-1}t_2t_2$

$\Rightarrow (2\ 3)t_0t_1t_2t_3t_2^{-1}(0\ 1)t_2(0\ 1)(0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2^{-1}$

$\Rightarrow (2\ 3)t_0t_1t_2t_3t_2^{-1}t_2^{(0\ 1)}(0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2^{-1}$

$\Rightarrow (2\ 3)t_0t_1t_2t_3t_2^{-1}t_2(0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2^{-1}$

$\Rightarrow (2\ 3)t_0t_1t_2t_3(0\ 1) = t_0t_1t_2^{-1}t_3^{-1}t_2^{-1}$, which implies that $Nt_0t_1t_2^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_3N$. That is, $[01\bar{2}\bar{3}\bar{2}] = [0123]$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1N$, and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

45. We next consider the double coset $Nt_0t_1t_0^{-1}t_1N$.

Let $[01\bar{0}1]$ denote the double coset $Nt_0t_1t_0^{-1}t_1N$.

Note that $N^{(01\bar{0}1)} \geq N^{01\bar{0}1} = \langle(2\ 3)\rangle \cong S_2$. Thus $|N^{(01)}| \geq |S_2| = 2$ and so, by Lemma 1.4, $|Nt_0t_1t_0^{-1}t_1N| = \frac{|N|}{|N^{(01\bar{0}1)}|} \leq \frac{24}{1} = 12$.

Therefore, the double coset $[01\bar{0}1]$ has at most twelve distinct single cosets.

Moreover, $N^{(01\bar{0}1)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2, 3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2\}$.

But note that $Nt_0t_1t_0^{-1}t_1t_1^{-1}N = Nt_0t_1t_0^{-1}eN = Nt_0t_1t_0^{-1}N$ and $Nt_0t_1t_0^{-1}t_1t_1N = Nt_0t_1t_0^{-1}t_1^2N = Nt_0t_1t_0^{-1}t_1^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$.
Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1^{-1}t^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_0^{-1}t_1t_0$
$\Rightarrow t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_1^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$.
Since $(0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
and $t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$,
we conclude that $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$.
Now, $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$

$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1} = [t_0t_1t_0^{-1}t_1]^{(0\ 1)}$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1}t_1 = (0\ 1)t_0t_1t_0^{-1}t_1(0\ 1)t_1$
$\Rightarrow t_0^{-1}t_1t_0^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1(0\ 1)t_1(0\ 1)(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1t_1^{(0\ 1)}(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1t_0(0\ 1)$, which implies that
$Nt_0^{-1}t_1t_0^{-1}N = Nt_0t_1t_0^{-1}t_1t_0N$.
Therefore, $Nt_0^{-1}t_1t_0^{-1}N = Nt_0t_1t_0^{-1}t_1t_0N$. That is, $[01\bar{0}10] = [\bar{0}1\bar{0}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$.
Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1^{-1}t^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_0^{-1}t_1t_0$
$\Rightarrow t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_1^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$.
Since $(0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
and $t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$,
we conclude that $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$.

Now, $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1} = [t_0t_1t_0^{-1}t_1]^{(0\ 1)}$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1^{-1}t_1 = (0\ 1)t_0t_1t_0^{-1}t_1(0\ 1)t_1$
$\Rightarrow t_0^{-1}t_1t_0^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1(0\ 1)t_1(0\ 1)(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1t_1^{(0\ 1)}(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1} = (0\ 1)t_0t_1t_0^{-1}t_1t_0(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = (0\ 1)t_0t_1t_0^{-1}t_1t_0(0\ 1)t_1$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = (0\ 1)t_0t_1t_0^{-1}t_1t_0(0\ 1)t_1(0\ 1)(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = (0\ 1)t_0t_1t_0^{-1}t_1t_0t_1^{(0\ 1)}(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = (0\ 1)t_0t_1t_0^{-1}t_1t_0t_0(0\ 1)$
$\Rightarrow t_0^{-1}t_1t_0^{-1}t_1 = (0\ 1)t_0t_1t_0^{-1}t_1t_0^{-1}(0\ 1)$, which implies that
$Nt_0^{-1}t_1t_0^{-1}t_1N = Nt_0t_1t_0^{-1}t_1t_0^{-1}N$.
Therefore, $Nt_0^{-1}t_1t_0^{-1}t_1N = Nt_0t_1t_0^{-1}t_1t_0^{-1}N$. That is, $[\bar{0}1\bar{0}1] = [01\bar{0}1\bar{0}]$.

Finally, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_1t_2N =$
$Nt_0t_1t_2t_0^{-1}t_2N$ and $Nt_0t_1t_0^{-1}t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

46. We next consider the double coset $Nt_0t_1t_0^{-1}t_2N$.

Let $[01\bar{0}2]$ denote the double coset $Nt_0t_1t_0^{-1}t_2N$.

Note that $N^{(01\bar{0}2)} \geq N^{01\bar{0}2} = \langle e \rangle$. Thus $\left|N^{(01\bar{0}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_0^{-1}t_2N\right| = \frac{|N|}{\left|N^{(01\bar{0}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{0}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{0}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_0^{-1}t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_0^{-1}t_2t_2^{-1}N = Nt_0t_1t_0^{-1}eN = Nt_0t_1t_0^{-1}N$ and $Nt_0t_1t_0^{-1}t_2t_2N =$
$Nt_0t_1t_0^{-1}t_2^2N = Nt_0t_1t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2t_0N =$

$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1t_0^{-1}t_2t_0^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}N$, and $Nt_0t_1t_0^{-1}t_2t_1^{-1}N =$ $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1t_0^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_0^{-1}t_2t_1N$, $Nt_0t_1t_0^{-1}t_2t_3N$, and $Nt_0t_1t_0^{-1}t_2t_3^{-1}N$.

47. We next consider the double coset $Nt_0t_1t_0^{-1}t_2^{-1}N$.

Let $[01\bar{0}\bar{2}]$ denote the double coset $Nt_0t_1t_0^{-1}t_2^{-1}N$.

Note that $N^{(01\bar{0}\bar{2})} \geq N^{01\bar{0}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(01\bar{0}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_0^{-1}t_2^{-1}N\right| = \frac{|N|}{|N^{(01\bar{0}\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0t_1t_0^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_0^{-1}t_2^{-1}t_2N = Nt_0t_1t_0^{-1}eN = Nt_0t_1t_0^{-1}N$ and $Nt_0t_1t_0^{-1}t_2^{-1}t_2^{-1}N$ $= Nt_0t_1t_0^{-1}t_2^{-2}N = Nt_0t_1t_0^{-1}t_2N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow t_0(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_0^{(0\ 1)(2\ 3)}t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_1t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$

$\Rightarrow [(0\ 1)(2\ 3)t_1t_3t_0t_1t_0^{-1}]^{(1\ 2)} = [t_0t_2t_0^{-1}t_1^{-1}t_0]^{(1\ 2)}$

$\Rightarrow (0\ 2)(1\ 3)t_2t_3t_0t_2t_0^{-1} = t_0t_1t_0^{-1}t_2^{-1}t_0$

$\Rightarrow (0\ 2)(1\ 3)[t_0t_1t_2t_0t_2^{-1}]^{(0\ 2)(1\ 3)} = t_0t_1t_0^{-1}t_2^{-1}t_0$

$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1t_2t_0t_2^{-1}](0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0$
$\Rightarrow et_0t_1t_2t_0t_2^{-1}(0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0$, which implies that
$Nt_0t_1t_2t_0t_2^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_0N$. That is, $[01\bar{0}\bar{2}0] = [0120\bar{2}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_0(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0^{(0\ 1)(2\ 3)}t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_1t_3t_0t_1t_0^{-1} = t_0t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow [(0\ 1)(2\ 3)t_1t_3t_0t_1t_0^{-1}]^{(1\ 2)} = [t_0t_2t_0^{-1}t_1^{-1}t_0]^{(1\ 2)}$
$\Rightarrow (0\ 2)(1\ 3)t_2t_3t_0t_2t_0^{-1} = t_0t_1t_0^{-1}t_2^{-1}t_0$
$\Rightarrow (0\ 2)(1\ 3)[t_0t_1t_2t_0t_2^{-1}]^{(0\ 2)(1\ 3)} = t_0t_1t_0^{-1}t_2^{-1}t_0$
$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)[t_0t_1t_2t_0t_2^{-1}](0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0$
$\Rightarrow et_0t_1t_2t_0t_2^{-1}(0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0$
$\Rightarrow et_0t_1t_2t_0t_2^{-1}(0\ 2)(1\ 3)t_0 = t_0t_1t_0^{-1}t_2^{-1}t_0t_0$
$\Rightarrow et_0t_1t_2t_0t_2^{-1}(0\ 2)(1\ 3)t_0(0\ 2)(1\ 3)(0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0^{-1}$
$\Rightarrow et_0t_1t_2t_0t_2^{-1}t_0^{(0\ 2)(1\ 3)}(0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0^{-1}$
$\Rightarrow et_0t_1t_2t_0t_2^{-1}t_2(0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0^{-1}$
$\Rightarrow et_0t_1t_2t_0(0\ 2)(1\ 3) = t_0t_1t_0^{-1}t_2^{-1}t_0^{-1}$, which imples that
$Nt_0t_1t_2t_0N = Nt_0t_1t_0^{-1}t_2^{-1}t_0^{-1}N$. That is, $[01\bar{0}\bar{2}\bar{0}] = [0120]$.

Finally, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2^{-1}t_1N = Nt_0t_1t_2^{-1}t_0N$ and $Nt_0t_1t_0^{-1}t_2^{-1}t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_0^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_0^{-1}t_2^{-1}t_3N$ and $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

48. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$.

Let $[\bar{0}\bar{1}2\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}2\bar{0})} \geq N^{\bar{0}\bar{1}2\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N| = \frac{|N|}{|N^{(\bar{0}\bar{1}2\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2eN = Nt_0^{-1}t_1^{-1}t_2N$ and $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_1N = Nt_0t_1t_2^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_2N = Nt_0t_1t_0^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_2^{-1}N = Nt_0t_1t_0^{-1}t_2t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_3N = Nt_0t_1t_2^{-1}t_3N$, and $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_3t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

49. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1N$.

Let $[\bar{0}\bar{1}21]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1N$.

Note that $N^{(\bar{0}\bar{1}21)} \geq N^{\bar{0}\bar{1}21} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}21)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2t_1N| = \frac{|N|}{|N^{(\bar{0}\bar{1}21)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}21]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}21)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2eN = Nt_0^{-1}t_1^{-1}t_2N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_1N = Nt_0^{-1}t_1^{-1}t_2t_1^2N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_2N = Nt_0^{-1}t_1t_2^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$.

50. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$.

Let $[\bar{0}\bar{1}2\bar{1}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$.

Note that $N^{(\bar{0}\bar{1}2\bar{1})} \geq N^{\bar{0}\bar{1}2\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}2\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2eN = Nt_0^{-1}t_1^{-1}t_2N$ and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_2N = Nt_0t_1t_0^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_2^{-1}N = Nt_0t_1t_2t_0^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_3N = Nt_0t_1t_2t_0t_2^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_3N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

51. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3N$.

Let $[\bar{0}\bar{1}23]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3N$.

Note that $N^{(\bar{0}\bar{1}23)} \geq N^{\bar{0}\bar{1}23} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}23)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}23)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}23]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}23)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2eN = Nt_0^{-1}t_1^{-1}t_2N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_3N$ $= Nt_0^{-1}t_1^{-1}t_2t_3^2N = Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_1^{-1}N =$ $Nt_0t_1^{-1}t_2^{-1}t_0t_3N$.

And, similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_2^{-1}t_3^{-1}t_0t_1t_0]^{(0\ 2\ 1\ 3)} = [t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_2t_3t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1^{-1}t_2t_3t_2]^{(0\ 1)} = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$, which implies that $Nt_0^{-1}t_1^{-1}t_2t_3t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$. That is, $[\bar{0}\bar{1}232] = [\bar{0}\bar{1}\bar{2}\bar{3}]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1^{-1}t_2t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_1N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$.

52. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

Let $[\bar{0}\bar{1}2\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}2\bar{3})} \geq N^{\bar{0}\bar{1}2\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}2\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2eN = Nt_0^{-1}t_1^{-1}t_2N$ and $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_0N = Nt_0t_1t_2^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_1N = Nt_0t_1t_2^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

53. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$.

Let $[\bar{0}\bar{1}\bar{2}0]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}0)} \geq N^{\bar{0}\bar{1}\bar{2}0} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_2N = Nt_0t_1t_0^{-1}t_2t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

54. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{0})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$.

Moreover, by relation (7.1), $(0\ 1\ 2)t_0t_2t_1t_0t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1\ 2)[t_0t_1t_2t_0t_1]^{(1\ 2)} = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1\ 2)(1\ 2)t_0t_1t_2t_0t_1(1\ 2) = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)t_0t_1t_2t_0t_1(1\ 2) = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$, which implies that $Nt_0t_1t_2t_0t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$. That is, $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}] = [01201]$.

Similarly, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_2N = Nt_0t_1t_2^{-1}t_1t_3^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1t_0^{-1}t_2N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

55. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$.

Let $[\bar{0}\bar{1}\bar{2}1]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}1)} \geq N^{\bar{0}\bar{1}\bar{2}1} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}1)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_2N = Nt_0t_1t_2t_0^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$.

56. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$.

Let $[\bar{0}\bar{1}\bar{2}3]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}3)} \geq N^{\bar{0}\bar{1}\bar{2}3} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}3)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_0N = Nt_0^{-1}t_1^{-1}t_2t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

57. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{3})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_2^{-1}t_3^{-1}t_0t_1t_0]^{(0\ 2\ 1\ 3)} = [t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_2t_3t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1^{-1}t_2t_3t_2]^{(0\ 1)} = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1)t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1)t_2(0\ 1)(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2t_2^{(0\ 1)}(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2t_2(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2^{-1}(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2$, which implies that $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2N$. That is, $[\bar{0}\bar{1}23\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{3}2]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_3^{-1}t_0t_1t_0 = t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_2^{-1}t_3^{-1}t_0t_1t_0]^{(0\ 2\ 1\ 3)} = [t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_2t_3t_2 = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1^{-1}t_2t_3t_2]^{(0\ 1)} = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1)t_2^{-1} = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(0\ 1)t_2^{-1}(0\ 1)(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2(t_2^{-1})^{(0\ 1)}(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3t_2t_2^{-1}(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1^{-1}t_2t_3(0\ 1) = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}$, which implies that
$Nt_0^{-1}t_1^{-1}t_2t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1}N$. That is, $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{2}] = [\bar{0}\bar{1}23]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

58. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_1N$.

Let $[\bar{0}1\bar{0}1]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_1N$.

Now, with the help of MAGMA, we know that $(0\ 2)(1\ 3)t_3t_1^{-1}t_2t_1^{-1} = t_0t_1^{-1}t_0t_2^{-1}$
$\Rightarrow t_1^{-1}(0\ 2)(1\ 3)t_3t_1^{-1}t_2t_1^{-1} = t_1^{-1}t_0t_1^{-1}t_0t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)t_1^{-1}(0\ 2)(1\ 3)t_3t_1^{-1}t_2t_1^{-1} = t_1^{-1}t_0t_1^{-1}t_0t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)(t_1^{-1})^{(0\ 2)(1\ 3)}t_3t_1^{-1}t_2t_1^{-1} = t_1^{-1}t_0t_1^{-1}t_0t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)t_3^{-1}t_3t_1^{-1}t_2t_1^{-1} = t_1^{-1}t_0t_1^{-1}t_0t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)t_1^{-1}t_2t_1^{-1} = t_1^{-1}t_0t_1^{-1}t_0t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)t_1^{-1}t_2t_1^{-1}t_2 = t_1^{-1}t_0t_1^{-1}t_0t_2^{-1}t_2$
$\Rightarrow (0\ 2)(1\ 3)t_1^{-1}t_2t_1^{-1}t_2 = t_1^{-1}t_0t_1^{-1}t_0$
$\Rightarrow [(0\ 2)(1\ 3)t_1^{-1}t_2t_1^{-1}t_2]^{(0\ 1)} = [t_1^{-1}t_0t_1^{-1}t_0]^{(0\ 1)}$
$\Rightarrow (1\ 2)(0\ 3)t_0^{-1}t_2t_0^{-1}t_2 = t_0^{-1}t_1t_0^{-1}t_1$
and, moreover, $(1\ 2)(0\ 3)t_0^{-1}t_2t_0^{-1}t_2 = t_0^{-1}t_1t_0^{-1}t_1$
$\Rightarrow [(1\ 2)(0\ 3)t_0^{-1}t_2t_0^{-1}t_2]^{(2\ 3)} = [t_0^{-1}t_1t_0^{-1}t_1]^{(2\ 3)}$
$\Rightarrow (1\ 3)(0\ 2)t_0^{-1}t_3t_0^{-1}t_3 = t_0^{-1}t_1t_0^{-1}t_1$.
Therefore, since $(1\ 2)(0\ 3)t_0^{-1}t_2t_0^{-1}t_2 = t_0^{-1}t_1t_0^{-1}t_1$ and $(1\ 3)(0\ 2)t_0^{-1}t_3t_0^{-1}t_3 =$

$t_0^{-1}t_1t_0^{-1}t_1$, we have that $(1\ 3)(0\ 2)t_0^{-1}t_3t_0^{-1}t_3 = t_0^{-1}t_1t_0^{-1}t_1 = (1\ 2)(0\ 3)t_0^{-1}t_2t_0^{-1}t_2$, and therefore the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1t_0^{-1}t_1 = Nt_0t_2^{-1}t_0t_2^{-1} = Nt_0t_3^{-1}t_0t_3^{-1}$.

That is, in terms of our short-hand notation,

$$\bar{0}1\bar{0}1 \sim \bar{0}2\bar{0}2 \sim \bar{0}3\bar{0}3.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}1\bar{0}1]$:

$$\bar{0}1\bar{0}1 \sim \bar{0}2\bar{0}2 \sim \bar{0}3\bar{0}3, \qquad \bar{1}0\bar{1}0 \sim \bar{1}2\bar{1}2 \sim \bar{1}3\bar{1}3,$$

$$\bar{2}1\bar{2}1 \sim \bar{2}0\bar{2}0 \sim \bar{2}3\bar{2}3, \qquad \bar{3}1\bar{3}1 \sim \bar{3}2\bar{3}2 \sim \bar{3}0\bar{3}0$$

Since each of the twelve single cosets has three names, the double coset $[\bar{0}1\bar{0}1]$ must have at most four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}1)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2, 3\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}t_1t_0^{-1}eN = Nt_0^{-1}t_1t_0^{-1}N$ and $Nt_0^{-1}t_1t_0^{-1}t_1t_1N = Nt_0^{-1}t_1t_0^{-1}t_1^2N = Nt_0^{-1}t_1t_0^{-1}t_1^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$.
Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1^{-1}t^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_0^{-1}t_1t_0$
$\Rightarrow t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_1^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$.
Since $(0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
and $t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$,
we conclude that $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$.
Now, $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_1t_0t_1^{-1}t_0$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1^{-1}t_0t_1^{-1}t_0$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1}t_1^{-1}t_1 = t_1^{-1}t_0t_1^{-1}t_0t_1$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}t_0t_1$
$\Rightarrow [t_1t_0^{-1}t_1t_0^{-1}]^{(0\ 1)} = [t_1^{-1}t_0t_1^{-1}t_0t_1]^{(0\ 1)}$
$\Rightarrow t_0t_1^{-1}t_0t_1^{-1} = t_0^{-1}t_1t_0^{-1}t_1t_0$, which implies that
$Nt_0t_1^{-1}t_0t_1^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_1t_0N$. That is, $[\bar{0}1\bar{0}10] = [0\bar{1}0\bar{1}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$.
Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1^{-1}t^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1} = t_0^{-1}t_1$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_0^{-1}t_1t_0$
$\Rightarrow t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1^{-1}(0\ 1)(2\ 3)t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_1^{-1})^{(0\ 1)(2\ 3)}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_0t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$
$\Rightarrow t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$.
Since $(0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0 = t_0^{-1}t_1t_0^{-1}t_1^{-1}$
and $t_1t_0t_1^{-1}t_0 = (0\ 1)(2\ 3)t_1^{-1}t_0^{-1}t_1t_0$,
we conclude that $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$.
Now, $t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_0t_1^{-1}t_0$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1t_1t_0t_1^{-1}t_0$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1}t_1^{-1} = t_1^{-1}t_0t_1^{-1}t_0$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1}t_1^{-1}t_1 = t_1^{-1}t_0t_1^{-1}t_0t_1$
$\Rightarrow t_1t_0^{-1}t_1t_0^{-1} = t_1^{-1}t_0t_1^{-1}t_0t_1$
$\Rightarrow [t_1t_0^{-1}t_1t_0^{-1}]^{(0\ 1)} = [t_1^{-1}t_0t_1^{-1}t_0t_1]^{(0\ 1)}$
$\Rightarrow t_0t_1^{-1}t_0t_1^{-1} = t_0^{-1}t_1t_0^{-1}t_1t_0$
$\Rightarrow t_0t_1^{-1}t_0t_1^{-1}t_0 = t_0^{-1}t_1t_0^{-1}t_1t_0t_0$
$\Rightarrow t_0t_1^{-1}t_0t_1^{-1}t_0 = t_0^{-1}t_1t_0^{-1}t_1t_0^{-1}$, which implies that
$Nt_0t_1^{-1}t_0t_1^{-1}t_0N = Nt_0^{-1}t_1t_0^{-1}t_1t_0^{-1}N$. That is, $[0\bar{1}0\bar{1}0] = [\bar{0}1\bar{0}1\bar{0}]$.

Therefore, we need not consider additional double coset of the form $Nt_0^{-1}t_1t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

59. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2N$.

Let $[\bar{0}1\bar{0}2]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2N$.

Note that $N^{(\bar{0}1\bar{0}2)} \geq N^{\bar{0}1\bar{0}2} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{0}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_0^{-1}t_2N\right| = \frac{|N|}{|N^{(\bar{0}1\bar{0}2)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{0}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_0^{-1}t_2t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}eN = Nt_0^{-1}t_1t_0^{-1}N$ and $Nt_0^{-1}t_1t_0^{-1}t_2t_2N = Nt_0^{-1}t_1t_0^{-1}t_2^2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_0^{-1}t_2t_0N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1t_0^{-1}t_2t_1^{-1}N = Nt_0^{-1}t_1t_2^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_2t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2t_1N$, $Nt_0^{-1}t_1t_0^{-1}t_2t_3N$, and $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}N$.

60. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$.

Let $[\bar{0}1\bar{0}\bar{2}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$.

Note that $N^{(\bar{0}1\bar{0}\bar{2})} \geq N^{\bar{0}1\bar{0}\bar{2}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}1\bar{0}\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N| = \frac{|N|}{|N^{(\bar{0}1\bar{0}\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_0^{-1}t_2^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_2N = Nt_0^{-1}t_1t_0^{-1}eN = Nt_0^{-1}t_1t_0^{-1}N$ and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-2}N = Nt_0^{-1}t_1t_0^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_0N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2t_0N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$, and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_2N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$ and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

61. We next consider the double coset $Nt_0^{-1}t_1t_2t_0N$.

Let $[\bar{0}120]$ denote the double coset $Nt_0^{-1}t_1t_2t_0N$.

Note that $N^{(\bar{0}120)} \geq N^{\bar{0}120} = \langle e\rangle$. Thus $\left|N^{(\bar{0}120)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1t_2t_0N| = \frac{|N|}{|N^{(\bar{0}120)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}120]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}120)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_0t_0^{-1}N = Nt_0^{-1}t_1t_2eN = Nt_0^{-1}t_1t_2N$ and $Nt_0^{-1}t_1t_2t_0t_0N = Nt_0^{-1}t_1t_2t_0^2N = Nt_0^{-1}t_1t_2t_0^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0]^{(1\ 2)} = [t_0^{-1}t_2t_0^{-1}t_1^{-1}]^{(1\ 2)}$
$\Rightarrow (0\ 2)(1\ 3)t_2^{-1}t_3t_0t_2t_0 = t_0^{-1}t_1t_0^{-1}t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)[t_0^{-1}t_1t_2t_0t_2]^{(0\ 2)(1\ 3)} = t_0^{-1}t_1t_0^{-1}t_2^{-1}$
$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)t_0^{-1}t_1t_2t_0t_2(0\ 2)(1\ 3) = t_0^{-1}t_1t_0^{-1}t_2^{-1}$
$\Rightarrow et_0^{-1}t_1t_2t_0t_2(0\ 2)(1\ 3) = t_0^{-1}t_1t_0^{-1}t_2^{-1}$, which implies that
$Nt_0^{-1}t_1t_2t_0t_2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$. That is, $[\bar{0}1202] = [\bar{0}1\bar{0}\bar{2}]$.

Similarly, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_0t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N$ and $Nt_0^{-1}t_1t_2t_0t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0^{-1}t_1t_2t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1t_2t_0t_1N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}N$, and $Nt_0^{-1}t_1t_2t_0t_3^{-1}N$.

62. We next consider the double coset $Nt_0^{-1}t_1t_2t_0^{-1}N$.

Let $[\bar{0}12\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_2t_0^{-1}N$.

Note that $N^{(\bar{0}12\bar{0})} \geq N^{\bar{0}12\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}12\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4,

$|Nt_0^{-1}t_1t_2t_0^{-1}N| = \frac{|N|}{|N^{(\bar{0}12\bar{0})}|} \leq \frac{24}{1} = 24.$

Therefore, the double coset $[\bar{0}12\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}12\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_0^{-1}t_0N = Nt_0^{-1}t_1t_2eN = Nt_0^{-1}t_1t_2N$ and $Nt_0^{-1}t_1t_2t_0^{-1}t_0^{-1}N$ $= Nt_0^{-1}t_1t_2t_0^{-2}N = Nt_0^{-1}t_1t_2t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_0^{-1}t_1N =$ $Nt_0t_1t_2t_3^{-1}t_0N$, $Nt_0^{-1}t_1t_2t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$, $Nt_0^{-1}t_1t_2t_0^{-1}t_2N =$ $Nt_0t_1^{-1}t_0^{-1}t_2N$, $Nt_0^{-1}t_1t_2t_0^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_0^{-1}t_3N =$ $Nt_0t_1t_2t_3^{-1}t_1N$, and $Nt_0^{-1}t_1t_2t_0^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

63. We next consider the double coset $Nt_0^{-1}t_1t_2t_3N$.

Let $[\bar{0}123]$ denote the double coset $Nt_0^{-1}t_1t_2t_3N$.

Note that $N^{(\bar{0}123)} \geq N^{\bar{0}123} = \langle e \rangle$. Thus $|N^{(\bar{0}123)}| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1t_2t_3N| = \frac{|N|}{|N^{(\bar{0}123)}|} \leq \frac{24}{1} = 24.$

Therefore, the double coset $[\bar{0}123]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}123)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_3^{-1}N = Nt_0^{-1}t_1t_2eN = Nt_0^{-1}t_1t_2N$ and $Nt_0^{-1}t_1t_2t_3t_3N =$ $Nt_0^{-1}t_1t_2t_3^2N = Nt_0^{-1}t_1t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_0^{-1}N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

And, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_3^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_3^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_3t_0t_1t_0 = t_3^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_3t_0t_1t_0 = t_3^{-1}t_2t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_2^{-1}t_3t_0t_1t_0]^{(0\ 2\ 1\ 3)} = [t_3^{-1}t_2t_0^{-1}t_1^{-1}]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_2t_3t_2 = t_0^{-1}t_1t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1t_2t_3t_2]^{(0\ 1)} = t_0^{-1}t_1t_2^{-1}t_3^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)t_0^{-1}t_1t_2t_3t_2(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3t_2(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}$, which implies that
$Nt_0^{-1}t_1t_2t_3t_2N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$. That is, $[\bar{0}1232] = [\bar{0}1\bar{2}\bar{3}]$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2t_3t_0N$, $Nt_0^{-1}t_1t_2t_3t_1N$, $Nt_0^{-1}t_1t_2t_3t_1^{-1}N$, and $Nt_0^{-1}t_1t_2t_3t_2^{-1}N$.

64. We next consider the double coset $Nt_0^{-1}t_1t_2t_3^{-1}N$.

Let $[\bar{0}12\bar{3}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3^{-1}N$.

Note that $N^{(\bar{0}12\bar{3})} \geq N^{\bar{0}12\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}12\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}12\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}12\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}12\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2t_3^{-1}t_3N = Nt_0^{-1}t_1t_2eN = Nt_0^{-1}t_1t_2N$ and $Nt_0^{-1}t_1t_2t_3^{-1}t_3^{-1}N$ $= Nt_0^{-1}t_1t_2t_3^{-2}N = Nt_0^{-1}t_1t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0^{-1}t_1t_2t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_1t_3N$, $Nt_0^{-1}t_1t_2t_3^{-1}t_1^{-1}N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$, and $Nt_0^{-1}t_1t_2t_3^{-1}t_2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}N$.

65. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_0N$.

Let $[\bar{0}1\bar{2}0]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_0N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1t_2^{-1}t_0 = Nt_1^{-1}t_2t_3^{-1}t_1 = Nt_2^{-1}t_3t_0^{-1}t_2$ $= Nt_3^{-1}t_0t_1^{-1}t_3$.

That is, in terms of our short-hand notation,

$$\bar{0}1\bar{2}0 \sim \bar{1}2\bar{3}1 \sim \bar{2}3\bar{0}2 \sim \bar{3}0\bar{1}3.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}1\bar{2}0]$:

$$\bar{0}1\bar{2}0 \sim \bar{1}2\bar{3}1 \sim \bar{2}3\bar{0}2 \sim \bar{3}0\bar{1}3, \qquad \bar{1}0\bar{2}1 \sim \bar{0}2\bar{3}0 \sim \bar{2}3\bar{1}2 \sim \bar{3}1\bar{0}3,$$

$$\bar{2}1\bar{0}2 \sim \bar{1}0\bar{3}1 \sim \bar{0}3\bar{2}0 \sim \bar{3}2\bar{1}3, \qquad \bar{3}1\bar{2}3 \sim \bar{1}2\bar{0}1 \sim \bar{2}0\bar{3}2 \sim \bar{0}3\bar{1}0,$$

$$\bar{0}2\bar{1}0 \sim \bar{2}1\bar{3}2 \sim \bar{1}3\bar{0}1 \sim \bar{3}0\bar{2}3, \qquad \bar{0}1\bar{3}0 \sim \bar{1}3\bar{2}1 \sim \bar{3}2\bar{0}3 \sim \bar{2}0\bar{1}2$$

Since each of the twenty-four single cosets has four names, the double coset $[\bar{0}1\bar{2}0]$ must have at most six distinct single cosets.

Now, $N^{(\bar{0}1\bar{2}0)}$ has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2^{-1}t_0t_i^{\pm 1}$, $i = 0$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_0t_0^{-1}N = Nt_0^{-1}t_1t_2^{-1}eN = Nt_0^{-1}t_1t_2^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_0t_0N = Nt_0^{-1}t_1t_2^{-1}t_0^2N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

66. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$.

Let $[\bar{0}1\bar{2}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$.

Note that $N^{(\bar{0}1\bar{2}\bar{0})} \geq N^{\bar{0}1\bar{2}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{2}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}1\bar{2}\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{2}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{2}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1t_2^{-1}eN = Nt_0^{-1}t_1t_2^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-2}N = Nt_0^{-1}t_1t_2^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_2N = Nt_0^{-1}t_1t_2t_0^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2N$, and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

67. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_3N$.

Let $[\bar{0}1\bar{2}3]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_3N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1t_2^{-1}t_3 = Nt_1^{-1}t_3t_2^{-1}t_0 = Nt_3^{-1}t_0t_2^{-1}t_1$.

That is, in terms of our short-hand notation,

$$\bar{0}1\bar{2}3 \sim \bar{1}3\bar{2}0 \sim \bar{3}0\bar{2}1.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}1\bar{2}3]$:

$$\bar{0}1\bar{2}3 \sim \bar{1}3\bar{2}0 \sim \bar{3}0\bar{2}1, \qquad \bar{1}0\bar{2}3 \sim \bar{0}3\bar{2}1 \sim \bar{3}1\bar{2}0, \qquad \bar{2}1\bar{0}3 \sim \bar{1}3\bar{0}2 \sim \bar{3}2\bar{0}1,$$

$$\bar{0}1\bar{3}2 \sim \bar{1}2\bar{3}0 \sim \bar{2}0\bar{3}1, \qquad \bar{0}2\bar{1}3 \sim \bar{2}3\bar{1}0 \sim \bar{3}0\bar{1}2, \qquad \bar{1}2\bar{0}3 \sim \bar{2}3\bar{0}1 \sim \bar{3}1\bar{0}2,$$

$$\bar{2}0\bar{1}3 \sim \bar{0}3\bar{1}2 \sim \bar{3}2\bar{1}0, \qquad \bar{2}1\bar{3}0 \sim \bar{1}0\bar{3}2 \sim \bar{0}2\bar{3}1$$

Since each of the twenty-four single cosets has three names, the double coset $[\bar{0}1\bar{2}3]$ must have at most eight distinct single cosets.

Now, $N^{(\bar{0}1\bar{2}3)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 3\}$, $\{2\}$, $\{\bar{0}, \bar{1}, \bar{3}\}$, and $\{\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2^{-1}t_3t_i^{\pm 1}$, $i \in \{2,3\}$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_3t_3^{-1}N = Nt_0^{-1}t_1t_2^{-1}eN = Nt_0^{-1}t_1t_2^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_3t_3N = Nt_0^{-1}t_1t_2^{-1}t_3^2N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$

$\Rightarrow t_2(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0 = t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0 = t_3^{-1}t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_3t_0^{-1}t_1t_0 = t_3^{-1}t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_3t_0^{-1}t_1t_0 = t_3^{-1}t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_3t_0^{-1}t_1t_0 = t_3^{-1}t_2t_1t_0^{-1}t_1^{-1}$

$\Rightarrow [(0\ 1)(2\ 3)t_2^{-1}t_3t_0^{-1}t_1t_0]^{(1\ 2)(0\ 3)} = [t_3^{-1}t_2t_1t_0^{-1}t_1^{-1}]^{(1\ 2)(0\ 3)}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_3^{-1}t_2t_3 = t_0^{-1}t_1t_2t_3^{-1}t_2^{-1}$

$\Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1t_2^{-1}t_3t_2]^{(0\ 1)(2\ 3)} = t_0^{-1}t_1t_2t_3^{-1}t_2^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}t_1t_2^{-1}t_3t_2(0\ 1)(2\ 3) = t_0^{-1}t_1t_2t_3^{-1}t_2^{-1}$

$\Rightarrow et_0^{-1}t_1t_2^{-1}t_3t_2(0\ 1)(2\ 3) = t_0^{-1}t_1t_2t_3^{-1}t_2^{-1}$, which implies that $Nt_0^{-1}t_1t_2^{-1}t_3t_2N = Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}N$. That is, $[\bar{0}1\bar{2}32] = [\bar{0}12\bar{3}\bar{2}]$.

Finally, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2^{-1}t_3t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

68. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$.

Let $[\bar{0}1\bar{2}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}1\bar{2}\bar{3})} \geq N^{\bar{0}1\bar{2}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{2}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N| = \frac{|N|}{|N^{(\bar{0}1\bar{2}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{2}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{2}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length five given by $w = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1t_2^{-1}eN = Nt_0^{-1}t_1t_2^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_3^{-2}N = Nt_0^{-1}t_1t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$, and $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow [(0\ 1)(2\ 3)t_2^{-1}t_3t_0t_1t_0^{-1}]^{(0\ 2\ 1\ 3)} = [t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_2t_3t_2^{-1} = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1t_2t_3t_2^{-1}]^{(0\ 1)} = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)t_0^{-1}t_1t_2t_3t_2^{-1}(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3t_2^{-1}(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$, which implies that
$Nt_0^{-1}t_1t_2t_3t_2^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_2N$. That is, $[\bar{0}12\bar{3}2] = [\bar{0}123\bar{2}]$.

Finally, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0t_0 = t_0^{-1}t_1^{-1}t_0$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow t_3^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)(t_3^{-1})^{(0\ 1)(2\ 3)}t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow (0\ 1)(2\ 3)t_2^{-1}t_3t_0t_1t_0^{-1} = t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0$
$\Rightarrow [(0\ 1)(2\ 3)t_2^{-1}t_3t_0t_1t_0^{-1}]^{(0\ 2\ 1\ 3)} = [t_3^{-1}t_2t_0^{-1}t_1^{-1}t_0]^{(0\ 2\ 1\ 3)}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_0t_2t_3t_2^{-1} = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (0\ 1)(2\ 3)[t_0^{-1}t_1t_2t_3t_2^{-1}]^{(0\ 1)} = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)t_0^{-1}t_1t_2t_3t_2^{-1}(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3t_2^{-1}(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3t_2^{-1}(0\ 1)t_2 = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2t_2$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3t_2^{-1}(0\ 1)t_2(0\ 1)(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3t_2^{-1}t_2^{(0\ 1)}(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3t_2^{-1}t_2(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2^{-1}$
$\Rightarrow (2\ 3)t_0^{-1}t_1t_2t_3(0\ 1) = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_2^{-1}$, which implies that
$Nt_0^{-1}t_1t_2t_3N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_2^{-1}N$. That is, $[\bar{0}1\bar{2}\bar{3}\bar{2}] = [\bar{0}123]$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

69. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1N$.

Let $[0\bar{1}201]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1N$.

Note that $N^{(0\bar{1}201)} \geq N^{0\bar{1}201} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}201)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}201)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}201]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}201)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_0t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_1^{-1}N = Nt_0t_1^{-1}t_2t_0eN = Nt_0t_1^{-1}t_2t_0N$ and $Nt_0t_1^{-1}t_2t_0t_1t_1N = Nt_0t_1^{-1}t_2t_0t_1^2N = Nt_0t_1^{-1}t_2t_0t_1^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow t_1(0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0 = t_1t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0 = t_1t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_1^{-1}t_3t_0t_1t_0 = t_1t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_0t_1^{-1}t_3t_0t_1t_0 = t_1t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow [(0\ 1)(2\ 3)t_0t_1^{-1}t_3t_0t_1t_0]^{(0\ 1)} = [t_1t_0^{-1}t_2t_0^{-1}t_1^{-1}]^{(0\ 1)}$

$\Rightarrow (0\ 1)(2\ 3)t_1t_0^{-1}t_3t_1t_0t_1 = t_0t_1^{-1}t_2t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)[t_0t_1^{-1}t_2t_0t_1t_0]^{(0\ 1)(2\ 3)} = t_0t_1^{-1}t_2t_1^{-1}t_0^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_2t_0t_1t_0(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_1^{-1}t_0^{-1}$

$\Rightarrow et_0t_1^{-1}t_2t_0t_1t_0(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_1^{-1}t_0^{-1}$, which implies that $Nt_0t_1^{-1}t_2t_0t_1t_0N = Nt_0t_1^{-1}t_2t_1^{-1}t_0^{-1}N$. That is, $[0\bar{1}2010] = [0\bar{1}2\bar{1}\bar{0}]$.

Therefore, we conclude that there are five distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_2N$, $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_3N$, and $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$.

70. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}N$.

Let $[0\bar{1}20\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}N$.

Note that $N^{(0\bar{1}20\bar{1})} \geq N^{0\bar{1}20\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_1N = Nt_0t_{\bar{1}}^{-1}t_2t_0eN = Nt_0t_{\bar{1}}^{-1}t_2t_0N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_0N =$ $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3N$, $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_3N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_{\bar{3}}^{-1}N =$ $Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_0N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_2N$, $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_3N$.

71. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3N$.

 Let $[0\bar{1}203]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3N$.

 Note that $N^{(0\bar{1}203)} \geq N^{0\bar{1}203} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}203)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_0t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}203)}\right|} \leq \frac{24}{1} = 24$.

 Therefore, the double coset $[0\bar{1}203]$ has at most twenty-four distinct single cosets.

 Moreover, $N^{(0\bar{1}203)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

 Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_2t_0t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

 But note that $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_{\bar{3}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_0eN = Nt_0t_{\bar{1}}^{-1}t_2t_0N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_3N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_3^2N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{3}}^{-1}N$.

 Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_0N =$ $Nt_0t_{\bar{1}}^{-1}t_2t_3N$, $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_{\bar{0}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_3t_1N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_{\bar{2}}^{-1}N =$ $Nt_{\bar{0}}^{-1}t_1t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_{\bar{3}}^{-1}N$.

 Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_1N$, $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_{\bar{1}}^{-1}N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_2N$.

72. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{3}}^{-1}N$.

 Let $[0\bar{1}20\bar{3}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{3}}^{-1}N$.

Note that $N^{(0\bar{1}20\bar{3})} \geq N^{0\bar{1}20\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_0eN = Nt_0t_1^{-1}t_2t_0N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-2}N = Nt_0t_1^{-1}t_2t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$.

73. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}N$.

Let $[0\bar{1}21\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}N$.

Note that $N^{(0\bar{1}21\bar{0})} \geq N^{0\bar{1}21\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}21\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}21\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}21\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}21\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_1eN = Nt_0t_1^{-1}t_2t_1N$ and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_0^{-2}N = Nt_0t_1^{-1}t_2t_1t_0N = Nt_0t_1^{-1}t_2t_0t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_2N = Nt_0t_1t_2^{-1}t_0^{-1}t_1N$, and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_2^{-1}N = Nt_0t_1t_2^{-1}t_3t_1N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_2t_1t_0^{-1}t_1N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_0^{-1}t_3N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

74. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3N$.

Let $[0\bar{1}213]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3N$.

Note that $N^{(0\bar{1}213)} \geq N^{0\bar{1}213} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}213)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_1t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}213)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}213]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}213)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_2t_1t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_3^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_1eN = Nt_0t_{\bar{1}}^{-1}t_2t_1N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_3N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3^2N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_1N = Nt_0t_{\bar{1}}^{-1}t_2t_3N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_0N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_0^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_2N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_2^{-1}N$.

75. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3^{-1}N$.

Let $[0\bar{1}21\bar{3}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3^{-1}N$.

Note that $N^{(0\bar{1}21\bar{3})} \geq N^{0\bar{1}21\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}21\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_1t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}21\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}21\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}21\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_2t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_3N = Nt_0t_{\bar{1}}^{-1}t_2t_1eN = Nt_0t_{\bar{1}}^{-1}t_2t_1N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_{\bar{3}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_1N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_1t_{\bar{0}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{3}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_2N = Nt_0t_1t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_1N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}N = Nt_0t_1t_{\bar{2}}^{-1}t_0t_1N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_{\bar{0}}^{-1}N$.

76. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0N$.

Let $[0\bar{1}2\bar{1}0]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0 = Nt_1t_{\bar{3}}^{-1}t_2t_{\bar{3}}^{-1}t_1 = Nt_3t_{\bar{0}}^{-1}t_2t_{\bar{0}}^{-1}t_3$.

That is, in terms of our short-hand notation,

$$0\bar{1}2\bar{1}0 \sim 1\bar{3}2\bar{3}1 \sim 3\bar{0}2\bar{0}3.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}2\bar{1}0]$:

$$0\bar{1}2\bar{1}0 \sim 1\bar{3}2\bar{3}1 \sim 3\bar{0}2\bar{0}1, \qquad 1\bar{0}2\bar{0}1 \sim 0\bar{3}2\bar{3}0 \sim 3\bar{1}2\bar{1}3,$$

$$2\bar{1}0\bar{1}2 \sim 1\bar{3}0\bar{3}1 \sim 3\bar{2}0\bar{2}3, \qquad 0\bar{1}3\bar{1}0 \sim 1\bar{2}3\bar{2}1 \sim 2\bar{0}3\bar{0}2,$$

$$0\bar{2}1\bar{2}0 \sim 2\bar{3}1\bar{3}2 \sim 3\bar{0}1\bar{0}3, \qquad 1\bar{2}0\bar{2}1 \sim 2\bar{3}0\bar{3}2 \sim 3\bar{1}0\bar{1}3,$$

$$2\bar{0}1\bar{0}2 \sim 0\bar{3}1\bar{3}0 \sim 3\bar{2}1\bar{2}3, \qquad 2\bar{1}3\bar{1}2 \sim 1\bar{0}3\bar{0}1 \sim 0\bar{2}3\bar{2}0$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}2\bar{1}0]$ must have at most eight distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{1}0)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,1,3\}$, $\{2\}$, $\{\bar{0},\bar{1},\bar{3}\}$, and $\{\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0t_i^{\pm 1}$, $i \in \{0,2\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0t_{\bar{0}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}eN = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0t_0N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^2N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0t_2N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0t_{\bar{2}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{3}}^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

77. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}N$.

Let $[0\bar{1}2\bar{1}\bar{0}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}N$.

Note the $N^{(0\bar{1}2\bar{1}\bar{0})} \geq N^{0\bar{1}2\bar{1}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2\bar{1}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}2\bar{1}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2\bar{1}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{1}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_0N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}eN = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-2}N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_1N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_1t_0^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_1N$, $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_2N = Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_{\bar{2}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_0t_2t_3N$, $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_3N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_0^{-1}N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_{\bar{3}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_0^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

78. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_3N$.

Let $[0\bar{1}2\bar{1}3]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_3N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_3 = Nt_1t_0^{-1}t_3t_0^{-1}t_2 = Nt_2t_{\bar{3}}^{-1}t_1t_{\bar{3}}^{-1}t_0 = Nt_3t_{\bar{2}}^{-1}t_0t_{\bar{2}}^{-1}t_1$.

That is, in terms of our short-hand notation,

$$0\bar{1}2\bar{1}3 \sim 1\bar{0}3\bar{0}2 \sim 2\bar{3}1\bar{3}0 \sim 3\bar{2}0\bar{2}1.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}2\bar{1}3]$:

$$0\bar{1}2\bar{1}3 \sim 1\bar{0}3\bar{0}2 \sim 2\bar{3}1\bar{3}0 \sim 3\bar{2}0\bar{2}1, \qquad 1\bar{0}2\bar{0}3 \sim 0\bar{1}3\bar{1}2 \sim 2\bar{3}0\bar{3}1 \sim 3\bar{2}1\bar{2}0,$$

$$2\bar{1}0\bar{1}3 \sim 1\bar{2}3\bar{2}0 \sim 0\bar{3}1\bar{3}2 \sim 3\bar{0}2\bar{0}1, \qquad 3\bar{1}2\bar{1}0 \sim 1\bar{3}0\bar{3}2 \sim 2\bar{0}1\bar{0}3 \sim 0\bar{2}3\bar{2}1,$$

$$1\bar{2}0\bar{2}3 \sim 2\bar{1}3\bar{1}0 \sim 0\bar{3}2\bar{3}1 \sim 3\bar{0}1\bar{0}2, \qquad 1\bar{3}2\bar{3}0 \sim 3\bar{1}0\bar{1}2 \sim 2\bar{0}3\bar{0}1 \sim 0\bar{2}1\bar{2}3$$

Since each of the twenty-four single cosets has four names, the double coset $[0\bar{1}2\bar{1}3]$ must have at most six distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{1}3)}$ has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_1^{-1}t_3t_i^{\pm 1}$, $i = 3$.

But note that $Nt_0t_1^{-1}t_2t_1^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2t_1^{-1}eN = Nt_0t_1^{-1}t_2t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_1^{-1}t_3t_3N = Nt_0t_1^{-1}t_2t_1^{-1}t_3^2N = Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

79. We next consider the double coset $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$.

Let $[0\bar{1}2\bar{1}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}2\bar{1}\bar{3})} \geq N^{0\bar{1}2\bar{1}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2\bar{1}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}2\bar{1}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2\bar{1}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{1}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_1^{-1}eN = Nt_0t_1^{-1}t_2t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2t_1^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_1N =$

$Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_3t_1N$, $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$, and $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

80. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_1N$.

Let $[0\bar{1}231]$ denote the double coset $Nt_0t_1^{-1}t_2t_3t_1N$.

Note that $N^{(0\bar{1}231)} \geq N^{0\bar{1}231} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}231)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}231)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}231]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}231)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_1t_1^{-1}N = Nt_0t_1^{-1}t_2t_3eN = Nt_0t_1^{-1}t_2t_3N$ and $Nt_0t_1^{-1}t_2t_3t_1t_1N = Nt_0t_1^{-1}t_2t_3t_1^2N = Nt_0t_1^{-1}t_2t_3t_1^{-1}N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$

$\Rightarrow t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2(0\ 1)(2\ 3)t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_3t_0t_1t_0 = t_2t_0^{-1}t_1^{-1}$

$\Rightarrow t_0^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0^{-1}(0\ 1)(2\ 3)t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(t_0^{-1})^{(0\ 1)(2\ 3)}t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0 = t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow t_3(0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0 = t_3t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_1^{-1}t_3t_0t_1t_0 = t_3t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_1^{-1}t_3t_0t_1t_0 = t_3t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow (0\ 1)(2\ 3)t_2t_1^{-1}t_3t_0t_1t_0 = t_3t_0^{-1}t_2t_0^{-1}t_1^{-1}$

$\Rightarrow [(0\ 1)(2\ 3)t_2t_1^{-1}t_3t_0t_1t_0]^{(0\ 1\ 3)} = [t_3t_0^{-1}t_2t_0^{-1}t_1^{-1}]^{(0\ 1\ 3)}$

$\Rightarrow (0\ 2)(1\ 3)t_2t_3^{-1}t_0t_1t_3t_1 = t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$

$\Rightarrow (0\ 2)(1\ 3)[t_0t_1^{-1}t_2t_3t_1t_3]^{(0\ 2)(1\ 3)} = t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$
$\Rightarrow (0\ 2)(1\ 3)(0\ 2)(1\ 3)t_0t_1^{-1}t_2t_3t_1t_3(0\ 2)(1\ 3) = t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$
$\Rightarrow et_0t_1^{-1}t_2t_3t_1t_3(0\ 2)(1\ 3) = t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$, which implies that $Nt_0t_1^{-1}t_2t_3t_1t_3N = Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$. That is, $[0\bar{1}2313] = [0\bar{1}2\bar{1}\bar{3}]$.

Similarly, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_1t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_3t_1t_0N$, $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_2N$, and $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$.

81. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}N$.

Let $[0\bar{1}23\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}N$.

Note that $N^{(0\bar{1}23\bar{2})} \geq N^{0\bar{1}23\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}23\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}23\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}23\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}23\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_3t_2^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_3eN = Nt_0t_1^{-1}t_2t_3N$ and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-2}N = Nt_0t_1^{-1}t_2t_3t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_3N = Nt_0t_1^{-1}t_0t_2t_3N$ and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$, and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$.

82. We next consider the double coset $Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}N$.

Let $[0\bar{1}2\bar{3}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1} = Nt_1t_3^{-1}t_2t_0^{-1}t_2^{-1} = Nt_3t_0^{-1}t_2t_1^{-1}t_2^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}2\bar{3}\bar{2} \sim 1\bar{3}2\bar{0}\bar{2} \sim 3\bar{0}2\bar{1}\bar{2}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}2\bar{3}\bar{2}]$:

$$0\bar{1}2\bar{3}\bar{2} \sim 1\bar{3}2\bar{0}\bar{2} \sim 3\bar{0}2\bar{1}\bar{2} \qquad 1\bar{0}2\bar{3}\bar{2} \sim 0\bar{3}2\bar{1}\bar{2} \sim 3\bar{1}2\bar{0}\bar{2} \qquad 2\bar{1}0\bar{3}\bar{0} \sim 1\bar{3}0\bar{2}\bar{0} \sim 3\bar{2}0\bar{1}\bar{0}$$

$$1\bar{2}0\bar{3}\bar{0} \sim 2\bar{3}0\bar{1}\bar{0} \sim 3\bar{1}0\bar{2}\bar{0} \qquad 2\bar{0}1\bar{3}\bar{1} \sim 0\bar{3}1\bar{2}\bar{1} \sim 3\bar{2}1\bar{0}\bar{1} \qquad 0\bar{2}1\bar{3}\bar{1} \sim 2\bar{3}1\bar{0}\bar{1} \sim 3\bar{0}1\bar{2}\bar{1}$$

$$0\bar{1}3\bar{2}\bar{3} \sim 1\bar{2}3\bar{0}\bar{3} \sim 2\bar{0}3\bar{1}\bar{3} \qquad 0\bar{2}3\bar{1}\bar{3} \sim 2\bar{1}3\bar{0}\bar{3} \sim 1\bar{0}3\bar{2}\bar{3}$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}2\bar{3}\bar{2}]$ must have at most eight distinct single cosets.

Moreover, $N^{(0\bar{1}2\bar{3}\bar{2})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 3\}$, $\{2\}$, $\{\bar{0}, \bar{1}, \bar{3}\}$, and $\{\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_3^{-1}eN = Nt_0t_1^{-1}t_2t_3^{-1}N$ and $Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2t_3^{-1}t_2N$.

Moreover, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_3(0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$

$\Rightarrow [(0\ 1)(2\ 3)t_2t_3^{-1}t_0^{-1}t_1t_0]^{(0\ 3)(1\ 2)} = [t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}]^{(0\ 3)(1\ 2)}$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0^{-1}t_3^{-1}t_2t_3 = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0t_1^{-1}t_2^{-1}t_3t_2]^{(0\ 1)(2\ 3)} = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3)t_3 = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2t_3^{(0\ 1)(2\ 3)}(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2t_2(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2^{-1}(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3$, which implies that
$Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3N$. That is, $[0\bar{1}2\bar{3}\bar{2}3] = [0\bar{1}23\bar{2}]$.

Similarly, by relation (7.2), $(0\ 1)(2\ 3)t_0t_1t_0 = t_0^{-1}t_1^{-1}$
$\Rightarrow t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_1(0\ 1)(2\ 3)t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_1^{(0\ 1)(2\ 3)}t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0t_0t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_2^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_2^{-1}(0\ 1)(2\ 3)t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(t_2^{-1})^{(0\ 1)(2\ 3)}t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0 = t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow t_3(0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_3(0\ 1)(2\ 3)t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_3^{(0\ 1)(2\ 3)}t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow (0\ 1)(2\ 3)t_2t_3^{-1}t_0^{-1}t_1t_0 = t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}$
$\Rightarrow [(0\ 1)(2\ 3)t_2t_3^{-1}t_0^{-1}t_1t_0]^{(0\ 3)(1\ 2)} = [t_3t_2^{-1}t_1t_0^{-1}t_1^{-1}]^{(0\ 3)(1\ 2)}$
$\Rightarrow (0\ 1)(2\ 3)t_1t_0^{-1}t_3^{-1}t_2t_3 = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow (0\ 1)(2\ 3)[t_0t_1^{-1}t_2^{-1}t_3t_2]^{(0\ 1)(2\ 3)} = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow (0\ 1)(2\ 3)(0\ 1)(2\ 3)t_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3)t_3^{-1} = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3^{-1}$
$\Rightarrow et_0t_1^{-1}t_2^{-1}t_3t_2(0\ 1)(2\ 3)t_3^{-1}(0\ 1)(2\ 3)(0\ 1)(2\ 3) = t_0t_1^{-1}t_2t_3^{-1}t_2^{-1}t_3^{-1}$

$\Rightarrow et_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_3t_2(t_{\bar{3}}^{-1})^{(0\ 1)(2\ 3)}(0\ 1)(2\ 3) = t_0t_{\bar{1}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}$
$\Rightarrow et_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_3t_2t_{\bar{2}}^{-1}(0\ 1)(2\ 3) = t_0t_{\bar{1}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}$
$\Rightarrow et_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_3(0\ 1)(2\ 3) = t_0t_{\bar{1}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}$, which implies that
$Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_3N = Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}N$. That is, $[0\bar{1}2\bar{3}\bar{2}\bar{3}] = [0\bar{1}\bar{2}3]$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

83. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1N$.

Let $[0\bar{1}\bar{2}01]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1N$.

Note that $N^{(0\bar{1}\bar{2}01)} \geq N^{0\bar{1}\bar{2}01} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}01)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}01)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}01]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}01)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0eN = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_1N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1^2N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_{\bar{1}}^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_0N = Nt_0t_1t_2t_{\bar{3}}^{-1}N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_{\bar{0}}^{-1}N = Nt_0t_1t_2t_{\bar{3}}^{-1}t_1N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_2N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_{\bar{2}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_3N$, and $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_{\bar{3}}^{-1}N$.

84. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_{\bar{1}}^{-1}N$.

Let $[0\bar{1}\bar{2}0\bar{1}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_{\bar{1}}^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}0\bar{1})} \geq N^{0\bar{1}\bar{2}0\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}0\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_{\bar{1}}^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}0\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}0\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}0\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0eN = Nt_0t_1^{-1}t_2^{-1}t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_2N = Nt_0t_1t_2t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_2^{-1}N =$ $Nt_0t_1^{-1}t_2t_0t_1t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_3^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_3t_0N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

85. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_3N$.

Let $[0\bar{1}\bar{2}03]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_3N$.

Note that $N^{(0\bar{1}\bar{2}03)} \geq N^{0\bar{1}\bar{2}03} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}03)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}03)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}03]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}03)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0eN = Nt_0t_1^{-1}t_2^{-1}t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_1N = Nt_0^{-1}t_1^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_1^{-1}N =$ $Nt_0^{-1}t_1^{-1}t_2t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_2^{-1}N =$ $Nt_0t_1t_2^{-1}t_0t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

86. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Let $[0\bar{1}\bar{2}0\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}0\bar{3})} \geq N^{0\bar{1}\bar{2}0\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}0\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N| = \frac{|N|}{|N^{(0\bar{1}\bar{2}0\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}0\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}0\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0eN = Nt_0t_1^{-1}t_2^{-1}t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2t_0t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_2N = Nt_0t_1t_2t_3^{-1}t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

87. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$.

Let $[0\bar{1}\bar{2}\bar{0}1]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}1)} \geq N^{0\bar{1}\bar{2}\bar{0}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}1)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

88. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{1})} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2t_0t_1N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$.

89. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$.

Let $[0\bar{1}\bar{2}\bar{0}2]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}2)} \geq N^{0\bar{1}\bar{2}\bar{0}2} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_0N = Nt_0t_1^{-1}t_0^{-1}t_2N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_0^{-1}N = Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$.

Therefore, we conclude that there are four distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$.

90. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$.

Let $[0\bar{1}\bar{2}\bar{0}3]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}3)} \geq N^{0\bar{1}\bar{2}\bar{0}3} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0N = Nt_0^{-1}t_1t_2t_0t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2^{-1}N = Nt_0t_1t_2t_3^{-1}t_1N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$.

91. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{3})} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}0\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_2N = Nt_0t_1t_2t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_1N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

92. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_0N$.

Let $[0\bar{1}\bar{2}10]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_0N$.

Note that $N^{(0\bar{1}\bar{2}10)} \geq N^{0\bar{1}\bar{2}10} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}10)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}10)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}10]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}10)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_1t_0t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_3N = Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_3^{-1}N = Nt_0^{-1}t_1t_2t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_0t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

93. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$.

Let $[0\bar{1}\bar{2}1\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}1\bar{0})} \geq N^{0\bar{1}\bar{2}1\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}1\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}1\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}1\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}1\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_2N = Nt_0t_1t_2t_3^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$.

94. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$.

Let $[0\bar{1}\bar{2}1\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}1\bar{2})} \geq N^{0\bar{1}\bar{2}1\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}1\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}1\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}1\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}1\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_1t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_0N =$ $Nt_0t_1t_2t_3^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_1N = Nt_0t_1^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_1^{-1}N =$ $Nt_0t_1t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_3^{-1}N = Nt_0t_1t_0^{-1}t_2t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

95. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3N$.

Let $[0\bar{1}\bar{2}13]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3N$.

