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Blow-up behavior of solutions for some ordinary and partial differential equations

Sarah Y. Bahk

BLOW-UP BEHAVIOR OF SOLUTIONS FOR SOME ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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September 2008

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Abstract

There are two parts in this project.

In part I, we consider the Riccati initial-value problem:

$$y'(t) = ay^2 + by + c,$$
$$y(0) = d,$$

where $a, b, c,$ and d are real numbers and $t \geq 0$ represents time. We determine conditions on the constants a, b, c and d that are necessary and sufficient for $y(t)$ to approach either $+\infty$ or $-\infty$ as t approaches some finite value t_b.

In part II, we consider blow-up property of solutions for the degenerate semilinear parabolic initial-boundary value problem:

$$\xi^q u_\tau - u_{\xi\xi} = f(u(\xi_0, \tau)) \text{ for } 0 < \xi < a, 0 < \tau < \sigma,$$
$$u(\xi, 0) = u_0(\xi) \geq 0 \text{ for } 0 \leq \xi \leq a,$$
$$u(0, \tau) = 0 = u_\xi(a, \tau) \text{ for } \tau > 0.$$

Here $a, \sigma,$ and q are constants with $a > 0, 0 < \sigma \leq \infty$, and $q \geq 0$. Also, let ξ_0 be some fixed point in $(0, a)$. It is assumed that $f \in C^2([0,\infty)), f(0) \geq 0, f' > 0, f'' \geq 0$. We will show that for sufficiently large initial function $u_0(\xi)$ solution of the above initial-boundary value problem blows up in finite time and the blow-up set is the entire interval $[0, a]$.

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Chapter 1

Introduction

There are two parts in this project.

In part I, we consider the blow-up property of solutions for Riccati equation. Count Jacopolo Francesco Riccati (May 28, 1676 - April 15, 1754) is famous for introducing and researching solvability of the equation that is now known as Riccati equation:

$$y'(t) = a(t)y^2 + b(t)y + c(t). \tag{1.1}$$

The matrix form of this equation is very important in modern times since it is used extensively in design problems in filtering and control [Bit91]. Even though the Riccati equation (1.1) is not solvable in general, numerous methods are developed for finding solutions for special cases of this equation [PZ03].

We consider the Riccati initial-value problem:

$$\begin{cases} y'(t) = ay^2 + by + c, \\ y(0) = d, \end{cases} \tag{1.2}$$

where a, b, c, and d are real numbers and $t \geq 0$ represents time. We determine conditions on the constants a, b, c,and d that are necessary and sufficient for $y(t)$ to approach either $+\infty$ or $-\infty$ as t approaches some finite value t_b. We provide exact values for the time t_b for the cases when $4ac - b^2$ is positive, negative, or zero. We are interested in the first occurrence of blow-up. We do not consider behavior of $y(t)$ for $t > t_b$.

In part II, we consider blow-up property of solutions for the degenerate semilinear parabolic initial-boundary value problem:

$$\begin{cases} \xi^q u_\tau - u_{\xi\xi} = f(u(\xi_0,\tau)) & \text{for} \quad 0<\xi<a, \quad 0<\tau<\sigma, \\ u(\xi,0) = u_0(\xi) \geq 0 & \text{for} \quad 0 \leq \xi \leq a, \\ u(0,\tau) = 0 = u_\xi(a,\tau) & \text{for} \quad \tau > 0. \end{cases}$$

Here a, σ, and q are constants with $a > 0, 0 < \sigma \leq \infty$, and $q \geq 0$. u_τ means the first order partial derivative of $u(\xi,\tau)$ with respect to τ, and $u_{\xi\xi}$ means the second order partial derivative of $u(\xi,\tau)$ with respect to ξ. Also let ξ_0 be some fixed point in $(0,a)$. Because the reaction term $f(u(\xi_0,\tau))$ depends on the fixed value ξ_0 in $(0,a)$, we say that this problem has localized nonlinear reaction. Let $\xi{=}ax, \tau{=}a^{q+2}t, D{=}(0,1), \Omega{=}D \times (0,T), \bar{D}$ and $\bar{\Omega}$ be the closures of D and Ω respectively, $x_0 = \xi_0/a$, and $Lu = x^q u_t - u_{xx}$. We can see that $x_0 = \xi_0/a$ is a fixed point in $(0,1)$. Since $\xi = ax$, we have $u_x = du/dx = (du/d\xi)\cdot(d\xi/dx) = u_\xi \cdot a$, then $u_\xi = u_x/a$. Also, $u_{xx} = d^2u/dx^2 = (d/dx)\cdot(du/dx) = (d/dx)\,(au_\xi)$, then we have $a\cdot(d/dx)\,(u_\xi) = a\cdot(du_\xi/d\xi)\cdot(d\xi/dx) = a\cdot u_{\xi\xi}\cdot a = a^2\cdot u_{\xi\xi}$, then we have $u_{\xi\xi} = u_{xx}/a^2$. Now since $\tau = a^{q+2}t$, so $u_t = du/dt = (du/d\tau)\cdot(d\tau/dt) = u_\tau\cdot a^{q+2}$, then we have $u_\tau = u_t/a^{q+2}$. And since $x_0 = \xi_0/a$, then $\xi_0 = a\cdot x_0$. Now we have this:

$$\begin{aligned} \xi^q u_\tau - u_{\xi\xi} &= (ax)^q \frac{u_t}{a^{q+2}} - \frac{u_{xx}}{a^2} \\ &= \frac{x^q u_t - u_{xx}}{a^2}. \end{aligned}$$

Since $\xi = ax$, and $0 < \xi < a$, then $0 < ax < a$, so we have $0 < x < 1$. Also, since $\tau = a^{q+2}t$, and $0 < \tau < \sigma$, then $0 < a^{q+2}t < \sigma$, so $0 < t < \sigma/\left(a^{q+2}\right) = T$.
So $f\left(u(\xi_0,\tau)\right) = f(u(ax_0, a^{q+t}t)) = [x^q\cdot u_t(ax_0, a^{q+2}t) - u_{xx}(ax_0, a^{q+2}t)]/a^2 = f\left(u(x_0,t)\right)$ in $\Omega, (0 < x < 1, 0 < t < T)$. Then, $\left(x^q u_t - u_{xx}\right)/a^2 = f\left(u(x_0,t)\right)$, hence $x^q u_t - u_{xx} = a^2 f(u(x_0,t))$.

The above problem is transformed into

$$\begin{cases} Lu = a^2 f(u(x_0,t)) & \text{in} \quad \Omega, \\ u(x,0) = u_0(x) \geq 0 & \text{on} \quad \bar{D}, \\ u(0,t) = 0 = u_x(1,t) & \text{for} \quad 0<t<T, \end{cases} \tag{1.3}$$

where $T{=}\sigma/a^{q+2} \leq \infty$. We assume that $f \in C^2([0,\infty)), f(0) \geq 0, f' > 0, f'' \geq 0$, and there exists a positive number r such that

$$f(u) \geq u^{1+r} \quad \text{for} \quad u \geq 1. \tag{1.4}$$

The function $u_0(x) \in C^{2+\alpha}(\bar{D})$ is required to satisfy the compatibility conditions $u_0(0)=$ 0 and $u_0'(1) = 0$, where $0 < \alpha < 1$. The solution of the problem (1.3) is said to blow-up at $x = \bar{x}$ and $t = t_b$ if there exists a sequence $\{(x_n, t_n)\} \to (\bar{x}, t_b)$ and $lim_{n\to\infty} u(x_n, t_n) \to \infty$. The blow-up of u is complete at t_b if at t_b, u blows up at every point $x \in \bar{D}$.

The complete blow-up of the solution of the problem (1.3) with $u_x(1,t)$ replaced by $u(1,t)$ was investigated by Chan and Yang [CY00]. In Chapter 3, we show existence of a unique classical solution of the problem (1.3) for any $q \geq 0$. If $T < \infty$, then $u(x_0, t)$ is unbounded in $(0, T)$. A criterion for u to blow up in a finite time is also given, and a nonlinear integral equation in terms of Green's function is used to show that the localized nonlinear reaction leads to the complete blow-up of u for any $q \geq 0$.

Chapter 2

Riccati Problems with Constant Coefficients

We will start by stating the theorem which is very important for the Riccati problems.

Theorem 2.1. *The following is true for the solution $y(t)$ of (1.2):*
1. Let $4ac - b^2 > 0$. If $a > 0$, then $y(t) \to +\infty$, while if $a < 0$, then $y(t) \to -\infty$.
2. Let $4ac - b^2 = 0$. If $a > 0$, and $d > -\frac{b}{2a}$, then $y(t) \to +\infty$. If $a < 0$, and $d < -\frac{b}{2a}$, then $y(t) \to -\infty$. Otherwise, $y(t)$ is bounded for any finite $t > 0$. In particular, if $d = -\frac{b}{2a}$, then $y(t) \equiv d$.
3. Let $4ac - b^2 < 0$. If $a > 0$ and $d > \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to +\infty$. If $a < 0$ and $d < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to -\infty$. Otherwise, $y(t)$ is bounded for any finite $t > 0$. In particular, if $d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \equiv d$.
4. If $a = 0$, then $y(t)$ is bounded for all $t > 0$.

Proof. 1. Let $4ac - b^2 > 0$. A solution of the initial value problem (1.2) can be found using separation of variables and the table of integrals [LHE06]:

$$y(t) = \frac{\sqrt{4ac - b^2}}{2a} \tan\left[\frac{t\sqrt{4ac - b^2}}{2} + \arctan\left(\frac{b + 2ad}{\sqrt{4ac - b^2}}\right)\right] - \frac{b}{2a}. \tag{2.1}$$

We can find the blow-up time t_b by solving the following equation for t:

$$\frac{t\sqrt{4ac-b^2}}{2} + \arctan\left(\frac{b+2ad}{\sqrt{4ac-b^2}}\right) = \frac{\pi}{2},$$

$$t_b = \frac{\pi}{\sqrt{4ac-b^2}} - \frac{2}{\sqrt{4ac-b^2}}\arctan\left[\frac{b+2ad}{\sqrt{4ac-b^2}}\right].$$

Also,

$$-\frac{\pi}{2} < \arctan\left[\frac{b+2ad}{\sqrt{4ac-b^2}}\right] < \frac{\pi}{2}.$$

This implies that t_b is always positive and solution $y(t)$ of (1.2) is guaranteed to blow-up as t approaches t_b. Also, from equation (2.1) we notice that if $a > 0$, then $y(t) \to +\infty$, while if $a < 0$, then $y(t) \to -\infty$. Changing initial value d cannot prevent blow-up from occurring. However, d influences the blow-up time t_b. For example, if $a < 0$, then decreasing d will accelerate the blow-up. If $a > 0$, then increasing d will accelerate the blow-up.

2. Let $4ac - b^2 = 0$. Using separation of variables, we obtain:

$$\int \frac{dy}{a\left(y + \frac{b}{2a}\right)^2} = \int dt.$$

Integration leads to the following solution:

$$y(t) = \frac{2ad+b}{a(2-2adt-bt)} - \frac{b}{2a}. \tag{2.2}$$

To find the blow-up time we will set the denominator of the first term in (2.2) equal to 0:

$$2 - 2adt - bt = 0,$$

$$t_b = \frac{2}{2ad+b}.$$

From the inequality

$$t_b = \frac{2}{2ad+b} > 0,$$

and from (2.2) we obtain the following: if $a > 0$ and $d > -\frac{b}{2a}$, then $y(t) \to +\infty$, while if $a < 0$ and $d < -\frac{b}{2a}$, then $y(t) \to -\infty$, Initial value d is very important since certain values can prevent blow-up from occurring. Also, d influences the blow-up time t_b. If blow-up occurs for some value d, then decreasing d (if $a < 0$) or increasing d (if $a > 0$) will accelerate the blow-up. Also, if $d = -\frac{b}{2a}$, then $y(t) \equiv d$ satisfies initial-value problem (1.2). By the existence and uniqueness theorem for first order initial-value problem [BDH02], (1.2) has a unique solution. Therefore, in this special case $y(t)$ is bounded for all finite $t > 0$.

