SERRIN'S TYPE OVERDETERMINED PROBLEMS IN CONVEX CONES

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ABSTRACT. We consider overdetermined problems of Serrin's type in convex cones for (possibly) degenerate operators in the Euclidean space as well as for a suitable generalization to space forms. We prove rigidity results by showing that the existence of a solution implies that the domain is a spherical sector.

1. INTRODUCTION

Given a bounded domain $E \subset \mathbb{R}^N$, $N \geq 2$, the classical Serrin's overdetermined problem [40] asserts that there exists a solution to

$$\begin{cases} \Delta u = -1 & \text{in } E\,, \\ u = 0 & \text{on } \partial E\,, \\ \partial_\nu u = -c & \text{on } \partial E\,, \end{cases} \tag{1}$$

for some constant $c > 0$, if and only if $E = B_R(x_0)$ is a ball of radius R centered at some point x_0. Moreover, the solution u is radial and it is given by

$$u(x) = \frac{R^2 - |x - x_0|^2}{2N}\,, \tag{2}$$

with $R = Nc$. Here, ν denotes the outward normal to $\partial\Omega$.

The starting observation of this manuscript is the following. Let Σ be an open cone in $\mathbb{R}^N$ with vertex at the origin O, i.e.

$$\Sigma = \{tx \,:\, x \in \omega,\, t \in (0, +\infty)\}$$

for some open domain $\omega \subset \mathbb{S}^{N-1}$. We notice that if x_0 is chosen appropriately then u given by (2) is still the solution to

$$\begin{cases} \Delta u = -1 & \text{in } B_R(x_0) \cap \Sigma\,, \\ u = 0 \text{ and } \partial_\nu u = -c & \text{on } \partial B_R(x_0) \setminus \overline{\Sigma}\,, \\ \partial_\nu u = 0 & \text{on } B_R(x_0) \cap \partial\Sigma\,. \end{cases}$$

More precisely, x_0 may coincide with O or it may be just a point of $\partial\Sigma \setminus \{O\}$ and, in this case, $B_R(x_0) \cap \Sigma$ is half a sphere lying over a flat portion of $\partial\Sigma$. Hence, it is natural to look for a characterization of symmetry in this direction, as done in [35] (see below for a more detailed description).

In order to properly describe the results, we introduce some notation. Given an open cone Σ such that $\partial\Sigma \setminus \{O\}$ is smooth, we consider a bounded domain $\Omega \subset \Sigma$ and denote by Γ_0 its relative boundary, i.e.

$$\Gamma_0 = \partial\Omega \cap \Sigma\,,$$

and we set

$$\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}_0\,.$$

We assume that $\mathcal{H}_{N-1}(\Gamma_1) > 0$, $\mathcal{H}_{N-1}(\Gamma_0) > 0$ and that Γ_0 is a smooth $(N-1)$-dimensional manifold, while $\partial\Gamma_0 = \partial\Gamma_1 \subset \partial\Omega \setminus \{O\}$ is a smooth $(N-2)$-dimensional manifold. Following

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[35], such a domain Ω is called a *sector-like domain*. In the following, we shall write $\nu = \nu_x$ to denote the exterior unit normal to $\partial\Omega$ wherever is defined (that is for $x \in \Gamma_0 \cup \Gamma_1 \setminus \{O\}$).

Under the assumption that Σ is a convex cone, in [35] it is proved that if Ω is a sector-like domain and there exists a classical solution $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\})$ to

$$\begin{cases} \Delta u = -1 & \text{in } \Omega\,, \\ u = 0 \text{ and } \partial_\nu u = -c & \text{on } \Gamma_0\,, \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}\,, \end{cases} \tag{3}$$

and such that

$$u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)\,,$$

then

$$\Omega = B_R(x_0) \cap \Sigma$$

for some $x_0 \in \mathbb{R}^N$ and u is given by (2). Differently from the original paper of Serrin [40], the method of moving planes is not helpful (at least when applied in a standard way) and the rigidity result in [35] is proved by using two alternative approaches. One is based on integral identities and it is inspired from [5], the other one uses a P-function approach as in [42].

In this paper, we generalize the rigidity result for Serrin's problem in [35] in two directions. The former is by considering more general operators than the Laplacian in the Euclidean space, where the operators may be of degenerate type. Here, the generalization is not trivial due to the lack of regularity of the solution (the operator may be degenerate) as well as to other technical details which are not present in the linear case.

The latter is by considering an analogous problem in space forms, i.e. the hyperbolic space and the (hemi)sphere. The operator that we consider is linear and it is interesting since it has been shown that it is a helpful generalization of the torsion problem to space forms ([15], [36], [37]).

Overdetermined problems for quasilinear and possible degenerate operators have attracted a lot of interest in the last decades, see for instance [26, 25, 19, 27, 38]. As Fosdick and Serrin noticed in [40] and [24], Serrin's overdetermined problem for quasilinear elliptic operators is also interesting for possible applications to the study of steady rectilinear motion of viscous incompressible fluids and incompressible non-Newtonian fluids (see also [26]), and in the theory of torsion of a solid straight bar. Roughly speaking, a rigidity result as the one given by Serrin proves that *the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section* or that *when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section.* There are other possible applications for Serrin's type rigidity results, and we refer to [25, Introduction] for connections to capillarity theory, torsional creep, Born-Infeld theory and other applications to quantum-physics.

As explained in [35], the study of Serrin's overdetermined problem in convex cones is related to relative isoperimetric inequality and Alexandrov soap bubble theorem. In this manuscript we extend this study to non-Euclidean manifolds, in particular to space forms. The study of isoperimetric inequality and Alexandrov theorem in non-Euclidean manifolds has recently attracted a lot of interest in the geometric analysis community (see [31, 36, 7] and references therein). We believe that, by taking inspiration from our results and the ones in [31, 36], one can study Alexandrov theorem and relative isoperimetric inequalities for sector-like domains in more general Riemannian settings.

The study of rigidity problems in convex cones appears also in the context of critical points for Sobolev inequality (which in turns can be related to Yamabe problem), see [12, 32]. Indeed, the study started in this manuscript served as inspiration for [12], where we characterized, together with A. Figalli, the solutions of critical anisotropic p-Laplace type equations in convex cones.

We also mention that the approach used in this paper originated from [5], which in turns has been later used for proving quantitative estimates for Serrin's overdetermined problem in [6]. As for the symmetry result, this approach is also useful when considering quantitative versions of Alexandrov soap bubble theorem, in particular to describe the appearance of bubbling [13].

More general operators in the Euclidean space. Let Ω be a sector like domain in $\mathbb{R}^N$ and let $f : [0,+\infty) \to [0,+\infty)$ be such that

$$\begin{gathered} f \in C^1([0,\infty)) \cap C^3((0,\infty)) \text{ with } f(0) = f'(0) = 0,\ f''(s) > 0 \text{ for } s > 0 \\ \text{and } \lim_{s\to+\infty} \frac{f(s)}{s} = +\infty\,. \end{gathered} \tag{4}$$

We consider the following mixed boundary value problem

$$\begin{cases} L_f u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases} \tag{5}$$

where the operator L_f is given by

$$L_f u = \operatorname{div}\left(f'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right), \tag{6}$$

and the equation $L_f u = -1$ is understood in the sense of distributions

$$\int_\Omega \frac{f'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, dx$$

for any

$$\varphi \in T(\Omega) := \{\varphi \in C^1(\Omega) \,:\, \varphi \equiv 0 \ \ on\ \Gamma_0\}.$$

Notice that the operator L_f may be of degenerate type.

We notice that the solution to $L_f u = -1$ in $B_R(x_0)$ (a ball of radius R centered at x_0) such that $u = 0$ on $\partial B_R(x_0)$ is radial and it is given by

$$u(x) = \int_{|x-x_0|}^R g'\left(\frac{s}{N}\right) ds\,, \tag{7}$$

where g denotes the Fenchel conjugate of f (see for instance [17] or [25]), i.e.

$$g = \sup\{st - f(s) \,:\, s \geq 0\}$$

(hence for us g' is the inverse function of f'). Our first main result is the following.