Note that $N^{(0\bar{1}\bar{2}13)} \geq N^{0\bar{1}\bar{2}13} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}13)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}13)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}13]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}13)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_1t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_0N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_0^{-1}N = Nt_0^{-1}t_1t_2t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_1^{-1}N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$.

96. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$.

Let $[0\bar{1}\bar{2}1\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}1\bar{3})} \geq N^{0\bar{1}\bar{2}1\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}1\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}1\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}1\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}1\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_0N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_1^{-1}N = Nt_0t_1t_2t_0t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2N = Nt_0t_1t_2t_0^{-1}t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$.

97. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0N$.

Let $[0\bar{1}\bar{2}30]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0N$.

Note that $N^{(0\bar{1}\bar{2}30)} \geq N^{0\bar{1}\bar{2}30} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}30)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}30)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}30]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}30)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$.

98. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$.

Let $[0\bar{1}\bar{2}3\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}3\bar{0})} \geq N^{0\bar{1}\bar{2}3\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}3\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}3\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}3\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}3\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1N = Nt_0t_1t_2t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2t_0t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$.

99. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1N$.

Let $[0\bar{1}\bar{2}31]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1N$.

Note that $N^{(0\bar{1}\bar{2}31)} \geq N^{0\bar{1}\bar{2}31} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}31)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}31)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}31]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}31)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_3N = Nt_0t_1t_2t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_1N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$.

100. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$.

Let $[0\bar{1}\bar{2}3\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}3\bar{1})} \geq N^{0\bar{1}\bar{2}3\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}3\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}3\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}3\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}3\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_0N = Nt_0t_1t_2t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

101. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Let $[0\bar{1}\bar{2}3\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}3\bar{2})} \geq N^{0\bar{1}\bar{2}3\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}3\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}3\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}3\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}3\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_3N = Nt_0t_1^{-1}t_0t_2N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0t_2t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

102. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Let $[0\bar{1}\bar{2}\bar{3}0]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}0)} \geq N^{0\bar{1}\bar{2}\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_1N = Nt_0^{-1}t_1t_2t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_1^{-1}N = Nt_0^{-1}t_1t_2t_3t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2N = Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_3N = Nt_0^{-1}t_1t_2t_0t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$.

103. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}\bar{0})} \geq N^{0\bar{1}\bar{2}\bar{3}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{3}\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$.

104. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Let $[0\bar{1}\bar{2}\bar{3}1]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}1)} \geq N^{0\bar{1}\bar{2}\bar{3}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{3}1)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0N = Nt_0t_1t_0^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_3^{-1}N = Nt_0t_1t_0^{-1}t_2t_1N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$.

105. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}\bar{1})} \geq N^{0\bar{1}\bar{2}\bar{3}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

106. We next consider the double coset $Nt_0t_1^{-1}t_0t_1^{-1}t_0N$.

Let $[0\bar{1}0\bar{1}0]$ denote the double coset $Nt_0t_1^{-1}t_0t_1^{-1}t_0N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_0t_1^{-1}t_0 = Nt_0t_2^{-1}t_0t_2^{-1}t_0 = Nt_0t_3^{-1}t_0t_3^{-1}t_0 = Nt_1t_0^{-1}t_1t_0^{-1}t_1 = Nt_1t_2^{-1}t_1t_2^{-1}t_1 = Nt_1t_3^{-1}t_1t_3^{-1}t_1 = Nt_2t_1^{-1}t_2t_1^{-1}t_2 = Nt_2t_0^{-1}t_2t_0^{-1}t_2 = Nt_2t_3^{-1}t_2t_3^{-1}t_2 = Nt_3t_1^{-1}t_3t_1^{-1}t_3 = Nt_3t_2^{-1}t_3t_2^{-1}t_3 = Nt_3t_0^{-1}t_3t_0^{-1}t_3$.

That is, in terms of our short-hand notation,

$$0\bar{1}0\bar{1}0 \sim 0\bar{2}0\bar{2}0 \sim 0\bar{3}0\bar{3}0 \sim 1\bar{0}1\bar{0}1 \sim 1\bar{2}1\bar{2}1 \sim 1\bar{3}1\bar{3}1 \sim$$

$$2\bar{1}2\bar{1}2 \sim 2\bar{0}2\bar{0}2 \sim 2\bar{3}2\bar{3}2 \sim 3\bar{1}3\bar{1}3 \sim 3\bar{2}3\bar{2}3 \sim 3\bar{0}3\bar{0}3.$$

Since each of the twelve single cosets has twelve names, the double coset $[0\bar{1}0\bar{1}0]$ must have at most one distinct single coset.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1}0\bar{1}0)} \geq N^{0\bar{1}0\bar{1}0} = \langle(2\ 3)\rangle \cong S_2$. In fact, with the help of MAGMA, we know that $N(t_0t_1^{-1}t_0t_1^{-1}t_0)^{(0\ 1)} = Nt_1t_0^{-1}t_1t_0^{-1}t_1 = Nt_0t_1^{-1}t_0t_1^{-1}t_0$ implies that $(0\ 1) \in N^{(0\bar{1}0\bar{1}0)}$, and $N(t_0t_1^{-1}t_0t_1^{-1}t_0)^{(0\ 2)} = Nt_2t_1^{-1}t_2t_1^{-1}t_2 = Nt_0t_1^{-1}t_0t_1^{-1}t_0$ implies that $(0\ 2) \in N^{(0\bar{1}0\bar{1}0)}$, and $N(t_0t_1^{-1}t_0t_1^{-1}t_0)^{(0\ 3)} = Nt_3t_1^{-1}t_3t_1^{-1}t_3 = Nt_0t_1^{-1}t_0t_1^{-1}t_0$ implies that $(0\ 3) \in N^{(0\bar{1}0\bar{1}0)}$. Therefore, $(0\ 1), (0\ 2), (0\ 3) \in N^{(0\bar{1}0\bar{1}0)}$, and so $N^{(0\bar{1}0\bar{1}0)} \geq \langle(0\ 1), (0\ 2), (0\ 3)\rangle \cong S_4$. Thus $\left|N^{(0\bar{1}0\bar{1}0)}\right| \geq |S_4| = 24$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0t_1^{-1}t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}0\bar{1}0)}\right|} \leq \frac{24}{24} = 1$.

Therefore, as we concluded earlier, the double coset $[0\bar{1}0\bar{1}0]$ has at most one distinct single coset.

Moreover, $N^{(0\bar{1}0\bar{1}0)}$ has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_0t_1^{-1}t_0t_i^{\pm 1}$, $i = 0$.

But note that $Nt_0t_1^{-1}t_0t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_0t1^{-1}eN = Nt_0t_1^{-1}t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_0t_1^{-1}t_0t_0N = Nt_0t_1^{-1}t_0t_1^{-1}t_0^2N = Nt_0t_1^{-1}t_0t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

107. We next consider the double coset $Nt_0t_1^{-1}t_0t_2t_3N$.

Let $[0\bar{1}023]$ denote the double coset $Nt_0t_1^{-1}t_0t_2t_3N$.

Note that $N^{(0\bar{1}023)} \geq N^{0\bar{1}023} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}023)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0t_2t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}023)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}023]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}023)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_0t_2t_3t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0t_2t_3t_3^{-1}N = Nt_0t_1^{-1}t_0t_2eN = Nt_0t_1^{-1}t_0t_2N$ and $Nt_0t_1^{-1}t_0t_2t_3t_3N = Nt_0t_1^{-1}t_0t_2t_3^2N = Nt_0t_1^{-1}t_0t_2t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0t_2t_3t_0^{-1}N =$ $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_0t_2t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_0t_2t_3t_1^{-1}N$ $= Nt_0t_1t_2t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_0t_2t_3t_2N = Nt_0t_1^{-1}t_2t_1^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_0t_2t_3t_2^{-1}N =$ $Nt_0t_1^{-1}t_2t_3t_2^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_0t_2t_3t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_0t_2t_3t_0N$.

108. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3N$.

Let $[0\bar{1}\bar{0}23]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3N$.

Note that $N^{(0\bar{1}\bar{0}23)} \geq N^{0\bar{1}\bar{0}23} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{0}23)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}23)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}23]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}23)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_0^{-1}t_2t_3t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2eN = Nt_0t_1^{-1}t_0^{-1}t_2N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_3N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0^{-1}N =$ $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}N$.

109. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$.

Let $[0\bar{1}\bar{0}2\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}2\bar{3})} \geq N^{0\bar{1}\bar{0}2\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}2\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}2\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}2\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}2\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_3N = Nt_0t_1^{-1}t_0^{-1}t_2eN = Nt_0t_1^{-1}t_0^{-1}t_2N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_0N = Nt_0t_1t_2t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1N = Nt_0t_1t_2t_0^{-1}t_3N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$.

110. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N$.

Let $[0\bar{1}\bar{0}\bar{2}1]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}1)} \geq N^{0\bar{1}\bar{0}\bar{2}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}\bar{2}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_1N = Nt_0t_1^{-1}t_0^{-1}t_2t_1^2N = Nt_0t_1^{-1}t_0^{-1}t_2t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_0N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_2N = Nt_0^{-1}t_1t_2t_0N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_1t_3N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_1t_3^{-1}N$.

111. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}N$.

Let $[0\bar{1}\bar{0}\bar{2}\bar{1}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}\bar{1})} \geq N^{0\bar{1}\bar{0}\bar{2}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{0}\bar{2}\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_1N = Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}eN = Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N = Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_2t_1t_0N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_2N = Nt_{\bar{0}}^{-1}t_1t_{\bar{0}}^{-1}t_2t_{\bar{0}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}N = Nt_{\bar{0}}^{-1}t_1t_{\bar{0}}^{-1}t_2N$, and $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_3N = Nt_{\bar{0}}^{-1}t_1t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_{\bar{1}}^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_0N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_3^{-1}N$.

112. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_3N$.

Let $[0\bar{1}\bar{0}\bar{2}3]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_3N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}3)} \geq N^{0\bar{1}\bar{0}\bar{2}3} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_3N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{0}\bar{2}3)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_0N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

113. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$.

Let $[0\bar{1}\bar{0}\bar{2}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}\bar{3})} \geq N^{0\bar{1}\bar{0}\bar{2}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}eN = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_0N = Nt_0t_1^{-1}t_2t_3t_1t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$.

114. We next consider the double coset $Nt_0t_1t_2t_0t_1N$.

Let $[01201]$ denote the double coset $Nt_0t_1t_2t_0t_1N$.

Note that $N^{(01201)} \geq N^{01201} = \langle e \rangle$. Thus $\left|N^{(01201)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_0t_1N| = \frac{|N|}{|N^{(01201)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [01201] has at most twenty-four distinct single cosets.

Moreover, $N^{(01201)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0t_1t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_1t_1^{-1}N = Nt_0t_1t_2t_0eN = Nt_0t_1t_2t_0N$ and $Nt_0t_1t_2t_0t_1t_1N = Nt_0t_1t_2t_0t_1^2N = Nt_0t_1t_2t_0t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_1t_0N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_0t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0t_1t_2t_0t_1t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_0t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, and $Nt_0t_1t_2t_0t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2t_0t_1t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_0t_1t_3N$.

115. We next consider the double coset $Nt_0t_1t_2t_0t_1^{-1}N$.

Let $[0120\bar{1}]$ denote the double coset $Nt_0t_1t_2t_0t_1^{-1}N$.

Note that $N^{(0120\bar{1})} \geq N^{0120\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0120\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0t_1^{-1}N\right| = \frac{|N|}{|N^{(0120\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0120\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0120\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0t_1^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_1^{-1}t_1N = Nt_0t_1t_2t_0eN = Nt_0t_1t_2t_0N$ and $Nt_0t_1t_2t_0t_1^{-1}t_1^{-1}N = Nt_0t_1t_2t_0t_1^{-2}N = Nt_0t_1t_2t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_1^{-1}t_0N = Nt_0t_1^{-1}t_0t_2t_3N$, $Nt_0t_1t_2t_0t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}N =$

$Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_0t_1^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, and $Nt_0t_1t_2t_0t_1^{-1}t_3^{-1}N$ $= Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2t_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_0t_1^{-1}t_2N$.

116. We next consider the double coset $Nt_0t_1t_2t_0t_2^{-1}N$.

Let [0120$\bar{2}$] denote the double coset $Nt_0t_1t_2t_0t_2^{-1}N$.

Note that $N^{(0120\bar{2})} \geq N^{0120\bar{2}} = \langle e\rangle$. Thus $\left|N^{(0120\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0t_2^{-1}N\right| = \frac{|N|}{|N^{(0120\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [0120$\bar{2}$] has at most twenty-four distinct single cosets.

Moreover, $N^{(0120\bar{2})}$ has eight orbits on $T = \{t_0,t_1,t_2,t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_0t_2^{-1}t_2N = Nt_0t_1t_2t_0eN = Nt_0t_1t_2t_0N$ and $Nt_0t_1t_2t_0t_2^{-1}t_2^{-1}N$ $= Nt_0t_1t_2t_0t_2^{-2}N = Nt_0t_1t_2t_0t_2N = Nt_0t_1t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_2^{-1}t_0N =$ $Nt_0t_1t_2^{-1}t_1t_3N$, $Nt_0t_1t_2t_0t_2^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$, $Nt_0t_1t_2t_0t_2^{-1}t_1N =$ $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$, $Nt_0t_1t_2t_0t_2^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_0t_2^{-1}t_3N$ $= Nt_0t_1^{-1}t_2t_1t_3t_2N$, and $Nt_0t_1t_2t_0t_2^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

117. We next consider the double coset $Nt_0t_1t_2t_0t_3N$.

Let [01203] denote the double coset $Nt_0t_1t_2t_0t_3N$.

Note that $N^{(01203)} \geq N^{01203} = \langle e\rangle$. Thus $\left|N^{(01203)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_0t_3N| = \frac{|N|}{|N^{(01203)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [01203] has at most twenty-four distinct single cosets.

Moreover, $N^{(01203)}$ has eight orbits on $T = \{t_0,t_1,t_2,t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_3t_3^{-1}N = Nt_0t_1t_2t_0eN = Nt_0t_1t_2t_0N$ and $Nt_0t_1t_2t_0t_3t_3N$ $= Nt_0t_1t_2t_0t_3^2N = Nt_0t_1t_2t_0t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_3t_0N =$ $Nt_0t_1t_2t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_0t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_0t_3t_1^{-1}N =$ $Nt_0t_1^{-1}t_2t_1t_3t_0N$, and $Nt_0t_1t_2t_0t_3t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2t_0t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_0t_3t_0^{-1}N$ and $Nt_0t_1t_2t_0t_3t_2N$.

118. We next consider the double coset $Nt_0t_1t_2t_0t_3^{-1}N$.

Let [0120$\bar{3}$] denote the double coset $Nt_0t_1t_2t_0t_3^{-1}N$.

Note that $N^{(0120\bar{3})} \geq N^{0120\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0120\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0t_3^{-1}N\right| = \frac{|N|}{|N^{(0120\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [0120$\bar{3}$] has at most twenty-four distinct single cosets.

Moreover, $N^{(0120\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_3^{-1}t_3N = Nt_0t_1t_2t_0eN = Nt_0t_1t_2t_0N$ and $Nt_0t_1t_2t_0t_3^{-1}t_3^{-1}N$ $= Nt_0t_1t_2t_0t_3^{-2}N = Nt_0t_1t_2t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_3^{-1}t_1N =$ $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$, $Nt_0t_1t_2t_0t_3^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_2N =$ $Nt_0t_1^{-1}t_2t_0t_3t_2N$, and $Nt_0t_1t_2t_0t_3^{-1}t_2^{-1}N = Nt_0t_1t_2^{-1}t_0t_3N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2t_0t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_0t_3^{-1}t_0N$ and $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$.

119. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_1N$.

Let [012$\bar{0}$1] denote the double coset $Nt_0t_1t_2t_0^{-1}t_1N$.

Note that $N^{(012\bar{0}1)} \geq N^{012\bar{0}1} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_1N\right| = \frac{|N|}{|N^{(012\bar{0}1)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_1t_1^{-1}N = Nt_0t_1t_2t_0^{-1}eN = Nt_0t_1t_2t_0^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_1t_1N = Nt_0t_1t_2t_0^{-1}t_1^2N = Nt_0t_1t_2t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_1t_0N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_1t_2N = Nt_0t_1^{-1}t_2t_0t_1t_2N$, $Nt_0t_1t_2t_0^{-1}t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, and $Nt_0t_1t_2t_0^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

120. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_2N$.

Let $[012\bar{0}2]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_2N$.

Note that $N^{(012\bar{0}2)} \geq N^{012\bar{0}2} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_2N\right| = \frac{|N|}{\left|N^{(012\bar{0}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0^{-1}t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_2t_2^{-1}N = Nt_0t_1t_2t_0^{-1}eN = Nt_0t_1t_2t_0^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_2t_2N = Nt_0t_1t_2t_0^{-1}t_2^2N = Nt_0t_1t_2t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_2t_0N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2t_0^{-1}N = Nt_0t_1t_0^{-1}t_1N$, $Nt_0t_1t_2t_0^{-1}t_2t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2t_3N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$, and $Nt_0t_1t_2t_0^{-1}t_2t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2t_0^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_0^{-1}t_2t_1N$.

121. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_2^{-1}N$.

Let $[012\bar{0}\bar{2}]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_2^{-1}N$.

Note that $N^{(012\bar{0}\bar{2})} \geq N^{012\bar{0}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(012\bar{0}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_2N = Nt_0t_1t_2t_0^{-1}eN = Nt_0t_1t_2t_0^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-2}N = Nt_0t_1t_2t_0^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_0N = Nt_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_0N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$, and $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2t_3t_1^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$ and $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$.

122. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_3N$.

Let $[012\bar{0}3]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_3N$.

Note that $N^{(012\bar{0}3)} \geq N^{012\bar{0}3} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_3N\right| = \frac{|N|}{\left|N^{(012\bar{0}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_3t_3^{-1}N = Nt_0t_1t_2t_0^{-1}eN = Nt_0t_1t_2t_0^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_3t_3N = Nt_0t_1t_2t_0^{-1}t_3^2N = Nt_0t_1t_2t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_3t_0N = Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3t_1N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3t_2N = Nt_0^{-1}t_1t_2t_3t_1N$, and $Nt_0t_1t_2t_0^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_3t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

123. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}N$.

Let $[012\bar{0}\bar{3}]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}N$.

Note that $N^{(012\bar{0}\bar{3})} \geq N^{012\bar{0}\bar{3}} = \langle e\rangle$. Thus $\left|N^{(012\bar{0}\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(012\bar{0}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_0^{-1}t_3^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_3N = Nt_0t_1t_2t_0^{-1}eN = Nt_0t_1t_2t_0^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-2}N = Nt_0t_1t_2t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_3t_0N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$, and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$.

124. We next consider the double coset $Nt_0t_1t_2t_3t_0N$.

Let $[01230]$ denote the double coset $Nt_0t_1t_2t_3t_0N$.

Note that $N^{(01230)} \geq N^{01230} = \langle e\rangle$. Thus $\left|N^{(01230)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_3t_0N| = \frac{|N|}{|N^{(01230)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01230]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01230)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_3t_0t_0^{-1}N = Nt_0t_1t_2t_3eN = Nt_0t_1t_2t_3N$ and $Nt_0t_1t_2t_3t_0t_0N = Nt_0t_1t_2t_3t_0^2N = Nt_0t_1t_2t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_0t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$, $Nt_0t_1t_2t_3t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_2N$, $Nt_0t_1t_2t_3t_0t_3N = Nt_0t_1t_2t_0^{-1}t_3^{-1}N$, and $Nt_0t_1t_2t_3t_0t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2t_3t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_3t_0t_2N$ and $Nt_0t_1t_2t_3t_0t_2^{-1}N$.

125. We next consider the double coset $Nt_0t_1t_2t_3t_0^{-1}N$.

Let $[0123\bar{0}]$ denote the double coset $Nt_0t_1t_2t_3t_0^{-1}N$.

Note that $N^{(0123\bar{0})} \geq N^{0123\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0123\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0123\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0123\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0123\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_3t_0^{-1}t_0N = Nt_0t_1t_2t_3eN = Nt_0t_1t_2t_3N$ and $Nt_0t_1t_2t_3t_0^{-1}t_0^{-1}N = Nt_0t_1t_2t_3t_0^{-2}N = Nt_0t_1t_2t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_0^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_3t_1N$, $Nt_0t_1t_2t_3t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$, and $Nt_0t_1t_2t_3t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1t_2t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_3t_0^{-1}t_3N$, and $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$.

126. We next consider the double coset $Nt_0t_1t_2t_3t_1N$.

Let [01231] denote the double coset $Nt_0t_1t_2t_3t_1N$.

Note that $N^{(01231)} \geq N^{01231} = \langle e \rangle$. Thus $\left|N^{(01231)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_3t_1N| = \frac{|N|}{\left|N^{(01231)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [01231] has at most twenty-four distinct single cosets.

Moreover, $N^{(01231)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3t_1t_1^{-1}N = Nt_0t_1t_2t_3eN = Nt_0t_1t_2t_3N$ and $Nt_0t_1t_2t_3t_1t_1N = Nt_0t_1t_2t_3t_1^2N = Nt_0t_1t_2t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, $Nt_0t_1t_2t_3t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_3t_1t_2^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$, $Nt_0t_1t_2t_3t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$, and $Nt_0t_1t_2t_3t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2t_3t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_3t_1t_0N$.

127. We next consider the double coset $Nt_0t_1t_2t_3t_2^{-1}N$.

Let $[0123\bar{2}]$ denote the double coset $Nt_0t_1t_2t_3t_2^{-1}N$.

Note that $N^{(0123\bar{2})} \geq N^{0123\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0123\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0123\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0123\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0123\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3t_2^{-1}t_2N = Nt_0t_1t_2t_3eN = Nt_0t_1t_2t_3N$ and $Nt_0t_1t_2t_3t_2^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_2^{-2}N = Nt_0t_1t_2t_3t_2N = Nt_0t_1t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_2^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$, $Nt_0t_1t_2t_3t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_3t_2^{-1}t_1N =$

$Nt_0t_1^{-1}t_2t_1t_3t_0N$, $Nt_0t_1t_2t_3t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_3t_2^{-1}t_3N = Nt_0t_1t_2^{-1}t_1t_0N$, and $Nt_0t_1t_2t_3t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

128. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_0N$.

Let $[012\bar{3}0]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_0N$.

Note that $N^{(012\bar{3}0)} \geq N^{012\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_0N\right| = \frac{|N|}{\left|N^{(012\bar{3}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_0t_0^{-1}N = Nt_0t_1t_2t_3^{-1}eN = Nt_0t_1t_2t_3^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_0t_0N = Nt_0t_1t_2t_3^{-1}t_0^2N = Nt_0t_1t_2t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_0t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_0t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$, $Nt_0t_1t_2t_3^{-1}t_0t_2^{-1}N = Nt_0^{-1}t_1t_2t_0^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_0t_3N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$, and $Nt_0t_1t_2t_3^{-1}t_0t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

129. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_0^{-1}N$.

Let $[012\bar{3}\bar{0}]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_0^{-1}N$.

Note that $N^{(012\bar{3}\bar{0})} \geq N^{012\bar{3}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(012\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_0N = Nt_0t_1t_2t_3^{-1}eN = Nt_0t_1t_2t_3^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-2}N = Nt_0t_1t_2t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0t_3N$, $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_2N = Nt_0t_1t_2^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_3N = Nt_0t_1t_2t_0t_3t_0^{-1}N$, and $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_0t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$.

130. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_1N$.

Let $[012\bar{3}1]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_1N$.

Note that $N^{(012\bar{3}1)} \geq N^{012\bar{3}1} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_1N\right| = \frac{|N|}{\left|N^{(012\bar{3}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_1t_1^{-1}N = Nt_0t_1t_2t_3^{-1}eN = Nt_0t_1t_2t_3^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_1t_1N = Nt_0t_1t_2t_3^{-1}t_1^{2}N = Nt_0t_1t_2t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_1t_0N = Nt_0t_1^{-1}t_2t_0t_3t_2N$, $Nt_0t_1t_2t_3^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_1t_2N = Nt_0t_1t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1t_2t_3^{-1}t_1t_2^{-1}N = Nt_0^{-1}t_1t_2t_0^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, and $Nt_0t_1t_2t_3^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

131. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_2N$.

Let $[012\bar{3}2]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_2N$.

Note that $N^{(012\bar{3}2)} \geq N^{012\bar{3}2} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_2N\right| = \frac{|N|}{\left|N^{(012\bar{3}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3^{-1}t_2t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_2t_2^{-1}N = Nt_0t_1t_2t_3^{-1}eN = Nt_0t_1t_2t_3^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_2t_2N = Nt_0t_1t_2t_3^{-1}t_2^2N = Nt_0t_1t_2t_3^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_2t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$, $Nt_0t_1t_2t_3^{-1}t_2t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$, and $Nt_0t_1t_2t_3^{-1}t_2t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2t_3^{-1}t_2t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_3^{-1}t_2t_1N$.

132. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_2^{-1}N$.

Let $[012\bar{3}\bar{2}]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_2^{-1}N$.

Note that $N^{(012\bar{3}\bar{2})} \geq N^{012\bar{3}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(012\bar{3}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2t_3^{-1}t_2^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_2N = Nt_0t_1t_2t_3^{-1}eN = Nt_0t_1t_2t_3^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1t_2t_3^{-1}t_2^{-2}N = Nt_0t_1t_2t_3^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_1^{-1}N =$

$Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_3N = Nt_0t_1t_2^{-1}t_1t_0^{-1}N$, and $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_3^{-1}N$ $= Nt_0t_1t_2^{-1}t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

133. We next consider the double coset $Nt_0t_1t_2^{-1}t_0t_1N$.

Let $[01\bar{2}01]$ denote the double coset $Nt_0t_1t_2^{-1}t_0t_1N$.

Note that $N^{(01\bar{2}01)} \geq N^{01\bar{2}01} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}01)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0t_1N\right| = \frac{|N|}{\left|N^{(01\bar{2}01)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}01]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}01)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_0t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_0t_1t_1^{-1}N = Nt_0t_1t_2^{-1}t_0eN = Nt_0t_1t_2^{-1}t_0N$ and $Nt_0t_1t_2^{-1}t_0t_1t_1N = Nt_0t_1t_2^{-1}t_0t_1^2N = Nt_0t_1t_2^{-1}t_0t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0t_1t_0N =$ $Nt_0t_1^{-1}t_2t_1t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_0t_1t_0^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_0t_1t_2N =$ $Nt_0t_1t_2t_0t_1^{-1}t_2N$, $Nt_0t_1t_2^{-1}t_0t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_0t_1t_3N =$ $Nt_0t_1^{-1}t_2^{-1}t_0t_3N$, and $Nt_0t_1t_2^{-1}t_0t_1t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

134. We next consider the double coset $Nt_0t_1t_2^{-1}t_0t_3N$.

Let $[01\bar{2}03]$ denote the double coset $Nt_0t_1t_2^{-1}t_0t_3N$.

Note that $N^{(01\bar{2}03)} \geq N^{01\bar{2}03} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}03)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0t_3N\right| = \frac{|N|}{\left|N^{(01\bar{2}03)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}03]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}03)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_0t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_0t_3t_3^{-1}N = Nt_0t_1t_2^{-1}t_0eN = Nt_0t_1t_2^{-1}t_0N$ and $Nt_0t_1t_2^{-1}t_0t_3t_3N = Nt_0t_1t_2^{-1}t_0t_3^2N = Nt_0t_1t_2^{-1}t_0t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0t_3t_0N = Nt_0t_1t_2t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_0t_3t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_0t_3t_1N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_0t_3t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3N$, $Nt_0t_1t_2^{-1}t_0t_3t_2N = Nt_0t_1t_2t_0t_3^{-1}N$, and $Nt_0t_1t_2^{-1}t_0t_3t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

135. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_1N$.

Let $[01\bar{2}\bar{0}1]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_1N$.

Note that $N^{(01\bar{2}\bar{0}1)} \geq N^{01\bar{2}\bar{0}1} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{0}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}t_1N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{0}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}eN = Nt_0t_1t_2^{-1}t_0^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_1N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^2N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_0N = Nt_0t_1t_2^{-1}t_3t_1N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_2N = Nt_0t_1t_2^{-1}t_3t_0N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_3N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$, and $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

136. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Let $[01\bar{2}\bar{0}\bar{1}]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Note that $N^{(01\bar{2}\bar{0}\bar{1})} \geq N^{01\bar{2}\bar{0}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{0}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N\right| = \frac{|N|}{|N^{(01\bar{2}\bar{0}\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_1N = Nt_0t_1t_2^{-1}t_0^{-1}eN = Nt_0t_1t_2^{-1}t_0^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-2}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_1t_0N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_2N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_2^{-1}N = Nt_0t_1t_2t_3^{-1}t_2^{-1}N$, and $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1t_2t_3t_0t_2^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$.

137. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3N$.

Let $[01\bar{2}\bar{0}3]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3N$.

Note that $N^{(01\bar{2}\bar{0}3)} \geq N^{01\bar{2}\bar{0}3} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{0}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}t_3N\right| = \frac{|N|}{|N^{(01\bar{2}\bar{0}3)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_3^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}eN = Nt_0t_1t_2^{-1}t_0^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_3N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^2N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1^{-1}N$

$= Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_2N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$, and $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$.

138. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Let $[01\bar{2}\bar{0}\bar{3}]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Note that $N^{(01\bar{2}\bar{0}\bar{3})} \geq N^{01\bar{2}\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{0}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_3N = Nt_0t_1t_2^{-1}t_0^{-1}eN = Nt_0t_1t_2^{-1}t_0^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-2}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_3t_0N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3N$, and $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_2^{-1}t_0t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

139. We next consider the double coset $Nt_0t_1t_2^{-1}t_1t_0N$.

Let $[01\bar{2}10]$ denote the double coset $Nt_0t_1t_2^{-1}t_1t_0N$.

Note that $N^{(01\bar{2}10)} \geq N^{01\bar{2}10} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}10)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_1t_0N\right| = \frac{|N|}{\left|N^{(01\bar{2}10)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}10]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}10)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_1t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_1t_0t_0^{-1}N = Nt_0t_1t_2^{-1}t_1eN = Nt_0t_1t_2^{-1}t_1N$ and $Nt_0t_1t_2^{-1}t_1t_0t_0N = Nt_0t_1t_2^{-1}t_1t_0^2N = Nt_0t_1t_2^{-1}t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_1t_0t_1N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_0t_2^{-1}N = Nt_0t_1t_2t_3t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_0t_3N = Nt_0t_1t_2t_3^{-1}t_2t_1N$, and $Nt_0t_1t_2^{-1}t_1t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_1t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

140. We next consider the double coset $Nt_0t_1t_2^{-1}t_1t_0^{-1}N$.

Let $[01\bar{2}1\bar{0}]$ denote the double coset $Nt_0t_1t_2^{-1}t_1t_0^{-1}N$.

Note that $N^{(01\bar{2}1\bar{0})} \geq N^{01\bar{2}1\bar{0}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}1\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{2}1\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}1\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}1\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_0N = Nt_0t_1t_2^{-1}t_1eN = Nt_0t_1t_2^{-1}t_1N$ and $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_1t_0^{-2}N = Nt_0t_1t_2^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_1N = Nt_0t_1t_2t_0t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2N$, $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_2N = Nt_0t_1t_2^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_2^{-1}N = Nt_0t_1t_2t_3^{-1}t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$, and $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_0t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

141. We next consider the double coset $Nt_0t_1t_2^{-1}t_1t_3N$.

Let $[01\bar{2}13]$ denote the double coset $Nt_0t_1t_2^{-1}t_1t_3N$.

Note that $N^{(01\bar{2}13)} \geq N^{01\bar{2}13} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}13)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_1t_3N\right| = \frac{|N|}{|N^{(01\bar{2}13)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}13]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}13)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_1t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_1t_3t_3^{-1}N = Nt_0t_1t_2^{-1}t_1eN = Nt_0t_1t_2^{-1}t_1N$ and $Nt_0t_1t_2^{-1}t_1t_3t_3N = Nt_0t_1t_2^{-1}t_1t_3^2N = Nt_0t_1t_2^{-1}t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_1t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_1N$, $Nt_0t_1t_2^{-1}t_1t_3t_1N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3t_2N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$, and $Nt_0t_1t_2^{-1}t_1t_3t_2^{-1}N = Nt_0t_1t_2t_0t_2^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2^{-1}t_1t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2^{-1}t_1t_3t_0N$.

142. We next consider the double coset $Nt_0t_1t_2^{-1}t_1t_3^{-1}N$.

Let $[01\bar{2}1\bar{3}]$ denote the double coset $Nt_0t_1t_2^{-1}t_1t_3^{-1}N$.

Note that $N^{(01\bar{2}1\bar{3})} \geq N^{01\bar{2}1\bar{3}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}1\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_1t_3^{-1}N\right| = \frac{|N|}{|N^{(01\bar{2}1\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}1\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}1\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_3N = Nt_0t_1t_2^{-1}t_1eN = Nt_0t_1t_2^{-1}t_1N$ and $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-2}N = Nt_0t_1t_2^{-1}t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$, $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_1N = Nt_0t_1t_2t_0t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_1^{-1}N$

$= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$, $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_2N = Nt_0t_1t_0^{-1}t_2N$, and $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_2^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form
$Nt_0t_1t_2^{-1}t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$.

143. We next consider the double coset $Nt_0t_1t_2^{-1}t_3t_0N$.

Let $[01\bar{2}30]$ denote the double coset $Nt_0t_1t_2^{-1}t_3t_0N$.

Note that $N^{(01\bar{2}30)} \geq N^{01\bar{2}30} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}30)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3t_0N\right| = \frac{|N|}{\left|N^{(01\bar{2}30)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}30]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}30)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_3t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_3t_0t_0^{-1}N = Nt_0t_1t_2^{-1}t_3eN = Nt_0t_1t_2^{-1}t_3N$ and
$Nt_0t_1t_2^{-1}t_3t_0t_0N = Nt_0t_1t_2^{-1}t_3t_0^2N = Nt_0t_1t_2^{-1}t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3t_0t_1^{-1}N =$
$Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_3t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_3t_0t_2^{-1}N$
$= Nt_0t_1t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_3t_0t_3N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, and $Nt_0t_1t_2^{-1}t_3t_0t_3^{-1}N$
$= Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Therefore, we conclude that there is one distinct double coset of the form
$Nt_0t_1t_2^{-1}t_3t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_3t_0t_1N$.

144. We next consider the double coset $Nt_0t_1t_2^{-1}t_3t_1N$.

Let $[01\bar{2}31]$ denote the double coset $Nt_0t_1t_2^{-1}t_3t_1N$.

Note that $N^{(01\bar{2}31)} \geq N^{01\bar{2}31} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}31)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3t_1N\right| = \frac{|N|}{\left|N^{(01\bar{2}31)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}31]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}31)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_3t_1t_1^{-1}N = Nt_0t_1t_2^{-1}t_3eN = Nt_0t_1t_2^{-1}t_3N$ and $Nt_0t_1t_2^{-1}t_3t_1t_1N = Nt_0t_1t_2^{-1}t_3t_1^2N = Nt_0t_1t_2^{-1}t_3t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3t_1t_0N = Nt_0t_1^{-1}t_2t_0t_3t_2N$, $Nt_0t_1t_2^{-1}t_3t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_3t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_3t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0N$, $Nt_0t_1t_2^{-1}t_3t_1t_3N = Nt_0t_1^{-1}t_2t_1t_0^{-1}N$, and $Nt_0t_1t_2^{-1}t_3t_1t_3^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

145. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_0N$.

Let $[01\bar{2}\bar{3}0]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_0N$.

Note that $N^{(01\bar{2}\bar{3}0)} \geq N^{01\bar{2}\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3^{-1}t_0N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{3}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_3^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_0^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}eN = Nt_0t_1t_2^{-1}t_3^{-1}N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_0N = Nt_0t_1t_2^{-1}t_3^{-1}t_0^2N = Nt_0t_1t_2^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_2^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$, and $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_1^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1N$.

146. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1N$.

Let $[01\bar{2}\bar{3}1]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1N$.

Note that $N^{(01\bar{2}\bar{3}1)} \geq N^{01\bar{2}\bar{3}1} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{3}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3^{-1}t_1N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{3}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{3}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{3}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_3^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_1^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}eN = Nt_0t_1t_2^{-1}t_3^{-1}N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_1N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^2N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_0N = Nt_0^{-1}t_1t_2t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_2N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_2^{-1}N = Nt_0^{-1}t_1t_2t_3t_0N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_3N = Nt_0t_1t_2^{-1}t_0t_1N$, and $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

147. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

Let $[01\bar{2}\bar{3}\bar{1}]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

Note that $N^{(01\bar{2}\bar{3}\bar{1})} \geq N^{01\bar{2}\bar{3}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{3}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{3}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_1N = Nt_0t_1t_2^{-1}t_3^{-1}eN = Nt_0t_1t_2^{-1}t_3^{-1}N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-2}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_0N = Nt_0t_1t_2t_3^{-1}t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_3N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}N$, and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_3N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

148. We next consider the double coset $Nt_0t_1t_0^{-1}t_2t_1N$.

Let $[01\bar{0}21]$ denote the double coset $Nt_0t_1t_0^{-1}t_2t_1N$.

Note that $N^{(01\bar{0}21)} \geq N^{01\bar{0}21} = \langle e\rangle$. Thus $\left|N^{(01\bar{0}21)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_0^{-1}t_2t_1N\right| = \frac{|N|}{\left|N^{(01\bar{0}21)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{0}21]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{0}21)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_0^{-1}t_2t_1t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_0^{-1}t_2t_1t_1^{-1}N = Nt_0t_1t_0^{-1}t_2eN = Nt_0t_1t_0^{-1}t_2N$ and $Nt_0t_1t_0^{-1}t_2t_1t_1N = Nt_0t_1t_0^{-1}t_2t_1^2N = Nt_0t_1t_0^{-1}t_2t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2t_1t_0N = Nt_0t_1t_2t_3t_0^{-1}t_3N$, $Nt_0t_1t_0^{-1}t_2t_1t_0^{-1}N = Nt_0t_1t_2^{-1}t_3t_0t_1N$, $Nt_0t_1t_0^{-1}t_2t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$, $Nt_0t_1t_0^{-1}t_2t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$, $Nt_0t_1t_0^{-1}t_2t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$, and $Nt_0t_1t_0^{-1}t_2t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_0^{-1}t_2t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

149. We next consider the double coset $Nt_0t_1t_0^{-1}t_2t_3N$.

Let $[01\bar{0}23]$ denote the double coset $Nt_0t_1t_0^{-1}t_2t_3N$.

Note that $N^{(01\bar{0}23)} \geq N^{01\bar{0}23} = \langle e\rangle$. Thus $\left|N^{(01\bar{0}23)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_0^{-1}t_2t_3N\right| = \frac{|N|}{\left|N^{(01\bar{0}23)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{0}23]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{0}23)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_0^{-1}t_2t_3t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_0^{-1}t_2t_3t_3^{-1}N = Nt_0t_1t_0^{-1}t_2eN = Nt_0t_1t_0^{-1}t_2N$ and $Nt_0t_1t_0^{-1}t_2t_3t_3N = Nt_0t_1t_0^{-1}t_2t_3^2N = Nt_0t_1t_0^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2t_3t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0t_1t_0^{-1}t_2t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1t_0^{-1}t_2t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_0^{-1}t_2t_3t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, and $Nt_0t_1t_0^{-1}t_2t_3t_2N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_0^{-1}t_2t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$.

150. We next consider the double coset $Nt_0t_1t_0^{-1}t_2t_3^{-1}N$.

Let $[01\bar{0}2\bar{3}]$ denote the double coset $Nt_0t_1t_0^{-1}t_2t_3^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_0^{-1}t_2t_3^{-1} = Nt_1t_3t_1^{-1}t_2t_0^{-1} = Nt_3t_0t_3^{-1}t_2t_1^{-1}$.

That is, in terms of our short-hand notation,

$$01\bar{0}2\bar{3} \sim 13\bar{1}2\bar{0} \sim 30\bar{3}2\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[01\bar{0}2\bar{3}]$:

$$01\bar{0}2\bar{3} \sim 13\bar{1}2\bar{0} \sim 30\bar{3}2\bar{1} \qquad 10\bar{1}2\bar{3} \sim 03\bar{0}2\bar{1} \sim 31\bar{3}2\bar{0} \qquad 21\bar{2}0\bar{3} \sim 13\bar{1}0\bar{2} \sim 32\bar{3}0\bar{1}$$

$$01\bar{0}3\bar{2} \sim 12\bar{1}3\bar{0} \sim 20\bar{2}3\bar{1} \qquad 02\bar{0}1\bar{3} \sim 23\bar{2}1\bar{0} \sim 30\bar{3}1\bar{2} \qquad 12\bar{1}0\bar{3} \sim 23\bar{2}0\bar{1} \sim 31\bar{3}0\bar{2}$$

$$20\bar{2}1\bar{3} \sim 03\bar{0}1\bar{2} \sim 32\bar{3}1\bar{0} \qquad 21\bar{2}3\bar{0} \sim 10\bar{1}3\bar{2} \sim 02\bar{0}3\bar{1}$$

Since each of the twenty-four single cosets has three names, the double coset $[01\bar{0}2\bar{3}]$ must have at most eight distinct single cosets.

Now, $N^{(01\bar{0}2\bar{3})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,1,3\}$, $\{2\}$, $\{\bar{0},\bar{1},\bar{3}\}$, and $\{\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_0^{-1}t_2t_3^{-1}t_i^{\pm 1}$, $i \in \{2,3\}$.

But note that $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_3N = Nt_0t_1t_0^{-1}t_2eN = Nt_0t_1t_0^{-1}t_2N$ and $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_3^{-1}N = Nt_0t_1t_0^{-1}t_2t_3^{-2}N = Nt_0t_1t_0^{-1}t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2N = Nt_0t_1t_0^{-1}t_2^{-1}t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}N$.

151. We next consider the double coset $Nt_0t_1t_0^{-1}t_2^{-1}t_3N$.

Let $[01\bar{0}\bar{2}3]$ denote the double coset $Nt_0t_1t_0^{-1}t_2^{-1}t_3N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_0^{-1}t_2^{-1}t_3 = Nt_1t_3t_1^{-1}t_2^{-1}t_0 = Nt_3t_0t_3^{-1}t_2^{-1}t_1$.

That is, in terms of our short-hand notation,

$$01\bar{0}\bar{2}3 \sim 13\bar{1}\bar{2}0 \sim 30\bar{3}\bar{2}1.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[01\bar{0}\bar{2}3]$:

$$01\bar{0}\bar{2}3 \sim 13\bar{1}\bar{2}0 \sim 30\bar{3}\bar{2}1 \qquad 10\bar{1}\bar{2}3 \sim 03\bar{0}\bar{2}1 \sim 31\bar{3}\bar{2}0 \qquad 21\bar{2}\bar{0}3 \sim 13\bar{1}\bar{0}2 \sim 32\bar{3}\bar{0}1$$

$$01\bar{0}\bar{3}2 \sim 12\bar{1}\bar{3}0 \sim 20\bar{2}\bar{3}1 \qquad 02\bar{0}\bar{1}3 \sim 23\bar{2}\bar{1}0 \sim 30\bar{3}\bar{1}2 \qquad 12\bar{1}\bar{0}3 \sim 23\bar{2}\bar{0}1 \sim 31\bar{3}\bar{0}2$$

$$20\bar{2}\bar{1}3 \sim 03\bar{0}\bar{1}2 \sim 32\bar{3}\bar{1}0 \qquad 21\bar{2}\bar{3}0 \sim 10\bar{1}\bar{3}2 \sim 02\bar{0}\bar{3}1$$

Since each of the twenty-four single cosets has three names, the double coset $[01\bar{0}\bar{2}3]$ must have at most eight distinct single cosets.

Now, $N^{(01\bar{0}\bar{2}3)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,1,3\}$, $\{2\}$, $\{\bar{0},\bar{1},\bar{3}\}$, and $\{\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_0^{-1}t_2^{-1}t_3t_i^{\pm1}$, $i \in \{2,3\}$.

But note that $Nt_0t_1t_0^{-1}t_2^{-1}t_3t_3^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}eN = Nt_0t_1t_0^{-1}t_2^{-1}N$ and $Nt_0t_1t_0^{-1}t_2^{-1}t_3t_3N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^2N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2^{-1}t_3t_2N = Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}N$ and $Nt_0t_1t_0^{-1}t_2^{-1}t_3t_2^{-1}N = Nt_0t_1t_0^{-1}t_2t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_0^{-1}t_2^{-1}t_3t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

152. We next consider the double coset $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Let $[01\bar{0}\bar{2}\bar{3}]$ denote the double coset $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Note that $N^{(01\bar{0}\bar{2}\bar{3})} \geq N^{01\bar{0}\bar{2}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(01\bar{0}\bar{2}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(01\bar{0}\bar{2}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{0}\bar{2}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{0}\bar{2}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_3N = Nt_0t_1t_0^{-1}t_2^{-1}eN = Nt_0t_1t_0^{-1}t_2^{-1}N$ and $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-2}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0N = Nt_0^{-1}t_1t_2t_3t_1^{-1}N$, $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_1N = Nt_0t_1t_2t_0t_1^{-1}N$, $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1t_2t_0t_1^{-1}t_2N$, $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$, and $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_0^{-1}t_2t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

153. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0N$.

Let $[\bar{0}\bar{1}210]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0N$.

Note that $N^{(\bar{0}\bar{1}210)} \geq N^{\bar{0}\bar{1}210} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}210)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_0N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}210)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}210]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}210)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_1t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1eN = Nt_0^{-1}t_1^{-1}t_2t_1N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_0N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^2N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_1N = Nt_0t_1t_2^{-1}t_0t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$.

154. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$.

Let $[\bar{0}\bar{1}21\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}21\bar{0})} \geq N^{\bar{0}\bar{1}21\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}21\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}21\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}21\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}21\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_1eN = Nt_0^{-1}t_1^{-1}t_2t_1N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_1N = Nt_0t_1t_2^{-1}t_1t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_2N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

155. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3N$.

Let $[\bar{0}\bar{1}213]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3N$.

Note that $N^{(\bar{0}\bar{1}213)} \geq N^{\bar{0}\bar{1}213} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}213)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}213)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}213]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}213)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_1t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1eN = Nt_0^{-1}t_1^{-1}t_2t_1N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_1N = Nt_0t_1t_2^{-1}t_0t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_2N = Nt_0^{-1}t_1t_0^{-1}t_2t_3N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

156. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$.

Let $[\bar{0}\bar{1}21\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}21\bar{3})} \geq N^{\bar{0}\bar{1}21\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}21\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}21\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}21\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}21\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1eN = Nt_0^{-1}t_1^{-1}t_2t_1N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_2N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$.

157. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$.

Let $[\bar{0}\bar{1}2\bar{1}0]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$.

Note that $N^{(\bar{0}\bar{1}2\bar{1}0)} \geq N^{\bar{0}\bar{1}2\bar{1}0} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2\bar{1}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}2\bar{1}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2\bar{1}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2\bar{1}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_0N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^2N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_1N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_1^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_3N = Nt_0t_1t_2t_0t_3t_2N$, and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

158. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Let $[\bar{0}\bar{1}2\bar{1}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}2\bar{1}\bar{0})} \geq N^{\bar{0}\bar{1}2\bar{1}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2\bar{1}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}2\bar{1}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2\bar{1}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2\bar{1}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1t_2t_0^{-1}t_2t_1N$, and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$.

159. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0N$.

Let $[\bar{0}\bar{1}230]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0N$.

Note that $N^{(\bar{0}\bar{1}230)} \geq N^{\bar{0}\bar{1}230} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}230)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3t_0N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}230)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}230]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}230)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_3t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3eN = Nt_0^{-1}t_1^{-1}t_2t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_0N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^2N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_1N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_3N = Nt_0t_1t_2^{-1}t_3t_1N$, and $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_3t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

160. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$.

Let $[\bar{0}\bar{1}23\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}23\bar{0})} \geq N^{\bar{0}\bar{1}23\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}23\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}23\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}23\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}23\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_3eN = Nt_0^{-1}t_1^{-1}t_2t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_3N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_2^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$.

161. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_1N$.

Let $[\bar{0}\bar{1}231]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_1N$.

Note that $N^{(\bar{0}\bar{1}231)} \geq N^{\bar{0}\bar{1}231} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}231)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3t_1N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}231)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}231]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}231)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3eN = Nt_0^{-1}t_1^{-1}t_2t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_1N = Nt_0^{-1}t_1^{-1}t_2t_3t_1^2N = Nt_0^{-1}t_1^{-1}t_2t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_0N = Nt_0t_1t_2^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_3N = Nt_0t_1t_2^{-1}t_1t_3N$, and $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_3t_0N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$.

162. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$.

Let $[\bar{0}\bar{1}23\bar{2}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$.

Note that $N^{(\bar{0}\bar{1}23\bar{2})} \geq N^{\bar{0}\bar{1}23\bar{2}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}23\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}23\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}23\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}23\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_3eN = Nt_0^{-1}t_1^{-1}t_2t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_3t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_3N = Nt_0t_1t_2t_0^{-1}t_2t_1N$, and $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

163. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$.

Let $[\bar{0}\bar{1}\bar{2}01]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}01)} \geq N^{\bar{0}\bar{1}\bar{2}01} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}01)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}01)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}01]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}01)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_0N = Nt_0^{-1}t_1t_2t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N = Nt_0t_1t_2t_0t_3t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

164. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}0\bar{1}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}0\bar{1})} \geq N^{\bar{0}\bar{1}\bar{2}0\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}0\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}0\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}0\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}0\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_2N = Nt_0t_1t_2t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

165. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$.

Let $[\bar{0}\bar{1}\bar{2}03]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}03)} \geq N^{\bar{0}\bar{1}\bar{2}03} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}03)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}03)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}03]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}03)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

166. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}0\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}0\bar{3})} \geq N^{\bar{0}\bar{1}\bar{2}0\bar{3}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}0\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}0\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}0\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}0\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_1N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_1t_3N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

167. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{0}1]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{0}1)} \geq N^{\bar{0}\bar{1}\bar{2}\bar{0}1} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{0}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{0}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N = Nt_0t_1t_0^{-1}t_2t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

168. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{0}3]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{0}3)} \geq N^{\bar{0}\bar{1}\bar{2}\bar{0}3} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0N = Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N = Nt_0t_1t_2t_3t_1N$,

$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N = Nt_0t_1t_2t_3t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2N$ $= Nt_0t_1t_2^{-1}t_3t_1N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_2N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$.

169. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_2N$ $= Nt_0t_1t_2t_0t_3N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_0t_3t_2N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

170. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$.

Let $[\bar{0}\bar{1}\bar{2}10]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}10)} \geq N^{\bar{0}\bar{1}\bar{2}10} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}10)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}10)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}10]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}10)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_1N =$ $Nt_0t_1t_2t_0t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_2N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_3^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

171. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$.

Let $[\bar{0}\bar{1}\bar{2}13]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}13)} \geq N^{\bar{0}\bar{1}\bar{2}13} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}13)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}13)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}13]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}13)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_1N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_2N = Nt_0^{-1}t_1t_2t_0t_3^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_2^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

172. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$.

Let $[\bar{0}\bar{1}\bar{2}31]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}31)} \geq N^{\bar{0}\bar{1}\bar{2}31} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}31)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}31)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}31]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}31)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N =$
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_2N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$,
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_3N$
$= Nt_0^{-1}t_1t_2t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_0N$.

Therefore, we conclude that there is one distinct double coset of the form
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$.

173. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$.

Let $[\bar{0}\bar{1}\bar{2}32]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}32)} \geq N^{\bar{0}\bar{1}\bar{2}32} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}32)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}32)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}32]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}32)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_0N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_1N = Nt_0^{-1}t_1^{-1}t_2t_3t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_1^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_3N = Nt_0^{-1}t_1^{-1}t_2t_3^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

174. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}3\bar{2}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}3\bar{2})} \geq N^{\bar{0}\bar{1}\bar{2}3\bar{2}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}3\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}3\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}3\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}3\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_2t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}t_3N$.

175. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{3}0]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{3}0)} \geq N^{\bar{0}\bar{1}\bar{2}\bar{3}0} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}0)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}\bar{3}0)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_1N = Nt_0t_1^{-1}t_2t_0t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_3t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_3N = Nt_0t_1^{-1}t_2t_0t_1t_2N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

176. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

177. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{3}1]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{3}1)} \geq N^{\bar{0}\bar{1}\bar{2}\bar{3}1} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{3}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{3}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0N = Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_3N = Nt_0^{-1}t_1t_2t_0t_3^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_3^{-1}N$ $= Nt_0^{-1}t_1t_2t_0N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$.

178. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N = Nt_0t_1t_2t_0t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N = Nt_0t_1t_2t_0t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

179. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-1}N$.

Let $[\bar{0}1\bar{0}2\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-1} = Nt_1^{-1}t_2t_1^{-1}t_3t_1^{-1} = Nt_2^{-1}t_3t_2^{-1}t_0t_2^{-1} = Nt_3^{-1}t_0t_3^{-1}t_1t_3^{-1}$.

That is, in terms of our short-hand notation,

$$\bar{0}1\bar{0}2\bar{0} \sim \bar{1}2\bar{1}3\bar{1} \sim \bar{2}3\bar{2}0\bar{2} \sim \bar{3}0\bar{3}1\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}1\bar{0}2\bar{0}]$:

$$\bar{0}1\bar{0}2\bar{0} \sim \bar{1}2\bar{1}3\bar{1} \sim \bar{2}3\bar{2}0\bar{2} \sim \bar{3}0\bar{3}1\bar{3}, \qquad \bar{1}0\bar{1}2\bar{1} \sim \bar{0}2\bar{0}3\bar{0} \sim \bar{2}3\bar{2}1\bar{2} \sim \bar{3}1\bar{3}0\bar{3},$$

$$\bar{2}1\bar{2}0\bar{2} \sim \bar{1}0\bar{1}3\bar{1} \sim \bar{0}3\bar{0}2\bar{0} \sim \bar{3}2\bar{3}1\bar{3}, \qquad \bar{3}1\bar{3}2\bar{3} \sim \bar{1}2\bar{1}0\bar{1} \sim \bar{2}0\bar{2}3\bar{2} \sim \bar{0}3\bar{0}1\bar{0},$$

$$\bar{1}3\bar{1}2\bar{1} \sim \bar{3}2\bar{3}0\bar{3} \sim \bar{2}0\bar{2}1\bar{2} \sim \bar{0}1\bar{0}3\bar{0}, \qquad \bar{3}0\bar{3}2\bar{3} \sim \bar{0}2\bar{0}1\bar{3} \sim \bar{2}1\bar{2}3\bar{2} \sim \bar{1}3\bar{1}0\bar{1}$$

Since each of the twenty-four single cosets has four names, the double coset $[\bar{0}1\bar{0}2\bar{0}]$ must have at most six distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}2\bar{0})}$ has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_0^{-1}t_2t_0^{-1}t_i^{\pm 1}$, $i = 0$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-1}t_0N = Nt_0^{-1}t_1t_0^{-1}t_2eN = Nt_0^{-1}t_1t_0^{-1}t_2N$ and $Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-2}N = Nt_0^{-1}t_1t_0^{-1}t_2t_0N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_2t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

180. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2t_3N$.

Let $[\bar{0}1\bar{0}23]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2t_3N$.

Note that $N^{(\bar{0}1\bar{0}23)} \geq N^{\bar{0}1\bar{0}23} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{0}23)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_0^{-1}t_2t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{0}23)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{0}23]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}23)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_0^{-1}t_2t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_3^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2eN = Nt_0^{-1}t_1t_0^{-1}t_2N$ and $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_3N = Nt_0^{-1}t_1t_0^{-1}t_2t_3^2N = Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_0N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3N$, $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_1N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$, $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$, and $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_2^{-1}N = Nt_0^{-1}t_1t_2t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_2t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

181. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}N$.

Let $[\bar{0}1\bar{0}2\bar{3}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1} = Nt_1^{-1}t_3t_1^{-1}t_2t_0^{-1} = Nt_3^{-1}t_0t_3^{-1}t_2t_1^{-1}$.

That is, in terms of our short-hand notation,

$$\bar{0}1\bar{0}2\bar{3} \sim \bar{1}3\bar{1}2\bar{0} \sim \bar{3}0\bar{3}2\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}1\bar{0}2\bar{3}]$:

$$\bar{0}1\bar{0}2\bar{3} \sim \bar{1}3\bar{1}2\bar{0} \sim \bar{3}0\bar{3}2\bar{1}, \qquad \bar{1}0\bar{1}2\bar{3} \sim \bar{0}3\bar{0}2\bar{1} \sim \bar{3}1\bar{3}2\bar{0},$$

$\bar{2}1\bar{2}0\bar{3} \sim \bar{1}3\bar{1}0\bar{2} \sim \bar{3}2\bar{3}0\bar{1}$, $\bar{0}1\bar{0}3\bar{2} \sim \bar{1}2\bar{1}3\bar{0} \sim \bar{2}0\bar{2}3\bar{1}$,

$\bar{0}2\bar{0}1\bar{3} \sim \bar{2}3\bar{2}1\bar{0} \sim \bar{3}0\bar{3}1\bar{2}$, $\bar{1}2\bar{1}0\bar{3} \sim \bar{2}3\bar{2}0\bar{1} \sim \bar{3}1\bar{3}0\bar{2}$,

$\bar{2}0\bar{2}1\bar{3} \sim \bar{0}3\bar{0}1\bar{2} \sim \bar{3}2\bar{3}1\bar{0}$, $\bar{2}1\bar{2}3\bar{0} \sim \bar{1}0\bar{1}3\bar{2} \sim \bar{0}2\bar{0}3\bar{1}$

Since each of the twenty-four single cosets has three names, the double coset $[\bar{0}1\bar{0}2\bar{3}]$ must have at most eight distinct single cosets.

Now, $N^{(\bar{0}1\bar{0}2\bar{3})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 3\}$, $\{2\}$, $\{\bar{0}, \bar{1}, \bar{3}\}$, and $\{\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_0^{-1}t_2t_3^{-1}t_i^{\pm 1}$, $i \in \{2, 3\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}t_3N = Nt_0^{-1}t_1t_0^{-1}t_2eN = Nt_0^{-1}t_1t_0^{-1}t_2N$ and $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-2}N = Nt_0^{-1}t_1t_0^{-1}t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}t_2N = Nt_0^{-1}t_1t_2^{-1}t_3N$ and $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

182. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Let $[\bar{0}1\bar{0}\bar{2}3]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Note that $N^{(\bar{0}1\bar{0}\bar{2}3)} \geq N^{\bar{0}1\bar{0}\bar{2}3} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{0}\bar{2}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{0}\bar{2}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{0}\bar{2}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}\bar{2}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_3^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}eN = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$ and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_3N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_0N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_1N = Nt_0t_1t_2t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$,

$Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_2N = Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}N$, and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_2^{-1}N$ $= Nt_0^{-1}t_1t_2t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

183. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Let $[\bar{0}1\bar{0}\bar{2}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}1\bar{0}\bar{2}\bar{3})} \geq N^{\bar{0}1\bar{0}\bar{2}\bar{3}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}1\bar{0}\bar{2}\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}1\bar{0}\bar{2}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{0}\bar{2}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}\bar{2}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}eN = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}N$ and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-2}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2N$ $= Nt_0^{-1}t_1t_2t_3t_2^{-1}N$, and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

184. We next consider the double coset $Nt_0^{-1}t_1t_2t_0t_1N$.

Let $[\bar{0}1201]$ denote the double coset $Nt_0^{-1}t_1t_2t_0t_1N$.