3. Let $4ac - b^2 < 0$. Let us notice that if

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

then $y \equiv d$ is the solution of the initial-value problem (1.2). Therefore, in this case solution is bounded for all $t > 0$. Now let us consider the case when

$$d \neq \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Using separation of variables and the table of integrals [LHE06], we have:

$$\int \frac{dy}{ay^2 + by + c} = \int dt,$$

$$\frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ay + b - \sqrt{b^2 - 4ac}}{2ay + b + \sqrt{b^2 - 4ac}} \right| = t + C_1, \tag{2.3}$$

where C_1 is a constant of integration. We can find C_1 substituting the initial condition $y(0) = d$ into the equation (2.3). We have:

$$C_1 = \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} \right|. \tag{2.4}$$

We will consider two possible cases:

$$\frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} > 0 \tag{2.5}$$

and

$$\frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} < 0. \tag{2.6}$$

Let us substitute (2.4) into (2.3) and solve for $y(t)$. In case (2.5), we can omit the absolute value symbol:

$$\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ay+b-\sqrt{b^2-4ac}}{2ay+b+\sqrt{b^2-4ac}}\right)=$$
$$t+\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ad+b-\sqrt{b^2-4ac}}{2ad+b+\sqrt{b^2-4ac}}\right).$$

In case (2.6), we have:

$$\frac{1}{\sqrt{b^2-4ac}}\ln\left(-\frac{2ay+b-\sqrt{b^2-4ac}}{2ay+b+\sqrt{b^2-4ac}}\right)=$$
$$t+\frac{1}{\sqrt{b^2-4ac}}\ln\left(-\frac{2ad+b-\sqrt{b^2-4ac}}{2ad+b+\sqrt{b^2-4ac}}\right).$$

In both cases (2.5) and (2.6), the solution of the problem (1.2) is given by the following formula:

$$y(t)=\frac{-bd+d\sqrt{b^2-4ac}-2c+(bd+d\sqrt{b^2-4ac}+2c)e^{t\sqrt{b^2-4ac}}}{2ad+b+\sqrt{b^2-4ac}-(2ad+b-\sqrt{b^2-4ac})e^{t\sqrt{b^2-4ac}}}. \tag{2.7}$$

To find the blow-up time we have to set the denominator equal to 0:

$$2ad+b+\sqrt{b^2-4ac}-(2ad+b-\sqrt{b^2-4ac})e^{t\sqrt{b^2-4ac}}=0. \tag{2.8}$$

Solving equation (2.8) for t, we obtain:

$$t_b=\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ad+b+\sqrt{b^2-4ac}}{2ad+b-\sqrt{b^2-4ac}}\right).$$

The blow-up time t_b must be positive, therefore, we have:

$$\frac{2ad+b+\sqrt{b^2-4ac}}{2ad+b-\sqrt{b^2-4ac}}>1. \tag{2.9}$$

Let us observe that if equation (2.6) holds, then (2.8) can never be satisfied, therefore, there is no blow-up. If equation (2.5) holds, then there are two possibilities:

$$\begin{cases} 2ad+b-\sqrt{b^2-4ac}>0 \\ 2ad+b+\sqrt{b^2-4ac}>0 \end{cases} \tag{2.10}$$

or

$$\begin{cases} 2ad + b - \sqrt{b^2 - 4ac} < 0 \\ 2ad + b + \sqrt{b^2 - 4ac} < 0. \end{cases} \tag{2.11}$$

Solving (2.10) and (2.9) simultaneously, we obtain conditions on d that lead to blow-up in finite time:

$$d > \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ if } a > 0,$$
$$d < \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ if } a < 0.$$

Solving (2.11) and (2.9) simultaneously, we obtain a contradiction which implies that there is no blow-up in this case. We notice that from (2.7) that if $a > 0$ and $d > \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to +\infty$. if $a < 0$ and $d < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to -\infty$. Initial value d is very important since certain values can prevent blow-up from occurring. Also, d influences the blow-up time t_b.

4. If $a = 0$, then the equation is linear. Using separation of variables, we obtain:

$$\int \frac{dy}{by + c} = \int dt,$$

$$y(t) = \frac{(bd + c)e^{bt} - c}{b}.$$

Function $y(t)$ is bounded for any finite time $t > 0$.

$\square$

Example 1 (case 3): Let us investigate the blow-up property of the solution for the initial-value problem

$$\begin{cases} y'(t) = -4y^2 + 5y - 1 \\ y(0) = d, \end{cases} \tag{2.12}$$

with three different values of d as indicated below.

First, we notice that $a = -4 < 0, b = 5, c = -1$, and $4ac - b^2 = -9 < 0$. Also, $(-b + \sqrt{b^2 - 4ac})/(2a) = 0.25$. According to the Theorem 1, we expect that solution of the problem (2.12) blows up for any $d < 0.25$ and is bounded otherwise. Solution of the problem (2.12) is given by the formula (2.7).

If $d = 2$, then we have

$$y(t) = \frac{-2 + 14e^{3t}}{-8 + 14e^{3t}}.$$

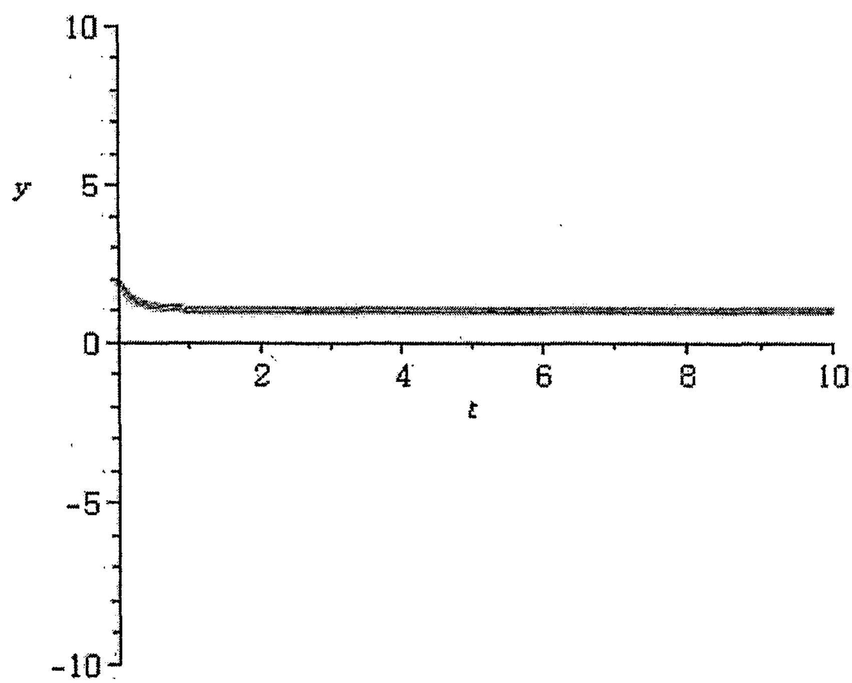

Figure 2.1: example of the case $4ac - b^2 < 0, \quad a < 0, \quad d > (-b + \sqrt{b^2 - 4ac})/(2a)$

This function is bounded for any finite time $t > 0$.

If $d = 0$, then we have

$$y(t) = \frac{1 - e^{3t}}{4 - e^{3t}}.$$

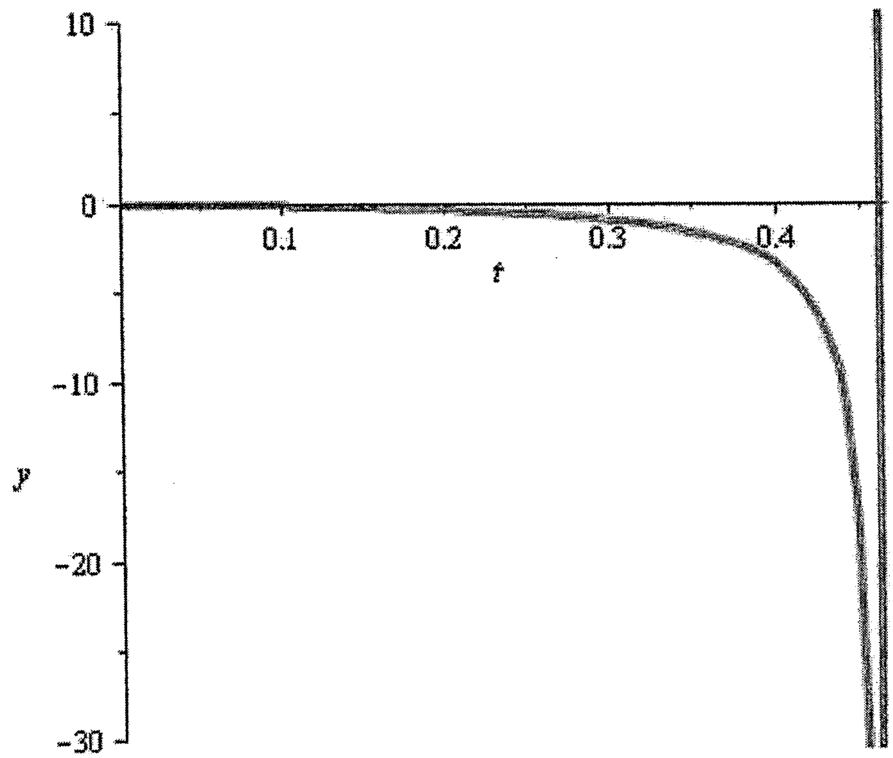

Figure 2.2: example of the case $4ac - b^2 < 0, \quad a < 0, \quad d < (-b + \sqrt{b^2 - 4ac})/(2a)$

Function $y(t) \to -\infty$ when $t_b = \ln(4)/3$.

Example 2 (case 3):

$$\begin{cases} y'(t) = 4y^2 + 5y + 1, \\ y(0) = d. \end{cases} \tag{2.13}$$

First, we notice that $a = 4 > 0, b = 5, c = 1, d = 0$, and $4ac - b^2 = -9 < 0$. Also, $(-b + \sqrt{b^2 - 4ac})/(2a) = -0.25$. According to the Theorem 1, we expect that solution of the problem (2.13) blows up for any $d > -0.25$ and is bounded otherwise. Solution of the problem (2.13) is given by the formula (2.7). If $d = 0$, then we have

$$y(t) = \frac{-2 + 2e^{3t}}{8 - 2e^{3t}}.$$

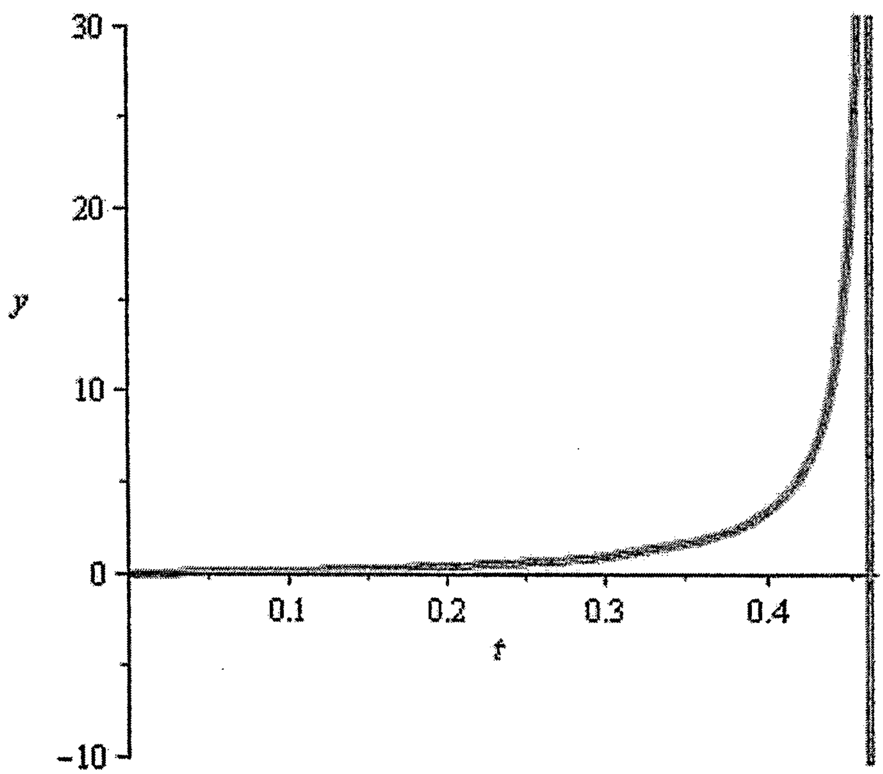

Figure 2.3: example of the case $4ac - b^2 < 0, \quad a > 0, \quad d > (-b + \sqrt{b^2 - 4ac})/(2a)$

Function $y(t) \to +\infty$ when $t_b = \ln(4)/3$.