Theorem 1. *Let f satisfy* (4)*. Let Σ be a convex cone such that $\Sigma \setminus \{O\}$ is smooth and let Ω be a sector-like domain in Σ. If there exists a solution $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega)$ to* (5) *such that*

$$\partial_\nu u = -c \ \ on\ \Gamma_0 \tag{8}$$

for some constant c, and satisfying

$$\frac{f'(|\nabla u|)}{|\nabla u|} \nabla u \in W^{1,2}(\Omega, \mathbb{R}^N)\,, \tag{9}$$

then there exists $x_0 \in \mathbb{R}^N$ such that $\Omega = \Sigma \cap B_R(x_0)$ with $c = g'(|\Omega|/|\Gamma_0|)$, $R = N|\Omega|/|\Gamma_0|$. Moreover u is given by (7)*, where x_0 is the origin or, if $\partial\Sigma$ contains flat regions, it is a point on $\partial\Sigma$.*

When $L_f = \Delta$ (i.e. $f(t) = t^2/2$), Theorem 1 is essentially Theorem 1.1 in [35]. Condition (9) holds locally in Ω for a large class of elliptic operators, such as the mean curvature operator ($f(t) = \sqrt{1+t^2}$), and for the $p-$Laplace operator ($f(t) = t^p/p$, $p > 1$), see [2, Theorem] and [10, Theorem 2.1]. We stress that the validity of (9) up to the boundary is more subtle, since it depends strongly on how Γ_0 and Γ_1 intersect. Some global results may be obtained by following the approach in [10], where (9) is proved for Dirichlet or Neumann boundary value problems of p-Laplace type in domains which are convex or satisfying minimal regularity assumptions on the boundary.

We observe that the overdetermined problem (5) with the condition (8) can be seen as a partially overdetermined problem (see for instance [20] and [21]), since we impose both Dirichlet and Neumann conditions only on a part of the boundary, namely Γ_0, while a sole homogeneous

Neumann boundary condition is assigned on Γ_1 (where, however, there is the strong assumption that it is contained in the boundary of a cone).

We notice that the proof of Theorem 1 still works when $\Gamma_1 = \emptyset$ (hence $\partial\Omega = \Gamma_0$). In this case we obtain the celebrated result of Serrin [40] for the operator L_f (see also [4], [5], [14], [25], [19], [26], [39], [42]). Moreover, the proof is also suitable to be adapted to the anisotropic counterpart of the overdetermined problem (5) and (8) by following the approach used in this manuscript and in [4] (see also [11] and [41]). We also mention that rigidity theorems in cones are related to the study of relative isoperimetric and Sobolev inequalities in cones, and we refer to [35] for a more detailed discussion (see also [3, 9, 23, 29, 32, 33]).

Serrin's problem in cones in space forms. A space form is a complete simply-connected Riemannian manifold (M, g) with constant sectional curvature K. Up to homotheties we may assume $K = 0,1,-1$: the case $K = 0$ corresponds to the Euclidean space $\mathbb{R}^N$, $K = -1$ is the hyperbolic space $\mathbb{H}^N$ and $K = 1$ is the unitary sphere with the round metric $\mathbb{S}^N$. More precisely, in the case $K = 1$ we consider the hemisphere $\mathbb{S}^N_+$. These three models can be described as warped product spaces $M = I \times \mathbb{S}^{N-1}$ equipped with the rotationally symmetric metric

$$g = dr^2 + h(r)^2\, g_{\mathbb{S}^{N-1}},$$

where $g_{\mathbb{S}^{N-1}}$ is the round metric on the $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$ and

- $h(r) = r$ in the Euclidean case ($K = 0$), with $I = [0, \infty)$;
- $h(r) = \sinh(r)$ in the hyperbolic case ($K = -1$), with $I = [0, \infty)$;
- $h(r) = \sin(r)$ in the spherical case ($K = 1$), with $I = [0, \pi/2)$ for $\mathbb{S}^N_+$.

By using the warping structure of the manifold, we denote by O the pole of the model and it is natural to define a *cone* Σ *with vertex at* $\{O\}$ as the set

$$\Sigma = \{tx \, : \, x \in \omega, \, t \in I\}$$

for some open domain $\omega \subset \mathbb{S}^{N-1}$. Moreover, we say that Σ is a *convex cone* if the second fundamental form II is nonnegative defined at every $p \in \partial\Sigma$.

Serrin's overdetermined problem for semilinear equations $\Delta u + f(u) = 0$ in space forms has been studied in [30] and [34] by using the method of moving planes. If one considers the corresponding problem for sector-like domains in space forms, the method of moving planes can not be used and one has to look for alternative approaches. As already mentioned, in the Euclidean space these approaches typically use integral identities and P-functions (see [5, 42]) and have the common feature that at a crucial step of the proof they use the fact that the radial solution attains the equality sign in a Cauchy-Schwartz inequality, which implies that the Hessian matrix $\nabla^2 u$ is proportional to the identity. Since the equivalent crucial step in space forms is to prove that the Hessian matrix of the solution is proportional to the metric, then the equation $\Delta u = -1$ is no more suitable (one can easily verify that in the radial case the Hessian matrix of the solution is not proportional to the metric) and a suitable equation to be considered is

$$\Delta u + NKu = -1 \tag{10}$$

as done in [15] and [36], [37]. It is clear that for $K = 0$, i.e. in the Euclidean case, the equation reduces to $\Delta u = -1$. For this reason, we believe that, in this setting, (10) is the natural generalization of the Euclidean $\Delta u = -1$ to space forms.

A Serrin's type rigidity result for (10) can be proved following Weinberger's approach by using a suitable P-function associated to (10) (see [15] and [37]). This approach is helpful for proving the following Serrin's type rigidity result for convex cones in space forms, which is the second main result of this paper.

Theorem 2. *Let (M, g) be the Euclidean space, hyperbolic space or the hemisphere. Let $\Sigma \subset M$ be a convex cone such that $\Sigma \setminus \{O\}$ is smooth and let Ω be a sector-like domain in Σ. Assume* *that there exists a solution* $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ *to*

$$\begin{cases} \Delta u + NKu = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases} \tag{11}$$

such that

$$\partial_\nu u = -c \ \text{ on } \ \Gamma_0 \tag{12}$$

for some constant c*. Then* $\Omega = \Sigma \cap B_R(x_0)$ *where* $B_R(x_0)$ *is a geodesic ball of radius* R *centered at* x_0 *and* u *is given by*

$$u(x) = \frac{H(R) - H(d(x,x_0))}{n\dot{h}(R)},$$

with

$$H(r) = \int_0^r h(s)ds$$

and where $d(x,x_0)$ *denotes the distance between* x *and* x_0*.*

Organization of the paper. The paper is organized as follows: in Section 2 we introduce some notation, we recall some basic facts about elementary symmetric function of a matrix and prove some preliminary result needed to prove Theorem 1. Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

2. Preliminary results for Theorem 1

In this section we collect some preliminary results which are needed in the proof of Theorem 1. Let f satisfy (4) and consider problem (5)

$$\begin{cases} L_f u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases}$$

where the operator L_f is given by

$$L_f u = \operatorname{div}\left(f'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right).$$

Definition 3. $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\})$ *is a solution to Problem* (5) *if*

$$\int_\Omega \frac{f'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, dx \tag{13}$$

for any

$$\varphi \in T(\Omega) := \{\varphi \in C^1(\Omega) \, : \, \varphi \equiv 0 \ \text{ on } \Gamma_0\}. \tag{14}$$