Note that $N^{(\bar{0}1201)} \geq N^{\bar{0}1201} = \langle e\rangle$. Thus $\left|N^{(\bar{0}1201)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_0t_1N\right| = \frac{|N|}{|N^{(\bar{0}1201)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1201]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1201)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_0t_1t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2t_0t_1t_1^{-1}N = Nt_0^{-1}t_1t_2t_0eN = Nt_0^{-1}t_1t_2t_0N$ and $Nt_0^{-1}t_1t_2t_0t_1t_1N = Nt_0^{-1}t_1t_2t_0t_1^2N = Nt_0^{-1}t_1t_2t_0t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_0t_1t_0N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$, $Nt_0^{-1}t_1t_2t_0t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1t_2t_0t_1t_3N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$, and $Nt_0^{-1}t_1t_2t_0t_1t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_0t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

185. We next consider the double coset $Nt_0^{-1}t_1t_2t_0t_1^{-1}N$.

Let $[\bar{0}120\bar{1}]$ denote the double coset $Nt_0^{-1}t_1t_2t_0t_1^{-1}N$.

Note that $N^{(\bar{0}120\bar{1})} \geq N^{\bar{0}120\bar{1}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}120\bar{1})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_0t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}120\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}120\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}120\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_0t_1^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_1N = Nt_0^{-1}t_1t_2t_0eN = Nt_0^{-1}t_1t_2t_0N$ and $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2t_0t_1^{-2}N = Nt_0^{-1}t_1t_2t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_0N = Nt_0t_1t_2t_0^{-1}t_2t_1N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_2N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$, and $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$.

186. We next consider the double coset $Nt_0^{-1}t_1t_2t_0t_3^{-1}N$.

Let $[\bar{0}120\bar{3}]$ denote the double coset $Nt_0^{-1}t_1t_2t_0t_3^{-1}N$.

Note that $N^{(\bar{0}120\bar{3})} \geq N^{\bar{0}120\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}120\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_0t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}120\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}120\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}120\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_0t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_3N = Nt_0^{-1}t_1t_2t_0eN = Nt_0^{-1}t_1t_2t_0N$ and $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2t_0t_3^{-2}N = Nt_0^{-1}t_1t_2t_0t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_1N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, and $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

187. We next consider the double coset $Nt_0^{-1}t_1t_2t_3t_0N$.

Let $[\bar{0}1230]$ denote the double coset $Nt_0^{-1}t_1t_2t_3t_0N$.

Note that $N^{(\bar{0}1230)} \geq N^{\bar{0}1230} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1230)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3t_0N\right| = \frac{|N|}{\left|N^{(\bar{0}1230)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1230]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1230)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_3t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_0t_0^{-1}N = Nt_0^{-1}t_1t_2t_3eN = Nt_0^{-1}t_1t_2t_3N$ and $Nt_0^{-1}t_1t_2t_3t_0t_0N = Nt_0^{-1}t_1t_2t_3t_0^2N = Nt_0^{-1}t_1t_2t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_0t_1N = Nt_0t_1t_2t_0t_1^{-1}t_2N$, $Nt_0^{-1}t_1t_2t_3t_0t_2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$, $Nt_0^{-1}t_1t_2t_3t_0t_3N = Nt_0t_1t_2^{-1}t_3^{-1}t_1N$, and $Nt_0^{-1}t_1t_2t_3t_0t_3^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}N$ and $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$.

188. We next consider the double coset $Nt_0^{-1}t_1t_2t_3t_1N$.

Let $[\bar{0}1231]$ denote the double coset $Nt_0^{-1}t_1t_2t_3t_1N$.

Note that $N^{(\bar{0}1231)} \geq N^{\bar{0}1231} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1231)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3t_1N\right| = \frac{|N|}{\left|N^{(\bar{0}1231)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1231]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1231)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_3t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_1t_1^{-1}N = Nt_0^{-1}t_1t_2t_3eN = Nt_0^{-1}t_1t_2t_3N$ and $Nt_0^{-1}t_1t_2t_3t_1t_1N = Nt_0^{-1}t_1t_2t_3t_1^2N = Nt_0^{-1}t_1t_2t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_1t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N$, $Nt_0^{-1}t_1t_2t_3t_1t_2N = Nt_0t_1t_2t_0t_1t_3N$, $Nt_0^{-1}t_1t_2t_3t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$, $Nt_0^{-1}t_1t_2t_3t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$, and $Nt_0^{-1}t_1t_2t_3t_1t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

189. We next consider the double coset $Nt_0^{-1}t_1t_2t_3t_1^{-1}N$.

Let $[\bar{0}123\bar{1}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3t_1^{-1}N$.

Note that $N^{(\bar{0}123\bar{1})} \geq N^{\bar{0}123\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}123\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}123\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}123\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}123\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_3t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_1N = Nt_0^{-1}t_1t_2t_3eN = Nt_0^{-1}t_1t_2t_3N$ and $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2t_3t_1^{-2}N = Nt_0^{-1}t_1t_2t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_0N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_0^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$, $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_3N = Nt_0t_1t_2t_0^{-1}t_2^{-1}N$, and $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

190. We next consider the double coset $Nt_0^{-1}t_1t_2t_3t_2^{-1}N$.

Let $[\bar{0}123\bar{2}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3t_2^{-1}N$.

Note that $N^{(\bar{0}123\bar{2})} \geq N^{\bar{0}123\bar{2}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}123\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3t_2^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}123\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}123\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}123\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_3t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_2N = Nt_0^{-1}t_1t_2t_3eN = Nt_0^{-1}t_1t_2t_3N$ and $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2t_3t_2^{-2}N = Nt_0^{-1}t_1t_2t_3t_2N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_0N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$, $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_3N$ $= Nt_0^{-1}t_1t_0^{-1}t_2t_3N$, and $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$.

191. We next consider the double coset $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$.

Let $[\bar{0}12\bar{3}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$.

Note that $N^{(\bar{0}12\bar{3}\bar{0})} \geq N^{\bar{0}12\bar{3}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}12\bar{3}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}12\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}12\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}12\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1t_2t_3^{-1}eN = Nt_0^{-1}t_1t_2t_3^{-1}N$ and $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-2}N = Nt_0^{-1}t_1t_2t_3^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$, $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_2N = Nt_0t_1t_2t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$, $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, and $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

192. We next consider the double coset $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}N$.

Let $[\bar{0}12\bar{3}\bar{2}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1} = Nt_1^{-1}t_2t_0t_3^{-1}t_0^{-1} = Nt_2^{-1}t_0t_1t_3^{-1}t_1^{-1}$.

That is, in terms of our short-hand notation,

$$\bar{0}12\bar{3}\bar{2} \sim \bar{1}20\bar{3}\bar{0} \sim \bar{2}01\bar{3}\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}12\bar{3}\bar{2}]$:

$$\bar{0}12\bar{3}\bar{2} \sim \bar{1}20\bar{3}\bar{0} \sim \bar{2}01\bar{3}\bar{1}, \qquad \bar{1}02\bar{3}\bar{2} \sim \bar{0}21\bar{3}\bar{1} \sim \bar{2}10\bar{3}\bar{0},$$

$$\bar{3}12\bar{0}\bar{2} \sim \bar{1}23\bar{0}\bar{3} \sim \bar{2}31\bar{0}\bar{1}, \qquad \bar{0}32\bar{1}\bar{2} \sim \bar{3}20\bar{1}\bar{0} \sim \bar{2}03\bar{1}\bar{3},$$

$$\bar{0}13\bar{2}\bar{3} \sim \bar{1}30\bar{2}\bar{0} \sim \bar{3}01\bar{2}\bar{1}, \qquad \bar{1}32\bar{0}\bar{2} \sim \bar{3}21\bar{0}\bar{1} \sim \bar{2}13\bar{0}\bar{3},$$

$$\bar{3}02\bar{1}\bar{2} \sim \bar{0}23\bar{1}\bar{3} \sim \bar{2}30\bar{1}\bar{0}, \qquad \bar{3}10\bar{2}\bar{0} \sim \bar{1}03\bar{2}\bar{3} \sim \bar{0}31\bar{2}\bar{1}$$

Since each of the twenty-four single cosets has three names, the double coset $[\bar{0}12\bar{3}\bar{2}]$ must have at most eight distinct single cosets.

Now, $N^{(\bar{0}12\bar{3}\bar{2})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2\}$, $\{3\}$, $\{\bar{0}, \bar{1}, \bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}t_2N = Nt_0^{-1}t_1t_2t_3^{-1}eN = Nt_0^{-1}t_1t_2t_3^{-1}N$ and $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-2}N = Nt_0^{-1}t_1t_2t_3^{-1}t_2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}t_3N = Nt_0^{-1}t_1t_0^{-1}t_2t_3^{-1}N$ and $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

193. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$.

Let $[\bar{0}1\bar{2}\bar{0}1]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$.

Note that $N^{(\bar{0}1\bar{2}\bar{0}1)} \geq N^{\bar{0}1\bar{2}\bar{0}1} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{2}\bar{0}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{2}\bar{0}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{2}\bar{0}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{2}\bar{0}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_1^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}eN = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_1N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^2N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_0N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_2^{-1}N = Nt_0t_1t_2t_3t_1t_0N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_3N = Nt_0t_1t_2t_3t_0^{-1}N$, and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

194. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Let $[\bar{0}1\bar{2}\bar{0}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Note that $N^{(\bar{0}1\bar{2}\bar{0}\bar{1})} \geq N^{\bar{0}1\bar{2}\bar{0}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{2}\bar{0}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{2}\bar{0}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{2}\bar{0}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{2}\bar{0}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_1N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}eN = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-2}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2t_3N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N = Nt_0t_1^{-1}t_2t_3t_1t_2N$, and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

195. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Let $[\bar{0}1\bar{2}\bar{0}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}1\bar{2}\bar{0}\bar{3})} \geq N^{\bar{0}1\bar{2}\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{2}\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{2}\bar{0}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{2}\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{2}\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}eN = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-2}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3N$ $= Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N =$ $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$, $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_3N$, and $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

196. We next consider the double coset $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

Let $[\bar{0}1\bar{2}\bar{3}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

Note that $N^{(\bar{0}1\bar{2}\bar{3}\bar{1})} \geq N^{\bar{0}1\bar{2}\bar{3}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{2}\bar{3}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{2}\bar{3}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{2}\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{2}\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length six given by $w = t_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_1N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}eN = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}N$ and $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-2}N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1N$ $= Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_0N =$ $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N = Nt_0t_1t_2t_3t_0t_2N$, $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$, $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$, and $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

197. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$.

Let $[0\bar{1}201\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$.

Note that $N^{(0\bar{1}201\bar{0})} \geq N^{0\bar{1}201\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}201\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}201\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}201\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}201\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_1eN = Nt_0t_1^{-1}t_2t_0t_1N$ and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-2}N = Nt_0t_1^{-1}t_2t_0t_1t_0N = Nt_0t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$.

Therefore, we conclude that there are three distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$, and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$.

198. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_2N$.

Let $[0\bar{1}2012]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_2N$.

Note that $N^{(0\bar{1}2012)} \geq N^{0\bar{1}2012} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2012)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}2012)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2012]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2012)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_2t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_1eN = Nt_0t_1^{-1}t_2t_0t_1N$ and $Nt_0t_1^{-1}t_2t_0t_1t_2t_2N = Nt_0t_1^{-1}t_2t_0t_1t_2^2N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_2t_0N =$ $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_2t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_0t_1t_2t_1N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_2t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_1t_2t_3N = Nt_0t_1t_2t_3t_0N$, and $Nt_0t_1^{-1}t_2t_0t_1t_2t_3^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

199. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$.

Let $[0\bar{1}201\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$.

Note that $N^{(0\bar{1}201\bar{2})} \geq N^{0\bar{1}201\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}201\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}201\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}201\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}201\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1eN = Nt_0t_1^{-1}t_2t_0t_1N$ and $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-2}N = Nt_0t_1^{-1}t_2t_0t_1t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_1N$ $= Nt_0t_1^{-1}t_0^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$, and $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}N$.

200. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_3N$.

Let $[0\bar{1}2013]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_3N$.

Note that $N^{(0\bar{1}2013)} \geq N^{0\bar{1}2013} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2013)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}2013)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2013]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2013)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_3t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1eN = Nt_0t_1^{-1}t_2t_0t_1N$ and $Nt_0t_1^{-1}t_2t_0t_1t_3t_3N = Nt_0t_1^{-1}t_2t_0t_1t_3^2N = Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_3t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_3t_1N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$, $Nt_0t_1^{-1}t_2t_0t_1t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_0t_1t_3t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_1t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

201. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$.

Let $[0\bar{1}201\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$.

Note that $N^{(0\bar{1}201\bar{3})} \geq N^{0\bar{1}201\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}201\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}201\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}201\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}201\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_1eN = Nt_0t_1^{-1}t_2t_0t_1N$ and $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_3^{-2}N = Nt_0t_1^{-1}t_2t_0t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_2N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

202. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$.

Let $[0\bar{1}20\bar{1}2]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$.

Note that $N^{(0\bar{1}20\bar{1}2)} \geq N^{0\bar{1}20\bar{1}2} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{1}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{1}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{1}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{1}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1^{-1}t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}eN = Nt_0t_1^{-1}t_2t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_2N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^2N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_0N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_1N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_3^{-1}N = Nt_0t_1t_2t_0t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

203. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$.

Let $[0\bar{1}20\bar{1}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$.

Note that $N^{(0\bar{1}20\bar{1}\bar{2})} \geq N^{0\bar{1}20\bar{1}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{1}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{1}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{1}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{1}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1^{-1}eN = Nt_0t_1^{-1}t_2t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$, and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$.

204. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$.

Let $[0\bar{1}20\bar{1}3]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$.

Note that $N^{(0\bar{1}20\bar{1}3)} \geq N^{0\bar{1}20\bar{1}3} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}20\bar{1}3)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{1}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{1}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{1}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_1^{-1}t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}eN = Nt_0t_1^{-1}t_2t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_3N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3^{2}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3^{-1}N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_0N =$ $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_1N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_2N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_2^{-1}N = Nt_0t_1t_2t_3t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

205. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3t_1N$.

Let $[0\bar{1}2031]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3t_1N$.

Note that $N^{(0\bar{1}2031)} \geq N^{0\bar{1}2031} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2031)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}2031)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2031]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2031)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3t_1t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3eN = Nt_0t_1^{-1}t_2t_0t_3N$ and $Nt_0t_1^{-1}t_2t_0t_3t_1t_1N = Nt_0t_1^{-1}t_2t_0t_3t_1^2N = Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3t_1t_0N =$
$Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$, $Nt_0t_1^{-1}t_2t_0t_3t_1t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$,
$Nt_0t_1^{-1}t_2t_0t_3t_1t_2N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_3t_1t_2^{-1}N = Nt_0t_1t_2t_3t_0^{-1}N$,
$Nt_0t_1^{-1}t_2t_0t_3t_1t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, and $Nt_0t_1^{-1}t_2t_0t_3t_1t_3^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

206. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$.

Let $[0\bar{1}203\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$.

Note that $N^{(0\bar{1}203\bar{1})} \geq N^{0\bar{1}203\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}203\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}203\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}203\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}203\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_3t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_3eN = Nt_0t_1^{-1}t_2t_0t_3N$ and $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_1^{-2}N = Nt_0t_1^{-1}t_2t_0t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_2^{-1}N =$ $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_3N = Nt_0t_1^{-1}t_2t_3t_1t_2N$, and $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

207. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3t_2N$.

Let $[0\bar{1}2032]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3t_2N$.

Note that $N^{(0\bar{1}2032)} \geq N^{0\bar{1}2032} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}2032)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}2032)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2032]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2032)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_3t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3t_2t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3eN = Nt_0t_1^{-1}t_2t_0t_3N$ and $Nt_0t_1^{-1}t_2t_0t_3t_2t_2N = Nt_0t_1^{-1}t_2t_0t_3t_2^2N = Nt_0t_1^{-1}t_2t_0t_3t_2^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3t_2t_0N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_0t_3t_2t_0^{-1}N = Nt_0t_1t_2^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2t_0t_3t_2t_1N$ $= Nt_0t_1t_2^{-1}t_0t_3N$, $Nt_0t_1^{-1}t_2t_0t_3t_2t_1^{-1}N = Nt_0t_1t_2t_0t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3t_2t_3N$ $= Nt_0t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_3t_2t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

208. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$.

Let $[0\bar{1}20\bar{3}1]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$.

Note that $N^{(0\bar{1}20\bar{3}1)} \geq N^{0\bar{1}20\bar{3}1} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}20\bar{3}1)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{3}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{3}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{3}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_1t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}eN = Nt_0t_1^{-1}t_2t_0t_3^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_1N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_0N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_0^{-1}N = Nt_0t_1t_2t_3t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_2N = Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_2^{-1}N = Nt_0^{-1}t_1t_2t_3t_0N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$.

209. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$.

Let $[0\bar{1}20\bar{3}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$.

Note that $N^{(0\bar{1}20\bar{3}\bar{1})} \geq N^{0\bar{1}20\bar{3}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{3}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{3}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_3^{-1}eN = Nt_0t_1^{-1}t_2t_0t_3^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2N = Nt_0t_1t_2t_0t_1^{-1}t_2N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_3N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_3^{-1}N$.

210. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$.

Let $[0\bar{1}20\bar{3}2]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$.

Note that $N^{(0\bar{1}20\bar{3}2)} \geq N^{0\bar{1}20\bar{3}2} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{3}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N\right| = \frac{|N|}{|N^{(0\bar{1}20\bar{3}2)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{3}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{3}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}eN = Nt_0t_1^{-1}t_2t_0t_3^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_0N = Nt_0t_1t_2^{-1}t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_0^{-1}N = Nt_0t_1t_2t_0t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_1N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

211. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$.

Let $[0\bar{1}20\bar{3}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$.

Note that $N^{(0\bar{1}20\bar{3}\bar{2})} \geq N^{0\bar{1}20\bar{3}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{3}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}20\bar{3}\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{3}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{3}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}eN = Nt_0t_1^{-1}t_2t_0t_3^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_0N =$ $Nt_0t_1t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_1N$ $= Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_3^{-1}N$ $= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

212. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$.

Let $[0\bar{1}21\bar{0}1]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$.

Note that $N^{(0\bar{1}21\bar{0}1)} \geq N^{0\bar{1}21\bar{0}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}21\bar{0}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}21\bar{0}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}21\bar{0}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}21\bar{0}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_1t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_0^{-1}eN = Nt_0t_1^{-1}t_2t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_1N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1^2N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_0N =$ $Nt_0t_1^{-1}t_2t_3t_1N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2t_1^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_2N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_3^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

213. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Let $[0\bar{1}21\bar{0}3]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Note that $N^{(0\bar{1}21\bar{0}3)} \geq N^{0\bar{1}21\bar{0}3} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}21\bar{0}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N\right| = \frac{|N|}{|N^{(0\bar{1}21\bar{0}3)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}21\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}21\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_1t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_0^{-1}eN = Nt_0t_1^{-1}t_2t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_3N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^2N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_0N =$ $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_1N = Nt_0t_1t_2t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_1^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$, and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

214. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

Let $[0\bar{1}21\bar{0}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}21\bar{0}\bar{3})} \geq N^{0\bar{1}21\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}21\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}21\bar{0}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}21\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}21\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_1t_0^{-1}eN = Nt_0t_1^{-1}t_2t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_0N =$ $Nt_0t_1t_2t_3t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_0N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_1N$

$= Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$,
$Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$, and $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_2^{-1}N$
$= Nt_0t_1t_2^{-1}t_3t_0t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

215. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_3t_0N$.

Let $[0\bar{1}2130]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_3t_0N$.

Note that $N^{(0\bar{1}2130)} \geq N^{0\bar{1}2130} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2130)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_3t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}2130)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2130]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2130)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_1t_3t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_3t_0t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3eN = Nt_0t_1^{-1}t_2t_1t_3N$ and $Nt_0t_1^{-1}t_2t_1t_3t_0t_0N = Nt_0t_1^{-1}t_2t_1t_3t_0^2N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_3t_0t_1N =$
$Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$,
$Nt_0t_1^{-1}t_2t_1t_3t_0t_2N = Nt_0t_1t_2t_0t_3N$, $Nt_0t_1^{-1}t_2t_1t_3t_0t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$,
$Nt_0t_1^{-1}t_2t_1t_3t_0t_3N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, and $Nt_0t_1^{-1}t_2t_1t_3t_0t_3^{-1}N$
$= Nt_0t_1t_2t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_3t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

216. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Let $[0\bar{1}213\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Note that $N^{(0\bar{1}213\bar{0})} \geq N^{0\bar{1}213\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}213\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}213\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}213\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}213\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_1t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_3eN = Nt_0t_1^{-1}t_2t_1t_3N$ and $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-2}N = Nt_0t_1^{-1}t_2t_1t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, and $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$.

217. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_3t_2N$.

Let $[0\bar{1}2132]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_3t_2N$.

Note that $N^{(0\bar{1}2132)} \geq N^{0\bar{1}2132} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2132)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_3t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}2132)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2132]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2132)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_1t_3t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_3t_2t_2^{-1}N = Nt_0t_1^{-1}t_2t_1t_3eN = Nt_0t_1^{-1}t_2t_1t_3N$ and $Nt_0t_1^{-1}t_2t_1t_3t_2t_2N = Nt_0t_1^{-1}t_2t_1t_3t_2^2N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_3t_2t_0N =$ $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_2t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_2t_1N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_2t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_2t_3N = Nt_0t_1t_2^{-1}t_1t_0^{-1}N$, and $Nt_0t_1^{-1}t_2t_1t_3t_2t_3^{-1}N = Nt_0t_1t_2t_0t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_3t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

218. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}N$.

Let $[0\bar{1}213\bar{2}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}N$.

Note that $N^{(0\bar{1}213\bar{2})} \geq N^{0\bar{1}213\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}213\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}213\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}213\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}213\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_2N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3eN = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_{\bar{2}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_0N =$ $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_3N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}N = Nt_0t_1t_2t_0t_3t_2N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_3t_1t_0N$, $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_3N = Nt_0t_1t_{\bar{2}}^{-1}t_1t_3N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}N = Nt_0t_1t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_{\bar{1}}^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_1N$.

219. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0N$.

Let $[0\bar{1}21\bar{3}0]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0N$.

Note that $N^{(0\bar{1}21\bar{3}0)} \geq N^{0\bar{1}21\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}21\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}21\bar{3}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}21\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}21\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_{\bar{0}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}eN = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_0N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0^2N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_1N = Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_3N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$, and $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2N$.

220. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$.

Let $[0\bar{1}21\bar{3}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$.

Note that $N^{(0\bar{1}21\bar{3}\bar{0})} \geq N^{0\bar{1}21\bar{3}\bar{0}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}21\bar{3}\bar{0})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}21\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}21\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}21\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_3^{-1}eN = Nt_0t_1^{-1}t_2t_1t_3^{-1}N$ and $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0t_2N$, $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_3N = Nt_0t_1t_2t_3t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

221. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_1t_0N$.

Let $[0\bar{1}2310]$ denote the double coset $Nt_0t_1^{-1}t_2t_3t_1t_0N$.

Note that $N^{(0\bar{1}2310)} \geq N^{0\bar{1}2310} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}2310)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_1t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}2310)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}2310]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2310)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_3t_1t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_1t_0t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_1eN = Nt_0t_1^{-1}t_2t_3t_1N$ and $Nt_0t_1^{-1}t_2t_3t_1t_0t_0N = Nt_0t_1^{-1}t_2t_3t_1t_0^2N = Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_1t_0t_1N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_3t_1t_0t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2N$, and $Nt_0t_1^{-1}t_2t_3t_1t_0t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2t_3t_1t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2t_3t_1t_0t_2N$.

222. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$.

Let $[0\bar{1}231\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$.

Note that $N^{(0\bar{1}231\bar{0})} \geq N^{0\bar{1}231\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}231\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}231\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}231\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}231\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_3t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_3t_1eN = Nt_0t_1^{-1}t_2t_3t_1N$ and $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_0^{-2}N = Nt_0t_1^{-1}t_2t_3t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$, and $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

223. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_1t_2N$.

Let [$0\bar{1}2312$] denote the double coset $Nt_0t_1^{-1}t_2t_3t_1t_2N$.

Note that $N^{(0\bar{1}2312)} \geq N^{0\bar{1}2312} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}2312)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_1t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}2312)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [$0\bar{1}2312$] has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}2312)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_3t_1t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_1t_2t_2^{-1}N = Nt_0t_1^{-1}t_2t_3t_1eN = Nt_0t_1^{-1}t_2t_3t_1N$ and $Nt_0t_1^{-1}t_2t_3t_1t_2t_2N = Nt_0t_1^{-1}t_2t_3t_1t_2^2N = Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_1t_2t_0N =$ $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_2t_0^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_2t_1N$ $= Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_2t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_3t_1t_2t_3N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_3t_1t_2t_3^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_1t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

224. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$.

Let [$0\bar{1}231\bar{2}$] denote the double coset $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$.

Note that $N^{(0\bar{1}231\bar{2})} \geq N^{0\bar{1}231\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}231\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}231\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [$0\bar{1}231\bar{2}$] has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}231\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_3t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_3t_1eN = Nt_0t_1^{-1}t_2t_3t_1N$ and $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_2^{-2}N = Nt_0t_1^{-1}t_2t_3t_1t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_0N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_1N = Nt_0t_1^{-1}t_0t_2t_3t_0N$, $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_1^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$, $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, and $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

225. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$.

Let $[0\bar{1}23\bar{2}0]$ denote the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$.

Note that $N^{(0\bar{1}23\bar{2}0)} \geq N^{0\bar{1}23\bar{2}0} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}23\bar{2}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}23\bar{2}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}23\bar{2}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}23\bar{2}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_3t_2^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}eN = Nt_0t_1^{-1}t_2t_3t_2^{-1}N$ and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_0N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^2N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_1N =$ $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

226. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$.

Let $[0\bar{1}23\bar{2}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$.

Note that $N^{(0\bar{1}23\bar{2}\bar{0})} \geq N^{0\bar{1}23\bar{2}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}23\bar{2}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}23\bar{2}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}23\bar{2}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}23\bar{2}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_3t_2^{-1}eN = Nt_0t_1^{-1}t_2t_3t_2^{-1}N$ and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_1N =$ $Nt_0t_1t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_3N = Nt_0t_1^{-1}t_2t_1t_3t_0N$, and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

227. We next consider the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$.

Let $[0\bar{1}23\bar{2}1]$ denote the double coset $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$.

Note that $N^{(0\bar{1}23\bar{2}1)} \geq N^{0\bar{1}23\bar{2}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}23\bar{2}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}23\bar{2}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}23\bar{2}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}23\bar{2}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2t_3t_2^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}eN = Nt_0t_1^{-1}t_2t_3t_2^{-1}N$ and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_1N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^2N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_0N =$ $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_1N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_3N = Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$, and $Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

228. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}N$.

Let $[0\bar{1}23\bar{2}\bar{1}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}N$.

Note that $N^{(0\bar{1}23\bar{2}\bar{1})} \geq N^{0\bar{1}23\bar{2}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}23\bar{2}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}23\bar{2}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}23\bar{2}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}23\bar{2}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_1N = Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}eN = Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{1}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_0N =$
$Nt_0t_1t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_3N$, $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}N = Nt_0t_1t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_3t_1N$,
$Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_2N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_3t_{\bar{2}}^{-1}t_1N$, $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}N$
$= Nt_0t_{\bar{1}}^{-1}t_2t_3t_1t_2N$, $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_3N = Nt_0t_{\bar{1}}^{-1}t_2t_0t_3t_1N$, and
$Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{3}}^{-1}N = Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_2t_1t_0t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_3t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

229. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_2N$.

Let $[0\bar{1}\bar{2}012]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_2N$.

Note that $N^{(0\bar{1}\bar{2}012)} \geq N^{0\bar{1}\bar{2}012} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}012)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}012)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}012]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}012)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_0t_1t_2t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_0N =$ $Nt_0^{-1}t_1t_2t_0t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_1N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_3t_1t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$.

230. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$.

Let $[0\bar{1}\bar{2}01\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}01\bar{2})} \geq N^{0\bar{1}\bar{2}01\bar{2}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}01\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}01\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}01\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}01\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_0N =$ $Nt_0t_1t_2t_3t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_1N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

231. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3N$.

Let $[0\bar{1}\bar{2}013]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3 = Nt_1t_0^{-1}t_3^{-1}t_1t_0t_2$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}013 \sim 1\bar{0}\bar{3}102.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}013]$:

$$0\bar{1}\bar{2}013 \sim 1\bar{0}\bar{3}102, \quad 1\bar{0}\bar{2}103 \sim 0\bar{1}\bar{3}012, \quad 2\bar{1}\bar{0}213 \sim 1\bar{2}\bar{3}120,$$

$$3\bar{1}\bar{2}310 \sim 1\bar{3}\bar{0}132, \quad 0\bar{2}\bar{1}023 \sim 2\bar{0}\bar{3}201, \quad 0\bar{3}\bar{2}031 \sim 3\bar{0}\bar{1}302,$$

$$1\bar{2}\bar{0}123 \sim 2\bar{1}\bar{3}210, \quad 2\bar{0}\bar{1}203 \sim 0\bar{2}\bar{3}021, \quad 1\bar{3}\bar{2}130 \sim 3\bar{1}\bar{0}312,$$

$$3\bar{0}\bar{2}301 \sim 0\bar{3}\bar{1}032, \quad 2\bar{3}\bar{0}231 \sim 3\bar{2}\bar{1}320, \quad 2\bar{3}\bar{1}230 \sim 3\bar{2}\bar{0}321$$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}\bar{2}013]$ must have at most twelve distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1}\bar{2}013)} \geq N^{0\bar{1}\bar{2}013} = \langle e \rangle$. In fact, with the help of MAGMA, we know that $N(t_0t_1^{-1}t_2^{-1}t_0t_1t_3)^{(0\ 1)(2\ 3)} = Nt_1t_0^{-1}t_3^{-1}t_1t_0t_2 = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3$ implies that $(0\ 1)(2\ 3) \in N^{(0\bar{1}\bar{2}013)}$, and so $N^{(0\bar{1}\bar{2}013)} \geq \langle (0\ 1)(2\ 3) \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}013)}\right| \geq |\langle (0\ 1)(2\ 3) \rangle| = 2$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}013)}\right|} \leq \frac{24}{2} = 12$.

Therefore, as we concluded earlier, the double coset $[0\bar{1}\bar{2}013]$ has at most twelve distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}013)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,1\}$, $\{2,3\}$, $\{\bar{0},\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0t_1t_3t_i^{\pm 1}$, $i \in \{0,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3t_1^{-1}N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

232. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

Let $[0\bar{1}\bar{2}01\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}01\bar{3})} \geq N^{0\bar{1}\bar{2}01\bar{3}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}01\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}01\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}01\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}01\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_0N =$ $Nt_0t_1t_2^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_2t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_1N$ $= Nt_0t_1t_2^{-1}t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_2N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

233. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

Let $[0\bar{1}\bar{2}0\bar{1}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}0\bar{1}\bar{0})} \geq N^{0\bar{1}\bar{2}0\bar{1}\bar{0}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}0\bar{1}\bar{0})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}0\bar{1}\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}0\bar{1}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}0\bar{1}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0N$

$= Nt_0t_1^{-1}t_2^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_2N = Nt_0t_1t_2t_0t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_3N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

234. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

Let $[0\bar{1}\bar{2}03\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}03\bar{0})} \geq N^{0\bar{1}\bar{2}03\bar{0}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}03\bar{0})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}03\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}03\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}03\bar{0})}$ has eight orbits on $T = \{t_0,t_1,t_2,t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0t_3eN = Nt_0t_1^{-1}t_2^{-1}t_0t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0N$ $= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

235. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}1\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}1\bar{0})} \geq N^{0\bar{1}\bar{2}\bar{0}1\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}1\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}1\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}1\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}1\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0N$
$= Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_1N =$
$Nt_0t_1^{-1}t_2^{-1}t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_3^{-1}t_2N$,
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_2N = Nt_0t_1t_2t_0t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_2^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_3N = Nt_0t_1t_2t_3t_2^{-1}N$, and
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

236. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$.

Let $[0\bar{1}\bar{2}\bar{0}12]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}12)} \geq N^{0\bar{1}\bar{2}\bar{0}12} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}12)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}12)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}12]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}12)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_0N =$ $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_1^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_3N = Nt_0t_1^{-1}t_2^{-1}t_3t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

237. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$.

Let $[0\bar{1}\bar{2}\bar{0}13]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}13)} \geq N^{0\bar{1}\bar{2}\bar{0}13} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}13)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}13)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}13]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}13)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_1^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$.

238. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}1\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}1\bar{3})} \geq N^{0\bar{1}\bar{2}\bar{0}1\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}1\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}1\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}1\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}1\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_3t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_0t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_2N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N$.

239. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{1}0]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{1}0)} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{1}0} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{1}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{1}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{1}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}\bar{1}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_1N = Nt_0t_1t_2t_3^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

240. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{1}3]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{1}3)} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{1}3} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{1}3)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}\bar{1}3)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{1}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}\bar{1}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_0N = Nt_0^{-1}t_1^{-1}t_2t_3t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_0^{-1}N = Nt_0t_1t_2^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_2N = Nt_0t_1t_2t_0t_3t_2N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

241. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3})} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_0N = Nt_0t_1t_2t_3t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

242. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$.

Let $[0\bar{1}\bar{2}\bar{0}21]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}21)} \geq N^{0\bar{1}\bar{2}\bar{0}21} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}21)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}21)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}21]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}21)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_0N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_2N = Nt_0^{-1}t_1t_2t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

243. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}2\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}2\bar{1})} \geq N^{0\bar{1}\bar{2}\bar{0}2\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}2\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}2\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}2\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}2\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_0N =$ $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_0t_2t_3t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_3N = Nt_0t_1t_2^{-1}t_0^{-1}t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

244. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$.

Let $[0\bar{1}\bar{2}\bar{0}23]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}23)} \geq N^{0\bar{1}\bar{2}\bar{0}23} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}23)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}23)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}23]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}23)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3t_0N =$ $Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3^{-1}N$,

$Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_3t_1N = Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{3}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_3t_{\bar{1}}^{-1}N$ $= Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_3t_{\bar{2}}^{-1}N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_3t_2N = Nt_0t_1t_2t_3t_1N$, and $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_3t_{\bar{2}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_1t_{\bar{0}}^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

245. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}2\bar{3}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}2\bar{3})} \geq N^{0\bar{1}\bar{2}\bar{0}2\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}2\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}2\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}2\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}2\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_3N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2eN = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2N$ and $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{3}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-2}N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_0N =$ $Nt_0t_{\bar{1}}^{-1}t_2t_0t_{\bar{1}}^{-1}t_2N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{0}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_2t_3t_0N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_1N = Nt_0t_1t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_0N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{1}}^{-1}N$ $= Nt_0t_1t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_0t_1N$, $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_2N = Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_0N$, and $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}N = Nt_0t_1t_2t_{\bar{3}}^{-1}t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_2t_{\bar{3}}^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

246. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_3t_{\bar{0}}^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}3\bar{0}]$ denote the double coset $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_3t_{\bar{0}}^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}3\bar{0})} \geq N^{0\bar{1}\bar{2}\bar{0}3\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}3\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_3t_{\bar{0}}^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}3\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}3\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}3\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0N$ $= Nt_0^{-1}t_1t_2t_0t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_2N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_3N = Nt_0t_1^{-1}t_0t_2t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0t_2t_3t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

247. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$.

Let [0$\bar{1}\bar{2}\bar{0}$31] denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}31)} \geq N^{0\bar{1}\bar{2}\bar{0}31} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}31)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}31)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [0$\bar{1}\bar{2}\bar{0}$31] has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}31)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_0N =$ $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

248. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}3\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}3\bar{1})} \geq N^{0\bar{1}\bar{2}\bar{0}3\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}3\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}3\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}3\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}3\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_0N =$ $Nt_0t_1t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_3t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_2N = Nt_0t_1^{-1}t_2t_1t_3t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

249. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{3}0]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{3}0)} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}\bar{3}0)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_1N =$ $Nt_0t_1^{-1}t_2t_0t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_3N = Nt_0t_1^{-1}t_0t_2t_3t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

250. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{3}1]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{3}1)} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{3}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{3}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{3}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{3}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}\bar{3}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_i^{\pm 1}$; $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_0N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_2N = Nt_0t_1t_2^{-1}t_3t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

251. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})} \geq N^{0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N =$ $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_2N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_3N$ $= Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$.

252. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$.

Let $[0\bar{1}\bar{2}1\bar{0}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}1\bar{0}\bar{3})} \geq N^{0\bar{1}\bar{2}1\bar{0}\bar{3}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}1\bar{0}\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}1\bar{0}\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}1\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}1\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3N$ $= Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_0N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_2N = Nt_0^{-1}t_1t_2t_0t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

253. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$.

Let $[0\bar{1}\bar{2}132]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$.

Note that $N^{(0\bar{1}\bar{2}132)} \geq N^{0\bar{1}\bar{2}132} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}132)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}132)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}132]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}132)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_1t_3t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3eN = Nt_0t_1^{-1}t_2^{-1}t_1t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_2N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2^2N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_0N =$ $Nt_0^{-1}t_1t_2t_3t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_3N = Nt_0t_1t_0^{-1}t_2t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$.

254. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$.

Let $[0\bar{1}\bar{2}1\bar{3}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}1\bar{3}\bar{2})} \geq N^{0\bar{1}\bar{2}1\bar{3}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}1\bar{3}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}1\bar{3}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}1\bar{3}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}1\bar{3}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2N$
$= Nt_0t_1t_2t_0^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_0N =$
$Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$,
$Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_3N$
$= Nt_0t_1t_2t_0t_1^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0t_2t_3N$.

Therefore, we conclude that there is one distinct double coset of the form
$Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1^{-1}N$.

255. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$.

Let $[0\bar{1}\bar{2}301]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$.

Note that $N^{(0\bar{1}\bar{2}301)} \geq N^{0\bar{1}\bar{2}301} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}301)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}301)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}301]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}301)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3t_0t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0eN = Nt_0t_1^{-1}t_2^{-1}t_3t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_0N =$
$Nt_0^{-1}t_1t_2t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_0^{-1}N = Nt_0t_1t_2t_0t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_2N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$,
$Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1t_3^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_3t_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

256. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$.

Let $[0\bar{1}\bar{2}30\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}30\bar{1})} \geq N^{0\bar{1}\bar{2}30\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}30\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}30\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}30\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}30\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0eN = Nt_0t_1^{-1}t_2^{-1}t_3t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_0^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_1t_2N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_3^{-1}N = Nt_0t_1t_2t_3t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

257. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$.

Let $[0\bar{1}\bar{2}302]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2 = Nt_2t_1^{-1}t_0^{-1}t_3t_2t_0$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}302 \sim 2\bar{1}\bar{0}320.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}302]$:

$$0\bar{1}\bar{2}302 \sim 2\bar{1}\bar{0}320, \qquad 1\bar{0}\bar{2}312 \sim 2\bar{0}\bar{1}321, \qquad 3\bar{1}\bar{2}032 \sim 2\bar{1}\bar{3}023,$$

$0\bar{2}\bar{1}302 \sim 1\bar{2}\bar{0}310$, $\quad 0\bar{3}\bar{2}102 \sim 2\bar{3}\bar{0}120$, $\quad 0\bar{1}\bar{3}203 \sim 3\bar{1}\bar{0}230$,

$1\bar{3}\bar{2}012 \sim 2\bar{3}\bar{1}021$, $\quad 3\bar{0}\bar{2}132 \sim 2\bar{0}\bar{3}123$, $\quad 0\bar{2}\bar{3}103 \sim 3\bar{2}\bar{0}130$,

$0\bar{3}\bar{1}201 \sim 1\bar{3}\bar{0}210$, $\quad 1\bar{2}\bar{3}013 \sim 3\bar{2}\bar{1}031$, $\quad 3\bar{0}\bar{1}231 \sim 1\bar{0}\bar{3}213$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}\bar{2}302]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}302)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,2\}$, $\{1\}$, $\{3\}$, $\{\bar{0},\bar{2}\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3t_0t_2t_i^{\pm 1}$, $i \in \{1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0eN = Nt_0t_1^{-1}t_2^{-1}t_3t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2^2N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2t_1N =$
$Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$,
$Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2t_3N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2t_3^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

258. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}3\bar{0}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}3\bar{0}\bar{1})} \geq N^{0\bar{1}\bar{2}3\bar{0}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}3\bar{0}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N| = \frac{|N|}{|N^{(0\bar{1}\bar{2}3\bar{0}\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}3\bar{0}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}3\bar{0}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1N$ $= Nt_0t_1t_2t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_2N = Nt_0^{-1}t_1t_2t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_2^{-1}N$ $= Nt_0t_1t_2t_0^{-1}t_2t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

259. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Let $[0\bar{1}\bar{2}310]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Note that $N^{(0\bar{1}\bar{2}310)} \geq N^{0\bar{1}\bar{2}310} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{2}310)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}310)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}310]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}310)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3t_1t_0t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1eN = Nt_0t_1^{-1}t_2^{-1}t_3t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_1N =$ $Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_3N = Nt_0^{-1}t_1^{-1}t_2t_3t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

260. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$.

Let $[0\bar{1}\bar{2}31\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}31\bar{0})} \geq N^{0\bar{1}\bar{2}31\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}31\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}31\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}31\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}31\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_3t_1eN = Nt_0t_1^{-1}t_2^{-1}t_3t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_2t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

261. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$.

Let $[0\bar{1}\bar{2}312]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2 = Nt_0t_2^{-1}t_1^{-1}t_3t_2t_1$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}312 \sim 0\bar{2}\bar{1}321.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}312]$:

$$0\bar{1}\bar{2}312 \sim 0\bar{2}\bar{1}321, \qquad 1\bar{0}\bar{2}302 \sim 1\bar{2}\bar{0}320, \qquad 2\bar{1}\bar{0}310 \sim 2\bar{0}\bar{1}301,$$

$$3\bar{1}\bar{2}012 \sim 3\bar{2}\bar{1}021, \qquad 0\bar{3}\bar{2}132 \sim 0\bar{2}\bar{3}123, \qquad 0\bar{1}\bar{3}213 \sim 0\bar{3}\bar{1}231,$$

$1\bar{3}\bar{2}032 \sim 1\bar{2}\bar{3}023$, $\quad 3\bar{0}\bar{2}102 \sim 3\bar{2}\bar{0}120$, $\quad 2\bar{1}\bar{3}013 \sim 2\bar{3}\bar{1}031$,

$3\bar{1}\bar{0}210 \sim 3\bar{0}\bar{1}201$, $\quad 1\bar{3}\bar{0}230 \sim 1\bar{0}\bar{3}203$, $\quad 2\bar{0}\bar{3}103 \sim 2\bar{3}\bar{0}130$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}\bar{2}312]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}312)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1,2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1},\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3t_1t_2t_i^{\pm 1}$, $i \in \{0, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1eN = Nt_0t_1^{-1}t_2^{-1}t_3t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2^2N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2t_0N =$
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$,
$Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2t_3N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2t_3^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

262. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}0\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}0\bar{2})} \geq N^{0\bar{1}\bar{2}\bar{3}0\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}0\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}0\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}0\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}0\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2N$
$= Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_0N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_3N = Nt_0t_1t_2^{-1}t_0^{-1}t_1N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

263. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{0}1]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1 = Nt_0t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}t_2 = Nt_0t_3^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{3}\bar{0}1 \sim 0\bar{2}\bar{3}\bar{1}\bar{0}2 \sim 0\bar{3}\bar{1}\bar{2}\bar{0}3.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{3}\bar{0}1]$:

$$0\bar{1}\bar{2}\bar{3}\bar{0}1 \sim 0\bar{2}\bar{3}\bar{1}\bar{0}2 \sim 0\bar{3}\bar{1}\bar{2}\bar{0}3, \qquad 1\bar{0}\bar{2}\bar{3}\bar{1}0 \sim 1\bar{2}\bar{3}\bar{0}\bar{1}2 \sim 1\bar{3}\bar{0}\bar{2}\bar{1}3,$$

$$2\bar{1}\bar{0}\bar{3}\bar{2}1 \sim 2\bar{0}\bar{3}\bar{1}\bar{2}0 \sim 2\bar{3}\bar{1}\bar{0}\bar{2}3, \qquad 3\bar{1}\bar{2}\bar{0}\bar{3}1 \sim 3\bar{2}\bar{0}\bar{1}\bar{3}2 \sim 3\bar{0}\bar{1}\bar{2}\bar{3}0,$$

$$0\bar{1}\bar{3}\bar{2}\bar{0}1 \sim 0\bar{3}\bar{2}\bar{1}\bar{0}3 \sim 0\bar{2}\bar{1}\bar{3}\bar{0}2, \qquad 2\bar{1}\bar{3}\bar{0}\bar{2}1 \sim 2\bar{3}\bar{0}\bar{1}\bar{2}3 \sim 2\bar{0}\bar{1}\bar{3}\bar{2}0,$$

$$3\bar{1}\bar{0}\bar{2}\bar{3}1 \sim 3\bar{0}\bar{2}\bar{1}\bar{3}0 \sim 3\bar{2}\bar{1}\bar{0}\bar{3}2, \qquad 1\bar{0}\bar{3}\bar{2}\bar{1}0 \sim 1\bar{3}\bar{2}\bar{0}\bar{1}3 \sim 1\bar{2}\bar{0}\bar{3}\bar{1}2$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{0}1]$ must have at most eight distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{3}\bar{0}1)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2, 3\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1t_0N = Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

264. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1})} \geq N^{0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1t_0^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2N = Nt_0t_1t_2^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1}N = Nt_0t_1t_2t_0t_1^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

265. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2})} \geq N^{0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

266. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}1\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}1\bar{0})} \geq N^{0\bar{1}\bar{2}\bar{3}1\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}1\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}1\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}1\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}1\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0N$ $= Nt_0t_1t_0^{-1}t_2t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_3N$ $= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_3^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}N$.

267. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}1\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}1\bar{2})} \geq N^{0\bar{1}\bar{2}\bar{3}1\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}1\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}1\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}1\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}1\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2N$ $= Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_0^{-1}N = Nt_0t_1t_2t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_1N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

268. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{1}0]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0 = Nt_2t_1^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2 = Nt_3t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{3}\bar{1}0 \sim 2\bar{1}\bar{3}\bar{0}\bar{1}2 \sim 3\bar{1}\bar{0}\bar{2}\bar{1}3.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}0]$:

$$0\bar{1}\bar{2}\bar{3}\bar{1}0 \sim 2\bar{1}\bar{3}\bar{0}\bar{1}2 \sim 3\bar{1}\bar{0}\bar{2}\bar{1}3, \qquad 1\bar{0}\bar{2}\bar{3}\bar{0}1 \sim 2\bar{0}\bar{3}\bar{1}\bar{0}2 \sim 3\bar{0}\bar{1}\bar{2}\bar{0}3,$$

$$2\bar{1}0\bar{3}\bar{1}2 \sim 0\bar{1}\bar{3}2\bar{1}0 \sim 3\bar{1}\bar{2}0\bar{1}3, \qquad 0\bar{2}\bar{1}\bar{3}20 \sim 1\bar{2}\bar{3}0\bar{2}1 \sim 3\bar{2}\bar{0}\bar{1}23,$$

$$0\bar{3}\bar{2}\bar{1}30 \sim 2\bar{3}\bar{1}0\bar{3}2 \sim 1\bar{3}0\bar{2}31, \qquad 0\bar{2}\bar{3}\bar{1}20 \sim 3\bar{2}\bar{1}0\bar{2}3 \sim 1\bar{2}0\bar{3}\bar{2}1,$$

$$0\bar{3}\bar{1}\bar{2}30 \sim 1\bar{3}\bar{2}0\bar{3}1 \sim 2\bar{3}0\bar{1}32, \qquad 1\bar{0}\bar{3}\bar{2}01 \sim 3\bar{0}\bar{2}\bar{1}03 \sim 2\bar{0}\bar{1}\bar{3}02$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}0]$ must have at most eight distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{3}\bar{1}0)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 2, 3\}$, $\{1\}$, $\{\bar{0}, \bar{2}, \bar{3}\}$, and $\{\bar{1}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N$ $= Nt_0^{-1}t_1t_2t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N$.

269. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{1}2]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}\bar{1}2)} \geq N^{0\bar{1}\bar{2}\bar{3}\bar{1}2} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{1}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{1}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}\bar{1}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N =$ $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_1N$,

$Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_1N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_3N = Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

270. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2})} \geq N^{0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}eN = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_0N =$ $Nt_0t_1t_2t_0t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1t_2t_0t_1t_3N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_1N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_3N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N = Nt_0t_1t_2t_3t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

271. We next consider the double coset $Nt_0t_1^{-1}t_0t_2t_3t_0N$.

Let $[0\bar{1}0230]$ denote the double coset $Nt_0t_1^{-1}t_0t_2t_3t_0N$.

Note that $N^{(0\bar{1}0230)} \geq N^{0\bar{1}0230} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}0230)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0t_2t_3t_0N\right| = \frac{|N|}{|N^{(0\bar{1}0230)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}0230]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}0230)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0t_2t_3t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0t_2t_3t_0t_0^{-1}N = Nt_0t_1^{-1}t_0t_2t_3eN = Nt_0t_1^{-1}t_0t_2t_3N$ and $Nt_0t_1^{-1}t_0t_2t_3t_0t_0N = Nt_0t_1^{-1}t_0t_2t_3t_0^2N = Nt_0t_1^{-1}t_0t_2t_3t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0t_2t_3t_0t_1N =$ $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_0t_2t_3t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N$, $Nt_0t_1^{-1}t_0t_2t_3t_0t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$, $Nt_0t_1^{-1}t_0t_2t_3t_0t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_0t_2t_3t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1^{-1}N$, and $Nt_0t_1^{-1}t_0t_2t_3t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0t_2t_3t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

272. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$.

Let $[0\bar{1}\bar{0}230]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$.

Note that $N^{(0\bar{1}\bar{0}230)} \geq N^{0\bar{1}\bar{0}230} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{0}230)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}230)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}230]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}230)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2t_3t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_0^{-1}N = Nt_0t_1^{-1}t_0{-}1t_2t_3eN = Nt_0t_1^{-1}t_0{-}1t_2t_3N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_0N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0^2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_1N =$ $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_1^{-1}N = Nt_0^{-1}t_1t_2t_0t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_2^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

273. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$.

Let $[0\bar{1}\bar{0}23\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}23\bar{1})} \geq N^{0\bar{1}\bar{0}23\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}23\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}23\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}23\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}23\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_1N = Nt_0t_1^{-1}t_0{-}1t_2t_3eN = Nt_0t_1^{-1}t_0{-}1t_2t_3N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_0N =$ $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_1t_3t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_2N = Nt_0t_1^{-1}t_2t_3t_1t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_3t_1t_0t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_3N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_3^{-1}N = Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

274. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}N$.

Let $[0\bar{1}\bar{0}23\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1} = Nt_1t_0^{-1}t_1^{-1}t_3t_2t_3^{-1} = Nt_2t_3^{-1}t_2^{-1}t_1t_0t_1^{-1}$ $= Nt_3t_2^{-1}t_3^{-1}t_0t_1t_0^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{0}23\bar{2} \sim 1\bar{0}\bar{1}32\bar{3} \sim 2\bar{3}\bar{2}10\bar{1} \sim 3\bar{2}\bar{3}01\bar{0}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{0}23\bar{2}]$:

$$0\bar{1}\bar{0}23\bar{2} \sim 1\bar{0}\bar{1}32\bar{3} \sim 2\bar{3}\bar{2}10\bar{1} \sim 3\bar{2}\bar{3}01\bar{0}, \qquad 1\bar{0}\bar{1}23\bar{2} \sim 0\bar{1}\bar{0}32\bar{3} \sim 2\bar{3}\bar{2}01\bar{0} \sim 3\bar{2}\bar{3}10\bar{1},$$

$$2\bar{1}\bar{2}03\bar{0} \sim 1\bar{2}\bar{1}30\bar{3} \sim 0\bar{3}\bar{0}12\bar{1} \sim 3\bar{0}\bar{3}21\bar{2}, \qquad 3\bar{1}\bar{3}20\bar{2} \sim 1\bar{3}\bar{1}02\bar{0} \sim 2\bar{0}\bar{2}13\bar{1} \sim 0\bar{2}\bar{0}31\bar{3},$$

$$0\bar{2}\bar{0}13\bar{1} \sim 2\bar{0}\bar{2}31\bar{3} \sim 1\bar{3}\bar{1}20\bar{2} \sim 3\bar{1}\bar{3}02\bar{0}, \qquad 0\bar{3}\bar{0}21\bar{2} \sim 3\bar{0}\bar{3}12\bar{1} \sim 2\bar{1}\bar{2}30\bar{3} \sim 1\bar{2}\bar{1}03\bar{0}$$

Since each of the twenty-four single cosets has four names, the double coset $[0\bar{1}\bar{0}23\bar{2}]$ must have at most six distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}23\bar{2})}$ has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}t_i^{\pm 1}$, $i = 2$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3eN = Nt_0t_1^{-1}t_0^{-1}t_2t_3N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2N$ $= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2t_3t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

275. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{0}2\bar{3}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}2\bar{3}\bar{1})} \geq N^{0\bar{1}\bar{0}2\bar{3}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}2\bar{3}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N| = \frac{|N|}{|N^{(0\bar{1}\bar{0}2\bar{3}\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}2\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}2\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_0{-}1t_2t_3^{-1}eN = Nt_0t_1^{-1}t_0{-}1t_2t_3^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1N$ $= Nt_0t_1t_2t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_3N = Nt_0t_1t_2t_3t_1t_0N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

276. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$.

Let $[0\bar{1}\bar{0}2\bar{3}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1} = Nt_0t_1^{-1}t_0^{-1}t_3t_2^{-1}t_3^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{0}2\bar{3}\bar{2} \sim 0\bar{1}\bar{0}3\bar{2}\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{0}2\bar{3}\bar{2}]$:

$$0\bar{1}\bar{0}2\bar{3}\bar{2} \sim 0\bar{1}\bar{0}3\bar{2}\bar{3}, \quad 1\bar{0}\bar{1}2\bar{3}\bar{2} \sim 1\bar{0}\bar{1}3\bar{2}\bar{3}, \quad 2\bar{1}\bar{2}0\bar{3}\bar{0} \sim 2\bar{1}\bar{2}3\bar{0}\bar{3},$$

$$3\bar{1}\bar{3}2\bar{0}\bar{2} \sim 3\bar{1}\bar{3}0\bar{2}\bar{0}, \quad 0\bar{2}\bar{0}1\bar{3}\bar{1} \sim 0\bar{2}\bar{0}3\bar{1}\bar{3}, \quad 0\bar{3}\bar{0}2\bar{1}\bar{2} \sim 0\bar{3}\bar{0}1\bar{2}\bar{1},$$

$$1\bar{2}\bar{1}0\bar{3}\bar{0} \sim 1\bar{2}\bar{1}3\bar{0}\bar{3}, \quad 2\bar{0}\bar{2}1\bar{3}\bar{1} \sim 2\bar{0}\bar{2}3\bar{1}\bar{3}, \quad 1\bar{3}\bar{1}2\bar{0}\bar{2} \sim 1\bar{3}\bar{1}0\bar{2}\bar{0},$$

$$3\bar{0}\bar{3}2\bar{1}\bar{2} \sim 3\bar{0}\bar{3}1\bar{2}\bar{1}, \quad 3\bar{2}\bar{3}0\bar{1}\bar{0} \sim 3\bar{2}\bar{3}1\bar{0}\bar{1}, \quad 2\bar{3}\bar{2}1\bar{0}\bar{1} \sim 2\bar{3}\bar{2}0\bar{1}\bar{0}$$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}\bar{0}2\bar{3}\bar{2}]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}\bar{0}2\bar{3}\bar{2})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}eN = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2N$ $= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_0N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$, and $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_1^{-1}N$ $= Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

277. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$.

Let $[0\bar{1}\bar{0}\bar{2}13]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}13)} \geq N^{0\bar{1}\bar{0}\bar{2}13} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}13)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{0}\bar{2}13)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}13]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}13)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_3^{-1}N = Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1eN = Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_0N =$ $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_1^{-1}N$ $= Nt_0t_1t_2t_3t_1t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

278. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$.

Let $[0\bar{1}\bar{0}\bar{2}1\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}1\bar{3})} \geq N^{0\bar{1}\bar{0}\bar{2}1\bar{3}} = \langle e\rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}1\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{0}\bar{2}1\bar{3})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}1\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}1\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_3N = Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1eN = Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-2}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_0N =$ $Nt_0t_1t_2t_3^{-1}t_2N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_2t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_1N = Nt_0t_1t_2t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_1^{-1}N$ $= Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_3N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

279. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$.

Let $[0\bar{1}\bar{0}\bar{2}\bar{1}0]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}\bar{1}0)} \geq N^{0\bar{1}\bar{0}\bar{2}\bar{1}0} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{1}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{1}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}\bar{1}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}\bar{1}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1^{-1}eN$ $= Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_0N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0^2N$ $= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_1N =$ $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_2N = Nt_0^{-1}t_1t_0^{-1}t_2t_3N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_2^{-1}N$ $= Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_3N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$, and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

280. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$.

Let $[0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3})} \geq N^{0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1^{-1}eN$
$= Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-2}N$
$= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_0N =$
$Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_0t_3^{-1}t_0N$,
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_1^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3N$, and
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

281. We next consider the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$.

Let $[0\bar{1}\bar{0}\bar{2}\bar{3}1]$ denote the double coset $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$.

Note that $N^{(0\bar{1}\bar{0}\bar{2}\bar{3}1)} \geq N^{0\bar{1}\bar{0}\bar{2}\bar{3}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{3}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{0}\bar{2}\bar{3}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{0}\bar{2}\bar{3}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{0}\bar{2}\bar{3}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_3^{-1}eN$
$= Nt_0t_1^{-1}t_0{-}1t_2^{-1}t_3^{-1}N$ and $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_1N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1^2N$
$= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_0N =$
$Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$,
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_2N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$
$= Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$, $Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_3N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$, and
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_3^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form
$Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

282. We next consider the double coset $Nt_0t_1t_2t_0t_1t_3N$.

Let [012013] denote the double coset $Nt_0t_1t_2t_0t_1t_3N$.

Note that $N^{(012013)} \geq N^{012013} = \langle e \rangle$. Thus $|N^{(012013)}| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_0t_1t_3N| = \frac{|N|}{|N^{(012013)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [012013] has at most twenty-four distinct single cosets.

Moreover, $N^{(012013)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0t_1t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_1t_3t_3^{-1}N = Nt_0t_1t_2t_0t_1eN = Nt_0t_1t_2t_0t_1N$ and
$Nt_0t_1t_2t_0t_1t_3t_3N = Nt_0t_1t_2t_0t_1t_3^2N = Nt_0t_1t_2t_0t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_1t_3t_0N =$
$Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$, $Nt_0t_1t_2t_0t_1t_3t_0^{-1}N = Nt_0^{-1}t_1t_2t_3t_1N$, $Nt_0t_1t_2t_0t_1t_3t_1N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_0t_1t_3t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$,
$Nt_0t_1t_2t_0t_1t_3t_2N = Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$, and $Nt_0t_1t_2t_0t_1t_3t_2^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form
$Nt_0t_1t_2t_0t_1t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

283. We next consider the double coset $Nt_0t_1t_2t_0t_1^{-1}t_2N$.

Let [$012\bar{0}\bar{1}2$]...

Let $[0120\bar{1}2]$ denote the double coset $Nt_0t_1t_2t_0t_1^{-1}t_2N$.