Example 3 (case 2):

$$\begin{cases} y'(t) = y^2 + 2y + 1, \\ y(0) = d. \end{cases} \tag{2.14}$$

First, we notice that $a = 1 > 0, b = 2, c = 1$, and $4ac - b^2 = 0$. Also, $-b/2a = -1$. According to the Theorem 1, we expect that solution of the problem (2.14) approaches $+\infty$ for any $d > -1$ and is bounded otherwise. Solution of the problem (2.14) is given by the formula (2.2). If $d = 0$, then we have

$$y(t) = \frac{1}{1-t} - 1.$$

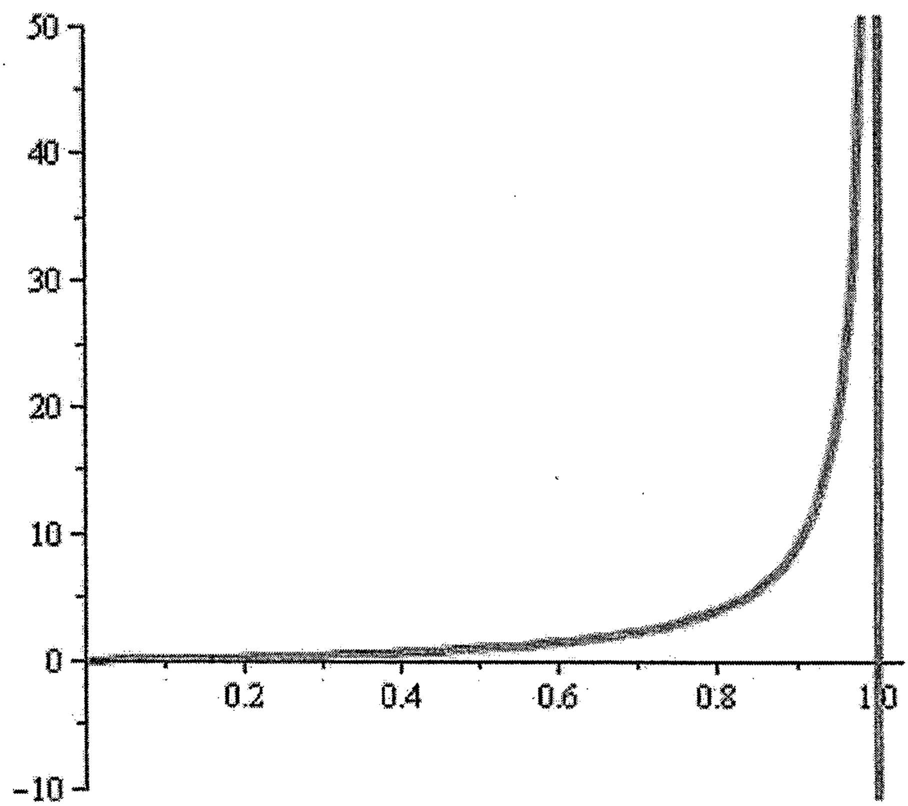

Figure 2.4: example of the case $4ac - b^2 = 0, \quad a > 0, \quad d > -b/2a$

Function $y(t) \rightarrow +\infty$ when $t_b = 1$.

$$\begin{cases} y'(t) = -y^2 - 2y - 1, \\ y(0) = d. \end{cases} \tag{2.15}$$

First, we notice that $a = -1 < 0, \quad b = -2, \quad c = -1$, and $4ac - b^2 = 0$. Also, $-b/2a = -1$. According to the Theorem 1, we expect that solution of the problem (2.15) approaches $-\infty$ for any $d < -1$ and is bounded otherwise. Solution of the problem (2.15) is given by the formula (2.2).

If $d = -1.5$, then we have

$$y(t) = \frac{1}{-1+t} - 1.$$

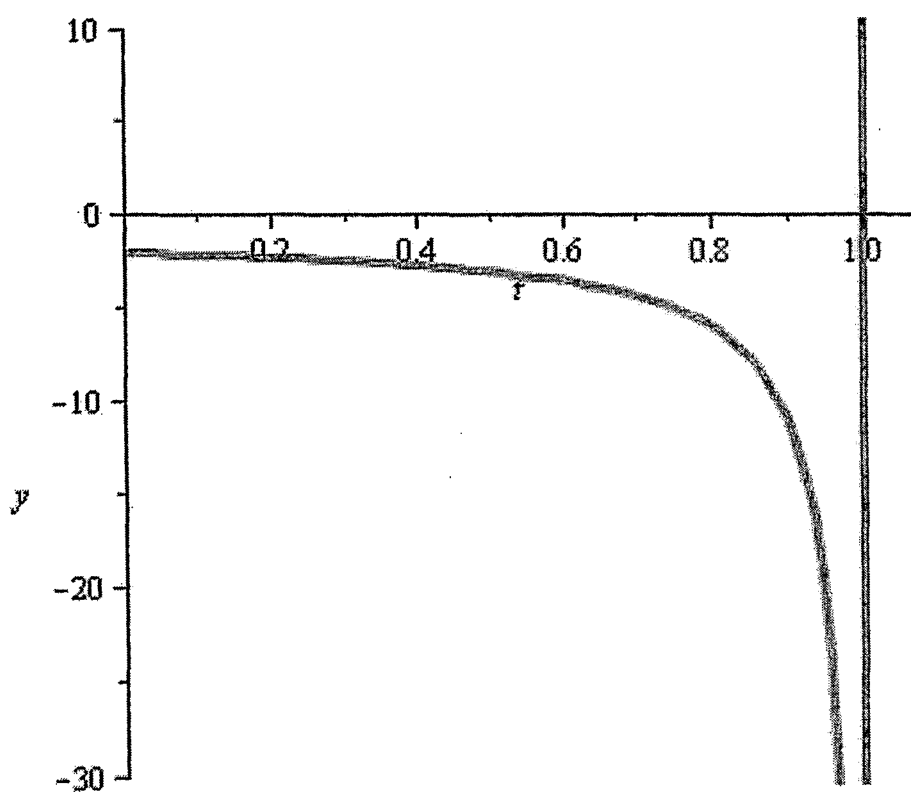

Figure 2.5: example of the case $4ac - b^2 = 0, \quad a < 0, \quad d < -b/2a$

Function $y(t) \to -\infty$ when $t_b = 1$.

Example 4 (case 1):

$$\begin{cases} y'(t) = 2y^2 + 3y + 2, \\ y(0) = d. \end{cases}$$

First we notice that $a = 2 > 0, b = 3, c = 2, d = 1$ and $4ac - b^2 = 7 > 0$. According to the Theorem 1, we expect $y(t) \to +\infty$ since $a > 0$. Changing initial value d cannot prevent blow-up from occurring.

$$y(t) = \frac{\sqrt{7}}{4}\tan\left[\frac{t\sqrt{7}}{2} + \arctan(\sqrt{7})\right] - \frac{3}{4}.$$

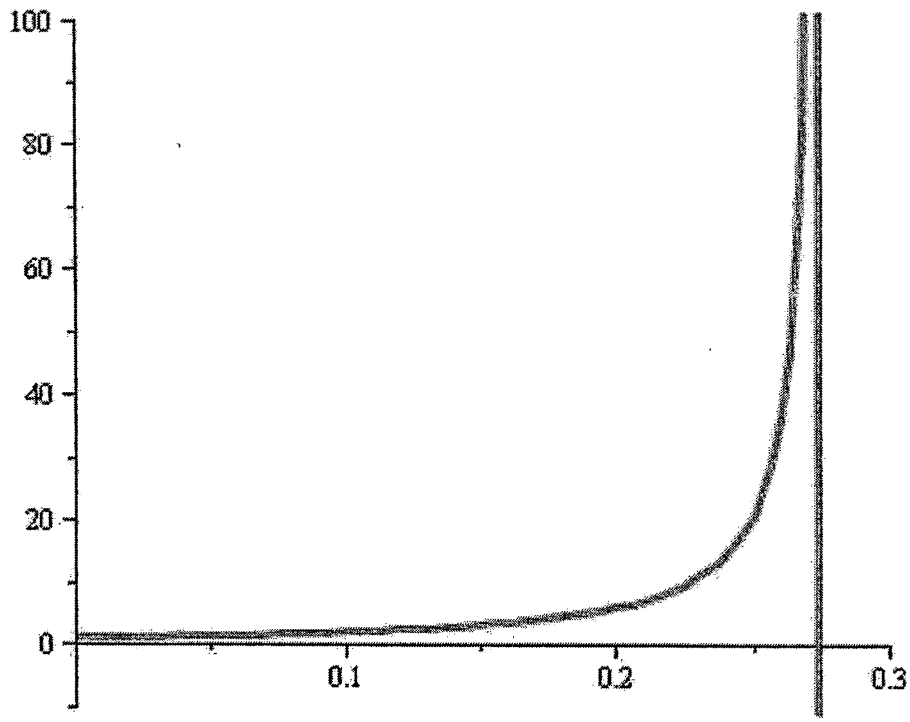

Figure 2.6: example of the case $4ac - b^2 > 0, \quad a > 0$

Function $y(t) \to +\infty$ when $t > 0$.

$$\begin{cases} y'(t) = -3y^2 + 2y - 2, \\ y(0) = d. \end{cases}$$

First we notice that $a = -3 < 0, b = 2, c = -2, d = 1$ and $4ac - b^2 = 20 > 0$. According to the Theorem 1, we expect $y(t) \to -\infty$ since $a < 0$. Changing initial value d cannot prevent blow-up from occurring.

$$y(t) = \frac{\sqrt{5}}{-3}\tan\left[t\sqrt{5} + \arctan\left(\frac{-2}{\sqrt{5}}\right)\right] + \frac{1}{3}.$$

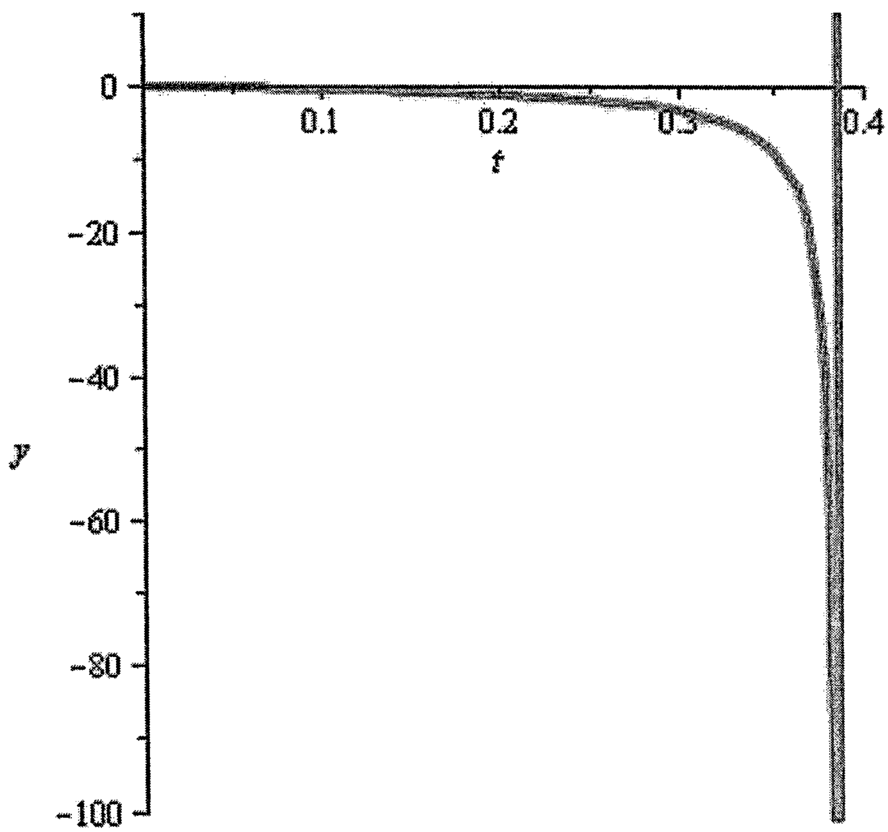

Figure 2.7: example of the case $4ac - b^2 > 0, \quad a < 0$

Function $y(t) \to -\infty$ when $t > 0$.