We observe some facts that will be useful in the following. Since the outward normal ν to Γ_0 is given by

$$\nu = -\frac{\nabla u}{|\nabla u|}|_{\Gamma_0}, \tag{15}$$

then (8) implies that

$$|\nabla u| = c \quad \text{on} \quad \Gamma_0. \tag{16}$$

Moreover we observe that the constant c in the statement is given by

$$c = g'\left(\frac{|\Omega|}{|\Gamma_0|}\right), \tag{17}$$

as it follows by integrating the equation $L_f u = -1$, by using the divergence theorem, formula (16) and the fact that $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$. We also notice that

$$x \cdot \nu = 0 \quad \text{on} \quad \Gamma_1. \tag{18}$$

It will be useful to write the operator L_f as the trace of a matrix. Let $V : \mathbb{R}^N \to \mathbb{R}$ be given by

$$V(\xi) = f(|\xi|) \quad \text{for } \xi \in \mathbb{R}^N, \tag{19}$$

and notice that

$$\begin{aligned} V_{\xi_i}(\xi) &:= \partial_{\xi_i} V(\xi) = f'(|\xi|)\frac{\xi_i}{|\xi|}\,, \\ V_{\xi_i\xi_j}(\xi) &:= \partial_{\xi_i\xi_j} V(\xi) = f''(|\xi|)\frac{\xi_i\xi_j}{|\xi|^2} - f'(|\xi|)\frac{\xi_i\xi_j}{|\xi|^3} + f'(|\xi|)\frac{\delta_{ij}}{|\xi|}. \end{aligned} \tag{20}$$

Hence, by setting

$$W = (w_{ij})_{i,j=1,\dots,N}$$

where

$$w_{ij}(x) = \partial_j V_{\xi_i}(\nabla u(x))\,, \tag{21}$$

we have

$$L_f(u) = \mathrm{Tr}(W). \tag{22}$$

Notice that at regular points, where $\nabla u \neq 0$, it holds that

$$W = \nabla^2_\xi V(\nabla u)\nabla^2 u\,. \tag{23}$$

Our approach to prove Theorem 1 is to write several integral identities and just one pointwise inequality, involving the matrix W. Writing the operator L_f as trace of W has the advantage that we can use the generalization of the so-called Newton's inequalities, as explained in the following subsection.

We mention that, unless otherwise specified, we adopt the Einstein convention of summation over repeated indices.

2.1. **Elementary symmetric functions of a matrix.** Given a matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$, for any $k = 1,\dots,N$ we denote by $S_k(A)$ the sum of all the principal minors of A of order k. In particular, $S_1(A) = \mathrm{Tr}(A)$ is the trace of A, and $S_n(A) = \det(A)$ is the determinant of A. We consider the case $k = 2$. By setting

$$S^2_{ij}(A) = -a_{ji} + \delta_{ij}\mathrm{Tr}(A), \tag{24}$$

we can write

$$S_2(A) = \frac{1}{2}\sum_{i,j} S^2_{ij}(A)a_{ij} = \frac{1}{2}((\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2))\,. \tag{25}$$

The elementary symmetric functions of a symmetric matrix A satisfy the so called Newton's inequalities. In particular, S_1 and S_2 are related by

$$S_2(A) \le \frac{N-1}{2N}(S_1(A))^2\,. \tag{26}$$

When the matrix $A = W$, with W given by (23), we have

$$S^2_{ij}(W) = -V_{\xi_j\xi_k}(\nabla u)u_{ki} + \delta_{ij}L_f u\,, \tag{27}$$

and $S^2_{ij}(W)$ is divergence free in the following (weak) sense

$$\frac{\partial}{\partial x_j} S^2_{ij}(W) = 0\,. \tag{28}$$

If V and u are sufficiently smooth, (28) was proved in [11, Equation (4.14)]. In Lemma 8 below we will prove (28) under weaker regularity assumptions on V and u by approximation (notice that (28) is implicitly written in (43), as follows from (25)).

We will need a generalization of (26) to not necessarily symmetric matrices, which is given by the following lemma.

Lemma 4 ([11], Lemma 3.2). *Let B and C be symmetric matrices in $\mathbb{R}^{N\times N}$, and let B be positive semidefinite. Set $A = BC$. Then the following inequality holds:*

$$S_2(A) \le \frac{N-1}{2N}\mathrm{Tr}(A)^2. \tag{29}$$

Moreover, if $\mathrm{Tr}(A) \neq 0$ and equality holds in (29), then

$$A = \frac{\mathrm{Tr}(A)}{N} Id,$$

and B is, in fact, positive definite.

2.2. **Some properties of solutions to** (5)**.** In this subsection we collect some properties of the solutions to (5). We assume that the solution is of class $C^1(\Omega) \cap W^{1,\infty}(\Omega)$. From standard regularity elliptic estimates one has that u is of class $C^{1,\alpha}(\Omega)$ and $C^{2,\alpha}$ where $\nabla u \neq 0$.

In the following two lemmas we show that $u > 0$ in $\Omega \cup \Gamma_1 \setminus \{O\}$ and we prove a Pohozaev-type identity.

Lemma 5. *Let f satisfy* (4) *and let u be a solution of* (5)*. Then*

$$u > 0 \quad in \quad \Omega \cup \Gamma_1 \setminus \{O\}. \tag{30}$$

Proof. We write $u = u^+ - u^-$ and use $\varphi = u^-$ as test function in (13):

$$0 \ge -\int_{\Omega\cap\{u<0\}} \frac{f'(|\nabla u|)}{|\nabla u|} |\nabla u^-|^2\, dx = \int_{\Omega\cap\{u<0\}} u^- \, dx \ge 0\,,$$

which implies that $u \ge 0$ in Ω. Moreover, if one assumes that $u(x_0) = 0$ at some point $x_0 \in \Omega \cup \Gamma_1 \setminus \{O\}$, then $\nabla u(x_0) = 0$. Since $x_0 \in \Omega \cup \Gamma_1 \setminus \{O\}$ and $\Gamma_1 \setminus \{O\}$ is smooth, there exists a ball $B_r \subset \Omega$ such that $x_0 \in \partial B_r$. Let v be the solution of

$$\begin{cases} L_f v = -1 & \text{in } B_r, \\ v = 0 & \text{on } \partial B_r\,. \end{cases}$$

By comparison principle we have that $v \le u$ in $\overline{B}_r$; from $\nabla u(x_0) = 0$ and since $\nabla v(x_0) \neq 0$ we get a contradiction. □

The following Pohozaev-type identity is a typical tool to prove symmetry results. In a similar setting as the one in this paper, a Pohozaev identity was proved in [26].

Lemma 6 (Pohozaev-type identity). *Let Ω be a sector-like domain and assume that f satisfies* (4)*. Let $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega)$ be a solution to* (5)*. Then the following integral identity*

$$\int_\Omega [(N+1)u - Nf(|\nabla u|)]\, dx = \int_{\Gamma_0} [f'(|\nabla u|)|\nabla u| - f(|\nabla u|)] x \cdot \nu \, d\sigma \tag{31}$$

holds.

Proof. We argue by approximation. We first approximate f with functions f_ε such that

$$f_\varepsilon \in C^\infty([0,\infty)) \text{ with } f_\varepsilon(0) = f'_\varepsilon(0) = 0,\ f''_\varepsilon(s) > 0 \text{ for } s \ge 0\,, \tag{32}$$

and

$$f_\varepsilon \to f \quad \text{and} \quad f'_\varepsilon \to f' \quad \text{uniformly on compact sets of } [0,+\infty). \tag{33}$$

We notice that such an approximation exists as shown in [26, Section 3].

We recall that $V(\xi) = f(|\xi|)$ (see (19)) for $\xi \in \mathbb{R}^N$, and we define $V^\varepsilon : \mathbb{R}^N \to \mathbb{R}$ as

$$V^\varepsilon(\xi) := f_\varepsilon(|\xi|).$$

We notice that $\nabla_\xi V^\varepsilon$ and $\nabla_\xi V$ can be continuously extended to 0 at $\xi = 0$.

