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W-Symmetries of Ito stochastic differential equations

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Abstract

We discuss W-symmetries of Ito stochastic differential equations, introduced in a recent paper by Gaeta and Spadaro [J. Math. Phys. 2017]. In particular, we discuss the general form of acceptable generators for continuous (Lie-point) W-symmetry, arguing they are related to the (linear) conformal group, and how W-symmetries can be used in the integration of Ito stochastic equations along Kozlov theory for standard (deterministic or random) symmetries. It turns out this requires, in general, to consider more general classes of stochastic equations than just Ito ones.

1 Introduction

In a recent paper [13] we have discussed in general terms symmetry of (systems of) Stochastic Differential Equations in Ito form,

$$dx^i \ = \ f^i(x,t)\, dt \ + \ \sigma^i{}_k(x,t)\, dw^k \ . \tag{1}$$

(Here and below sum over dummy indices is routinely understood.) We have argued that albeit apparently one could consider general vector fields in $(x,t;w)$ space, i.e.

$$X \ = \ \varphi^i(x,t;w)\, \frac{\partial}{\partial x^i} \ + \ \tau(x,t;w)\, \frac{\partial}{\partial t} \ + \ h^k(x,t;w)\, \frac{\partial}{\partial w^k} \ , \tag{2}$$

some limitations are actually in order on the functions φ, τ, h, see below.

Based on these, we have proposed a classification of different types of symmetries, and investigated the determining equations characterizing them for a given Ito equation (1).

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In later work [10, 11], after clarifying how the relevant symmetries of an Ito SDE are preserved under a change of variables despite the non-geometric transformation properties of Ito equations under these [10], our discussion and classification were useful in order to extend the Kozlov theory [16, 17, 18, 19] relating symmetry and – complete or partial – integrability of SDEs. In particular, it was shown that the sufficient conditions identified by Kozlov for this in the case (according to our classification, see below) of *deterministic* symmetries, are also necessary; and the theory was also extended to *random* symmetries [11], albeit in this case we only treated scalar equations (we will fill this gap by treating the case of systems, in sect.5.2 below)[1]

The purpose of the present work is to discuss the extension of this approach and the results mentioned above to the other case allowed by our classification, i.e. to *W-symmetries*. These are symmetries directly acting – beside the x^i and t variables – on the Wiener process w^i as well [13].

We will also denote vector fields and symmetries *not* acting on (but possibly depending on) the w^i variables, as *standard* ones, for ease of reference; thus standard symmetries comprise both deterministic and random ones.

Let us now briefly sketch the *plan of the paper*. We will start by recalling, in Section 2, our discussion about the limitations to be put on (2) to get admissible symmetries, and hence our classification for the three types of admissible symmetries (deterministic, random, and W-symmetries) together with the relevant concept of *simple* symmetry. We will also briefly recall, in Section 3, how it is possible to use (deterministic or random) symmetries of Ito SDE despite the transformation properties of these.

We will then discuss, in Section 4, Kozlov theory for standard symmetries; in particular we will recall how the presence of simple symmetries – deterministic or random – allow to integrate a scalar SDE. In Section 5 we will consider systems; in particular, in 5.1 we recall how one can use deterministic symmetries (provided a Lie algebraic condition, analogous to the one met when dealing with deterministic equations, is satisfied) to partially integrate, i.e. reduce to a smaller dimension, a system of Ito equations. As mentioned above, our previous work only considered symmetry reduction (actually, in this case, integration) under random symmetries for the case of scalar equations; in Section 5.2 we will extend that discussion to the general case, i.e. systems of Ito equations (this result is new, but is a straightforward extension of those already present in the literature).

We will then be ready to introduce the most relevant – and original – part of our work, namely the extension of the theory developed so far for deterministic or random symmetries to the third case in our classification, i.e. for W-symmetries.

[1] It should be mentioned that in recent work [20], Kozlov considered the situation where the Ito equation admits a conserved quantity, and studied the consequence of this on the symmetries and their algebraic structure; it was also shown how, even in this case, there is a correspondence between the symmetries of the Ito and of the associated Stratonovich equation [21], confirming the results of [10].

This will first of all require again to discuss more precisely what kind of transformation could and should be considered, which is the subject of Section 6. After this, we will have to extend the discussion of Section 3 to the case of W-symmetries; this will be done in Section 7, and we will find that the extension is not complete.

After this we will finally be able to tackle the extension of Kozlov theory to W-symmetries, in Section 8. Again we will find that the extension is not complete; in particular we will see that albeit W-symmetries are of help in educing or integrating Stochastic Differential Equations, this will in general go though mapping an Ito equation into a more general type of stochastic equation. This also means that the existing results about multiple symmetry reduction cannot be applied in the case of multiple W-symmetries.

In the final Section 9 we will summarize and discuss our findings.

We also have two Appendices, devoted to the (simpler) special case of scalar equations. In Appendix A we derive, in the simplified one-dimensional setting, our basic result about the correspondence of W-symmetries for an Ito and the associated Stratonovich equations; in Appendix B we show that not all vector fields can be realized as nontrivial W-symmetries of stochastic equations, discussing in detail some one-dimensional examples.

The symbol $\odot$ will mark the end of a Remark or of an Example.

Acknowledgements

I thank C. Lunini, L. Peliti and F. Spadaro for interesting discussion on symmetries of SDEs in general, and on this research in particular; the communication by prof. R. Kozlov of a simple but significant Example (see Example 2 below) was also very useful to focus my ideas. A substantial part of this work was performed while I was in residence at SMRI over the summer 2018.

2 Standard symmetries of Ito equations

When we consider an Ito equation[2] (1), applying the vector field (2) produces a map

$$x^i \to \widetilde{x}^i = x^i + \varepsilon \varphi^i(x,t;w) \ , \ \ t \to \widetilde{t} = t + \varepsilon \tau(x,t;w) \ , \ \ w^i \to \widetilde{w}^k = w^k + \varepsilon h^k(x,t;w) \ ; \tag{3}$$

this in turn maps the Ito SDE (1) into a, generally different, SDE.

The point is that for general choices of φ^i, τ, h^k the new SDE is not even of Ito type, as discussed in detail in [13]. In order to ensure we remain within the framework of Ito equations, we should introduce several limitation on these

[2] By this we always mean possibly a system, unless otherwise specified. Note that – albeit this will in general not be used – we can also assume σ to be non-degenerate or, passing to suitable coordinates, we would have an Ito system coupled to deterministic ODEs.

coefficients[3]; in particular, leaving aside for a moment the coefficients h^k and hence the possibility to consider W-symmetries:

- The functions φ^i are unrestricted, beside the requirement to be smooth functions of their arguments.
- The function τ should (be smooth and) depend *only* on t, with moreover $\tau'(t) > 0$ (this guarantees the new variable $\widetilde{t}$ still represents time, albeit a rescaled one).

We will from now on always assume that these restrictions on τ are satisfied; we refer to these vector fields, and possibly symmetries, as the *admissible* ones.

Remark 1. Note that if $\tau \neq 0$, the rescaling of time will affect the Wiener processes w^i. More precisely, their expression $w^i = w^i(\widetilde{t})$ in terms of the new time $\widetilde{t}$ will differ from their expression $w^i = w^i(t)$ in terms of the pristine time variable. However, this difference amounts to a scalar factor, which is then absorbed in the coefficients ${\sigma^i}_k$ of the Ito equation; see [12, 13].

More precisely, in this case we get $w^i(t) \to \widetilde{w}^i(\widetilde{t})$ with

$$\widetilde{w}^i(\widetilde{t}) \ = \ \sqrt{1 + \varepsilon \tau'(t)} \ w^i(\widetilde{t}) \ ; \tag{4}$$

all in all, this amounts to the map

$$dw^k \ \to \ dw^k \ + \ \varepsilon \, \frac{1}{2} \left(\frac{d\tau}{dt} \right) dw^k \ := \ dw^k \ + \ \varepsilon \ \delta w^k \ . \tag{5}$$

See e.g. [13], sect. IIB, for details. $\odot$

With these limitations, and still keeping $h^k = 0$, we have a simple classification of maps and hence of possible symmetries.

- If the φ^i do not depend on the w^k variables, then we speak of *deterministic* vector fields; if they also effectively depend on the w^k, we speak of *random* vector field. Note that by assumption τ only depends on t, if it is present.
- If $\tau = 0$, we speak of *simple* vector fields; if $\tau \neq 0$ (but $\tau = \tau(t)$, $\tau'(t) > 0$) we speak of *general* vector fields.

Remark 2. We anticipate that the Kozlov theory relating symmetry of SDE to the possibility of reducing, and possibly completely integrating, them makes only use of *simple* symmetries (see however [20, 21]); hence the special interest of this seemingly restricted class. $\odot$

[3] We stress they are always supposed to be C^∞ (we will also say just *smooth*) functions of their argument; this guarantees we are dealing with proper – albeit possibly formal – vector fields in the $(x, t; w)$ space.

Remark 3. As well known, when dealing with *deterministic* differential equations, there is no such difference between symmetries acting on the time and on the spatial variables (and actually we can consider symmetries mixing time and the space variables). The reason for their different standing in the present context is readily understood: in fact, now t is in all cases a smooth variable, while x is a smooth variable as the spatial coordinate, but becomes a stochastic process when we look at solutions to the Ito equation (1); thus t and the x^i are inherently different, and it is no surprise that time should not be mixed with space variables, and that the presence of symmetries acting on them will have different consequences. $\odot$

3 Standard symmetry and change of variables

The possibility of using symmetries to solve or reduce *deterministic* equations rests ultimately on the fact that symmetries are preserved under changes of variables. This in turn follows immediately from the fact that symmetry vector fields are geometrical objects, and the same holds for the *solution manifold* $S_\Delta \subset J^n M$ representing a differential equation (or system) Δ of order n in the suitable Jet space [1, 6, 25, 26, 28].

It is not at all obvious that the same holds for Ito equations: as well known, they do not transform geometrically (i.e. under the chain rule), but in their own way – in fact, under the Ito rule.

This point was raised and solved in some recent work [10]. The approach followed there was to use the Stratonovich equation associated to a given Ito one; this transforms geometrically (this is its main advantage, together with the related time-inversion properties), so its symmetries are surely preserved under changes of variables. The determining equations for an Ito equation and for the associated Stratonovich one are different and give different solutions [13, 30]; but it is known that they have the same solutions if we restrict to either simple (deterministic or random) symmetries, or to general symmetries (2) with τ satisfying a certain third order compatibility condition identified by Unal [30]; this is automatically satisfied if τ only depends on t, i.e. for *admissible* symmetries according to our classification [13] recalled above.

In other words, we have the following result [10]; here "simple" refers again (as for symmetries) to the fact the t variable is unaffected; we will similarly denote as "simple maps" those not acting on t.

Proposition 1. *Admissible standard symmetries of an Ito equation (1) are preserved under (simple, deterministic) smooth changes of variables* $x^i = \Phi^i(y,t)$.

4 Standard symmetry and integrability of scalar Ito equations

With the result of the previous Section, we can start discussing symmetries of an Ito equation and its use.

First if all we note that – as an Ito equation lacks a geometrical interpretation – in this context symmetry will be an *algebraic* rather than a geometrical property. That is, we require that the map (3), i.e. the substitution $x^i \to x^i + \varepsilon \varphi^i$, $t \to t + \varepsilon \tau$, $w^k \to w^k + \varepsilon h^k$, leaves the Ito equation (1) invariant *at first order in* ε.

In the case of interest here, i.e. disregarding for the moment W-symmetries, and focusing on (deterministic or random) *simple* symmetries

$$X \ = \ \varphi^i (x,t:w) \ \partial / \partial x^i \ , \tag{6}$$

it can be proven [13] that they comply with the *determining equations* (for simple symmetries)

$$\begin{aligned} \partial_t \varphi^i \ + \ f^j \, \partial_j \varphi^i \ - \ \varphi^j \, \partial_j f^i \ &= \ - \frac{1}{2} \, \triangle \varphi^i \ , \qquad (7) \\ \widehat{\partial}_k \varphi^i \ + \ \sigma^j_{\ k} \, \partial_j \varphi^i \ - \ \varphi^j \, \partial_j \sigma^i_{\ k} \ &= \ 0 \ ; \qquad (8) \end{aligned}$$

here we have used the notation

$$\partial_t \ := \ \partial / \partial t \ , \ \ \partial_i \ := \ \partial / \partial x^i \ , \ \ \widehat{\partial}_k \ := \ \partial / \partial w^k \ ; \tag{9}$$

and the symbol $\triangle$ denotes the *Ito Laplacian*

$$\triangle u \ := \ \sum_{k=1}^n \frac{\partial^2 u}{\partial w^k \partial w^k} + \sum_{j,k=1}^n \left(\sigma \, \sigma^T \right)^{jk} \ \frac{\partial^2 u}{\partial x^j \, \partial x^k} + 2 \ \sum_{j,k=1}^n \sigma^{jk} \ \frac{\partial^2 u}{\partial x^j \, \partial w^k} \ . \tag{10}$$

These notations will be used routinely in the following.

Let us first consider the case of a scalar equation; then the presence of a simple symmetry guarantees that the equation can be explicitly integrated, i.e. transformed into an Ito integral. The result is constructive, in that the symmetry determines the appropriate change of variables.

This result holds for any standard simple symmetry, but in the case of random ones some additional condition should also be checked.