Proof. If $b(x,t) \equiv 0$, we can apply the strong maximum principle [Fri64] to obtain the conclusion immediately.

For the case $b(x,t)$ being nonnegative and nontrivial (not indentically zero), let η be a positive constant, and

$$V(x,t) = u(x,t) + \eta(1 + x^{1/2})e^{ct},$$

where c is a positive constant to be determined.

Let us verify $V(x,0) > 0$ on $\partial\Omega$. We have $V(x,0) = u(x,0) + \eta(1 + x^{1/2})e^0 > 0$, because we know that $u(x,0) \geq 0$, and $\eta(1 + x^{1/2})e^0 > 0$, thus $V(x,0) > 0$.

Since $u(0,t) = 0$ and $\eta(1+0)e^{ct} > 0$, so $V(0,t) = u(0,t) + \eta(1+0)e^{ct} > 0$. As we know $\xi = ax$, then we have $u_\xi = u_x/a$. We know $u_\xi(a,\tau) = 0$, for $\tau > 0$. By the substitution, we get $(u_x/a)(1,t) = 0$, for $0 < t < T$, finally we have $u_x(1,t) = 0$.

We have

$$\begin{aligned} V_x(x,t) &= u_x(x,t) + \frac{1}{2}\eta e^{ct} x^{1/2-1}, \\ &= u_x(x,t) + \frac{\eta e^{ct}}{2\sqrt{x}}, \end{aligned}$$

thus $V_x(1,t) = u_x(1,t) + \dfrac{\eta e^{ct}}{2} > 0$.

Let $\partial\Omega$ denote the parabolic boundary

$$(\{0,a\} \times (0,T)) \cup ([0,a] \times \{0\})$$

of Ω. Then $V(x,t) > 0$ on $\partial\Omega$, and $Lu - b(x,t)u(x_0,t) \geq 0$.

Now,

$$\begin{aligned} &LV - b(x,t)V(x_0,t) = L(u(x,t) + \eta(1+\sqrt{x})e^{ct}) - b(x,t)V(x_0,t) \\ &= x^q \frac{\partial}{\partial t}(u + \eta(1+\sqrt{x})e^{ct}) - \frac{\partial^2}{\partial x^2}(u + \eta(1+\sqrt{x})e^{ct}) - b(x,t)[u(x_0,t) + \eta(1 + x_0^{1/2})e^{ct}] \\ &= Lu - b(x,t)u(x_0,t) + L(\eta(1+\sqrt{x})e^{ct}) - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &\geq L(\eta(1+\sqrt{x})e^{ct}) - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &= \frac{x^q \partial(\eta(1+\sqrt{x})e^{ct})}{\partial t} - \frac{\partial^2(\eta(1+\sqrt{x})e^{ct})}{\partial x^2} - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &= x^q \eta(1+\sqrt{x})e^{ct} c - \frac{\partial}{\partial x}\left(\frac{\eta e^{ct}}{2\sqrt{x}}\right) - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &= \eta e^{ct}\left((cx^q(1+\sqrt{x})) - b(x,t)(1+\sqrt{x_0}) + \frac{1}{4x^{3/2}}\right). \end{aligned}$$

Let s denote the positive zero of

$$\frac{1}{4x^{3/2}} - (1 + x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t).$$

For $x = s$,

$$\begin{aligned}
&cx^q(1 + x^{1/2}) - b(x,t)(1 + x_0^{1/2}) + \frac{1}{4x^{3/2}} \\
&\geq cx^q(1 + x^{1/2}) - (1 + x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4x^{3/2}} \\
&= cs^q(1 + s^{1/2}) - (1 + x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4s^{3/2}} \\
&= cs^q(1 + s^{1/2}) - 0 > 0.
\end{aligned}$$

For $x < s$, since $cx^q(1 + x^{1/2}) \geq 0$, and $-(1 + x^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \dfrac{1}{4x^{3/2}} > 0$,

$$\begin{aligned}
&cx^q(1 + x^{1/2}) - b(x,t)(1 + x_0^{1/2}) + \frac{1}{4x^{3/2}} \\
&\geq cx^q(1 + x^{1/2}) - (1 + x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4x^{3/2}} > 0.
\end{aligned}$$

For $x > s$,

$$\begin{aligned}
&cx^q(1 + x^{1/2}) - b(x,t)(1 + x_0^{1/2}) + \frac{1}{4x^{3/2}} \\
&\geq cx^q(1 + x^{1/2}) - (1 + x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4x^{3/2}} \\
&> cs^q - (1 + x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) \geq 0.
\end{aligned}$$

If we choose $c \geq \dfrac{(1 + x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t)}{s^q}$, then $LV - b(x,t)V(x_0,t) > 0$ in Ω.

Suppose $V(x,t) \leq 0$ somewhere in Ω. Then, the set

$$\{t : V(x,t) \leq 0 \text{ for some } x \in D\}$$

is nonempty. Let $\bar{t}$ denote its infimum. Since $V(x,0) > 0$, we have $0 < \bar{t} < T$. Thus, there exists some $x_1 \in D$ such that $V(x_1,\bar{t}) = 0$, and $V_t(x_1,\bar{t}) \leq 0$. On the other hand, since $V(x,t)$ attains its local minimum at $(x_1,\bar{t})$, we have $V_{xx}(x_1,\bar{t}) \geq 0$. Since $\bar{t}$ is the infimum, we also have $V(x_0,\bar{t}) \geq 0$. Now we have,

$$\begin{gathered}
LV(x_1,\bar{t}) = x_1^q V_t(x_1,\bar{t}) - V_{xx}(x_1,\bar{t}) \leq 0, \\
x_1^q V_t(x_1,\bar{t}) - V_{xx}(x_1,\bar{t}) - b(x_1,\bar{t})V(x_0,\bar{t}) \leq 0, \\
0 \geq x_1^q V_t(x_1,\bar{t}) \geq LV(x_1,\bar{t}) - b(x_1,\bar{t})V(x_0,\bar{t}) > 0.
\end{gathered}$$

This contradiction shows that $V(x,t) = u(x,t) + \eta(1 + x^{1/2}) > 0$, in Ω.
As $\eta \to 0^+$, $u(x,t) \geq 0$.

□

3.2 Existence and Uniqueness

Let $\omega = D \times (0, t_0)$ for some positive number t_0, and $\bar{\omega}$ be its closure.

Lemma 3.2. *There exists a positive constant $t_0 < T$ such that the problem (1.3) has an upper solution $\mu_1(x,t) \in C^{2,1}(\bar{\omega})$.*

Note: An upper solution $\mu_1(x,t)$ has to satisfy the following:

$$\begin{aligned}
&L\mu_1 - a^2 f(\mu_1(x_0,t)) \geq 0 \quad \text{in} \quad \omega \\
&\mu_1(x,0) \geq u_0(x) \quad \text{on} \quad \bar{D} \\
&\mu_{1_x}(1,t) \geq 0, \quad t \in [0,t_0] \\
&\mu_1(0,t) \geq 0, \quad t \in [0,t_0].
\end{aligned}$$

Note: $\mu_1 \in C^{2,1}(\bar{\omega})$ means that $\mu_1, \mu_{1_x}, \mu_{1_{xx}}$ and μ_{1_t} are continuous on $\bar{\omega}$.

Proof. Let $k_1 = 1 + \max_{x \in \bar{D}}(u_0(x))$, and $k_2 (> 2a^2)$ be chosen sufficiently large such that

$$\gamma \equiv k_2 f(1 + k_1) > 2, \tag{3.2}$$

$$0 < \epsilon < \min\left\{1 - \frac{1}{2^{\frac{1}{\gamma-2}}}, \frac{1}{k_2}\sqrt{\frac{k_2 - 2a^2}{f(1+k_1)}}\right\}. \tag{3.3}$$

Since $2^{1/(\gamma-2)} > 1$ for any $\gamma > 2$, we have

$$0 < 1 - \frac{1}{2^{\frac{1}{\gamma-2}}} < 1.$$

Let $\mu_1(x,t) = \theta_1(x)\tau_1(t)$, where

$$\theta_1(x) = (1-x)^{\gamma} e^{\gamma x} + k_1, \tag{3.4}$$

$$\tau_1' = \epsilon^{-q} k_1^{-1} \left[\frac{\gamma(1+\gamma)e^{\gamma}}{\theta_1(x_0)} + a^2 \right] f(\theta_1(x_0)\tau_1), \quad \tau_1(0) = 1. \tag{3.5}$$

By the existence and uniqueness theorem for first order initial-value problem [BDH02], (3.5) has a unique solution. We note that in D,

$$\begin{aligned} \theta_1'(x) &= \gamma(1-x)^{\gamma-1}(-1)e^{\gamma x} + (1-x)^{\gamma}\gamma e^{\gamma x} \\ &= -\gamma(1-x)^{\gamma-1} x e^{\gamma x} < 0, \end{aligned}$$

and the function $\theta_1(x)$ is decreasing, which implies that

$$\theta_1(x) \leq 1 + k_1.$$

From (3.2),

$$\begin{aligned} &\gamma - \epsilon^2\gamma^2\tau_1(0) - \frac{a^2 f(\theta_1(x_0)\tau_1(0))}{(1-\epsilon)^{\gamma-2}} \\ &= \gamma - \epsilon^2\gamma^2 - \frac{a^2 f(\theta_1(x_0))}{(1-\epsilon)^{\gamma-2}} \\ &\geq k_2 f(1+k_1) - \epsilon^2 k_2^2 f^2(1+k_1) - \frac{a^2 f(1+k_1)}{\left(1 - 1 + \frac{1}{2^{\frac{1}{\gamma-2}}}\right)^{\gamma-2}} \\ &= f(1+k_1)(k_2 - 2a^2 - \epsilon^2 k_2^2 f(1+k_1)) \\ &> 0. \end{aligned}$$

We used the following: $\left(\frac{1}{2^{\frac{1}{\gamma-2}}}\right)^{\gamma-2} = \frac{1^{\gamma-2}}{\left(2^{\frac{1}{\gamma-2}}\right)^{\gamma-2}} = \frac{1}{2^{\frac{\gamma-2}{\gamma-2}}} = \frac{1}{2}$, and

$$\begin{aligned} &k_2 - 2a^2 - \epsilon^2 k_2^2 f(1+k_1) > 0, \\ &k_2 - 2a^2 > \epsilon^2 k_2^2 f(1+k_1), \\ &\frac{k_2 - 2a^2}{k_2^2 f(1+k_1)} > \epsilon^2, \\ &\epsilon < \sqrt{\frac{k_2 - 2a^2}{k_2^2 f(1+k_1)}} = \frac{1}{k_2}\sqrt{\frac{k_2 - 2a^2}{f(1+k_1)}} \end{aligned}$$

holds by (3.3).