Note that $N^{(0120\bar{1}2)} \geq N^{0120\bar{1}2} = \langle e \rangle$. Thus $\left|N^{(0120\bar{1}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0t_1^{-1}t_2N\right| = \frac{|N|}{\left|N^{(0120\bar{1}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0120\bar{1}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0120\bar{1}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0t_1^{-1}t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_1^{-1}t_2t_2^{-1}N = Nt_0t_1t_2t_0t_1^{-1}eN = Nt_0t_1t_2t_0t_1^{-1}N$ and $Nt_0t_1t_2t_0t_1^{-1}t_2t_2N = Nt_0t_1t_2t_0t_1^{-1}t_2^2N = Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_1^{-1}t_2t_0N =$ $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}N$, $Nt_0t_1t_2t_0t_1^{-1}t_2t_0^{-1}N = Nt_0^{-1}t_1t_2t_3t_0N$, $Nt_0t_1t_2t_0t_1^{-1}t_2t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_0t_1^{-1}t_2t_1^{-1}N$ $= Nt_0t_1t_2^{-1}t_0t_1N$, $Nt_0t_1t_2t_0t_1^{-1}t_2t_3N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}N$, and $Nt_0t_1t_2t_0t_1^{-1}t_2t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0t_1^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

284. We next consider the double coset $Nt_0t_1t_2t_0t_3t_0^{-1}N$.

Let $[01203\bar{0}]$ denote the double coset $Nt_0t_1t_2t_0t_3t_0^{-1}N$.

Note that $N^{(01203\bar{0})} \geq N^{01203\bar{0}} = \langle e \rangle$. Thus $\left|N^{(01203\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0t_3t_0^{-1}N\right| = \frac{|N|}{\left|N^{(01203\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01203\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01203\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_3t_0^{-1}t_0N = Nt_0t_1t_2t_0t_3eN = Nt_0t_1t_2t_0t_3N$ and $Nt_0t_1t_2t_0t_3t_0^{-1}t_0^{-1}N = Nt_0t_1t_2t_0t_3t_0^{-2}N = Nt_0t_1t_2t_0t_3t_0N$

$= Nt_0t_1t_2t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_3t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_0t_3t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_0t_3t_0^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$, $Nt_0t_1t_2t_0t_3t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_0t_3t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$, and $Nt_0t_1t_2t_0t_3t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

285. We next consider the double coset $Nt_0t_1t_2t_0t_3t_2N$.

Let [012032] denote the double coset $Nt_0t_1t_2t_0t_3t_2N$.

Note that the point stabilizer is $N^{012032} = \{n \in N \mid (t_0t_1t_2t_0t_3t_2)^n = t_0t_1t_2t_0t_3t_2\} =$ $\langle e \rangle$ and, moreover, that the coset stabilizer is $N^{(012032)} \geq N^{012032} = \langle e \rangle$. Thus $|N^{(012032)}| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_0t_3t_2N| = \frac{|N|}{|N^{(012032)}|} \leq \frac{24}{1} =$ 24.

Therefore, the double coset [012032] has at most twenty-four distinct single cosets.

Moreover, $N^{(012032)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0t_3t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_3t_2t_2^{-1}N = Nt_0t_1t_2t_0t_3eN = Nt_0t_1t_2t_0t_3N$ and $Nt_0t_1t_2t_0t_3t_2t_2N = Nt_0t_1t_2t_0t_3t_2^2N = Nt_0t_1t_2t_0t_3t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_3t_2t_0N =$ $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$, $Nt_0t_1t_2t_0t_3t_2t_0^{-1}N = Nt_0t_1t_2^{-1}t_1t_3t_0N$, $Nt_0t_1t_2t_0t_3t_2t_1N$ $= Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}N$, $Nt_0t_1t_2t_0t_3t_2t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1t_2t_0t_3t_2t_3N = Nt_0t_1t_2^{-1}t_1t_0^{-1}N$, and $Nt_0t_1t_2t_0t_3t_2t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0t_3t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

286. We next consider the double coset $Nt_0t_1t_2t_0t_3^{-1}t_0N$.

Let $[01203\bar{3}0]$ denote the double coset $Nt_0t_1t_2t_0t_3^{-1}t_0N$.

Note that $N^{(01203\bar{3}0)} \geq N^{01203\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(01203\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0t_3^{-1}t_0N\right| = \frac{|N|}{\left|N^{(01203\bar{3}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01203\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01203\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0t_3^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_3^{-1}t_0t_0^{-1}N = Nt_0t_1t_2t_0t_3^{-1}eN = Nt_0t_1t_2t_0t_3^{-1}N$ and $Nt_0t_1t_2t_0t_3^{-1}t_0t_0N = Nt_0t_1t_2t_0t_3^{-1}t_0^2N = Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_3^{-1}t_0t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_0t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_0t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_0t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$, and $Nt_0t_1t_2t_0t_3^{-1}t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0t_3^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

287. We next consider the double coset $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$.

Let $[01203\bar{3}\bar{0}]$ denote the double coset $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$.

Note that $N^{(01203\bar{3}\bar{0})} \geq N^{01203\bar{3}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(01203\bar{3}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(01203\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01203\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01203\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0t_3^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_0N = Nt_0t_1t_2t_0t_3^{-1}eN = Nt_0t_1t_2t_0t_3^{-1}N$ and $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1t_2t_0t_3^{-1}t_0^{-2}N = Nt_0t_1t_2t_0t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$, $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$, and $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

288. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_2t_1N$.

Let $[012\bar{0}21]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_2t_1N$.

Note that $N^{(012\bar{0}21)} \geq N^{012\bar{0}21} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}21)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_2t_1N\right| = \frac{|N|}{\left|N^{(012\bar{0}21)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}21]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}21)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0^{-1}t_2t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_2t_1t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_2eN = Nt_0t_1t_2t_0^{-1}t_2N$ and $Nt_0t_1t_2t_0^{-1}t_2t_1t_1N = Nt_0t_1t_2t_0^{-1}t_2t_1^2N = Nt_0t_1t_2t_0^{-1}t_2t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_2t_1t_0N =$ $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2t_1t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2t_1t_3N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$, and $Nt_0t_1t_2t_0^{-1}t_2t_1t_3^{-1}N = Nt_0^{-1}t_1t_2t_0t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_2t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

289. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$.

Let $[012\bar{0}\bar{2}1]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$.

Note that $N^{(012\bar{0}\bar{2}1)} \geq N^{012\bar{0}\bar{2}1} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}\bar{2}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N\right| = \frac{|N|}{\left|N^{(012\bar{0}\bar{2}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}\bar{2}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}\bar{2}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0^{-1}t_2^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}eN = Nt_0t_1t_2t_0^{-1}t_2^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_1N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^2N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_0N = Nt_0t_1t_2^{-1}t_3t_0N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_0^{-1}N = Nt_0t_1t_2^{-1}t_3t_0t_1N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_2N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, and $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

290. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$.

Let $[012\bar{0}\bar{2}\bar{1}]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$.

Note that $N^{(012\bar{0}\bar{2}\bar{1})} \geq N^{012\bar{0}\bar{2}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}\bar{2}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(012\bar{0}\bar{2}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}\bar{2}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}\bar{2}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_1N = Nt_0t_1t_2t_0^{-1}t_2^{-1}eN = Nt_0t_1t_2t_0^{-1}t_2^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-2}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_2N$, $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$, and $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

291. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$.

Let $[012\bar{0}\bar{3}0]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$.

Note that $N^{(012\bar{0}\bar{3}0)} \geq N^{012\bar{0}\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(012\bar{0}\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N\right| = \frac{|N|}{\left|N^{(012\bar{0}\bar{3}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{0}\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{0}\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0^{-1}t_3^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}eN = Nt_0t_1t_2t_0^{-1}t_3^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_0N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0^2N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0^{-1}N$
$= Nt_0t_1t_2t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_1N =$
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$,
$Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_2N = Nt_0t_1t_2t_3t_1N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_2^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_3N = Nt_0t_1t_2t_3^{-1}t_2t_1N$, and
$Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_3^{-1}N = Nt_0t_1t_2^{-1}t_3t_0t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

292. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$.

Let $[012\bar{0}\bar{3}1]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1 = Nt_0t_2t_1t_0^{-1}t_3^{-1}t_2$.

That is, in terms of our short-hand notation,

$$012\bar{0}\bar{3}1 \sim 021\bar{0}\bar{3}2.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[012\bar{0}\bar{3}1]$:

$$012\bar{0}\bar{3}1 \sim 021\bar{0}\bar{3}2, \qquad 102\bar{1}\bar{3}0 \sim 120\bar{1}\bar{3}2, \qquad 210\bar{2}\bar{3}1 \sim 201\bar{2}\bar{3}0,$$

$$312\bar{3}\bar{0}1 \sim 321\bar{3}\bar{0}2, \qquad 032\bar{0}\bar{1}3 \sim 023\bar{0}\bar{1}2, \qquad 013\bar{0}\bar{2}1 \sim 031\bar{0}\bar{2}3,$$

$$132\bar{1}\bar{0}3 \sim 123\bar{1}\bar{0}2, \qquad 302\bar{3}\bar{1}0 \sim 320\bar{3}\bar{1}2, \qquad 213\bar{2}\bar{0}1 \sim 231\bar{2}\bar{0}3,$$

$$310\bar{3}\bar{2}1 \sim 301\bar{3}\bar{2}0, \qquad 130\bar{1}\bar{2}3 \sim 103\bar{1}\bar{2}0, \qquad 203\bar{2}\bar{1}0 \sim 230\bar{2}\bar{1}3$$

Since each of the twenty-four single cosets has two names, the double coset $[012\bar{0}\bar{3}1]$ must have at most twelve distinct single cosets.

Now, $N^{(012\bar{0}\bar{3}1)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}, \bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_0^{-1}t_3^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}eN = Nt_0t_1t_2t_0^{-1}t_3^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_1N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1^2N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1^{-1}N$ $= Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_3N =$ $Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$.

Therefore, we conclude that there are two distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$.

293. We next consider the double coset $Nt_0t_1t_2t_3t_0t_2N$.

Let [012302] denote the double coset $Nt_0t_1t_2t_3t_0t_2N$.

Note that $N^{(012302)} \geq N^{012302} = \langle e \rangle$. Thus $|N^{(012302)}| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_3t_0t_2N| = \frac{|N|}{|N^{(012302)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [012302] has at most twenty-four distinct single cosets.

Moreover, $N^{(012302)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3t_0t_2t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_3t_0t_2t_2^{-1}N = Nt_0t_1t_2t_3t_0eN = Nt_0t_1t_2t_3t_0N$ and $Nt_0t_1t_2t_3t_0t_2t_2N = Nt_0t_1t_2t_3t_0t_2^2N = Nt_0t_1t_2t_3t_0t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_0t_2t_0N = Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_3t_0t_2t_0^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_3t_0t_2t_1N = Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$, $Nt_0t_1t_2t_3t_0t_2t_1^{-1}N = Nt_0^{-1}t_1t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3t_0t_2t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$, and $Nt_0t_1t_2t_3t_0t_2t_3^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3t_0t_2t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

294. We next consider the double coset $Nt_0t_1t_2t_3t_0t_2^{-1}N$.

Let $[01230\bar{2}]$ denote the double coset $Nt_0t_1t_2t_3t_0t_2^{-1}N$.

Note that $N^{(01230\bar{2})} \geq N^{01230\bar{2}} = \langle e\rangle$. Thus $\left|N^{(01230\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3t_0t_2^{-1}N\right| = \frac{|N|}{\left|N^{(01230\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01230\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01230\bar{2})}$ has eight orbits on $T = \{t_0,t_1,t_2,t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3t_0t_2^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_3t_0t_2^{-1}t_2N = Nt_0t_1t_2t_3t_0eN = Nt_0t_1t_2t_3t_0N$ and $Nt_0t_1t_2t_3t_0t_2^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0t_2^{-2}N = Nt_0t_1t_2t_3t_0t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_0t_2^{-1}t_0N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3t_0t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1t_2t_3t_0t_2^{-1}t_1N = Nt_0t_1^{-1}t_2t_3t_1t_0t_2N$, $Nt_0t_1t_2t_3t_0t_2^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$, $Nt_0t_1t_2t_3t_0t_2^{-1}t_3N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$, and $Nt_0t_1t_2t_3t_0t_2^{-1}t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3t_0t_2^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

295. We next consider the double coset $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$.

Let $[01230\bar{2}]$ denote the double coset $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$.

Note that $N^{(0123\bar{0}\bar{2})} \geq N^{0123\bar{0}\bar{2}} = \langle e\rangle$. Thus $\left|N^{(0123\bar{0}\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0123\bar{0}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0123\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0123\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3t_0^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_2N = Nt_0t_1t_2t_3t_0^{-1}eN = Nt_0t_1t_2t_3t_0^{-1}N$ and $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-2}N = Nt_0t_1t_2t_3t_0^{-1}t_2N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_0N =$ $Nt_0^{-1}t_1t_2t_3t_0N$, $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_1N$ $= Nt_0t_1t_2t_3t_0t_2N$, $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$, $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_3N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_2^{-1}N$, and $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_3^{-1}N$ $= Nt_0^{-1}t_1t_2t_0t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

296. We next consider the double coset $Nt_0t_1t_2t_3t_0^{-1}t_3N$.

Let $[0123\bar{0}3]$ denote the double coset $Nt_0t_1t_2t_3t_0^{-1}t_3N$.

Note that $N^{(0123\bar{0}3)} \geq N^{0123\bar{0}3} = \langle e\rangle$. Thus $\left|N^{(0123\bar{0}3)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3t_0^{-1}t_3N\right| = \frac{|N|}{\left|N^{(0123\bar{0}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0123\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0123\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3t_0^{-1}t_3t_3^{-1}N = Nt_0t_1t_2t_3t_0^{-1}eN = Nt_0t_1t_2t_3t_0^{-1}N$ and $Nt_0t_1t_2t_3t_0^{-1}t_3t_3N = Nt_0t_1t_2t_3t_0^{-1}t_3^2N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_0^{-1}t_3t_0N = Nt_0t_1t_2^{-1}t_3t_0t_1N$, $Nt_0t_1t_2t_3t_0^{-1}t_3t_0^{-1}N = Nt_0t_1t_0^{-1}t_2t_1N$, $Nt_0t_1t_2t_3t_0^{-1}t_3t_1N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_3t_0^{-1}t_3t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2t_3t_0^{-1}t_3t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$, and $Nt_0t_1t_2t_3t_0^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3t_0^{-1}t_3t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

297. We next consider the double coset $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$.

Let $[01230\bar{3}]$ denote the double coset $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N$.

Note that $N^{(0123\bar{0}\bar{3})} \geq N^{0123\bar{0}\bar{3}} = \langle e\rangle$. Thus $\left|N^{(0123\bar{0}\bar{3})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0123\bar{0}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0123\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0123\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3t_0^{-1}t_3^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_3N = Nt_0t_1t_2t_3t_0^{-1}eN = Nt_0t_1t_2t_3t_0^{-1}N$ and $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_3^{-2}N = Nt_0t_1t_2t_3t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$, $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_0N$, $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1^{-1}N$, and $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3t_0^{-1}t_3^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

298. We next consider the double coset $Nt_0t_1t_2t_3t_1t_0N$.

Let $[012310]$ denote the double coset $Nt_0t_1t_2t_3t_1t_0N$.

Note that $N^{(012310)} \geq N^{012310} = \langle e \rangle$. Thus $|N^{(012310)}| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0t_1t_2t_3t_1t_0N| = \frac{|N|}{|N^{(012310)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset [012310] has at most twenty-four distinct single cosets.

Moreover, $N^{(012310)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3t_1t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3t_1t_0t_0^{-1}N = Nt_0t_1t_2t_3t_1eN = Nt_0t_1t_2t_3t_1N$ and $Nt_0t_1t_2t_3t_1t_0t_0N = Nt_0t_1t_2t_3t_1t_0^2N = Nt_0t_1t_2t_3t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3t_1t_0t_1N =$ $Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1N$, $Nt_0t_1t_2t_3t_1t_0t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3t_1t_0t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2t_3t_1t_0t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$, $Nt_0t_1t_2t_3t_1t_0t_3N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$, and $Nt_0t_1t_2t_3t_1t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3t_1t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

299. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$.

Let $[012\bar{3}\bar{0}1]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$.

Note that $N^{(012\bar{3}\bar{0}1)} \geq N^{012\bar{3}\bar{0}1} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}\bar{0}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N\right| = \frac{|N|}{|N^{(012\bar{3}\bar{0}1)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}\bar{0}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}\bar{0}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3^{-1}t_0^{-1}t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_1^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}eN = Nt_0t_1t_2t_3^{-1}t_0^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_1N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1^2N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_0N =$ $Nt_0^{-1}t_1t_2t_0t_1N$, $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$,

$Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_2N = Nt_0t_1t_2^{-1}t_1t_3t_0N$, $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_2^{-1}N$
$= Nt_0t_1t_2t_0t_3t_2N$, and $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$: $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}N$.

300. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_2t_1N$.

Let $[012\bar{3}21]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_2t_1N$.

Note that $N^{(012\bar{3}21)} \geq N^{012\bar{3}21} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}21)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_2t_1N\right| = \frac{|N|}{\left|N^{(012\bar{3}21)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}21]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}21)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3^{-1}t_2t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_2t_1t_1^{-1}N = Nt_0t_1t_2t_3^{-1}t_2eN = Nt_0t_1t_2t_3^{-1}t_2N$ and
$Nt_0t_1t_2t_3^{-1}t_2t_1t_1N = Nt_0t_1t_2t_3^{-1}t_2t_1^2N = Nt_0t_1t_2t_3^{-1}t_2t_1^{-1}N$
$= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_2t_1t_0N =$
$Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2t_1t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$,
$Nt_0t_1t_2t_3^{-1}t_2t_1t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2t_1t_2^{-1}N = Nt_0t_1t_2^{-1}t_1t_0N$,
$Nt_0t_1t_2t_3^{-1}t_2t_1t_3N = Nt_0t_1t_2^{-1}t_3t_0t_1N$, and $Nt_0t_1t_2t_3^{-1}t_2t_1t_3^{-1}N$
$= Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3^{-1}t_2t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

301. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$.

Let $[012\bar{3}\bar{2}\bar{0}]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$.

Note that $N^{(012\bar{3}\bar{2}\bar{0})} \geq N^{012\bar{3}\bar{2}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(012\bar{3}\bar{2}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(012\bar{3}\bar{2}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[012\bar{3}\bar{2}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(012\bar{3}\bar{2}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_0N = Nt_0t_1t_2t_3^{-1}t_2^{-1}eN = Nt_0t_1t_2t_3^{-1}t_2^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-2}N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0N$ $= Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_1N =$ $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$, $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_3N$, $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$, $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$, and $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

302. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$.

Let $[01\bar{2}\bar{0}\bar{1}3]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$.

Note that $N^{(01\bar{2}\bar{0}\bar{1}3)} \geq N^{01\bar{2}\bar{0}\bar{1}3} = \langle e\rangle$. Thus $\left|N^{(01\bar{2}\bar{0}\bar{1}3)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{0}\bar{1}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}\bar{1}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0}\bar{1}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_3^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}eN = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_3N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3^2N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$ $= Nt_0t_1t_2t_3t_0t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_0N =$ $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_1N = Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_1^{-1}N$

$= Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_2N$
$= Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$, and $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_2^{-1}N = Nt_0^{-1}t_1t_2t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

303. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$.

Let $[01\bar{2}\bar{0}31]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$.

Note that $N^{(01\bar{2}\bar{0}31)} \geq N^{01\bar{2}\bar{0}31} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{0}31)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N\right| = \frac{|N|}{|N^{(01\bar{2}\bar{0}31)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}31]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0}31)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_0^{-1}t_3t_1t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3eN = Nt_0t_1t_2^{-1}t_0^{-1}t_3N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_1N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1^2N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1^{-1}N$
$= Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_0N =$
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$,
$Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_2N = Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_2^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_3N = Nt_0t_1t_2t_3t_0t_2^{-1}N$, and
$Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_3^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_0t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

304. We next consider the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Let $[01\bar{2}\bar{0}\bar{3}0]$ denote the double coset $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Note that $N^{(01\bar{2}\bar{0}\bar{3}0)} \geq N^{01\bar{2}\bar{0}\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{0}\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N\right| = \frac{|N|}{|N^{(01\bar{2}\bar{0}\bar{3}0)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{0}\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{0}\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}eN = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$ and $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_0N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0^2N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0^{-1}N$
$= Nt_0t_1t_2^{-1}t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_1N =$
$Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_1^{-1}N = Nt_0^{-1}t_1t_2t_0t_3^{-1}N$,
$Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_2N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_2^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_3N = Nt_0t_1t_2t_3^{-1}t_2t_1N$, and
$Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

305. We next consider the double coset $Nt_0t_1t_2^{-1}t_1t_3t_0N$.

Let $[01\bar{2}130]$ denote the double coset $Nt_0t_1t_2^{-1}t_1t_3t_0N$.

Note that $N^{(01\bar{2}130)} \geq N^{01\bar{2}130} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}130)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_1t_3t_0N\right| = \frac{|N|}{\left|N^{(01\bar{2}130)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}130]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}130)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_1t_3t_0t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_1t_3t_0t_0^{-1}N = Nt_0t_1t_2^{-1}t_1t_3eN = Nt_0t_1t_2^{-1}t_1t_3N$ and $Nt_0t_1t_2^{-1}t_1t_3t_0t_0N = Nt_0t_1t_2^{-1}t_1t_3t_0^2N = Nt_0t_1t_2^{-1}t_1t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_1t_3t_0t_1N =$
$Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3t_0t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$,
$Nt_0t_1t_2^{-1}t_1t_3t_0t_2N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3t_0t_2^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3t_0t_3N = Nt_0t_1t_2t_0t_3t_2N$, and
$Nt_0t_1t_2^{-1}t_1t_3t_0t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_1t_3t_0t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

306. We next consider the double coset $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$.

Let $[01\bar{2}1\bar{3}\bar{0}]$ denote the double coset $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$.

Note that $N^{(01\bar{2}1\bar{3}\bar{0})} \geq N^{01\bar{2}1\bar{3}\bar{0}} = \langle e\rangle$. Thus $\left|N^{(01\bar{2}1\bar{3}\bar{0})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{2}1\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}1\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}1\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0,t_1,t_2,t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_0N = Nt_0t_1t_2^{-1}t_1t_3^{-1}eN = Nt_0t_1t_2^{-1}t_1t_3^{-1}N$ and $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-2}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0N$ $= Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_1N =$ $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_1t_3t_0N$, $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0t_1t_2t_3t_0^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_3N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, and $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_2t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

307. We next consider the double coset $Nt_0t_1t_2^{-1}t_3t_0t_1N$.

Let $[01\bar{2}301]$ denote the double coset $Nt_0t_1t_2^{-1}t_3t_0t_1N$.

Note that $N^{(01\bar{2}301)} \geq N^{01\bar{2}301} = \langle e\rangle$. Thus $\left|N^{(01\bar{2}301)}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3t_0t_1N\right| = \frac{|N|}{\left|N^{(01\bar{2}301)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}301]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}301)}$ has eight orbits on $T = \{t_0,t_1,t_2,t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_3t_0t_1t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_3t_0t_1t_1^{-1}N = Nt_0t_1t_2^{-1}t_3t_0eN = Nt_0t_1t_2^{-1}t_3t_0N$ and $Nt_0t_1t_2^{-1}t_3t_0t_1t_1N = Nt_0t_1t_2^{-1}t_3t_0t_1^2N = Nt_0t_1t_2^{-1}t_3t_0t_1^{-1}N$ $= Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3t_0t_1t_0N =$ $Nt_0t_1t_0^{-1}t_2t_1N$, $Nt_0t_1t_2^{-1}t_3t_0t_1t_0^{-1}N = Nt_0t_1t_2t_3t_0^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_3t_0t_1t_2N$ $= Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_3t_0t_1t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_3t_0t_1t_3N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$, and $Nt_0t_1t_2^{-1}t_3t_0t_1t_3^{-1}N$ $= Nt_0t_1t_2t_3^{-1}t_2t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3t_0t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

308. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1N$.

Let $[01\bar{2}\bar{3}01]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1 = Nt_3t_2t_1^{-1}t_0^{-1}t_3t_2$.

That is, in terms of our short-hand notation,

$$01\bar{2}\bar{3}01 \sim 32\bar{1}\bar{0}32.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[01\bar{2}\bar{3}01]$:

$$01\bar{2}\bar{3}01 \sim 32\bar{1}\bar{0}32, \quad 10\bar{2}\bar{3}10 \sim 32\bar{0}\bar{1}32, \quad 21\bar{0}\bar{3}21 \sim 30\bar{1}\bar{2}30,$$

$$31\bar{2}\bar{0}31 \sim 02\bar{1}\bar{3}02, \quad 01\bar{3}\bar{2}01 \sim 23\bar{1}\bar{0}23, \quad 12\bar{0}\bar{3}12 \sim 30\bar{2}\bar{1}30,$$

$$20\bar{1}\bar{3}20 \sim 31\bar{0}\bar{2}31, \quad 13\bar{2}\bar{0}13 \sim 02\bar{3}\bar{1}02, \quad 10\bar{3}\bar{2}10 \sim 23\bar{0}\bar{1}23,$$

$$03\bar{1}\bar{2}03 \sim 21\bar{3}\bar{0}21, \quad 03\bar{2}\bar{1}03 \sim 12\bar{3}\bar{0}12, \quad 13\bar{0}\bar{2}13 \sim 20\bar{3}\bar{1}20$$

Since each of the twenty-four single cosets has two names, the double coset $[01\bar{2}\bar{3}01]$ must have at most twelve distinct single cosets.

Now, $N^{(01\bar{2}\bar{3}01)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,3\}$, $\{1,2\}$, $\{\bar{0},\bar{3}\}$, and $\{\bar{1},\bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_3^{-1}t_0t_1t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1t_1^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_0eN = Nt_0t_1t_2^{-1}t_3^{-1}t_0N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1t_1N = Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1^2N = Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1t_0N =$ $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1t_0^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_0t_1t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

309. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Let $[01\bar{2}\bar{3}\bar{1}2]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Note that $N^{(01\bar{2}\bar{3}\bar{1}2)} \geq N^{01\bar{2}\bar{3}\bar{1}2} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{3}\bar{1}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{3}\bar{1}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{3}\bar{1}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{3}\bar{1}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_2^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}eN = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_2N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^2N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0^{-1}N =$ $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_1N = Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_3N$ $= Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$, and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$.

310. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Let $[01\bar{2}\bar{3}\bar{1}\bar{2}]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Note that $N^{(01\bar{2}\bar{3}\bar{1}\bar{2})} \geq N^{01\bar{2}\bar{3}\bar{1}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(01\bar{2}\bar{3}\bar{1}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{2}\bar{3}\bar{1}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[01\bar{2}\bar{3}\bar{1}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{2}\bar{3}\bar{1}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_2N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}eN = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-2}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_0N = Nt_0^{-1}t_1t_2t_3t_2^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_3t_1^{-1}N$, and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

311. We next consider the double coset $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$.

Let $[01\bar{0}23\bar{2}]$ denote the double coset $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1} = Nt_0t_1t_0^{-1}t_3t_2t_3^{-1}$.

That is, in terms of our short-hand notation,

$$01\bar{0}23\bar{2} \sim 01\bar{0}32\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[01\bar{0}23\bar{2}]$:

$01\bar{0}23\bar{2} \sim 01\bar{0}32\bar{3}$,	$10\bar{1}23\bar{2} \sim 10\bar{1}32\bar{3}$,	$21\bar{2}03\bar{0} \sim 21\bar{2}30\bar{3}$,
$31\bar{3}20\bar{2} \sim 31\bar{3}02\bar{0}$,	$02\bar{0}13\bar{1} \sim 02\bar{0}31\bar{3}$,	$03\bar{0}21\bar{2} \sim 03\bar{0}12\bar{1}$,
$12\bar{1}03\bar{0} \sim 12\bar{1}30\bar{3}$,	$20\bar{2}13\bar{1} \sim 20\bar{2}31\bar{3}$,	$13\bar{1}20\bar{2} \sim 13\bar{1}02\bar{0}$,

$$30\bar{3}21\bar{2} \sim 30\bar{3}12\bar{1}, \qquad 23\bar{2}10\bar{1} \sim 23\bar{2}01\bar{0}, \qquad 32\bar{3}01\bar{0} \sim 32\bar{3}10\bar{1}$$

Since each of the twenty-four single cosets has two names, the double coset $[01\bar{0}23\bar{2}]$ must have at most twelve distinct single cosets.

Now, $N^{(01\bar{0}23\bar{2})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2, 3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_0^{-1}t_2t_3t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2\}$.

But note that $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_2N = Nt_0t_1t_0^{-1}t_2t_3eN = Nt_0t_1t_0^{-1}t_2t_3N$ and $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_2^{-1}N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-2}N = Nt_0t_1t_0^{-1}t_2t_3t_2N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$, $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$, and $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$.

312. We next consider the double coset $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}N$.

Let $[01\bar{0}2\bar{3}\bar{2}]$ denote the double coset $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1} = Nt_1t_2t_1^{-1}t_0t_3^{-1}t_0^{-1}$
$= Nt_2t_0t_2^{-1}t_1t_3^{-1}t_1^{-1} = Nt_1t_3t_1^{-1}t_2t_0^{-1}t_2^{-1} = Nt_3t_0t_3^{-1}t_2t_1^{-1}t_2^{-1} = Nt_2t_1t_2^{-1}t_3t_0^{-1}t_3^{-1}$
$= Nt_3t_1t_3^{-1}t_0t_2^{-1}t_0^{-1} = Nt_0t_2t_0^{-1}t_3t_1^{-1}t_3^{-1} = Nt_0t_3t_0^{-1}t_1t_2^{-1}t_1^{-1} = Nt_1t_0t_1^{-1}t_3t_2^{-1}t_3^{-1}$
$= Nt_2t_3t_2^{-1}t_0t_1^{-1}t_0^{-1} = Nt_3t_2t_3^{-1}t_1t_0^{-1}t_1^{-1}$.

That is, in terms of our short-hand notation,

$$01\bar{0}2\bar{3}\bar{2} \sim 12\bar{1}0\bar{3}\bar{0} \sim 20\bar{2}1\bar{3}\bar{1} \sim 13\bar{1}2\bar{0}\bar{2} \sim 30\bar{3}2\bar{1}\bar{2} \sim 21\bar{2}3\bar{0}\bar{3}$$

$$\sim 31\bar{3}0\bar{2}\bar{0} \sim 02\bar{0}3\bar{1}\bar{3} \sim 03\bar{0}1\bar{2}\bar{1} \sim 10\bar{1}3\bar{2}\bar{3} \sim 23\bar{2}0\bar{1}\bar{0} \sim 32\bar{3}1\bar{0}\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[01\bar{0}2\bar{3}\bar{2}]$:

$$01\bar{0}2\bar{3}\bar{2} \sim 12\bar{1}0\bar{3}\bar{0} \sim 20\bar{2}1\bar{3}\bar{1} \sim 13\bar{1}2\bar{0}\bar{2} \sim 30\bar{3}2\bar{1}\bar{2} \sim 21\bar{2}3\bar{0}\bar{3}$$

$$\sim 31\bar{3}0\bar{2}\bar{0} \sim 02\bar{0}3\bar{1}\bar{3} \sim 03\bar{0}1\bar{2}\bar{1} \sim 10\bar{1}3\bar{2}\bar{3} \sim 23\bar{2}0\bar{1}\bar{0} \sim 32\bar{3}1\bar{0}\bar{1},$$

$$1\bar{0}1\bar{2}3\bar{2} \sim 0\bar{2}0\bar{1}3\bar{1} \sim 2\bar{1}2\bar{0}3\bar{0} \sim 0\bar{3}0\bar{2}1\bar{2} \sim 3\bar{1}3\bar{2}0\bar{2} \sim 2\bar{0}2\bar{3}1\bar{3}$$

$$\sim 3\bar{0}3\bar{1}2\bar{1} \sim 1\bar{2}1\bar{3}0\bar{3} \sim 1\bar{3}1\bar{0}2\bar{0} \sim 0\bar{1}0\bar{3}2\bar{3} \sim 2\bar{3}2\bar{1}0\bar{1} \sim 3\bar{2}3\bar{0}1\bar{0}$$

Since each of the twenty-four single cosets has twelve names, the double coset $[01\bar{0}23\bar{2}]$ must have at most two distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(01\bar{0}23\bar{2})} \geq N^{01\bar{0}23\bar{2}} = \langle e \rangle$. In fact, our relations tell us that $N(t_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1})^{(0\ 1\ 2)} = Nt_1t_2t_1^{-1}t_0t_3^{-1}t_0^{-1} = Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}$, which implies that $(0\ 1\ 2) \in N^{(01\bar{0}23\bar{2})}$, and moreover $N(t_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1})^{(0\ 1)(2\ 3)}$
$= Nt_1t_0t_1^{-1}t_3t_2^{-1}t_3^{-1} = Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}$, which implies that $(0\ 1)(2\ 3) \in N^{(01\bar{0}23\bar{2})}$. Therefore, $(0\ 1\ 2), (0\ 1)(2\ 3) \in N^{(01\bar{0}23\bar{2})}$, and so $N^{(01\bar{0}23\bar{2})} \geq \langle (0\ 1\ 2), (0\ 1)(2\ 3) \rangle \cong A_4$. Thus $\left|N^{(01\bar{0}23\bar{2})}\right| \geq |A_4| = 12$ and so, by Lemma 1.4,
$\left|Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{0}23\bar{2})}\right|} \leq \frac{24}{12} = 2.$

Therefore, as we concluded earlier, the double coset $[01\bar{0}23\bar{2}]$ has at most two distinct single cosets.

Now, $N^{(01\bar{0}23\bar{2})}$ has two orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$.

Therefore, there are at most two double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}$, $i = 2$.

But note that $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}t_2N = Nt_0t_1t_0^{-1}t_2t_3^{-1}eN = Nt_0t_1t_0^{-1}t_2t_3^{-1}N$ and $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-2}N = Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2N$
$= Nt_0t_1t_0^{-1}t_2^{-1}t_3N.$

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_0^{-1}t_2t_3^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

313. We next consider the double coset $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Let $[01\bar{0}\bar{2}\bar{3}\bar{0}]$ denote the double coset $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Note that $N^{(01\bar{0}\bar{2}\bar{3}\bar{0})} \geq N^{01\bar{0}\bar{2}\bar{3}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(01\bar{0}\bar{2}\bar{3}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(01\bar{0}\bar{2}\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24.$

Therefore, the double coset $[01\bar{0}\bar{2}\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(01\bar{0}\bar{2}\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}eN = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-2}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0N$ $= Nt_0^{-1}t_1t_2t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N$, $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0t_2t_3t_0N$, $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}N$, $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$, $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, and $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

314. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$.

Let $[\bar{0}\bar{1}2103]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$.

Note that $N^{(\bar{0}\bar{1}2103)} \geq N^{\bar{0}\bar{1}2103} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2103)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}2103)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2103]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2103)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1t_0t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0eN = Nt_0^{-1}t_1^{-1}t_2t_1t_0N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^2N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_1N = Nt_0t_1^{-1}t_2t_3t_1t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_1^{-1}N$ $= Nt_0t_1^{-1}t_0t_2t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_2N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_3t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

315. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$.

Let $[\bar{0}\bar{1}210\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}210\bar{3})} \geq N^{\bar{0}\bar{1}210\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}210\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}210\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}210\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}210\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_0eN = Nt_0^{-1}t_1^{-1}t_2t_1t_0N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_0N =$ $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_0^{-1}N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_1N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_2N = Nt_0t_1t_2t_3t_0^{-1}t_2^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

316. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Let $[\bar{0}\bar{1}21\bar{0}3]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Note that $N^{(\bar{0}\bar{1}21\bar{0}3)} \geq N^{\bar{0}\bar{1}21\bar{0}3} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}21\bar{0}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}21\bar{0}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}21\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}21\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^2N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_0N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_1N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_1^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_2N = Nt_0^{-1}t_1t_2t_3t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

317. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

Let $[\bar{0}\bar{1}21\bar{0}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}21\bar{0}\bar{3})} \geq N^{\bar{0}\bar{1}21\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}21\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}21\bar{0}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}21\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}21\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_1^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

318. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$.

Let $[\bar{0}\bar{1}2130]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$.

Note that $N^{(\bar{0}\bar{1}2130)} \geq N^{\bar{0}\bar{1}2130}$ $= \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2130)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}2130)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2130]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2130)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1t_3t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3eN = Nt_0^{-1}t_1^{-1}t_2t_1t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_0N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_1N =$ $Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_2N = Nt_0t_1t_2t_0t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_2^{-1}N$ $= Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

319. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Let $[\bar{0}\bar{1}213\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}213\bar{0})} \geq N^{\bar{0}\bar{1}213\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}213\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}213\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}213\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}213\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_1t_3eN = Nt_0^{-1}t_1^{-1}t_2t_1t_3N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_1N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0t_3t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

320. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$.

Let $[\bar{0}\bar{1}21\bar{3}0]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$.

Note that $N^{(\bar{0}\bar{1}21\bar{3}0)} \geq N^{\bar{0}\bar{1}21\bar{3}0} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}21\bar{3}0)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}21\bar{3}0)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}21\bar{3}0]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}21\bar{3}0)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_0N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0^2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_1N =$ $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_2N = Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_3N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

321. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$.

Let $[\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3})} \geq N^{\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

322. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$.

Let $[\bar{0}\bar{1}23\bar{0}2]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$.

Note that $N^{(\bar{0}\bar{1}23\bar{0}2)} \geq N^{\bar{0}\bar{1}23\bar{0}2} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}23\bar{0}2)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}23\bar{0}2)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}23\bar{0}2]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}23\bar{0}2)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^2N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_0N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_1N = Nt_0t_1t_2t_3t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_3N = Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_3^{-1}N = Nt_0^{-1}t_1t_2t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

323. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$.

Let $[\bar{0}\bar{1}23\bar{0}\bar{2}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$.

Note that $N^{(\bar{0}\bar{1}23\bar{0}\bar{2})} \geq N^{\bar{0}\bar{1}23\bar{0}\bar{2}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}23\bar{0}\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}23\bar{0}\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}23\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}23\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}eN = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_1N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_3N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, and $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

324. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$.

Let $[\bar{0}\bar{1}231\bar{2}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$.

Note that $N^{(\bar{0}\bar{1}231\bar{2})} \geq N^{\bar{0}\bar{1}231\bar{2}} = \langle e\rangle$. Thus $\left|N^{(\bar{0}\bar{1}231\bar{2})}\right| \geq |\langle e\rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}231\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}231\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}231\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_1eN = Nt_0^{-1}t_1^{-1}t_2t_3t_1N$ and $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-2}N = Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2N$
$= Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_0N =$
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$,
$Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_1^{-1}N$
$= Nt_0t_1^{-1}t_2t_0t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}N$, and
$Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form
$Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

325. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$.

Let $[\bar{0}\bar{1}\bar{2}012]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}012)} \geq N^{\bar{0}\bar{1}\bar{2}012} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}012)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}012)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}012]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}012)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2^{-1}N$
$= Nt_0t_1t_2t_0t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_0N =$
$Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_0^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$,

$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_1^{-1}N$
$= Nt_0t_1t_2t_0t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_3N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$, and
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_3^{-1}N = Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

326. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}01\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1} = Nt_3^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}$.

That is, in terms of our short-hand notation,

$$\bar{0}\bar{1}\bar{2}01\bar{3} \sim \bar{3}\bar{1}\bar{2}31\bar{0}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}\bar{1}\bar{2}01\bar{3}]$:

$\bar{0}\bar{1}\bar{2}01\bar{3} \sim \bar{3}\bar{1}\bar{2}31\bar{0}$, $\bar{1}\bar{0}\bar{2}10\bar{3} \sim \bar{3}\bar{0}\bar{2}30\bar{1}$, $\bar{2}\bar{1}\bar{0}21\bar{3} \sim \bar{3}\bar{1}\bar{0}31\bar{2}$, $\bar{0}\bar{2}\bar{1}02\bar{3} \sim \bar{3}\bar{2}\bar{1}32\bar{0}$,

$\bar{0}\bar{3}\bar{2}03\bar{1} \sim \bar{1}\bar{3}\bar{2}13\bar{0}$, $\bar{0}\bar{1}\bar{3}01\bar{2} \sim \bar{2}\bar{1}\bar{3}21\bar{0}$, $\bar{1}\bar{2}\bar{0}12\bar{3} \sim \bar{3}\bar{2}\bar{0}32\bar{1}$, $\bar{2}\bar{0}\bar{1}20\bar{3} \sim \bar{3}\bar{0}\bar{1}30\bar{2}$,

$\bar{0}\bar{2}\bar{3}02\bar{1} \sim \bar{1}\bar{2}\bar{3}12\bar{0}$, $\bar{0}\bar{3}\bar{1}03\bar{2} \sim \bar{2}\bar{3}\bar{1}23\bar{0}$, $\bar{1}\bar{3}\bar{0}13\bar{2} \sim \bar{2}\bar{3}\bar{0}23\bar{1}$, $\bar{2}\bar{0}\bar{3}20\bar{1} \sim \bar{1}\bar{0}\bar{3}10\bar{2}$

Since each of the twenty-four single cosets has two names, the double coset $[\bar{0}\bar{1}\bar{2}01\bar{3}]$ must have at most twelve distinct single cosets.

Now, $N^{(\bar{0}\bar{1}\bar{2}01\bar{3})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 3\}$, $\{1\}$, $\{2\}$, $\{\bar{0}, \bar{3}\}$, $\{\bar{1}\}$, and $\{\bar{2}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3N$
$= Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_1N =$
$Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_2^{-1}N$,

$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_2N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_2^{-1}N$ $= Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

327. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0})} \geq N^{\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_1N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

328. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}03\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}03\bar{0})} \geq N^{\bar{0}\bar{1}\bar{2}03\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}03\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}N\right| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}03\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}03\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}03\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_1N =$ $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_2N = Nt_0t_1t_2t_0t_1t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_3N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1^{-1}t_0^{-1}t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_3t_0^{-1}t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

329. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N =$ $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_0N =$ $Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_3t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N$

$= Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2t_3t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

330. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0N = Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_3N = Nt_0t_1t_2t_0^{-1}t_2t_1N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

331. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}eN$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-2}N$
$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N =$
$Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2t_0t_1^{-1}N$,
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_2N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$
$= Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_3N = Nt_0t_1t_2t_0t_3t_0^{-1}N$, and
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

332. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Let $[\bar{0}\bar{1}\bar{2}310]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}310)} \geq N^{\bar{0}\bar{1}\bar{2}310} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}310)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N\right| = \frac{|N|}{\left|N^{(\bar{0}\bar{1}\bar{2}310)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}310]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}310)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1N$ and
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0^2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_1N =$
$Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_1^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$,
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_2N = Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_2^{-1}N$

$= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_1N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_3^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

333. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0N = Nt_0^{-1}t_1t_2t_3^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}N = Nt_0t_1t_2t_0t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_2^{-1}N$ $= Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

334. We next consider the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$.

Let $[\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2}]$ denote the double coset $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$.

Note that $N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2})} \geq N^{\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $|Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N| = \frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2})}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1eN = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1N$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-2}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_0N = Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_1^{-1}N$ $= Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1N$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_3N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

335. We next consider the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Let $[\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}]$ denote the double coset $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Note that $N^{(\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0})} \geq N^{\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}eN = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}N$ and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-2}N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2N = Nt_0t_1t_2t_3t_0t_2N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$

$= Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, and $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

336. We next consider the double coset $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$.

Let $[\bar{0}120\bar{1}\bar{3}]$ denote the double coset $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$.

Note that $N^{(\bar{0}120\bar{1}\bar{3})} \geq N^{\bar{0}120\bar{1}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}120\bar{1}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}120\bar{1}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}120\bar{1}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}120\bar{1}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_3N = Nt_0^{-1}t_1t_2t_0t_1^{-1}eN = Nt_0^{-1}t_1t_2t_0t_1^{-1}N$ and $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_3^{-1}N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-2}N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_0N =$ $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_1^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_2N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$, and $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1t_2t_3t_0t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

337. We next consider the double coset $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}N$.

Let $[\bar{0}12301\bar{1}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1} = Nt_0^{-1}t_2t_3t_1t_0t_2^{-1} = Nt_0^{-1}t_3t_1t_2t_0t_3^{-1}$.

That is, in terms of our short-hand notation,

$$\bar{0}1230\bar{1} \sim \bar{0}2310\bar{2} \sim \bar{0}3120\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[\bar{0}1230\bar{1}]$:

$$\bar{0}1230\bar{1} \sim \bar{0}2310\bar{2} \sim \bar{0}3120\bar{3}, \qquad \bar{1}0231\bar{0} \sim \bar{1}2301\bar{2} \sim \bar{1}3021\bar{3},$$

$$\bar{2}1032\bar{1} \sim \bar{2}0312\bar{0} \sim \bar{2}3102\bar{3}, \qquad \bar{3}1203\bar{1} \sim \bar{3}2013\bar{2} \sim \bar{3}0123\bar{0},$$

$$\bar{0}2130\bar{2} \sim \bar{0}1320\bar{1} \sim \bar{0}3210\bar{3}, \qquad \bar{1}2031\bar{2} \sim \bar{1}0321\bar{0} \sim \bar{1}3201\bar{3},$$

$$\bar{2}0132\bar{0} \sim \bar{2}1302\bar{1} \sim \bar{2}3012\bar{3}, \qquad \bar{3}0213\bar{0} \sim \bar{3}2103\bar{2} \sim \bar{3}1023\bar{1}$$

Since each of the twenty-four single cosets has three names, the double coset $[\bar{0}1230\bar{1}]$ must have at most eight distinct single cosets.

Now, $N^{(\bar{0}1230\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1,2,3\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1t_2t_3t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}t_1N = Nt_0^{-1}t_1t_2t_3t_0eN = Nt_0^{-1}t_1t_2t_3t_0N$ and $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2t_3t_0t_1^{-2}N = Nt_0^{-1}t_1t_2t_3t_0t_1N$ $= Nt_0t_1t_2t_0t_1^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}t_0N =$ $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1^{-1}N$ and $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3t_0t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

338. We next consider the double coset $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$.

Let $[\bar{0}1230\bar{2}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N$.

Note that $N^{(\bar{0}1230\bar{2})} \geq N^{\bar{0}1230\bar{2}}$ $= \langle e \rangle$. Thus $\left|N^{(\bar{0}1230\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}N\right| =$ $\frac{|N|}{\left|N^{(\bar{0}1230\bar{2})}\right|} \leq \frac{24}{1} = 24.$

Therefore, the double coset $[\bar{0}1230\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}1230\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1t_2t_3t_0t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_2N = Nt_0^{-1}t_1t_2t_3t_0eN = Nt_0^{-1}t_1t_2t_3t_0N$ and $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_2^{-1}N = Nt_0^{-1}t_1t_2t_3t_0t_2^{-2}N = Nt_0^{-1}t_1t_2t_3t_0t_2N$
$= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_0N =$
$Nt_0^{-1}t_1t_2^{-1}t_0^{-1}t_1^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_2N$,
$Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_1^{-1}N$
$= Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$, $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$, and
$Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3t_0t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

339. We next consider the double coset $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$.

Let $[\bar{0}123\bar{2}\bar{1}]$ denote the double coset $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N$.

Note that $N^{(\bar{0}123\bar{2}\bar{1})} \geq N^{\bar{0}123\bar{2}\bar{1}} = \langle e \rangle$. Thus $\left|N^{(\bar{0}123\bar{2}\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(\bar{0}123\bar{2}\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[\bar{0}123\bar{2}\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(\bar{0}123\bar{2}\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length seven given by $w = t_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_1N = Nt_0^{-1}t_1t_2t_3t_2^{-1}eN = Nt_0^{-1}t_1t_2t_3t_2^{-1}N$ and $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_1^{-1}N = Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-2}N = Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1N$
$= Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$.

Moreover, with the help of MAGMA, we know that $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_0N =$
$Nt_0t_1t_2t_3^{-1}t_2^{-1}t_0^{-1}N$, $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$,
$Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0N$, $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_2^{-1}N$
$= Nt_0t_1t_2t_0t_1t_3N$, $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_3N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$, and
$Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_3^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_1t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0^{-1}t_1t_2t_3t_2^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

340. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$.

Let $[0\bar{1}2010\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N$.

Note that $N^{(0\bar{1}201\bar{0}\bar{2})} \geq N^{0\bar{1}201\bar{0}\bar{2}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}201\bar{0}\bar{2})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}201\bar{0}\bar{2})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}201\bar{0}\bar{2}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}201\bar{0}\bar{2})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}eN = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2N$ $= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_0N =$ $Nt_0t_1^{-1}t_2t_1t_3t_2N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_2^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_2t_1N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_3N = Nt_0t_1t_2t_3t_0^{-1}t_3N$, and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_3^{-1}N = Nt_0t_1t_2^{-1}t_1t_3^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

341. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$.

Let $[0\bar{1}201\bar{0}3]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$.

Note that $N^{(0\bar{1}201\bar{0}3)} \geq N^{0\bar{1}201\bar{0}3} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}201\bar{0}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}201\bar{0}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}201\bar{0}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}201\bar{0}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_0t_1t_0^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}eN = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_3N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^2N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_1N = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_2N = Nt_0t_1t_2^{-1}t_1t_3t_0N$, and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

342. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$.

Let $[0\bar{1}201\bar{0}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N$.

Note that $N^{(0\bar{1}201\bar{0}\bar{3})} \geq N^{0\bar{1}201\bar{0}\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}201\bar{0}\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}201\bar{0}\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}201\bar{0}\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}201\bar{0}\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}eN = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_0N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_2N = Nt_0t_1t_2t_0t_3^{-1}t_0N$, and $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

343. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}N$.

Let $[0\bar{1}201\bar{2}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1} = Nt_2t_3^{-1}t_0t_2t_3t_0^{-1}t_1^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}201\bar{2}\bar{3} \sim 2\bar{3}023\bar{0}\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}201\bar{2}\bar{3}]$:

$0\bar{1}201\bar{2}\bar{3} \sim 2\bar{3}023\bar{0}\bar{1}$,	$1\bar{0}210\bar{2}\bar{3} \sim 2\bar{3}123\bar{1}\bar{0}$,	$2\bar{1}021\bar{0}\bar{3} \sim 0\bar{3}203\bar{2}\bar{1}$,
$3\bar{1}231\bar{2}\bar{0} \sim 2\bar{0}320\bar{3}\bar{1}$,	$0\bar{2}102\bar{1}\bar{3} \sim 1\bar{3}013\bar{0}\bar{2}$,	$0\bar{1}301\bar{3}\bar{2} \sim 3\bar{2}032\bar{0}\bar{1}$,
$1\bar{2}012\bar{0}\bar{3} \sim 0\bar{3}103\bar{1}\bar{2}$,	$2\bar{0}120\bar{1}\bar{3} \sim 1\bar{3}213\bar{2}\bar{0}$,	$3\bar{0}230\bar{2}\bar{1} \sim 2\bar{1}321\bar{3}\bar{0}$,
$3\bar{1}031\bar{0}\bar{2} \sim 0\bar{2}302\bar{3}\bar{1}$,	$1\bar{2}312\bar{3}\bar{0} \sim 3\bar{0}130\bar{1}\bar{2}$,	$1\bar{0}310\bar{3}\bar{2} \sim 3\bar{2}132\bar{1}\bar{0}$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}201\bar{2}\bar{3}]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}201\bar{2}\bar{3})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 2\}$, $\{1, 3\}$, $\{\bar{0}, \bar{2}\}$, and $\{\bar{1}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{0, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}eN = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3N$ $= Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}t_0N =$ $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$ and $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1t_2^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

344. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$.

Let $[0\bar{1}20\bar{1}\bar{2}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$.

Note that $N^{(0\bar{1}20\bar{1}\bar{2}\bar{0})} \geq N^{0\bar{1}20\bar{1}\bar{2}\bar{0}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{1}\bar{2}\bar{0})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{1}\bar{2}\bar{0})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{1}\bar{2}\bar{0}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{1}\bar{2}\bar{0})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}eN$
$= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_0^{-1}N$
$= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N =$
$Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$,
$Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1}N$
$= Nt_0t_1^{-1}t_2t_1t_3t_0N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, and
$Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N = Nt_0t_1t_2t_0t_3t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form
$Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

345. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$.

Let $[0\bar{1}20\bar{1}\bar{2}3]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N$.

Note that $N^{(0\bar{1}20\bar{1}\bar{2}3)} \geq N^{0\bar{1}20\bar{1}\bar{2}3} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{1}\bar{2}3)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{1}\bar{2}3)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{1}\bar{2}3]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{1}\bar{2}3)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}eN$
$= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_3N$
$= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3^2N = Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_0^{-1}t_3^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_0N =$
$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_2N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_0^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_0^{-1}N$,

$Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_1N = Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_1^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_2N$, $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_2N = Nt_0t_1t_2t_0t_3^{-1}t_0^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_3t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

346. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$.

Let $[0\bar{1}20\bar{3}1\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$.

Note that $N^{(0\bar{1}20\bar{3}1\bar{3})} \geq N^{0\bar{1}20\bar{3}1\bar{3}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}20\bar{3}1\bar{3})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}20\bar{3}1\bar{3})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}20\bar{3}1\bar{3}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}20\bar{3}1\bar{3})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1eN = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-2}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3N$ $= Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_0N =$ $Nt_0t_1t_2t_0t_3^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_1N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_1N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_1^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_2N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_2^{-1}N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

347. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$.

Let $[0\bar{1}20\bar{3}\bar{1}\bar{0}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1} = Nt_1t_0^{-1}t_2t_1t_3^{-1}t_0^{-1}t_1^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}20\bar{3}\bar{1}\bar{0} \sim 1\bar{0}21\bar{3}\bar{0}\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}20\bar{3}\bar{1}\bar{0}]$:

$$0\bar{1}20\bar{3}\bar{1}\bar{0} \sim 1\bar{0}21\bar{3}\bar{0}\bar{1}, \qquad 2\bar{1}02\bar{3}\bar{1}\bar{2} \sim 1\bar{2}01\bar{3}\bar{2}\bar{1}, \qquad 3\bar{1}23\bar{0}\bar{1}\bar{3} \sim 1\bar{3}21\bar{0}\bar{3}\bar{1},$$

$$0\bar{2}10\bar{3}\bar{2}\bar{0} \sim 2\bar{0}12\bar{3}\bar{0}\bar{2}, \qquad 0\bar{3}20\bar{1}\bar{3}\bar{0} \sim 3\bar{0}23\bar{1}\bar{0}\bar{3}, \qquad 0\bar{1}30\bar{2}\bar{1}\bar{0} \sim 1\bar{0}31\bar{2}\bar{0}\bar{1},$$

$$0\bar{2}30\bar{1}\bar{2}\bar{0} \sim 2\bar{0}32\bar{1}\bar{0}\bar{2}, \qquad 0\bar{3}10\bar{2}\bar{3}\bar{0} \sim 3\bar{0}13\bar{2}\bar{0}\bar{3}, \qquad 2\bar{1}32\bar{0}\bar{1}\bar{2} \sim 1\bar{2}31\bar{0}\bar{2}\bar{1},$$

$$3\bar{1}03\bar{2}\bar{1}\bar{3} \sim 1\bar{3}01\bar{2}\bar{3}\bar{1}, \qquad 2\bar{3}12\bar{0}\bar{3}\bar{2} \sim 3\bar{2}13\bar{0}\bar{2}\bar{3}, \qquad 3\bar{2}03\bar{1}\bar{2}\bar{3} \sim 2\bar{3}02\bar{1}\bar{3}\bar{2}$$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}20\bar{3}\bar{1}\bar{0}]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}20\bar{3}\bar{1}\bar{0})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}, \bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_0N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}eN$
$= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_0^{-1}N$
$= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-2}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0N = Nt_0t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_2N =$ $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_2^{-1}N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$, and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_3^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

348. We next consider the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}N$.

Let $[0\bar{1}20\bar{3}\bar{1}\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1} = Nt_2t_3^{-1}t_0t_2t_1^{-1}t_3^{-1}t_0^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}20\bar{3}\bar{1}\bar{2} \sim 2\bar{3}02\bar{1}\bar{3}\bar{0}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}20\bar{3}\bar{1}\bar{2}]$:

$$0\bar{1}20\bar{3}\bar{1}\bar{2} \sim 2\bar{3}02\bar{1}\bar{3}\bar{0}, \quad 1\bar{0}21\bar{3}\bar{0}\bar{2} \sim 2\bar{3}12\bar{0}\bar{3}\bar{1}, \quad 2\bar{1}02\bar{3}\bar{1}\bar{0} \sim 0\bar{3}20\bar{1}\bar{3}\bar{2},$$

$$3\bar{1}23\bar{0}\bar{1}\bar{2} \sim 2\bar{0}32\bar{1}\bar{0}\bar{3}, \quad 0\bar{2}10\bar{3}\bar{2}\bar{1} \sim 1\bar{3}01\bar{2}\bar{3}\bar{0}, \quad 0\bar{1}30\bar{2}\bar{1}\bar{3} \sim 3\bar{2}03\bar{1}\bar{2}\bar{0},$$

$$0\bar{2}30\bar{1}\bar{2}\bar{3} \sim 3\bar{1}03\bar{2}\bar{1}\bar{0}, \quad 0\bar{3}10\bar{2}\bar{3}\bar{1} \sim 1\bar{2}01\bar{3}\bar{2}\bar{0}, \quad 2\bar{1}32\bar{0}\bar{1}\bar{3} \sim 3\bar{0}23\bar{1}\bar{0}\bar{2},$$

$$2\bar{0}12\bar{3}\bar{0}\bar{1} \sim 1\bar{3}21\bar{0}\bar{3}\bar{2}, \quad 1\bar{0}31\bar{2}\bar{0}\bar{3} \sim 3\bar{2}13\bar{0}\bar{2}\bar{1}, \quad 1\bar{2}31\bar{0}\bar{2}\bar{3} \sim 3\bar{0}13\bar{2}\bar{0}\bar{1}$$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}20\bar{3}\bar{1}\bar{2}]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}20\bar{3}\bar{1}\bar{2})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 2\}$, $\{1, 3\}$, $\{\bar{0}, \bar{2}\}$, and $\{\bar{1}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}t_i^{\pm1}$, $i \in \{1, 2\}$.

But note that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}t_2N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}eN$
$= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}t_2^{-1}N$
$= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-2}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2N = Nt_0t_1t_2t_0t_1^{-1}t_2N.$

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}t_1N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$ and $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}t_1^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N.$

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_2^{-1}t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

349. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$.

Let $[0\bar{1}213\bar{0}\bar{3}]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1} = Nt_3t_1^{-1}t_2t_1t_0t_3^{-1}t_0^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}213\bar{0}\bar{3} \sim 3\bar{1}210\bar{3}\bar{0}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}213\bar{0}\bar{3}]$:

$0\bar{1}213\bar{0}\bar{3} \sim 3\bar{1}210\bar{3}\bar{0}$, $1\bar{0}203\bar{1}\bar{3} \sim 3\bar{0}201\bar{3}\bar{1}$, $2\bar{1}013\bar{2}\bar{3} \sim 3\bar{1}012\bar{3}\bar{2}$,

$0\bar{2}123\bar{0}\bar{3} \sim 3\bar{2}120\bar{3}\bar{0}$, $0\bar{3}231\bar{0}\bar{1} \sim 1\bar{3}230\bar{1}\bar{0}$, $0\bar{1}312\bar{0}\bar{2} \sim 2\bar{1}310\bar{2}\bar{0}$,

$1\bar{2}023\bar{1}\bar{3} \sim 3\bar{2}021\bar{3}\bar{1}$, $2\bar{0}103\bar{2}\bar{3} \sim 3\bar{0}102\bar{3}\bar{2}$, $0\bar{2}321\bar{0}\bar{1} \sim 1\bar{2}320\bar{1}\bar{0}$,

$0\bar{3}132\bar{0}\bar{2} \sim 2\bar{3}130\bar{2}\bar{0}$, $1\bar{3}032\bar{1}\bar{2} \sim 2\bar{3}031\bar{2}\bar{1}$, $2\bar{0}301\bar{2}\bar{1} \sim 1\bar{0}302\bar{1}\bar{2}$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}213\bar{0}\bar{3}]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}213\bar{0}\bar{3})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 3\}$, $\{1\}$, $\{2\}$, $\{\bar{0}, \bar{3}\}$, $\{\bar{1}\}$, and $\{\bar{2}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_i^{\pm 1}$, $i \in \{1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_3N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}eN = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}N$ and $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-2}N = Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3N$ $= Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_1N =$ $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$, and $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

350. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$.

