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AN INTRINSIC APPROACH TO THE NON-ABELIAN TENSOR PRODUCT

MAT\02

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Introduction

The notion of a *non-abelian tensor product of groups* first appeared in [15] where Brown and Loday generalised the following theorem from [2]:

Theorem 0.0.1. *Consider a CW-complex X, two subcomplexes A and B such that $X = A \cup B$, and denote $C = A \cap B$. If A, B and C are connected, (A, C) is $(p-1)$-connected, (B, C) is $(q-1)$-connected and C is simply connected, then the natural map*

$$\pi_p(A, C) \otimes_{\mathbb{Z}} \pi_q(B, C) \to \pi_{p+q-1}(X; A, B)$$

is an isomorphism.

They managed to remove the requirement of simple connectedness on C by using the new notion of *non-abelian tensor product* of two groups acting on each other, instead of the usual tensor product $\otimes_{\mathbb{Z}}$ of abelian groups. In particular, they took two groups M and N acting on each other and they defined their non-abelian tensor product $M \otimes N$ via an explicit presentation.

This led to the development of an algebraic theory based on this construction. Many results were obtained treating the properties which are satisfied by this non-abelian tensor product as well as some explicit calculations in particular classes of groups.

In order to state many of their results regarding this tensor product, Brown and Loday needed to require, as an additional condition, that the two groups acted on each other "compatibly". A key fact that we will need is that these compatibility conditions are equivalent to the existence of a group L and of two crossed modules structures $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$ such that the original actions are induced from ξ_M and ξ_N by composition with μ and ν.

Furthermore, they proved that the non-abelian tensor product is part of a so called *crossed square of groups*: this particular crossed square is the pushout of a specific diagram in the category **XSqr**(**Grp**) of crossed squares of groups. Note that crossed squares are a 2-dimensional version of crossed modules of groups. Indeed **XMod**(**Grp**) is equivalent to the category of groupoids, whereas **XSqr**(**Grp**) is equivalent to the category of double groupoids, that is groupoids of groupoids or crossed modules of crossed modules.

Following the idea of generalising the algebraic theory arising from the study of the non-abelian tensor product of groups, Ellis gave a definition of non-abelian tensor product of Lie algebras in [37], and obtained similar results. Further generalisations have been studied in the contexts of Leibniz algebras [43], restricted Lie algebras [60], Lie-Rinehart algebras [26], Hom-Lie algebras [20, 22], Hom-Leibniz algebras [21], Hom-Lie-Rinehart algebras [63], Lie superalgebras [41] and restricted Lie superalgebras [70].

The aim of our work is to build a general version of non-abelian tensor product, having the specific definitions in **Grp** and $\mathbf{Lie}_R$ as particular instances. In order to do so we first extend the concept of a pair of compatible actions (introduced in the case of groups by Brown and Loday [15] and in the case of Lie algebras by Ellis [37]) to semi-abelian categories (in the sense of [58]). This is indeed the most general environment in which we are able to talk about actions, thanks to [11, 5] where the authors introduced the concept of internal actions. In this general context, we give a diagrammatic definition of the compatibility conditions for internal actions, which specialises to the particular definitions known for groups and Lie algebras. We then give a new construction of the Peiffer product in this setting and we use these tools to show that in any semi-abelian category satisfying the *Smith-is-Huq* condition (denoted as (SH) from now on), asking that two actions are compatible is the same as requiring that these actions are induced from a pair of internal crossed modules over a common base object.

Thanks to this equivalence, in order to deal with the generalisation to the semi-abelian context of the non-abelian tensor product, we are able to use a pair of internal crossed modules over a common base object instead of a pair of compatible internal actions, whose formalism is far more intricate.

Now we fix a semi-abelian category $\mathbb{A}$ satisfying (SH) and we show that, for each pair of internal crossed modules $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$ over a common base object L, it is possible to construct an internal crossed square which is the pushout in $\mathbf{XSqr}(\mathbb{A})$ of the general version of the diagram used in Proposition 2.15 of [15] in the groups case. The non-abelian tensor product $M \otimes N$ is then defined as a piece of this internal crossed square. We show that if $\mathbb{A} = \mathbf{Grp}$ or $\mathbb{A} = \mathbf{Lie}_R$, this general construction coincides with the specific notions of non-abelian tensor products already known for groups and Lie algebras. We construct an L-crossed module structure on this $M \otimes N$, some additional universal properties are shown and by using these we prove that $- \otimes -$ is a bifunctor.