Also, $\tau_1(t)$ is an increasing function that blows up at

$$t_b = \frac{\epsilon^q k_1}{\gamma(1+\gamma)e^{\gamma} + a^2\theta_1(x_0)} \int_{\theta_1(x_0)}^{\infty} \frac{ds}{f(s)} < \infty.$$

Let t_2 denote the time such that

$$\gamma - \epsilon^2\gamma^2\tau_1(t_2) - \frac{a^2 f(\theta_1(x_0)\tau(t_2))}{(1-\epsilon)^{\gamma-2}} = 0, \tag{3.6}$$

and $t_0 = \min\{t_2; t_b - \kappa\}$ for some fixed small number $0 < \kappa \ll t_b$. Since $\tau_1(t)$ is an increasing function, it follows from (3.2), (3.4), and (3.6) that for any $x \in [0, \epsilon]$ and $t \leq t_0$,

$$\begin{aligned}
&L\mu_1 - a^2 f(\mu_1(x_0,t)) = x^q \mu_{1_t} - \mu_{1_{xx}} - a^2 f(\mu_1(x_0,t)) \\
&= x^q\theta_1(x)\tau_1'(t) - \theta_1''(x)\tau_1(t) - a^2 f(\theta_1(x_0)\tau_1) \\
&= \left(x^q\left((1-x)^\gamma e^{\gamma x} + k_1\right)\right)\left(\epsilon^{-q}k_1^{-1}\left[\frac{\gamma(1+\gamma)e^\gamma}{\theta_1(x_0)} + a^2\right] f(\theta_1(x_0)\tau_1)\right) \\
&-\left(\gamma(1-x)^{\gamma-2}(-1+\gamma x^2)e^{\gamma x}\right)\tau_1 - a^2 f(\theta_1(x_0)\tau_1) \\
&\geq -\gamma(1-x)^{\gamma-2}(-1+\gamma x^2)e^{\gamma x}\tau_1 - a^2 f(\theta_1(x_0)\tau_1) \\
&\geq (1-x)^{\gamma-2}e^{\gamma x}\left[\gamma\tau_1(0) - \gamma^2\tau_1(t_0)x^2 - \frac{a^2 f(\theta_1(x_0)\tau_1(t_0))}{e^{\gamma x}(1-x)^{\gamma-2}}\right] \\
&\geq (1-x)^{\gamma-2}e^{\gamma x}\left[\gamma - \gamma^2\tau_1(t_0)\epsilon^2 - \frac{a^2 f(\theta_1(x_0)\tau_1(t_0))}{(1-\epsilon)^{\gamma-2}}\right] \geq 0.
\end{aligned}$$

From (3.4) we get

$$\begin{aligned}
\theta_1''(x) &= -\gamma(\gamma-1)(1-x)^{\gamma-2}(-1)xe^{\gamma x} - \gamma(1-x)^{\gamma-1}e^{\gamma x} - \gamma(1-x)^{\gamma-1}x\gamma e^{\gamma x} \\
&= \gamma(1-x)^{\gamma-2}(-1+x^2\gamma)e^{\gamma x},
\end{aligned}$$

we obtain

$$\max_{x\in\bar{D}}\left|\theta_1''(x)\right| \leq \gamma(1+\gamma)e^\gamma.$$

Using $\tau_1' > 0$, and from (1.4), (3.2), and (3.6) we have $f(\theta_1(x_0)\tau_1) \geq \theta_1(x_0)\tau_1$, and for any $x \in [\epsilon, 1]$, and $t \leq t_0$,

$$\begin{aligned}
&L\mu_1 - a^2 f(\mu_1(x_0,t)) = x^q \mu_{1_t} - \mu_{1_{xx}} - a^2 f(\mu_1(x_0,t)) \\
&= x^q \theta_1(x)\tau_1'(t) - \theta_1''(x)\tau_1(t) - a^2 f(\mu_1(x_0,t)) \\
&= x^q \Big((1-x)^\gamma e^{\gamma x} + k_1\Big)\tau_1'(t) - \Big(\gamma(1-x)^{\gamma-2}(-1+\gamma x^2)e^{\gamma x}\Big)\tau_1(t) - a^2 f(\mu_1(x_0,t)) \\
&\geq x^q \Big((1-x)^\gamma e^{\gamma x} + k_1\Big)\tau_1'(t) - \Big(\gamma(1+\gamma)e^{\gamma}\Big)\tau_1(t) - a^2 f(\mu_1(x_0,t)) \\
&\geq \epsilon^q k_1 \tau_1' - \gamma(1+\gamma)e^\gamma \tau_1 - a^2 f(\theta_1(x_0)\tau_1) \\
&\geq \epsilon^q k_1 \tau_1' - \gamma(1+\gamma)e^\gamma \frac{f(\theta_1(x_0)\tau_1)}{\theta_1(x_0)} - a^2 f(\theta_1(x_0)\tau_1) \\
&= \epsilon^q k_1 \left\{\tau_1' - \epsilon^{-q}k_1^{-1}\left[\frac{\gamma(1+\gamma)e^\gamma}{\theta_1(x_0)} + a^2\right] f(\theta_1(x_0)\tau_1)\right\} \\
&= 0.
\end{aligned}$$

Since $\tau_1(0) = 1, \quad 0 \leq x \leq 1$, and $k_1 = 1 + \max_{x \in \bar{D}}(u_0(x))$,

$$\begin{aligned}
\theta_1(x) &= (1-x)^\gamma e^{\gamma x} + k_1 \\
&= (1-x)^\gamma e^{\gamma x} + 1 + \max_{x\in\bar{D}}(u_0(x)) > 0.
\end{aligned}$$

So, we have $\mu_1(x,0) = \theta_1(x)\tau_1(0) > u_0(x)$. Also, $\mu_{1_x}(1,t) = \theta_1'(1)\tau_1(t) = 0$, since $\theta_1'(1) = -\gamma(1-1)^{\gamma-1}e^\gamma = 0$. Then $\mu_1(0,t) = \theta_1(0)\tau_1(t) = ((1-0)^\gamma e^{\gamma 0} + k_1)\tau_1(t) > 0$. Since $L\mu_1 - a^2 f(\mu_1(x_0,t)) \geq 0$ and $Lu = a^2 f(u(x_0,t))$, using the Mean Value Theorem (Appendix A), we attain

$$\begin{aligned}
L(\mu_1 - u) &= L\mu_1 - Lu \\
&\geq a^2 f(\mu_1(x_0,t)) - a^2 f(u(x_0,t)) \\
&= a^2 f'(\zeta(x_0,t))(\mu_1(x_0,t) - u(x_0,t)),
\end{aligned}$$

for some $\zeta(x_0,t)$ between $\mu_1(x_0,t)$ and $u(x_0,t)$. By Lemma 3.1,

$$\mu_1(x,t)(\in C^{2,1}(\bar{\omega})) \text{ is an upper solution.}$$

□

Let $\rho(x)$ in $C^3[0,1]$ be an increasing function such that $\rho(x)$ is 0 for $x \leq 0$ and 1 for $x \geq 1$. Also, let δ be some positive constant with $\delta < x_0/2, \quad D_\delta = (\delta, 1), \quad \omega_\delta = D_\delta \times (0, t_0), \quad \bar{D}_\delta$ and $\bar{\omega_\delta}$ be the closures of D_δ and ω_δ respectively,

$$\rho_\delta = \begin{cases} 0 & \text{for } x \le \delta, \\ \rho(\frac{x}{\delta} - 1) & \text{for } \delta < x < 2\delta, \\ 1 & \text{for } x \ge 2\delta, \end{cases}$$

$$u_{0_\delta}(x) = \rho_\delta(x) u_0(x).$$

From

$$\frac{\partial u_{0_\delta}(x)}{\partial \delta} = \begin{cases} 0 & \text{for } x \le \delta, \\ -\frac{x}{\delta^2}\rho'\left(\frac{x}{\delta} - 1\right)u_0(x) & \text{for } \delta < x < 2\delta, \\ 0 & \text{for } x \ge 2\delta, \end{cases}$$

we have $\partial u_{0_\delta}(x)/\partial\delta \le 0$, and $u_{0_\delta}(x) \le u_0(x)$.

Let us consider the following problem,

$$\begin{cases} Lu_\delta = a^2 f(u_\delta(x_0, t)) & \text{in} \quad \omega_\delta, \\ u_\delta(x, 0) = u_{0_\delta}(x)(\ge 0) & \text{on} \quad \bar{D}_\delta, \\ u_\delta(\delta, t) = 0 = u_{\delta_x}(1, t) & \text{for} \quad 0 < t < t_0. \end{cases} \tag{3.7}$$

Existence of a classical solution for the problem (3.7) with $u_{\delta_x}(1,t) = 0$ replaced by $u_\delta(1,t) = 0$ has been established by Chan and Yang [CY00]. By using Theorem A.4.1 (instead of Theorem 4.2.2) of Ladde, Lakshmikantham and Vatsala [LLV85], and Theorem 5.3 (instead of Theorem 5.2) of Ladyženskaja, Solonnikov and Ural'ceva [LSU67], a proof similar to that of Theorem 3 of Chan and Yang [CY00] gives the following result.

Lemma 3.3. *The problem (3.7) has a unique nonnegative solution $u_\delta \in C^{2+\alpha,1+\alpha/2}(\bar{\omega}_\delta)$ such that $u_\delta(x,t) \le \mu_1(x,t)$.*

It is shown in Chan and Liu [CL98] that there exists solution $u_\delta \in C^{2+\alpha,1+\alpha/2}(\bar{\omega}_\delta)$ of (3.7).

Let $\lim_{\delta\to 0} u_\delta(x,t) = u(x,t)$. By using the singular index 3 (cf. Ladyženskaja, Solonnikov and Ural'ceva [LSU67]), a proof similar to that of Lemma 2 of Chan and Liu [CL98] gives the following result.

Theorem 3.4. *The problem (1.3) has a unique solution* $u(x,t) \in C(\bar{\omega}) \cap C^{2,1}((0,1] \times [0,t_0])$.

Since $0 \le u_\delta \le \mu_1, \lim_{\delta \to 0} u_\delta$ exists for all $(x,t) \in \bar{\omega}$. It is also shown that $u_{\delta_1}, (u_\delta)_t, (u_\delta)_x$, and $(u_\delta)_{xx}$ are equicontinuous (Appendix A) in $\bar{\omega}$. By the Ascoli-Arzela Theorem (Appendix A), the partial derivatives of u are the limits of the corresponding derivatives of u_δ. Therefore, $u(x,t) = \lim_{\delta \to 0} u_\delta$.

Let T be the supremum over t_0 for which the problem (1.3) has a unique solution $u \in C(\bar{\omega}) \cap C^{2,1}((0,1] \times [0,t_0])$. Then, it has a unique solution $u(x,t) \in C(\bar{D} \times [0,T)) \cap C^{2,1}((0,1] \times [0,T))$. We modify the proof of Theorem 2.5 by Floater [Flo91] to prove the following result.

Theorem 3.5. *If* $T < \infty$, *then* $u(x_0,t)$ *is unbounded in* $(0,T)$.

Proof. Let us suppose that $u(x_0,t)$ is bounded above by some positive constant M in Ω. We would like to show that u can be continued into a time interval $[0, T + \tilde{t}_0]$ for some positive $\tilde{t}_0$. To do so, let

$$K = \max\left\{ a^2 f(M), \quad \max_{x \in \bar{D}} \frac{2u_0(x)}{x(2k_3 + 1 - x)} \right\},$$

$$\begin{aligned} W(x) &= \frac{Kx(2k_3 + 1 - x)}{2} \\ &= Kxk_3 + \tfrac{1}{2}Kx - \tfrac{1}{2}Kx^2, \end{aligned}$$

$$\begin{aligned} &W'(x) = Kk_3 + \tfrac{1}{2}K - Kx, \\ &W''(x) = -K, \end{aligned}$$

where $k_3 \ge 1/2$. Since $LW = x^q W_t - W_{xx} = x^q \cdot 0 - (-K) = K$, we have

$$\begin{aligned} L(W-u) &= LW - Lu \\ &= K - a^2 f(u(x_0,t)) \\ &\ge a^2 f(M) - a^2 f(u(x_0,t)) \\ &= a^2 \Big(f(M) - f(u(x_0,t)) \Big) \ge 0 \quad \text{in} \quad \Omega. \end{aligned}$$

Also, we have

$$W(x) \geq \frac{\left(\max_{x\in\bar{D}}\dfrac{2u_0(x)}{x(2k_3+1-x)}\right)x(2k_3+1-x)}{2} \geq u_0(x).$$

Also,

$$W(0) = \frac{K\cdot 0(2k_3+1-0)}{2} = 0 = u(0,t),$$

and since $k_3 \geq \frac{1}{2}$,

$$W'(1) = Kk_3 + \tfrac{1}{2}K - K = K(k_3 - 0.5) \geq 0, \text{ and } u_x(1,t) = 0,$$

we have $W'(1) = K(k_3 - 0.5) \geq u_x(1,t)$ for $t > 0$.

By Lemma 3.1, $W(x)$ is an upper solution of $u(x,t)$ for $0 \leq t \leq T$.