Let $[0\bar{1}213\bar{2}1]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N$.

Note that $N^{(0\bar{1}213\bar{2}1)} \geq N^{0\bar{1}213\bar{2}1} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}213\bar{2}1)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1N\right| = \frac{|N|}{\left|N^{(0\bar{1}213\bar{2}1)}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}213\bar{2}1]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}213\bar{2}1)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_i^{\pm1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_1^{-1}N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}eN = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}N$ and $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_1N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1^2N = Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1^{-1}N$ $= Nt_0t_1^{-1}t_2t_3t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_0N =$ $Nt_0t_1^{-1}t_2t_0t_1t_0^{-1}t_3N$, $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_0^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_2N = Nt_0t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_0N$, $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_2^{-1}N$ $= Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1t_3^{-1}N$, $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_3N = Nt_0t_1^{-1}t_2t_3t_1t_2N$, and $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_3^{-1}N = Nt_0t_1^{-1}t_2t_3t_2^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_1t_3t_2^{-1}t_1t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

351. We next consider the double coset $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2N$.

Let $[0\bar{1}21\bar{3}02]$ denote the double coset $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2t_1t_3^{-1}t_0t_2 = Nt_1t_0^{-1}t_3t_0t_2^{-1}t_1t_3$.

That is, in terms of our short-hand notation,

$$0\bar{1}21\bar{3}02 \sim 1\bar{0}30\bar{2}13.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}21\bar{3}02]$:

$0\bar{1}21\bar{3}02 \sim 1\bar{0}30\bar{2}13$,	$1\bar{0}20\bar{3}12 \sim 0\bar{1}31\bar{2}03$,	$2\bar{1}01\bar{3}20 \sim 1\bar{2}32\bar{0}13$,
$3\bar{1}21\bar{0}32 \sim 1\bar{3}03\bar{2}10$,	$0\bar{2}12\bar{3}01 \sim 2\bar{0}30\bar{1}23$,	$0\bar{3}23\bar{1}02 \sim 3\bar{0}10\bar{2}31$,
$1\bar{2}02\bar{3}10 \sim 2\bar{1}31\bar{0}23$,	$2\bar{0}10\bar{3}21 \sim 0\bar{2}32\bar{1}03$,	$1\bar{3}23\bar{0}12 \sim 3\bar{1}01\bar{2}30$,
$3\bar{0}20\bar{1}32 \sim 0\bar{3}13\bar{2}01$,	$2\bar{3}03\bar{1}20 \sim 3\bar{2}12\bar{0}31$,	$2\bar{3}13\bar{0}21 \sim 3\bar{2}02\bar{1}30$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}21\bar{3}02]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}21\bar{3}02)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1\}$, $\{2, 3\}$, $\{\bar{0}, \bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_2t_i^{\pm 1}$, $i \in \{0, 2\}$.

But note that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_2t_{\bar{2}}^{-1}N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0eN = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0N$ and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_2t_2N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_2^2N = Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_{\bar{2}}^{-1}N$
$= Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_1t_{\bar{0}}^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_2t_0N =$ $Nt_0t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_{\bar{0}}^{-1}t_{\bar{1}}^{-1}N$, and $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_2t_{\bar{0}}^{-1}N = Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_{\bar{1}}^{-1}t_2t_1t_{\bar{3}}^{-1}t_0t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

352. We next consider the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_3t_1t_0t_2N$.

Let [$0\bar{1}23102$] denote the double coset $Nt_0t_{\bar{1}}^{-1}t_2t_3t_1t_0t_2N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_{\bar{1}}^{-1}t_2t_3t_1t_0t_2 = Nt_1t_{\bar{0}}^{-1}t_3t_2t_0t_1t_3$.

That is, in terms of our short-hand notation,

$$0\bar{1}23102 \sim 1\bar{0}32013.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset [$0\bar{1}23102$]:

$0\bar{1}23102 \sim 1\bar{0}32013$,	$1\bar{0}23012 \sim 0\bar{1}32103$,	$2\bar{1}03120 \sim 1\bar{2}30213$,
$3\bar{1}20132 \sim 1\bar{3}02310$,	$0\bar{2}13201 \sim 2\bar{0}31023$,	$0\bar{3}21302 \sim 3\bar{0}12031$,
$1\bar{2}03210 \sim 2\bar{1}30123$,	$2\bar{0}13021 \sim 0\bar{2}31203$,	$1\bar{3}20312 \sim 3\bar{1}02130$,
$3\bar{0}21032 \sim 0\bar{3}12301$,	$2\bar{3}01320 \sim 3\bar{2}10231$,	$1\bar{3}02310 \sim 3\bar{1}20132$

Since each of the twenty-four single cosets has two names, the double coset [$0\bar{1}23102$] must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}23102)}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1\}$, $\{2, 3\}$, $\{\bar{0}, \bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_{\bar{1}}^{-1}t_2t_3t_1t_0t_2t_i^{\pm 1}$, $i \in \{0, 2\}$.

But note that $Nt_0t_1^{-1}t_2t_3t_1t_0t_2t_2^{-1}N = Nt_0t_1^{-1}t_2t_3t_1t_0eN = Nt_0t_1^{-1}t_2t_3t_1t_0N$ and $Nt_0t_1^{-1}t_2t_3t_1t_0t_2t_2N = Nt_0t_1^{-1}t_2t_3t_1t_0t_2^2N = Nt_0t_1^{-1}t_2t_3t_1t_0t_2^{-1}N$
$= Nt_0t_1^{-1}t_0^{-1}t_2t_3t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2t_3t_1t_0t_2t_0N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$ and $Nt_0t_1^{-1}t_2t_3t_1t_0t_2t_0^{-1}N = Nt_0t_1t_2t_3t_0t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2t_3t_1t_0t_2t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

353. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$.

Let $[0\bar{1}\bar{2}0123]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N$.

Note that $N^{(0\bar{1}\bar{2}0123)} \geq N^{0\bar{1}\bar{2}0123} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}0123)}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3N\right| = \frac{|N|}{|N^{(0\bar{1}\bar{2}0123)}|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}0123]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}0123)}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_i^{\pm 1}$, $i \in \{0, 1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2eN = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2N$ and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_3N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3^2N = Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3^{-1}N$
$= Nt_0t_1^{-1}t_0^{-1}t_2t_3t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_0N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_0^{-1}N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_1^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$, and $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_2^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0t_1t_2t_3t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

354. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}13\bar{2}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1} = Nt_1t_0^{-1}t_3^{-1}t_2^{-1}t_0t_1t_3^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{0}13\bar{2} \sim 1\bar{0}\bar{3}\bar{2}01\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{0}13\bar{2}]$:

$0\bar{1}\bar{2}\bar{0}13\bar{2} \sim 1\bar{0}\bar{3}\bar{2}01\bar{3}$, $1\bar{0}\bar{2}\bar{3}01\bar{2} \sim 0\bar{1}\bar{3}\bar{2}10\bar{3}$, $2\bar{1}\bar{0}\bar{3}12\bar{0} \sim 1\bar{2}\bar{3}\bar{0}21\bar{3}$,

$3\bar{1}\bar{2}\bar{0}13\bar{2} \sim 1\bar{3}\bar{0}\bar{2}31\bar{0}$, $0\bar{2}\bar{1}\bar{3}20\bar{1} \sim 2\bar{0}\bar{3}\bar{1}02\bar{3}$, $0\bar{3}\bar{2}\bar{1}30\bar{2} \sim 3\bar{0}\bar{1}\bar{2}03\bar{1}$,

$1\bar{2}\bar{0}\bar{3}21\bar{0} \sim 2\bar{1}\bar{3}\bar{0}12\bar{3}$, $2\bar{0}\bar{1}\bar{3}02\bar{1} \sim 0\bar{2}\bar{3}\bar{1}20\bar{3}$, $1\bar{3}\bar{2}\bar{0}31\bar{2} \sim 3\bar{1}\bar{0}\bar{2}13\bar{0}$,

$3\bar{0}\bar{2}\bar{1}03\bar{2} \sim 0\bar{3}\bar{1}\bar{2}30\bar{1}$, $2\bar{3}\bar{0}\bar{1}32\bar{0} \sim 3\bar{2}\bar{1}\bar{0}23\bar{1}$, $1\bar{3}\bar{0}\bar{2}31\bar{0} \sim 3\bar{1}\bar{2}\bar{0}13\bar{2}$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}\bar{2}\bar{0}13\bar{2}]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{0}13\bar{2})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 2\}$, $\{1\}$, $\{3\}$, $\{\bar{0}, \bar{2}\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_i^{\pm 1}$, $i \in \{1, 2, 3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3eN$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_2^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2N = Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1N =$
$Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_3t_0t_2N$,
and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2t_0t_3^{-1}t_1^{-1}t_0^{-1}N$.

Therefore, we conclude that there is one distinct double coset of the form
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}N$.

355. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1} = Nt_2t_3^{-1}t_0^{-1}t_2^{-1}t_3t_1^{-1}t_3^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1} \sim 2\bar{3}\bar{0}\bar{2}3\bar{1}\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}]$:

$$0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1} \sim 2\bar{3}\bar{0}\bar{2}3\bar{1}\bar{3}, \qquad 1\bar{0}\bar{2}\bar{1}0\bar{3}\bar{0} \sim 2\bar{3}\bar{1}\bar{2}3\bar{0}\bar{3}, \qquad 2\bar{1}\bar{0}\bar{2}1\bar{3}\bar{1} \sim 0\bar{3}\bar{2}\bar{0}3\bar{1}\bar{3},$$

$$3\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1} \sim 2\bar{0}\bar{3}\bar{2}0\bar{1}\bar{0}, \qquad 0\bar{2}\bar{1}\bar{0}2\bar{3}\bar{2} \sim 1\bar{3}\bar{0}\bar{1}3\bar{2}\bar{3}, \qquad 0\bar{1}\bar{3}\bar{0}1\bar{2}\bar{1} \sim 3\bar{2}\bar{0}\bar{3}2\bar{1}\bar{2},$$

$$1\bar{2}\bar{0}\bar{1}2\bar{3}\bar{2} \sim 0\bar{3}\bar{1}\bar{0}3\bar{2}\bar{3}, \qquad 2\bar{0}\bar{1}\bar{2}0\bar{3}\bar{0} \sim 1\bar{3}\bar{2}\bar{1}3\bar{0}\bar{3}, \qquad 3\bar{0}\bar{2}\bar{3}0\bar{1}\bar{0} \sim 2\bar{1}\bar{3}\bar{2}1\bar{0}\bar{1},$$

$$3\bar{1}\bar{0}\bar{3}1\bar{2}\bar{1} \sim 0\bar{2}\bar{3}\bar{0}2\bar{1}\bar{2}, \qquad 1\bar{0}\bar{3}\bar{1}0\bar{2}\bar{0} \sim 3\bar{2}\bar{1}\bar{3}2\bar{0}\bar{2}, \qquad 1\bar{2}\bar{3}\bar{1}2\bar{0}\bar{2} \sim 3\bar{0}\bar{1}\bar{3}0\bar{2}\bar{0}$$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 2\}$, $\{1, 3\}$, $\{\bar{0}, \bar{2}\}$, and $\{\bar{1}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}eN$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}t_1^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

356. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$.

Let $[0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0 = Nt_2t_1^{-1}t_0^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0 \sim 2\bar{1}\bar{0}\bar{2}\bar{3}\bar{1}2.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0]$:

$$0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0 \sim 2\bar{1}\bar{0}\bar{2}\bar{3}\bar{1}2, \qquad 1\bar{0}\bar{2}\bar{1}\bar{3}\bar{0}1 \sim 2\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}2, \qquad 3\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}3 \sim 2\bar{1}\bar{3}\bar{2}\bar{0}\bar{1}2,$$

$$0\bar{2}\bar{1}\bar{0}\bar{3}\bar{2}0 \sim 1\bar{2}\bar{0}\bar{1}\bar{3}\bar{2}1, \qquad 0\bar{3}\bar{2}\bar{0}\bar{1}\bar{3}0 \sim 2\bar{3}\bar{0}\bar{2}\bar{1}\bar{3}2, \qquad 0\bar{1}\bar{3}\bar{0}\bar{2}\bar{1}0 \sim 3\bar{1}\bar{0}\bar{3}\bar{2}\bar{1}2,$$

$$1\bar{3}\bar{2}\bar{1}\bar{0}\bar{3}1 \sim 2\bar{3}\bar{1}\bar{2}\bar{0}\bar{3}2, \qquad 3\bar{0}\bar{2}\bar{3}\bar{1}\bar{0}3 \sim 2\bar{0}\bar{3}\bar{2}\bar{1}\bar{0}2, \qquad 0\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}0 \sim 3\bar{2}\bar{0}\bar{3}\bar{1}\bar{2}3,$$

$$0\bar{3}\bar{1}\bar{0}\bar{2}\bar{3}0 \sim 1\bar{3}\bar{0}\bar{1}\bar{2}\bar{3}1, \qquad 1\bar{2}\bar{3}\bar{1}\bar{0}\bar{2}1 \sim 3\bar{2}\bar{1}\bar{3}\bar{0}\bar{2}3, \qquad 3\bar{0}\bar{1}\bar{3}\bar{2}\bar{0}3 \sim 1\bar{0}\bar{3}\bar{1}\bar{2}\bar{0}1$$

Since each of the twenty-four single cosets has two names, the double coset $[0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0]$ must have at most twelve distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,2\}$, $\{1\}$, $\{3\}$, $\{\bar{0},\bar{2}\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_i^{\pm 1}$, $i \in \{0,1,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}eN$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_0N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0^2N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_1t_2^{-1}N.$

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_1N =$
$Nt_0t_1^{-1}t_2t_1t_3t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_2N$,
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N$, and
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_3^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form
$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

357. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$.

Let $[0\bar{1}\bar{2}132\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N$.

Note that $N^{(0\bar{1}\bar{2}132\bar{1})} \geq N^{0\bar{1}\bar{2}132\bar{1}} = \langle e \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}132\bar{1})}\right| \geq |\langle e \rangle| = 1$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}132\bar{1})}\right|} \leq \frac{24}{1} = 24$.

Therefore, the double coset $[0\bar{1}\bar{2}132\bar{1}]$ has at most twenty-four distinct single cosets.

Moreover, $N^{(0\bar{1}\bar{2}132\bar{1})}$ has eight orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most eight double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_i^{\pm1}$, $i \in \{0,1,2,3\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2eN = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2N$ and $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_1^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1N$ $= Nt_0t_1^{-1}t_2t_0t_1^{-1}t_2^{-1}t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_0N =$ $Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_2N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1^{-1}t_0^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_2^{-1}N$ $= Nt_0t_1^{-1}t_0^{-1}t_2t_3^{-1}t_1^{-1}N$, $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_3N = Nt_0t_1^{-1}t_2^{-1}t_3t_1t_0N$, and $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3t_0^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_1t_3t_2t_1^{-1}t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

358. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_1t_3^{-1}t_2^{-1}t_1^{-1} = Nt_1t_2^{-1}t_0^{-1}t_2t_3^{-1}t_0^{-1}t_2^{-1}$ $= Nt_2t_0^{-1}t_1^{-1}t_0t_3^{-1}t_1^{-1}t_0^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1} \sim 1\bar{2}\bar{0}2\bar{3}\bar{0}\bar{2} \sim 2\bar{0}\bar{1}0\bar{3}\bar{1}\bar{0}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}]$:

$$0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1} \sim 1\bar{2}\bar{0}2\bar{3}\bar{0}\bar{2} \sim 2\bar{0}\bar{1}0\bar{3}\bar{1}\bar{0}, \qquad 1\bar{0}\bar{2}0\bar{3}\bar{2}\bar{0} \sim 0\bar{2}\bar{1}2\bar{3}\bar{1}\bar{2} \sim 2\bar{1}\bar{0}1\bar{3}\bar{0}\bar{1},$$

$$2\bar{1}\bar{0}1\bar{3}\bar{0}\bar{1} \sim 1\bar{0}\bar{2}0\bar{3}\bar{2}\bar{0} \sim 0\bar{2}\bar{1}2\bar{3}\bar{1}\bar{2}, \qquad 3\bar{1}\bar{2}1\bar{0}\bar{2}\bar{1} \sim 1\bar{2}\bar{3}2\bar{0}\bar{3}\bar{2} \sim 2\bar{3}\bar{1}3\bar{0}\bar{1}\bar{3},$$

$$0\bar{3}\bar{2}3\bar{1}\bar{2}\bar{3} \sim 3\bar{2}\bar{0}2\bar{1}\bar{0}\bar{2} \sim 2\bar{0}\bar{3}0\bar{1}\bar{3}\bar{0}, \qquad 0\bar{1}\bar{3}1\bar{2}\bar{3}\bar{1} \sim 1\bar{3}\bar{0}3\bar{2}\bar{0}\bar{3} \sim 3\bar{0}\bar{1}0\bar{2}\bar{1}\bar{0},$$

$$1\bar{3}\bar{2}3\bar{0}\bar{2}\bar{3} \sim 3\bar{2}\bar{1}2\bar{0}\bar{1}\bar{2} \sim 2\bar{1}\bar{3}0\bar{0}\bar{3}\bar{1}, \qquad 3\bar{0}\bar{2}0\bar{1}\bar{2}\bar{0} \sim 0\bar{2}\bar{3}2\bar{1}\bar{3}\bar{2} \sim 2\bar{3}\bar{0}3\bar{1}\bar{0}\bar{3}$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}]$ must have at most eight distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 1, 2\}$, $\{3\}$, $\{\bar{0}, \bar{1}, \bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1^{-1} t_i^{\pm 1}$, $i \in \{1, 3\}$.

But note that $Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1^{-1} t_1 N = Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} eN$
$= Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} N$ and $Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1^{-1} t_1^{-1} N$
$= Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1^{-2} N = Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1 N = Nt_0^{-1} t_1^{-1} t_2^{-1} t_0 t_1^{-1} t_0^{-1} N$.

Moreover, with the help of MAGMA, we know that $Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1^{-1} t_3 N = Nt_0 t_1^{-1} t_2^{-1} t_3^{-1} t_0^{-1} t_1 N$ and $Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1^{-1} t_3^{-1} N = Nt_0^{-1} t_1 t_2 t_3 t_0 t_1^{-1} N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0 t_1^{-1} t_2^{-1} t_1 t_3^{-1} t_2^{-1} t_1^{-1} t_i^{\pm 1} N$, where $i \in \{0, 1, 2, 3\}$.

359. We next consider the double coset $Nt_0 t_1^{-1} t_2^{-1} t_3^{-1} t_1 t_0^{-1} t_1^{-1} N$.

Let $[0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}]$ denote the double coset $Nt_0 t_1^{-1} t_2^{-1} t_3^{-1} t_1 t_0^{-1} t_1^{-1} N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0 t_1^{-1} t_2^{-1} t_3^{-1} t_1 t_0^{-1} t_1^{-1} = Nt_0 t_2^{-1} t_3^{-1} t_1^{-1} t_2 t_0^{-1} t_2^{-1}$
$= Nt_0 t_3^{-1} t_1^{-1} t_2^{-1} t_3 t_0^{-1} t_3^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1} \sim 0\bar{2}\bar{3}\bar{1}2\bar{0}\bar{2} \sim 0\bar{3}\bar{1}\bar{2}3\bar{0}\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}]$:

$$0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1} \sim 0\bar{2}\bar{3}\bar{1}2\bar{0}\bar{2} \sim 0\bar{3}\bar{1}\bar{2}3\bar{0}\bar{3}, \quad 1\bar{0}\bar{2}\bar{3}0\bar{1}\bar{0} \sim 1\bar{2}\bar{3}\bar{0}2\bar{1}\bar{2} \sim 1\bar{3}\bar{0}\bar{2}3\bar{1}\bar{3},$$

$$2\bar{1}\bar{0}\bar{3}1\bar{2}\bar{1} \sim 2\bar{0}\bar{3}\bar{1}0\bar{2}\bar{0} \sim 2\bar{3}\bar{1}\bar{0}3\bar{2}\bar{3}, \quad 3\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1} \sim 3\bar{2}\bar{0}\bar{1}2\bar{3}\bar{2} \sim 3\bar{0}\bar{1}\bar{2}0\bar{3}\bar{0},$$

$$0\bar{2}\bar{1}\bar{3}2\bar{0}\bar{2} \sim 0\bar{1}\bar{3}\bar{2}1\bar{0}\bar{1} \sim 0\bar{3}\bar{2}\bar{1}3\bar{0}\bar{3}, \quad 1\bar{2}\bar{0}\bar{3}2\bar{1}\bar{2} \sim 1\bar{0}\bar{3}\bar{2}0\bar{1}\bar{0} \sim 1\bar{3}\bar{2}\bar{0}3\bar{1}\bar{3},$$

$$2\bar{0}\bar{1}\bar{3}0\bar{2}\bar{0} \sim 2\bar{1}\bar{3}\bar{0}1\bar{2}\bar{1} \sim 2\bar{3}\bar{0}\bar{1}3\bar{2}\bar{3}, \quad 3\bar{0}\bar{2}\bar{1}0\bar{3}\bar{0} \sim 3\bar{2}\bar{1}\bar{0}2\bar{3}\bar{2} \sim 3\bar{1}\bar{0}\bar{2}1\bar{3}\bar{1}$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}]$ must have at most eight distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2, 3\}$, $\{\bar{0}\}$, and $\{\bar{1}, \bar{2}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}eN$
$= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}t_1^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1N = Nt_0^{-1}t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N.$

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}t_0N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

360. We next consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1} = Nt_2t_1^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2t_1^{-1}$
$= Nt_3t_1^{-1}t_0^{-1}t_2^{-1}t_1^{-1}t_3t_1^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1} \sim 2\bar{1}\bar{3}\bar{0}\bar{1}2\bar{1} \sim 3\bar{1}\bar{0}\bar{2}\bar{1}3\bar{1}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1}]$:

$$0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1} \sim 2\bar{1}\bar{3}\bar{0}\bar{1}2\bar{1} \sim 3\bar{1}\bar{0}\bar{2}\bar{1}3\bar{1}, \qquad 1\bar{0}\bar{2}\bar{3}\bar{0}1\bar{0} \sim 2\bar{0}\bar{3}\bar{1}\bar{0}2\bar{0} \sim 3\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0},$$

$$2\bar{1}\bar{0}\bar{3}\bar{1}2\bar{1} \sim 0\bar{1}\bar{3}\bar{2}\bar{1}0\bar{1} \sim 3\bar{1}\bar{2}\bar{0}\bar{1}3\bar{1}, \qquad 3\bar{1}\bar{2}\bar{0}\bar{1}3\bar{1} \sim 2\bar{1}\bar{0}\bar{3}\bar{1}2\bar{1} \sim 0\bar{1}\bar{3}\bar{2}\bar{1}0\bar{1},$$

$$0\bar{2}\bar{1}\bar{3}\bar{2}0\bar{2} \sim 1\bar{2}\bar{3}\bar{0}\bar{2}1\bar{2} \sim 3\bar{2}\bar{0}\bar{1}\bar{2}3\bar{2}, \qquad 0\bar{3}\bar{2}\bar{1}\bar{3}0\bar{3} \sim 2\bar{3}\bar{1}\bar{0}\bar{3}2\bar{3} \sim 1\bar{3}\bar{0}\bar{2}\bar{3}1\bar{3},$$

$$1\bar{2}\bar{0}\bar{3}\bar{2}1\bar{2} \sim 0\bar{2}\bar{3}\bar{1}\bar{2}0\bar{2} \sim 3\bar{2}\bar{1}\bar{0}\bar{2}3\bar{2}, \qquad 2\bar{0}\bar{1}\bar{3}\bar{0}2\bar{0} \sim 1\bar{0}\bar{3}\bar{2}\bar{0}1\bar{0} \sim 3\bar{0}\bar{2}\bar{1}\bar{0}3\bar{0}$$

Since each of the twenty-four single cosets has three names, the double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1}]$ must have at most eight distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 2, 3\}$, $\{1\}$, $\{\bar{0}, \bar{2}, \bar{3}\}$, and $\{\bar{1}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0eN$
$= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}t_1^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1N = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}t_0N = Nt_0t_1^{-1}t_0t_2t_3t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}t_0^{-1}N = Nt_0t_1t_0^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1}t_i^{\pm1}N$, where $i \in \{0, 1, 2, 3\}$.

361. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$.

Let $[012\bar{0}\bar{3}10]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0 = Nt_0t_2t_1t_0^{-1}t_3^{-1}t_2t_0$.

That is, in terms of our short-hand notation,

$$012\bar{0}\bar{3}10 \sim 021\bar{0}\bar{3}20.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[012\bar{0}\bar{3}10]$:

$$\begin{array}{lll} 012\bar{0}\bar{3}10 \sim 021\bar{0}\bar{3}20, & 102\bar{1}\bar{3}01 \sim 120\bar{1}\bar{3}21, & 210\bar{2}\bar{3}12 \sim 201\bar{2}\bar{3}02, \\ 312\bar{3}\bar{0}13 \sim 321\bar{3}\bar{0}23, & 032\bar{0}\bar{1}30 \sim 023\bar{0}\bar{1}20, & 013\bar{0}\bar{2}10 \sim 031\bar{0}\bar{2}30, \\ 132\bar{1}\bar{0}31 \sim 123\bar{1}\bar{0}21, & 302\bar{3}\bar{1}03 \sim 320\bar{3}\bar{1}23, & 213\bar{2}\bar{0}12 \sim 231\bar{2}\bar{0}32, \\ 310\bar{3}\bar{2}13 \sim 301\bar{3}\bar{2}03, & 130\bar{1}\bar{2}31 \sim 103\bar{1}\bar{2}01, & 203\bar{2}\bar{1}02 \sim 230\bar{2}\bar{1}32 \end{array}$$

Since each of the twenty-four single cosets has two names, the double coset $[012\bar{0}\bar{3}10]$ must have at most twelve distinct single cosets.

Now, $N^{(012\bar{0}\bar{3}10)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}, \bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_i^{\pm1}$, $i \in \{0, 1, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1eN$
$= Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_0N$
$= Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^2N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_1N =$ $Nt_0t_1t_2t_3t_0t_2^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_1^{-1}N = Nt_0^{-1}t_1t_2t_0t_1^{-1}t_3^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}N$, and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_3^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

362. We next consider the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$.

Let $[012\bar{0}\bar{3}1\bar{0}]$ denote the double coset $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1} = Nt_0t_2t_1t_0^{-1}t_3^{-1}t_2t_0^{-1}$.

That is, in terms of our short-hand notation,

$$012\bar{0}\bar{3}1\bar{0} \sim 021\bar{0}\bar{3}2\bar{0}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[012\bar{0}\bar{3}1\bar{0}]$:

$$012\bar{0}\bar{3}1\bar{0} \sim 021\bar{0}\bar{3}2\bar{0}, \qquad 102\bar{1}\bar{3}0\bar{1} \sim 120\bar{1}\bar{3}2\bar{1}, \qquad 210\bar{2}\bar{3}1\bar{2} \sim 201\bar{2}\bar{3}0\bar{2},$$

$$312\bar{3}\bar{0}1\bar{3} \sim 321\bar{3}\bar{0}2\bar{3}, \qquad 032\bar{0}\bar{1}3\bar{0} \sim 023\bar{0}\bar{1}2\bar{0}, \qquad 013\bar{0}\bar{2}1\bar{0} \sim 031\bar{0}\bar{2}3\bar{0},$$

$$132\bar{1}\bar{0}3\bar{1} \sim 123\bar{1}\bar{0}2\bar{1}, \qquad 302\bar{3}\bar{1}0\bar{3} \sim 320\bar{3}\bar{1}2\bar{3}, \qquad 213\bar{2}\bar{0}1\bar{2} \sim 231\bar{2}\bar{0}3\bar{2},$$

$$310\bar{3}\bar{2}1\bar{3} \sim 301\bar{3}\bar{2}0\bar{3}, \qquad 130\bar{1}\bar{2}3\bar{1} \sim 103\bar{1}\bar{2}0\bar{1}, \qquad 203\bar{2}\bar{1}0\bar{2} \sim 230\bar{2}\bar{1}3\bar{2}$$

Since each of the twenty-four single cosets has two names, the double coset $[012\bar{0}\bar{3}1\bar{0}]$ must have at most twelve distinct single cosets.

Now, $N^{(012\bar{0}\bar{3}1\bar{0})}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1, 2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}, \bar{2}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_i^{\pm 1}$, $i \in \{0, 1, 3\}$.

But note that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_0N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1eN = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1N$ and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_0^{-1}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-2}N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_1N =$ $Nt_0^{-1}t_1^{-1}t_2t_1t_3t_0^{-1}N$, $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_1^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_1t_3^{-1}t_0N$,

$Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_3N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$, and $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_3^{-1}N$ $= Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$.

363. We next consider the double coset $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}N$.

Let $[012\bar{3}\bar{0}1\bar{3}]$ denote the double coset $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1} = Nt_3t_2t_1t_0^{-1}t_3^{-1}t_2t_0^{-1}$.

That is, in terms of our short-hand notation,

$$012\bar{3}\bar{0}1\bar{3} \sim 321\bar{0}\bar{3}2\bar{0}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[012\bar{3}\bar{0}1\bar{3}]$:

$012\bar{3}\bar{0}1\bar{3} \sim 321\bar{0}\bar{3}2\bar{0}$,	$102\bar{3}\bar{1}0\bar{3} \sim 320\bar{1}\bar{3}2\bar{1}$,	$210\bar{3}\bar{2}1\bar{3} \sim 301\bar{2}\bar{3}0\bar{2}$,
$312\bar{0}\bar{3}1\bar{0} \sim 021\bar{3}\bar{0}2\bar{3}$,	$032\bar{1}\bar{0}3\bar{1} \sim 123\bar{0}\bar{1}2\bar{0}$,	$013\bar{2}\bar{0}1\bar{2} \sim 231\bar{0}\bar{2}3\bar{0}$,
$120\bar{3}\bar{1}2\bar{3} \sim 302\bar{1}\bar{3}0\bar{1}$,	$201\bar{3}\bar{2}0\bar{3} \sim 310\bar{2}\bar{3}1\bar{2}$,	$132\bar{0}\bar{1}3\bar{0} \sim 023\bar{1}\bar{0}2\bar{1}$,
$031\bar{2}\bar{0}3\bar{2} \sim 213\bar{0}\bar{2}1\bar{0}$,	$103\bar{2}\bar{1}0\bar{2} \sim 230\bar{1}\bar{2}3\bar{1}$,	$130\bar{2}\bar{1}3\bar{2} \sim 203\bar{1}\bar{2}0\bar{1}$

Since each of the twenty-four single cosets has two names, the double coset $[012\bar{3}\bar{0}1\bar{3}]$ must have at most twelve distinct single cosets.

Now, $N^{(012\bar{3}\bar{0}1\bar{3})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 3\}$, $\{1, 2\}$, $\{\bar{0}, \bar{3}\}$, and $\{\bar{1}, \bar{2}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}t_i^{\pm 1}$, $i \in \{1, 3\}$.

But note that $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}t_3N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1eN = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1N$ and $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}t_3^{-1}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-2}N = Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3N$ $= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}t_1N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1t_2^{-1}N$ and $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}t_1^{-1}N = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2t_3^{-1}t_0^{-1}t_1t_3^{-1}t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

364. We next consider the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$.

Let $[01\bar{2}\bar{3}\bar{1}20]$ denote the double coset $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0 = Nt_2t_1t_0^{-1}t_3^{-1}t_1^{-1}t_0t_2$.

That is, in terms of our short-hand notation,

$$01\bar{2}\bar{3}\bar{1}20 \sim 21\bar{0}\bar{3}\bar{1}02.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[01\bar{2}\bar{3}\bar{1}20]$:

$$\begin{array}{lll}
01\bar{2}\bar{3}\bar{1}20 \sim 21\bar{0}\bar{3}\bar{1}02, & 10\bar{2}\bar{3}\bar{0}21 \sim 20\bar{1}\bar{3}\bar{0}12, & 12\bar{3}\bar{0}\bar{2}31 \sim 32\bar{1}\bar{0}\bar{2}13, \\
31\bar{2}\bar{0}\bar{1}23 \sim 21\bar{3}\bar{0}\bar{1}32, & 02\bar{1}\bar{3}\bar{2}10 \sim 12\bar{0}\bar{3}\bar{2}01, & 03\bar{2}\bar{1}\bar{3}20 \sim 23\bar{0}\bar{1}\bar{3}02, \\
01\bar{3}\bar{2}\bar{1}30 \sim 31\bar{0}\bar{2}\bar{1}03, & 13\bar{2}\bar{0}\bar{3}21 \sim 23\bar{1}\bar{0}\bar{3}12, & 30\bar{2}\bar{1}\bar{0}23 \sim 20\bar{3}\bar{1}\bar{0}32, \\
02\bar{3}\bar{1}\bar{2}30 \sim 32\bar{0}\bar{1}\bar{2}03, & 03\bar{1}\bar{2}\bar{3}10 \sim 13\bar{0}\bar{2}\bar{3}01, & 10\bar{3}\bar{2}\bar{0}31 \sim 30\bar{1}\bar{2}\bar{0}13
\end{array}$$

Since each of the twenty-four single cosets has two names, the double coset $[01\bar{2}\bar{3}\bar{1}20]$ must have at most twelve distinct single cosets.

Now, $N^{(01\bar{2}\bar{3}\bar{1}20)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0,2\}$, $\{1\}$, $\{3\}$, $\{\bar{0},\bar{2}\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_i^{\pm 1}$, $i \in \{0,1,3\}$.

But note that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_0^{-1}N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2eN$
$= Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2N$ and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_0N$
$= Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0^2N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2t_3t_0^{-1}t_2^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_1N =$ $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_1^{-1}N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$, $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_3N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$, and $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_3^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0t_i^{\pm1}N$, where $i \in \{0,1,2,3\}$.

365. We next consider the double coset $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$.

Let $[01\bar{0}23\bar{2}1]$ denote the double coset $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1 = Nt_0t_1t_0^{-1}t_3t_2t_3^{-1}t_1$.

That is, in terms of our short-hand notation,

$$01\bar{0}23\bar{2}1 \sim 01\bar{0}32\bar{3}1.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[01\bar{0}23\bar{2}1]$:

$$01\bar{0}23\bar{2}1 \sim 01\bar{0}32\bar{3}1, \quad 10\bar{1}23\bar{2}0 \sim 10\bar{1}32\bar{3}0, \quad 21\bar{2}03\bar{0}1 \sim 21\bar{2}30\bar{3}1,$$

$$31\bar{3}20\bar{2}1 \sim 31\bar{3}02\bar{0}1, \quad 02\bar{0}13\bar{1}2 \sim 02\bar{0}31\bar{3}2, \quad 03\bar{0}21\bar{2}3 \sim 03\bar{0}12\bar{1}3,$$

$$12\bar{1}03\bar{0}2 \sim 12\bar{1}30\bar{3}2, \quad 20\bar{2}13\bar{1}0 \sim 20\bar{2}31\bar{3}0, \quad 13\bar{1}20\bar{2}3 \sim 13\bar{1}02\bar{0}3,$$

$$30\bar{3}21\bar{2}0 \sim 30\bar{3}12\bar{1}0, \quad 23\bar{2}01\bar{0}3 \sim 23\bar{2}10\bar{1}3, \quad 32\bar{3}01\bar{0}2 \sim 32\bar{3}10\bar{1}2$$

Since each of the twenty-four single cosets has two names, the double coset $[01\bar{0}23\bar{2}1]$ must have at most twelve distinct single cosets.

Now, $N^{(01\bar{0}23\bar{2}1)}$ has six orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Therefore, there are at most six double cosets of the form NwN, where w is a word of length eight given by $w = t_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_i^{\pm1}$, $i \in \{0,1,2\}$.

But note that $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_1^{-1}N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}eN = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}N$ and $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_1N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1^2N = Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1^{-1}N$ $= Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0^{-1}N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_0N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$, $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_0^{-1}N = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0t_1t_3^{-1}N$, $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_2N = Nt_0^{-1}t_1^{-1}t_2t_1t_0t_3^{-1}N$, and $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_2^{-1}N$ $= Nt_0^{-1}t_1^{-1}t_2t_1t_0^{-1}t_3N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1t_0^{-1}t_2t_3t_2^{-1}t_1t_i^{\pm 1}N$, where $i \in \{0,1,2,3\}$.

366. We finally consider the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}N$.

Let $[0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}]$ denote the double coset $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}N$.

Now, with the help of MAGMA, we know that that the following right cosets, or single cosets, are equivalent: $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1} = Nt_2t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_0^{-1}t_1^{-1}$ $= Nt_0t_3^{-1}t_2^{-1}t_0^{-1}t_3t_1t_2^{-1}t_3^{-1} = Nt_2t_3^{-1}t_0^{-1}t_2^{-1}t_3t_1t_0^{-1}t_3^{-1}$.

That is, in terms of our short-hand notation,

$$0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1} \sim 2\bar{1}\bar{0}\bar{2}13\bar{0}\bar{1} \sim 0\bar{3}\bar{2}\bar{0}31\bar{2}\bar{3} \sim 2\bar{3}\bar{0}\bar{2}31\bar{0}\bar{3}.$$

By conjugating the equivalence relation above with the elements of S_4, we determine that the following single cosets are equivalent in the double coset $[0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}]$:

$$0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1} \sim 2\bar{1}\bar{0}\bar{2}13\bar{0}\bar{1} \sim 0\bar{3}\bar{2}\bar{0}31\bar{2}\bar{3} \sim 2\bar{3}\bar{0}\bar{2}31\bar{0}\bar{3},$$

$$1\bar{0}\bar{2}\bar{1}03\bar{2}\bar{0} \sim 2\bar{0}\bar{1}\bar{2}03\bar{1}\bar{0} \sim 1\bar{3}\bar{2}\bar{1}30\bar{2}\bar{3} \sim 2\bar{3}\bar{1}\bar{2}30\bar{1}\bar{3},$$

$$3\bar{1}\bar{2}\bar{3}10\bar{2}\bar{1} \sim 2\bar{1}\bar{3}\bar{2}10\bar{3}\bar{1} \sim 3\bar{0}\bar{2}\bar{3}01\bar{2}\bar{0} \sim 2\bar{0}\bar{3}\bar{2}01\bar{3}\bar{0},$$

$$0\bar{2}\bar{1}\bar{0}23\bar{1}\bar{2} \sim 1\bar{2}\bar{0}\bar{1}23\bar{0}\bar{2} \sim 0\bar{3}\bar{1}\bar{0}32\bar{1}\bar{3} \sim 1\bar{3}\bar{0}\bar{1}32\bar{0}\bar{3},$$

$$0\bar{1}\bar{3}\bar{0}12\bar{3}\bar{1} \sim 3\bar{1}\bar{0}\bar{3}12\bar{0}\bar{1} \sim 0\bar{2}\bar{3}\bar{0}21\bar{3}\bar{2} \sim 3\bar{2}\bar{0}\bar{3}21\bar{0}\bar{2},$$

$$1\bar{0}\bar{3}\bar{1}02\bar{3}\bar{0} \sim 3\bar{0}\bar{1}\bar{3}02\bar{1}\bar{0} \sim 1\bar{2}\bar{3}\bar{1}20\bar{3}\bar{2} \sim 3\bar{2}\bar{1}\bar{3}20\bar{1}\bar{2}$$

Since each of the twenty-four single cosets has four names, the double coset $[0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}]$ must have at most six distinct single cosets.

An alternative approach for determining the order of the double coset is as follows: We note that $N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})} \geq N^{0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}} = \langle e \rangle$. But, with the help of MAGMA, we know that $N(t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1})^{(0\ 2)} = Nt_2t_1^{-1}t_0^{-1}t_2^{-1}t_1t_3t_0^{-1}t_1^{-1}$ $= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}$ implies that $(0\ 2) \in N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})}$, and $N(t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1})^{(1\ 3)} = Nt_0t_3^{-1}t_2^{-1}t_0^{-1}t_3t_1t_2^{-1}t_3^{-1} = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}$ implies that $(1\ 3) \in N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})}$. Therefore, $(0\ 2), (1\ 3) \in N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})}$, and so $N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})} \geq \langle (0\ 2), (1\ 3) \rangle$. Thus $\left|N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})}\right| \geq |\langle (0\ 2), (1\ 3) \rangle| = 4$ and so, by Lemma 1.4, $\left|Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}N\right| = \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})}\right|} \leq \frac{24}{4} = 6$.

Therefore, as we concluded earlier, the double coset $[0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}]$ has at most six distinct single cosets.

Now, $N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})}$ has four orbits on $T = \{t_0, t_1, t_2, t_3\}$: $\{0, 2\}$, $\{1, 3\}$, $\{\bar{0}, \bar{2}\}$, and $\{\bar{1}, \bar{3}\}$.

Therefore, there are at most four double cosets of the form NwN, where w is a word of length nine given by $w = t_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_i^{\pm 1}$, $i \in \{0, 1\}$.

But note that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_1N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}eN$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_1^{-1}N$
$= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-2}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1N = Nt_0t_1t_2t_0^{-1}t_3^{-1}t_1t_0N$.

Moreover, with the help of MAGMA, we know that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_0N = Nt_0t_1t_2^{-1}t_3^{-1}t_1^{-1}t_2t_0N$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_0^{-1}N = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1}t_0N$.

Therefore, we conclude that there are no distinct double cosets of the form $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_i^{\pm 1}N$, where $i \in \{0, 1, 2, 3\}$. In fact, since neither of the double cosets

$$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_0N,$$

$$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_1N,$$

$$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_0^{-1}N,$$

$$Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_3t_2^{-1}t_1^{-1}t_1^{-1}N$$

is distinct, there are no distict double cosets of the form NwN, where w is a word of length nine or greater, in G. Our manual double coset enumeration of G over S_4 is therefore complete.

In total, therefore, there are at most 366 distinct double cosets of N in G and at most 7920 distinct right (single) cosets of N in G.

Double Cosets of N in G. Below, in our short-form notation, we list the 366 distinct double cosets of N in G, along with the names of equivalent double cosets. The first name listed for each double coset will be considered its cannonical name:

1. [*]
2. [0]

3. $[\bar{0}]$
4. $[01] = [\bar{0}\bar{1}\bar{0}]$
5. $[0\bar{1}]$
6. $[\bar{0}1] = [0\bar{1}\bar{0}\bar{1}]$
7. $[\bar{0}\bar{1}] = [\bar{0}10]$
8. $[01\bar{0}] = [\bar{0}\bar{1}0] = [0\bar{1}010]$
9. $[012] = [0\bar{1}\bar{2}\bar{1}]$
10. $[01\bar{2}] = [0\bar{1}212]$
11. $[0\bar{1}0] = [0\bar{1}2\bar{1}2] = [0\bar{1}0\bar{1}2]$
12. $[0\bar{1}\bar{0}] = [\bar{0}10]$
13. $[0\bar{1}2] = [0\bar{1}2\bar{0}1] = [0\bar{1}2\bar{0}2] = [0\bar{1}2\bar{0}3] = [0\bar{1}2\bar{3}0] = [0\bar{1}2\bar{3}1]$
14. $[0\bar{1}\bar{2}] = [0121]$
15. $[\bar{0}1\bar{0}] = [0\bar{1}\bar{0}1] = [01\bar{0}10] = [\bar{0}1\bar{0}1\bar{2}]$
16. $[\bar{0}12] = [\bar{0}\bar{1}\bar{2}\bar{1}]$
17. $[\bar{0}1\bar{2}] = [\bar{0}\bar{1}212] = [\bar{0}1\bar{0}2\bar{1}] = [\bar{0}1\bar{2}0\bar{1}] = [\bar{0}1\bar{2}0\bar{2}] = [\bar{0}1\bar{2}0\bar{3}] = [\bar{0}1\bar{2}3\bar{0}] = [\bar{0}1\bar{2}3\bar{1}]$
18. $[\bar{0}\bar{1}2] = [01\bar{2}\bar{0}\bar{2}]$
19. $[\bar{0}\bar{1}\bar{2}] = [\bar{0}121]$
20. $[0\bar{1}20] = [0\bar{1}2\bar{0}\bar{1}] = [0\bar{1}2\bar{0}\bar{2}] = [0\bar{1}2\bar{0}3] = [0\bar{1}\bar{2}02] = [0\bar{1}\bar{0}\bar{2}\bar{0}]$
21. $[0\bar{1}2\bar{0}]$
22. $[0\bar{1}21] = [01\bar{2}\bar{1}] = [0\bar{1}20\bar{3}\bar{0}]$
23. $[0\bar{1}2\bar{1}] = [0\bar{1}0\bar{2}] = [0\bar{1}2\bar{1}0\bar{1}] = [0\bar{1}2\bar{1}0\bar{3}] = [0\bar{1}2\bar{1}3\bar{0}] = [0\bar{1}2\bar{1}3\bar{1}] = [0\bar{1}2\bar{1}3\bar{2}]$
24. $[0\bar{1}23] = [0\bar{1}2\bar{3}\bar{0}] = [0\bar{1}2\bar{3}\bar{1}] = [0\bar{1}\bar{2}\bar{3}\bar{2}] = [0\bar{1}20\bar{1}\bar{0}] = [0\bar{1}2030] = [0\bar{1}2131]$
25. $[0\bar{1}2\bar{3}] = [0\bar{1}2\bar{1}0\bar{2}]$
26. $[0\bar{1}\bar{2}0] = [0\bar{1}20\bar{2}] = [0\bar{1}\bar{0}\bar{2}0]$
27. $[0\bar{1}\bar{2}\bar{0}] = [0\bar{1}020]$
28. $[0\bar{1}\bar{2}1] = [012\bar{1}] = [0\bar{1}02\bar{1}]$
29. $[0\bar{1}\bar{2}3] = [0\bar{1}2\bar{3}\bar{2}\bar{0}] = [0\bar{1}2\bar{3}\bar{2}\bar{1}] = [0\bar{1}2\bar{3}\bar{2}\bar{3}]$
30. $[0\bar{1}\bar{2}\bar{3}] = [0\bar{1}232]$
31. $[0\bar{1}01] = [01\bar{0}\bar{1}] = [0\bar{1}0\bar{1}\bar{2}] = [01\bar{2}1\bar{2}] = [0\bar{1}\bar{2}1\bar{2}1]$
32. $[0\bar{1}0\bar{1}] = [\bar{0}1\bar{0}10] = [0\bar{1}0\bar{1}0\bar{1}] = [0\bar{1}0\bar{1}0\bar{2}] = [0\bar{1}0\bar{1}0\bar{3}]$
33. $[0\bar{1}02] = [0\bar{1}2\bar{1}\bar{2}] = [0\bar{1}\bar{2}0\bar{2}] = [0\bar{1}\bar{2}12] = [0\bar{1}\bar{2}3\bar{2}3]$
34. $[0\bar{1}\bar{0}2] = [\bar{0}12\bar{0}2] = [\bar{0}1\bar{2}\bar{0}\bar{2}] = [0\bar{1}2\bar{1}3\bar{2}] = [0\bar{1}\bar{2}\bar{0}20]$

35. $[0\bar{1}\bar{0}\bar{2}] = [0\bar{1}202] = [0\bar{1}\bar{2}0\bar{2}]$
36. $[0120] = [01\bar{0}\bar{2}\bar{0}]$
37. $[012\bar{0}] = [0\bar{1}\bar{2}313]$
38. $[0123] = [01\bar{2}\bar{3}\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{3}2]$
39. $[012\bar{3}] = [0\bar{1}\bar{2}010]$
40. $[01\bar{2}0] = [01\bar{0}\bar{2}1] = [\bar{0}\bar{1}2\bar{0}1] = [\bar{0}\bar{1}2\bar{3}1] = [012\bar{0}\bar{2}\bar{0}]$
41. $[01\bar{2}\bar{0}] = [\bar{0}\bar{1}20]$
42. $[01\bar{2}1] = [0\bar{1}21\bar{2}] = [0\bar{1}012] = [0\bar{1}\bar{2}1\bar{2}\bar{1}]$
43. $[01\bar{2}3] = [\bar{0}\bar{1}2\bar{0}3] = [\bar{0}\bar{1}2\bar{3}0] = [012\bar{3}\bar{2}\bar{3}] = [01\bar{2}1\bar{0}2]$
44. $[01\bar{2}\bar{3}] = [01232] = [012\bar{3}\bar{0}2]$
45. $[01\bar{0}1] = [\bar{0}1\bar{0}\bar{1}] = [0\bar{1}01\bar{0}] = [\bar{0}\bar{1}2\bar{1}2] = [\bar{0}1\bar{0}12] = [012\bar{0}2\bar{0}]$
46. $[01\bar{0}2] = [\bar{0}\bar{1}2\bar{0}2] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{2}] = [01\bar{2}1\bar{3}2] = [01\bar{0}2\bar{3}0] = [01\bar{0}2\bar{3}1]$
47. $[01\bar{0}\bar{2}] = [01202] = [01\bar{2}0\bar{2}] = [012\bar{0}\bar{2}0] = [01\bar{0}\bar{2}3\bar{0}] = [01\bar{0}\bar{2}3\bar{1}]$
48. $[\bar{0}\bar{1}2\bar{0}] = [01\bar{2}0\bar{1}] = [01\bar{2}\bar{0}2] = [01\bar{2}3\bar{1}] = [01\bar{0}2\bar{1}]$
49. $[\bar{0}\bar{1}21] = [\bar{0}1\bar{2}\bar{1}] = [\bar{0}1\bar{0}21]$
50. $[\bar{0}\bar{1}2\bar{1}] = [01\bar{0}1\bar{2}] = [0120\bar{2}\bar{0}] = [012\bar{0}20] = [01\bar{2}132]$
51. $[\bar{0}\bar{1}23] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{2}] = [0\bar{1}\bar{2}031]$
52. $[\bar{0}\bar{1}2\bar{3}] = [01\bar{2}0\bar{3}] = [01\bar{2}3\bar{0}] = [\bar{0}\bar{1}2\bar{1}0\bar{2}] = [\bar{0}\bar{1}\bar{2}323]$
53. $[\bar{0}\bar{1}\bar{2}0] = [0\bar{1}\bar{2}1\bar{2}3] = [01\bar{0}21\bar{2}]$
54. $[\bar{0}\bar{1}\bar{2}\bar{0}] = [01\bar{0}20] = [012012] = [01\bar{2}1\bar{3}\bar{2}]$
55. $[\bar{0}\bar{1}\bar{2}1] = [\bar{0}12\bar{1}] = [\bar{0}1\bar{0}\bar{2}1] = [0\bar{1}\bar{2}3\bar{0}2] = [0\bar{1}\bar{2}3\bar{1}2] = [012\bar{0}2\bar{3}]$
56. $[\bar{0}\bar{1}\bar{2}3] = [0\bar{1}\bar{2}0\bar{1}3] = [0\bar{1}\bar{2}0\bar{3}1] = [\bar{0}\bar{1}230\bar{1}]$
57. $[\bar{0}\bar{1}\bar{2}\bar{3}] = [\bar{0}\bar{1}232]$
58. $[\bar{0}1\bar{0}1] = [0\bar{1}0\bar{1}\bar{0}] = [01\bar{0}1\bar{0}] = [0\bar{1}0\bar{1}01] = [0\bar{1}0\bar{1}02] = [0\bar{1}0\bar{1}03]$
59. $[\bar{0}1\bar{0}2] = [\bar{0}1\bar{2}1] = [\bar{0}\bar{1}21\bar{2}] = [0\bar{1}\bar{0}\bar{2}1\bar{2}] = [\bar{0}1\bar{0}2\bar{0}1] = [\bar{0}1\bar{0}2\bar{0}2] = [\bar{0}1\bar{0}2\bar{0}3] = [\bar{0}1\bar{0}2\bar{3}0]$
$= [\bar{0}1\bar{0}2\bar{3}1]$
60. $[\bar{0}1\bar{0}\bar{2}] = [\bar{0}\bar{1}\bar{2}1\bar{2}] = [\bar{0}1202] = [0\bar{1}\bar{0}\bar{2}1\bar{2}] = [012\bar{0}23]$
61. $[\bar{0}120] = [\bar{0}1\bar{0}\bar{2}\bar{0}] = [0\bar{1}\bar{0}\bar{2}12] = [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{3}]$
62. $[\bar{0}12\bar{0}] = [0\bar{1}\bar{0}2\bar{1}] = [\bar{0}1\bar{2}\bar{0}2] = [012\bar{3}0\bar{2}] = [012\bar{3}1\bar{2}] = [01\bar{2}\bar{3}10] = [\bar{0}\bar{1}\bar{2}313]$
63. $[\bar{0}123] = [\bar{0}1\bar{2}\bar{3}\bar{2}] = [0\bar{1}\bar{2}\bar{3}01]$
64. $[\bar{0}12\bar{3}] = [0\bar{1}\bar{2}10\bar{3}] = [0\bar{1}\bar{2}13\bar{0}] = [\bar{0}\bar{1}\bar{2}010] = [\bar{0}1\bar{0}\bar{2}3\bar{2}] = [\bar{0}12\bar{3}\bar{2}0] = [\bar{0}12\bar{3}\bar{2}1]$
65. $[\bar{0}1\bar{2}0]$

66. $[\bar{0}1\bar{2}\bar{0}] = [0\bar{1}\bar{0}21] = [\bar{0}12\bar{0}\bar{2}] = [\bar{0}1\bar{2}01] = [\bar{0}1\bar{2}02] = [\bar{0}1\bar{2}03] = [\bar{0}\bar{1}21\bar{3}\bar{2}]$
67. $[\bar{0}1\bar{2}3] = [\bar{0}1\bar{0}2\bar{3}2] = [\bar{0}12\bar{3}\bar{2}\bar{3}]$
68. $[\bar{0}1\bar{2}\bar{3}] = [\bar{0}1232] = [\bar{0}1\bar{2}30] = [\bar{0}1\bar{2}31] = [\bar{0}\bar{1}21\bar{0}\bar{2}] = [\bar{0}1\bar{2}\bar{0}10] = [\bar{0}1\bar{2}\bar{0}\bar{3}\bar{0}]$
69. $[0\bar{1}201] = [0\bar{1}2\bar{1}\bar{0}\bar{1}]$
70. $[0\bar{1}20\bar{1}] = [0\bar{1}230] = [0\bar{1}213\bar{1}] = [\bar{0}\bar{1}\bar{2}301]$
71. $[0\bar{1}203] = [0\bar{1}23\bar{1}] = [\bar{0}1\bar{2}\bar{0}\bar{3}2]$
72. $[0\bar{1}20\bar{3}] = [0\bar{1}210]$
73. $[0\bar{1}21\bar{0}] = [0\bar{1}20\bar{3}0] = [0\bar{1}2\bar{1}\bar{0}3] = [01\bar{2}313] = [01\bar{2}\bar{0}1\bar{0}]$
74. $[0\bar{1}213] = [0\bar{1}23\bar{0}] = [0\bar{1}20\bar{1}0]$
75. $[0\bar{1}21\bar{3}] = [0\bar{1}2\bar{1}\bar{3}0] = [01\bar{2}\bar{3}1\bar{3}] = [01\bar{2}010] = [0\bar{1}201\bar{0}\bar{1}]$
76. $[0\bar{1}2\bar{1}0] = [0\bar{1}2\bar{3}2]$
77. $[0\bar{1}2\bar{1}\bar{0}] = [0\bar{1}2010] = [0\bar{1}21\bar{0}\bar{1}] = [0\bar{1}2\bar{1}01] = [0\bar{1}2\bar{1}03] = [0\bar{1}23\bar{2}\bar{3}] = [0\bar{1}0232]$
78. $[0\bar{1}2\bar{1}3]$
79. $[0\bar{1}2\bar{1}\bar{3}] = [0\bar{1}\bar{0}20] = [0\bar{1}21\bar{3}\bar{1}] = [0\bar{1}2\bar{1}30] = [0\bar{1}2\bar{1}31] = [0\bar{1}2\bar{1}32] = [0\bar{1}2313]$
$= [0\bar{1}\bar{2}\bar{0}2\bar{0}] = [0\bar{1}201\bar{0}1] = [0\bar{1}21\bar{0}1\bar{0}]$
80. $[0\bar{1}231] = [0\bar{1}203\bar{0}] = [0\bar{1}2\bar{1}\bar{3}\bar{1}] = [0\bar{1}21\bar{0}10]$
81. $[0\bar{1}23\bar{2}] = [0\bar{1}\bar{2}\bar{3}2] = [0\bar{1}2\bar{1}\bar{0}2] = [0\bar{1}023\bar{2}]$
82. $[0\bar{1}2\bar{3}\bar{2}] = [0\bar{1}\bar{2}32] = [0\bar{1}2\bar{1}02]$
83. $[0\bar{1}\bar{2}01] = [012\bar{3}\bar{1}] = [0\bar{1}\bar{2}013\bar{2}]$
84. $[0\bar{1}\bar{2}0\bar{1}] = [\bar{0}\bar{1}\bar{2}3\bar{0}] = [0\bar{1}\bar{2}10\bar{2}] = [012\bar{0}1\bar{2}] = [\bar{0}\bar{1}2301] = [0\bar{1}20121]$
85. $[0\bar{1}\bar{2}03] = [\bar{0}\bar{1}23\bar{1}] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{3}] = [01\bar{2}013] = [\bar{0}\bar{1}210\bar{1}]$
86. $[0\bar{1}\bar{2}0\bar{3}] = [\bar{0}\bar{1}\bar{2}3\bar{1}] = [012\bar{3}1\bar{0}] = [\bar{0}\bar{1}\bar{2}13\bar{2}] = [\bar{0}120\bar{3}2] = [0\bar{1}20323]$
87. $[0\bar{1}\bar{2}\bar{0}1] = [0\bar{1}\bar{2}0\bar{1}\bar{0}\bar{1}] = [0\bar{1}\bar{2}03\bar{0}3]$
88. $[0\bar{1}\bar{2}\bar{0}\bar{1}] = [0\bar{1}\bar{2}101] = [0\bar{1}\bar{2}\bar{3}0\bar{3}] = [\bar{0}12012]$
89. $[0\bar{1}\bar{2}\bar{0}2] = [0\bar{1}02\bar{0}] = [0\bar{1}\bar{0}2\bar{0}] = [0\bar{1}2\bar{1}\bar{3}2]$
90. $[0\bar{1}\bar{2}\bar{0}3] = [0\bar{1}\bar{2}1\bar{0}\bar{2}] = [012\bar{3}13] = [\bar{0}120\bar{3}\bar{0}]$
91. $[0\bar{1}\bar{2}\bar{0}\bar{3}] = [0123\bar{1}] = [0\bar{1}\bar{2}303]$
92. $[0\bar{1}\bar{2}10] = [\bar{0}12\bar{3}0] = [0\bar{1}\bar{2}0\bar{1}0] = [0\bar{1}\bar{2}\bar{0}\bar{1}\bar{0}]$
93. $[0\bar{1}\bar{2}1\bar{0}] = [0\bar{1}\bar{2}\bar{0}32] = [012\bar{3}1\bar{3}] = [012313] = [\bar{0}1\bar{0}\bar{2}3\bar{0}] = [0\bar{1}\bar{2}\bar{0}23\bar{2}]$
94. $[0\bar{1}\bar{2}1\bar{2}] = [0\bar{1}01\bar{2}] = [0\bar{1}021] = [01\bar{2}12] = [\bar{0}\bar{1}\bar{2}0\bar{2}] = [012\bar{3}2\bar{3}] = [01\bar{0}212] = [0\bar{1}\bar{2}\bar{0}1\bar{0}1]$
95. $[0\bar{1}\bar{2}13] = [\bar{0}12\bar{3}1] = [0\bar{1}\bar{2}3\bar{1}3] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{3}] = [\bar{0}\bar{1}\bar{2}01\bar{0}] = [\bar{0}\bar{1}\bar{2}0\bar{3}2]$
96. $[0\bar{1}\bar{2}1\bar{3}] = [012010] = [012\bar{0}1\bar{0}] = [012\bar{0}\bar{3}2] = [\bar{0}\bar{1}\bar{2}10\bar{1}] = [\bar{0}1\bar{0}\bar{2}3\bar{1}]$