4.1 Deterministic symmetries

We start with the case of simple deterministic symmetries. Here we have the following result, due to Kozlov [16] (see also [11]):

Proposition 2. *The scalar SDE*

$$dy \ = \ \widetilde{f}(y,t) \ dt \ + \ \widetilde{\sigma}(y,t) \ dw \tag{11}$$

can be transformed by a simple deterministic map $y = y(x,t)$ into

$$dx \ = \ f(t)\, dt \ + \ \sigma(t)\, dw \ , \tag{12}$$

and hence explicitly integrated in Ito sense, if and only if *it admits a simple deterministic symmetry.*

If the generator of the latter is $X = \varphi(y,t)\partial_y$, then the change of variables $y = F(x,t)$ transforming (11) into (12) is the inverse to the map $x = \Phi(y,t)$ identified by

$$\Phi(y,t) \ = \ \int \frac{1}{\varphi(y,t)} \ dy \ .$$

Remark 4. We stress that here the "only if" refers to the transformation *by a deterministic map.* We will see in a moment that the transformation is possible also in case there is no deterministic symmetry but a random symmetry is present; but in this case this is achieved by a *random map* rather than a deterministic one. See also Remark 6 in this sense. ⊙

Remark 5. Note that (12) provides immediately the solution in the new variable,

$$x(t) \ = \ x(t_0) \ + \ \int_{t_0}^t f(s)\, ds \ + \ \int_{t_0}^t \sigma(s)\, dw(s) \ ;$$

in order to obtain the solution in the original variable we should of course use $y = F(x,t)$. ⊙

Example 1. The Ito equation [10]

$$dy \ = \ \left[e^{-y} \ - \ (1/2)\, e^{-2y} \right] \, dt \ + \ e^{-y} \, dw$$

admits the vector field $X = e^{-y}\partial_y$ as a symmetry generator. By the associate change of variables

$$x \ = \ \int \frac{1}{\varphi(y)} \ dy \ = \ \int e^y \, dy \ = \ \exp[y] \ + \ K$$

the vector field reads $X = \partial_x$, and the initial equation reads

$$dx \ = \ dt \ + \ dw \ ;$$

this is readily integrated.[4] ⊙

[4]More precisely, we get $x(t) = c_0 + t + w(t)$, and hence $y(t) = \log[x(t) - K]$; a suitable choice of the arbitrary (integration) constant K guarantees existence of the solution $y(t)$ for sufficient times $t > 0$ with probability one. In the following Examples we will not discuss the map of solutions back into the original variables.

4.2 Random symmetries

The difference between the case where deterministic symmetries are considered and the one where the considered symmetries are random ones, lies in that in the case of random symmetries the associated random change of variables could change the Ito equation into a random system of different nature. This problem accounts for the appearance of an extra condition, absent when one is only considering deterministic simple symmetries. Here we just give the relevant result, referring to [11] for a comprehensive discussion.

Proposition 3. *Let the Ito equation*

$$dy \ = \ F(y,t)\, dt \ + \ S(y,t)\, dw \tag{13}$$

admit the simple random vector field $X = \varphi(y,t,w)\partial_y$ as Lie-point symmetry; define $\gamma(y,t,w) := \partial_w(1/\varphi)$.

If the functions $F(y,t)$, $S(y,t)$ and $\gamma(y,t,w)$ satisfy the relation

$$S\,\gamma_t \ + \ S_t\,\gamma \ = \ F\,\gamma_w \ + \ (1/2)\,\left[S\,\gamma_{ww} \ + \ S^2\,\gamma_{yw} \right] \ , \tag{14}$$

then the equation (13) can be mapped by a simple random change of variables into an integrable Ito equation

$$dx \ = \ f(t)\, dt \ + \ \sigma(t)\, dw \ . \tag{15}$$

Conversely, let the Ito equation (13) be reducible to the integrable form (15) by a simple random change of variables $x = \Phi(y,t;w)$. Then necessarily (13) admits $X = [\Phi_y(y,t,w)]^{-1}\,\partial_y := \varphi(y,t,w)\partial_y$ as a symmetry vector field, and (14) is satisfied with $\gamma = \partial_w(1/\varphi)$.

Remark 6. As mentioned above, see Remark 4, it is possible that an equation can be integrated by a (random) change of variables, albeit it has no deterministic simple symmetry; in this case it should, as stated by Proposition 3, have a random simple symmetry. $\odot$

Example 2. A simple example of this situation is provided by the scalar Ito equation[5]

$$dx \ = \ e^x\, dt \ + \ dw \ . \tag{16}$$

This has no simple deterministic symmetry (it has the deterministic symmetry $X_0 = \partial_t$, but this is not simple and hence cannot be used for integration), but has a simple random symmetry, $X = \exp[x-w]\partial_x$, which can be used for integration. In fact, the X-related new variable is

$$y \ = \ \int \frac{1}{e^{x-w}}\ dx \ = \ - \, e^{w-x} \ , \tag{17}$$

[5]This example was communicated to me by prof. Kozlov (personal communication), whom I warmly thank.

and in terms of this we have

$$dy = e^w \, dt \ , \tag{18}$$

hence

$$y(t) = y(t_0) + \int_{t_0}^t e^{w(s)} \, ds \ .$$

It may be noted that the equation also has a W-symmetry, $X_1 = \partial_w$; it turns out this is not of acceptable type, as discussed in Sect.6 below. $\odot$

Remark 7. Note also that eq.(18) is *not* of Ito type; correspondingly eq.(14), meant now for (16) and γ associated to X, is *not* satisfied: the l.h.s. vanishes, while the r.h.s. yields $e^w[1+(1/2)e^{-x}]$. This notwithstanding, eq.(18) is readily integrated, and hence the change of variables (17) allows to integrate the original equation (16). This suggest that our theory can be extended, allowing for transformations to non-Ito equation and hence for symmetries such that the compatibility condition (14) is not satisfied. We will not dwell in this direction in the present work, but we will find that this situation is rather generic when dealing, in later Sections, with W-symmetries. $\odot$

5 Systems

The scope of Proposition 2 is quite limited, in that it only concerns *scalar* equations. On the basis of what is achieved in the case of deterministic equations, we would expect that if a (simple) symmetry is present for a multi-dimensional system, then the latter can be reduced to one of lower dimension – in this case we also say it can be "partially integrated". In fact, this is the case also for SDEs, as stated by the following Proposition 4.

Needless to say, in the case of multi-dimensional systems one could have several (simple) symmetries, and – in principles – multiple reduction is possible. Once again, in the case of deterministic equations this is the case only if the symmetries (more precisely, only for those of the symmetries which) have a suitable algebraic structure, i.e. which span a *solvable Lie algebra* acting with regular orbits [1, 6, 25, 26, 28], and one would expect the same kind of condition is required also in the analysis of SDEs, as indeed is the case.

5.1 Partial integrability and multiple deterministic symmetries

It appears that only deterministic symmetries have been considered so far in discussing systems. We will provide a discussion of multiple reduction by random symmetries in Section 5.2.

Again the relevant results in this direction have been obtained by Kozlov [17, 18] (see also [11, 23]). We will be quoting from [11].

Proposition 4. *Suppose the system (1) admits an r-parameter* solvable *Lie algebra $\mathcal{G}$ of simple deterministic symmetries, with generators*

$$X_{(k)} = \sum_{i=1}^{n} \varphi^i_{(k)}(x,t) \ \frac{\partial}{\partial x_i} \qquad (k = 1, ..., r) \ , \tag{19}$$

acting regularly with r-dimensional orbits.

Then it can be reduced to a system of $m = (n-r)$ equations,

$$dy^i \ = \ g^i(y^1, ..., y^m; t)\, dt \ + \ \sigma^i{}_k(y^1, ..., y^m; t)\, dw^k \qquad (i,k = 1, ..., m) \tag{20}$$

and r "reconstruction equations", the solutions of which can be obtained by quadratures from the solution of the reduced $(n-r)$-order system.

Remark 8. It is convenient to label the different generators $X_{(k)}$ of the symmetry Lie algebra $\mathcal{G}$ according to the Lie structure of this; thus the element $\mathcal{G}^q$ in the derived series of $\mathcal{G}$ will be the span of $\{X_{(q)}, X_{(q+1)}, ..., X_{(r)}\}$. We recall that the derived series is defined as $\mathcal{G}^1 = \mathcal{G}$, and $\mathcal{G}^{q+1} = [\mathcal{G}^q, \mathcal{G}^q]$.

Then the reduction should be performed using sequentially the symmetries $X_{(1)}$, $X_{(2)}$, ... $X_{(r)}$, i.e. respecting the Lie algebraic structure of $\mathcal{G}$. In this way we obtain a sequence of reduced equations E_k, where E_0 is the original system, E_q the one obtained after reduction by $X_{(1)}, ..., X_{(q)}$, and the reduced system mentioned in Proposition 4 coincides with E_r. $\odot$

Remark 9. It follows immediately from Proposition 4 that, in particular, for $r = n$ the general solution of the system can be found by quadratures. Note that here, and in the statement of Proposition 4, this means performing Ito integrals. $\odot$

Remark 10. In Kozlov's original paper [17] (see Example 4.2 in there) this result is applied to *any* linear two-dimensional system of SDEs; see there for a detailed discussion and results. $\odot$

Remark 11. In view of Proposition 1 (i.e. ultimately of the coincidence of admissible symmetries for an Ito and the associated Stratonovich systems), the proof of Proposition 4 can be obtained following the same approach as for deterministic differential equations, apart from the obvious difference that now quadratures correspond to Ito integrals. An explicit proof (with details) is provided in [11, 17, 18, 23]. $\odot$

5.2 Systems and random symmetries

The approach discussed above, i.e. Kozlov theory, is based on performing changes of variables related to (simple) symmetries of the Ito equation under study. If these symmetries are random ones, we will have to consider (simple) random changes of variables; this introduces an additional problem, as we are then not guaranteed to remain within the class of Ito equations (see the discussion in Section 4.2, in particular Remark 7).

Let us consider a general vector Ito equation (1). If we operate a general simple random change of variables, i.e. pass to consider coordinates

$$y^i \ = \ \Phi^i(x,t;w) \ , \tag{21}$$

leaving the time coordinate t and the Wiener processes $w^k(t)$ unaffected, the equation (1) is mapped into a new equation

$$dy^i \ = \ F^i \, dt \ + \ S^i_{\ k} \, dw^k \ , \tag{22}$$

where the new coefficients F and S are given by

$$F^i \ = \ \frac{\partial \Phi^i}{\partial t} \ + \ f^j \, \frac{\partial \Phi^i}{\partial x^j} \ + \ \frac{1}{2} \, \Delta(\Phi^i) \ , \tag{23}$$

$$S^i_{\ k} \ = \ \frac{\partial \Phi^i}{\partial w^k} \ + \ \sigma^j_{\ k} \, \frac{\partial \Phi^i}{\partial x^j} \ ; \tag{24}$$

see [11, 13] for details of the computation.

It should be stressed that albeit the F and S are given here as functions of the old variables x, as f, σ and Φ^i all depend of them, they should be thought as functions of the new coordinates y^i through the change of variables inverse to (21), which we write as

$$x^i \ = \ \Theta^i(y,t;w) \ . \tag{25}$$

The point is that in general the F, S can and will depend not only on the (y,t) variables, but on the Wiener processes as well (both through the explicit w-dependence of the Φ^i and through the dependence of the Θ^i on the w^k). If this happens, the new equation (22) will *not* be of Ito type (see however Remark 7 in this respect).

Thus we will have a transformed equation (22) again of Ito type if and only if the additional conditions

$$\widehat{\partial}_m F^i \ = \ 0 \ = \ \widehat{\partial}_m S^i_{\ k} \tag{26}$$

are satisfied for all choices of i and k and for all m. In view of the explicit expressions for F and S (see above), these conditions are also written as

$$\left(\partial_t \ + \ f^j \, \partial_j \ + \ \frac{1}{2} \, \Delta \right) \ \left(\widehat{\partial}_m \, \Phi^i \right) \ = \ 0 \ , \tag{27}$$

$$\left(\widehat{\partial}_k \ + \ \sigma^j_{\ k} \, \partial_j \right) \ \left(\widehat{\partial}_m \, \Phi^i \right) \ = \ 0 \tag{28}$$

(these represent a generalization of the similar condition for the case of a scalar equation, determined in [11]).

In other words, we should require that all the components of the gradient of Φ^i w.r.t. the Wiener coordinates w^m belong to the intersection of the kernels of the linear differential operators[6]

$$L_0 \ := \ \partial_t \ + \ f^j \, \partial_j \ + \ \frac{1}{2} \Delta \quad ; \quad L_k \ := \ \widehat{\partial}_k \ + \ \sigma^j_{\ k} \, \partial_j \ . \tag{29}$$

Needless to say, (29) is always satisfied when Φ does not depend on the w^k variables, i.e. for deterministic changes of variables.

We can then easily extend Proposition 4 to the following one, in which we make free use of the notation established in Remark 8.

Proposition 5. *Suppose the system (1) admits an r-parameter* solvable *Lie algebra $\mathcal{G}$ of simple – deterministic or random – symmetries, with generators*

$$\mathbf{X}_{(k)} = \sum_{i=1}^n \varphi^i_k(x,t) \ \frac{\partial}{\partial x_i} \qquad (k = 1, ..., r) \ , \tag{30}$$

acting regularly with r-dimensional orbits. Let these be labeled according to the derived series for $\mathcal{G}$, and let E_q be the equation obtained after reduction by the first q symmetries.

Suppose moreover that, with $\Phi^i_{(k)}$ the maps (21) describing the change of variables associated to the symmetries $X_{(k)}$, the equations (27), (28) are satisfied for equation E_{k-1}.

Then the system (1) can be reduced to a system of $m = (n-r)$ Ito equations,

$$dy^i \ = \ g^i(y^1, ..., y^m; t) \, dt \ + \ \sigma^i_{\ k}(y^1, ..., y^m; t) \, dw^k \qquad (i, k = 1, ..., m) \tag{31}$$

and r "reconstruction equations", the solutions of which can be obtained by (stochastic) quadratures from the solution of the reduced $(n-r)$-order system.