We approximate Ω by domains Ω_δ obtained by chopping off a δ-tubular neighborhood of $\partial\Gamma_0$ and a δ-neighborhood of O. For $n \in \mathbb{N}$, we consider $u^n_\delta \in C^\infty(\Omega_\delta) \cap C^1(\bar\Omega_\delta)$ such that

$$u^n_\delta \to u \ \text{ in } \ C^1(\overline{\Omega}_\delta),$$

as n goes to infinity (see for instance [8, Section 2.6]).
Since

$$\mathrm{div}\,(x\cdot\nabla u_\delta^n\nabla_\xi V^\varepsilon(\nabla u_\delta^n)) = x\cdot\nabla u_\delta^n \mathrm{div}\,(\nabla_\xi V^\varepsilon(\nabla u_\delta^n)) + \nabla(x\cdot\nabla u_\delta^n)\cdot\nabla_\xi V^\varepsilon(\nabla u_\delta^n)$$

and from

$$\begin{aligned}\nabla(x\cdot\nabla u_\delta^n)\cdot\nabla_\xi V^\varepsilon(\nabla u_\delta^n) =& \nabla u_\delta^n\cdot\nabla_\xi V^\varepsilon(\nabla u_\delta^n) + x\nabla^2(u_\delta^n)\cdot\nabla_\xi V^\varepsilon(\nabla u_\delta^n)\\ =& \mathrm{div}\,(u_\delta^n\nabla_\xi V^\varepsilon(\nabla u_\delta^n)) - u_\delta^n\mathrm{div}\,(\nabla_\xi V^\varepsilon(\nabla u_\delta^n))\\ &+ \mathrm{div}(xV^\varepsilon(\nabla u_\delta^n)) - NV^\varepsilon(\nabla u_\delta^n)\,,\end{aligned}$$

we obtain

$$\mathrm{div}\,(\varphi_n\nabla_\xi V^\varepsilon(\nabla u_\delta^n) - xV^\varepsilon(\nabla u_\delta^n)) = \varphi_n\mathrm{div}\,(\nabla_\xi V^\varepsilon(\nabla u_\delta^n)) - NV^\varepsilon(\nabla u_\delta^n)\,, \tag{34}$$

where

$$\varphi_n(x) = x\cdot\nabla u_\delta^n(x) - u_\delta^n(x)\,.$$

Moreover, from the divergence theorem we have

$$\int_{\Omega_\delta}\nabla_\xi V^\varepsilon(\nabla u_\delta^n)\cdot\nabla\varphi_n\,dx = -\int_{\Omega_\delta}\varphi_n\mathrm{div}\,(\nabla_\xi V^\varepsilon(\nabla u_\delta^n))\,dx + \int_{\partial\Omega_\delta}\varphi_n\nabla_\xi V^\varepsilon(\nabla u_\delta^n)\cdot\nu\,d\sigma\,. \tag{35}$$

We are going to apply the divergence theorem in Ω_δ; to this end we set

$$\Gamma_{0,\delta} = \Gamma_0\cap\partial\Omega_\delta\,,\quad \Gamma_{1,\delta} = \Gamma_1\cap\partial\Omega_\delta \quad\text{and}\quad \Gamma_\delta = \partial\Omega_\delta\setminus(\Gamma_{0,\delta}\cup\Gamma_{1,\delta})\,.$$

From (35) and by integrating (34) in Ω_δ we obtain

$$\begin{aligned}\int_{\Omega_\delta}\nabla_\xi V^\varepsilon(\nabla u_\delta^n)\cdot\nabla\varphi_n\,dx =& -N\int_{\Omega_\delta}V^\varepsilon(\nabla u_\delta^n)\,dx - \int_{\Omega_\delta}\mathrm{div}\,(\varphi_n\nabla_\xi V^\varepsilon(\nabla u_\delta^n))\,dx\\ &+ \int_{\Omega_\delta}\mathrm{div}\,(xV^\varepsilon(\nabla u_\delta^n))\,dx\,,\end{aligned}$$

and from $x\cdot\nu = 0$ on $\Gamma_{1,\delta}$, we find

$$\begin{aligned}\int_{\Omega_\delta}\nabla_\xi V^\varepsilon(\nabla u_\delta^n)\cdot\nabla\varphi_n\,dx =& -N\int_{\Omega_\delta}V^\varepsilon(\nabla u_\delta^n)\,dx - \int_{\Gamma_{0,\delta}\cup\Gamma_{1,\delta}}\varphi_n\nabla_\xi V^\varepsilon(\nabla u_\delta^n)\cdot\nu\,d\sigma\\ &+ \int_{\Gamma_{0,\delta}}V^\varepsilon(\nabla u_\delta^n)x\cdot\nu\,d\sigma\\ &- \int_{\Gamma_\delta}[\varphi_n\nabla_\xi V^\varepsilon(\nabla u_\delta^n) - xV^\varepsilon(\nabla u_\delta^n)]\cdot\nu\,d\sigma\,.\end{aligned}$$

By taking the limit as $\varepsilon\to0$ and then as $n\to\infty$, using that $\nabla u\cdot\nu = 0$ on $\Gamma_{1,\delta}$ (since $\partial_\nu u = 0$ on Γ_1), we obtain

$$\begin{aligned}\int_{\Omega_\delta}\nabla_\xi V(\nabla u)\cdot\nabla\varphi\,dx =& -N\int_{\Omega_\delta}V(\nabla u)\,dx - \int_{\Gamma_{0,\delta}}\varphi\nabla_\xi V(\nabla u)\cdot\nu\,d\sigma + \int_{\Gamma_{0,\delta}}V(\nabla u)x\cdot\nu\,d\sigma\\ &- \int_{\Gamma_\delta}[\varphi\nabla_\xi V(\nabla u) - xV(\nabla u)]\cdot\nu\,d\sigma\end{aligned} \tag{36}$$

where we let

$$\varphi(x) = x\cdot\nabla u(x) - u(x)\,. \tag{37}$$

Now, we take the limit as $\delta\to0$. Since $u\in W^{1,\infty}(\Omega)$ and $\mathcal{H}_{N-1}(\Gamma_\delta)$ goes to 0 as $\delta\to0$, we have that the last term in (36) vanishes and we obtain

$$\int_\Omega\nabla_\xi V(\nabla u)\cdot\nabla\varphi\,dx = -N\int_\Omega V(\nabla u)\,dx - \int_{\Gamma_0}\varphi\nabla_\xi V(\nabla u)\cdot\nu\,d\sigma + \int_{\Gamma_0}V(\nabla u)x\cdot\nu\,d\sigma\,,$$

i.e. (in terms of f)

$$\int_\Omega\frac{f'(|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla\varphi\,dx = -N\int_\Omega f(|\nabla u|)\,dx - \int_{\Gamma_0}\varphi\frac{f'(|\nabla u|)}{|\nabla u|}\partial_\nu u\,d\sigma + \int_{\Gamma_0}f(|\nabla u|)x\cdot\nu\,d\sigma.$$

Since u satisfies (13), we get

$$\int_\Omega \varphi \, dx = -N \int_\Omega f(|\nabla u|)\, dx - \int_{\Gamma_0} \varphi \frac{f'(|\nabla u|)}{|\nabla u|} \partial_\nu u \, d\sigma + \int_{\Gamma_0} f(|\nabla u|) x \cdot \nu \, d\sigma. \tag{38}$$