Once we have the non-abelian tensor product among our tools, we are also able to state the new definition of *weak crossed square*: the idea behind this is to generalise the explicit presentations of crossed squares given for groups in [48, 15] and for Lie algebras in [36, 23]. These equivalent definitions, which (contrarily to the semi-abelian one) do not rely on the formalism of internal groupoids but include some set-theoretic constructions, are shown to be equivalent to the implicit ones, where, by definition, crossed squares are crossed modules of crossed modules and hence normalisations of double groupoids. Our idea is to give an alternative explicit description of crossed squares of groups (resp. Lie algebras) using the non-abelian tensor product, so that it does not involve anymore the so-called *crossed pairing* (resp. *Lie pairing*), which is not a morphism in the base category but only a set-theoretic function; in its place we use a morphism from the non-abelian tensor product which is more suitable for generalisations. Doing so, the explicit definitions can be summarised by saying that a crossed square is a commutative square of crossed modules, compatible with an additional crossed module structure on the diagonal, and endowed with a morphism out of the non-abelian tensor product.

Our definition of weak crossed squares is based on the one of non-abelian tensor product and plays the role of the explicit version of the definition of internal crossed squares: in particular we proved that it restricts to the explicit definitions for groups and Lie algebras and hence that in these cases weak crossed squares are equivalent to crossed squares. So far we have shown that any internal crossed square is automatically a weak crossed square, but we are currently missing precise conditions on the base category under which the converse is true: this means that any internal crossed square can be described explicitly as a particular weak crossed square, but this is not a complete characterisation.

In order to give a direct application of our non-abelian tensor product construction, we focus on universal central extensions in the category $\mathbf{XMod}_L(\mathbb{A})$: in [25] Casas and Van der Linden studied the theory of universal central extensions in semi-abelian categories, using the general notion of central extension (with respect to a Birkhoff subcategory) given by Janelidze and Kelly in [56]. We are mainly interested in one of their results, namely the following theorem.

Theorem 0.0.2. *Given a Birkhoff subcategory $\mathbb{B}$ of a semi-abelian category $\mathbb{X}$ with enough projectives, the following holds:*

$$\textit{An object of } \mathbb{X} \textit{ is } \mathbb{B}\textit{-perfect iff it admits a universal } \mathbb{B}\textit{-central extension.} \tag{1}$$

In [35] Edalatzadeh considered the category $\mathbb{X} = \mathbf{XMod}_L(\mathbf{Lie}_R)$ and crossed modules with vanishing aspherical commutator as Birkhoff subcategory $\mathbb{B} = \mathbf{AAXMod}_L(\mathbf{Lie}_R)$. Since the category $\mathbf{XMod}_L(\mathbf{Lie}_R)$ is not semi-abelian (because not pointed) the existing theory does not apply in this situation: nevertheless he managed to prove the same result as the one stated in (1) and furthermore he gave an explicit construction of the universal $\mathbb{B}$-central extensions by using the non-abelian tensor product of Lie algebras.

Using our general definition of non-abelian tensor product of L-crossed modules as given in the third chapter, we are able to extend Edalatzadeh's results to $\mathbf{XMod}_L(\mathbb{A})$ (with Birkhoff subcategory $\mathbf{AAXMod}_L(\mathbb{A})$) for each semi-abelian category $\mathbb{A}$ satisfying the (SH) condition: this is a useful application of the construction of the non-abelian tensor product, which again manages to express in this more general setting exactly the same properties as in its known particular instances.

Furthermore, taking $\mathbb{B} = \mathbf{Ab}(\mathbb{A})$ as Birkhoff subcategory of $\mathbf{XMod}(\mathbb{A})$, we are able to show that, whenever the category $\mathbb{A}$ has enough projectives, our generalisation of Edalatzadeh's work is partly a consequence of Theorem 0.0.2, reframing Edalatzadeh's result within the "standard" theory of universal central extensions in the semi-abelian context.