Taking $W(x)$ as the initial function at $t = T$, we can construct, as in Lemma 3.2, an upper solution $\tilde{\mu}_1(x,t)$ of $u(x,t)$ on $\bar{D}\times[T, T+\tilde{t}_0]$ for some positive $\tilde{t}_0$. Thus, u can be continued into a time interval $[0, T+\tilde{t}_0]$. This contradicts the definition of T. Hence, the theorem is proved. □

Theorem 3.6. *If $u_0(x)$ is sufficiently large in the neighborhood of x_0, then u blows up in a finite time.*

Proof. Let

$$\begin{aligned}
\theta_2(x) &= (x - x_0 + \epsilon)^2(x - x_0 - \epsilon)^2,\\
&= (x^2 - 2x_0x + x_0^2 - \epsilon^2)(x^2 - 2x_0x + x_0^2 - \epsilon^2)\\
&= x^4 + x^3(-2x_0 - 2x_0) + x^2(x_0^2 - \epsilon^2 + 4x_0^2 + x_0^2 - \epsilon^2)\\
&+ x(-2x_0^3 + 2x\epsilon^2x_0 - 2x_0^3 + 2\epsilon^2x_0) + (x_0^4 - \epsilon^2x_0^2 - \epsilon^2x_0^2 + \epsilon^4)\\
&= x^4 + x^3(-4x_0) + x^2(6x_0^2 - 2\epsilon^2) + x(-4x_0^3 + 4\epsilon^2x_0) + (x_0^4 - 2\epsilon^2x_0^2 + \epsilon^4).
\end{aligned}$$

Also,

$$\theta_2'(x) = 4x^3 + 3x^2(-4x_0) + 2x(6x_0^2 - 2\epsilon^2) + (-4x_0^3 + 4\epsilon^2x_0),$$

and

$$\begin{aligned} \theta_2''(x) &= 12x^2 - 24x_0x + 12x_0^2 - 4\epsilon^2 \\ &= 4(3x^2 - 6x_0x + 3x_0^2 - \epsilon^2). \end{aligned}$$

Also, we have

$$\tau_2' + \frac{4}{(x_0+\epsilon)^q\epsilon^2}\tau_2 = \frac{a^2\epsilon^{4r}}{(x_0+\epsilon)^q}\tau_2^{1+r}, \tau_2(0) \equiv \tau_0 > \max\left\{\frac{1}{\epsilon^4}, \left(\frac{4}{a^2\epsilon^{4r+2}}\right)^{\frac{1}{r}}\right\}, \tag{3.8}$$

where ϵ is sufficiently small such that $0 < x_0 - \epsilon < x_0 + \epsilon < 1$, and r is given in (1.4).

We note that

$$\theta_2''(x) = 4(3x^2 - 6x_0x + 3x_0^2 - \epsilon^2)$$

is a quadratic function with vertex at $x = x_0$, $a = 12, b = -24x_0$, and $c = (3x_0^2 - \epsilon^2)4$. Now,

$$\begin{gathered} \theta_2'' = 0 \text{ at } x = x_0 - \frac{\sqrt{3}}{3}\epsilon \text{ and } x = x_0 + \frac{\sqrt{3}}{3}\epsilon, \\ \theta_2'' < 0 \text{ for } x \in \left(x_0 - \frac{\sqrt{3}}{3}\epsilon, x_0 + \frac{\sqrt{3}}{3}\epsilon\right), \\ \theta_2'' > 0 \text{ for } x \in \left(x_0 - \epsilon, x_0 - \frac{\sqrt{3}}{3}\right), \quad x \in \left(x_0 + \frac{\sqrt{3}}{3}, x_0 + \epsilon\right), \\ \theta_2(x_0) = (x_0 - x_0 + \epsilon)^2(x_0 - x_0 - \epsilon)^2 = \epsilon^4 \end{gathered}$$

and $\theta_2(x)$ attains its maximum ϵ^4 at $x = x_0$.

Let $\mu_2(x,t) = \theta_2(x)\tau_2(t)$. Then for $x \in (x_0 - \epsilon, x_0 + \epsilon)$, since $f(u) \geq u^{1+r}, u \geq 1$, and $f(\epsilon^4\tau_2) \geq (\epsilon^4\tau_2)^{1+r}$ provided $\epsilon^4\tau_2 \geq 1$, and $\epsilon^4\tau_2 \geq \epsilon^4(1/\epsilon^4) = 1$ from (3.8), since $x_0 + \epsilon$ is maximum, $\theta_2(x_0) = \epsilon^4$, and $\tau_2' + \frac{4}{(x_0+\epsilon)^q\epsilon^2}\tau_2 = \frac{a^2\epsilon^{4r}}{(x_0+\epsilon)^q}\tau_2^{1+r}$,

$$\begin{aligned} & L\mu_2 - a^2 f(\mu_2(x_0,t)) \\ &= L(\theta_2(x)\tau_2(t)) - a^2 f(\mu_2(x_0,t)) \\ &= x^q(x - x_0 + \epsilon)^2(x - x_0 - \epsilon)^2\tau_2' + (-12x^2 + 24x_0x - 12x_0^2 + 4\epsilon^2)\tau_2 - a^2 f(\theta_2(x_0)\tau_2) \\ &\leq x^q(x - x_0 + \epsilon)^2(x - x_0 - \epsilon)^2\tau_2' + 4\epsilon^2\tau_2 - a^2 f(\theta_2(x_0)\tau_2) \\ &\leq (x_0 + \epsilon)^q\epsilon^4\tau_2' + 4\epsilon^2\tau_2 - a^2 f(\epsilon^4\tau_2) \\ &\leq (x_0 + \epsilon)^q\epsilon^4\tau_2' + 4\epsilon^2\tau_2 - a^2\epsilon^{4(1+r)\tau_2^{1+r}} \\ &= (x_0 + \epsilon)^q\epsilon^4\left[\tau_2' + \frac{4}{(x_0+\epsilon)^q\epsilon^2}\tau_2 - \frac{a^2\epsilon^{4r}}{(x_0+\epsilon)^q}\tau_2^{1+r}\right] \\ &= 0. \end{aligned}$$

For $x \in [x_0 - \epsilon, x_0 + \epsilon], \mu_2(x, 0) \geq 0$. Also, $\mu_2(x_0 - \epsilon, t) = 0 = \mu_2(x_0 + \epsilon, t)$. In (3.8), let $z = \tau_2^{-r}$. We obtain a linear equation,

$$z' - \frac{4r}{(x_0 + \epsilon)^q \epsilon^2} z + \frac{a^2 r \epsilon^{4r}}{(x_0 + \epsilon)^q} = 0.$$

For solving this, we start from (3.8)

$$\tau_2' + \frac{4}{(x_0 + \epsilon)^q \epsilon^2} \tau_2 = \frac{a^2 \epsilon^{4r}}{(x_0 + \epsilon)^q} \tau_2^{1+r}.$$

Let $y = \tau_2, a = \dfrac{4}{(x_0 + \epsilon)^q \epsilon^2}$, and $b = \dfrac{a^2 \epsilon^{4r}}{(x_0 + \epsilon)^q}$, then $y' + ay = by^{1+r}$. We have $\dfrac{dy}{dx} + ay = by^{1+r}$.

Now, $P(x) = a, Q(x) = b, n = 1 + r$, and $v = y^{1-1-r} = y^{-r}$, then since $y = v^{-\frac{1}{r}}$, we have $by^{1+r} = b(v^{-\frac{1}{r}})^{1+r} = bv^{-\frac{1}{r}-1}$, and $dy/dx = (dy/dv)(dv/dx) = -1/rv^{-\frac{1}{r}-1} dv/dx$,

$$-\frac{1}{r} v^{-\frac{1}{r}-1} \frac{dv}{dx} + av^{-\frac{1}{r}} = bv^{-\frac{1}{r}-1},$$

if we divide by $-1/r(v)^{-1/r-1}$ to both sides, then we have

$$\frac{dv}{dx} - arv = -br.$$

Now, we have

$$\begin{aligned}
&\rho(x) = e^{\int -ar dx} = e^{-arx}, \\
&D_x(e^{-arx} v) = -br(e^{-arx}), \\
&e^{-arx} v = \textstyle\int -br(e^{-arx}) dx = \dfrac{b}{a}(e^{-arx}) + c, \\
&e^{-arx} y^{-r} = \dfrac{b}{a}(e^{-arx}) + c, \\
&y^{-r} = \dfrac{b}{a} + \dfrac{c}{e^{-arx}}.
\end{aligned}$$

Since $\tau_2 = y, a = \dfrac{4}{(x_0 + \epsilon)^q \epsilon^2}, b = \dfrac{a^2 \epsilon^{4r}}{(x_0 + \epsilon)^q}$, and $x = t$.

$$\begin{aligned}
\tau_2^{-r} &= \frac{\frac{a^2 \epsilon^{4r}}{(x_0 + \epsilon)^q}}{\frac{4}{(x_0 + \epsilon)^q \epsilon^2}} + \frac{c}{e^{-\frac{4rt}{(x_0+\epsilon)^q \epsilon^2}}} \\
&= \frac{a^2 \epsilon^{4r+2}}{4} + ce^{\frac{4rt}{(x_0+\epsilon)^q \epsilon^2}},
\end{aligned}$$

$$\tau_2 = \left[\frac{a^2\epsilon^{4r+2}}{4} + ce^{\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}}\right]^{-\frac{1}{r}},$$

since $e^{\left(\frac{4\cdot r\cdot 0}{(x_0+\epsilon)^q\epsilon^2}\right)} = e^0 = 1$, now we have

$$\begin{aligned}
\tau_0 = \tau_2(0) &= \left[\frac{a^2\epsilon^{4r+2}}{4} + c\right]^{-\frac{1}{r}},\\
\tau_2^{-r} &= \frac{a^2\epsilon^{4r+2}}{4} + c,\\
\tau_2^{-r} - \frac{a^2\epsilon^{4r+2}}{4} &= c,\\
\tau_2 &= \left[\frac{a^2\epsilon^{4r+2}}{4} + \left(\frac{1}{\tau_0^r} - \frac{a^2\epsilon^{4r+2}}{4}\right)e^{\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}}\right]^{-\frac{1}{r}}.
\end{aligned}$$

Therefore, $\tau_2 = \left\{\left[\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} + \frac{1}{\tau_0^r} - \frac{a^2\epsilon^{4r+2}}{4}\right]e^{\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}}\right\}^{-\frac{1}{r}}$.

To find the blow-up time,

$$\begin{aligned}
&\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} + \frac{1}{\tau_0^r} - \frac{a^2\epsilon^{4r+2}}{4} = 0,\\
&\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} = -\frac{1}{\tau_0^r} + \frac{a^2\epsilon^{4r+2}}{4},\\
&\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} = \frac{-4 + a^2\tau_0^r\epsilon^{4r+2}}{4\tau_0^r},\\
&e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} = \frac{\frac{-4 + a^2\tau_0^r\epsilon^{4r+2}}{4\tau_0^r}}{\frac{a^2\epsilon^{4r+2}}{4}} = \frac{-4 + a^2\tau_0^r\epsilon^{4r+2}}{\tau_0^r a^2\epsilon^{4r+2}} = 1 - \frac{4}{a^2\tau_0^r\epsilon^{4r+2}},\\
&\frac{-4rt}{(x_0+\epsilon)^q\epsilon^2} = \ln\left(1 - \frac{4}{a^2\tau_0^r\epsilon^{4r+2}}\right).
\end{aligned}$$

Therefore, $t = \frac{-(x_0+\epsilon)^q\epsilon^2}{4r}\ln\left(1 - \frac{4}{a^2\tau_0^r\epsilon^{4r+2}}\right)$.

Since $\tau_2(0) \equiv \tau_0 > \max\left\{\frac{1}{\epsilon^4}, \left(\frac{4}{a^2\epsilon^{4r+2}}\right)^{\frac{1}{r}}\right\}$, then $\frac{4}{a^2\tau_0^r\epsilon^{4r+2}} < 1$.

Therefore, $\tau_2(t)$ blows up at the time $t_{b_1} = -\frac{(x_0+\epsilon)^q\epsilon^2}{4r}\ln\left(1 - \frac{4}{a^2\tau_0^r\epsilon^{2+4r}}\right) > 0$.
Since

$$Lu = a^2 f(u(x_0,t)) = a^2 f'(\xi)u(x_0,t),$$

where ξ lies between 0 and $u(x_0,t)$, it follows from Lemma 3.1 that $u(x,t) \geq 0$. Therefore, if we choose $u_0(x)$ such that $u_0(x) \geq \mu_2(x,0)$ on $[x_0-\epsilon, x_0+\epsilon]$, then by Lemma 1 of

Chan and Yang [CY00], $u(x,t) \geq \mu_2(x,t)$. Hence, the solution $u(x,t)$ of the problem (1.3) blows up no later than t_{b_1} whenever $u_0(x) \geq \mu_2(x,0)$.