97. $[0\bar{1}\bar{2}30] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{0}] = [\bar{0}\bar{1}\bar{2}031] = [0\bar{1}\bar{2}302\bar{0}]$

98. $[0\bar{1}\bar{2}3\bar{0}] = [\bar{0}\bar{1}\bar{2}1\bar{3}] = [012\bar{3}0\bar{1}] = [\bar{0}12010] = [0\bar{1}\bar{2}\bar{0}21\bar{2}]$

99. $[0\bar{1}\bar{2}31] = [012\bar{0}\bar{1}] = [0\bar{1}\bar{2}\bar{0}123] = [0\bar{1}\bar{2}312\bar{1}]$

100. $[0\bar{1}\bar{2}3\bar{1}] = [\bar{0}\bar{1}\bar{2}1\bar{0}] = [0\bar{1}\bar{2}13\bar{2}] = [012\bar{0}\bar{3}\bar{2}] = [\bar{0}12313]$

101. $[0\bar{1}\bar{2}3\bar{2}] = [0\bar{1}02\bar{3}] = [0\bar{1}2\bar{3}\bar{2}0] = [0\bar{1}2\bar{3}\bar{2}1] = [0\bar{1}2\bar{3}\bar{2}3] = [\bar{0}\bar{1}\bar{2}032] = [\bar{0}\bar{1}\bar{2}10\bar{3}]$
$= [\bar{0}\bar{1}\bar{2}13\bar{0}] = [0\bar{1}\bar{2}\bar{0}121]$

102. $[0\bar{1}\bar{2}\bar{3}0] = [\bar{0}123\bar{0}] = [0\bar{1}\bar{2}\bar{0}\bar{1}2] = [\bar{0}1201\bar{2}] = [0\bar{1}203\bar{1}\bar{0}]$

103. $[0\bar{1}\bar{2}\bar{3}\bar{0}] = [0\bar{1}\bar{2}030] = [0\bar{1}2013\bar{0}] = [0\bar{1}\bar{2}\bar{3}\bar{0}1\bar{3}] = [0\bar{1}\bar{2}\bar{3}\bar{0}1\bar{2}]$

104. $[0\bar{1}\bar{2}\bar{3}1] = [01\bar{0}213] = [01\bar{0}23\bar{0}] = [0\bar{1}213\bar{0}\bar{1}] = [0\bar{1}\bar{2}1323\bar{3}]$

105. $[0\bar{1}\bar{2}\bar{3}\bar{1}] = [0\bar{1}\bar{2}131] = [\bar{0}\bar{1}\bar{2}0\bar{3}\bar{2}] = [\bar{0}12310] = [0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{2}] = [0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{3}]$

106. $[0\bar{1}0\bar{1}0] = [\bar{0}1\bar{0}1\bar{0}]$

107. $[0\bar{1}023] = [0\bar{1}2\bar{1}0\bar{2}] = [0\bar{1}23\bar{2}3] = [0\bar{1}\bar{2}3\bar{2}\bar{3}] = [0120\bar{1}0] = [0\bar{1}\bar{2}\bar{0}3\bar{0}3] = [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{3}]$

108. $[0\bar{1}\bar{0}23] = [0\bar{1}\bar{0}\bar{2}\bar{3}\bar{2}] = [0\bar{1}201\bar{2}1] = [0\bar{1}\bar{2}\bar{0}31\bar{0}] = [0\bar{1}\bar{0}23\bar{2}0] = [0\bar{1}\bar{0}23\bar{2}1] = [0\bar{1}\bar{0}23\bar{2}3]$

109. $[0\bar{1}\bar{0}2\bar{3}] = [0\bar{1}\bar{0}\bar{2}3\bar{2}] = [012\bar{0}1\bar{3}] = [012\bar{0}3\bar{1}] = [0\bar{1}\bar{2}3\bar{1}20] = [0\bar{1}\bar{0}2\bar{3}\bar{2}3]$

110. $[0\bar{1}\bar{0}\bar{2}1] = [\bar{0}1\bar{0}\bar{2}0] = [\bar{0}120\bar{2}] = [0\bar{1}20\bar{3}\bar{2}\bar{3}] = [0\bar{1}\bar{2}\bar{0}\bar{1}02]$

111. $[0\bar{1}\bar{0}\bar{2}\bar{1}] = [\bar{0}1\bar{0}20] = [\bar{0}\bar{1}2102] = [\bar{0}1\bar{0}2\bar{0}\bar{2}] = [\bar{0}1\bar{0}2\bar{0}\bar{3}] = [\bar{0}1\bar{0}2\bar{0}\bar{1}] = [\bar{0}1\bar{2}3\bar{1}3]$

112. $[0\bar{1}\bar{0}\bar{2}3] = [0\bar{1}\bar{0}2\bar{3}2] = [\bar{0}\bar{1}\bar{2}0\bar{1}3] = [\bar{0}\bar{1}\bar{2}0\bar{3}1] = [0\bar{1}20\bar{1}\bar{2}\bar{1}] = [0\bar{1}\bar{2}301\bar{2}] = [0\bar{1}\bar{0}2\bar{3}\bar{2}\bar{3}]$

113. $[0\bar{1}\bar{0}\bar{2}\bar{3}] = [0\bar{1}\bar{0}232] = [0\bar{1}2310\bar{3}] = [0\bar{1}\bar{2}0121] = [0\bar{1}\bar{0}23\bar{2}\bar{0}] = [0\bar{1}\bar{0}23\bar{2}\bar{1}] = [0\bar{1}\bar{0}23\bar{2}\bar{3}]$

114. $[01201] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}] = [0\bar{1}\bar{2}1\bar{3}\bar{1}] = [\bar{0}\bar{1}\bar{2}101] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}0]$

115. $[0120\bar{1}] = [0\bar{1}023\bar{1}] = [01\bar{0}\bar{2}\bar{3}1] = [0\bar{1}21\bar{0}31] = [0\bar{1}\bar{2}1\bar{3}\bar{2}3] = [0\bar{1}\bar{2}\bar{3}1\bar{2}\bar{0}]$

116. $[0120\bar{2}] = [01\bar{0}\bar{2}0] = [\bar{0}12\bar{1}3] = [01\bar{2}1\bar{0}1] = [01\bar{2}13\bar{2}] = [01\bar{2}1\bar{3}1] = [0\bar{1}20\bar{3}2\bar{0}]$
$= [0\bar{1}21323\bar{3}]$

117. $[01203] = [012\bar{3}\bar{0}\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}2] = [0\bar{1}21302] = [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{0}]$

118. $[0120\bar{3}] = [01\bar{2}032] = [\bar{0}\bar{1}\bar{2}3\bar{1}0] = [0\bar{1}20\bar{1}2\bar{3}] = [0\bar{1}2032\bar{1}]$

119. $[012\bar{0}1] = [0\bar{1}\bar{2}0\bar{1}2] = [0\bar{1}\bar{2}31\bar{3}] = [0\bar{1}\bar{2}1\bar{3}2] = [0\bar{1}\bar{0}2\bar{3}0] = [0\bar{1}2012\bar{1}] = [0\bar{1}\bar{2}\bar{3}\bar{1}2\bar{0}]$

120. $[012\bar{0}2] = [01\bar{0}12] = [\bar{0}\bar{1}2\bar{1}\bar{2}] = [\bar{0}\bar{1}\bar{2}12] = [\bar{0}1\bar{0}\bar{2}\bar{1}] = [\bar{0}\bar{1}23\bar{2}3]$

121. $[012\bar{0}\bar{2}] = [01\bar{2}02] = [01\bar{0}\bar{2}\bar{1}] = [\bar{0}123\bar{1}3] = [0\bar{1}\bar{2}132\bar{0}]$

122. $[012\bar{0}3] = [0\bar{1}\bar{2}3\bar{1}0] = [0\bar{1}\bar{0}2\bar{3}1] = [\bar{0}1231\bar{3}] = [0\bar{1}\bar{0}\bar{2}1\bar{3}1] = [0120\bar{3}\bar{0}\bar{3}]$

123. $[012\bar{0}\bar{3}] = [0\bar{1}\bar{2}1\bar{3}\bar{0}] = [012303] = [\bar{0}1\bar{0}\bar{2}31] = [\bar{0}12\bar{3}\bar{0}2] = [012\bar{0}\bar{3}1\bar{2}]$

124. $[01230] = [012\bar{0}\bar{3}\bar{0}] = [0\bar{1}20123] = [0\bar{1}\bar{2}\bar{3}0\bar{1}\bar{3}]$

125. $[0123\bar{0}] = [\bar{0}1\bar{2}\bar{0}13] = [0\bar{1}20\bar{1}3\bar{2}] = [0\bar{1}2031\bar{2}]$

126. $[01231] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{2}] = [0\bar{1}\bar{2}1\bar{0}\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{0}31] = [0\bar{1}\bar{2}\bar{0}232] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}] = [012\bar{0}\bar{3}02]$

127. $[0123\bar{2}] = [01\bar{2}\bar{3}2] = [01\bar{2}10\bar{2}] = [0\bar{1}20\bar{3}1\bar{0}] = [0\bar{1}21\bar{0}30] = [0\bar{1}2130\bar{3}] = [0\bar{1}21\bar{3}\bar{0}3]$
$= [0\bar{1}\bar{2}\bar{0}1\bar{0}3]$

128. $[012\bar{3}0] = [\bar{0}12\bar{0}1] = [0\bar{1}\bar{2}3\bar{0}1] = [\bar{0}\bar{1}\bar{2}0\bar{1}2] = [\bar{0}\bar{1}\bar{2}31\bar{3}] = [0123\bar{0}\bar{3}\bar{0}]$

129. $[012\bar{3}\bar{0}] = [01\bar{2}\bar{3}\bar{0}] = [012030] = [01\bar{2}030]$

130. $[012\bar{3}1] = [\bar{0}12\bar{0}3] = [0\bar{1}\bar{2}01\bar{0}] = [0\bar{1}\bar{2}\bar{0}3\bar{2}] = [0\bar{1}\bar{2}0\bar{3}2] = [0\bar{1}\bar{2}1\bar{0}2] = [01\bar{2}\bar{3}1\bar{0}]$
$= [0\bar{1}2032\bar{3}]$

131. $[012\bar{3}2] = [0\bar{1}\bar{2}1\bar{2}0] = [0\bar{1}\bar{2}\bar{0}1\bar{0}\bar{1}] = [0\bar{1}\bar{2}\bar{0}\bar{1}01] = [0\bar{1}\bar{2}\bar{0}2\bar{3}\bar{2}] = [0\bar{1}\bar{0}\bar{2}1\bar{3}0]$

132. $[012\bar{3}\bar{2}] = [01\bar{2}32] = [01\bar{2}\bar{0}\bar{1}\bar{2}] = [01\bar{2}1\bar{0}\bar{2}] = [01\bar{2}\bar{3}\bar{1}0] = [\bar{0}\bar{1}23\bar{0}\bar{3}]$

133. $[01\bar{2}01] = [\bar{0}\bar{1}2\bar{0}\bar{1}] = [0\bar{1}21\bar{3}\bar{2}] = [0\bar{1}\bar{2}03\bar{2}] = [01\bar{2}\bar{3}13] = [\bar{0}\bar{1}2101] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}2]$
$= [0120\bar{1}2\bar{1}]$

134. $[01\bar{2}03] = [\bar{0}\bar{1}2\bar{3}\bar{1}] = [0120\bar{3}\bar{2}] = [012\bar{3}\bar{0}\bar{1}] = [01\bar{2}\bar{0}3\bar{2}] = [\bar{0}\bar{1}2131] = [0\bar{1}20321]$

135. $[01\bar{2}\bar{0}1] = [0\bar{1}21\bar{0}2] = [01\bar{2}30\bar{2}] = [01\bar{2}31\bar{3}] = [0\bar{1}201\bar{2}\bar{0}] = [0\bar{1}\bar{2}\bar{0}3\bar{1}0] = [0\bar{1}\bar{2}\bar{3}0\bar{2}3]$

136. $[01\bar{2}\bar{0}\bar{1}] = [012\bar{3}\bar{2}0] = [01\bar{2}101] = [0\bar{1}20\bar{3}\bar{2}\bar{0}] = [01230\bar{2}0]$

137. $[01\bar{2}\bar{0}3] = [\bar{0}\bar{1}2\bar{1}0\bar{1}] = [\bar{0}\bar{1}\bar{2}32\bar{0}] = [0\bar{1}23\bar{2}\bar{1}0] = [0\bar{1}\bar{2}01\bar{3}1] = [0\bar{1}\bar{2}\bar{0}2\bar{1}3]$

138. $[01\bar{2}\bar{0}\bar{3}] = [01\bar{2}031] = [01\bar{2}303] = [\bar{0}\bar{1}213\bar{1}] = [0\bar{1}203\bar{1}\bar{2}] = [0\bar{1}\bar{0}\bar{2}\bar{3}10]$

139. $[01\bar{2}10] = [0123\bar{2}3] = [01\bar{2}\bar{0}\bar{1}\bar{0}] = [0\bar{1}20\bar{3}\bar{2}0] = [0\bar{1}\bar{2}01\bar{3}0] = [0\bar{1}\bar{2}\bar{0}1\bar{0}\bar{3}] = [012\bar{3}21\bar{2}]$

140. $[01\bar{2}1\bar{0}] = [01\bar{2}3\bar{2}] = [0120\bar{2}\bar{3}] = [012\bar{3}\bar{2}3] = [\bar{0}\bar{1}2\bar{1}0\bar{3}] = [0\bar{1}21323] = [0120323]$

141. $[01\bar{2}13] = [\bar{0}\bar{1}2\bar{1}\bar{3}] = [0120\bar{2}0] = [01\bar{2}\bar{3}1\bar{3}] = [\bar{0}\bar{1}2313] = [0\bar{1}213\bar{2}3]$

142. $[01\bar{2}1\bar{3}] = [01\bar{0}2\bar{0}] = [\bar{0}\bar{1}\bar{2}\bar{0}2] = [0120\bar{2}\bar{1}] = [\bar{0}\bar{1}2\bar{1}0\bar{1}] = [0\bar{1}20\bar{3}20]$

143. $[01\bar{2}30] = [\bar{0}\bar{1}23\bar{0}] = [01\bar{2}\bar{0}12] = [01\bar{2}\bar{0}3\bar{0}] = [0\bar{1}\bar{2}\bar{0}3\bar{1}\bar{0}] = [012\bar{0}\bar{2}10]$

144. $[01\bar{2}31] = [\bar{0}\bar{1}2\bar{0}\bar{3}] = [0\bar{1}21\bar{0}\bar{2}] = [01\bar{2}\bar{0}10] = [\bar{0}\bar{1}2303] = [\bar{0}\bar{1}\bar{2}\bar{0}32] = [0\bar{1}2032\bar{0}]$
$= [0\bar{1}\bar{2}\bar{0}\bar{1}3\bar{0}]$

145. $[01\bar{2}\bar{3}0] = [012\bar{3}0\bar{2}] = [\bar{0}\bar{1}2310] = [\bar{0}\bar{1}\bar{2}32\bar{1}] = [0\bar{1}23\bar{2}\bar{0}1] = [0\bar{1}\bar{2}\bar{0}2\bar{3}1] = [0\bar{1}\bar{2}\bar{0}3\bar{1}\bar{2}]$
$= [01\bar{2}\bar{3}01\bar{2}]$

146. $[01\bar{2}\bar{3}1] = [\bar{0}12\bar{0}\bar{3}] = [0\bar{1}21\bar{3}2] = [012\bar{3}12] = [01\bar{2}01\bar{0}] = [\bar{0}12303] = [0123\bar{0}\bar{2}\bar{0}]$

147. $[01\bar{2}\bar{3}\bar{1}] = [012\bar{3}\bar{2}\bar{1}] = [01\bar{2}131] = [\bar{0}\bar{1}23\bar{0}3] = [0\bar{1}213\bar{2}\bar{3}]$

148. $[01\bar{0}21] = [\bar{0}\bar{1}2\bar{0}\bar{2}] = [\bar{0}\bar{1}\bar{2}02] = [0\bar{1}\bar{2}1\bar{2}\bar{3}] = [0\bar{1}\bar{2}\bar{3}1\bar{3}] = [0\bar{1}\bar{2}1323] = [0123\bar{0}3\bar{0}]$
$= [01\bar{2}3010]$

149. $[01\bar{0}23] = [0\bar{1}\bar{2}\bar{3}10] = [01\bar{0}2\bar{3}\bar{0}] = [01\bar{0}2\bar{3}\bar{1}] = [01\bar{0}\bar{2}\bar{3}\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}12] = [0\bar{1}\bar{2}\bar{3}0\bar{1}\bar{0}]$
$= [01\bar{0}23\bar{2}3]$

150. $[01\bar{0}2\bar{3}] = [01\bar{0}\bar{2}3\bar{2}] = [01\bar{0}2\bar{3}\bar{2}0] = [01\bar{0}2\bar{3}21] = [01\bar{0}2\bar{3}\bar{2}3]$

151. $[01\bar{0}\bar{2}3] = [01\bar{0}2\bar{3}2] = [01\bar{0}2\bar{3}\bar{2}\bar{0}] = [01\bar{0}2\bar{3}\bar{2}\bar{1}] = [01\bar{0}2\bar{3}\bar{2}\bar{3}]$

152. $[01\bar{0}\bar{2}\bar{3}] = [0120\bar{1}\bar{2}] = [01\bar{0}232] = [01\bar{0}\bar{2}30] = [01\bar{0}\bar{2}31] = [\bar{0}123\bar{1}\bar{0}] = [01\bar{0}23\bar{2}\bar{3}]$

153. $[\bar{0}\bar{1}210] = [0\bar{1}\bar{2}032] = [0\bar{1}\bar{0}\bar{2}\bar{1}\bar{0}] = [01\bar{2}01\bar{3}]$

154. $[\bar{0}\bar{1}21\bar{0}] = [\bar{0}1\bar{2}\bar{3}0] = [\bar{0}1\bar{2}\bar{0}1\bar{0}] = [01\bar{2}130\bar{2}] = [01\bar{2}1\bar{3}01]$

155. $[\bar{0}\bar{1}213] = [01\bar{2}03\bar{1}] = [01\bar{2}\bar{0}\bar{3}2] = [\bar{0}1\bar{0}23\bar{0}] = [0\bar{1}\bar{0}2\bar{1}\bar{3}2]$

156. $[\bar{0}\bar{1}21\bar{3}] = [\bar{0}1\bar{2}\bar{0}3] = [0\bar{1}20\bar{3}23] = [0\bar{1}\bar{2}\bar{0}2\bar{1}\bar{0}] = [0\bar{1}\bar{2}\bar{3}\bar{0}21]$

157. $[\bar{0}\bar{1}2\bar{1}0] = [\bar{0}\bar{1}2\bar{3}2] = [01\bar{2}1\bar{0}3] = [01\bar{2}1\bar{3}0] = [\bar{0}\bar{1}\bar{2}32\bar{3}] = [012032\bar{3}]$

158. $[\bar{0}\bar{1}2\bar{1}\bar{0}] = [01\bar{2}\bar{0}32] = [\bar{0}\bar{1}23\bar{2}\bar{3}] = [0\bar{1}\bar{2}\bar{0}2\bar{1}\bar{3}] = [012\bar{0}210] = [012\bar{3}\bar{2}\bar{0}\bar{3}]$

159. $[\bar{0}\bar{1}230] = [\bar{0}\bar{1}\bar{2}30] = [0\bar{1}\bar{2}0\bar{1}\bar{3}] = [01\bar{2}3\bar{1}\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{0}] = [0\bar{1}\bar{2}\bar{0}\bar{1}30] = [0\bar{1}\bar{2}3103]$

160. $[\bar{0}\bar{1}23\bar{0}] = [012\bar{3}\bar{2}1] = [01\bar{2}\bar{3}\bar{1}\bar{0}] = [0\bar{1}\bar{0}\bar{2}13\bar{0}] = [01\bar{2}\bar{3}010] = [01\bar{2}\bar{3}013]$

161. $[\bar{0}\bar{1}231] = [0\bar{1}\bar{2}03\bar{1}] = [01\bar{2}13\bar{0}] = [01\bar{2}\bar{3}0\bar{3}] = [\bar{0}\bar{1}\bar{2}321] = [0\bar{1}23\bar{2}1\bar{0}]$

162. $[\bar{0}\bar{1}23\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{3}2] = [\bar{0}\bar{1}2\bar{1}\bar{0}2] = [0\bar{1}\bar{2}\bar{3}\bar{0}23] = [012\bar{0}21\bar{0}] = [\bar{0}\bar{1}2103\bar{0}] = [\bar{0}\bar{1}21\bar{0}\bar{3}0]$
$= [\bar{0}\bar{1}2130\bar{3}]$

163. $[\bar{0}\bar{1}\bar{2}01] = [\bar{0}12\bar{3}\bar{1}] = [0\bar{1}\bar{2}130] = [0\bar{1}\bar{2}1\bar{0}\bar{3}\bar{0}] = [0120\bar{3}\bar{0}3] = [\bar{0}\bar{1}\bar{2}01\bar{3}0]$

164. $[\bar{0}\bar{1}\bar{2}0\bar{1}] = [0\bar{1}\bar{0}\bar{2}3\bar{1}] = [012\bar{3}0\bar{3}] = [\bar{0}\bar{1}\bar{2}10\bar{2}] = [0\bar{1}\bar{2}3012] = [0123\bar{0}\bar{3}0]$

165. $[\bar{0}\bar{1}\bar{2}03] = [0\bar{1}\bar{2}30\bar{2}] = [0\bar{1}\bar{2}3\bar{2}\bar{0}] = [\bar{0}\bar{1}\bar{2}103] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{3}] = [0\bar{1}\bar{2}3020]$

166. $[\bar{0}\bar{1}\bar{2}0\bar{3}] = [0\bar{1}\bar{2}13\bar{1}] = [0\bar{1}\bar{0}\bar{2}\bar{3}\bar{0}] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}3] = [0\bar{1}20\bar{1}\bar{2}1] = [0\bar{1}231\bar{0}3] = [0\bar{1}\bar{2}\bar{0}23\bar{0}]$

167. $[\bar{0}\bar{1}\bar{2}\bar{0}1] = [01201\bar{2}] = [01\bar{0}23\bar{1}] = [\bar{0}1\bar{0}\bar{2}31] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}0] = [\bar{0}\bar{1}213\bar{0}\bar{2}] = [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}\bar{1}]$

168. $[\bar{0}\bar{1}\bar{2}\bar{0}3] = [01231\bar{0}] = [01\bar{2}31\bar{0}] = [0\bar{1}20320] = [0\bar{1}231\bar{0}\bar{1}]$

169. $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}] = [0120\bar{3}\bar{2}] = [\bar{0}\bar{1}2302] = [0\bar{1}\bar{2}310\bar{3}] = [\bar{0}\bar{1}21\bar{0}\bar{3}\bar{1}]$

170. $[\bar{0}\bar{1}\bar{2}10] = [0\bar{1}\bar{2}1\bar{3}1] = [0\bar{1}\bar{2}3\bar{1}\bar{2}] = [0\bar{1}\bar{2}3\bar{2}\bar{0}] = [01201\bar{0}] = [\bar{0}\bar{1}\bar{2}0\bar{3}\bar{2}] = [\bar{0}\bar{1}\bar{2}0\bar{1}0]$

171. $[\bar{0}\bar{1}\bar{2}13] = [0\bar{1}\bar{2}0\bar{3}0] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}] = [0\bar{1}\bar{2}\bar{3}\bar{2}1] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{3}] = [\bar{0}120\bar{3}\bar{2}] = [0\bar{1}\bar{2}012\bar{1}]$
$= [0\bar{1}\bar{0}\bar{2}1\bar{3}2]$

172. $[\bar{0}\bar{1}\bar{2}31] = [\bar{0}12\bar{0}\bar{1}] = [0\bar{1}\bar{2}0\bar{3}\bar{1}] = [012\bar{3}02] = [0\bar{1}\bar{2}0130] = [0\bar{1}\bar{2}0131] = [0\bar{1}\bar{2}\bar{0}312]$
$= [012\bar{3}\bar{2}\bar{0}\bar{2}]$

173. $[\bar{0}\bar{1}\bar{2}32] = [\bar{0}\bar{1}2\bar{3}\bar{2}] = [01\bar{2}\bar{0}30] = [01\bar{2}\bar{3}03] = [\bar{0}\bar{1}2\bar{1}02] = [\bar{0}\bar{1}231\bar{0}] = [0\bar{1}\bar{2}01\bar{3}\bar{1}]$

174. $[\bar{0}\bar{1}\bar{2}3\bar{2}] = [012\bar{0}2\bar{1}] = [0\bar{1}\bar{2}\bar{0}21\bar{3}] = [0\bar{1}\bar{2}\bar{0}23\bar{1}] = [0\bar{1}\bar{0}\bar{2}132] = [\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}1]$.

175. $[\bar{0}\bar{1}\bar{2}\bar{3}0] = [0\bar{1}20\bar{1}\bar{3}] = [0\bar{1}2012\bar{0}] = [0\bar{1}20313] = [0\bar{1}\bar{2}0\bar{3}\bar{0}\bar{1}] = [012\bar{0}\bar{2}\bar{1}2]$

176. $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}] = [\bar{0}\bar{1}\bar{2}030] = [0\bar{1}231\bar{2}0] = [0\bar{1}\bar{2}30\bar{1}0] = [0123\bar{0}\bar{3}\bar{2}] = [01\bar{2}\bar{0}31\bar{2}]$

177. $[\bar{0}\bar{1}\bar{2}\bar{3}1] = [\bar{0}1203] = [\bar{0}12\bar{3}\bar{0}\bar{3}] = [\bar{0}\bar{1}23\bar{0}\bar{2}\bar{0}]$

178. $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}] = [0120\bar{3}\bar{1}] = [\bar{0}\bar{1}\bar{2}131] = [0\bar{1}20\bar{1}23] = [0\bar{1}\bar{0}\bar{2}1\bar{3}\bar{2}] = [0120131] = [\bar{0}\bar{1}\bar{2}03\bar{0}\bar{2}]$

179. $[\bar{0}1\bar{0}\bar{2}\bar{0}] = [0\bar{1}\bar{0}\bar{2}\bar{1}2]$

180. $[\bar{0}1\bar{0}23] = [\bar{0}\bar{1}2132] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{1}] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{2}] = [\bar{0}123\bar{2}3] = [\bar{0}1\bar{2}\bar{0}\bar{1}\bar{0}]$

$= [0\bar{1}\bar{0}\bar{2}\bar{1}02] = [0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}\bar{2}]$

181. $[\bar{0}1\bar{0}2\bar{3}] = [\bar{0}1\bar{2}3\bar{2}] = [\bar{0}12\bar{3}\bar{2}3]$

182. $[\bar{0}1\bar{0}\bar{2}3] = [\bar{0}12\bar{3}2] = [0\bar{1}\bar{2}1\bar{0}3] = [0\bar{1}\bar{2}1\bar{3}0] = [012\bar{0}\bar{3}\bar{2}] = [\bar{0}12\bar{3}\bar{2}\bar{0}] = [\bar{0}12\bar{3}\bar{2}\bar{1}]$

183. $[\bar{0}1\bar{0}\bar{2}\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{0}1\bar{0}] = [\bar{0}1\bar{0}232] = [\bar{0}123\bar{2}\bar{3}] = [0\bar{1}\bar{2}\bar{0}3\bar{0}\bar{2}] = [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}1]$

184. $[\bar{0}1201] = [0\bar{1}\bar{2}\bar{0}1\bar{2}] = [0\bar{1}\bar{2}3\bar{0}\bar{3}] = [\bar{0}\bar{1}\bar{2}303] = [0\bar{1}\bar{2}\bar{0}212] = [0\bar{1}\bar{0}230\bar{1}] = [012\bar{3}\bar{0}10]$

185. $[\bar{0}120\bar{1}] = [0\bar{1}\bar{2}1\bar{0}\bar{3}2] = [0\bar{1}\bar{2}3\bar{0}\bar{2}1] = [012\bar{0}21\bar{3}] = [0123\bar{0}\bar{2}\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{1}\bar{0}]$

186. $[\bar{0}120\bar{3}] = [0\bar{1}\bar{2}0\bar{3}\bar{0}] = [0\bar{1}\bar{2}\bar{0}30] = [\bar{0}\bar{1}\bar{2}\bar{3}13] = [\bar{0}\bar{1}\bar{2}132] = [0\bar{1}23\bar{2}\bar{1}0] = [01\bar{2}\bar{0}30\bar{1}]$

187. $[\bar{0}1230] = [0\bar{1}\bar{2}\bar{3}0\bar{1}] = [01\bar{2}\bar{3}1\bar{2}] = [0\bar{1}20\bar{3}1\bar{2}] = [0120\bar{1}2\bar{0}] = [0123\bar{0}\bar{2}0] = [\bar{0}1230\bar{1}2]$
$= [\bar{0}1230\bar{1}3]$

188. $[\bar{0}1231] = [0\bar{1}\bar{2}3\bar{1}\bar{0}] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}] = [012\bar{0}32] = [0\bar{1}\bar{2}3010] = [0\bar{1}\bar{2}\bar{3}\bar{1}02] = [0\bar{1}\bar{2}\bar{3}\bar{1}03]$
$= [012013\bar{0}]$

189. $[\bar{0}123\bar{1}] = [012\bar{0}\bar{2}\bar{3}] = [01\bar{0}\bar{2}\bar{3}0] = [0\bar{1}\bar{2}1320] = [\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}\bar{2}] = [\bar{0}\bar{1}21\bar{0}32]$

190. $[\bar{0}123\bar{2}] = [\bar{0}1\bar{2}\bar{3}2] = [\bar{0}1\bar{0}23\bar{2}] = [\bar{0}1\bar{0}\bar{2}\bar{3}2] = [01\bar{2}\bar{0}1\bar{3}\bar{2}] = [01\bar{2}\bar{3}\bar{1}\bar{2}0] = [\bar{0}\bar{1}23\bar{0}2\bar{3}]$

191. $[\bar{0}12\bar{3}\bar{0}] = [0\bar{1}\bar{2}103] = [012\bar{0}\bar{3}\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{3}10] = [0\bar{1}\bar{2}30\bar{1}\bar{2}] = [0\bar{1}\bar{2}\bar{3}\bar{1}23] = [012\bar{0}\bar{3}12]$

192. $[\bar{0}12\bar{3}\bar{2}] = [\bar{0}1\bar{2}32] = [\bar{0}1\bar{0}2\bar{3}\bar{2}] = [\bar{0}1\bar{0}\bar{2}32]$

193. $[\bar{0}1\bar{2}\bar{0}1] = [\bar{0}1\bar{2}\bar{3}\bar{0}] = [0123\bar{0}\bar{1}] = [\bar{0}\bar{1}21\bar{0}2] = [0\bar{1}20312] = [0\bar{1}\bar{0}2\bar{3}\bar{1}\bar{3}] = [0123101]$

194. $[\bar{0}1\bar{2}\bar{0}\bar{1}] = [\bar{0}1\bar{0}231] = [0\bar{1}2312\bar{0}] = [0\bar{1}\bar{0}\bar{2}\bar{1}0\bar{2}] = [0\bar{1}\bar{0}\bar{2}\bar{3}1\bar{3}] = [0123\bar{0}\bar{3}1] = [\bar{0}1230\bar{2}0]$

195. $[\bar{0}1\bar{2}\bar{0}\bar{3}] = [\bar{0}1\bar{2}\bar{3}1] = [0\bar{1}203\bar{2}] = [\bar{0}\bar{1}21\bar{3}2] = [0\bar{1}20\bar{1}30] = [0\bar{1}\bar{2}\bar{3}0\bar{2}\bar{1}]$

196. $[\bar{0}1\bar{2}\bar{3}\bar{1}] = [0\bar{1}\bar{0}\bar{2}\bar{1}3] = [\bar{0}1\bar{2}\bar{0}30] = [0\bar{1}201\bar{3}2] = [0\bar{1}\bar{0}23\bar{1}3] = [0120\bar{3}\bar{0}\bar{2}] = [012302\bar{1}]$

197. $[0\bar{1}201\bar{0}] = [0\bar{1}21\bar{3}1] = [0\bar{1}2\bar{1}\bar{0}1] = [0\bar{1}2\bar{1}\bar{3}\bar{0}] = [0\bar{1}\bar{0}\bar{2}\bar{1}0\bar{1}]$

198. $[0\bar{1}2012] = [0\bar{1}\bar{2}0\bar{1}\bar{2}] = [012\bar{0}12] = [01230\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{3}03] = [0\bar{1}\bar{2}30\bar{1}3] = [012\bar{0}2\bar{1}\bar{2}]$

199. $[0\bar{1}201\bar{2}] = [0\bar{1}\bar{0}23\bar{0}] = [01\bar{2}\bar{0}13] = [0\bar{1}\bar{2}30\bar{2}\bar{3}] = [0\bar{1}\bar{0}\bar{2}31\bar{2}] = [0\bar{1}201\bar{2}\bar{3}1]$

200. $[0\bar{1}2013] = [0\bar{1}\bar{2}\bar{3}0\bar{2}] = [0\bar{1}20\bar{3}\bar{1}\bar{3}] = [0\bar{1}\bar{2}\bar{0}301] = [0\bar{1}\bar{2}31\bar{0}2] = [\bar{0}\bar{1}2312\bar{1}]$

201. $[0\bar{1}201\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}] = [0\bar{1}23\bar{2}13] = [0\bar{1}\bar{2}\bar{3}0\bar{2}\bar{0}] = [0123021] = [01\bar{2}\bar{0}\bar{1}31] = [0\bar{1}201\bar{0}3\bar{1}]$

202. $[0\bar{1}20\bar{1}2] = [0120\bar{3}1] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}] = [0\bar{1}23\bar{2}\bar{0}\bar{2}] = [0\bar{1}\bar{2}\bar{0}2\bar{3}0] = [0\bar{1}\bar{2}\bar{3}1\bar{2}3] = [0\bar{1}\bar{0}230\bar{3}]$

203. $[0\bar{1}20\bar{1}\bar{2}] = [0\bar{1}\bar{0}\bar{2}30] = [\bar{0}\bar{1}\bar{2}0\bar{3}\bar{1}] = [0\bar{1}\bar{2}1\bar{0}\bar{3}1] = [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{0}]$

204. $[0\bar{1}20\bar{1}3] = [\bar{0}1\bar{2}\bar{0}\bar{3}\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{3}0\bar{1}] = [0123\bar{0}2] = [0\bar{1}21\bar{3}\bar{0}1] = [0\bar{1}\bar{2}\bar{3}0\bar{2}1] = [01\bar{2}\bar{0}\bar{1}3\bar{0}]$

205. $[0\bar{1}2031] = [\bar{0}1\bar{2}\bar{0}1\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{3}0\bar{2}] = [0123\bar{0}1] = [0\bar{1}23\bar{2}\bar{1}3] = [0\bar{1}\bar{2}03\bar{0}1] = [\bar{0}\bar{1}21032\bar{2}]$

206. $[0\bar{1}203\bar{1}] = [01\bar{2}\bar{0}\bar{3}1] = [0\bar{1}\bar{2}\bar{3}02] = [0\bar{1}2312\bar{3}] = [0\bar{1}\bar{0}\bar{2}\bar{3}1\bar{0}] = [01\bar{2}\bar{3}\bar{1}23]$

207. $[0\bar{1}2032] = [\bar{0}1\bar{2}\bar{0}\bar{3}\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{2}] = [01\bar{2}310] = [01\bar{2}0\bar{3}\bar{2}] = [012\bar{3}10] = [0120\bar{3}2]$
$= [0\bar{1}\bar{2}0\bar{3}\bar{2}]$

208. $[0\bar{1}20\bar{3}1] = [0123\bar{2}0] = [\bar{0}12302] = [0\bar{1}21\bar{3}\bar{0}\bar{3}] = [01\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}]$

209. $[0\bar{1}20\bar{3}\bar{1}] = [0\bar{1}20131] = [0\bar{1}21\bar{0}\bar{3}\bar{0}] = [0\bar{1}\bar{2}\bar{0}30\bar{1}] = [0120\bar{1}2\bar{3}] = [0\bar{1}20\bar{3}\bar{1}\bar{0}1]$
$= [0\bar{1}20\bar{3}\bar{1}\bar{2}0]$

210. $[0\bar{1}20\bar{3}2] = [0120\bar{2}1] = [01\bar{2}1\bar{3}\bar{1}] = [\bar{0}\bar{1}21\bar{3}\bar{1}] = [0\bar{1}\bar{2}\bar{0}2\bar{1}0] = [0\bar{1}\bar{2}\bar{3}1\bar{0}3] = [01\bar{0}\bar{2}\bar{3}\bar{0}\bar{3}]$

211. $[0\bar{1}20\bar{3}\bar{2}] = [0\bar{1}\bar{0}\bar{2}10] = [01\bar{2}10\bar{1}] = [01\bar{2}\bar{0}\bar{1}0] = [0\bar{1}\bar{2}\bar{0}\bar{1}0\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}3] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}\bar{3}]$

212. $[0\bar{1}21\bar{0}1] = [0\bar{1}2\bar{1}\bar{0}3] = [0\bar{1}2\bar{1}\bar{3}1] = [0\bar{1}231\bar{3}] = [0\bar{1}\bar{2}01\bar{2}1] = [0\bar{1}\bar{2}\bar{3}\bar{1}2\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{0}30\bar{2}]$
$= [0\bar{1}20\bar{3}1\bar{3}1]$

213. $[0\bar{1}21\bar{0}3] = [01201\bar{3}] = [0\bar{1}20\bar{3}\bar{1}0] = [0\bar{1}\bar{2}\bar{0}1\bar{3}2] = [0\bar{1}\bar{2}310\bar{2}] = [0\bar{1}\bar{2}\bar{3}1\bar{2}0] = [0\bar{1}20\bar{3}\bar{1}\bar{0}\bar{1}]$

214. $[0\bar{1}21\bar{0}\bar{3}] = [0123\bar{2}\bar{1}] = [0\bar{1}21303] = [0\bar{1}23\bar{2}0\bar{2}] = [0\bar{1}\bar{2}\bar{0}3\bar{1}\bar{2}] = [0\bar{1}\bar{2}\bar{3}\bar{1}21] = [01\bar{2}3012]$

215. $[0\bar{1}2130] = [0123\bar{2}1] = [01203\bar{1}] = [0\bar{1}21\bar{0}\bar{3}\bar{0}] = [0\bar{1}23\bar{2}\bar{0}3] = [0\bar{1}\bar{2}\bar{0}1\bar{3}0] = [0\bar{1}20\bar{1}\bar{2}\bar{0}\bar{2}]$

216. $[0\bar{1}213\bar{0}] = [0\bar{1}\bar{2}\bar{3}12] = [0\bar{1}21\bar{3}0\bar{3}] = [0\bar{1}\bar{2}\bar{0}31\bar{0}] = [\bar{0}\bar{1}203\bar{0}1] = [0\bar{1}213\bar{0}\bar{3}0]$

217. $[0\bar{1}2132] = [0120\bar{2}3] = [01\bar{2}1\bar{0}\bar{1}] = [0\bar{1}\bar{2}\bar{0}3\bar{1}2] = [0120\bar{2}\bar{1}\bar{0}] = [\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}\bar{0}] = [0\bar{1}201\bar{0}\bar{2}0]$

218. $[0\bar{1}213\bar{2}] = [01\bar{2}13\bar{1}] = [01\bar{2}\bar{3}\bar{1}3] = [0\bar{1}23101] = [0\bar{1}\bar{2}\bar{0}\bar{1}3\bar{2}] = [0120321]$

219. $[0\bar{1}21\bar{3}0] = [0\bar{1}213\bar{0}3] = [0\bar{1}231\bar{2}\bar{3}] = [0\bar{1}\bar{2}03\bar{0}2] = [0\bar{1}\bar{2}\bar{3}1\bar{0}2] = [0\bar{1}213\bar{0}\bar{3}\bar{0}]$
$= [0\bar{1}21\bar{3}02\bar{3}]$

220. $[0\bar{1}21\bar{3}\bar{0}] = [0123\bar{2}\bar{0}] = [0\bar{1}20\bar{1}\bar{3}\bar{1}] = [0\bar{1}20\bar{3}10] = [0123020] = [01\bar{2}\bar{0}\bar{1}30] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}\bar{2}]$

221. $[0\bar{1}2310] = [0\bar{1}\bar{0}\bar{2}\bar{3}0] = [0\bar{1}213\bar{2}\bar{1}] = [0\bar{1}\bar{2}012\bar{1}] = [0\bar{1}\bar{0}23\bar{1}2] = [0\bar{1}23102\bar{3}]$

222. $[0\bar{1}231\bar{0}] = [\bar{0}\bar{1}\bar{2}0\bar{3}\bar{0}] = [\bar{0}\bar{1}\bar{2}030] = [0\bar{1}\bar{2}\bar{0}230] = [\bar{0}\bar{1}\bar{2}0120] = [0\bar{1}20\bar{1}\bar{2}\bar{3}\bar{0}]$

223. $[0\bar{1}2312] = [\bar{0}1\bar{2}\bar{0}\bar{1}3] = [0\bar{1}203\bar{1}3] = [0\bar{1}23\bar{2}\bar{1}\bar{2}] = [01\bar{2}\bar{3}\bar{1}2\bar{3}] = [\bar{0}1230\bar{2}\bar{0}] = [0\bar{1}213\bar{2}13]$

224. $[0\bar{1}231\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}] = [0\bar{1}21\bar{3}01] = [0\bar{1}\bar{2}03\bar{0}\bar{2}] = [0\bar{1}0230\bar{2}] = [01\bar{2}\bar{0}312] = [\bar{0}121031]$

225. $[0\bar{1}23\bar{2}0] = [0\bar{1}21\bar{0}\bar{3}1] = [0\bar{1}\bar{2}01\bar{2}\bar{3}] = [0\bar{1}\bar{2}01\bar{3}2] = [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{0}] = [0\bar{1}\bar{2}\bar{3}1\bar{2}1] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}]$

226. $[0\bar{1}23\bar{2}\bar{0}] = [01\bar{2}\bar{3}0\bar{2}] = [0\bar{1}20\bar{1}20] = [0\bar{1}2130\bar{1}] = [0\bar{1}\bar{2}\bar{0}3\bar{1}2] = [0\bar{1}\bar{2}\bar{3}1\bar{2}\bar{3}] = [0\bar{1}20\bar{1}\bar{2}\bar{0}2]$

227. $[0\bar{1}23\bar{2}1] = [\bar{0}\bar{1}2312] = [0\bar{1}201\bar{3}\bar{0}] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}\bar{0}] = [0\bar{1}\bar{2}\bar{3}0\bar{2}0] = [0\bar{1}201\bar{0}\bar{3}0]$

228. $[0\bar{1}23\bar{2}\bar{1}] = [01\bar{2}\bar{0}3\bar{1}] = [0\bar{1}2031\bar{0}] = [0\bar{1}23121] = [\bar{0}121032] = [0\bar{1}213\bar{2}1\bar{3}]$

229. $[0\bar{1}\bar{2}012] = [0\bar{1}\bar{0}\bar{2}\bar{3}\bar{0}] = [\bar{0}120\bar{3}\bar{1}] = [0\bar{1}23103] = [0\bar{1}\bar{0}2302] = [01\bar{2}\bar{0}\bar{3}01]$

230. $[0\bar{1}\bar{2}01\bar{2}] = [0\bar{1}21\bar{0}1\bar{3}] = [0\bar{1}23\bar{2}03] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}] = [0\bar{1}\bar{2}\bar{3}\bar{1}21] = [0\bar{1}\bar{2}3\bar{0}\bar{2}3] = [0123\bar{0}\bar{3}\bar{2}]$

231. $[0\bar{1}\bar{2}013] = [\bar{0}\bar{1}\bar{2}31\bar{2}] = [012\bar{3}\bar{2}\bar{0}2]$

232. $[0\bar{1}\bar{2}01\bar{3}] = [01\bar{2}\bar{0}\bar{3}\bar{0}] = [01\bar{2}10\bar{3}] = [\bar{0}\bar{1}\bar{2}320] = [0\bar{1}23\bar{2}0\bar{1}] = [0\bar{1}\bar{2}0132] = [0\bar{1}\bar{2}\bar{0}130]$
$= [012\bar{3}212]$

233. $[0\bar{1}\bar{2}0\bar{1}\bar{0}] = [0\bar{1}\bar{2}\bar{0}10] = [0\bar{1}\bar{2}102] = [0120\bar{3}\bar{0}\bar{1}] = [012\bar{0}\bar{2}\bar{1}3] = [\bar{0}\bar{1}\bar{2}3101] = [0\bar{1}20\bar{1}\bar{2}\bar{0}3]$

234. $[0\bar{1}\bar{2}03\bar{0}] = [0\bar{1}\bar{2}\bar{0}1\bar{2}] = [0\bar{1}\bar{2}\bar{3}\bar{0}3] = [\bar{0}\bar{1}\bar{2}\bar{3}02] = [0\bar{1}2031\bar{3}] = [0\bar{1}21\bar{3}0\bar{1}] = [0\bar{1}231\bar{2}\bar{3}]$

235. $[0\bar{1}\bar{2}\bar{0}1\bar{0}] = [0\bar{1}\bar{2}1\bar{2}\bar{0}] = [012\bar{3}23] = [012\bar{3}\bar{2}\bar{3}] = [01\bar{2}102] = [0\bar{1}20\bar{1}\bar{0}1] = [0120\bar{3}0\bar{3}]$
$= [\bar{0}\bar{1}\bar{2}0121]$

236. $[0\bar{1}\bar{2}\bar{0}12] = [0\bar{1}\bar{2}31\bar{2}] = [0\bar{1}\bar{2}3\bar{2}\bar{1}] = [\bar{0}\bar{1}\bar{2}130] = [0\bar{1}\bar{2}03\bar{0}\bar{3}] = [0\bar{1}\bar{2}3121] = [0\bar{1}\bar{0}2\bar{3}\bar{1}2]$
$= [0\bar{1}20\bar{1}\bar{2}3\bar{1}]$

237. $[0\bar{1}\bar{2}\bar{0}13] = [0\bar{1}23\bar{2}01] = [0\bar{1}\bar{2}01\bar{3}\bar{2}] = [0\bar{1}\bar{2}\bar{0}3\bar{1}\bar{3}] = [01\bar{2}\bar{0}31\bar{0}] = [0\bar{1}\bar{2}01232]$
$= [0\bar{1}\bar{2}\bar{0}13\bar{2}0]$

238. $[0\bar{1}\bar{2}\bar{0}1\bar{3}] = [012031] = [0\bar{1}21\bar{0}3\bar{2}] = [0\bar{1}2130\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{1}3\bar{1}] = [0\bar{1}\bar{2}3102] = [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}3]$

239. $[0\bar{1}\bar{2}\bar{0}\bar{1}0] = [0\bar{1}\bar{2}10\bar{1}] = [0\bar{1}\bar{0}\bar{2}1\bar{0}] = [012\bar{3}\bar{2}\bar{0}] = [0\bar{1}20\bar{3}\bar{2}3] = [0\bar{1}\bar{2}\bar{0}21\bar{0}] = [0\bar{1}\bar{2}\bar{0}2\bar{3}2]$
$= [0\bar{1}201\bar{0}\bar{2}1]$

240. $[0\bar{1}\bar{2}\bar{0}\bar{1}3] = [01\bar{2}312] = [\bar{0}\bar{1}230\bar{3}] = [0\bar{1}213\bar{2}0] = [0\bar{1}\bar{2}\bar{0}1\bar{3}1] = [012032\bar{1}] = [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}\bar{3}]$

241. $[0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}] = [0\bar{1}\bar{0}\bar{2}3\bar{1}] = [0\bar{1}\bar{2}\bar{0}313] = [0\bar{1}\bar{0}\bar{2}131] = [012310\bar{3}] = [0\bar{1}\bar{2}0123\bar{1}]$

242. $[0\bar{1}\bar{2}\bar{0}21] = [0\bar{1}\bar{2}3\bar{0}3] = [\bar{0}\bar{1}\bar{2}3\bar{2}0] = [\bar{0}1201\bar{0}] = [0\bar{1}\bar{2}\bar{0}\bar{1}03] = [0\bar{1}\bar{0}\bar{2}13\bar{2}] = [0\bar{1}201\bar{0}\bar{2}\bar{1}]$

243. $[0\bar{1}\bar{2}\bar{0}2\bar{1}] = [01\bar{2}\bar{0}3\bar{2}] = [\bar{0}\bar{1}2\bar{1}\bar{0}1] = [\bar{0}\bar{1}21\bar{3}1] = [0\bar{1}20\bar{3}\bar{2}\bar{3}] = [0\bar{1}\bar{2}\bar{0}3\bar{0}\bar{3}] = [0\bar{1}02303]$

244. $[0\bar{1}\bar{2}\bar{0}23] = [0\bar{1}\bar{2}1\bar{0}1] = [01231\bar{3}] = [\bar{0}\bar{1}\bar{2}3\bar{2}1] = [\bar{0}\bar{1}\bar{2}0\bar{3}0] = [0\bar{1}231\bar{0}\bar{3}] = [\bar{0}\bar{1}2\bar{1}\bar{0}3\bar{1}]$

245. $[0\bar{1}\bar{2}\bar{0}2\bar{3}] = [012\bar{3}20] = [01\bar{2}\bar{3}0\bar{1}] = [0\bar{1}20\bar{1}2\bar{1}] = [0\bar{1}\bar{2}\bar{0}\bar{1}0\bar{1}] = [0\bar{1}\bar{0}2303] = [01\bar{2}\bar{3}012]$

246. $[0\bar{1}\bar{2}\bar{0}3\bar{0}] = [0\bar{1}023\bar{0}] = [\bar{0}\bar{1}\bar{0}\bar{2}\bar{3}0] = [\bar{0}120\bar{3}0] = [0\bar{1}201\bar{0}30] = [0\bar{1}213\bar{2}\bar{1}\bar{0}]$

247. $[0\bar{1}\bar{2}\bar{0}31] = [0\bar{1}\bar{0}231] = [\bar{0}\bar{1}\bar{2}31\bar{0}] = [0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}\bar{1}] = [0\bar{1}\bar{2}01231]$

248. $[0\bar{1}\bar{2}\bar{0}3\bar{1}] = [01\bar{2}\bar{0}1\bar{2}] = [01\bar{2}302] = [0\bar{1}2132\bar{0}] = [0\bar{1}\bar{2}\bar{0}31\bar{3}] = [\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}0] = [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}0]$
$= [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}2]$

249. $[0\bar{1}\bar{2}\bar{0}\bar{3}0] = [0\bar{1}\bar{2}30\bar{3}] = [0\bar{1}2013\bar{1}] = [0\bar{1}20\bar{3}\bar{1}3] = [0\bar{1}\bar{2}\bar{0}2\bar{1}2] = [0\bar{1}0230\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}\bar{1}]$
$= [0\bar{1}201\bar{0}\bar{3}1]$

250. $[0\bar{1}\bar{2}\bar{0}\bar{3}1] = [0\bar{1}21\bar{0}\bar{3}2] = [0\bar{1}213\bar{0}2] = [0\bar{1}\bar{2}\bar{0}3\bar{1}3] = [01\bar{2}301\bar{2}] = [\bar{0}\bar{1}\bar{2}03\bar{0}\bar{1}] = [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}\bar{0}]$
$= [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}\bar{2}]$

251. $[0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}] = [01\bar{2}\bar{3}02] = [0\bar{1}23\bar{2}\bar{0}\bar{1}] = [0\bar{1}\bar{2}\bar{0}131] = [\bar{0}\bar{1}231\bar{2}0] = [0\bar{1}\bar{2}0123\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0\bar{2}]$

252. $[0\bar{1}\bar{2}1\bar{0}\bar{3}] = [\bar{0}\bar{1}\bar{2}013] = [\bar{0}\bar{1}\bar{0}\bar{2}30] = [\bar{0}120\bar{1}\bar{2}] = [0\bar{1}20\bar{1}\bar{2}\bar{3}] = [0123\bar{0}\bar{2}3] = [\bar{0}\bar{1}\bar{2}01\bar{3}\bar{0}]$

253. $[0\bar{1}\bar{2}132] = [0\bar{1}\bar{2}\bar{3}13] = [0\bar{1}\bar{2}3\bar{1}\bar{3}] = [012\bar{0}\bar{2}3] = [01\bar{0}21\bar{3}] = [\bar{0}123\bar{1}\bar{3}] = [0\bar{1}20\bar{1}\bar{2}\bar{0}\bar{1}]$

254. $[0\bar{1}\bar{2}1\bar{3}\bar{2}] = [0\bar{1}0231] = [0120\bar{1}\bar{0}] = [012\bar{0}10] = [0\bar{1}20\bar{1}\bar{2}0] = [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}\bar{3}] = [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}0]$
$= [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}2]$

255. $[0\bar{1}\bar{2}301] = [0\bar{1}\bar{0}\bar{2}31] = [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{3}] = [\bar{0}1231\bar{2}] = [0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}\bar{1}] = [0120130] = [\bar{0}\bar{1}\bar{2}3103]$

256. $[0\bar{1}\bar{2}30\bar{1}] = [012301] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}] = [\bar{0}12\bar{3}\bar{0}1] = [0\bar{1}2012\bar{3}] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}] = [0123\bar{0}\bar{3}2]$

257. $[0\bar{1}\bar{2}302] = [\bar{0}\bar{1}\bar{2}03\bar{1}] = [0\bar{1}\bar{0}2\bar{3}\bar{2}1] = [012\bar{0}\bar{3}1\bar{3}] = [0\bar{1}20\bar{3}\bar{1}\bar{0}\bar{3}] = [0\bar{1}\bar{2}\bar{0}13\bar{2}3]$

258. $[0\bar{1}\bar{2}3\bar{0}\bar{1}] = [012\bar{3}01] = [\bar{0}120\bar{1}\bar{0}] = [0\bar{1}\bar{2}01\bar{2}\bar{0}] = [0\bar{1}\bar{2}3101] = [012\bar{0}213] = [0123\bar{0}32]$
$= [0\bar{1}\bar{2}132\bar{1}\bar{3}]$

259. $[0\bar{1}\bar{2}310] = [\bar{0}\bar{1}\bar{2}30\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}30] = [0\bar{1}21\bar{0}32] = [0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{2}] = [0\bar{1}\bar{2}3\bar{0}\bar{2}\bar{0}] = [0\bar{1}\bar{2}132\bar{1}3]$

260. $[0\bar{1}\bar{2}31\bar{0}] = [0\bar{1}2013\bar{2}] = [0120\bar{3}\bar{0}1] = [012\bar{0}212] = [\bar{0}\bar{1}231\bar{2}1] = [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}\bar{3}] = [0\bar{1}20\bar{1}\bar{2}3\bar{2}]$

261. $[0\bar{1}\bar{2}312] = [0\bar{1}\bar{2}\bar{0}12\bar{3}] = [0\bar{1}\bar{0}2\bar{3}\bar{2}\bar{0}] = [\bar{0}\bar{1}201\bar{3}1] = [0\bar{1}213\bar{0}\bar{3}2] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0\bar{1}]$

262. $[0\bar{1}\bar{2}\bar{3}0\bar{2}] = [01\bar{2}\bar{0}1\bar{3}] = [\bar{0}1\bar{2}\bar{0}\bar{3}1] = [0\bar{1}201\bar{2}0] = [0\bar{1}201\bar{3}0] = [0\bar{1}20\bar{1}3\bar{0}] = [0\bar{1}203\bar{1}0]$
$= [0\bar{1}23\bar{2}1\bar{3}]$

263. $[0\bar{1}\bar{2}\bar{3}\bar{0}1] = [\bar{0}1230\bar{1}\bar{0}] = [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}3]$

264. $[0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}] = [01\bar{2}01\bar{2}] = [01\bar{0}231] = [\bar{0}1\bar{2}\bar{0}1\bar{2}] = [0\bar{1}\bar{2}\bar{3}012] = [0\bar{1}\bar{2}\bar{3}013] = [0120\bar{1}21]$
$= [\bar{0}\bar{1}21\bar{3}0\bar{1}] = [0\bar{1}21\bar{3}020] = [0\bar{1}21\bar{3}021]$

265. $[0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}] = [\bar{0}\bar{1}21\bar{3}\bar{0}] = [\bar{0}\bar{1}23\bar{2}\bar{1}] = [0\bar{1}20130] = [0\bar{1}23\bar{2}12] = [\bar{0}\bar{1}21030] = [0\bar{1}201\bar{0}\bar{3}\bar{0}]$

266. $[0\bar{1}\bar{2}\bar{3}1\bar{0}] = [01\bar{0}230] = [0\bar{1}20\bar{3}\bar{2}\bar{1}] = [0\bar{1}21\bar{3}0\bar{2}] = [01\bar{0}\bar{2}\bar{3}\bar{0}3] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}\bar{1}] = [0\bar{1}21\bar{3}023]$
$= [0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}3] = [0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}2]$

267. $[0\bar{1}\bar{2}\bar{3}1\bar{2}] = [0120\bar{1}3] = [0\bar{1}20\bar{1}2\bar{0}] = [0\bar{1}21\bar{0}3\bar{1}] = [0\bar{1}213\bar{0}1] = [0\bar{1}23\bar{2}0\bar{3}] = [0\bar{1}23\bar{2}\bar{0}2]$
$= [0\bar{1}\bar{2}01\bar{2}3]$

268. $[0\bar{1}\bar{2}\bar{3}\bar{1}0] = [\bar{0}12310\bar{0}] = [0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}\bar{0}]$

269. $[0\bar{1}\bar{2}\bar{3}\bar{1}2] = [0\bar{1}\bar{0}2\bar{3}\bar{0}] = [012\bar{0}13] = [\bar{0}12\bar{3}\bar{0}\bar{1}] = [0\bar{1}21\bar{0}13] = [0\bar{1}\bar{2}01\bar{2}\bar{1}] = [0\bar{1}\bar{2}30\bar{1}2]$

270. $[0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}] = [01201\bar{3}] = [012312] = [0\bar{1}21\bar{0}\bar{3}\bar{1}] = [0\bar{1}23\bar{2}02] = [012\bar{0}\bar{3}0\bar{2}]$

271. $[0\bar{1}0230] = [0\bar{1}231\bar{2}1] = [0\bar{1}\bar{2}\bar{0}2\bar{1}\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{3}03] = [0\bar{1}\bar{2}\bar{0}3\bar{0}\bar{3}] = [01\bar{0}\bar{2}\bar{3}\bar{0}\bar{1}] = [\bar{0}\bar{1}2103\bar{1}]$
$= [0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1}0] = [0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1}2] = [0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1}3]$