There are two non-trivial consequences of this fact. First of all, besides the existence of the universal $\mathbb{B}$-central extension for each $\mathbb{B}$-perfect object in $\mathbf{XMod}(\mathbb{A})$, we are also able to give its explicit construction by using the non-abelian tensor product: notice that this construction is completely unrelated to what has been done in [25]. Secondly, this construction of universal $\mathbb{B}$-central extensions is valid even when $\mathbb{A}$ (and consequently also $\mathbf{XMod}(\mathbb{A})$) does not have enough projectives, whereas within the general theory this is a key requirement for (1) to hold.

Structure of the text

In the first chapter we recall the basic categorical notions and tools that we are going to use in the rest of the text: after a recap on the semi-abelian context, we recall some definitions and results regarding internal actions, the cosmash product and commutator theory; finally we give a quick description of the categories of internal points, reflexive (multiplicative) graphs, categories, groupoids and crossed modules.

In the second chapter we study compatible actions of groups and Lie algebras, reframing them in an internal language which is more suitable for generalisations (this change of perspective and its consequences are presented in [31]). We extend the notion of compatibility to internal actions in the context of semi-abelian categories with (SH). We give a new construction of the Peiffer product, which specialises to the definitions known for groups and Lie algebras and we use it to prove the main result of this chapter: an equivalence between pairs of compatible actions and pairs of crossed modules over a common base object. We also study the Peiffer product in its own right, in terms of its universal properties, and prove its equivalence with existing definitions in specific cases. All the results on semi-abelian categories contained in this chapter can be found in the submitted paper [33].

The third chapter is mainly devoted to the construction of the non-abelian tensor product: by imitating the reasoning that appears in Proposition 2.15 of [15] we are able to define the non-abelian tensor product of two coterminal crossed modules in any semi-abelian category that satisfy (SH): this construction and the following results will appear in the paper in preparation [34]. We show that this construction has as particular cases the non-abelian tensor products already existing for groups and Lie algebras. Through this new construction we are also able to state the new definition of *weak crossed squares*: these objects are designed to generalise to the semi-abelian context an explicit approach to crossed squares (possible in **Grp**, $\mathbf{Lie}_R$ and many other categories) which aims to describe them not as normalisations of double groupoids, but as commutative squares of crossed modules endowed with an additional explicit structure. We were not able to find precise categorical conditions on the base category $\mathbb{A}$ such that weak crossed squares and crossed squares would coincide, but we show this equivalence in some particular cases.

In the last chapter an application of the previous construction is presented: in the context of internal crossed modules over a fixed base object in a given semi-abelian category, we use the non-abelian tensor product in order to prove that an object is perfect (in an appropriate sense) if and only if it admits a universal central extension. This extends results of Brown and Loday ([15], in the case of groups) and Edalatzadeh ([35], in the case of Lie algebras). We also explain how those results can be understood in terms of categorical Galois theory: Edalatzadeh's interpretation in terms of quasi-pointed categories applies, but a more straightforward approach based on the theory developed in a pointed setting by Casas and Van der Linden [25] works as well. All the material in this chapter is part of the paper in preparation [32].

Further developments and open questions

Starting from what is presented in this thesis, there are many aspects that would benefit from further investigation.

- For what concerns compatible actions, the (SH) condition made calculations much easier, but it is probably not stricly necessary. It should be checked that even without this additional requirement things work smoothly.

- Many other particular definitions of compatible actions have been given in different settings: one should check that all the ones that fall under the semi-abelian context are indeed a particular case of Definition 2.3.1.

- We also need to understand what are the additional conditions on the base category $\mathbb{A}$ in order for the conditions in Definition 2.3.1 to collapse to simpler ones as it happens in **Grp** and $\mathbf{Lie}_R$, where, for example, the existence of the coproduct action is always guaranteed.

- As for the non-abelian tensor product, we would like to know whether there is a way to explicitly compute it, at least in more concrete cases like when $\mathbb{A}$ is a semi-abelian variety of algebras, or when the involved actions are trivial: the latter case is closely related to currently unpublished work on intrinsic tensor products by Hartl and Van der Linden.