□

3.3 Complete Blow-Up

Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.3) is determined by the following system: for x and ξ in D, and t and τ in $(-\infty,\infty)$,

$$\begin{aligned} &LG = \delta(x-\xi)\delta(t-\tau), \\ &G(x,t;\xi,\tau) = 0 \quad \text{for} \quad t < \tau, \\ &G(0,t;\xi,\tau) = 0 = G_x(1,t;\xi,\tau), \end{aligned}$$

where $\delta(x)$ is the Dirac delta function.

Definition of the *Dirac delta function* δ:

We define the *delta function*, or more accurately the *delta distribution* [McO97], in $\mathbf{R}^n$ to be object $\delta(x)$ so that formally

$$\int_{\mathbf{R}^n} \delta(x)v(x)dx = v(0)$$

for every test function $v \in C_0^\infty(\mathbf{R}^n)$. $C_0^\infty(\mathbf{R}^n)$ denotes the space of continuous functions with continuous derivatives on $\mathbf{R}^n$ whose support is a compact subset of $\mathbf{R}^n$. The support of a continuous function $f(x)$ defined on $\mathbf{R}^n$ is the closure of the set of points where $f(x)$ is nonzero [McO97]. In the one-dimensional case we find $H'(x) = \delta(x)$, where the Heaviside function of a single real variable is:

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Notice that we can take any number of distributional derivatives of $\delta(x)$. Also, we can translate the singularity in $\delta(x)$ to any point $\mu \in \mathbf{R}^n$ by letting $\delta_\mu(x) = \delta(x-\mu)$ so that a change of variables $y = x - \mu$ yields

$$\int_{\mathbf{R}^n} \delta_\mu v(x)dx = \int_{\mathbf{R}^n} \delta(x-\mu)v(x)dx = \int_{\mathbf{R}^n} \delta(y)v(y+\mu)dx = v(\mu).$$

Lemma 3.7. *If h(x,t) is nontrivial (not indentically zero) such that* $0 \leq h(x,t) \leq 1$ *and* $h(x,t) \in C^{\infty}(\boldsymbol{R}^2)$, *then for any finite* θ, *the degenerate linear parabolic problem,*

$$\begin{cases} Lv = h(x,t) \quad in \quad D \times (0,\theta], \\ v(x,0) = 0 \quad on \quad \bar{D}, \\ v(0,t) = 0 = v_x(1,t) \quad for \quad 0 < t \leq \theta, \end{cases} \tag{3.9}$$

has a unique solution $v(x,t) \in C(\bar{D} \times [0,\theta]) \cap C^{2,1}((0,1] \times [0,\theta])$.

Proof. Let δ be some positive constant such that $\delta < x_0$. We consider the problem,

$$\begin{cases} Lv_\delta = h(x,t) \quad \text{in} \quad D_\delta \times (0,\theta], \\ v_\delta(x,0) = 0 \quad \text{on} \quad \bar{D}_\delta, \\ v_\delta(\delta,t) = 0 = v_{\delta_x}(1,t) \quad \text{for} \quad 0 < t \leq \theta. \end{cases} \tag{3.10}$$

We would like to construct an upper solution $\mu_3(x,t) \in C^{2,1}(\bar{D} \times [0,\theta])$ in the form $\theta_3(x)\tau_3(t)$ for v and all v_δ given by (3.9) and (3.10) respectively. Let $\theta_3(x) = xe^{1-x}$, $\theta_3'(x) = e^{1-x} - xe^{1-x}$, $\theta_3''(x) = e^{1-x}(x-2)$, and ϵ be a fixed positive number less than x_0, and

$$\tau_3' = \epsilon^{-q}(1+\epsilon^{-1})\tau_3, \tau_3(0) = 1.$$

It follows that

$$\tau_3(t) = e^{\epsilon^{-q}(1+\epsilon^{-1})t} \geq 0,$$

which is increasing and bounded for any $t \leq \theta$.

Let $\mu_3(x,t) = \theta_3(x)\tau_3(t)$. Then,

$$\begin{aligned} \mu_3(x,t) &= \theta_3(x)\tau_3(t) \\ &= (xe^{1-x})(e^{\epsilon^{-q}(1+\epsilon^{-1})t}) \in C^{2,1}(\bar{D} \times [0,\theta]). \end{aligned}$$

For any $x \in [0,\epsilon]$, and any $t \leq \theta$,

$$\begin{aligned} L\mu_3 - h(x,t) &= L(\theta_3(x)\tau_3(t)) - h(x,t) \\ &= x^q(xe^{1-x})\epsilon^{-q}(1+\epsilon^{-1})\tau_3 - e^{1-x}(x-2)\tau_3 - h(x,t) \\ &\geq -e^{1-x}(x-2)\tau_3 - 1 \\ &\geq e^{1-x}\tau_3\left(-\epsilon + 2 - \frac{1}{e^{1-x}\tau_3}\right) \\ &\geq e^{1-x}\tau_3(-\epsilon + 2 - 1) \\ &= (1-\epsilon)e^{1-x}\tau_3 > 0. \end{aligned}$$

Since $\tau_3(t)$ is an increasing function, we have for any $x \in (\epsilon, 1]$ and any $t \le \theta$,

$$\begin{aligned} L\mu_3 - h(x,t) &= x^q(xe^{1-x})\epsilon^{-q}(1+\epsilon^{-1})\tau_3 - e^{1-x}(x-2)\tau_3 - h(x,t) \\ &\ge \epsilon^q xe^{1-x}\tau_3' - xe^{1-x}\tau_3 - 1 \\ &= \epsilon^q xe^{1-x}\left(\tau_3' - \epsilon^{-q}\tau_3 - \frac{\epsilon^{-q}\tau_3}{e^{1-x}x\tau_3}\right) \\ &\ge \epsilon^q xe^{1-x}\left(\tau_3' - \epsilon^{-q}\tau_3 - \frac{\epsilon^{-q}\tau_3}{e^{1-x}\epsilon\tau_3}\right) \\ &\ge \epsilon^q xe^{1-x}(\tau_3' - \epsilon^{-q}\tau_3 - \epsilon^{-q-1}\tau_3) \\ &= 0. \end{aligned}$$

Since $\mu_3(x,0) = \theta_3(x)\tau_3(0) = xe^{1-x}\cdot 1 = xe^{1-x} \ge 0$, $\mu_3(0,t) = \theta_3(0)\tau_3(t) = 0$, and $\mu_{3_x}(1,t) = \theta_3'(1)\tau_3(t) = (e^{1-1} - e^{1-1})\tau_3(t) = 0$, it follows from the strong maximum principle [Fri64] and the parabolic version of Hopf's lemma [Fri64] that $\mu_3(x,t)$ is an upper solution for all v_δ and v.

We note that $x^{-q} \in C^{\alpha,\alpha/2}(\bar{D}_\delta \times [0,\theta])$, $|x^{-q}h| \le \delta^{-q}$ for $(x,t,v_\delta) \in \bar{D}_\delta \times [0,\theta] \times R$, and $v_{0_\delta}(x) = 0 \in C^{2+\alpha}(\bar{D}_\delta)$.

Definition of *Hölder Continuity*:

If $0 \le \alpha \le 1$ and u is defined and continuous in a neighborhood U of x_0, then we can say that u is *Hölder continuous* at x_0 with exponent α if

$$[u]_{\alpha;x_0} \equiv \sup_{x\in U}\frac{|u(x)-u(x_0)|}{|x-x_0|^\alpha} < \infty.$$

Here $x_0 \in \bar{D}_\delta = [\delta, 1]$.

Let us prove that $x^{-q} \in C^{\alpha,\alpha/2}(\bar{D}_\delta \times [0,\theta])$. By the mean value theorem, $f(u) - f(v) = f'(\xi)(u-v)$. Let $f(x) = x^{-q}$, and $f(x_0) = x_0^{-q}$,

$$\begin{aligned} \left|x^{-q} - x_0^{-q}\right| &= |f'(\xi)(x-x_0)| \\ &= \left|-q\xi^{-q-1}(x-x_0)\right| \\ &= \frac{q}{\xi^{q+1}}(x-x_0) < \infty. \end{aligned}$$

So

$$\begin{aligned} [u]_{\alpha;x_0} &\equiv \sup_{x\in U}\frac{|u(x)-u(x_0)|}{|x-x_0|^\alpha} \\ &\equiv \sup_{x\in U}\frac{\left|x^{-q}-x_0^{-q}\right|}{|x-x_0|^\alpha} < \infty. \end{aligned}$$

By Theorem A.4.1 of Ladde, Lakshmikantham and Vatsala [LLV85] (Appendix A), the problem (3.10) has a unique solution $v_\delta \in C^{2+\alpha,1+\alpha/2}(\bar{D}_\delta \times [0,\theta])$.

By the strong maximum principle (Appendix A), $v_\delta \geq 0$ on $\bar{D}_\delta \times [0,\theta]$. Let (A), (B), (C) hold. $Lu \leq 0$ means $u \geq 0$, so $Lv_\delta \leq 0$ means $v_\delta \geq 0$, on $\bar{D}_\delta \times [0,\theta]$.
For $\delta_1 \leq \delta_2$,

$$\begin{gathered} L(v_{\delta_1} - v_{\delta_2}) = 0 \text{ for } x \in (\delta_2, 1) \\ v_{\delta_1}(x,0) = v_{\delta_2}(x,0) \text{ for } x \in [\delta_2, 1] \\ v_{\delta_1}(\delta_2,t) - v_{\delta_2}(\delta_2,t) \geq 0, \quad v_{\delta_{1_x}}(1,t) - v_{\delta_{2_x}}(1,t) = 0, \quad 0 < t \leq \theta. \end{gathered}$$

By Lemma 3.1, $v_{\delta_1} \geq v_{\delta_2}$ on $\bar{D}_{\delta_2} \times [0,\theta]$, Thus, $\lim_{\delta\to 0} v_\delta$ exists, since v_δ is bounded by upper solution μ_3 and is monotone because $v_{\delta_1} \geq v_{\delta_2}$.

Let $v(x,t) = \lim_{\delta\to 0} v_\delta(x,t)$. A proof similar to that of Theorem 3.5 shows that $v(x,t)$ is a solution of the problem (3.9), and $v(x,t) \in C(\bar{D} \times [0,\theta]) \cap C^{2,1}((0,1] \times [0,\theta])$.

Let's prove that $v(x,t)$ is unique. Let $y_1 = v_1 - v_2$, and $y_2 = v_2 - v_1$. We have

$$\begin{cases} Lv_1 = x^q v_{1_t} - v_{1_{xx}} = h(x,t) \quad \text{in} \quad D \times (0,\theta], \\ v_1(x,0) = 0 \quad \text{on} \quad \bar{D}. \\ v_1(0,t) = 0 = v_{1_x}(1,t) \quad \text{for} \quad 0 < t \leq \theta. \end{cases} \tag{3.11}$$

$$\begin{cases} Lv_2 = x^q v_{2_t} - v_{2_{xx}} = h(x,t) \quad \text{in} \quad D \times (0,\theta], \\ v_2(x,0) = 0 \quad \text{on} \quad \bar{D}. \\ v_2(0,t) = 0 = v_{2_x}(1,t) \quad \text{for} \quad 0 < t \leq \theta. \end{cases} \tag{3.12}$$

Subtracting equations (3.12) from (3.11), we get

$$\begin{cases} Ly_1 = L(v_1 - v_2) = 0 \\ v_1(x,0) - v_2(x,0) = 0, \\ v_1(0,t) - v_2(0,t) = 0, \\ v_{1_x}(1,t) - v_{2_x}(1,t) = 0. \end{cases}$$

Subtracting equations (3.11) from (3.12), we get

$$\begin{cases} Ly_2 = L(v_2 - v_1) = 0 \\ v_2(x,0) - v_1(x,0) = 0, \\ v_2(0,t) - v_1(0,t) = 0, \\ v_{2_x}(1,t) - v_{1_x}(1,t) = 0. \end{cases}$$

By Lemma 3.1, $y_1 \geq 0$, then $v_1 - v_2 \geq 0$, so $v_1 \geq v_2$, and $y_2 \geq 0$, then $v_2 - v_1 \geq 0$, so $v_2 \geq v_1$, hence $v_1 \geq v_2$ and $v_2 \geq v_1$, thus $v_1 = v_2$. Therefore, $v(x,t)$ is a unique solution. □

Lemma 3.8. *Given any $x \in (0,1]$ and any finite time θ, there exist positive constants k_4 (depending on x and θ) and k_5 (depending on θ) such that*

$$k_4 < \int_0^1 G(x,t;\xi,\tau)d\xi \quad \text{for} \quad 0 \leq t \leq \theta,$$
$$\int_0^1 G(x_0,t;\xi,\tau)d\xi < k_5 \quad \text{for} \quad 0 \leq t \leq \theta.$$

Proof. By Lemma 3.7, the problem,

$$Lv = 1 \text{ in } D \times (0,\theta], v(x,0) = 0 \text{ on } \bar{D}, v(0,t) = 0 = v_x(1,t) \text{ for } 0 < t \leq \theta,$$

has a unique solution $v \in C(\bar{D} \times [0,\theta]) \cap C^{2,1}((0,1] \times [0,\theta])$. The adjoint operator L^* of L is given by

$$L^* u = -x^q u_t - u_{xx}.$$

Definition: The *operator* L^* is called the *adjoint* of L, and is an m-th order linear differential operator with continuous coefficients.