From (37) and since $u = 0$ on Γ_0 and $\partial_\nu u = 0$ on Γ_1, we have

$$\int_\Omega \varphi \, dx = -(N+1) \int_\Omega u \, dx$$

and

$$\int_{\Gamma_0} \varphi \frac{f'(|\nabla u|)}{|\nabla u|} \partial_\nu u \, d\sigma = \int_{\Gamma_0} f'(|\nabla u|) |\nabla u| x \cdot \nu \, d\sigma \,, \tag{39}$$

where we used the expresion of the unit exterior normal on Γ_0 given by (15). From (39) and (38) we obtain

$$-(N+1) \int_\Omega u \, dx + N \int_\Omega f(|\nabla u|)\, dx = -\int_{\Gamma_0} f'(|\nabla u|)|\nabla u| x \cdot \nu \, d\sigma + \int_{\Gamma_0} f(|\nabla u|) x \cdot \nu \, d\sigma.$$

which is (31), and the proof is complete. □

We conclude this subsection by exploiting the boundary condition $\partial_\nu u = 0$ on Γ_1. Before doing this, we need to recall some notation from differential geometry (see also [18, Appendix A]). We denote by D the standard Levi-Civita connection. Recall that, given an $(N-1)$-dimensional smooth orientable submanifold M of $\mathbb{R}^N$ we define the *tangential gradient* of a smooth function $f : M \to \mathbb{R}$ with respect to M as

$$\nabla^T f(x) = \nabla f(x) - \nu \cdot \nabla f(x) \nu$$

for $x \in M$, where ∇f denotes the usual gradient of f in $\mathbb{R}^N$ and ν is the outward unit normal at x to M. Moreover, we recall that the *second fundamental form* of M is the bilinear and symmetric form defined on $TM \times TM$ as

$$\mathrm{II}(v, w) = D\nu(v) w \cdot \nu \,;$$

a submanifold is called *convex* if the second fundamental form is non-negative definite.

Lemma 7. *Let u be the solution to* (5)*. Then*

$$\nabla_\xi V(\nabla u) \cdot \nu = 0 \quad on \quad \Gamma_1 \,, \tag{40}$$

and

$$\nabla(\nabla_\xi V(\nabla u) \cdot \nu) \cdot \nabla u = 0 \quad on \quad \Gamma_1 \,.[1] \tag{41}$$

Proof. Since $\partial_\nu u = 0$ on Γ_1, we immediately find (40). By taking the tangential derivative in (40) we get

$$0 = \nabla^T (\nabla_\xi V(\nabla u) \cdot \nu) = \nabla(\nabla_\xi V(\nabla u) \cdot \nu) - \nu \cdot \nabla(\nabla_\xi V(\nabla u) \cdot \nu) \nu \quad \text{on} \quad \Gamma_1 \,.$$

By taking the scalar product with ∇u we obtain

$$0 = \nabla(\nabla_\xi V(\nabla u) \cdot \nu) \cdot \nabla u - \nu \cdot \nabla(\nabla_\xi V(\nabla u) \cdot \nu) \partial_\nu u \,,$$

and since $\partial_\nu u = 0$ on Γ_1, we find (41). □

[1]We remark that (41) is understood to be zero at points where $\nabla u = 0$.

2.3. **Integral Identities for S_2.** In this Subsection we prove some integral inequalities involving $S_2(W)$ and the solution to problem (5).

Lemma 8. *Let $\Omega \subset \mathbb{R}^N$ be a sector-like domain and assume that f satisfies (4). Let $u \in W^{1,\infty}(\Omega)$ be a solution of (5) such that (9) holds. Then the following inequality*

$$2\int_\Omega S_2(W)u\,dx \geq -\int_\Omega S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j u\,dx \tag{42}$$

holds. Moreover the equality sign holds in (42) if and only if $\mathrm{II}(\nabla^T u, \nabla^T u) = 0$ on Γ_1.

Proof. We split the proof in two steps.

Step 1: the following identity

$$2\int_\Omega S_2(W)\phi\,dx = -\int_\Omega S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j\phi\,dx\,, \tag{43}$$

holds for every $\phi \in C^1_0(\Omega)$.

For $t > 0$ we set $\Omega_t = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > t\}$. Let $\phi \in C^1_0(\Omega)$ be a test function and let $\varepsilon_0 > 0$ be such that $\Omega_{\varepsilon_0} \subset \Omega$ and $supp(\phi) \subset \Omega_{\varepsilon_0}$. For $\varepsilon < \varepsilon_0$ sufficiently small, we set

$$a^i(x) = V_{\xi_i}(\nabla u(x)) \quad \text{for every } i = 1, \dots, N,\, x \in \Omega.$$

From (9) we have that $a^i \in W^{1,2}(\Omega)$, $i = 1, \dots, N$. With this notation, the elements $w_{ij} = \partial_j V_{\xi_i}(\nabla u)$ of the matrix W are given by

$$w_{ij} = \partial_j a^i\,.$$

Let $\{\rho_\varepsilon\}$ be a family of mollifiers and define $a^i_\varepsilon = a^i * \rho_\varepsilon$. Let $W^\varepsilon = (w^\varepsilon_{ij})_{i,j=1,\dots,N}$ where $w^\varepsilon_{ij} = \partial_j a^i_\varepsilon$, and notice that

$$a^i_\varepsilon \to a^i \quad \text{in } W^{1,2}(\Omega_{\varepsilon_0}) \quad \text{and} \quad W^\varepsilon \to W \quad \text{in } L^2(\Omega_{\varepsilon_0})\,,$$

as $\varepsilon \to 0$. Moreover, since

$$\mathrm{Tr}\, W^\varepsilon(x) = \int_{\mathbb{R}^N} \rho_\varepsilon(y)\mathrm{Tr}\, W(x-y)\,dy$$

and $\mathrm{Tr}\, W = -1$, we have that

$$\mathrm{Tr}\, W^\varepsilon(x) = -1 \tag{44}$$

for every $x \in \Omega_\varepsilon$.

Let $i, j = 1, \dots, N$ be fixed. We have

$$\begin{aligned} w^\varepsilon_{ji}w^\varepsilon_{ij} &= \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) - a^i_\varepsilon\partial_j\partial_i a^j_\varepsilon \\ &= \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) - a^i_\varepsilon\partial_i\partial_j a^j_\varepsilon \\ &= \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) - a^i_\varepsilon\partial_i w^\varepsilon_{jj}\,, \end{aligned}$$

for every $x \in \Omega_\varepsilon$, and by summing up over $j = 1, \dots, N$, using (44) (hence $\partial_i \sum_j w^\varepsilon_{jj} = 0$), we obtain

$$\begin{aligned} \sum_j w^\varepsilon_{ji}w^\varepsilon_{ij} &= \sum_j \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) \\ &= w^\varepsilon_{ii}\mathrm{Tr}W^\varepsilon - \sum_j \partial_j(S^2_{ij}(W^\varepsilon)a^i_\varepsilon), \quad x \in \Omega_\varepsilon. \end{aligned}$$

By summing over $i = 1, \dots, N$, from (25) and (28) we have

$$2S_2(W^\varepsilon) = \sum_{i,j} \partial_j(S^2_{ij}(W^\varepsilon)a^i_\varepsilon)\,, \quad x \in \Omega_\varepsilon. \tag{45}$$

Since

$$\int_{\Omega_{\varepsilon_0}} \partial_j(S^2_{ij}(W^\varepsilon)a^i_\varepsilon)\phi\,dx + \int_{\Omega_{\varepsilon_0}} S^2_{ij}(W^\varepsilon)a^i_\varepsilon\partial_j\phi\,dx = \int_{\partial\Omega_{\varepsilon_0}} S^2_{ij}(W^\varepsilon)a^i_\varepsilon\nu_j\phi\,d\sigma = 0\,,$$

from (45) and by letting ε to zero, we obtain (43).