- Once again, there are many definitions of non-abelian tensor products in specific semi-abelian categories: a theoretically interesting but tedious step would be to verify whether these are particular instances of Definition 3.2.5 as it happens for groups and Lie algebras. This question applies also to different existing notions of tensor products which are apparently unrelated with the non-abelian one.

- As regards internal crossed squares and weak crossed squares, some additional categorical conditions on $\mathbb{A}$ are probably needed in order to have the equivalence between these two notions (and consequently an explicit description for internal crossed squares). The author is currently studying the case of semi-abelian varieties of algebras in order to understand if in this simpler context the two already coincide.

Notation

We will use the following notation for standard categorical objects:

- the kernel of the map f is (K_f, k_f);

- the cokernel of the map f is (C_f, c_f);

- the equaliser of the maps f and g is $(E_{(f,g)}, e_{(f,g)})$;

- the coequaliser of the maps f and g is $(C_{(f,g)}, c_{(f,g)})$;

- the kernel pair of the map f is $(Kp(f), r_0, r_1)$;
- the pullback of the map f along the map g is $g^*(f)$;
- the pushout of the map f along the map g is $g_*(f)$;
- in diagrams we will use the notations $A \rightarrowtail B$, $A \triangleright\!\!\longrightarrow B$ and $A \twoheadrightarrow B$ respectively for monomorphism, normal monomorphisms (which in our context concide with kernels) and regular epimorphisms (including cokernels).

Chapter 1

Preliminaries

1.1 Basic categorical tools

1.1.1 Categorical context and basic results

Our base category $\mathbb{A}$ will almost always be "semi-abelian", but since we will also deal with the category $\mathbf{XMod}_L(\mathbb{A})$ of crossed module in $\mathbb{A}$ over a fixed object, which is not semi-abelian, we need to introduce a wider range of categorical notions included in the "semi-abelian context". The idea behind semi-abelian categories is to express a categorical generalisation of groups in the same way as abelian categories generalise abelian groups. This problem was first introduced by Mac Lane in [62] and after many results based on different requirements, a definition of semi-abelian category was given in [58], including those previous results as part of this new theory. We will later recall some of the useful constructions and definitions that are possible in a semi-abelian category, such as internal actions, internal crossed modules and different types of commutators. Let us start by introducing the ingredients that we need for the definition of a semi-abelian category.

Definition 1.1.1. A category $\mathbb{A}$ is said to be *pointed* if it has an initial object 0 which is also terminal, that is $0 \cong 1$. We call a map $f\colon A \to B$ *the zero map* (and we denote it as 0) if it is the (necessarily unique) map from A to B factorising through the zero object 0. In a pointed category we are able to define the *kernel* of a map f as the equaliser of f and 0. Similarly we define the *cokernel* of f as the coequaliser of f and 0.

Lemma 1.1.2 (Lemma 4.2.4 in [3])**.** *Let $\mathbb{A}$ be a pointed category and consider the diagram*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\gamma"'] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
K_{f'} \arrow[r, tail, "k_{f'}"'] & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

where $K_{f'}$ is the kernel of f'. When β is a monomorphism, K is the kernel of f iff the left-hand side square is a pullback. $\square$

Definition 1.1.3. A category $\mathbb{A}$ is said to be *quasi-pointed* if it has an initial object 0, a terminal object 1 and the unique morphism $0 \to 1$ is a monomorphism. In this context there is no zero morphism between two given objects, but we can define kernels and cokernels as follows: the kernel $f\colon A \to B$ is the pullback of f along the (necessarily unique) map from 0 to B, whereas the cokernel of f is the pushout of f along the (unique if it exists) map from A to 0.

$$\begin{tikzcd} K_f \arrow[r, tail, "k_f"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "c_f"] \\ 0 \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & C_f \end{tikzcd}$$

Notice that in a quasi-pointed category, it suffices to have pullback in order to be able to compute the kernel of each map, but only few objects can end up being kernels: indeed in order for A to be a kernel, it has to admit a map from A to the initial object, which is not a trivial requirement. Conversely even if a quasi-pointed category has all pushouts, we may not be able to compute every cokernel: indeed the pushout that we want to construct to build the cokernel of f involves a map from the domain of f to the initial object and if this doesn't exist, then also the desired cokernel is not defined (notice that this is due to the non existence of the diagram that we want to compute the pushout of, so this doesn't imply that the category in exam is not cocomplete).