Let $Lu = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u$, where $a_\alpha \in C^{|\alpha|}(\Omega)$, $u \in C^m(\Omega)$, and $v \in C_0^m(\Omega)$. Then $\int_\Omega (D^\alpha u) v dx = (-1)^m \int_\Omega u D^\alpha v dx$, where $m = |\alpha|$,

$$L^* v = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) v).$$

Therefore, $\int_\Omega (Lu) v dx = \int_\Omega u (L^* v) dx$ [McO97].

Using Green's second identity, we obtain

$$v(x,t) = \int_0^t \int_0^1 G(x,\xi,t-\tau) d\xi d\tau = \int_0^t \int_0^1 G(x,\xi,t) d\xi d\tau.$$

This gives

$$v_t = \int_0^1 G(x,\xi,t) d\xi,$$

corresponding to the problem

$$\begin{cases} Lv_t = 0 \quad \text{in} \quad \Omega, \\ v_t(x,0) = x^{-q} \quad \text{for} \quad 0 < x \leq 1, \\ v_t(0,t) = 0 = v_{t_x}(1,t) \quad \text{for} \quad 0 < t < \infty. \end{cases}$$

By using the strong maximum principle [Fri64] and the parabolic version of Hopf's lemma [Fri64] (Appendix A), $v_t > 0$ in $(0,1] \times [0,\theta]$. Thus for any finite time θ, there exists a positive constant k_4 (depending on x and θ) such that

$$\int_0^1 G(x,\xi,t)d\xi > k_4 \text{ for } 0 \leq t \leq \theta.$$

Since $v(x,t) \in C(\bar{\Omega}) \cap C^{2,1}((0,1] \times [0,\theta])$, there exists a positive constant k_5 (depending on θ) such that

$$\int_0^1 G(x_0,t;\xi)d\xi < k_5 \text{ for } 0 \leq t \leq \theta.$$

□

Our next result gives the complete blow-up of the solution u.

Theorem 3.9. *If the solution of the problem (1.3) blows up in a finite time t_b, then the blow-up set is $\bar{D}$.*

Proof. By Green's second identity,

$$u(x,t) \;=\; \int_0^t \int_0^1 a^2 f(u(x_0,\tau))G(x,\xi,t-\tau)d\xi d\tau + \int_0^t \xi^q u_0(\xi)G(x,\xi,t)d\xi,$$

for any $t < t_b$. If the solution of the problem (1.3) blows up in a finite time t_b, then by Theorem 3.5, u blows up at the point x_0. We know that maximum of ξ^q is 1, maximum of $\int_0^1 u_0(\xi)d\xi$ is $\max_{x\in\bar{D}} u_0(x)$, $\int_0^1 G(x_0,t;\xi)d\xi < k_5$, for $0 \leq t \leq \theta$, and $a^2 \int_0^t \int_0^1 G(x_0,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \leq k_5 a^2 \int_0^t f(u(x_0,t-\tau))d\tau$. It follows from Lemma 3.8 that for any $t < t_b$,

$$\begin{aligned} u(x_0,t) &= \int_0^1 \xi^q G(x_0,\xi,t)u_0(\xi)d\xi + a^2 \int_0^t \int_0^1 G(x_0,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \\ &\leq k_5 \Big(\max_{x\in\bar{D}} u_0(x) + a^2 \int_0^t f(u(x_0,t-\tau))d\tau \Big). \end{aligned}$$

Since $u(x_0,t) \to \infty$ as $t \to t_b^-$, we have

$$\int_0^{t_b} f(u(x_0,t_b-\tau))d\tau = \infty.$$

On the other hand, for any $(x,t) \in (0,1] \times [0,t_b)$, since $\int_0^1 G(x,\xi,t)d\xi > k_4$ for $0 \le t \le \theta$,

we have

$$\begin{aligned} u(x,t) &= \int_0^1 \xi^q G(x,\xi,t)u_0(\xi)d\xi + a^2\int_0^t\int_0^1 G(x,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \\ &> a^2\int_0^t\int_0^1 G(x,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \ge a^2k_4\int_0^t f(u(x_0,t-\tau))d\tau, \end{aligned}$$

which tends to infinity as t approaches t_b^-. For $x=0$, we can always find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n) \to (0,t_b)$ and $\lim_{n\to\infty} u(x_n,t_n) \to \infty$. Thus, the blow-up set is $\bar{D}$.

□

Appendix A

1. **The Maximum Principle** [Fri64]

   Consider the operator

$$Nu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t}$$

   in an $(n+1)$-dimensional domain Ω with the following assumptions:

   (A) N is parabolic in Ω, i.e., for every $(x,t) \in \Omega$ and for any real vector $\xi \neq 0, \sum a_{ij}(x,t)\xi_i\xi_j > 0$;

   (B) the coefficients of N are continuous functions in Ω;

   (C) $c(x,t) \leq 0$ in Ω.

   The functions u are always assumed to have two continuous x-derivatives and one continuous t-derivative in Ω.

   Definition A.1. *Notation: For any point $P^0 = (x^0, t^0)$ in D, we denote by $S(P^0)$ the set of all points Q in D which can be connected to P^0 by a simple continuous curve in D along which the t-coordinate is nondecreasing from Q to P^0. By $C(P^0)$, we denote the component (in $t = t^0$) of $D \cap \{t = t^0\}$ which contains P^0. Note that $S(P^0) \supset C(P^0)$.*

   Theorem A.2. *Let (A), (B), (C) hold. If $Lu \geq 0 \quad (Lu \leq 0)$ in D and if u has in D a positive maximum (negative minimum) which is attained at a point $P^0(x^0, t^0)$, then $u(P) = u(P^0)$ for all $P \in S(P^0)$.*

2. **Extensions of the Maximum Principle** [Fri64]

Theorem A.3. *Let (A), (B) hold. If* $u \le 0 \quad (u \ge 0)$ *in* $S(P^0), Lu \ge 0 \quad (Lu \le 0)$ *in* $S(P^0)$ *and* $u(P^0) = 0$, *then* $u \equiv 0$ *in* $S(P^0)$.

3. **Hopf's Lemma** [Fri64]

Definition A.4. *Let* $P^0 = (x^0, t^0)$ *be a point on the boundary* $\partial\Omega$ *of a domain* Ω. *If there exists a closed ball* B *with center* $(\bar{x}, \bar{t})$ *such that* $B \subset \bar{\Omega}, B \cap \partial\Omega = \{P^0\}$, *and if* $\bar{x} \neq x^0$, *then we say that* P^0 *has the inside strong sphere property.*

Lemma A.5. *Let* P^0 *have the inside strong sphere property. Assume further that, for some neighborhood* V *of* $P^0, u < M$ *in* $D \cap V$. *Then, for any non-tangential inward direction* τ,

$$\frac{\partial u}{\partial \tau} \equiv lim_{\Delta\tau \to 0} inf\left(\frac{\Delta u}{\Delta \tau}\right) < 0 \quad \text{at } P^0.$$

By a non-tangential inward direction we mean direction pointing from P^0 *into the interior of the ball* B *whose boundary touches* ∂D *at* P^0.

4. **The Mean Value Theorem** [LHE06]

Theorem A.6. *Let* f *be a continuous function on* $[a, b]$ *that is differentiable on* (a, b). *Then there exists at least one point* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

5. **Parabolic Equations** [LLV85]

Let $\mathfrak{L}$ be a differential operator defined by

$$\mathfrak{L} = -N,$$

where N is defined on page 36. Let $f \in C^{\alpha/2,\alpha}[[0,T] \times \bar{\Omega} \times R \times R^m, R]$, that is $f(t, x, u, y)$ is Hölder continuous in t and (x, u, y) with exponent $\alpha/2$ and α, respectively, where $0 < \alpha < 1$.

Consider the linear second order parabolic initial boundary value problem (IBVP for short)

$$\begin{cases} \mathfrak{L}u = F(t,x), \quad (t,x) \in Q_T, \\ (Bu)(t,x) = \phi(t,x), \quad (t,x) \in \Gamma_T, \\ u(0,x) = \phi_0(x), \quad x \in \bar{\Omega}, \end{cases}$$

where B is defined by $Bu = p(t,x)u + q(t,x)du/d\nu$ and $du/d\nu$ stands for the normal derivative of u; Ω is a bounded domain; $Q_T = (0,T) \times \Omega$; $\Gamma_T = (0,T) \times \partial\Omega$; and $D_x u = (\partial u/\partial x_1, ..., \partial u/\partial x_m)$.

Theorem A.7. *Assume that*

(a1) a_{ij}, b_i, c *and* $F \in C^{\alpha/2,\alpha}[\bar{\Omega}_T, R], c(t,x) \le 0$ *and* $\mathfrak{L}$ *is strictly uniformly parabolic in* Q_T*;*

(a2) $p, q \in C^{(1+\alpha)/2,1+\alpha}[\bar{\Gamma}_T, R], p$ *and* q *are nonnegative functions which do not vanish simultaneously;*

(a3) $\partial\Omega$ *belongs to class* $C^{2+\alpha}$*;*

(a4) $\phi \in C^{(1+\alpha)/2,1+\alpha}[\bar{\Gamma}_T, R]$ *and* $\phi_0 \in C^{2+\alpha}[\bar{\Omega}, R]$*;*

(a5) the IBVP

$$\begin{cases} \mathfrak{L}u = F(t,x), \quad (t,x) \in Q_T, \\ (Bu)(t,x) = \phi(t,x), \quad (t,x) \in \Gamma_T, \\ u(0,x) = \phi_0(x), \quad x \in \bar{\Omega}, \end{cases}$$

satisfies the compatibility condition of order $[(1+\alpha)/2]$.

Then this linear parabolic IBVP has a unique solution u *such that* $u \in C^{1+\alpha/2,2+\alpha}[\bar{Q}_T, R]$. .

6. **The Ascoli-Arzela Theorem** [Col88]

Definition A.8. *: **Equicontinuity***

A set of functions $\{\psi_j(s)\}$ *is said to be equicontinuous on an interval* $[0,l]$ *if for every* $\epsilon > 0$ *there exists a number* $\delta = \delta(\epsilon)$ *independent of* j *such that*

$$|\psi_j(s_1) - \psi_j(s_2)| < \epsilon \quad for \quad s_1, s_2 \in [0,l], \quad |s_1 - s_2| < \delta.$$

Definition A.9. *A set of functions* $\{\psi_j(s)\}$ *defined on* $[0, l]$ *is said to be uniformly bounded if there exists a constant* M *independent of* j *such that*

$$max_{0 \leq s \leq l} |\psi_j(s)| \leq M.$$

Theorem A.10. *Let* $\{\psi_j(s)\}$ *be a set of uniformly bounded and equicontinuous functions defined on an interval* $[0, l]$*. Then there exists a subsequence of* $\{\psi_j(s)\}$ *that is uniformly convergent on* $[0, l]$*.*

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