Step 2. Let $\delta > 0$ and consider a cut-off funtion $\eta^\delta \in C_c^\infty(\Omega)$ such that $\eta^\delta = 1$ in Ω_δ and $|\nabla \eta^\delta| \leq \frac{C}{\delta}$ in $\Omega \setminus \Omega_\delta$ for some constant C not depending on δ. By taking $\phi(x) = u(x)\eta^\delta(x)$ for $x \in \Omega$ in (43) we obtain

$$2\int_\Omega S_2(W)u\eta^\delta\,dx = -\int_\Omega S_{ij}^2(W)V_{\xi_i}(\nabla u)\partial_j u\eta^\delta\,dx - \int_\Omega S_{ij}^2(W)V_{\xi_i}(\nabla u)u\partial_j(\eta^\delta)\,dx\,. \tag{46}$$

From (9) we have that $W \in L^2(\Omega)$ and the dominated convergence Theorem yields

$$2\int_\Omega S_2(W)u\eta^\delta\,dx \to 2\int_\Omega S_2(W)u\,dx\,, \tag{47}$$

as $\delta \to 0$. Analogously,

$$\int_\Omega S_{ij}^2(W)V_{\xi_i}(\nabla u)\partial_j u\eta^\delta\,dx \to \int_\Omega S_{ij}^2(W)V_{\xi_i}(\nabla u)\partial_j u\,dx\,, \tag{48}$$

as $\delta \to 0$.

Now, we consider the last term in (46). We write Ω in the following way:

$$\Omega = A_0^\delta \cup A_1^\delta\,, \tag{49}$$

where

$$A_0^\delta = \{x \in \Omega\,:\,\mathrm{dist}(x,\Gamma_0) \leq \delta\} \quad \text{and} \quad A_1^\delta = \Omega \setminus A_0^\delta.$$

Since $u = 0$ on Γ_0, we get that

$$u(x) \leq ||u||_{W^{1,\infty}(\Omega)}\,\mathrm{dist}(x,\Gamma_0) \leq ||u||_{W^{1,\infty}(\Omega)}\,\delta$$

for every $x \in A_0^\delta$ and we obtain

$$\left|\int_{A_0^\delta} S_{ij}^2(W)V_{\xi_i}(\nabla u)u\partial_j(\eta^\delta)\,dx\right| \leq C_1|A_0^\delta|\,,$$

where C_1 is a constant depending on $||u||_{W^{1,\infty}(\Omega)}$ and $\|W\|_{L^2(\Omega)}$, which implies that

$$\lim_{\delta\to 0}\int_{A_0^\delta} S_{ij}^2(W)V_{\xi_i}(\nabla u)u\partial_j(\eta^\delta)\,dx = 0\,. \tag{50}$$

Now we show that

$$\lim_{\delta\to 0}\int_{A_1^\delta} S_{ij}^2(W(x))V_{\xi_i}(\nabla u(x))u(x)\partial_j(\eta^\delta)(x)\,dx \geq 0\,. \tag{51}$$

By choosing δ small enough, a point $x \in A_1^\delta$ can be written in the following way: $x = \bar{x} + t\nu(\bar{x})$ where $\bar{x} = \bar{x}(x) \in \Gamma_1$ and $t = |x - \bar{x}|$ with $0 < t < \delta$. Moreover, by using a standard approximation argument, η^δ can be chosen in such a way that $\eta^\delta(x) = \frac{1}{\delta}\mathrm{dist}(x,\Gamma_1)$ for any $x \in A_1^\delta$, so that

$$\nabla \eta^\delta(x) = -\frac{1}{\delta}\nu(\bar{x})\,, \tag{52}$$

for every $x \in A_1^\delta \setminus \Omega_\delta$. For simplicity of notation we set $F = (F_1, \ldots, F_N)$, where

$$F_j(x) = u(x)S_{ij}^2(W(x))V_{\xi_i}(\nabla u(x)) \tag{53}$$

for $j = 1, \ldots, N$, and hence

$$\int_{A_1^\delta} S_{ij}^2(W)V_{\xi_i}(\nabla u)u\partial_j(\eta^\delta)\,dx = \int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx\,. \tag{54}$$

Since $\nabla\eta^\delta = 0$ in Ω_δ and $\nabla\eta^\delta(x) = -\frac{1}{\delta}\nu(\bar{x})$, for every $x \in A_1^\delta \setminus \Omega_\delta$, we have

$$\begin{aligned}\int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx &= -\frac{1}{\delta}\int_{A_1^\delta\setminus\Omega_\delta} F(x)\cdot\nu(\bar{x})\,dx \\ &= -\frac{1}{\delta}\int_0^\delta dt\int_{\{x\in A_1^\delta\,:\,\mathrm{dist}(x,\Gamma_1)=t\}} F(x)\cdot\nu(\bar{x})\,d\sigma\end{aligned}$$

where we used coarea formula. Since we are in a *small* δ-tubular neighborhood of (part of) Γ_1, we can parametrize $A_1^\delta \setminus \Omega_\delta$ over (part of) Γ_1 as from [28, Formula 14.98] we obtain that

$$\int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx = -\frac{1}{\delta}\int_0^\delta dt\int_{\Gamma_1} F(\bar{x}+t\nu(\bar{x}))\cdot\nu(\bar{x})|\det(Dg)|\,d\sigma\,. \tag{55}$$

We notice that, by using this notation, proving (51) is equivalent to prove

$$\lim_{\delta\to 0}\int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx \geq 0\,, \tag{56}$$

for $\delta > 0$ sufficiently small.

From (52), (53) and the definition of S^2_{ij} (24), we have

$$\begin{aligned} F(x)\cdot\nu(\bar{x}) &= -\delta_{ij}V_{\xi_i}(\nabla u(x))u(x)\nu_j(\bar{x}) - w_{ji}(x)V_{\xi_i}(\nabla u(x))u(x)\nu_j(\bar{x}) \\ &= -\left\{\delta_{ij}V_{\xi_i}(\nabla u(x))u(x)\nu_j(\bar{x}) + u(x)\frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}w_{ji}(x)\partial_i u(x)\nu_j(\bar{x})\right\} \end{aligned}$$

for almost every $x = \bar{x}+t\nu(\bar{x}) \in A_1^\delta\setminus\Omega_\delta$, with $0\leq t\leq\delta$. Since

$$w_{ij}\nu_i\partial_j u = \partial_j(V_{\xi_i}(\nabla u)\nu_i)\partial_j u - V_{\xi_i}(\nabla u)\partial_j\nu_i\partial_j u\,,$$

we have

$$\begin{aligned} F(x)\cdot\nu(\bar{x}) = -u(x)\nabla_\xi V(\nabla u(x))\cdot\nu(\bar{x}) - u(x)\frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\times \\ \left\{\nabla(\nabla_\xi V(\nabla u(x))\cdot\nu(\bar{x}))\cdot\nabla u(x) - \frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\partial_j\nu_i(\bar{x})\partial_j u(x)\partial_i u(x)\right\} \end{aligned} \tag{57}$$

for almost every $x = \bar{x}+t\nu(\bar{x}) \in A_1^\delta\setminus\Omega_\delta$, with $0\leq t\leq\delta$. Let

$$\Gamma_1^{\delta,t} = \{x\in A_1^\delta \,:\, \mathrm{dist}(x,\Gamma_1) = t\}\,.$$

We notice that if $x\in\Gamma_1^{\delta,t}$ then $\nu(\bar{x}) = \nu^t(x)$ where $\nu^t(x)$ is the outward normal to $\Gamma_1^{\delta,t}$ at x. Hence

$$\partial_j\nu_i(\bar{x})\partial_j u(x)\partial_i u(x) = \mathrm{II}_x^{\delta,t}(\nabla^T u(x),\nabla^T u(x)) \tag{58}$$

where $\mathrm{II}_x^{\delta,t}$ is the second fundamental form of $\Gamma_1^{\delta,t}$ at x. Since Σ is a convex cone then the second fundamental form of $\Gamma_1\setminus\{O\}$ is non-negative definite. This implies that the second fundamental form of $\Gamma_1^{\delta,t}$ is non-negative definite for t sufficiently small [28, Appendix 14.6] and hence

$$\partial_j\nu_i(\bar{x})\partial_j u(x)\partial_i u(x)\geq 0\,. \tag{59}$$

From (59) and (57) we obtain

$$F(x)\cdot\nu(\bar{x}) \geq -u(x)\nabla_\xi V(\nabla u(x))\cdot\nu(\bar{x}) - u(x)\frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\nabla(\nabla_\xi V(\nabla u(x))\cdot\nu(\bar{x}))\cdot\nabla u(x) \tag{60}$$

for almost every $x = \bar{x}+t\nu(\bar{x}) \in A_1^\delta\setminus\Omega_\delta$, with $0\leq t\leq\delta$. We use (60) in the right-hand side of (55) and, by taking the limit as $\delta\to 0$, we obtain

$$\lim_{\delta\to 0}\int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx \geq -\int_{\Gamma_1} u\left(\nabla_\xi V(\nabla u)\cdot\nu + \frac{f'(|\nabla u|)}{|\nabla u|}\nabla(\nabla_\xi V(\nabla u)\cdot\nu)\cdot\nabla u\right)d\sigma\,.$$

From (40) and (41) we find (56), and hence (51). From (46), (47), (48), (49), (50) and (51), we obtain (42).