Proof. If equations (27) and (28) are satisfied, we are guaranteed Ito equations are mapped into Ito equations, thus the application of each map associated to symmetries $X_{(k)}$ transform the equation E_{k-1} (and in particular the original system (1)) into one of the same nature (and dimension). Moreover Proposition 1 guarantees that after the application of the map, $X_{(k)}$ is still a symmetry of the new system.

Thus the new system is still of Ito type but with r.h.s. not depending on one of the x^i variables, and if the maps are performed in the proper order, i.e. following the Lie algebraic structure of the symmetry algebra, at each step we eliminate an additional variable with no risk of reintroducing dependencies on previously eliminated ones.

Alternatively, once we are guaranteed to remain within the class of Ito equations, we can deal with the associated Stratonovich ones, which admit the same symmetries [10, 30] and transform according to the standard chain rule. We

[6] In [13] we have proposed the name "Misawa vector fields", see there for the motivation of such a name, for $Y_k = L_k$ and $Y_0 = L_0 - (1/2)\Delta$.

can then proceed as in the case of deterministic equations, and reach the same conclusion, modulo the substitution of standard integrals by stochastic ones in the reconstruction equations. $\triangle$

Remark 12. Note that the conditions (27) and (28) should be checked at each step of the reduction procedure. We do not have determined a criterion to establish *apriori* – i.e. just on the original system – if this will be the case, at least to some order. We also remark that albeit we have seen that reduction can be effective even if it leads us outside the realm of Ito equations (see Remarks 6 and 7), if this is the case for intermediate equations we are not guaranteed the symmetries will be preserved in the reduction procedure. In fact, this results rests on the relation between symmetries of Ito and the equivalent Stratonovich equation, and the matter has not been investigated (neither here nor elsewhere in the literature) for non-Ito equations. $\odot$

6 W-maps and W-symmetries

The definition of simple symmetries can be too restrictive even for obviously invariant systems. We will consider one of these, i.e. the isotropic (linear or non linear) stochastic oscillators, as a motivating example.

6.1 Stochastic oscillators

Consider e.g. the isotropic "stochastic harmonic oscillator" ($i = 1, ..., n$):

$$dx^i \ = \ -x^i \, dt \ + \ dw^i \ , \tag{32}$$

or more generally the system

$$dx^i \ = \ -F(|\mathbf{x}|^2) \, x^i \, dt \ + \ S(|\mathbf{x}|^2) dw^i \ , \tag{33}$$

where F and S are scalar functions of $|\mathbf{x}|^2 = (x^1)^2 + ... + (x^n)^2$ alone, and there are as many independent Wiener processes as x variables. In view of our definition, this is *not* rotationally invariant (under standard, deterministic or random, maps): this is due to the fact that we can rotate the $\mathbf{x} = (x^i, ..., x^n)$ vector, but we are *not* allowed to rotate at the same time also the $\mathbf{w} = (w^i, ..., w^n)$ one.

On the other hand, *if* we consider maps also acting on the w variables, then (33) *is* obviously invariant under simultaneous (identical) rotations in the x *and* in the w variables spaces.

Similarly, (32) – but not (33), in general – is also invariant under a simultaneous identical scaling of the x^i and the w^i variables, $x^i \to \lambda x^i$, $w^i \to \lambda w^i$. (This Example will be considered in more detail below.)

6.2 W maps

It is thus natural to consider also maps acting on the Wiener processes themselves, $w \to z$. At first sight, as we want to have again standard independent Wiener processes (we want to have, in the end, an equation of the same type as the original one, let alone this being exactly the same), the z^i can be at most of the form

$$z^i \ = \ R^i_{\ j} \ w^j \ , \tag{34}$$

with z^m independent unit Wiener processes and R a *constant orthogonal* matrix, $R \in \mathrm{O}(n)$. (This point of view was used in [13]; the discussion given there should be amended with the considerations to follow.)

But this is not entirely correct. In fact, as for rescalings of time, we can allow to obtain a non-standard Wiener process provided the non-standard nature amounts to a scalar (not necessarily constant) factor, which can then be adsorbed by the coefficients S^i_k.

On the other hand, it is essential to preserve the independence of the Wiener processes; in geometrical terms this means that the transformation must preserve the (right) angle between the different Wiener processes.

In other words, we must consider *conformal* transformations, possibly depending on the $(\mathbf{x}, t)$ variables. As we will see, these will be subject to further constraints.

Remark 13. It is well known that the conformal group in a d-dimensional space (with $d \neq 2$; here we will not dwell into the special properties of the conformal group for $d = 2$) is made of translations, certain linear transformations, more precisely orthogonal ones and dilations, and certain quadratic transformations, also known as *special conformal maps* (which are singular in the origin). In our context, obviously translations should be discarded (they would produce stochastic processes which have non-zero average increment and hence are not even a martingale); and we are not willing to admit singular maps or to forbid the process to go through zero. So we are left with linear conformal maps alone.

It is also known that the conformal group in $d \neq 2$ dimensions is isomorphic to the group $SO(d{+}1, 1)$, which gives a practical way to tackle the case of general conformal group, albeit the isomorphism can introduce some computational difficulties. $\odot$

In the present work, for the reasons sketched in Remark 13, we will *not* consider translations nor special conformal maps, and restrict our attention to the simplest sector of linear conformal maps, i.e. rotations and dilations. It will turn out that these have different standings in the present context, see Section 7.3 below.

We will thus consider in general transformations of the type

$$x^i = \Phi^i(y, \theta; z) \ , \quad t = \Theta(\theta) \ , \quad w^k = R^k_{\ m}(y, \theta) \, z^m \ ,$$

with $R = R(y, \theta) \in O(n) \times \mathbf{R}_+$, which we will denote as the linear conformal group. Moreover, we should require $\Theta'(t) > 0$ for all t.

Actually, we know that in Kozlov theory only vector fields with no component along t are of interest, so from now on we will assume $\Theta(t) \equiv t$, and the considered transformations will just be (with again $R \in O(n) \times \mathbf{R}_+$)

$$\begin{aligned} x^i &= \Phi^i(y,t;z) \ , \\ w^k &= h(y,t;z) \ = \ R^k_{\ m}(y,t)\, z^m \ . \end{aligned} \tag{35}$$

We will refer to these as *linear W-maps*, and correspondingly we may have *linear W-symmetries*. Note that "linear" only refers to the action on the sector of the w variables.

The inverse of this map will be written as

$$\begin{aligned} y^i &= \Psi^i(x,t;w) \ , \\ z^k &= A^k_{\ m}(x,t)\, w^m \ ; \end{aligned} \tag{36}$$

here A is again in the linear conformal group. As mentioned above, the new Wiener processes should be independent, i.e. we should require

$$dz^i \cdot dz^j \ = \ \delta^{ij} \ \zeta(x,t)\, dt \ ;$$

it is essential that ζ should not depend on the w, albeit it is – in principles, but see below – allowed to depend on the x,t variables.

By a straightforward application of Ito rule (and, in the last step, restricting to the solutions to (1)) and up to terms of order $o(dt)$, we get

$$\begin{aligned} dz^p &= (\partial_k A^p_{\ j})\, w^j \ dx^k \ + \ (\partial_t A^p_{\ j})\, w^j \ dt \ + \ A^p_{\ j} \ dw^j \ + \ \frac{1}{2} \Delta(A^p_{\ j} w^j) \ dt \ ; \\ dz^p \cdot dz^q &= [A^p_{\ j} A^q_{\ k}] \ (dw^j \cdot dw^k) \ + \ [(\partial_i A^p_{\ j}) w^j (\partial_k A^q_{\ \ell}) w^\ell] \ (dx^i \cdot dx^k) \\ & \quad + [(\partial_i A^p_{\ j}) w^j \ A^q_{\ k}] \ (dx^i \cdot dw^k) \ + \ [A^p_{\ i} \, (\partial_j A^q_{\ \ell} \, w^\ell] \ (dw^i \cdot dx^j) \\ &= [A^p_{\ j} A^q_{\ k}] \ \delta^{jk} \ dt \ + \ [(\partial_i A^p_{\ j}) w^j (\partial_k A^q_{\ \ell}) w^\ell] \ (\sigma^i_{\ r} \sigma^k_{\ s} \delta^{rs}) \ dt \\ & \quad + [(\partial_i A^p_{\ j}) w^j \ A^q_{\ k}] \ (\sigma^i_{\ r} \delta^{kr}) \ dt \ + \ [A^p_{\ k} \, (\partial_j A^q_{\ \ell} \, w^\ell] \ (\sigma^j_{\ s} \delta^{ks}) \ dt \ . \end{aligned}$$

Thus, in order to have $\zeta = \zeta(x,t)$ as required, we must impose that *the A matrices do not depend on the spatial variables x*, i.e. $A = A(t)$; and hence, recalling that $R = A^{-1}$, also that *the R do not depend on the new spatial variables y*, i.e. $R = R(t)$.

We are thus reduced to consider maps (35) of the form

$$\begin{aligned} x^i &= \Phi^i(y,t;z) \ , \\ w^k &= h^m(t,z) \ = \ R^k_{\ m}(t)\, z^m \ , \end{aligned} \tag{37}$$

with $R(t)$ in the linear conformal group.

Note that with this form of h^m, we immediately have

$$\Delta(h^m) \ = \ 0 \ . \tag{38}$$

Moreover, expressing dw^k in terms of the new variables we get

$$dw^k \ = \ R^k_m \, dz^m \ + \ (\partial_t R^k_m) \, z^m \, dt \ . \tag{39}$$

This is not acceptable: in fact we know that a Wiener process has $dw \simeq \sqrt{dt}$. Thus the last term in (39) must be zero, i.e. $\partial_t R^k_m = 0$, i.e. R can not depend on t.

Thus in conclusion, summarizing our discussion in a formal statement, we have:

Lemma 1. *Acceptable W-maps (35) have*

$$h^m = R^m_{\ k} w^k \ , \tag{40}$$

and therefore

$$dw^k \ = \ R^k_m \, dz^m \ . \tag{41}$$

With this, (37) further reduce to

$$\begin{aligned} x^i \ &= \ \Phi^i(y,t;z) \ , \\ w^k \ &= \ A^k_{\ m} \, z^m \ , \end{aligned} \tag{42}$$

with A a constant matrix in the linear conformal group.

When we consider infinitesimal maps, we have

$$\begin{aligned} x^i \ &= \ \Phi^i(y,t;z) \ = \ y^i \ + \ \varepsilon \, \varphi^i(y,t;z) \ , \\ w^k \ &= \ A^k_{\ m} \, z^m \ = \ z^k \ + \ \varepsilon \, R^k_{\ m} \, z^m \ , \end{aligned}$$

hence these will be generated by vector fields

$$X \ = \ \varphi^i(x,t;w) \, \partial_i \ + \ \left(R^k_{\ m} \, w^m \right) \, \widehat{\partial}_k \ . \tag{43}$$

6.3 W symmetries

We will then proceed as usual in order to determine the effect of the map (42) on the Ito equation (1). It follows from (42) that (we stress that ∂_i, $\widehat{\partial}_k$ and Δ are now defined w.r.t. the new set of variables)

$$dx^i \ = \ (\partial_t \Phi^i) \, dt \ + \ (\partial_j \Phi^i) \, dy^j \ + \ \frac{1}{2} (\Delta \Phi^i) \, dt \ + \ (\widehat{\partial}_m \Phi^i) \, dz^m \ . \tag{44}$$

Comparing this with the Ito equation under study (1) and writing for ease of notation

$$M^i_{\ j} \ := \ \frac{\partial \Phi^i}{\partial y^j} \ , \tag{45}$$

we readily obtain

$$M^i_{\ j} \, dy^j \ = \ \left(\widetilde{f}^i - \partial_t \Phi^i - \frac{1}{2} \Delta \Phi^i \right) \, dt \ + \ \widetilde{\sigma}^i_{\ k} \, dw^k \ - \ (\widehat{\partial}_m \Phi^i) \, dz^m \ . \tag{46}$$

We stress that now f and σ should be thought as functions of the new variables; thus we introduced

$$\widetilde{f}^i(y,t;z) \ := \ f^i[\Phi(y,t;z),t] \ , \ \ \widetilde{\sigma}^i{}_k(y,t:z) \ := \ \sigma^i{}_k[\Phi(y,t;z),t] \ .$$

Note that by assumption M is invertible, as the map (42) provides a change of variables; we will denote the inverse of M by Λ,

$$\Lambda^i{}_j \ = \ \frac{\partial y^i}{\partial x^j} \ . \tag{47}$$

In (46) we should still express dw^k in the new variables, i.e. use (41). Inserting this into (46) we get

$$M^i{}_j \ dy^j \ = \ \left(\widetilde{f}^i - \partial_t \Phi^i - \frac{1}{2} \Delta \Phi^i \right) \ dt \ + \ \left[(\widehat{\partial}_m \Phi^i) \ + \ \widetilde{\sigma}^i{}_k \, R^k{}_m \right] \ dz^m \ . \tag{48}$$

Multiplying by $M^{-1} = \Lambda$, (48) yields finally

$$\begin{aligned} dy^i \ &= \ \Lambda^i_j \left[\widetilde{f}^j - \partial_t \Phi^j - \frac{1}{2} \Delta \Phi^i \right] \ dt \ + \ \Lambda^i_j \ \left[\widehat{\partial}_m \Phi^j + \widetilde{\sigma}^j_k R^k_m \right] \ dz^m \\ &:= \ F^i \, dt \ + \ S^i{}_m \, dz^m \ , \end{aligned} \tag{49}$$

where we have of course introduced the compact notation

$$F^i \ = \ \Lambda^i_j \left(\widetilde{f}^j - \partial_t \Phi^j - \frac{1}{2} \Delta \Phi^j \right) \ , \ \ S^i{}_m \ = \ \Lambda^i_j \left[\widehat{\partial}_m \Phi^j + \widetilde{\sigma}^j_k R^k_m \right] \ . \tag{50}$$

Remark 14. For (49) to be again an Ito equation, we need that both the conditions

$$(\partial F^i / \partial z^\ell) = 0 \ , \quad (\partial S^i_m / \partial z^\ell) = 0 \tag{51}$$

hold, for all i, m, ℓ. These equations provide the further limitation on the form of Φ and h, i.e. R, mentioned above. On the other hand, as already remarked (see in particular Remarks 6 and 7), the requirement to stay within the class of Ito equations can be too restrictive for a number of concrete applications. $\odot$

6.4 Split W-symmetries

We note that in the case we have a change of variables of the simpler form

$$\begin{aligned} x^i \ &= \ \Phi^i(y,t) \ , \\ w^k \ &= \ R^k{}_m \, z^m \ , \end{aligned} \tag{52}$$

i.e. when the change of variables does not mix the spatial variables and the Wiener processes, the situation is substantially simpler.