Example 1.1.4. The most basic example of a quasi-pointed category which is not pointed is the category **Set**. Here the initial object is the empty set $\varnothing$, whereas the terminal object is the singleton $\{*\}$ and obviously the unique map $\varnothing \to \{*\}$ is a monomorphism. If we consider an arrow $f\colon A \to B$, its kernel is always given by $\varnothing$ itself because it is the unique set which admits an arrow entering $\varnothing$. Conversely we are able to compute its cokernel iff $A = \varnothing$, and in this case we obtain that $(C_f, c_f) = (B, 1_B)$.

The next one is a common property of many important classes of categories, such as abelian categories, toposes (also quasi-toposes) and obviously semi-abelian categories. The key idea behind regularity is the existence of a good notion of factorisation for each map: in particular in the abelian context this means that every morphism can be decomposed as a cokernel followed by a kernel, but this is too strict as a requirement for the semi-abelian case.

Definition 1.1.5. We say that a category $\mathbb{A}$ is *regular* if

- it is finitely complete (or *LEX*),
- it has coequalisers of kernel pairs and
- regular epimorphism are pullback stable.

An equivalent way to state this is saying that $\mathbb{A}$ is regular if

- it is finitely complete (or *LEX*),

- any arrow factorises as a regular epimorphism followed by a monomorphism and
- these factorisations are pullback stable.

Therefore in a regular category $\mathbb{A}$ each morphism has a factorisation $f = m \circ e$ as a regular epimorphism followed by a monomorphism (from now on denoted as REM-factorisation) which moreover is functorial and unique up to isomorhisms: in particular we can find this factorisation through the following construction

$$\begin{tikzcd} Kp(f) \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & A \arrow[rr, "f"] \arrow[dr, two heads, "c_{(r_0,r_1)}"'] & & B \\ & & I(f) \arrow[ur, dotted] & \end{tikzcd}$$

where $e = c_{(r_0,r_1)}$ and m is the dotted map induced by the universal property of the coequaliser $c_{(r_0,r_1)}$. We call the object $I(f)$ *direct image* of f.

Example 1.1.6. It is trivial to see that **Set** is a regular category, as for **Grp**. A counter-example is given by the category **Top**, in which regular epimorphisms are not pullback stable.

Regular categories are a suitable environment where to study relations in an abstract context.

Definition 1.1.7 ([1])**.** A regular category $\mathbb{A}$ is said to be *(Barr) exact* if every equivalence relation is effective (i.e. a kernel pair).

Example 1.1.8. Any abelian category, **Set**, **Grp** and also the category **HComp** of compact Hausdorff spaces are examples of (Barr) exact categories. On the other hand two classical example of regular categories which are not exact are given by the category $\mathbf{Ab}_{TF}$ of torsion-free abelian groups and the category **Grp**(**Top**) of topological groups (that is internal groups in **Top**).

One more ingredient of semi-abelian categories is called *Bourn protomodularity* and it has been first defined in [6].

Lemma 1.1.9. *Consider the following diagram in any category*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(*_1)"] & B \arrow[r, "g"] \arrow[d, "\beta"'] \arrow[dr, phantom, "(*_2)"] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd} \tag{1.1}$$

where the square $*_2$ is a pullback. Then the square $*_1$ is a pullback if and only if the outer rectangle is a pullback. □

Definition 1.1.10 ([6])**.** A lex category $\mathbb{A}$ is said to be *(Bourn) protomodular* if the converse of the previous lemma holds whenever β is a split epimorphism: looking at (1.1) this means that if both the outer rectangle and the square $(*_1)$ are pullbacks, then also $(*_2)$ is a pullback.