□

3. Proof of Theorem 1

Proof of Theorem 1. We divide the proof in two steps. We first show that

$$W = -\frac{1}{N} Id \quad \text{a.e. in } \Omega. \tag{61}$$

and

$$\mathrm{II}(\nabla^T u, \nabla^T u) = 0 \quad \text{on } \Gamma_1\,, \tag{62}$$

and then we exploit (61) in order to prove that u is indeed radial.

Step 1. Let g be the Fenchel conjugate of f (in our case $g' = (f')^{-1}$), using (20) we get that

$$\begin{aligned}\operatorname{div}\left(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)\right) =& g'(|\nabla_\xi V(\nabla u)|)\nabla|\nabla_\xi V(\nabla u)| V_\xi(\nabla u) + g(|\nabla_\xi V(\nabla u)|)\mathrm{Tr}(W)\\ =& g'(f'(|\nabla u|))\frac{V_{\xi_i}(\nabla u)}{|\nabla_\xi V(\nabla u)|}\partial_j(V_{\xi_i}(\nabla u))V_{\xi_j}(\nabla u)\\ &+ g(f'(|\nabla u|))\mathrm{Tr}(W)\,,\end{aligned}$$

a.e. in Ω, where we used (20). Since $\partial_j V_{\xi_i}(\nabla u) = w_{ij}$ and $g' = (f')^{-1}$, we obtain

$$\operatorname{div}\left(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)\right) = \partial_i u w_{ij} V_{\xi_j}(\nabla u) + g(f'(|\nabla u|))\mathrm{Tr}(W)$$

a.e. in Ω, and using again (20) we find

$$\operatorname{div}\left(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)\right) = \frac{f'(|\nabla u|)}{|\nabla u|}\partial_i u w_{ij}\partial_j u + g(f'(|\nabla u|))\mathrm{Tr}(W)$$

a.e. in Ω. Since

$$g(f'(t)) = tf'(t) - f(t) \tag{63}$$

and $\mathrm{Tr}(W) = -1$, we obtain

$$\operatorname{div}\left(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)\right) = \frac{f'(|\nabla u|)}{|\nabla u|}\partial_i u w_{ij}\partial_j u + f(|\nabla u|) - |\nabla u| f'(|\nabla u|) \tag{64}$$

a.e. in Ω.

Since (27), (20) and (22) yield

$$-S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j u = \frac{f'(|\nabla u|)}{|\nabla u|} w_{ji}\partial_i u \partial_j u + f'(|\nabla u|)|\nabla u|\,,$$

a.e. in Ω, from (64) we obtain

$$-S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j u = \operatorname{div}\left(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)\right) + 2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\,, \tag{65}$$

a.e. in Ω.

From Lemma 8 and (65), we obtain

$$\begin{aligned}2\int_\Omega S_2(W)u\,dx &\geq -\int_\Omega S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j u\,dx\\ &= \int_{\partial\Omega} g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)\cdot\nu\,d\sigma + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,.\end{aligned}$$

From (20) and (40) we find

$$2\int_\Omega S_2(W)u\,dx \geq \int_{\Gamma_0} g(|\nabla_\xi V(\nabla u)|)\frac{f'(|\nabla u|)}{|\nabla u|}\partial_\nu u\,d\sigma + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,.$$

From (20) and (8) we have

$$2\int_\Omega S_2(W)u\,dx \geq -g(f'(c))f'(c)|\Gamma_0| + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx$$

and from (63) we obtain

$$2\int_\Omega S_2(W)u\,dx \geq -[cf'(c) - f(c)]f'(c)|\Gamma_0| + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,. \tag{66}$$

From the Pohozaev identity (31) and (16) we get

$$(N+1)\int_\Omega u\,dx - N\int_\Omega f(|\nabla u|)\,dx = (f'(c)c - f(c))N|\Omega|\,;$$

which we use in (66) to obtain

$$2\int_\Omega S_2(W)u\,dx \geq -\frac{f'(c)|\Gamma_0|}{N|\Omega|}\int_\Omega [(N+1)u - Nf(|\nabla u|)]\,dx \\ + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,. \quad (67)$$

We notice that from (17) we have

$$|\Omega| = f'(c)|\Gamma_0|,$$

and from (67) we obtain

$$2\int_\Omega S_2(W)u\,dx \geq -\frac{N+1}{N}\int_\Omega u\,dx + 2\int_\Omega f'(|\nabla u|)|\nabla u|\,dx\,. \quad (68)$$

By using u as a test function in (13) we have that

$$\int_\Omega u\,dx = \int_\Omega f'(|\nabla u|)|\nabla u|\,dx\,,$$

and from (68) we find

$$2\int_\Omega S_2(W)u\,dx \geq \frac{N-1}{N}\int_\Omega u\,dx\,. \quad (69)$$

From (29) and using the fact that $\mathrm{Tr}(W) = L_f u = -1$, we get that also the reverse inequality

$$\frac{N-1}{N}\int_\Omega u\,dx \geq \int_\Omega 2S_2(W)u\,dx \quad (70)$$

holds. From (69) and (70), we conclude that the equality sign must hold in (69) and (70). From Lemma 4 we have that

$$W = \frac{\mathrm{Tr}(W)}{N}Id$$

a.e. in Ω, and since $\mathrm{Tr}(W) = -1$ we obtain (61). Moreover, Lemma 8 yields (62).

Step 2: u is a radial function. From (61) we have that

$$-\frac{1}{N}\delta_{ij} = \partial_j V_{\xi_i}(\nabla u(x))\,,$$

for every $i,j = 1,\dots,N$, which implies that there exists $x_0 \in \mathbb{R}^N$ such that

$$\nabla_\xi V(\nabla u(x)) = -\frac{1}{N}(x - x_0),$$

i.e. according to (20)

$$\frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\nabla u(x) = -\frac{1}{N}(x - x_0)\,.$$

Hence

$$\nabla u(x) = -g'\left(\frac{|x-x_0|}{N}\right)\frac{x-x_0}{|x-x_0|} \quad \text{in } \Omega\,.$$

Since $u = 0$ on Γ_0, we obtain (7) and in particular u is radial with respect to x_0. Moreover, from (62) we find that x_0 must be the origin or, if $\partial\Sigma$ contains flat regions, a point on $\partial\Sigma$. □

4. Cones in space forms: proof of Theorem 2

The goal of this section is to give an easily readable proof of Theorem 2. More precisely we assume more regularity on the solution than the one actually assumed in Theorem 2 in order to give a coincise and clear idea of the proof in this setting, and we omit the technical details which are, in fact, needed. A rigorous treatment of the argument described below can be done by adapting the (technical) details in Section 3 and in [35].