In fact, now M, Λ, $\widetilde{f}$ and $\widetilde{\sigma}$ are all independent of z, and both equations (51) are always satisfied.

We have thus identified a simple class of W-maps, (52), which is guaranteed to map Ito equations into Ito equations.

As in this case the spatial variables and the Wiener process transform independently of each other, we will refer to this class of maps as *split W-maps*; our discussion above shows that:

Lemma 2. *Split W-maps transform Ito equations into Ito equations.*

If some split W-maps leave a given equation invariant, we will speak of *split W-symmetries*.

Remark 15. We anticipate and stress that the (Kozlov-type) change of variables associated to a split W-symmetry rectifying it – see Section 8 – is in general *not* a split W-map. $\odot$

Remark 16. It is interesting to note that vector fields, including symmetries, transform in a specially simple way under split W-maps. In fact, in general

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} + \frac{\partial z^m}{\partial x^i} \frac{\partial}{\partial z^m} , \\ \frac{\partial}{\partial w^k} &= \frac{\partial y^j}{\partial w^k} \frac{\partial}{\partial y^j} + \frac{\partial z^m}{\partial w^k} \frac{\partial}{\partial z^m} ; \end{aligned}$$

actually, as we have seen that in admissible maps z's do not depend on x's,

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} , \quad \frac{\partial}{\partial w^k} = \frac{\partial y^j}{\partial w^k} \frac{\partial}{\partial y^j} + \frac{\partial z^m}{\partial w^k} \frac{\partial}{\partial z^m} . \tag{53}$$

Moreover, for split W-maps we also have $\partial y/\partial w = 0$, and in general $\partial y/\partial x = \Lambda$; hence these reduce to

$$\frac{\partial}{\partial x^i} = \Lambda^j_{\ i} \frac{\partial}{\partial y^j} , \quad \frac{\partial}{\partial w^k} = \frac{\partial z^m}{\partial w^k} \frac{\partial}{\partial z^m} . \tag{54}$$

Note also that, with $Q = R^{-1}$, it follows from (42) – hence for all kind of W-maps – that $\partial z^m/\partial w^k = Q^m_{\ k}$; thus (53) and (54) are respectively

$$\frac{\partial}{\partial x^i} = \Lambda^j_{\ i} \frac{\partial}{\partial y^j} , \quad \frac{\partial}{\partial w^k} = \frac{\partial y^j}{\partial w^k} \frac{\partial}{\partial y^j} + Q^m_{\ k} \frac{\partial}{\partial z^m} \tag{55}$$

in the general case; and

$$\frac{\partial}{\partial x^i} = \Lambda^j_{\ i} \frac{\partial}{\partial y^j} , \quad \frac{\partial}{\partial w^k} = Q^m_{\ k} \frac{\partial}{\partial z^m} . \tag{56}$$

in the split one. $\odot$

Remark 17. Note that the simultaneous rotations in x and in w space considered in Sect.6.1 correspond to a split W-symmetry. $\odot$

7 W-symmetries of Ito versus associated Stratonovich equations

The main tool allowing for an effective description and use of standard (deterministic or random) admissible symmetries of Ito equation is the result identifying these with symmetries of the associated Stratonovich equation; see Section 3 above.

We thus wonder if a similar result also holds for W-symmetries; or at least for split W-symmetries, or under some additional condition. The present Section provides an answer to this question.

7.1 Determining equations for W-symmetries of Ito equations

We consider the Ito equation (1) and act on it by the simple vector field

$$X \ = \ \varphi^i(x,t;w)\,\partial_i \ + \ h^k(x,t;w)\,\widehat{\partial}_k \tag{57}$$

(note that at the moment we are not restricting the form of h^k; see Remark 19 below about this).

The action of X is described by

$$x^i \ \to \ x^i \ + \ \varepsilon\,\varphi^i(x,t;w) \ , \quad w^k \ \to \ w^k \ + \ \varepsilon\,h^k(x,t;w) \ , \tag{58}$$

while t remains unaffected.

With standard computations, using also (1) itself, we obtain that, at first order in ε,

$$\begin{aligned}
dx^i \quad &\to \quad dx^i \ + \ \varepsilon \ \left[(\partial_t\varphi^i)dt + (\partial_j\varphi^i)dx^j + (\widehat{\partial}_k\varphi^i)dw^k + \frac{1}{2}\Delta(\varphi^i)dt\right] \\
&= \quad dx^i \ + \ \varepsilon \ \left[\left(\partial_t\varphi^i + f^j\partial_j\varphi^i + \frac{1}{2}\Delta\varphi^i\right)\,dt \ + \ \left(\widehat{\partial}_k\varphi^i + \sigma^j{}_k\partial_j\varphi^i\right)\,dw^k\right] \ , \\
dw^k \quad &\to \quad dw^k \ + \ \varepsilon \ \left[(\partial_t h^k)dt + (\partial_j h^k)dx^j + (\widehat{\partial}_m h^k)dw^m + \frac{1}{2}(\Delta h^k)dt\right] \\
&= \quad dw^k \ + \ \varepsilon \ \left[\left(\partial_t h^k + f^j\partial_j h^k + \frac{1}{2}\Delta h^k\right)\,dt \ + \ \left(\widehat{\partial}_m h^k + \sigma^j{}_m\partial_j h^k\right)\,dw^m\right] \ ; \\
f^i \quad &\to \quad f^i \ + \ \varepsilon\,\varphi^j\,\partial_j f^i \ , \\
\sigma^i{}_k \quad &\to \quad \sigma^i{}_k \ + \ \varepsilon\,\varphi^j\,\partial_j\sigma^i{}_k \ .
\end{aligned}$$

With these and some standard computations, it is easy to check that the condition for the equation to remain invariant is that the following equations hold for all i and k:

$$\partial_t\varphi^i \ + \ (f^j\partial_j\varphi^i - \varphi^j\partial_j f^i) \ + \ \frac{1}{2}\,\Delta\varphi^i \quad = \quad \sigma^i{}_k\,\left(\partial_t h^k + f^j\partial_j h^k + \frac{1}{2}\Delta h^k\right) \tag{59}$$

$$\widehat{\partial}_k\varphi^i \ + \ (\sigma^j{}_k\partial_j\varphi^i - \varphi^j\partial_j\sigma^i{}_k) \quad = \quad \sigma^i{}_m\,\left(\widehat{\partial}_k h^m + \sigma^j{}_k\partial_j h^m\right) \ . \tag{60}$$

These were obtained for a *general* h; but we have seen in Lemma 1 that h is of the form (40), hence the above equations further reduce and we have:

Lemma 3 *The* determining equations *for (general simple) W-symmetries of the Ito equation (1) are*

$$\begin{aligned} \partial_t \varphi^i \ + \ (f^j \partial_j \varphi^i \ - \ \varphi^j \partial_j f^i) \ + \ \frac{1}{2} \Delta \varphi^i \ &= \ 0 \ , \qquad (61) \\ \widehat{\partial}_k \varphi^i \ + \ ({\sigma^j}_k \partial_j \varphi^i \ - \ \varphi^j \partial_j {\sigma^i}_k) \ - \ {\sigma^i}_m \ R^m_k \ &= \ 0 \ . \qquad (62) \end{aligned}$$

7.2 Determining equations for W-symmetries of Stratonovich equations

The computations are pretty much similar – at the exception of using the chain rule rather than the Ito one – when considering the Stratonovich equation

$$dx^i \ = \ b^i(x,t) \, dt \ + \ {\sigma^i}_k(x,t) \circ dw^k \ ; \qquad (63)$$

in this case we obtain the determining equations for (general simple) W-symmetries of the Stratonovich equation (63) in the form

$$\begin{aligned} \partial_t \varphi^i \ \ &+ \ \ (b^j \partial_j \varphi^i \ - \ \varphi^j \partial_j b^i) \ = \ {\sigma^i}_k \ \left(\partial_t h^k \ + \ b^j \partial_j h^k \right) \ , \qquad (64) \\ \widehat{\partial}_k \varphi^i \ \ &+ \ \ ({\sigma^j}_k \partial_j \varphi^i \ - \ \varphi^j \partial_j {\sigma^i}_k) \ = \ {\sigma^i}_m \ \left(\widehat{\partial}_k h^m \ + \ {\sigma^j}_k \partial_j h^m \right) \ . \qquad (65) \end{aligned}$$

.

We note immediately that (65) coincides with (62), and this with *general* form of h^k. For h as dictated by Lemma 1, i.e. as in (40), the above equations are simplified, and we get:

Lemma 4. *The* determining equations *for (general simple) W-symmetries of the Stratonovich equation (63) are*

$$\begin{aligned} \partial_t \varphi^i \ \ &+ \ \ (b^j \partial_j \varphi^i \ - \ \varphi^j \partial_j b^i) \ = \ 0 \ , \qquad (66) \\ \widehat{\partial}_k \varphi^i \ \ &+ \ \ ({\sigma^j}_k \partial_j \varphi^i \ - \ \varphi^j \partial_j {\sigma^i}_k) \ = \ {\sigma^i}_m \ R^m_k \ . \qquad (67) \end{aligned}$$

.

7.3 The relation between symmetries of an Ito and of the associated Stratonovich equations

In order to compare symmetries of the equations (1) and (63), we should require that (63) is just the equation associated to (1). As well known, this amounts to requiring that

$$f^i(x,t) \ = \ b^i(x,t) \ + \ \rho^i(x,t) \ ; \quad \rho^i \ = \ \frac{1}{2} \, (\partial_k \sigma^{ij}) \, {\sigma^k}_j \ . \qquad (68)$$

Using this, (59) is rewritten as

$$\begin{aligned} \partial_t \varphi^i &+ (b^j \partial_j \varphi^i - \varphi^j \partial_j b^i) + (\rho^j \partial_j \varphi^i - \varphi^j \partial_j \rho^i) + \frac{1}{2} \Delta \varphi^i \\ &= \sigma^i_{\ k} \left(\partial_t h^k + b^j \partial_j h^k + \rho^j \partial_j h^k + \frac{1}{2} \Delta h^k \right) . \end{aligned}$$

Subtracting (64) from this, we get

$$(\rho^j \partial_j \varphi^i - \varphi^j \partial_j \rho^i) + \frac{1}{2} \Delta \varphi^i = \sigma^i_{\ k} \left(\rho^j \partial_j h^k + \frac{1}{2} \Delta h^k \right) ; \tag{69}$$

recalling the definition of ρ in terms of σ, see (68), this reads

$$(\partial_k \sigma^{jm}) \, \sigma^k_{\ m} \, (\partial_j \varphi^i) - \varphi^j \partial_j [(\partial_k \sigma^{im}) \, \sigma^k_{\ m}] + \Delta \varphi^i = \sigma^i_{\ k} \left[(\partial_q \sigma^{jm}) \, \sigma^q_{\ m} \, (\partial_j h^k) + \Delta h^k \right] . \tag{70}$$

Moreover, if we restrict to linear W-symmetries, $h = Rz$, the terms in the square bracket on the r.h.s.[7] both vanish, and we are just left with

$$(\partial_k \sigma^{jm}) \, \sigma^k_{\ m} \, (\partial_j \varphi^i) - \varphi^j \partial_j [(\partial_k \sigma^{im}) \, \sigma^k_{\ m}] + \Delta \varphi^i = 0 , \tag{71}$$

which we also write as

$$\Delta(\varphi^i) = \Sigma(\varphi^i) , \tag{72}$$

having of course defined

$$\Sigma(\varphi^i) := \varphi^j \partial_j [(\partial_k \sigma^{im}) \, \sigma^k_{\ m}] - (\partial_k \sigma^{jm}) \, \sigma^k_{\ m} \, (\partial_j \varphi^i) . \tag{73}$$

We stress that in order to have the same symmetries for the Ito and the associated Stratonovich equations, it is not required that (71) holds in general, but only that it holds when (67) is also satisfied. That is, we can substitute in (71) – in particular, in the term $\Delta \varphi^i$ – for the derivatives of φ^i (involving derivation w.r.t. at least one Wiener process) according to (67) and its differential consequences. Recalling that σ does not depend on the w^k and that R is constant, we get:

$$\begin{aligned} \widehat{\partial}_k \varphi^i &= (\varphi^j \partial_j \sigma^i_{\ k} - \sigma^j_{\ k} \partial_j \varphi^i) + \sigma^i_{\ m} \, R^m_k , \\ \partial_\ell \widehat{\partial}_k \varphi^i &= \Big((\partial_\ell \varphi^j)(\partial_j \sigma^i_{\ k}) + \varphi^j (\partial_\ell \partial_j \sigma^i_{\ k}) - (\partial_\ell \sigma^j_{\ k})(\partial_j \varphi^i) - \sigma^j_{\ k} (\partial_\ell \partial_j \varphi^i) \Big) \\ &\quad + (\partial_\ell \sigma^i_{\ m}) \, R^m_k ; \\ \widehat{\partial}_m \widehat{\partial}_k \varphi^i &= \Big((\widehat{\partial}_m \varphi^j)(\partial_j \sigma^i_{\ k}) - \sigma^\ell_{\ k} (\widehat{\partial}_m \partial_\ell \varphi^i) \Big) \\ &= [(\varphi^p \partial_p \sigma^j_{\ m} - \sigma^p_{\ m} \partial_p \varphi^j) + \sigma^j_{\ q} \, R^q_{\ m}](\partial_j \sigma^i_{\ k}) \\ &\quad - \sigma^j_{\ k} \left[(\partial_j \varphi^p)(\partial_p \sigma^i_{\ m}) + \varphi^p (\partial_j \partial_p \sigma^i_{\ m}) - (\partial_j \sigma^p_{\ m})(\partial_p \varphi^i) \right. \\ &\quad \left. - \sigma^p_{\ m} (\partial_j \partial_p \varphi^i) + (\partial_j \sigma^i_{\ q}) \, R^q_{\ m} \right] . \end{aligned}$$

[7] These are the only ones depending on h, so apparently we obtain a condition which is independent of h when this satisfies the condition set in Lemma 1; but a dependence on h will be introduced when we restrict to solutions to the common set of determining equations (62), (67), see below.