272. $[0\bar{1}\bar{0}230] = [\bar{0}12013] = [0\bar{1}201\bar{2}\bar{1}] = [0\bar{1}20\bar{1}21] = [0\bar{1}\bar{2}012\bar{3}] = [0\bar{1}\bar{2}\bar{0}2\bar{3}\bar{0}] = [012\bar{3}0\bar{1}\bar{0}]$

273. $[0\bar{1}\bar{0}23\bar{1}] = [\bar{0}1\bar{2}\bar{3}\bar{1}\bar{0}] = [0\bar{1}23102\bar{2}] = [0\bar{1}\bar{2}\bar{0}310] = [0120\bar{3}\bar{0}2] = [01\bar{2}1301] = [0\bar{1}201\bar{0}\bar{3}\bar{2}]$
$= [0\bar{1}231023]$

274. $[0\bar{1}\bar{0}23\bar{2}] = [0\bar{1}\bar{0}\bar{2}\bar{3}2]$

275. $[0\bar{1}\bar{0}2\bar{3}\bar{1}] = [012\bar{0}31] = [\bar{0}1\bar{2}\bar{0}12] = [0\bar{1}\bar{2}\bar{0}12\bar{0}] = [012310\bar{1}] = [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}2] = [0\bar{1}20\bar{1}\bar{2}31]$
$= [0\bar{1}\bar{2}1321\bar{1}\bar{2}]$

276. $[0\bar{1}\bar{0}2\bar{3}\bar{2}] = [0\bar{1}\bar{0}\bar{2}32] = [0\bar{1}\bar{2}302\bar{1}] = [0\bar{1}\bar{2}3123] = [012\bar{0}\bar{3}13] = [\bar{0}\bar{1}\bar{2}01\bar{3}\bar{1}]$

277. $[0\bar{1}\bar{0}\bar{2}13] = [\bar{0}123\bar{0}1] = [\bar{0}1\bar{2}3\bar{2}\bar{0}] = [0\bar{1}\bar{2}\bar{0}\bar{1}3\bar{0}] = [0\bar{1}\bar{2}\bar{0}213] = [0123103] = [01\bar{2}301\bar{0}]$
$= [01\bar{2}\bar{3}01\bar{3}]$

278. $[0\bar{1}\bar{0}\bar{2}1\bar{3}] = [012\bar{0}3\bar{0}] = [0123\bar{2}\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}3] = [\bar{0}\bar{1}\bar{2}13\bar{1}] = [0120\bar{3}\bar{0}3]$

279. $[0\bar{1}\bar{0}\bar{2}\bar{1}0] = [\bar{0}\bar{1}2102\bar{2}] = [\bar{0}1\bar{0}23\bar{1}] = [\bar{0}1\bar{2}\bar{0}\bar{1}0] = [0\bar{1}201\bar{0}2] = [0\bar{1}20\bar{3}1\bar{3}\bar{2}] = [0\bar{1}213\bar{2}12]$

280. $[0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}] = [\bar{0}\bar{1}213\bar{2}] = [\bar{0}1\bar{0}230] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}3] = [0\bar{1}\bar{2}3013] = [0120\bar{3}02] = [\bar{0}\bar{1}\bar{2}310\bar{3}]$
$= [0\bar{1}201\bar{0}\bar{3}\bar{2}]$

281. $[0\bar{1}\bar{0}\bar{2}31] = [01\bar{2}\bar{0}\bar{3}\bar{1}] = [\bar{0}1\bar{2}\bar{0}\bar{1}2] = [0\bar{1}2012\bar{3}] = [0\bar{1}203\bar{1}2] = [0\bar{1}\bar{2}\bar{0}\bar{1}3\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}2]$
$= [0123\bar{0}\bar{3}\bar{1}] = [0\bar{1}201\bar{2}\bar{3}\bar{1}]$

282. $[012013] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}] = [\bar{0}12312] = [0\bar{1}\bar{2}301\bar{0}] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{0}] = [\bar{0}\bar{1}21302] = [\bar{0}\bar{1}\bar{2}030\bar{2}]$
$= [\bar{0}123\bar{2}\bar{1}\bar{2}]$

283. $[0120\bar{1}2] = [01\bar{2}012] = [01\bar{0}\bar{2}\bar{3}\bar{1}] = [\bar{0}12301] = [0\bar{1}20\bar{3}\bar{1}2] = [0\bar{1}\bar{2}\bar{3}0\bar{1}\bar{2}] = [\bar{0}1230\bar{1}\bar{2}]$
$= [\bar{0}1230\bar{1}\bar{3}] = [0\bar{1}20\bar{3}\bar{1}\bar{2}\bar{0}]$

284. $[01203\bar{0}] = [012\bar{3}\bar{0}3] = [\bar{0}\bar{1}\bar{2}01\bar{2}] = [0\bar{1}\bar{2}0\bar{1}\bar{0}2] = [\bar{0}\bar{1}2130\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}3] = [0\bar{1}20\bar{1}\bar{2}\bar{0}\bar{3}]$

285. $[012032] = [01\bar{2}\bar{1}\bar{0}\bar{3}] = [\bar{0}\bar{1}2\bar{1}03] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{2}] = [0\bar{1}213\bar{2}\bar{0}] = [0\bar{1}\bar{2}\bar{0}\bar{1}32] = [012\bar{3}\bar{0}1\bar{2}]$
$= [01\bar{2}1303]$

286. $[0120\bar{3}0] = [0\bar{1}\bar{2}0\bar{1}\bar{0}2] = [0\bar{1}\bar{0}\bar{2}\bar{1}3\bar{0}] = [\bar{0}\bar{1}\bar{2}012\bar{1}] = [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}] = [0\bar{1}201\bar{0}\bar{3}2]$
$= [0\bar{1}20\bar{3}1\bar{3}0]$

287. $[01203\bar{0}] = [012\bar{0}30] = [\bar{0}1\bar{2}\bar{3}\bar{1}0] = [0\bar{1}\bar{2}31\bar{0}\bar{1}] = [0\bar{1}\bar{0}23\bar{1}\bar{3}] = [0\bar{1}\bar{0}\bar{2}1\bar{3}\bar{1}] = [0\bar{1}20\bar{1}\bar{2}32]$

288. $[012\bar{0}21] = [\bar{0}123\bar{2}3] = [\bar{0}\bar{1}2\bar{1}\bar{0}\bar{2}] = [\bar{0}\bar{1}\bar{2}3\bar{2}\bar{3}] = [\bar{0}120\bar{1}0] = [0\bar{1}\bar{2}3\bar{0}\bar{2}\bar{1}] = [0\bar{1}\bar{2}31\bar{0}\bar{3}]$
$= [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}3]$

289. $[012\bar{0}\bar{2}1] = [01\bar{2}30\bar{1}] = [\bar{0}\bar{1}21301] = [\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}\bar{2}] = [\bar{0}\bar{1}\bar{2}03\bar{0}3] = [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2}\bar{1}]$

290. $[012\bar{0}\bar{2}\bar{1}] = [\bar{0}\bar{1}\bar{2}3\bar{0}\bar{3}] = [0\bar{1}20120] = [0\bar{1}21321] = [0\bar{1}\bar{2}0\bar{1}\bar{0}3] = [\bar{0}\bar{1}\bar{2}310\bar{1}] = [0\bar{1}201\bar{0}\bar{2}\bar{0}]$

291. $[012\bar{0}\bar{3}0] = [01230\bar{3}] = [01231\bar{2}] = [0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}3] = [01\bar{2}\bar{3}21\bar{3}] = [01\bar{2}3013] = [\bar{0}\bar{1}21\bar{0}31]$
$= [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2}\bar{0}]$

292. $[012\bar{0}\bar{3}1] = [\bar{0}12\bar{3}\bar{0}\bar{2}] = [0\bar{1}\bar{2}3021] = [0\bar{1}\bar{0}2\bar{3}\bar{2}\bar{1}]$

293. $[012302] = [\bar{0}1\bar{2}\bar{3}\bar{1}2] = [0\bar{1}201\bar{3}\bar{2}] = [0\bar{1}21\bar{3}\bar{0}\bar{2}] = [0123\bar{0}\bar{2}1] = [\bar{0}\bar{1}210\bar{3}\bar{2}] = [\bar{0}\bar{1}\bar{0}\bar{2}\bar{3}\bar{0}2]$

294. $[01230\bar{2}] = [01\bar{2}\bar{0}\bar{1}\bar{3}] = [01\bar{2}\bar{0}313] = [\bar{0}120\bar{1}\bar{3}\bar{2}] = [0\bar{1}23102\bar{0}] = [0\bar{1}23102\bar{1}]$
$= [012\bar{0}\bar{3}101] = [012\bar{0}\bar{3}102]$

295. $[0123\bar{0}\bar{2}] = [01\bar{2}\bar{3}12] = [\bar{0}120\bar{1}2] = [\bar{0}1230\bar{3}] = [0\bar{1}20\bar{1}32] = [0\bar{1}\bar{2}1\bar{0}\bar{3}\bar{2}] = [012302\bar{3}]$
$= [\bar{0}\bar{1}210\bar{3}2]$

296. $[0123\bar{0}3] = [01\bar{0}210] = [0\bar{1}\bar{2}01\bar{2}0] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}\bar{3}] = [01\bar{2}1\bar{3}\bar{0}\bar{2}] = [01\bar{2}301\bar{0}] = [0\bar{1}201\bar{0}\bar{2}3]$

297. $[0123\bar{0}\bar{3}] = [012\bar{3}03] = [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{3}01] = [\bar{0}1\bar{2}\bar{0}\bar{1}\bar{2}] = [0\bar{1}\bar{2}30\bar{1}0] = [0\bar{1}\bar{0}\bar{2}313]$

298. $[012310] = [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{1}] = [\bar{0}1\bar{2}\bar{0}1\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{1}30] = [0\bar{1}\bar{0}2\bar{3}\bar{1}3] = [0\bar{1}\bar{0}\bar{2}13\bar{1}] = [\bar{0}123\bar{0}21]$
$= [\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}\bar{0}]$

299. $[012\bar{3}\bar{0}1] = [01\bar{2}03\bar{0}] = [\bar{0}1201\bar{3}] = [0\bar{1}\bar{0}2301] = [0120320] = [01\bar{2}130\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}\bar{2}]$
$= [012\bar{3}\bar{0}1\bar{3}0]$

300. $[012\bar{3}21] = [01\bar{2}103] = [0\bar{1}\bar{2}01\bar{3}\bar{0}] = [0\bar{1}\bar{0}\bar{2}1\bar{3}\bar{0}] = [012\bar{0}\bar{3}03] = [01\bar{2}\bar{0}\bar{3}03] = [01\bar{2}1\bar{3}\bar{0}\bar{3}]$
$= [01\bar{2}301\bar{3}]$

301. $[012\bar{3}\bar{2}\bar{0}] = [01\bar{2}\bar{0}\bar{1}2] = [\bar{0}\bar{1}2\bar{1}\bar{0}3] = [\bar{0}\bar{1}\bar{2}312] = [0\bar{1}\bar{2}013\bar{0}] = [0\bar{1}\bar{2}013\bar{1}] = [\bar{0}120\bar{1}\bar{3}\bar{0}]$
$= [\bar{0}123\bar{2}\bar{1}0]$

302. $[01\bar{2}\bar{0}\bar{1}3] = [\bar{0}123\bar{2}0] = [0\bar{1}201\bar{3}\bar{1}] = [0\bar{1}20\bar{1}31] = [0\bar{1}21\bar{3}\bar{0}\bar{1}] = [01230\bar{2}\bar{0}] = [01\bar{2}\bar{3}\bar{1}\bar{2}0] = [0\bar{1}201\bar{0}31]$

303. $[01\bar{2}\bar{0}31] = [\bar{0}\bar{1}\bar{2}\bar{3}02] = [0\bar{1}231\bar{2}\bar{0}] = [0\bar{1}\bar{2}\bar{0}132] = [01230\bar{2}\bar{1}] = [0\bar{1}23\bar{2}\bar{1}\bar{0}] = [0\bar{1}231020] = [0\bar{1}231021] = [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{0}]$

304. $[01\bar{2}\bar{0}\bar{3}0] = [01\bar{2}30\bar{3}] = [\bar{0}120\bar{3}1] = [0\bar{1}\bar{2}012\bar{0}] = [012\bar{3}21\bar{0}] = [01\bar{2}1\bar{3}\bar{0}3] = [\bar{0}\bar{1}\bar{2}31\bar{2}3]$

305. $[01\bar{2}130] = [\bar{0}\bar{1}21\bar{0}1] = [\bar{0}\bar{1}231\bar{3}] = [0\bar{1}\bar{0}23\bar{1}\bar{0}] = [012032\bar{0}] = [012\bar{3}\bar{0}12] = [01\bar{2}1\bar{3}\bar{0}\bar{1}] = [0\bar{1}201\bar{0}32]$

306. $[01\bar{2}1\bar{3}\bar{0}] = [\bar{0}\bar{1}21\bar{0}\bar{1}] = [\bar{0}\bar{1}2\bar{1}01] = [0123\bar{0}31] = [012\bar{3}210] = [01\bar{2}\bar{0}\bar{3}0\bar{3}] = [01\bar{2}1302] = [0\bar{1}201\bar{0}\bar{2}\bar{3}]$

307. $[01\bar{2}301] = [01\bar{0}21\bar{0}] = [0\bar{1}21\bar{0}\bar{3}\bar{2}] = [0\bar{1}\bar{2}\bar{0}312] = [012\bar{0}\bar{2}1\bar{0}] = [012\bar{0}\bar{3}0\bar{3}] = [0123\bar{0}30] = [012\bar{3}213]$

308. $[01\bar{2}\bar{3}01] = [\bar{0}\bar{1}230\bar{1}] = [0\bar{1}\bar{2}\bar{0}2\bar{3}\bar{1}] = [0\bar{1}\bar{0}\bar{2}130]$

309. $[01\bar{2}\bar{3}\bar{1}2] = [\bar{0}\bar{1}230\bar{2}3] = [\bar{0}\bar{1}231\bar{2}\bar{3}] = [\bar{0}\bar{1}\bar{2}0123] = [\bar{0}1230\bar{2}\bar{3}] = [0\bar{1}201\bar{2}\bar{3}0] = [0\bar{1}201\bar{2}\bar{3}2] = [01\bar{2}\bar{3}\bar{1}20\bar{2}]$

310. $[01\bar{2}\bar{3}\bar{1}\bar{2}] = [\bar{0}123\bar{2}\bar{0}] = [0\bar{1}203\bar{1}\bar{3}] = [0\bar{1}20\bar{3}13] = [0\bar{1}23123] = [01\bar{2}\bar{0}\bar{1}32]$

311. $[01\bar{0}23\bar{2}] = [01\bar{0}\bar{2}\bar{3}2] = [0\bar{1}20\bar{3}1\bar{0}\bar{2}] = [0\bar{1}213\bar{0}31] = [012\bar{0}\bar{3}1\bar{0}3]$

312. $[01\bar{0}2\bar{3}\bar{2}] = [01\bar{0}\bar{2}32]$

313. $[01\bar{0}\bar{2}3\bar{0}] = [\bar{0}123\bar{1}0] = [0\bar{1}20\bar{3}21] = [0\bar{1}\bar{2}31\bar{0}\bar{3}] = [0\bar{1}02301] = [\bar{0}\bar{1}210\bar{3}1] = [0\bar{1}20\bar{3}\bar{1}\bar{2}\bar{1}] = [0\bar{1}20\bar{3}\bar{1}\bar{2}\bar{3}] = [0\bar{1}\bar{2}\bar{3}1\bar{0}1\bar{0}] = [0\bar{1}\bar{2}\bar{3}1\bar{0}1\bar{2}] = [0\bar{1}\bar{2}\bar{3}1\bar{0}1\bar{3}]$

314. $[\bar{0}\bar{1}2103] = [\bar{0}\bar{1}23\bar{2}1] = [0\bar{1}20310] = [0\bar{1}231\bar{2}\bar{1}] = [0\bar{1}23\bar{2}\bar{1}\bar{3}] = [0\bar{1}\bar{2}\bar{3}0\bar{2}\bar{3}] = [0\bar{1}02302]$

315. $[\bar{0}\bar{1}210\bar{3}] = [0123023] = [0123\bar{0}\bar{2}\bar{1}] = [01\bar{0}\bar{2}3\bar{0}\bar{2}] = [\bar{0}\bar{1}21\bar{0}3\bar{0}] = [0\bar{1}20\bar{3}\bar{1}21] = [0\bar{1}20\bar{3}\bar{1}23] = [01\bar{0}23\bar{2}12] = [01\bar{0}23\bar{2}13]$

316. $[\bar{0}\bar{1}21\bar{0}3] = [\bar{0}123\bar{1}\bar{2}] = [012\bar{0}30\bar{1}] = [\bar{0}\bar{1}210\bar{3}0] = [\bar{0}\bar{1}\bar{2}01\bar{3}2] = [\bar{0}\bar{1}\bar{2}31\bar{2}0] = [01\bar{0}23\bar{2}1\bar{2}] = [01\bar{0}23\bar{2}1\bar{3}]$

317. $[\bar{0}\bar{1}21\bar{0}\bar{3}] = [\bar{0}\bar{1}23\bar{2}\bar{0}] = [\bar{0}\bar{1}\bar{2}\bar{0}31] = [\bar{0}\bar{1}21303] = [\bar{0}\bar{1}23\bar{0}2\bar{0}] = [0\bar{1}\bar{2}132\bar{1}0]$

318. $[\bar{0}\bar{1}2130] = [\bar{0}\bar{1}23\bar{2}0] = [012013\bar{2}] = [012\bar{0}\bar{2}1\bar{3}] = [\bar{0}\bar{1}21\bar{0}3\bar{0}] = [\bar{0}\bar{1}\bar{2}31\bar{2}1] = [\bar{0}123\bar{2}1\bar{2}]$

319. $[\bar{0}\bar{1}213\bar{0}] = [\bar{0}\bar{1}\bar{2}013] = [01203\bar{0}2] = [\bar{0}\bar{1}21\bar{3}0\bar{3}] = [\bar{0}\bar{1}\bar{2}03\bar{1}\bar{3}] = [012\bar{0}\bar{3}1\bar{0}1] = [012\bar{0}\bar{3}1\bar{0}2]$

320. $[\bar{0}\bar{1}21\bar{3}0] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}3] = [0\bar{1}\bar{2}\bar{3}0\bar{2}\bar{1}] = [\bar{0}\bar{1}213\bar{0}3] = [\bar{0}\bar{1}\bar{2}31\bar{0}2] = [\bar{0}1230\bar{2}\bar{1}] = [0\bar{1}21\bar{3}02\bar{0}] = [0\bar{1}21\bar{3}02\bar{1}] = [012\bar{0}\bar{3}1\bar{0}\bar{1}] = [012\bar{0}\bar{3}1\bar{0}\bar{2}]$

321. $[\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}] = [\bar{0}\bar{1}\bar{2}3\bar{2}\bar{1}] = [0\bar{1}21320] = [0\bar{1}\bar{2}\bar{0}3\bar{1}\bar{2}] = [0\bar{1}\bar{2}\bar{0}231] = [012\bar{0}\bar{2}12] = [012\bar{3}\bar{2}\bar{0}2] = [\bar{0}\bar{1}\bar{2}03\bar{0}\bar{3}]$

322. $[\bar{0}\bar{1}23\bar{0}2] = [\bar{0}123\bar{2}1] = [012310\bar{2}] = [\bar{0}\bar{1}21\bar{0}3\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}\bar{0}] = [0\bar{1}\bar{2}132\bar{1}\bar{0}]$

323. $[\bar{0}\bar{1}23\bar{0}\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{3}12] = [01\bar{2}\bar{3}\bar{1}2\bar{0}] = [\bar{0}120\bar{1}\bar{3}\bar{1}] = [0\bar{1}\bar{2}01230] = [01\bar{2}\bar{3}\bar{1}202]$

324. $[\bar{0}\bar{1}231\bar{2}] = [0\bar{1}20132] = [0\bar{1}23\bar{2}10] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}\bar{0}] = [0\bar{1}\bar{2}31\bar{0}\bar{2}] = [01\bar{2}\bar{3}\bar{1}23] = [0\bar{1}201\bar{2}\bar{3}\bar{0}]$
$= [0\bar{1}201\bar{2}\bar{3}\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}02]$

325. $[\bar{0}\bar{1}\bar{2}012] = [0\bar{1}231\bar{0}\bar{2}] = [0\bar{1}\bar{2}\bar{0}1\bar{0}\bar{2}] = [0120\bar{3}\bar{0}\bar{3}] = [0120\bar{3}03] = [01\bar{2}\bar{3}\bar{1}2\bar{1}] = [\bar{0}1230\bar{2}\bar{3}]$
$= [0\bar{1}20\bar{1}\bar{2}30]$

326. $[\bar{0}\bar{1}\bar{2}01\bar{3}] = [0\bar{1}\bar{2}1\bar{0}\bar{3}0] = [0\bar{1}\bar{2}312\bar{3}] = [0\bar{1}\bar{0}\bar{2}\bar{3}\bar{2}0] = [01\bar{2}\bar{3}\bar{1}201] = [01\bar{0}2\bar{3}\bar{2}1\bar{0}]$

327. $[\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}] = [\bar{0}\bar{1}\bar{2}\bar{0}10] = [\bar{0}\bar{1}\bar{2}102] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{1}] = [0\bar{1}\bar{2}1\bar{3}\bar{2}1] = [0\bar{1}\bar{0}\bar{2}\bar{3}\bar{1}\bar{0}] = [0\bar{1}\bar{2}132\bar{1}2]$
$= [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}\bar{0}] = [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}\bar{2}]$

328. $[\bar{0}\bar{1}\bar{2}0\bar{3}\bar{0}] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}3] = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}2] = [0\bar{1}213\bar{0}\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{3}10] = [01201\bar{3}\bar{1}] = [012\bar{0}\bar{2}1\bar{2}]$
$= [\bar{0}\bar{1}2\bar{1}\bar{0}\bar{3}2]$

329. $[\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}] = [\bar{0}123\bar{1}2] = [0123102] = [\bar{0}\bar{1}21\bar{0}\bar{3}\bar{2}] = [\bar{0}\bar{1}21\bar{3}\bar{0}2] = [\bar{0}\bar{1}23\bar{0}\bar{2}\bar{1}] = [\bar{0}\bar{1}\bar{2}31\bar{0}\bar{2}]$
$= [\bar{0}123\bar{2}\bar{1}3]$

330. $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{0}] = [0\bar{1}21\bar{0}12] = [0\bar{1}231\bar{0}1] = [0\bar{1}\bar{2}\bar{0}\bar{3}02] = [0\bar{1}\bar{2}31\bar{0}3] = [012\bar{0}\bar{2}1\bar{2}] = [0\bar{1}201\bar{0}\bar{3}\bar{1}]$
$= [0\bar{1}20\bar{3}1\bar{3}\bar{1}]$

331. $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}] = [\bar{0}120\bar{1}3] = [0120\bar{3}\bar{0}\bar{2}] = [01\bar{2}\bar{3}\bar{0}13] = [\bar{0}\bar{1}21\bar{0}\bar{3}1] = [\bar{0}\bar{1}21\bar{3}\bar{0}1] = [01\bar{2}\bar{3}\bar{0}1\bar{3}\bar{0}]$

332. $[\bar{0}\bar{1}\bar{2}310] = [0\bar{1}\bar{2}0\bar{1}\bar{0}\bar{3}] = [0\bar{1}\bar{2}031\bar{2}] = [0\bar{1}\bar{2}301\bar{3}] = [0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}1] = [012\bar{0}\bar{2}\bar{1}3] = [\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}1]$
$= [\bar{0}123\bar{2}\bar{1}\bar{3}]$

333. $[\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}] = [\bar{0}12\bar{3}\bar{0}3] = [0\bar{1}20\bar{3}\bar{2}\bar{1}] = [0120\bar{3}01] = [\bar{0}\bar{1}21\bar{3}\bar{0}\bar{2}] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}3] = [\bar{0}1230\bar{2}1]$
$= [0\bar{1}20\bar{3}1\bar{3}\bar{0}]$

334. $[\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2}] = [012\bar{0}\bar{2}13] = [012\bar{0}\bar{3}01] = [0\bar{1}\bar{2}\bar{0}\bar{3}0\bar{2}] = [\bar{0}\bar{1}21\bar{0}\bar{3}\bar{1}] = [\bar{0}\bar{1}2130\bar{1}] = [\bar{0}\bar{1}23\bar{0}\bar{2}0]$
$= [01\bar{2}\bar{3}\bar{0}1\bar{3}1] = [01\bar{2}\bar{3}\bar{0}1\bar{3}2]$

335. $[\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}] = [0\bar{1}20\bar{3}\bar{2}1] = [0\bar{1}21\bar{3}\bar{0}2] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{0}2] = [0\bar{1}\bar{2}\bar{3}1\bar{0}1] = [01230\bar{2}\bar{0}] = [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}\bar{3}]$
$= [0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}\bar{2}] = [0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}\bar{3}]$

336. $[\bar{0}120\bar{1}\bar{3}] = [01230\bar{2}\bar{3}] = [01\bar{2}\bar{3}\bar{2}\bar{0}1] = [\bar{0}\bar{1}23\bar{0}\bar{2}1] = [\bar{0}123\bar{2}\bar{1}\bar{0}] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0] = [0\bar{1}\bar{2}0123\bar{0}]$
$= [012\bar{0}\bar{3}10\bar{1}] = [012\bar{0}\bar{3}10\bar{2}]$

337. $[\bar{0}1230\bar{1}] = [0\bar{1}\bar{2}\bar{3}\bar{0}10] = [0120\bar{1}20] = [0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}\bar{3}]$

338. $[\bar{0}1230\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}] = [0\bar{1}20\bar{3}12] = [0\bar{1}23120] = [01\bar{2}\bar{3}\bar{1}21] = [\bar{0}\bar{1}21\bar{3}02] = [\bar{0}\bar{1}\bar{2}012\bar{3}]$
$= [\bar{0}\bar{1}\bar{2}31\bar{0}\bar{2}]$

339. $[\bar{0}123\bar{2}\bar{1}] = [0120132] = [01\bar{2}\bar{3}\bar{2}\bar{0}\bar{1}] = [\bar{0}\bar{1}2130\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}] = [\bar{0}\bar{1}23\bar{0}\bar{2}3] = [\bar{0}\bar{1}\bar{2}3102]$
$= [\bar{0}120\bar{1}\bar{3}0]$

340. $[0\bar{1}201\bar{0}\bar{2}] = [0\bar{1}2132\bar{1}] = [0\bar{1}\bar{2}\bar{0}1\bar{0}\bar{3}] = [0\bar{1}\bar{2}\bar{0}210] = [0\bar{1}\bar{0}\bar{2}\bar{1}01] = [0123\bar{0}\bar{3}\bar{1}]$
$= [012\bar{0}\bar{2}\bar{1}0] = [01\bar{2}1\bar{3}\bar{0}2]$

341. $[0\bar{1}201\bar{0}3] = [0\bar{1}201\bar{3}1] = [0\bar{1}\bar{2}0\bar{3}0\bar{1}] = [0\bar{1}\bar{0}23\bar{1}0] = [01\bar{2}130\bar{1}] = [01\bar{2}\bar{0}\bar{1}3\bar{1}]$
$= [0\bar{1}213\bar{2}10]$
342. $[0\bar{1}2010\bar{3}] = [0\bar{1}23\bar{2}1\bar{2}] = [0\bar{1}\bar{2}\bar{0}30\bar{2}] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{2}0] = [0\bar{1}\bar{0}\bar{2}\bar{1}\bar{3}0] = [0120\bar{3}0\bar{2}] = [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}1]$
343. $[0\bar{1}201\bar{2}\bar{3}] = [0\bar{1}\bar{0}\bar{2}\bar{3}12] = [01\bar{2}\bar{3}\bar{1}2\bar{3}] = [\bar{0}\bar{1}231\bar{2}\bar{3}]$
344. $[0\bar{1}20\bar{1}\bar{2}\bar{0}] = [0\bar{1}23\bar{2}\bar{0}\bar{3}] = [0\bar{1}21301] = [0\bar{1}\bar{2}0\bar{1}\bar{0}\bar{2}] = [0\bar{1}\bar{2}1321] = [0\bar{1}\bar{2}1\bar{3}\bar{2}0] = [01203\bar{0}1]$
345. $[0\bar{1}20\bar{1}\bar{2}3] = [0\bar{1}231\bar{0}\bar{2}] = [0\bar{1}\bar{2}\bar{0}120] = [0\bar{1}\bar{2}1\bar{0}\bar{3}\bar{1}] = [0\bar{1}\bar{2}31\bar{0}1] = [0\bar{1}\bar{0}\bar{2}\bar{3}\bar{1}\bar{2}]$
$= [0120\bar{3}\bar{0}\bar{1}] = [\bar{0}\bar{1}\bar{2}012\bar{0}]$
346. $[0\bar{1}20\bar{3}1\bar{3}] = [0\bar{1}21\bar{0}1\bar{2}] = [0\bar{1}\bar{0}\bar{2}\bar{1}03] = [0120\bar{3}\bar{0}\bar{1}] = [01\bar{2}\bar{3}\bar{1}\bar{2}1] = [\bar{0}\bar{1}\bar{2}\bar{0}3\bar{0}2]$
$= [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{0}1] = [0\bar{1}213\bar{2}1\bar{2}]$
347. $[0\bar{1}20\bar{3}\bar{1}\bar{0}] = [0\bar{1}21\bar{0}30] = [0\bar{1}\bar{2}3023] = [01\bar{0}23\bar{2}0] = [0\bar{1}213\bar{0}\bar{3}\bar{1}] = [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{3}]$
348. $[0\bar{1}20\bar{3}\bar{1}\bar{2}] = [0120\bar{1}23] = [01\bar{0}\bar{2}\bar{3}\bar{0}2] = [\bar{0}\bar{1}210\bar{3}\bar{1}]$
349. $[0\bar{1}213\bar{0}\bar{3}] = [0\bar{1}21\bar{3}03] = [0\bar{1}\bar{2}312\bar{0}] = [01\bar{0}23\bar{2}\bar{0}] = [0\bar{1}20\bar{3}\bar{1}\bar{0}2] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}01]$
350. $[0\bar{1}213\bar{2}1] = [0\bar{1}2310\bar{1}] = [0\bar{1}2312\bar{1}] = [0\bar{1}23\bar{2}\bar{1}2] = [0\bar{1}\bar{2}\bar{0}3\bar{0}1] = [0\bar{1}\bar{0}\bar{2}\bar{1}0\bar{3}]$
$= [0\bar{1}201\bar{0}3\bar{0}] = [0\bar{1}20\bar{3}1\bar{3}2]$
351. $[0\bar{1}21\bar{3}02] = [0\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{3}] = [0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{2}] = [\bar{0}\bar{1}21\bar{3}01]$
352. $[0\bar{1}23102] = [0\bar{1}\bar{0}23\bar{1}\bar{2}] = [01230\bar{2}1] = [01\bar{2}\bar{0}31\bar{3}]$
353. $[0\bar{1}\bar{2}0123] = [0\bar{1}\bar{2}\bar{0}13\bar{1}] = [0\bar{1}\bar{2}\bar{0}\bar{1}\bar{3}1] = [0\bar{1}\bar{2}\bar{0}31\bar{3}] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}3] = [0\bar{1}\bar{0}230\bar{2}]$
$= [\bar{0}\bar{1}23\bar{0}\bar{2}\bar{1}] = [\bar{0}120\bar{1}\bar{3}1]$
354. $[0\bar{1}\bar{2}\bar{0}13\bar{2}] = [0\bar{1}\bar{2}302\bar{3}] = [01\bar{2}\bar{0}310] = [0\bar{1}20\bar{3}\bar{1}\bar{0}3] = [012\bar{0}\bar{3}10\bar{3}] = [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}3]$
355. $[0\bar{1}\bar{2}\bar{0}1\bar{3}\bar{1}] = [0\bar{1}\bar{2}\bar{0}\bar{1}31] = [0\bar{1}\bar{2}\bar{0}3\bar{1}\bar{3}] = [0\bar{1}\bar{2}\bar{0}313]$
356. $[0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}0] = [0\bar{1}\bar{2}3120] = [\bar{0}\bar{1}2312\bar{0}] = [0\bar{1}213\bar{0}\bar{3}\bar{2}] = [01\bar{2}\bar{3}\bar{1}203] = [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}\bar{0}]$
$= [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}\bar{2}]$
357. $[0\bar{1}\bar{2}132\bar{1}] = [0\bar{1}\bar{2}310\bar{1}] = [0\bar{1}\bar{2}3\bar{0}\bar{2}0] = [0\bar{1}\bar{0}2\bar{3}\bar{1}0] = [\bar{0}\bar{1}21\bar{0}\bar{3}\bar{2}] = [\bar{0}\bar{1}23\bar{0}20]$
$= [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}\bar{2}] = [0\bar{1}20\bar{1}\bar{2}01]$
358. $[0\bar{1}\bar{2}1\bar{3}\bar{2}\bar{1}] = [0\bar{1}\bar{2}\bar{3}\bar{0}1\bar{0}] = [\bar{0}\bar{1}\bar{2}0\bar{1}\bar{0}3] = [\bar{0}1230\bar{1}\bar{0}]$
359. $[0\bar{1}\bar{2}\bar{3}1\bar{0}\bar{1}] = [0\bar{1}\bar{2}\bar{3}\bar{1}01] = [\bar{0}1\bar{0}\bar{2}\bar{3}\bar{0}1]$
360. $[0\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1}] = [0\bar{1}0230\bar{1}] = [01\bar{0}\bar{2}\bar{3}\bar{0}1] = [0\bar{1}\bar{2}31\bar{0}\bar{1}0]$
361. $[012\bar{0}\bar{3}10] = [01230\bar{2}\bar{3}] = [\bar{0}120\bar{1}\bar{3}2] = [0\bar{1}\bar{2}\bar{0}13\bar{2}1] = [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}\bar{3}]$
362. $[012\bar{0}\bar{3}1\bar{0}] = [01\bar{0}23\bar{2}\bar{1}] = [\bar{0}\bar{1}213\bar{0}\bar{3}] = [\bar{0}\bar{1}21\bar{3}03]$
363. $[012\bar{3}\bar{0}1\bar{3}] = [01\bar{2}\bar{0}\bar{3}02] = [\bar{0}\bar{1}\bar{2}\bar{3}1\bar{2}\bar{3}] = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}2]$
364. $[01\bar{2}\bar{3}\bar{1}20] = [\bar{0}\bar{1}23\bar{0}\bar{2}\bar{3}] = [\bar{0}\bar{1}\bar{2}01\bar{3}\bar{2}] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}\bar{0}\bar{3}] = [01\bar{0}23\bar{2}10] = [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}0]$
$= [0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}2]$

365. $[01\bar{0}23\bar{2}1] = [\bar{0}\bar{1}210\bar{3}\bar{0}] = [\bar{0}\bar{1}21\bar{0}30] = [\bar{0}\bar{1}\bar{2}01\bar{3}2] = [012\bar{0}\bar{3}1\bar{0}\bar{3}] = [01\bar{2}\bar{3}\bar{1}20\bar{1}]$

366. $[0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1}] = [0\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}03] = [012\bar{0}\bar{3}103] = [01\bar{2}\bar{3}\bar{1}20\bar{3}]$

7.3 Cayley Diagram of G Over S_4

The Cayley diagram of G over S_4 is sketched broadly in Figures 7.1 and 7.2 and illustrated in detail in Figures 7.3 through 7.17. In Figures 7.1 and 7.2, the labels Z, A, B, C, D, E, F, G, and H indicate the double cosets represented by words of length 0, 1, 2, 3, 4, 5, 6, 7, and 8 letters, respectively. Likewise, in Figures 7.3 through 7.17, the label Z1 denotes the double coset represented by a word of length zero, the labels A1 and A2 denote the double cosets represented by words of length one, the labels B1,...,B4 denote the double cosets represented by words of length two, the labels C1,...,C12 denote the double cosets represented by words of length three, the labels D1,...,D49 denote the double cosets represented by words of length four, the labels E1,...,E128 denote the double cosets represented by words of length five, the labels F1,...,F143 denote the double cosets represented by words of length six, the labels G1,...,G26 denote the double cosets represented by words of length seven, and the label H1 denotes the double coset represented by a word of length eight. For a more detailed explanation of the meaning of the component parts of a Cayley diagram, see Section 2.3.