Before starting the proof we declare some notations we use in the statement of Theorem 2 and we are going to adopt in the following. Given a N-dimensional Riemannian manifold (M,g), we denote by D the Levi-Civita connection of g. Moreover given a C^2-map $u : M \to \mathbb{R}$, we denote by ∇u the gradient of u, i.e. the dual field of the differential of u with respect to g, and by $\nabla^2 u = Ddu$ the Hessian of u. We denote by Δ the Laplace-Betrami operator induced by g; Δu can be defined as the trace of $\nabla^2 u$ with respect to g. Given a vector field X on an oriented Riemannian manifold (M,g), we denote by $\mathrm{div}X$ the divergence of X with respect to g. If $\{e_k\}$ is a local orthonormal frame on (M,g), then

$$\mathrm{div}X = \sum_{k=1}^{N} g(D_{e_k}X, e_k)\,;$$

notice that, if u is a C^1-map and if X is a C^1 vector field on M, we have the following *integration by parts* formula

$$\int_\Omega g(\nabla u, \nu)\, dx = -\int_\Omega u\mathrm{div}X\, dx + \int_{\partial\Omega} ug(X,\nu)\, d\sigma\,,$$

where ν is the outward normal to $\partial\Omega$ and Ω is a bounded domain which is regular enough. Here and in the following, dx and $d\sigma$ denote the volume form of g and the induced $(N-1)$-dimensional Hausdorff measure, respectively.

Proof of Theorem 2. We divide the proof in four steps.

Step 1: the P-function. Let u be the solution to problem (11) and, as in [15], we consider the P-function defined by

$$P = |\nabla u|^2 + \frac{2}{N}u + Ku^2\,.$$

Following [15, Lemma 2.1], P is a subharmonic function and, since $u = 0$ on Γ_0 and from (16), we have that $P = c^2$ on Γ_0. Moreover,

$$\nabla P = 2\nabla^2 u \nabla u + \frac{2}{n}\nabla u + 2Ku\nabla u\,. \tag{71}$$

From the convexity assumption of the cone Σ, we have that

$$g(\nabla^2 u\nabla u, \nu) \leq 0\,. \tag{72}$$

Indeed, since $u_\nu = 0$ on Γ_1 and by arguing as done for (41), we obtain that

$$0 = g(\nabla u_\nu, \nabla u) = g(\nabla^2 u\nabla u, \nu) + \mathrm{II}(\nabla u, \nabla u) \geq g(\nabla^2 u \nabla u, \nu) \quad \text{on} \quad \Gamma_1\,,$$

which is (72). From (71) and (72) we obtain

$$\partial_\nu P = 2g(\nabla^2 u\nabla u, \nu) + \frac{2}{n}\partial_\nu u + 2Ku\partial_\nu u \leq 0 \quad \text{on } \Gamma_1 \setminus \{O\}\,.$$

Hence, the function P satisfies:

$$\begin{cases} \Delta P \geq 0 & \text{in } \Omega,\\ P = c^2 & \text{on } \Gamma_0\\ \partial_\nu P \leq 0 & \text{on } \Gamma_1 \setminus \{O\}\,. \end{cases}$$

Moreover, again from [15, Lemma 2.1], we have that

$$\Delta P = 0 \quad \text{if and only if} \quad \nabla^2 u = \left(-\frac{1}{N} - Ku\right) g\,. \tag{73}$$

Step 2: we have

$$P \le c^2 \quad in\ \Omega. \tag{74}$$

Indeed, we multiply $\Delta P \ge 0$ by $(P-c^2)^+$ and by integrating by parts we obtain

$$0 \ge \int_{\Omega\cap\{P>c^2\}} |\nabla P|^2\, dx - \int_{\partial\Omega} (P-c^2)^+ \partial_\nu P\, d\sigma\,.$$

Since $P = c^2$ on Γ_0 and $\partial_\nu P \le 0$ on Γ_1 we obtain that

$$0 \ge \int_{\Omega\cap\{P>c^2\}} |\nabla P|^2\, dx \ge 0$$

and hence $P \le c^2$.

Step 3: $P = c^2$. By contradiction, we assume that $P < c^2$ in Ω. Since $\dot h > 0$, we have

$$c^2 \int_\Omega \dot h\, dx > \int_\Omega \dot h |\nabla u|^2\, dx + \frac{2}{N}\int_\Omega \dot h u\, dx + K\int_\Omega \dot h u^2\, dx\,.$$

Since

$$\operatorname{div}(\dot h u \nabla u) = \dot h |\nabla u|^2 + \dot h u \Delta u + \ddot h u \partial_r u$$

and

$$\ddot h = -Kh\,,$$

and from $u = 0$ on Γ_0 and $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$, we have that

$$\begin{aligned} c^2 \int_\Omega \dot h\, dx &> -\int_\Omega \dot h u \Delta u\, dx - \int_\Omega \ddot h u \partial_r u\, dx + \frac{2}{N}\int_\Omega \dot h u\, dx + K\int_\Omega \dot h u^2\, dx \\ &= (N+1)K\int_\Omega \dot h u^2\, dx + \left(1+\frac{2}{N}\right)\int_\Omega \dot h u\, dx + K\int_\Omega h u \partial_r u\, dx\,. \end{aligned}$$

From $\operatorname{div}(h\partial_r) = N\dot h$ we have

$$\operatorname{div}(u^2 h \partial_r) = N\dot h u^2 + 2hu\partial_r u\,,$$

and from $u = 0$ on Γ_0 and $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$ we obtain

$$c^2 \int_\Omega \dot h\, dx > \left(1+\frac{2}{N}\right)\left(\int_\Omega \dot h u\, dx - K\int_\Omega h u \partial_r u\, dx\right)\,. \tag{75}$$

Now we show that if u is a solution of (11) satisfying (12) then the equality sign holds in (75). Indeed, let $X = h\partial_r$ be the radial vector field and, by integrating formula (2.8) in [15], we get

$$-\frac{c^2}{N}\int_{\partial\Omega} g(X,\nu)\, d\sigma + \frac{N+2}{N}\int_\Omega \dot h u\, dx - (N-2)K\int_\Omega \dot h u^2\, dx + \left(\frac{2}{N}-3\right)K\int_\Omega u g(X,\nabla u)\, dx = 0\,.$$

Since $\operatorname{div}X = N\dot h$ we obtain

$$c^2 \int_\Omega \dot h\, dx = \frac{N+2}{N}\int_\Omega \dot h u\, dx - (N-2)K\int_\Omega \dot h u^2\, dx + \left(\frac{2}{N}-3\right)K\int_\Omega u g(X,\nabla u)\, dx\,,$$

i.e.

$$c^2 \int_\Omega \dot h\, dx = \left(1+\frac{2}{N}\right)\left(\int_\Omega \dot h u\, dx - K\int_\Omega h u \partial_r u\, dx\right)\,,$$

where we used that $u = 0$ on Γ_0, $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$ and $g(X,\nu) = 0$ on Γ_1. From (75) we find a contradiction and hence $P \equiv c^2$ in Ω.

Step 4: u is radial. Since P is constant, then $\Delta P = 0$ and from (73) we find that u satisfies the following Obata-type problem

$$\begin{cases} \nabla^2 u = (-\frac{1}{N} - Ku)g & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}\,. \end{cases} \tag{76}$$

We notice that the maximum and the minimum of u can not be both achieved on Γ_0 since otherwise we would have that $u \equiv 0$. Hence, at least one between the maximum and the minimum of u is achieved at a point $p \in \Omega \cup \Gamma_1$. Let $\gamma : I \to M$ be a unit speed maximal geodesic satisfying $\gamma(0) = p$ and let $f(s) = u(\gamma(s))$. From the first equation of (76) it follows

$$f''(s) = -\frac{1}{N} - Kf(s)\,.$$

Moreover, the definition of f and the fact that $\nabla u(p) = 0$ yield

$$f'(0) = 0 \quad \text{and} \quad f(0) = u(p),$$

and therefore

$$f(s) = \left(u(p) - \frac{1}{N}\right) H(s) - \frac{1}{N}.$$

This implies that u has the same expression along any geodesic strating from p, and hence u depends only on the distance from p. This means that $\Omega = \Sigma \cap B_R$ where B_R is a geodesic ball and u depends only on the distance from the center of B_R. □

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