We can now insert these expressions into the explicit form (10) of $\Delta\varphi$. [8]. One obtains in this way

$$\begin{aligned}
\Delta\varphi^i \ &= \ \sigma^{jk}\sigma^m_{\ k}\partial_j\partial_m\varphi^i \ + \ 2\sigma^{\ell k}\left[(\partial_\ell\varphi^j)(\partial_j\sigma^i_{\ k}) + \varphi^j(\partial_\ell\partial_j\sigma^i_{\ k})\right. \\
&\quad \left. -(\partial_\ell\sigma^j_{\ k})(\partial_j\varphi^i) - \sigma^j_{\ k}(\partial_\ell\partial_j\varphi^i) + (\partial_\ell\sigma^i_{\ p})R^p_{\ k}\right] \\
&\quad + \delta^{\ell k}\left[\varphi^p(\partial_p\sigma^j_{\ \ell}) - \sigma^p_{\ \ell}(\partial_p\varphi^j) + \sigma^j_{\ q}R^q_{\ \ell}\right](\partial_j\sigma^i_{\ k}) \\
&\quad - \delta^{\ell k}\,\sigma^j_{\ k}\left[(\partial_j\varphi^p)(\partial_p\sigma^i_{\ \ell}) + \varphi^p(\partial_j\partial_p\sigma^i_{\ \ell}) - (\partial_j\sigma^p_{\ \ell})(\partial_p\varphi^i)\right. \\
&\quad \left. -\sigma^p_{\ \ell}(\partial_j\partial_p\varphi^i) + (\partial_j\sigma^i_{\ q})R^q_{\ \ell}\right] \\
&= \ \left[\sigma^{\ell k}(\partial_\ell\partial_j\sigma^i_{\ k}) + (\partial_j\sigma^{\ell k})(\partial_\ell\sigma^i_{\ k})\right]\varphi^j \ - \ \sigma^{jk}(\partial_j\sigma^p_{\ k})(\partial_p\varphi^i) \\
&\quad + \ \left[\sigma^{\ell k}(\partial_\ell\sigma^i_{\ p}) - \sigma^\ell_{\ p}(\partial_\ell\sigma^{ik})\right] \ R^p_{\ k} \\
&= \ \varphi^j\,\partial_j[\sigma^{\ell k}(\partial_\ell\sigma^i_{\ k})] \ - \ [\sigma^{\ell k}(\partial_\ell\sigma^j_{\ k})]\,\partial_j\varphi^i \ + \ \sigma^{\ell k}(\partial_\ell\sigma^{ip})\,[R_{pk} - R_{kp}] \\
&= \ \Sigma(\varphi^i) \ + \ \sigma^{\ell k}(\partial_\ell\sigma^{ip})\,[R_{pk} + R_{kp}] \ .
\end{aligned}$$

In the last step we have used the definition (73) of $\Sigma(\varphi^i)$.

We denote by

$$\mathcal{R}(\sigma) \ := \ \sigma^{\ell k}(\partial_\ell\sigma^{ip})\,[R_{pk} + R_{kp}] \tag{74}$$

the term depending on R in the final result above; note that $\mathcal{R}(\sigma)$ is identically zero for constant σ. In general (71) is satisfied if and only if $\mathcal{R}(\sigma) = 0$ on solutions to (67). (We recall that our computation was indeed based on restricting to solutions to (67).)

As already mentioned, the generators of the linear conformal groups correspond to rotations and dilations; it is also well known that the generators of rotations are skew-symmetric matrices, while the generator of dilations is a diagonal matrix.[9]

We conclude immediately that if only the rotation part is present in R, then $\mathcal{R}(\varphi^i) = 0$, while if the dilations are also present we have in general, unless σ satisfies

$$\left[\sigma^{jm}\,(\partial_j\sigma^i_{\ q}) \ + \ \sigma^j_{\ q}\,(\partial_j\sigma^{im})\right] \ R^q_{\ m} \ = \ 0 \ , \tag{75}$$

that $\mathcal{R}(\varphi^i) \neq 0$ and hence we *can* have a difference between symmetries of the Ito and the associated Stratonovich equations. This will be explicitly shown to be the case in Example 5 below.

Note that (75) is always satisfied (for whatever R) if and only if σ is constant w.r.t. the spatial variables x^i. Actually, as we know that the only "dangerous" situation is that with R diagonal (generating dilations), it is immediate to check that in this case we have $\mathcal{R} = 0$ if and only if σ is spatially constant.

We also recall that we have considered maps not acting on time; thus it is not surprising that time derivatives of σ do not appear to play a role.

[8] Note that here we will raise and lower indices – also inside spatial derivatives – making use of the assumption we are working in an Euclidean space; it would be interesting to study if one obtains different results in a general Riemannian manifold.

[9] We recall that we are considering infinitesimal (near-identity) maps, see (58); hence we are dealing with the connected component of the identity in the group, and with generators.

We can summarize our discussion as follows

Theorem 1. *All the rotation linear W-symmetries of an Ito equation are also symmetries of the associated Stratonovich equation, and viceversa. Dilation W-symmetries of an Ito equation are also symmetries of the associated Stratonovich equation (and vice versa) if and only if the diffusion matrix is spatially constant.*

Corollary 1. *If the diffusion matrix $\sigma^i{}_k$ in (1) is constant w.r.t. space variables, then all W-symmetries of the Ito equation are also symmetries of the corresponding Stratonovich equation.*

It is also a simple consequence of the above Theorem 1 that Proposition 1 extends to this kind of symmetries:

Theorem 2. *Rotation linear W-symmetries of an Ito equation (1) are preserved under changes of variables defined by an admissible W-map $x^i = \Phi^i(y,t;z)$, $w^k = R^k{}_m z^m$. The same applies to all linear W-symmetries if the diffusion matrix is constant w.r.t. space variables.*

Remark 18. We stress that the limitation (to linear structure) only regards the symmetry vector field, while – as clear from our previous discussion – it does not affect the form of the considered W-map. On the other hand, in practice we will consider W-maps associated to W-symmetries, and as these are linear (we recall that here "linear" only refers to the w component of the symmetry and map) the W-maps will also be linear, at the exception of dilations. $\odot$

Remark 19. As stressed above, see Remarks 6 and 7, in some cases one may wish to consider equations more general than Ito ones. Similarly, one may wish to consider maps such that the random processes underlying the x stochastic processes are allowed to be more general than Wiener ones. This is why we have taken the seemingly odd choice of performing a part of our computations considering general $h(x,t;w)$ functions, albeit in the present paper we are only interested in the linear case $h = Rz$. $\odot$

7.4 Examples

We will now consider some explicit Examples illustrating our results and in particular Theorem 1. As we deal with time-autonomous equations, we will consider time-independent symmetries, thus slightly simplifying the discussion and the (intermediate) explicit formulas.

Example 3. We consider the scalar Ito equation

$$dx \ = \ \lambda x \, dt \ + \ \mu \, dw \ , \tag{76}$$

with λ and μ nonzero real constants. In this case (as always for a constant diffusion coefficient) the associated Stratonovich equation reads just in the same way, i.e. $b(x,t) = f(x,t) = \lambda x$.

As for the determining equations, those for the Ito equation, i.e. (61) and (62), read

$$\lambda\, x\, \varphi_x \ - \ \lambda \varphi \ + \ \frac{1}{2} \Delta(\varphi) \ = \ 0 \ , \quad \varphi_w \ + \ \mu\, \varphi_x \ = \ \mu\, R \ ;$$

while those for the associated Stratonovich equation, i.e. (66) and (67), read

$$\lambda\, x\, \varphi_x \ - \ \lambda \varphi \ = \ 0 \ , \quad \varphi_w \ + \ \mu\, \varphi_x \ = \ \mu\, R \ .$$

Thus the second equation in the two sets is the same (as always), while the first ones are different. However, when we restrict to solutions of the second equation, i.e. to

$$\varphi \ = \ R\, x \ + \ \Theta\, [\zeta] \ , \qquad \zeta := w - x/\mu \ ,$$

the two equations coincide. Hence the symmetries of the Ito and of the associated Stratonovich equation coincide, as stated by our Theorem 1.

Actually, one finds immediately that enforcing also the first equation requires $\Theta = 0$, thus the symmetries reduce to the obvious scaling one, $(x, w) \to (sx, sw)$. This one-parameter group ($s \in \mathbf{R}_+$) is generated by the vector field

$$X \ = \ x\, \partial_x \ + \ w\, \partial_w \ .$$

Note that here we have $\phi = x$, $R = 1$. $\odot$

Example 4. Consider more generally the scalar Ito equation

$$dx \ = \ \lambda\, x \ dt \ + \ \mu\, x^\alpha \ dw \ ; \tag{77}$$

here again λ, μ are nonzero real constants. For $\alpha = 0$ this reduces to the previous Example, so we assume the real constant α is also nonzero.

It is clear that this equation is invariant under the scalings $(x, w) \to (sx, s^{1-\alpha} w)$, generated by the vector field

$$X \ = \ x\, \partial_x \ + \ (1 - \alpha)\, \partial_w \ ;$$

note the case $\alpha = 1$ is not of interest here, as it does not correspond to a W-symmetry.

In this case the associated Stratonovich equation is

$$dx \ = \ \left(\lambda x \ - \ \frac{1}{2}\, \alpha\, \mu^2\, x^{(2\alpha - 1)} \right) \ dt \ + \ \mu\, x^\alpha \ dw \ . \tag{78}$$

It is quite obvious that this equation is invariant under the scaling mentioned above if and only if $2\alpha - 1 = 1$, i.e. for the "uninteresting case" $\alpha = 1$.

The second equation in the set of determining ones is, in both cases,

$$\varphi_w \ + \ \mu\, x^\alpha \varphi_x \ - \ \alpha\, \mu\, x^{\alpha - 1}\, \varphi \ = \ \mu\, x^\alpha\, R \ . \tag{79}$$

The most general solution to this equation is

$$\varphi(x,w) \ = \ \frac{R}{\alpha-1}\, x \ + \ \Theta \left[\frac{x + (\alpha-1)\mu x^\alpha w}{(\alpha-1)\mu x^\alpha} \right] \ ,$$

with Θ an arbitrary function. When we consider (61) we obtain that Θ must be zero. We are thus left with vector fields of the form $X = [R/(\alpha-1)](x\partial_x + (\alpha-1)w\partial_w)$; we can of course choose $R = \alpha - 1$, hence $\varphi = x$, and we are left with the symmetry generator

$$X \ = \ x\, \partial_x \ + \ (\alpha-1) w\, \partial_w \ . \tag{80}$$

Direct substitution in (66) shows that (unless $\alpha = 1$) this is *not* a symmetry for the associated Stratonovich equation. In fact, that equation reduces now to the identity

$$\alpha\, (\alpha - 1)\, \mu^2 \ x^{2\alpha-1} \ = \ 0 \ ,$$

which is satisfied only in the cases we have excluded ($\mu = 0$, $\alpha = 0$, $\alpha = 1$). ⊙

Example 5. In the one-dimensional case one will most frequently find scaling symmetries, if any, but it is possible to build some (admittedly, rather artificial) example which admits a nonlinear $\varphi(x)$. To this aim, consider the equation

$$dx \ = \ f(x)\, dt \ + \ \sigma(x)\, dw$$

with the functions

$$\begin{aligned} f(x) \ &:= \ c_1\, x^2 \ + \ x^2 \ \left(x \exp[2/x] \ - \ 2\, \mathtt{Ei}[2/x] \right) \ , \\ \sigma(x) \ &:= \ c_1\, x^2 \ \exp[1/x] \ , \end{aligned}$$

with $\mathtt{Ei}(z)$ denoting the exponential integral function

$$\mathtt{Ei}(z) \ = \ - \int_{-z}^{\infty} \frac{e^{-t}}{t}\, dt$$

(the principal value is taken here).

In this case we have (only) the W-symmetry vector field

$$X \ = \ x^2\, \partial_x \ + \ w\, \partial_w \ .$$

Obviously this example was built by reverse engineering, i.e. assigning φ, R and looking at the determining equations as equations for φ, σ.

One can check that this X is not a symmetry for the associated Stratonovich equation. ⊙

Example 6. We will now consider a multi-dimensional generalization of Example 3, i.e. the stochastic linear oscillator (no sum on i in this Example)

$$dx^i \ = \ \lambda_i\, x^i \ dt \ + \ \mu_i\, dw^i \quad (i = 1, ..., n) \ . \tag{81}$$

It is clear that this will have scaling symmetries, and in the isotropic case $\lambda_i = \lambda$, $\mu_i = \mu$ also rotation symmetries (with partial rotation symmetries in case of partially isotropic oscillator). In this Example we will not assume any relation between the constants; in the next Example we will consider the isotropic linear oscillator.