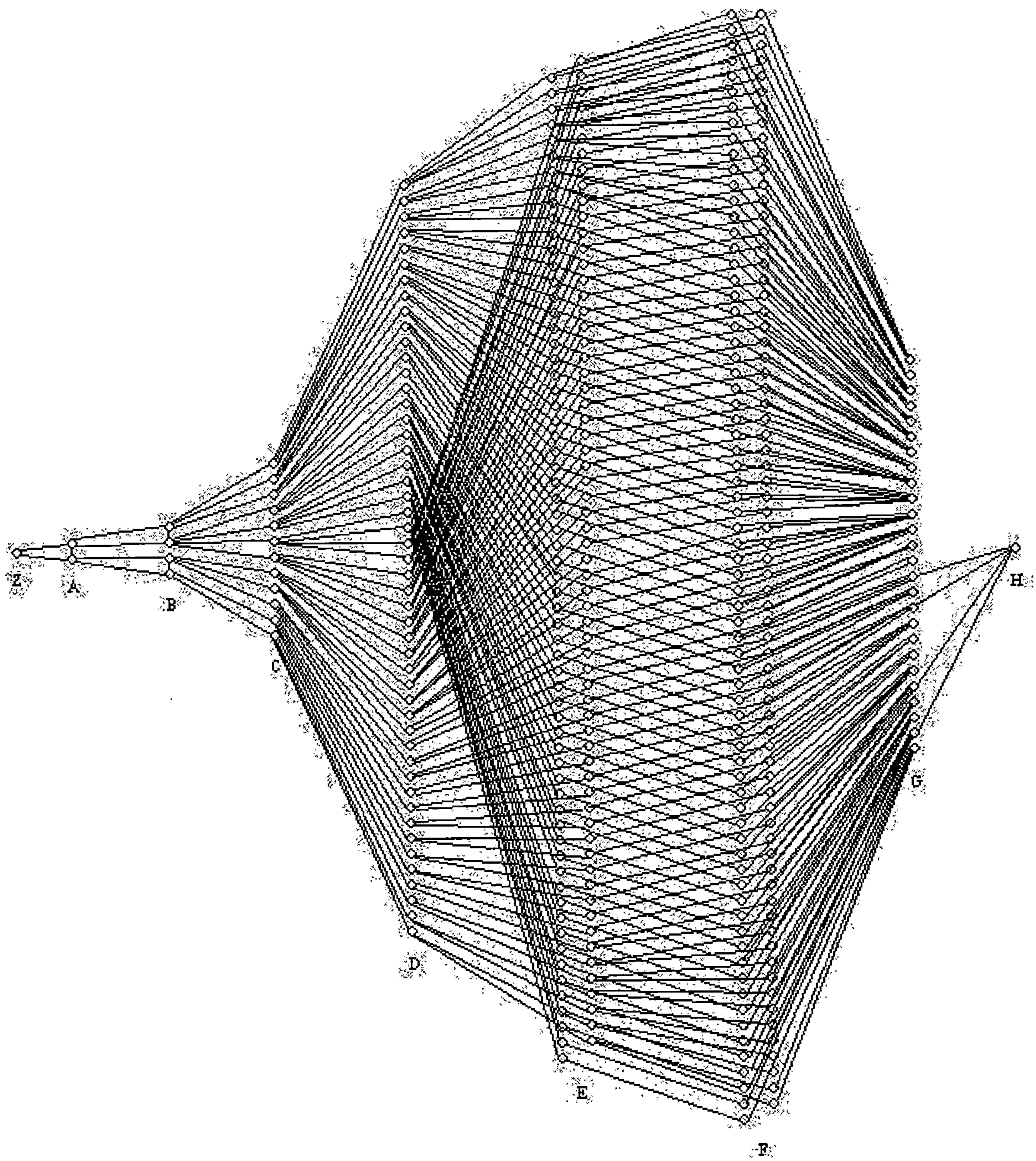

Figure 7.1: A Rough Sketch of the Cayley Diagram of G Over S_4

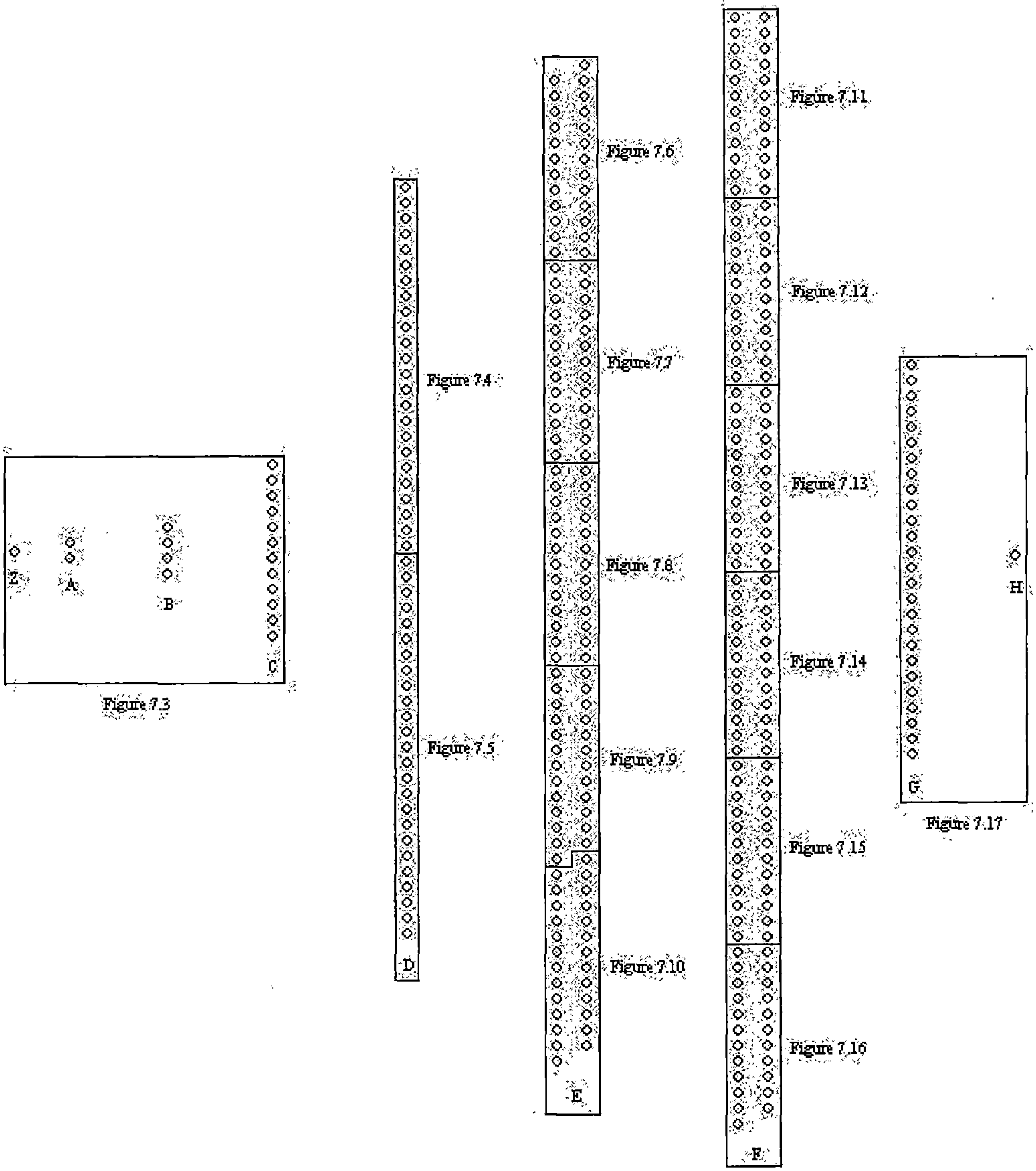

Figure 7.2: Our Breakdown of the Cayley Diagram of G Over S_4

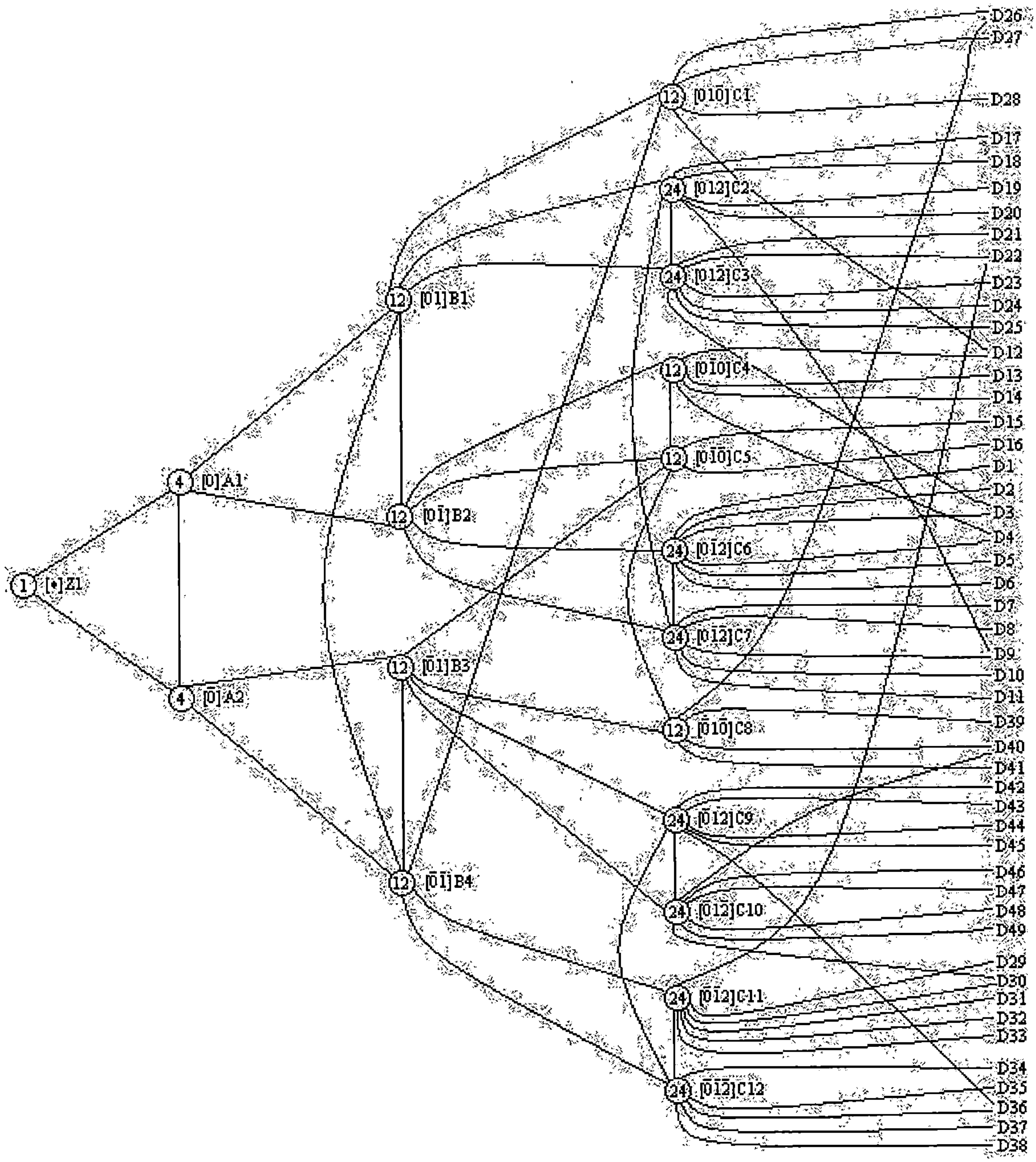

Figure 7.3: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 0, 1, 2, and 3

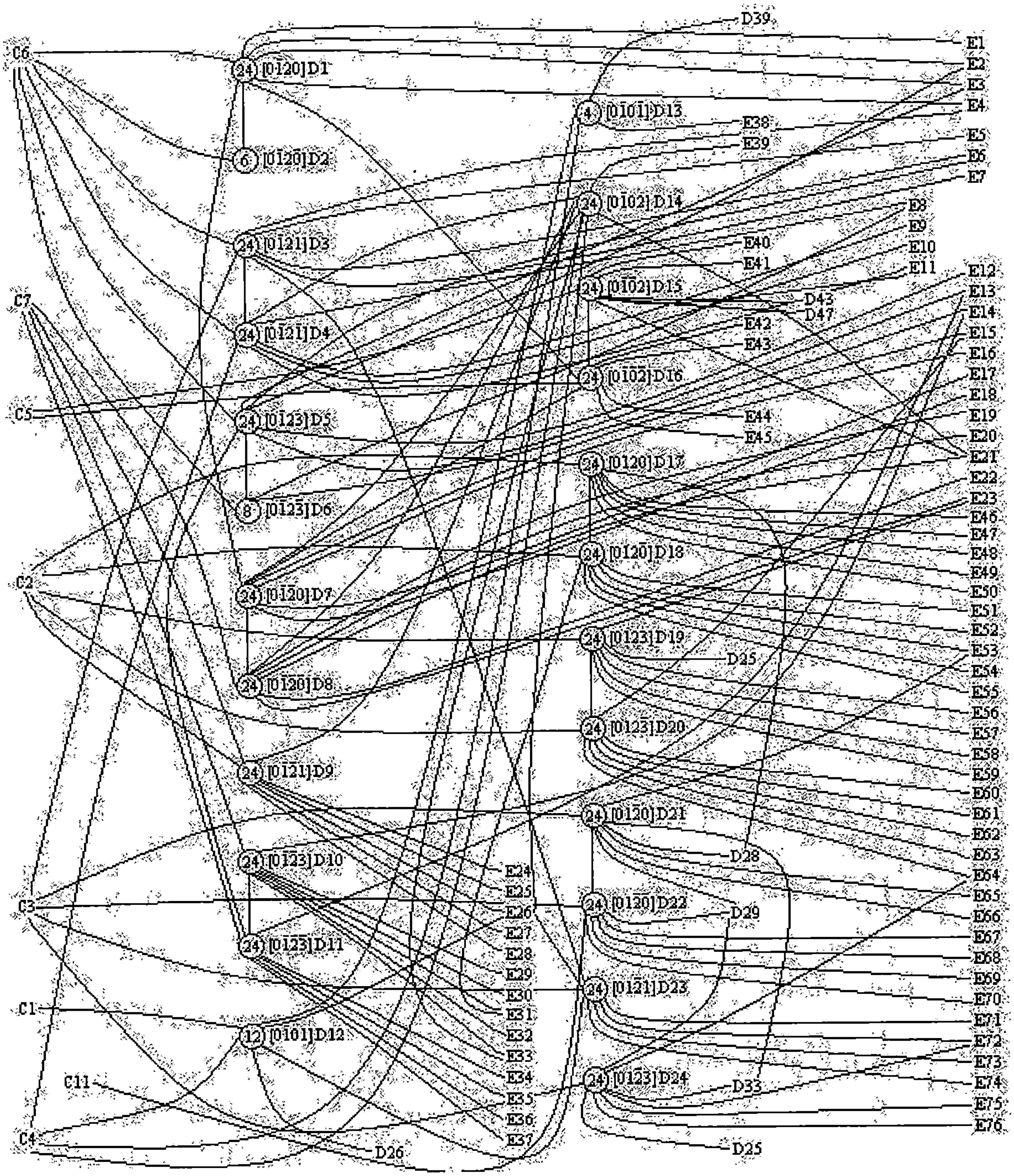

Figure 7.4: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 4

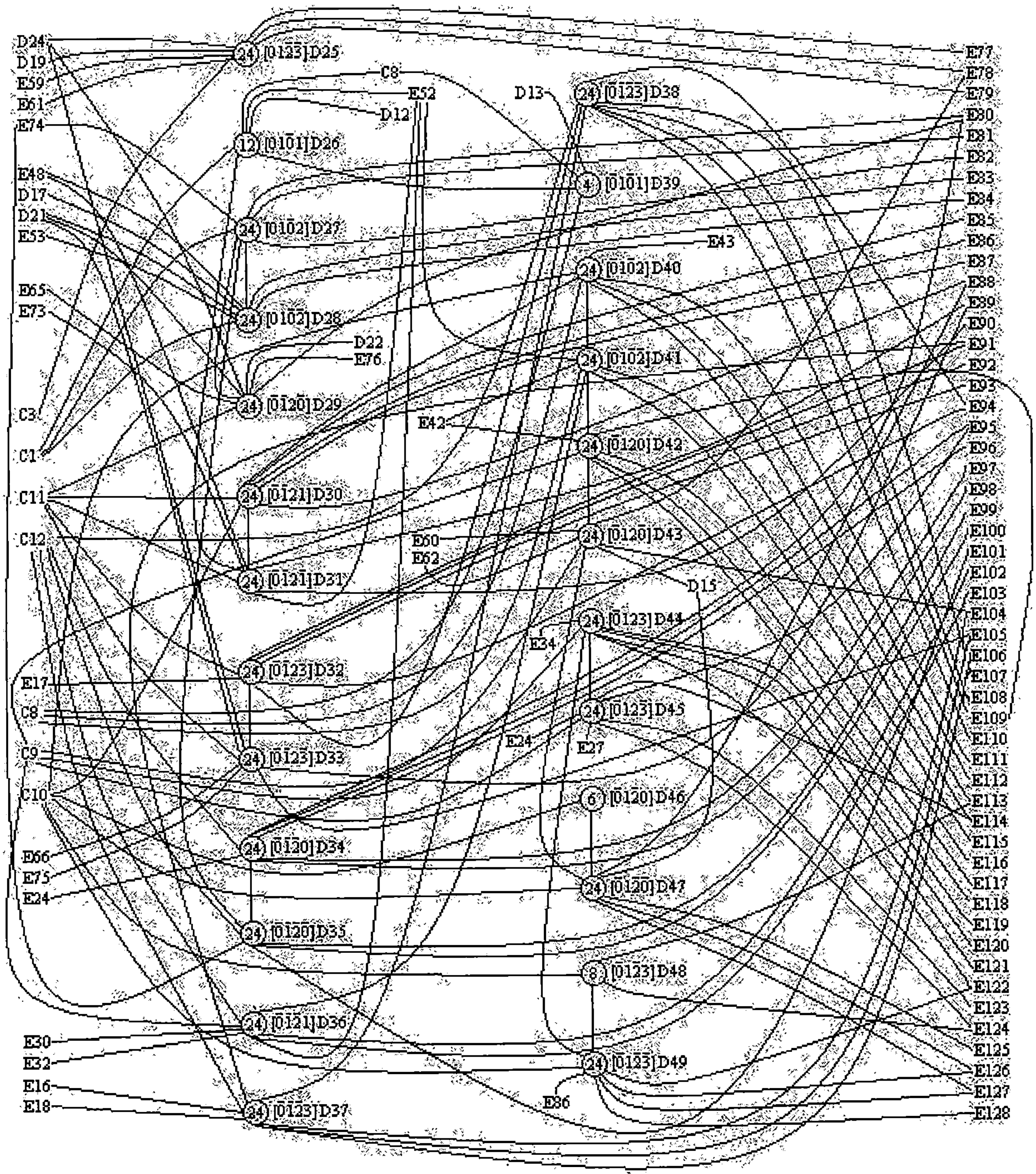

Figure 7.5: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 4

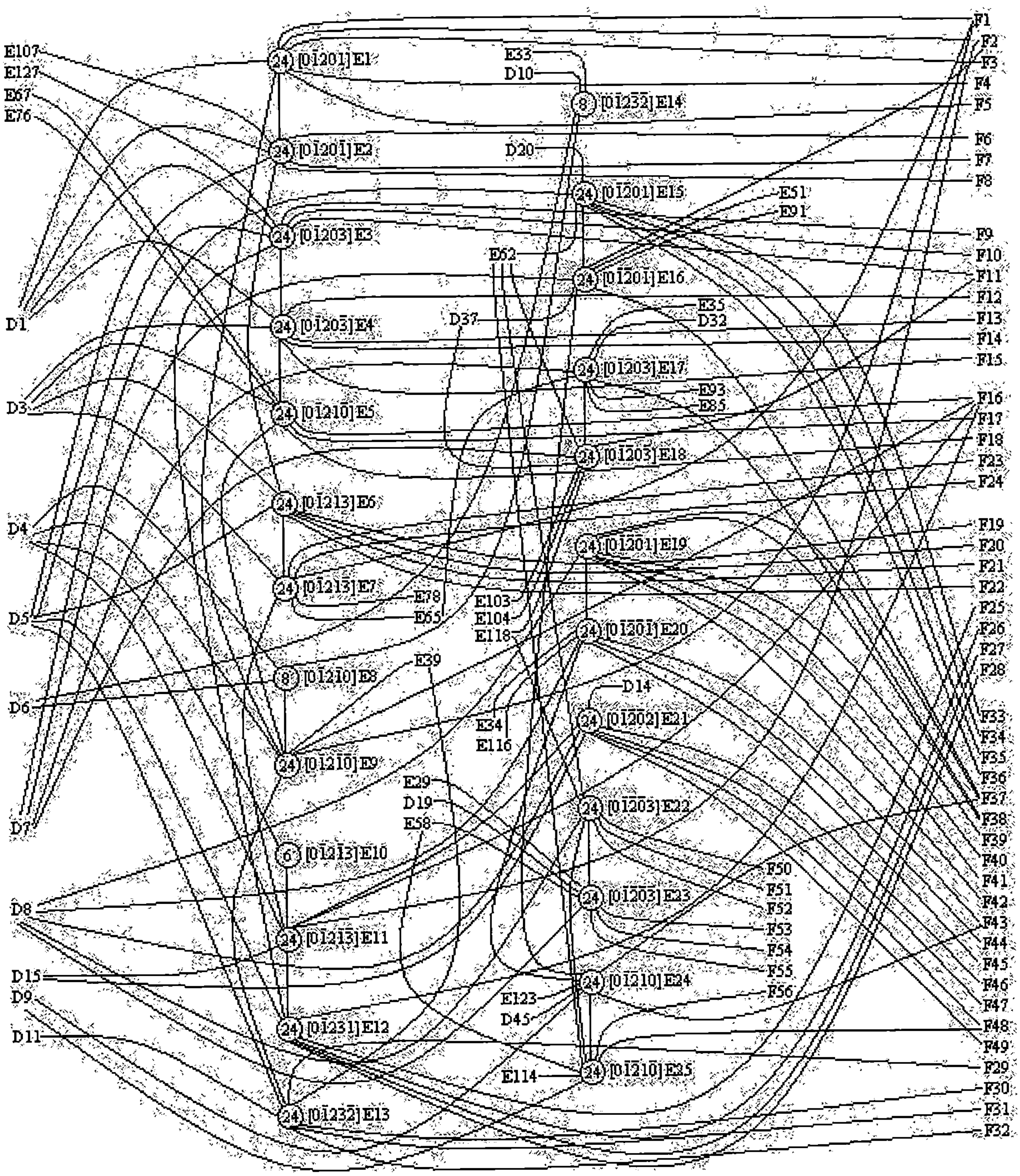

Figure 7.6: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 5

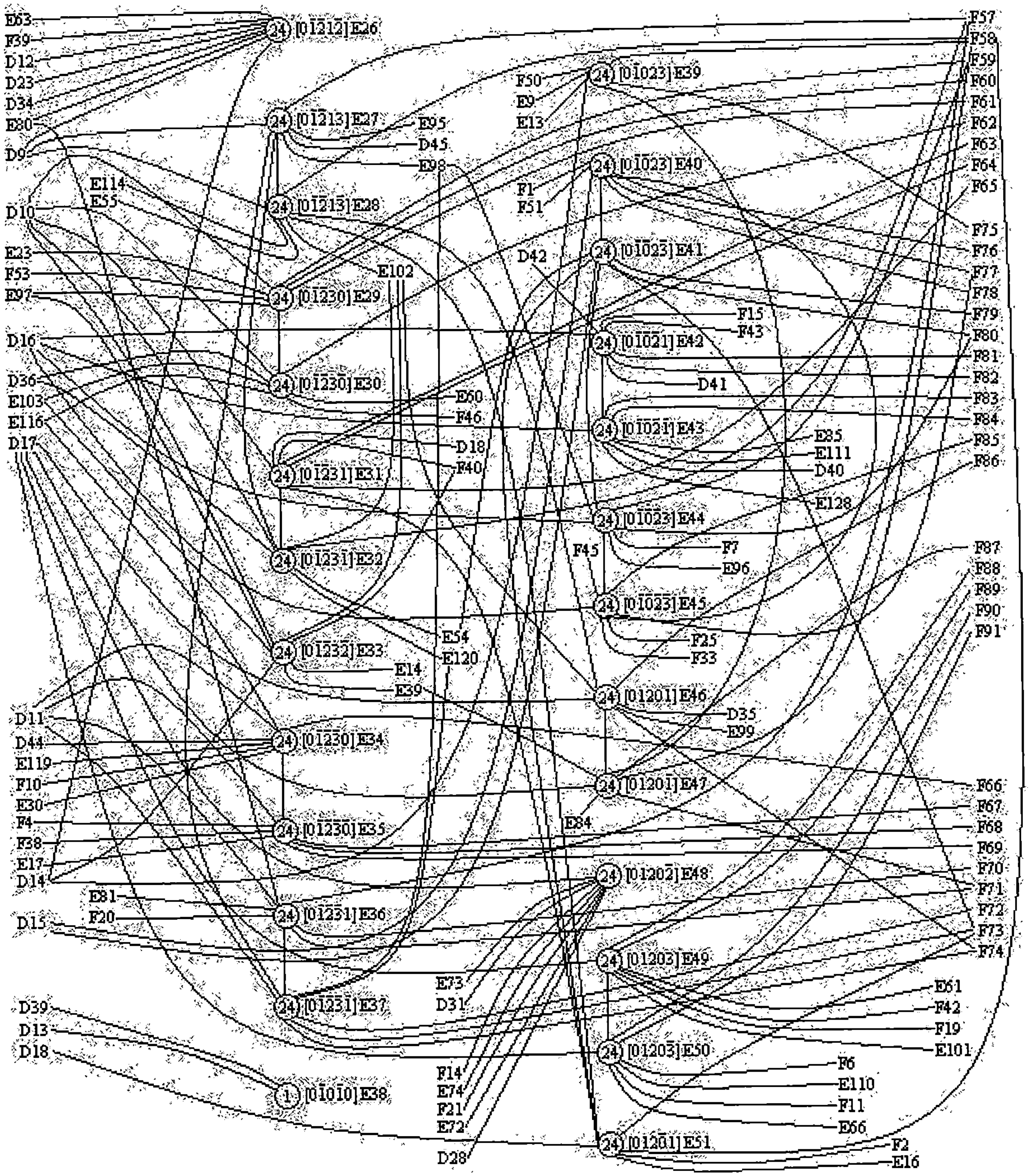

Figure 7.7: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 5

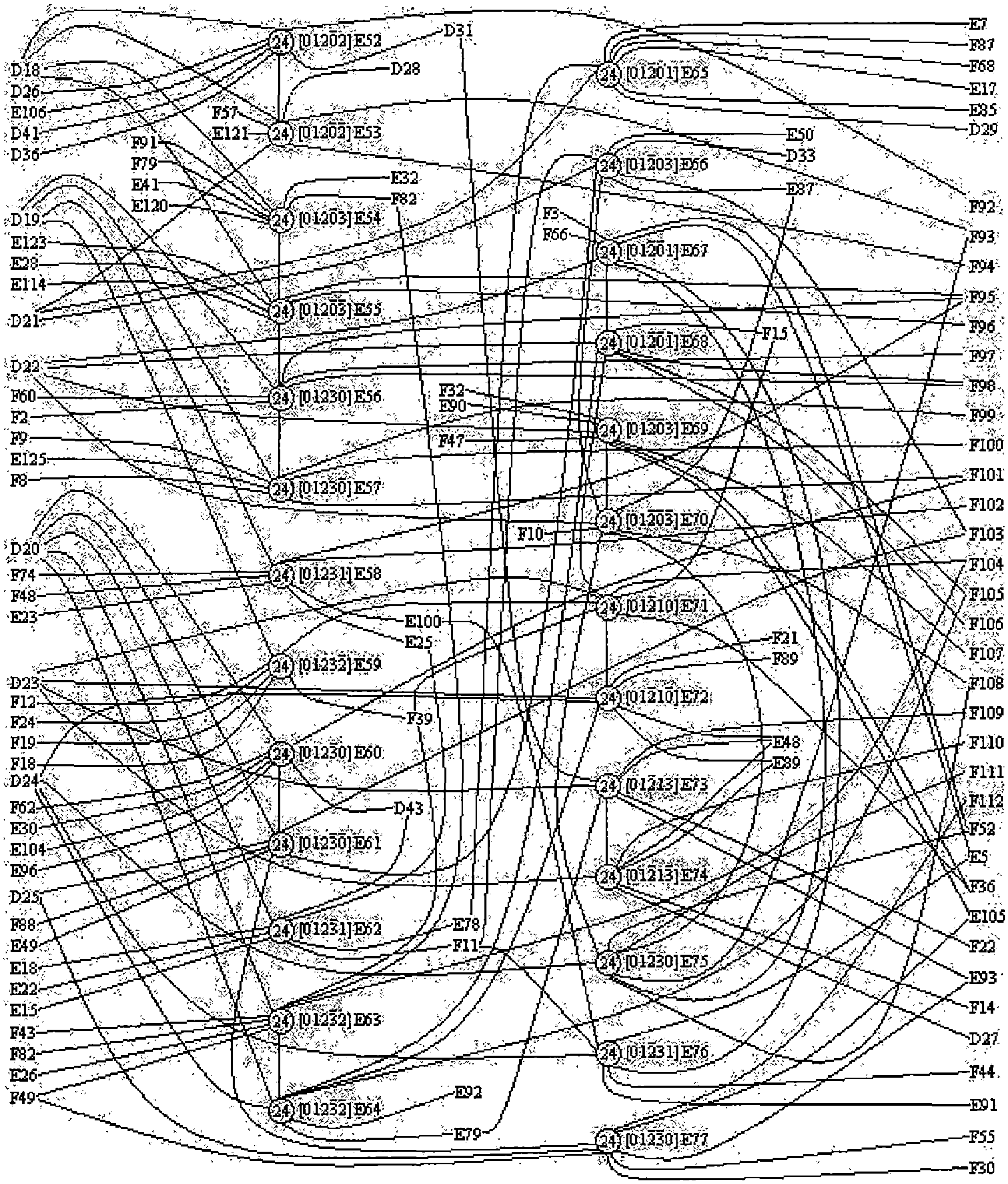

Figure 7.8: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 5

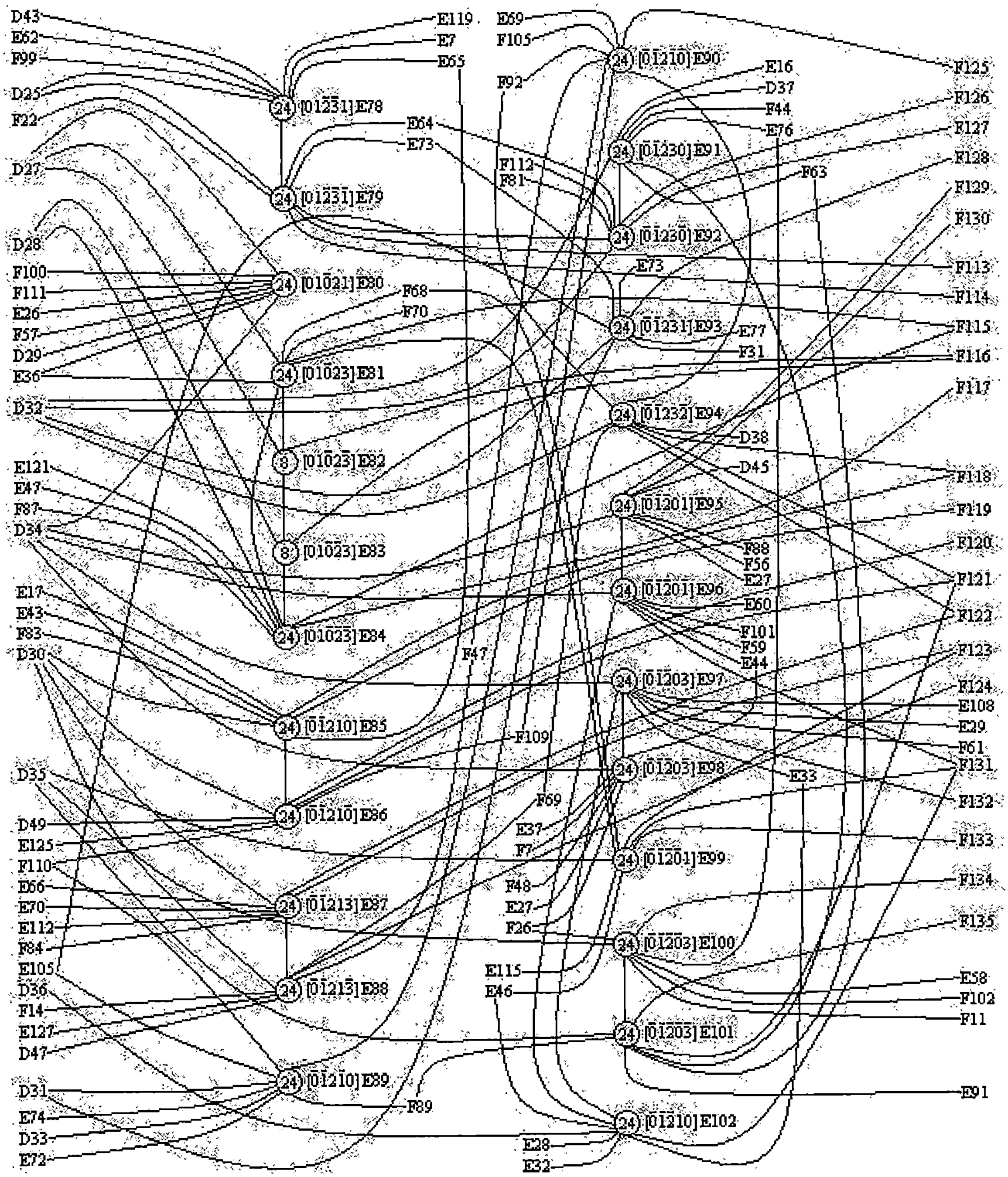

Figure 7.9: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 5

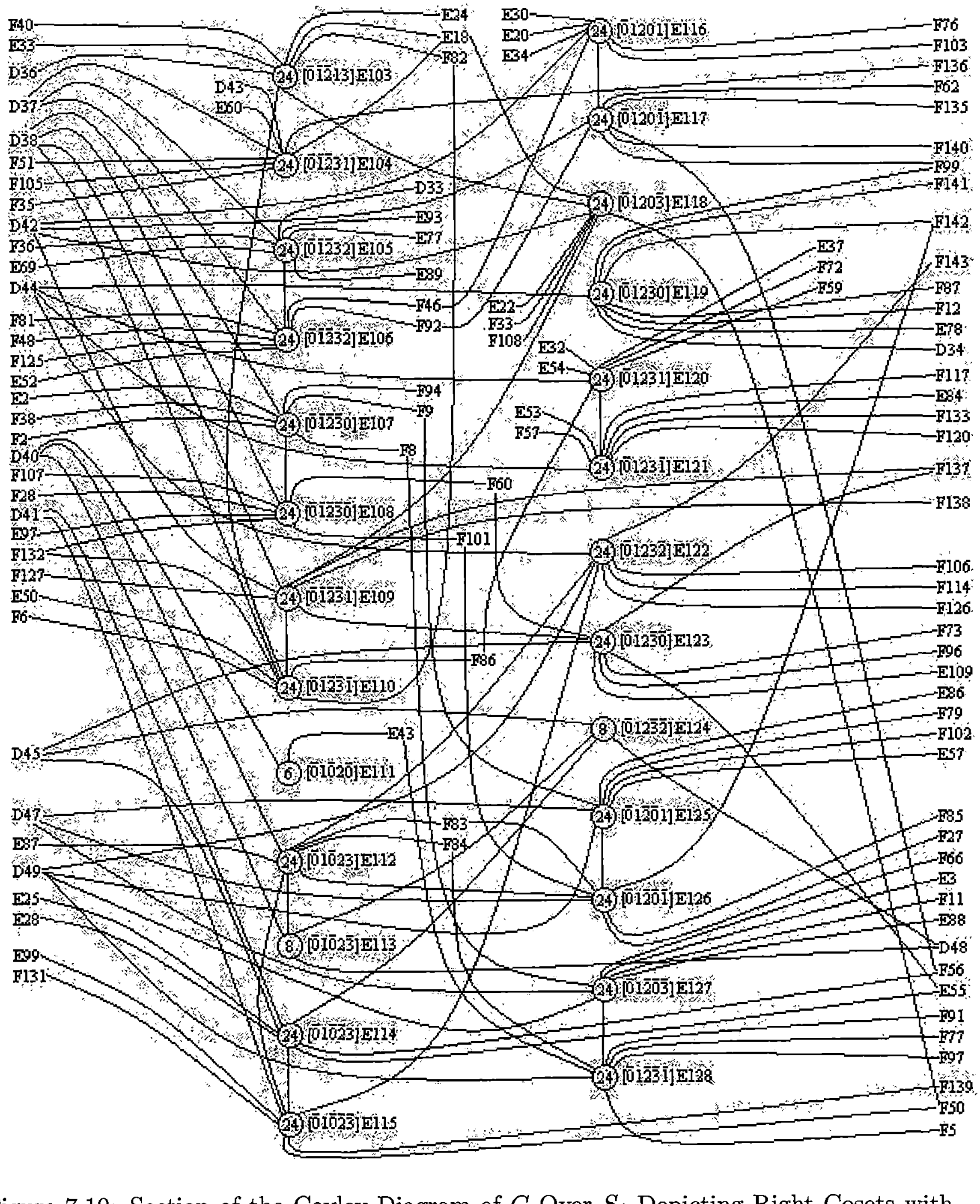

Figure 7.10: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 5

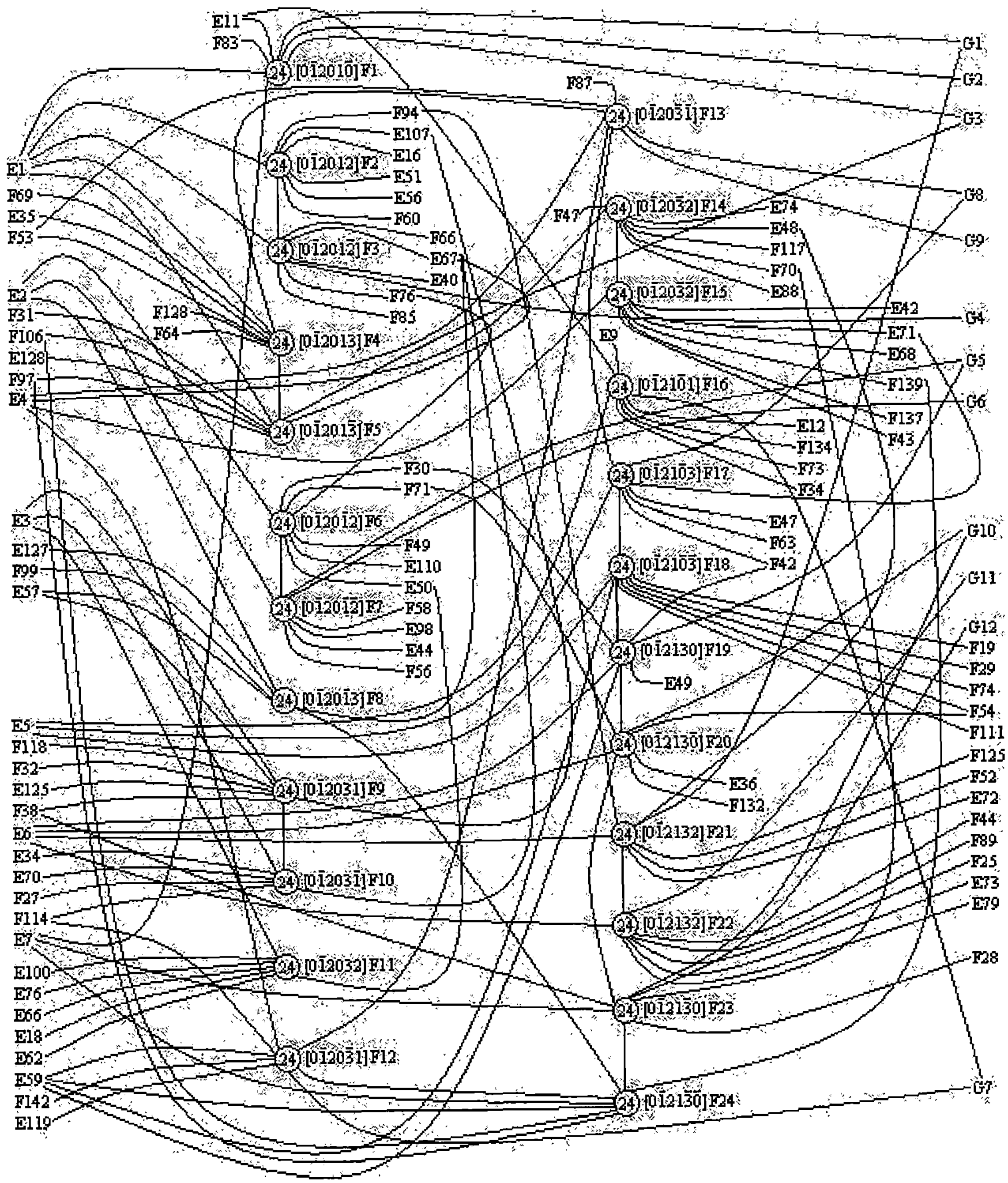

Figure 7.11: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 6

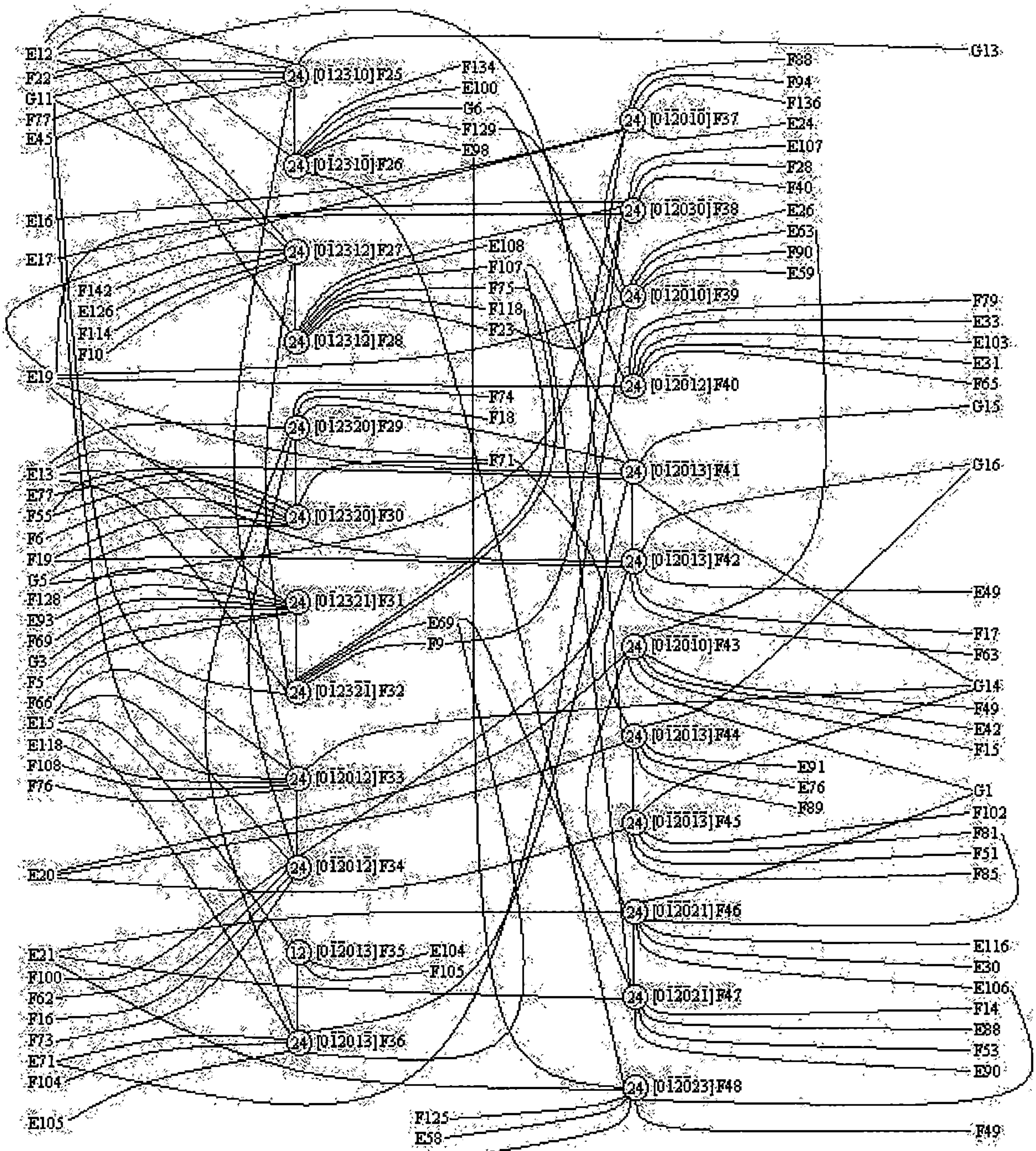

Figure 7.12: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 6

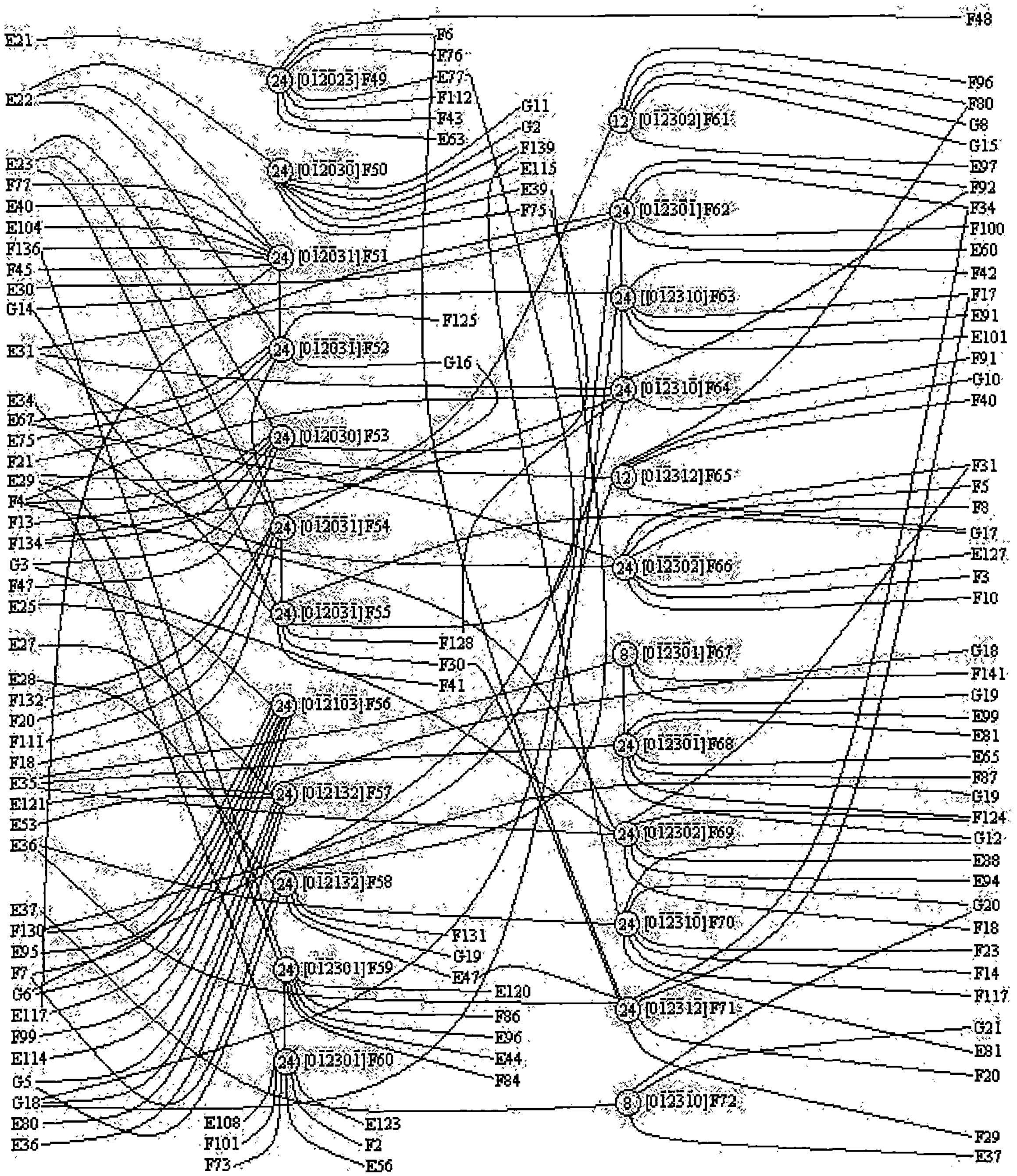

Figure 7.13: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 6

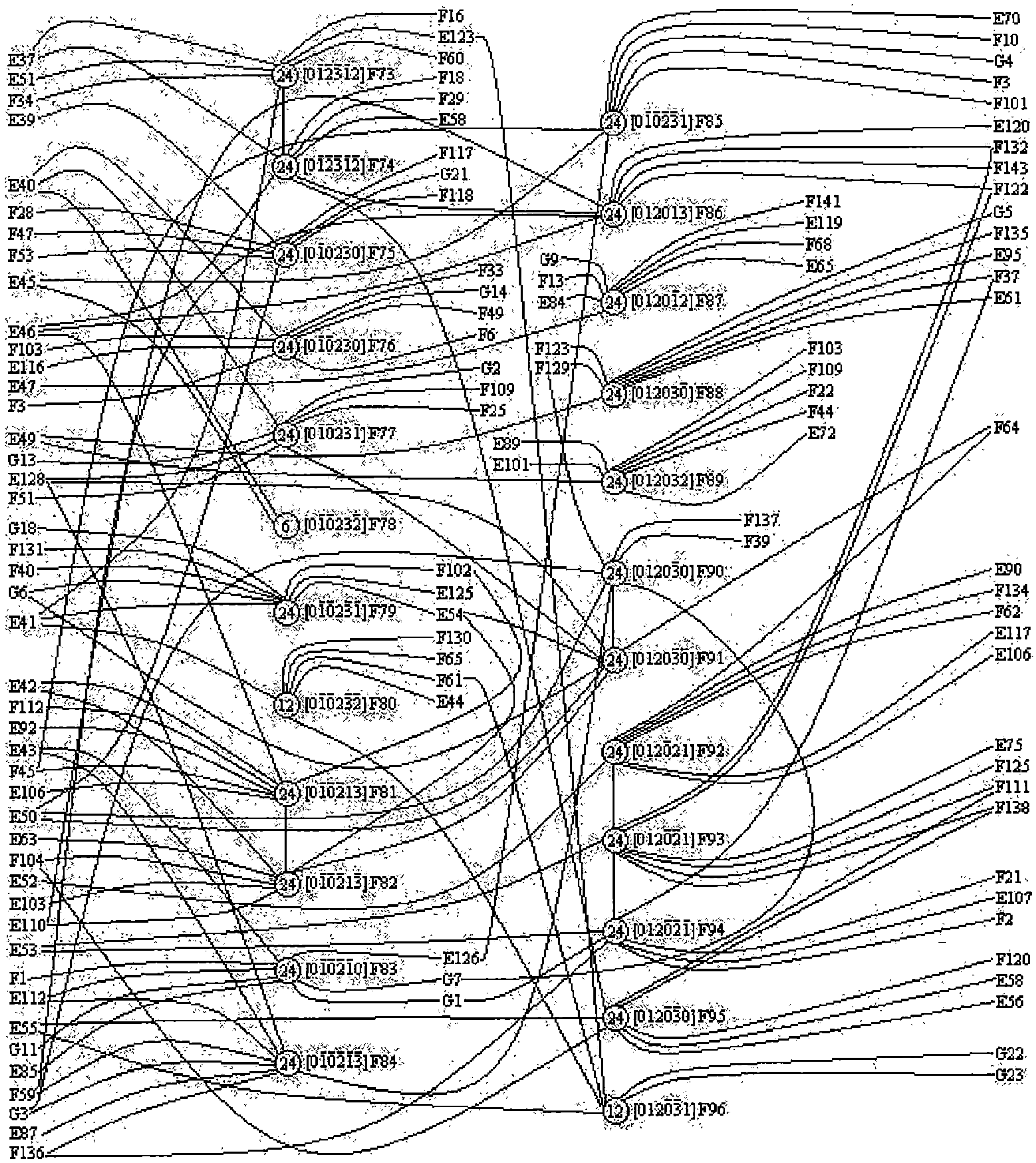

Figure 7.14: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 6

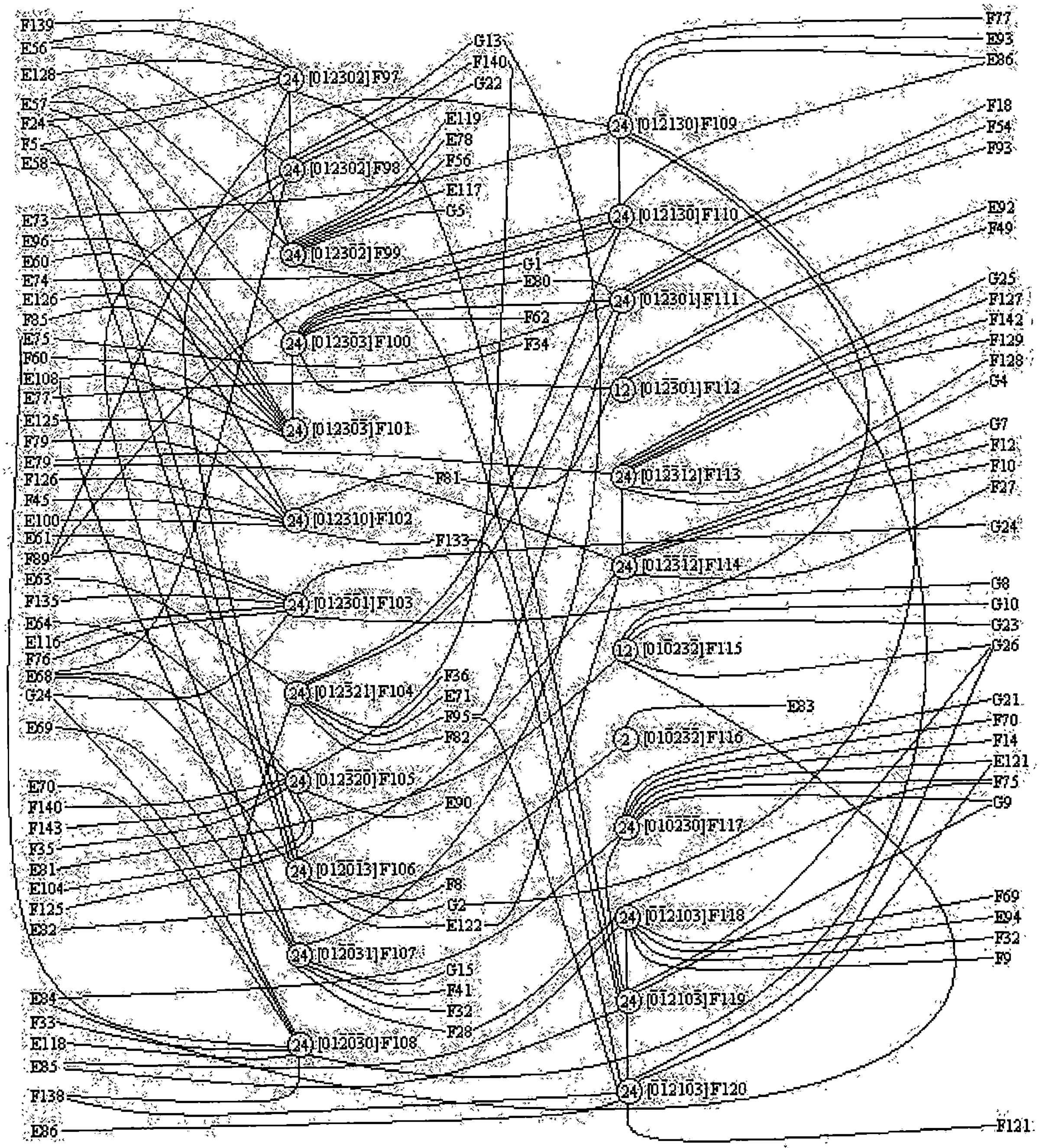

Figure 7.15: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 6

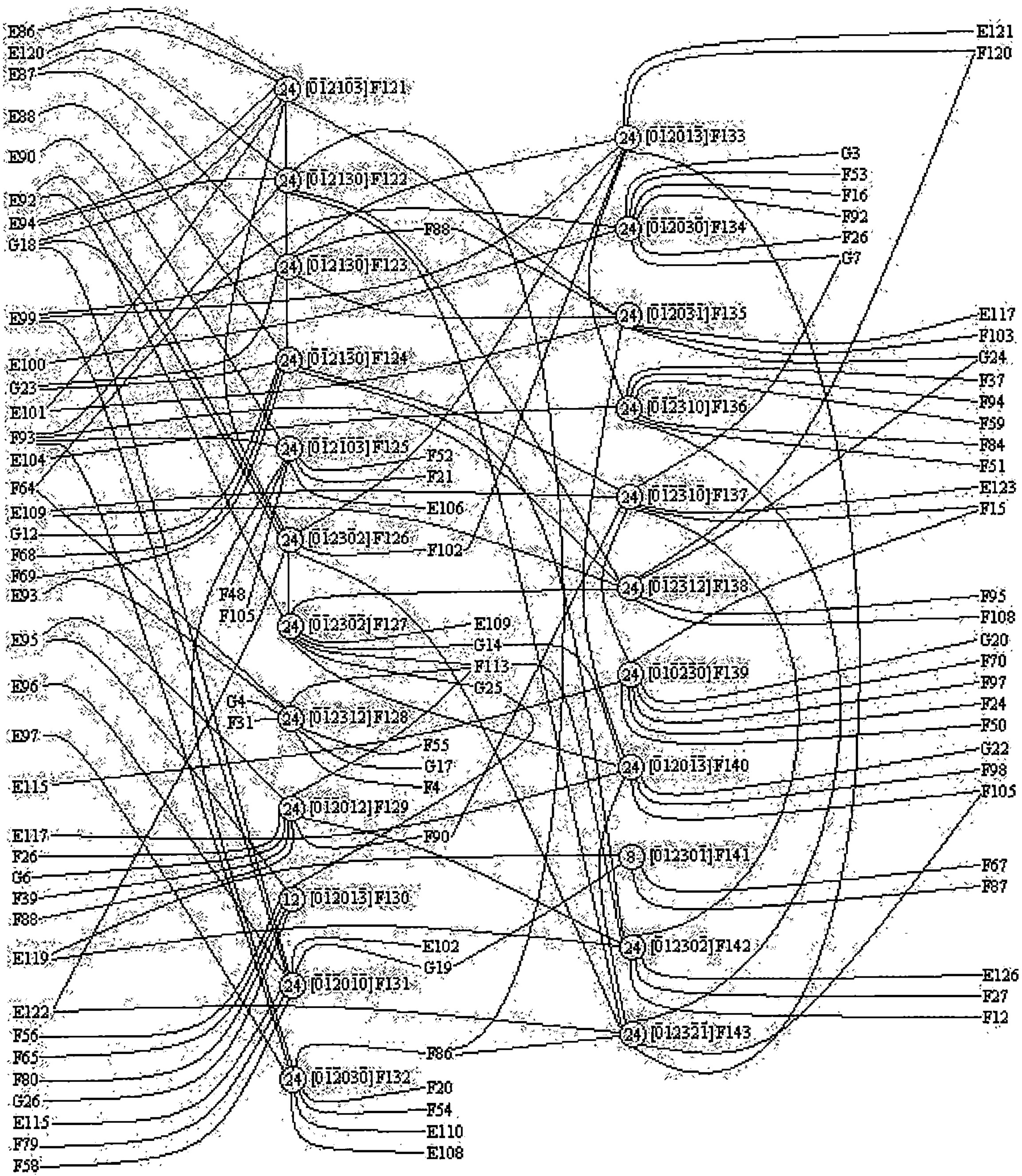

Figure 7.16: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 6

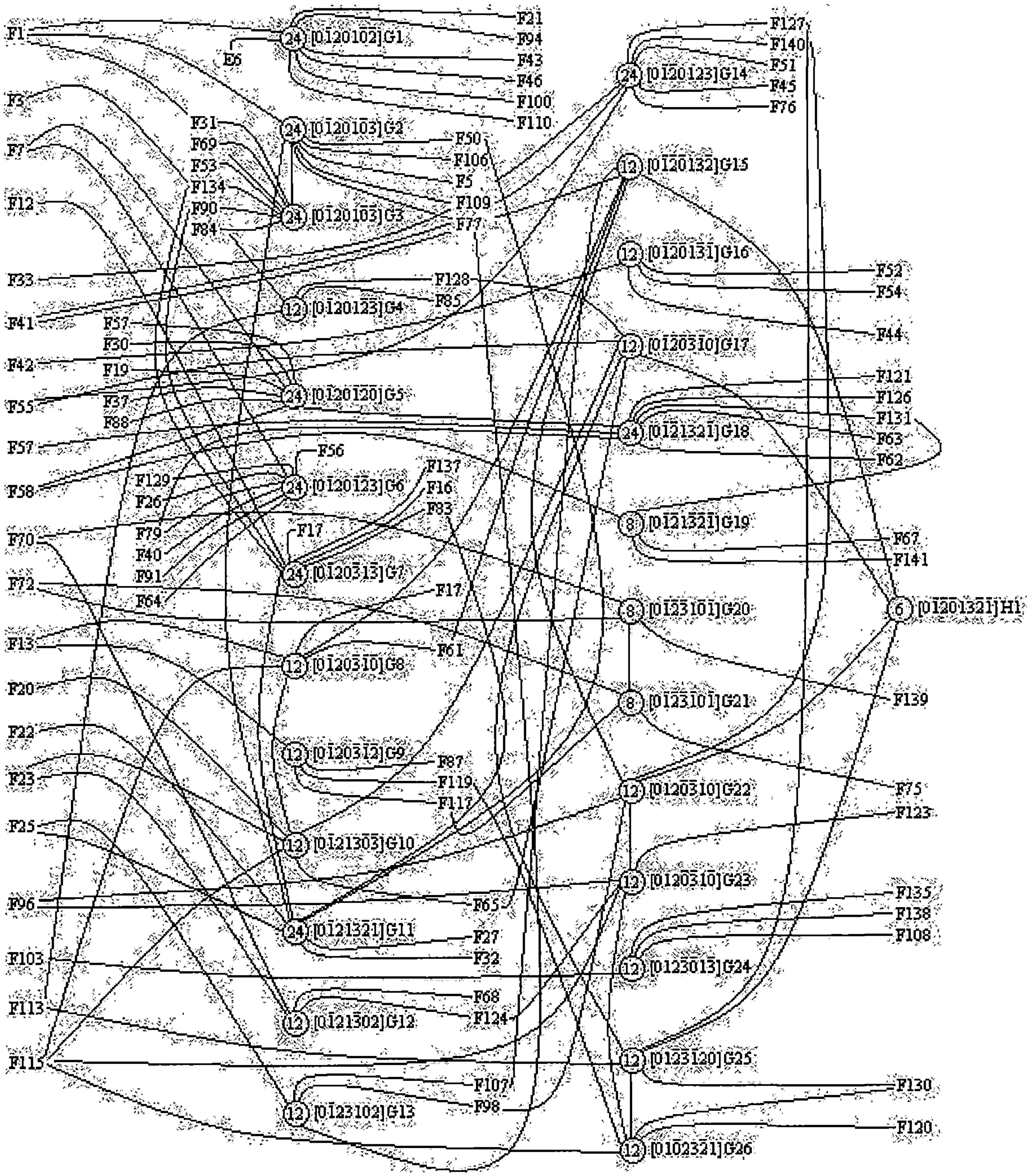

Figure 7.17: Section of the Cayley Diagram of G Over S_4 Depicting Right Cosets with Words of Length 7 and 8

7.4 Action of the Symmetric Generators and the Generators of S_4 on the Right Cosets of G Over S_4

Let X denote the set of all (7920) distinct right cosets of N in G. We define a mapping $\phi : G \longrightarrow S_X$ so that ϕ maps a generator $g \in G$ to its action (by right multiplication) on X. That is, we define ϕ so that $\phi(g) = \widehat{\phi}(g) : X \to X$. By way of the process described in Subsection 1.4.3, we may find the action $\phi(t) \sim \phi(t_0)$ of the symmetric generator $t \sim t_0$ on the set of right cosets of N in G, the action $\phi(x) \sim \phi((0\ 1\ 2\ 3))$ of the generator $x \sim (0\ 1\ 2\ 3)$ on the set of right cosets of N in G, and the action $\phi(y) \sim \phi((2\ 3))$ of the generator $y \sim (2\ 3)$ on the set of right cosets of N in G. Since there are 7920 right cosets of N in G, these actions may be written as permutations on 7920 letters. With the help of MAGMA (see [BCP97]), we have labeled each of the 7920 right cosets with a number between 1 and 7920.

Having labeled each of the 7920 right cosets, we may write the action $\phi(t) \sim \phi(t_0)$ of the symmetric generator $t \sim t_0$ on the right cosets of N in G as a permutation on 7920 letters:

$\phi(t) \sim \phi(t_0) = (1\ 2\ 3)(4\ 9\ 10)(5\ 11\ 12)(6\ 14\ 15)(7\ 16\ 17)(8\ 18\ 19)(13\ 28\ 29)(20\ 43\ 44)$ $(21\ 33\ 46) \cdots (7907\ 7918\ 7909)(7908\ 7910\ 7919)(7913\ 7920\ 7914)$.

Similarly, having labeled each of the 7920 right cosets, we may also write the action $\phi(x) \sim \phi((0\ 1\ 2\ 3))$ of the generator $x \sim (0\ 1\ 2\ 3)$ on the right cosets of N in G as a permutation on 7920 letters:

$\phi(x) \sim \phi((0\ 1\ 2\ 3)) = (2\ 4\ 8\ 5)(3\ 6\ 13\ 7)(9\ 20\ 42\ 21)(10\ 22\ 47\ 23)(11\ 24\ 53\ 25)(12\ 26\ 57\ 27)$ $(14\ 30\ 66\ 31) \cdots (7851\ 7864\ 7909\ 7880)(7915\ 7916)(7917\ 7918\ 7920\ 7919)$.

Finally, having labeled each of the 7920 right cosets, we may write the action $\phi(y) \sim \phi((2\ 3))$ of the generator $y \sim (2\ 3)$ on the right cosets of N in G as a permutation on 7920 letters:

$\phi(y) \sim \phi((2\ 3)) = (5\ 8)(7\ 13)(11\ 18)(12\ 19)(16\ 28)(17\ 29)(20\ 38)(21\ 45)(22\ 40)(23\ 50)$

(30 62) $\cdots$ (7904 7913)(7905 7911)(7912 7914)(7915 7917)(7916 7920).

7.5 Proof of Isomorphism between G and Aut(M_{12})

We now demonstrate that $G \cong \text{Aut}(M_{12})$.

Proof. To prove that $G \cong \text{Aut}(M_{12})$, we must first show that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and that $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 190080$ (from which we can conclude $G \cong \langle \phi(x), \phi(y), \phi(t) \rangle$), and we must next show that $\langle \phi(x), \phi(y), \phi(t) \rangle \cong \text{Aut}(M_{12})$ (from which we can conclude $\text{Aut}(M_{12})$ is a homomorphic image of G and $G \cong \text{Aut}(M_{12})$).

We first show $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of G and $|G| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 190080$. From our construction of G using manual double coset enumeration of $\bar{G}$ over S_4, illustrated by the Cayley Diagram in Figures 7.1 through 7.17, we concluded that group G defined by the symmetric presentation must contain a homomorphic image of $N \cong S_4$ whose index $[G : N]$ is at most 7920:

$$[G:N] = \frac{|N|}{\left|N^{(*)}_{[*]}\right|} + \frac{|N|}{\left|N^{(0)}\right|} + \frac{|N|}{\left|N^{(\bar{0})}\right|} + \frac{|N|}{\left|N^{(01)}\right|} + \frac{|N|}{\left|N^{(0\bar{1})}\right|} + \frac{|N|}{\left|N^{(\bar{0}\bar{1})}\right|} + \frac{|N|}{\left|N^{(\bar{0}1)}\right|}$$

$$+ \frac{|N|}{\left|N^{(0\bar{1}0)}\right|} + \frac{|N|}{\left|N^{(0\bar{1}2)}\right|} + \frac{|N|}{\left|N^{(0\bar{1}\bar{2})}\right|} + \cdots + \frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{0}13\bar{2}\bar{1})}\right|} \leq$$

$$\frac{24}{24} + \frac{24}{6} + \frac{24}{6} + \frac{24}{2} + \frac{24}{2} + \frac{24}{2} + \frac{24}{2} + \frac{24}{2} + \frac{24}{1} + \frac{24}{1} + \cdots + \frac{24}{4} =$$

$$1 + 4 + 4 + 12 + 12 + 12 + 12 + 12 + 24 + 24 + \cdots + 6 = 7920.$$

That is, $[G : N] = \frac{|G|}{|N|} \leq 7920$. Since the index of N in G is at most 7920, and since $|G| = \frac{|G|}{|N|} \cdot |N|$, the order of the homomorphic image group G is at most 190080:

$$|G| = \frac{|G|}{|N|} \cdot |N| \leq 7920 \cdot |N| = 7920 \cdot 24 = 190080 \Rightarrow |G| \leq 190080$$

We now consider $\langle \phi(x), \phi(y), \phi(t) \rangle$. Note that $\langle \phi(x), \phi(y), \phi(t) \rangle$ is a group generated by the permutation representations of the generators x, y, and t and, as such, it is a subgroup of the symmetric group S_{7920} acting on the seven thousand, nine hundred twenty right cosets of N in G. Let $G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle$. Now, G_1 is a homomorphic image of G and, therefore, $G_1 \leq G$ and $|G_1| \leq |G|$. Moreover, it is easily verified that $|G_1| = 190080$. Therefore, $|G| \geq |G_1| = 190080$. Therefore, $190080 \leq |G| \leq 190080$. That is, $|G| = 190080$, and so, since G_1 is a homomorphic image of G and $|G| = |G_1|$, we conclude $G \cong G_1 = \langle \phi(x), \phi(y), \phi(t) \rangle$. Moreover, with the help of MAGMA (see [BCP97]), we know that the elements $c = \phi(y^{x^2} t^{x^3} t t^x t^x t^{x^3} t^{x^3})$ and $d = \phi((xy^{x^2}) t t^x t^x t^{x^3} t^{x^3} t^{x^2} t t^{x^3})$ in G_1 satisfy the following known presentation of $\mathrm{Aut}(M_{12})$, or $M_{12}\colon 2$:

$$\langle c, d \mid c^2 = d^3 = (cd)^{12} = (cd)^5 [c, d] (cd^{-1})^3 cd [c, d^{-1}]^2 cdcd (cd^{-1})^3 [c, d^{-1}] = e \rangle.$$

Therefore, $M_{12}\colon 2 \leq G_1$. But $|M_{12}\colon 2| = |G_1| = 190080$. Hence, $G_1 \cong M_{12}\colon 2$, that is, $G_1 \cong \mathrm{Aut}(M_{12})$. Finally, since $G \cong G_1$, we conclude $G \cong \mathrm{Aut}(M_{12})$.

□

7.6 Converting an Element of G from its Permutation Representation to its Symmetric Representation

To illustrate how a permutation representation of an element of $\mathrm{Aut}(M_{12})$ on 7920 letters may be converted to its symmetric representation form, we consider the following example:

Example 7.1. Let $g \in G \cong \mathrm{Aut}(M_{12})$ and let $p = \phi(g) = \phi((0\ 1)(2\ 3)) \phi(t_0) \phi(t_1^{-1}) \phi(t_2) \phi(t_1^{-1}) \phi(t_3^{-1})$ be the permutation representation of g on 7920 letters. Then $N^p = N t_0 t_1^{-1} t_2 t_1^{-1} t_3^{-1}$. Moreover, since $N^p = Np$ and $N^p = N t_0 t_1^{-1} t_2 t_1^{-1} t_3^{-1}$, we have that $Np = N t_0 t_1^{-1} t_2 t_1^{-1} t_3^{-1}$. Now, $Np = N t_0 t_1^{-1} t_2 t_1^{-1} t_3^{-1}$ implies that $p \in N t_0 t_1^{-1} t_2 t_1^{-1} t_3^{-1}$ which implies that $p \sim \pi t_0 t_1^{-1} t_2 t_1^{-1} t_3^{-1}$ for some $\pi \in N \cong S_4$ or, more precisely, $p = \phi(\pi t_0 t_1^{-1} t_2 t_1^{-1} t_3^{-1}) = \phi(\pi) \phi(t_0) \phi(t_1^{-1}) \phi(t_2) \phi(t_1^{-1}) \phi(t_3^{-1})$ for some $\pi \in N \cong S_4$.

To determine $\pi \in N \cong S_4$, we note first that

$$p = \phi(\pi) \phi(t_0) \phi(t_1^{-1}) \phi(t_2) \phi(t_1^{-1}) \phi(t_3^{-1}) \Rightarrow$$

$$p (\phi(t_3^{-1}))^{-1} (\phi(t_1^{-1}))^{-1} (\phi(t_2))^{-1} (\phi(t_1^{-1}))^{-1} (\phi(t_0))^{-1} = \phi(\pi) \Rightarrow$$

$$p \phi((t_3^{-1})^{-1}) \phi((t_1^{-1})^{-1}) \phi(t_2^{-1}) \phi((t_1^{-1})^{-1}) \phi(t_0^{-1}) = \phi(\pi) \Rightarrow$$

$$p\phi(t_3)\phi(t_1)\phi(t_2^{-1})\phi(t_1)\phi(t_0^{-1}) = \phi(\pi).$$

We then calculate the action of $\pi \sim \phi(\pi) = p\phi(t_3)\phi(t_1)\phi(t_2^{-1})\phi(t_1)\phi(t_0^{-1})$ on the symmetric generators $\{t_i \mid i \in \{0,1,2,3\}\}$. The element $\pi \sim \phi(\pi) = p\phi(t_3)\phi(t_1)\phi(t_2^{-1})\phi(t_1)\phi(t_0^{-1})$ acts on the right cosets Nt_0, Nt_1, Nt_2, and Nt_3 via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$. By this mapping, the element $\phi(\pi)$ acts as $(0\ 1)(2\ 3)$ on the right cosets Nt_0, Nt_1, Nt_2, and Nt_3, and so $\phi(\pi)$ is the permutation representation of $\pi = (0\ 1)(2\ 3) \in S_4$ on 7920 letters. Therefore, $\pi = (0\ 1)(2\ 3)$ and $w = t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$, and so the symmetric representation of g is $(0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$.

With the computer algebra system MAGMA (see [BCP97]), we can use a similar algorithm to convert $g \in G \cong \mathrm{Aut}(M_{12})$ from its permutation representation $p = \phi(g) = \phi((0\ 1)(2\ 3))\phi(t_0)\phi(t_1^{-1})\phi(t_2)\phi(t_1^{-1})\phi(t_3^{-1})$ to its symmetric representation g. The MAGMA code for this algorithm is provided below:

```
G<x,y,t>:=Group<x,y,t| x^4, y^2, (y*x)^3, t^3, (t,y), (t^x,y), (y*x*t)^(10),
((x^2*y)^2*t)^5>;
f, G1, k := CosetAction(G,sub<G|x,y>);
S4:=SymmetricGroup(4); xx:=S4!(1,2,3,4); yy:=S4!(2,3); N:=sub<S4|xx,yy>;
IN:=sub<G1|f(x),f(y)>;
prodim := function(pt, Q, I)
/* Return the image of pt under permutations Q[I] applied sequentially. */
v := pt;
for i in I do v := v^(Q[i]); end for; return v; end function;
ts := [ (t^(x^i)) @ f : i in [1 .. 4] ];
cst := [null : i in [1 .. 7920]] where null is [Integers() | ];
ConvertPermutationToSymmetric:= function(G1,N,p)
ww:= cst[1^p];
tt:= p*&*[G1 | ts[ww[#ww - 1 + 1]]: i in [1 .. #ww]];
zz:= N![rep j : j in [1 .. 4] | (1^ts[i])^tt eq 1^ts[j] : i in [1 .. 4]];
return <zz, ww>; end function;
p:= f((x^2 * y)^2)*f(t)*f((t^x)^(-1))*f(t^(x^2)) *f((t^x)^(-1))*f((t^(x^3))^(-1));
ConvertPermutationToSymmetric(G1,N,p);
```

Note that the elements $x \sim (0\ 1\ 2\ 3)$ and $y \sim (2\ 3)$ in this algorithm act on the right cosets N in G via the mapping $f : G \longrightarrow G1$ defined by $f(x, Nw) = Nw^x$, and the symmetric generators $t_0 \sim t, t_1^{-1} \sim (t^x)^{-1}, t_2 \sim t^{x^2}$, and $t_3^{-1} \sim (t^{x^3})^{-1}$ act on the right cosets of N in G via the mapping $f : G \longrightarrow G1$ defined by $f(t_i, Nw) = Nwt_i$. For this reason, in this case, the permutation representation of g on 7920 letters is given by $p = f(g) = f((x^2y)^2)f(t)f((t^x)^{-1})f(t^{x^2})f((t^x)^{-1})f((t^{x^3})^{-1})$. With the help of MAGMA (see [BCP97]), we find $\pi = (0\ 1)(2\ 3) \sim (x^2y)^2$ and $w = t_0t_1^{-1}t_2t_1^{-1}t_3^{-1} \sim t(t^x)^{-1}t^{x^2}(t^x)^{-1}(t^{x^3})^{-1}$, and so we determine, as before, that the symmetric representation of g is $g = (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_3^{-1} \sim (x^2y)^2t(t^x)^{-1}t^{x^2}(t^x)^{-1}(t^{x^3})^{-1}$.

7.7 Converting an Element of G from its Symmetric Representation to its Permutation Representation

To illustrate how an element of $\mathrm{Aut}(M_{12})$ in symmetric representation form may be converted to its permutation representation on 7920 letters, we consider the following example:

Example 7.2. Let $g \in G \cong \mathrm{Aut}(M_{12})$ have the symmetric representation $g = (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$. To determine the permutation representation $p = \phi(g)$ of g, we first calculate the action of $\pi = (0\ 1)(2\ 3)$ on the right cosets of N in G. The element $\pi = (0\ 1)(2\ 3)$ acts on the right cosets N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(\pi, Nw) = Nw^{\pi}$ In this sense, $\phi(\pi)$ is the permutation representation of π on 7920 letters.

We next calculate: the action of the symmetric generator t_0 on the right cosets of N in G, the action of the symmetric generator t_1^{-1} on the right cosets of N in G, the action of the symmetric generator t_2 on the right cosets of N in G, and the action of the symmetric generator t_3^{-1} on the right cosets of N in G. The symmetric generators $\{t_i^{\pm 1} \mid i \in \{0,1,2,3\}\}$ act on the right cosets of N in G via the mapping $\phi : G \longrightarrow S_X$ defined by $\phi(t_i, Nw) = Nwt_i$. In this sense, $\phi(t_0), \phi(t_1^{-1}), \phi(t_2)$, and $\phi(t_3^{-1})$ are the permutation representations of t_0, t_1^{-1}, t_2, and t_3^{-1} on 7920 letters, respectively. The permutation representation of $g = (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$ is therefore $p = \phi(g) = \phi((0\ 1)(2\ 3))\phi(t_0)\phi(t_1^{-1})\phi(t_2)\phi(t_1^{-1})\phi(t_3^{-1})$.

With the computer algebra system MAGMA (see [BCP97]), we can use a similar algorithm to convert $g \in G \cong \mathrm{Aut}(M_{12})$ from its symmetric representation $g = (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_3^{-1}$ to its permutation representation $p = \phi(g)$. The MAGMA code for this algorithm is provided below:

```
S4:=SymmetricGroup(4); xx:=S4!(1,2,3,4); yy:=S4!(2,3); N:=sub<S4| xx,yy>;
NN<x,y>:=Group<x,y| x^4, y^2, (y*x)^3>;
Sch:=SchreierSystem(NN,sub<NN—Id(NN)>);
ArrayP:=[Id(N): i in [1..24]];
for i in [2..24] do P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for j in [1..#Sch[i]] do if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^(-1); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if; end for;
PP:=Id(N); for k in [1..#P] do PP:=PP*P[k]; end for; ArrayP[i]:=PP; end for; end for;
for i in [1..24] do if ArrayP[i] eq N!(4,1)(2,3) then print Sch[i]; end if; end for;
> (x^2 * y)^2
G<x,y,t>:=Group<x,y,t| x^4, y^2, (y*x)^3, t^3, (t,y), (t^x,y), (y*x*t)^(10),
((x^2*y)^2*t)^5>;
f, G1, k := CosetAction(G,sub<G| x,y>);
IN:=sub<G1| f(x),f(y)>;
f((x^2 * y)^2)*f(t)*f((t^x)^(-1))*f(t^(x^2))*f((t^x)^(-1)) *f((t^(x^3))^(-1));
```

Note that the element $\pi = (0\ 1)(2\ 3) \sim (x^2y)^2$ in this algorithm acts on the right cosets N in G via the mapping $f : G \longrightarrow G1$ defined by $f(\pi, Nw) = Nw^{\pi}$, and the symmetric generators $t_0 \sim t, t_1^{-1} \sim (t^x)^{-1}, t_2 \sim t^{x^2}$, and $t_3^{-1} \sim (t^{x^3})^{-1}$ act on the right cosets of N in G via the mapping $f : G \longrightarrow G1$ defined by $f(t_i, Nw) = Nwt_i$. In this sense, $f((x^2y)^2)$ is the permutation representation of $\pi \sim (x^2y)^2$ on 7920 letters, and $f(t), f((t^x)^{-1}), f(t^{x^2})$, and $f((t^{x^3})^{-1})$ are the permutation representations of $t_0 \sim t, t_1^{-1} \sim (t^x)^{-1}, t_2 \sim t^{x^2}$, and $t_3^{-1} \sim (t^{x^3})^{-1}$, respectively. The permutation representation of $g = (0\ 1)(2\ 3)t_0t_1^{-1}t_2t_1^{-1}t_3^{-1} \sim (x^2y)^2t(t^x)^{-1}t^{x^2}(t^x)^{-1}(t^{x^3})^{-1}$ is therefore
$p = f(g) = f((x^2y)^2)f(t)f((t^x)^{-1})f(t^{x^2})f((t^x)^{-1})f((t^{x^3})^{-1})$.

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