Let us discuss in detail the case $n = 2$, assuming all diffusion constants μ_i appearing in the system are nonzero; the general case would be not too different. We will write all indices as lower ones for typographical convenience and in order to avoid any possible confusion.

In this $n = 2$ case the second set of determining equations (common to the Ito and the Stratonovich case) is solved by the method of characteristics and yields the general solutions

$$\begin{aligned} \varphi_1 &= R_{11}\, x_1 \;+\; \frac{\mu_1}{\mu_2}\, R_{12}\, x_2 \;+\; \psi_1(z_1,z_2)\ , &(82)\\ \varphi_2 &= \frac{\mu_2}{\mu_1} R_{21}\, x_1 \;+\; R_{22}\, x_2 \;+\; \psi_2(z_1,z_2)\ , &(83)\end{aligned}$$

where we have written

$$z_1 \;:=\; w_1 \;-\; \frac{x_1}{\mu_1}\ , \quad z_2 \;:=\; w_2 \;-\; \frac{x_2}{\mu_2}\ ,$$

and ψ_i are arbitrary smooth functions of their arguments (z_1, z_2).

Now the first set of determining equations for the Ito equations reads

$$\begin{aligned} \lambda_1\,\psi_1 \;&+\; \left(\frac{\lambda_1(\partial\psi_1/\partial z_1)}{\mu_1}\right) x_1 \\ &+\; \left(\frac{\lambda_1\mu_1^2 R_{12} \;-\; \lambda_2\mu_1^2 R_{12} \;+\; \lambda_2\mu_1(\partial\psi_1/\partial z_2)}{\mu_1\,\mu_2}\right) x_2 \;=\; 0\ , &(84)\\ \lambda_2\,\psi_2 \;&-\; \left(\frac{(\lambda_1-\lambda_2)\mu_2 R_{21} - \lambda_1(\partial\psi_2/\partial z_1)}{\mu_1}\right) x_1 \\ &+\; \left(\frac{\lambda_2(\partial\psi_2/\partial z_2)}{\mu_2}\right) x_2 \;=\; 0\ . &(85)\end{aligned}$$

These two (uncoupled) equations are again solved by the method of characteristics. Recalling that $\psi_i = \psi_i(z_1, z_2)$ we readily get that for $\lambda_1 \neq 0 \neq \lambda_2$ we necessarily have $\psi_1 = 0 = \psi_2$. With these, we are reduced to

$$\begin{aligned} \frac{\mu_2}{\mu_1}\,(\lambda_2-\lambda_1)\,R_{12} &= 0\ , &(86)\\ \frac{\mu_2}{\mu_1}\,(\lambda_1-\lambda_2)\,R_{21} &= 0\ ; &(87)\end{aligned}$$

thus in the case $\lambda_1 \neq \lambda_2$ we necessarily have $R_{12} = R_{21} = 0$. Finally, the coefficient of the symmetry vector field are

$$\varphi^1 \;=\; R_{11}\,x_1\ ,\ \ \varphi^2 \;=\; R_{22}\,x_2\ .$$

Thus we have two scaling symmetries:

$$Y_1 \ = \ x_1 \, \partial_1 \ + \ w_1 \, \widehat{\partial}_1 \ , \ \ Y_2 \ = \ x_2 \, \partial_2 \ + \ w_2 \, \widehat{\partial}_2 \ .$$

It is more convenient to consider the sum and difference of these two, providing

$$\begin{aligned} X_1 &= Y_1 + Y_2 \ = \ x_1 \, \partial_1 \ + \ x_2 \, \partial_2 \ + \ w_1 \, \widehat{\partial}_1 \ + \ w_2 \widehat{\partial}_2 \ , && (88) \\ X_2 &= Y_1 - Y_2 \ = \ x_1 \, \partial_1 \ - \ x_2 \, \partial_2 \ + \ w_1 \, \widehat{\partial}_1 \ - \ w_2 \widehat{\partial}_2 \ , && (89) \\ &. && (90) \end{aligned}$$

The R matrices associated to these are respectively

$$R_1 \ = \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad R_2 \ = \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The first one generates scalings $(w_1, w_2) \mapsto (s w_1, s w_2)$ and hence conformal maps, while the second one generates the one parameter group $(w_1, w_2) \mapsto (s w_1, s^{-1} w_2)$ and hence does not correspond to conformal maps.

Thus X_1 is an acceptable W-symmetry generator, while X_2 fails to preserve the independence of the Wiener processes and is therefore *not* an acceptable W-symmetry generator.

One can check by explicit computation that X_1, X_2 (or more precisely the corresponding vectors φ and matrices R) also satisfy the determining equations for symmetries of the associated Stratonovich equation. ⊙

Example 7. We will now consider the *isotropic* stochastic linear oscillator

$$d x^i \ = \ \lambda \, x^i \ dt \ + \ \mu \, d w^i \quad (i = 1, ..., n) \ . \tag{91}$$

Now after solving the common set of determining equations we have (we write again all indices as lower ones to avoid confusion)

$$\varphi_1 \ = \ \psi_1 \ + \ R_{11} \, x_1 \ + \ R_{12} \, x_2 \ , \quad \varphi_2 \ = \ \psi_2 \ + \ R_{21} x_1 \ + \ R_{22} \, x_2 \ . \tag{92}$$

Plugging this into the equations (61), (62) we obtain again that $\psi_i = 0$, but now the final result is that

$$\varphi_1 \ = \ R_{11} \, x_1 \ + \ R_{22} \, x_2 \ , \quad \varphi_2 \ = \ R_{21} x_1 \ + \ R_{22} \, x_2 \ . \tag{93}$$

This leaves us with four symmetry vector fields,

$$\begin{aligned} Y_1 &= x_1 \, \partial_1 \ + \ w_1 \, \widehat{\partial}_1 \ , \ \ Y_2 \ = \ x_2 \, \partial_2 \ + \ w_2 \, \widehat{\partial}_2 \ ; \\ Y_3 &= x_2 \, \partial_1 \ + \ w_2 \, \widehat{\partial}_1 \ , \ \ Y_4 \ = \ x_1 \, \partial_2 \ + \ w_1 \, \widehat{\partial}_2 \ . \end{aligned}$$

Again it is more convenient to consider sum and differences of these, i.e.

$$\begin{aligned} X_1 &= Y_1 \ + \ Y_2 \ = \ x_1 \, \partial_1 \ + \ x_2 \, \partial_2 \ + \ w_1 \, \widehat{\partial}_1 \ + \ w_2 \, \widehat{\partial}_2 \ , \\ X_2 &= Y_1 \ - \ Y_2 \ = \ x_1 \, \partial_1 \ - \ x_2 \, \partial_2 \ + \ w_1 \, \widehat{\partial}_1 \ - \ w_2 \, \widehat{\partial}_2 \ ; \\ X_3 &= Y_3 \ + \ Y_4 \ = \ x_2 \, \partial_1 \ + \ x_1 \, \partial_2 \ + \ w_2 \, \widehat{\partial}_1 \ + \ w_1 \, \widehat{\partial}_2 \ , \\ X_4 &= Y_4 \ - \ Y_4 \ = \ x_2 \, \partial_1 \ - \ x_1 \, \partial_2 \ + \ w_2 \, \widehat{\partial}_1 \ - \ w_1 \, \widehat{\partial}_2 \ . \end{aligned}$$

The first two are the scaling symmetries always present and discussed in the general case, while X_4 generates equal rotations in the (x_1, x_2) and in the (w_1, w_2) planes, and X_3 generates equal *hyperbolic* rotations in the (x_1, x_2) and in the (w_1, w_2) planes.

It is immediate to check that X_1 and X_4 generates groups of conformal transformations (in particular, in the (w_1, w_2) plane), while this is not the case for X_2 and X_3.

Thus in view of our general discussion only X_1 and X_4 are acceptable W-symmetry generators.

We note that the Lie algebra structure of the X_i fields is as in the following commutator table, where as usual the entry (i, j) represents $[X_i, X_j]$:

	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	$-2X_4$	$-2X_3$
X_3	0	$2X_4$	0	$2X_2$
X_4	0	$2X_3$	$-2X_2$	0

Note that the acceptable W-symmetries $\{X_1, X_4\}$ span a Lie subalgebra, as they should (moreover, in the case under study this is Abelian).

Again the symmetry vector fields are also symmetries for the associated Stratonovich equation. $\odot$

Example 8. We will now consider a generalization of Example 7 with non constant diffusion matrix. We deal with the isotropic stochastic non-linear oscillator

$$dx^i \ = \ \alpha(|x|^2) \, x^i \ dt \ + \ \beta(|x|^2) \, dw^i \quad (i = 1, ..., n) \ ; \tag{94}$$

in general (that is, unless α and β are actually both constant functions) this has no scaling symmetries, but retains rotational symmetries. Again we just consider the case $n = 2$, so that now rotations are generated by the single vector field

$$X \ = \ -x^2 \, \partial_1 \ + \ x^1 \, \partial_2 \ - \ w^2 \, \widehat{\partial}_1 \ + \ w^1 \, \widehat{\partial}_2 \ = \ J^i_{\ k} \left(x^k \, \partial_i \ + \ w^k \, \widehat{\partial}_i \right) \ , \tag{95}$$

with the same notation as above.

Here we will not look for the most general solution to the determining equations, but just note that – as can be checked by direct computation, the X above is an acceptable W-asymmetry generator. In fact, choosing

$$\varphi \ = \ \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \ , \quad R \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the Ito determining equations (61), (62) are satisfied.

One can also check, again by direct computation, that in this case the Stratonovich determining equations (66), (67) are also satisfied. Note that in this case the diffusion matrix

$$\sigma \ = \ \begin{pmatrix} \mu[x_1^2 + x_2^2] & 0 \\ 0 & \mu[x_1^2 + x_2^2] \end{pmatrix}$$

is non constant, but R is skew-symmetric.

The same result is obtained if only one of the two function α, β is non constant, as can be checked by explicit computations (in these cases determining the most general solution to the determining equations is rather simple, and one finds indeed only the rotations given above). $\odot$

Example 9. It may be interesting to look again at the linear isotropic stochastic oscillator

$$dx^i \ = \ - K \ x^i \, dt \ + \ \sigma \, dw^i \tag{96}$$

with K and σ real constants in arbitrary dimension ($i = 1, ..., n$), but using the "general" notation h^k for the $\widehat{\partial}_k$ component of the symmetry vector field. Then the Ito determining equations read

$$\begin{aligned} \partial_t (\varphi^i - \sigma h^i) \ &= \ K \sum_{j=1}^n \partial_j (\varphi^i - \sigma h^i) \ - \ \frac{1}{2} \Delta (\varphi^i - \sigma h^i) \\ \widehat{\partial}_k (\varphi^i - \sigma h^i) \ &= \ - \sigma \, \partial_k (\varphi^i - \sigma h^i) \ . \end{aligned}$$

Thus in this case we can pass to consider $\psi^i := \varphi^i - \sigma h^i$, hence be reduced to considering a single set of functions $\psi^i (x, t; w)$, as for standard (in general, random) symmetries. Doing this we obtain the determining equations

$$\begin{aligned} \partial_t \psi^i \ &= \ K \sum_{j=1}^n \partial_j \psi^i \ - \ \frac{1}{2} \Delta \psi^i \\ \widehat{\partial}_k \psi^i \ &= \ - \sigma \, \partial_k \psi^i \ , \end{aligned}$$

which are just the determining equations for standard symmetries of (96). $\odot$

Example 10. We have seen that isotropic nonlinear stochastic oscillators admit (only) rotation symmetries; in particular, in the two-dimensional case we have the single W.symmetry vector field (95). We can look at the inverse problem, i.e. identifying Ito equations admitting rotation symmetries; we will again confine ourselves to the two-dimensional case.

This just requires to look at the determining equations with φ and h (i.e. R) assigned, and f^i, ${\sigma^i}_k$ as unknown.

In this case, writing $r^2 = x_1^2 + x_2^2$, (62) yield

$$\sigma \ = \ \begin{pmatrix} \alpha(r^2, t) & -\beta(r^2, t) \\ \beta(r^2, t) & \alpha(r^2, t) \end{pmatrix} \ ,$$

with α and β smooth functions of their arguments. As for (61), these yield

$$f^i \ = \ G(r^2, t) \ x^i \ .$$

These are also the most general solutions to (66), (67). $\odot$

Example 11. All Examples considered so far yielded split W-symmetries; one could wonder if non-split W-symmetries are possible at all. The answer is positive, as shown by the trivial case of a constant coefficients scalar Ito equation

$$dx \ = \ A\,dt \ + \ B\,dw \ ; \tag{97}$$

note we must assume $B \neq 0$, or we would not have a stochastic equation.

Now the determining equations are

$$\begin{aligned} \varphi_t \ &+ \ A\,\varphi_x \ + \ \frac{1}{2}\left(\varphi_{ww} + 2B\varphi_{xw} + B^2\varphi_{xx}\right) \ = \ 0 \ , \\ \varphi_w \ &+ \ B\,\varphi_x \ - \ B\,R \ = \ 0 \ . \end{aligned}$$

The second equations yields immediately

$$\varphi \ = \ R\,x \ + \ \psi(z,t) \ , \quad z := w - x/B \ .$$

Plugging this into the first determining equation we get

$$\psi_t \ - \ (A/B)\,\psi_z \ + \ (A/B)R \ = \ 0 \ ,$$

which in turn yields

$$\psi(z,t) \ = \ A\,R\,t \ + \ \Theta(\zeta) \ , \quad \zeta = z + (A/B)t \ .$$

Thus, we always have the W-symmetries

$$X_\Theta \ = \ [x - At + \Theta(\zeta)]\ \partial_x \ + \ \partial_w \ ,$$

which is a non-split W-symmetry provided $\Theta \neq 0$; in particular, with the choice $\Theta(y) = BRy$ we get

$$X \ = \ (B\,w)\ \partial_x \ + \ \partial_w$$

i.e. a time-autonomous non-split nontrivial W-symmetry. (In Appendix B it will be shown that this is essentially the only example of non-split W-symmetries for scalar Ito equations.)

Note that (97) is solved as

$$x(t) \ = \ x(t_0) \ + \ A\,(t - t_0) \ + \ B\,[w(t) - w(t_0)] \ ,$$

so it is immediate to check that (the one-parameter group generated by) X maps solutions into solutions. $\odot$

8 Application of W-symmetries

The general idea behind the use of symmetries to simplify and/or solve differential equations (deterministic or stochastic) is to pass to *symmetry-adapted coordinates.*

This is also the case for Kozlov theorems, discussed in Sections 4 and 5; in fact, in this case one change coordinates so that the symmetry vector field is transformed into a vector field along one of the new coordinates, and the independence of the equation on this allows for a direct (partial, for systems) integration.

We will thus try to follow the same approach here. It will be quite clear, even from the simplest example of stochastic oscillators, that the outcome will be quite different from that seen in the case of standard (deterministic or stochastic) symmetries.

8.1 Scalar equations

The problem is already apparent if we consider one-dimensional systems, i.e. scalar equations. We will just cons8der autonomous equations, thus time will not need to be considered even in the functional dependencies (this will just simplify our notation, with no loss of generality, as the reader can easily check).

In our case of W-symmetries, the standard Kozlov change of coordinates

$$\xi \ = \ \int \frac{1}{\varphi(x,t,w)} dx \tag{98}$$

(which is guaranteed to map the Ito equation into an Ito equation) does not suffice to rectify the vector field

$$X \ = \ \varphi(x,t,w)\, \partial_x \ + \ Rw\, \partial_w \ ,$$

and hence guarantee integrability. In fact, now

$$X(\xi) \ = \ \varphi \frac{1}{\varphi} \ + \ Rw \int \left(\frac{\partial}{\partial w} \frac{1}{\varphi} \right) dx \ = \ 1 \ - \ R\, w \ \int \frac{\varphi_w}{\varphi^2}\, dx \ ,$$

and as the second term in general is not zero, we do not have $X = \partial_\xi$, hence the r.h.s. of the transformed equation is not independent of ξ and it cannot be explicitly integrated.

This is already apparent when considering Stratonovich equations, i.e. is not related to problems arising from applying the Ito rule when changing coordinates.

This is not surprising: as X has also a component along ∂_w, we should also change the w variable, i.e. pass from (x,w) to $(\xi(x,w), \zeta(x,w))$ variables in order to have $X = \partial_\xi$ and hence guarantee direct integration of the equation $d\xi = F dt + S d\zeta$ for ξ.

Example 12. Consider the Stratonovich equation (linear stochastic oscillator)

$$dx \ = \ \lambda\, x \ dt \ + \ \mu \circ dw \ , \tag{99}$$

with λ, μ real constants. This admits as symmetry generator the scaling vector field

$$X \ = \ x\, \partial_x \ + \ w\, \partial_w \ .$$

The Kozlov change of variable is then

$$\xi \ = \ \int \frac{1}{\varphi} dx \ = \ \int \frac{1}{x}\, dx \ = \ \log x \ ; \quad x = e^{\xi} \ . \tag{100}$$

In terms of this variable, we have of course

$$d\xi \ = \ \frac{1}{x} \ dx \ = \ \frac{1}{x} \ [\lambda x dt \ + \ \mu dw] \ ,$$

so our original equation (99) reads now

$$d\xi \ = \ \lambda \, dt \ + \ \mu \, e^{-\xi} \circ dw \ . \tag{101}$$

The vector field does now read

$$X \ = \ \partial_\xi \ + \ w \, \partial_w \ ,$$

and it is immediate to check this is indeed a symmetry of (101). The problem is that (101) can *not* be directly integrated.

As shown by the fact we are considering a Stratonovich equation, this is not even related to the Ito rule, but to the very nature of W-symmetries. $\odot$

8.2 Adapted variables

As hinted above, our strategy in using W-symmetries should be equal in principles, but slightly different in practice, to the one for standard symmetries. That is, we should pass from the old variables (x^i, w^k) ($i = 1, ..., n$, $k = 1, ..., m$; possibly with $m = n$) to new coordinates

$$\xi^i (x, t; w) \ , \quad \zeta^k (x, t; w) \tag{102}$$

such that in the new variables the symmetry vector field X – which we assume to be of the type identified in Sect.6, see (43) – reads

$$X \ = \ \frac{\partial}{\partial \xi^n} \ .$$

Now the equations will read as

$$d\xi^i \ = \ F^i \, dt \ + \ S^i_{\ k} \, d\zeta^k \qquad (i = 1, ..., n) \ ; \tag{103}$$

as X is a symmetry, we will have

$$\frac{\partial F^i}{\partial \xi^n} \ = \ 0 \ = \ \frac{\partial \mathcal{S}^i_{\ k}}{\partial \xi^n} \ . \tag{104}$$

Thus, *if* we are able to solve the reduced system

$$d\xi^i \ = \ F^i \, dt \ + \ S^i_{\ k} \, d\zeta^k \qquad (i = 1, ..., n-1) \ , \tag{105}$$

then the solution to the last equation

$$d\xi^n \;=\; F^n \, dt \;+\; S^n_{\ k} \, d\zeta^k \tag{106}$$

amounts to a direct (stochastic) integration.

This is only apparently identical to what happens for standard symmetries. Actually, a substantial difference arises now due to the more general form of the change of variables (102).

In fact, now

1. The functions F^i and $S^i_{\ k}$ appearing in (103) will in general depend not only on the (ξ, t) variables but *also* on the ζ^k;

2. The ζ^k will in general *not* be Wiener processes.

Each of these features makes that the new equation (103) is *not* of Ito type. As remarked above, see Remarks 6and 7, this in itself is not forbidding the reduced equation can be integrated and thus the W-symmetry reduction procedure maintains some interest (see also Section 8.3 below).

On the other hand, it should be noted that in the case of multiple symmetries we get out of what is covered by the presently existing theory. In fact, one can very well consider stochastic differential equations which are not of Ito (or Stratonovich) type [15]; but as soon as we deal with an equation which is not of Ito type, we cannot use the correspondence with the Stratonovich form in order to guarantee that symmetries will survive a change of variables[10], hence we cannot – at the present stage of our mathematical knowledge – ignite the recursion procedure which was able to guarantee multiple reduction for standard symmetries, i.e. in the frame of standard Kozlov theory [11].[11]

We summarize or discussion as a formal statement[12], which will then be illustrated by studying stochastic oscillators in the next subsection.

Theorem 3. *Let the Ito equation (1) admit a nontrivial W-symmetry with generator X. Passing to adapted variables (y, z) the equation is mapped into a system of stochastic differential equations*

$$dy^i \;=\; F^i \, dt \;+\; S^i_{\ k} \, dz^k \tag{107}$$

with

$$\frac{\partial F^i}{\partial y^n} \;=\; 0 \;=\; \frac{\partial S^i_{\ k}}{\partial y^n}$$

for all $i, k = 1, ..., n$; these are in general not of Ito type, i.e. the coefficients F^i and $S^i_{\ k}$ can depend on the driving stochastic processes z^k.

[10] Albeit one would expect this to be the case, at least for admissible symmetries.

[11] Needless to say, this is not a "no go" result, but rather calls for a study of a more general framework for the use of symmetry in the stochastic realm.

[12] We recall that the adapted variables (y, z) for a vector field X are those such that $X = (\partial/\partial y^n)$.

Corollary 2. *If the n-dimensional system of Ito equations (1) admits a non-trivial W-symmetry, it can be mapped into a system of stochastic differential equations (107) which decouples into an autonomous systems of $(n-1)$ equations plus a "reconstruction equation"*

$$dy^n \ = \ F^n[y^1, ..., y^{n-1}; z^1, ..., z^n] \, dt \ + \ S^n_k[y^1, ..., y^{n-1}; z^1, ..., z^n] \, dz^k \ . \quad (108)$$

8.3 Example. Stochastic oscillators

We will now apply the previous discussion to the simple but relevant case of *stochastic oscillators*, considering both dilation (scaling) and rotation symmetries. We will confine ourselves to the simplest cases, i.e. those in one and two spatial dimensions; these were already considered in the Examples of Section 7.4, so we will build on the computations performed there.

8.3.1 Scaling

We start by considering the linear stochastic oscillator (in one dimension, as this will suffice to point out the problem we need to discuss)

$$dx \ = \ \lambda \, x \, dt \ + \ \mu \, dw \ ; \quad (109)$$

here λ and μ are real constants. As discussed above (see Section 7.4) eq. (109) admits the simple scaling symmetry generator

$$X \ = \ x \, \partial_x \ + \ w \, \partial_w \ ; \quad (110)$$

which generates the one-parameter group of scalings $x \to sx$, $w \to sw$.

The invariant quantity under this is $\zeta = w/x$, and the vector field satisfies $X(\xi) = 1$ e.g. for $\xi = \log(x)$. We will thus pass to coordinates

$$\xi = \log(x) \ , \quad \zeta = w/x \ ;$$

the inverse change of variables is

$$x = e^\xi \ , \quad w = e^\xi \, \zeta \ .$$

In these coordinates, the symmetry vector field reads simply

$$X \ = \ \partial_\xi \ .$$

The equation (109) will now be written as

$$d\xi \ = \ F \, dt \ + \ S \, d\zeta \ , \quad (111)$$

with F and S functions which will now be determined.

Using the Ito rule we have[13]

$$
\begin{aligned}
dx &= e^\xi \, d\xi \ + \ \frac{1}{2} \left(e^\xi \, S^2 \right) \ dt \\
dw &= e^\xi \, \zeta \, d\xi \ + \ e^\xi \, d\zeta \ + \ \frac{1}{2} \left(2 S e^\xi \ + \ S^2 e^x i \zeta \right) \ dt \ .
\end{aligned}
$$

Thus (109) reads now, with simple algebra,

$$
d\xi \ = \ \left[\frac{\lambda + \mu S (1 + S\zeta) - S^2/2}{1 - \mu \zeta} \right] \ dt \ + \ \left(\frac{\mu}{1 - \mu \zeta} \right) \ d\zeta \ .
$$

This shows that

$$
S \ = \ \left(\frac{\mu}{1 - \mu \zeta} \right)
$$

and inserting this into the coefficient of dt we obtain that

$$
F \ = \ \left[\lambda \ + \ \frac{1}{2} \, \frac{\mu^2}{(1 - \mu\zeta)^2} \right] \ \left(\frac{1}{1 - \mu \zeta} \right) \ .
$$

The explicit expressions of F and S are not relevant; the important thing are their functional dependencies. That is, eq.(111) is more precisely rewritten as

$$
d\xi \ = \ F(\zeta) \, dt \ + \ S(\zeta) \, d\eta \ . \tag{112}
$$

We conclude that:

- The vector field X is still a symmetry of the transformed equation, as stated in our Theorem 2;
- But the transformed equation is *not* of Ito type, as the coefficients depend explicitly on the driving random process ζ.

This situation should be compared with that seen above in Example 2, see also Remark 6. Albeit the equation is not of Ito type, we immediately have

$$
\xi(t) \ = \ \int F[\zeta(t)] \, dt \ + \ \int S[\zeta(t)] \, d\zeta(t)
$$

and $\xi(t)$ is recovered by a stochastic integral.

Note that $\zeta(t)$ is in general *not* a Wiener process, as clear from the transformation law linking $\zeta(t) = w(t)/x(t)$ to the Wiener process $w(t)$.

Once we have determined $\xi(t)$ for a given realization of the stochastic process $\zeta(t)$, the $x(t)$ is immediately recovered as

$$
x(t) \ = \ \exp[\xi(t)] \ .
$$

We proceed exactly in the same way, apart from introducing some indices, in considering multi-dimensional linear stochastic oscillators and their scaling symmetries.

[13] The computation is performed here following the scheme suggested in [11]; other ways of performing the same computation are of course also possible.

8.3.2 Rotation

The same qualitative situation is found if we work in higher dimensions and consider an isotropic stochastic oscillator (in this case, linear or nonlinear) and its rotational symmetry.

Consider, for the sake of definiteness, the two-dimensional setting (i.e. $n = 2$; we write again all indices as lower ones to avoid any confusion) for the general equation (33). In this case the W-symmetry generator is

$$X \ = \ x_2 \partial_1 - x_1 \partial_2 + w_2 \widehat{\partial}_1 - w_1 \widehat{\partial}_2 \ . \tag{113}$$

We now want to consider adapted coordinates; in this case they are polar coordinates in both the x and the w space, i.e.

$$r = \sqrt{x_1^2 + x_2^2}, \ \vartheta = \arctan(x_2/x_1); \ z = \sqrt{w_1^2 + w_2^2} \ , \ \xi = \arctan(w_2/w_1) \ .$$

This corresponds, obviously, to

$$x_1 = r \cos(\vartheta), \ x_2 = r \sin(\vartheta); \ w_1 = z \ \cos(\xi), \ w_2 = z \ \sin(\xi) \ .$$

The vector field (113) reads now $X = \partial_\vartheta + \partial_\xi$.

With a standard application of Ito rule, we have

$$\begin{aligned} dr &= \frac{1}{2r} \left(S^2 + 2r^2 F \right) \, dt \ + \ 2Sr \left[\cos(\vartheta - \xi) \, dz \ + \ z \sin(\vartheta - \xi) \, d\xi \right] \ , \\ d\vartheta &= \frac{Sz}{r} \cos(\vartheta - \xi) \, d\xi \ - \ \frac{1}{r} \sin(\vartheta - \xi) \, dz \ . \end{aligned}$$

It is apparent that these are invariant under the simultaneous rotations $\vartheta \to \vartheta \ \delta$, $\xi \to \xi + \delta$; that is, under X.

With a further trivial change of variables, i.e. switching from ϑ to $\psi = \vartheta - \xi$, these become

$$\begin{aligned} dr &= \frac{1}{2r} \left(S^2 + 2r^2 F \right) \, dt \ + \ 2Sr \left[\cos(\psi) \, dz \ + \ z \sin(\psi) \, d\xi \right] \ , \\ d\psi &= \left(\frac{S}{r} z \cos\psi \ - \ 1 \right) \, d\xi \ - \ \frac{1}{r} \ \sin\psi \, dz \ . \end{aligned} \tag{114}$$

(These are immediately seen to be invariant under a shift in ξ, i.e. to admit the W-symmetry $X_0 = \partial / \partial \xi$; but this is not admissible in view of the discussion in Section 6.)

We stress that, due to the presence of z in the coefficient of the $d\xi$ terms on r.h.s., these equations (114) are *not* in Ito form. On the other hand, they can be integrated.

9 Discussion & Conclusions

In a previous work [13] we have classified admissible (on physical and mathematical basis) transformations of Ito stochastic differential equations, and hence types of possible symmetries of these. This classifications yielded three types of symmetries, i.e. standard deterministic symmetries, standard random ones, and W-symmetries. The first two types have been studied, by ourselves and different authors, in the literature [9, 10, 11, 13, 16, 17, 18, 19, 20, 21, 23, 27], while W-symmetries had so far lacked attention.

In this paper, we have first reviewed relevant notions and results in the recent literature devoted to symmetry of SDEs; in particular we have stressed the relevance of the relation between symmetries of an Ito equation and of the associated Stratonovich one (actually, this is an equality for admissible symmetries), and recalled how Kozlov theory makes use of symmetries to integrate – at least partially, in which case we actually have a reduction – Ito equations.

In the second and main part of the paper, we have studied W-symmetries. In particular, in Sect.6 we have determined the class of vector fields qualifying as admissible would-be W-symmetries, obtaining in particular that the action of the w variables must correspond to an origin-preserving action of the linear conformal group – thus essentially reduce to dilations and/or rotations. In Sect 7 we have established the determining equations for W-symmetries of Ito and Stratonovich SDEs, discussing the relation between their solutions. This turns out to be less trivial than for standard symmetries, and in particular symmetries acting as dilations in the w sector may not be shared by an Ito and the associated Stratonovich equation, as also shown in concrete simple examples.

Finally, in Sect.8 we have considered how W-symmetries can be concretely used in studying Ito SDEs. We have seen that the situation is different from the one which became familiar with standard symmetries. In fact, once one has determined a W-symmetry of a given Ito equation, one can reduce it to a partially integrable equation by passing to symmetry-adapted variables, but in general this produce a non-Ito equation. This means in particular that the existing theory – which only considers Ito equations – cannot be used for reduction under multiple symmetries, and thus calls for extension of the theory to a wider realm.

It should be recalled that more general (than Ito or Stratonovich) types of stochastic differential equations do not have an equally solid mathematical foundation; but they are nevertheless used in several physical (and chemical) contexts [15].

Appendix A. The one-dimensional case

In this Appendix we discuss the problem tackled in Section 7 in the simplified setting of scalar Ito equation. This will allow to avoid plethora of indices and get a more clear view of the reasoning and computations leading to our results there.

It follows from our general discussion that in the scalar (one-dimensional) case

$$dx \ = \ f(x,t)\, dt \ + \ \sigma(x,t)\, dw \tag{A.1}$$

the only nontrivial W-symmetries act on w as dilations. This case is of course specially simple, and it is worth looking at it specifically; we will use an obvious simplified notation, and this will provide a check of our general result in the simplest setting.

Now the Ito determining equations read

$$\varphi_t \ + \ f\, \varphi_x \ - \ \varphi\, f_x \ + \ \frac{1}{2} \Delta\varphi \ = \ 0 \ , \tag{A.2}$$

$$\varphi_w \ + \ \sigma\, \varphi_x \ - \ \varphi\, \sigma_x \ - \ \sigma\, R \ = \ 0 \ . \tag{A.3}$$

As for the Stratonovich determining equations for

$$dx \ = \ b(x,t)\, dt \ + \ \sigma(x,t) \circ dw \ , \tag{A.4}$$

these read

$$\begin{aligned} \varphi_t \ + \ b\, \varphi_x \ - \ \varphi\, b_x \ &= \ 0 \ , \\ \varphi_w \ + \ \sigma\, \varphi_x \ - \ \varphi\, \sigma_x \ - \ \sigma\, R \ &= \ 0 \ . \end{aligned}$$

The (common) second equation of these sets yields

$$\varphi_w \ = \ \varphi\, \sigma_x \ - \ \sigma\, \varphi_x \ + \ \sigma\, R \ .$$

This in turn provides

$$\begin{aligned} \varphi_{xw} \ &= \ \varphi_x\, \sigma_x \ + \ \varphi\, \sigma_{xx} \ - \ \sigma_x\, \varphi_x \ - \ \sigma\, \varphi_{xx} \ + \ \sigma_x\, R \ , \\ \varphi_{ww} \ &= \ \varphi_w\, \sigma_x \ - \ \sigma\, \varphi_{xw} \\ &= \ \varphi_w\, \sigma_x \ - \ \varphi\, \sigma\, \sigma_{xx} \ + \ \sigma^2\, \varphi_{xx} \ - \ \sigma\, \sigma_x\, R \ . \end{aligned}$$

With these, and some trivial algebra, we get

$$\Delta\varphi \ = \ \varphi\, \sigma_x^2 \ - \ \varphi_x\, \sigma\, \sigma_x \ + \ \varphi\, \sigma\, \sigma_{xx} \ + \ 2\, \sigma\, \sigma_x\, R \ . \tag{A.5}$$

On the other hand, recalling that for the associated Stratonovich equation $b = f - (1/2)\rho$, we readily get that the first determining equation in the Stratonovich case reads

$$\varphi_t \ + \ f\, \varphi_x \ - \ \varphi\, f_x \ + \ \frac{1}{2} \left[\varphi\, \sigma_x^2 \ + \ \varphi\, \sigma\, \sigma_{xx} \ - \ \varphi_x\, \sigma \sigma_x \right] \ = \ 0 \ .$$

This coincides with the first determining equation for the Ito equation if and only if

$$\Delta\varphi \ = \ \varphi\, \sigma_x^2 \ + \ \varphi\, \sigma\, \sigma_{xx} \ - \ \varphi_x\, \sigma \sigma_x \ . \tag{A.6}$$

As discussed above, it suffices that the equality holds when we restrict to solutions to the second (common) equation in the sets of determining equations.

Comparing (A.5) and (A.6) we see that the equations coincide, and hence symmetries of the Ito and of the associated Stratonovich equations also do, if and only if

$$\sigma \, \sigma_x \, R \ = \ 0 \ . \tag{A.7}$$

We do of course exclude the case $\sigma = 0$ (or the equation would not be a stochastic one), and also the case $R = 0$ as in that case we have a standard symmetry (which is trivial as a W-symmetry).

So in the end we have shown the following, which of course is a special case of the general result (Theorem 1) obtained above:

Lemma A.1. *In the one dimensional case the nontrivial W-symmetries of an Ito equation are shared by the associated Stratonovich equation if and only if the diffusion coefficient $\sigma(x,t)$ in eq.(A.1) satisfies $\sigma_x = 0$.*

Appendix B. Forbidden forms of W-symmetries

It turns out that W-symmetries can not take all forms, i.e. some forms of the W-symmetry generator $X = \varphi^i(x,t;w)\partial_i + (R^k_{\ m} w^m)\widehat{\partial}_k$ can not be realized.

In order to illustrate this, we will consider some situations assuming a given shape for X – i.e. for the coefficients φ^i – and showing there can be no Ito equation admitting such W-symmetry.

We will just consider some one-dimensional cases and restrict to the time-autonomous case (that is, f and σ, and hence also φ are assumed to be independent of t), which will help keeping computations simple and focus on the qualitative relevant point.

At the moment we are not able to provide general results on which shape of W-symmetries are possible or forbidden.

Example B.1. Let us make the ansatz

$$\varphi(x,t;w) \ = \ p(x,t) \ e^w \ ; \tag{B.1}$$

then differentiating the second determining equation (A.3) w.r.t. w we get

$$e^w \ \left[p \ + \ \sigma \, p_x \ - \ p \, \sigma_x \right] \ = \ 0 \ ,$$

and hence

$$\sigma(x,t) \ = \ h(t) \, p(x,t) \ + \ \int_0^x \frac{1}{p(y,t)} \, dy \ .$$

When plugging this into (A.3) itself, we get

$$R \ p(x,t) \ \left[h[t] \ + \ \int_0^x \frac{1}{p(y,t)} \, dy \right] \ = \ 0 \ .$$

But the only solutions to this is $R = 0$, in which case we have a standard symmetry (choosing $p(x,t) = 0$ gives a singular situation, and no symmetry anyway). ⊙

Example B.2. Let us now look for W-symmetry generators with

$$\varphi(x,t,w) \ = \ p(x,t) \ w^2 \ ; \tag{B.2}$$

in this case (A.3) reads

$$2 \, w \, p(x,t) \ - \ R \, \sigma(x,t) \ + \ w^2 \ \left[S[x,t] \, p_x(x,t) \ - \ p(x,t) \, \sigma_x(x,t) \right] \ = \ 0 \ .$$

The term linear in w enforces $p = 0$, hence there is no equation admitting W-symmetries of the form (B.2). $\odot$

Example B.3. Finally, we consider the separable ansatz

$$\varphi(x,t,w) \ = \ p(x) \ q(w) \ ; \tag{B.3}$$

note here we must assume $p \neq 0 \neq q$ to rule out trivial cases.

We know that there is at least one case admitting nontrivial W-symmetries of this form, see Example 11 in the main text. Here we show that is the *only* class of time-autonomous[14] scalar equations admitting W-symmetries of the form (B.3).

In fact, plugging the ansatz (B.3) into (A.3) and differentiating w.r.t. w, we get

$$p\, q'' \ + \ \sigma\, p_x\, q' \ - \ \sigma_x\, p\, q' \ = \ 0 \ ;$$

this also reads

$$\frac{p\, \sigma_x \ - \ p_x\, \sigma}{p} \ = \ \frac{q''}{q'} \ .$$

As the l.h.s. only depends on (x,t) and the r.h.s. only on w, it must be

$$\frac{p\, \sigma_x \ - \ p_x\, \sigma}{p} \ = \ K \ = \ \frac{q''}{q'} \tag{B.4}$$

for some constant K.

Let us first assume $K \neq 0$. Then the r.h.s. equality in (B.4) yields immediately

$$q(w) \ = \ c_1 \ e^{Kw} \ + \ c_2 \tag{B.5}$$

(note that we must require $c_1 \neq 0$, or we would have a split symmetry); while setting

$$\sigma(x) \ = \ [c_3 \ + \ r(x)] \ p(x) \tag{B.6}$$

the l.h.s. equality reads

$$p(x) \ = \ - \frac{K}{r'(x)} \ . \tag{B.7}$$

When we insert (B.5)–(B.7) into (A.3), and look at the coefficient of e^{Kw} in this, we get

$$c_1\, K^3 \ = \ 0 \ .$$

Thus we are left only with non-viable options: $c_1 = 0$ would give a split symmetry and $K = 0$ was excluded by assumption.

So we are forced to assume $K = 0$. Now the r.h.s. equality in (B.4) yields

$$q(w) \ = \ c_1\, w \ + \ c_2 \tag{B.8}$$

while the l.h.s. one provides

$$p(x) \ = \ c_3 \ \sigma(x,t) \ . \tag{B.9}$$

These were obtained from a differential consequence of (A.3); when we plug (B.8) and (B.9) into (A.3) itself, we obtain

$$c_3 \ = \ R/c_1 \ .$$

[14] A full discussion would also be possible admitting time dependencies, but it would be too long to report here.

We can now tackle (A.2), which is an expression linear in w (all the dependencies on w are now explicit). Thus it splits into two equations (corresponding to the vanishing of terms independent of w and of the coefficient of w); assuming $R \neq 0$ these read

$$\begin{aligned} R[2c_2 f \sigma_x + 2\sigma(c_1\sigma_x - c_2 f_x) + c_2\sigma^2\sigma_{xx}] &= 0 \\ R[2c_1 f \sigma_x - 2c_1 \sigma f_x + c_1\sigma^2\sigma_{xx}] &= 0 \ . \end{aligned}$$

Multiplying the first by c_1, the second by c_2 and taking the difference, we get

$$R\, c_1^2\, \sigma\, \sigma_x \ = \ 0 \ .$$

As c_1, R and σ can not vanish, we must have $\sigma_x = 0$, i.e. $\sigma(x) = \mu$ (with $\mu \neq 0$). With this, the two equations reduce to

$$c_2\, \mu\, f_x \ = \ 0 \ = \ c_1\, \mu\, f_x \ ,$$

$$f(x) \ = \ \lambda \ .$$

In conclusion, we have obtained

$$\begin{aligned} f(x) &= \lambda \ , \\ \sigma(x) &= \mu \ ; \\ \varphi &= \mu\, R\, w \ + \ k_1\, \mu\, R \ . \end{aligned}$$

Here $k_1 = (c_2/c_1)$ is an arbitrary constant, so we actually have two symmetry generators, i.e.

$$X_1 \ = \ \mu\, w\, \partial_x \ + \ \partial_w \ , \quad X_2 \ = \ k_1\, \mu\, \partial_x \ + \ \partial_w \ ;$$

note however that X_2 is a split W-symmetry. $\odot$

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