If $\mathbb{A}$ is quasi-pointed, protomodularity implies that every regular epimorphism is a cokernel. If moreover $\mathbb{A}$ is regular, protomodularity is equivalent to requiring the validity of the *(Regular) Short Five Lemma.*

Lemma 1.1.11 (Regular Short Five Lemma)**.** *Consider the diagram*

$$\begin{tikzcd} K_f \arrow[r, tail, "k_f"] \arrow[d, "\gamma"'] & A \arrow[r, two heads, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ K_{f'} \arrow[r, tail, "k_{f'}"'] & A' \arrow[r, two heads, "f'"'] & B' \end{tikzcd} \tag{1.2}$$

with both f and f' being regular epimorphisms. If γ and β are isomorphisms, then α is so.

When $\mathbb{A}$ is regular, it is also possible to prove that the requirement of having a split epimorphism in Definition 1.1.10 can be weakened as shown in the following.

Lemma 1.1.12 (Proposition 4.1.4 in [3])**.** *Let $\mathbb{A}$ be a regular category. Then $\mathbb{A}$ is protomodular iff whenever β in* (1.1) *is a regular epimorphism and both* $(*_1)$ *and the outer rectangle are pullbacks, then* $(*_2)$ *is a pullback too.* □

Definition 1.1.13 ([58])**.** A category $\mathbb{A}$ is said to be *semi-abelian* if it is pointed, (Barr) exact, protomodular and if it has binary coproducts. Since semi-abelian categories are regular and pointed by definition, requiring protomodularity amounts to requiring the Regular Short Five Lemma.

Every abelian category is semi-abelian, but the converse is false: a semi-abelian category is abelian iff it is additive iff its dual is again semi-abelian. Even if we require explicitly only binary coproducts to exist, semi-abelian categories are actually finally cocomplete. The principal example of semi-abelian category is **Grp**, but also **Rng** (that is the category of rings without unit) is so: notice that **Ring** is not semi-abelian since it is not pointed. If $\mathbb{A}$ is an exact category, then $\mathbf{Grp}(\mathbb{A})$ is semi-abelian as soon as it has finite coproducts. The categories of Lie algebras and crossed modules are semi-abelian as well, as their internal versions $\mathbf{Lie}_R(\mathbb{A})$ and $\mathbf{XMod}(\mathbb{A})$ where $\mathbb{A}$ is already semi-abelian.

We will sometimes use intermediate notions such as *Mal'cev*, *sequentiable* and *homogical* categories.

Definition 1.1.14 ([18])**.** A lex category $\mathbb{A}$ is said to be a *Mal'tsev category* if the following axiom holds:

$$\text{Any reflexive relation } R \text{ on } X \in \mathbb{A} \text{ is an equivalence relation.} \tag{M}$$

Proposition 1.1.15 ([3])**.** *Let $\mathbb{A}$ be a regular category. TFAE:*

- (M) *holds in $\mathbb{A}$;*
- *Any reflexive relation is symmetric;*

- *Any reflexive relation is transitive.* □

Definition 1.1.16 ([8])**.** A category $\mathbb{A}$ is said to be *sequentiable* if it is quasi-pointed, regular and protomodular. If $\mathbb{A}$ is also pointed we say that it is a *homological* category.

Example 1.1.17. Let $\mathbb{A}$ be a category and $X \in \mathbb{A}$ an object, then we define the *slice category* $\mathbb{A}/X$ as the category whose objects are arrows to X and morphisms between two arrows are commutative triangles

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

If $\mathbb{A}$ is a semi-abelian category, then for every choice of $X \in \mathbb{A}$ we know that $\mathbb{A}/X$ is sequentiable. It is quasi-pointed since $0 \to X$ is the initial object, $X \xrightarrow{1_X} X$ is the terminal object and the morphism

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$

is a monomorphism. Furthermore regularity and protomodularity are preserved by taking slice categories.

In the rest of this subsection we are going to state some useful properties that hold in the context of semi-abelian categories and that we will use throughout the text.

Lemma 1.1.18 ([8])**.** *Let $\mathbb{A}$ be a sequentiable category and consider the diagram*

$$\begin{tikzcd} K_f \arrow[r, tail, "k_f"] \arrow[d, "\gamma"'] & A \arrow[r, two heads, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ K_{f'} \arrow[r, tail, "k_{f'}"'] & A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

where f is a regular epimorphism and the objects on the left are the kernels of f and f'. Then the following hold:

- *if γ is an isomorphism, then the right-hand side square is a pullback;*
- *if β is an isomorphism, then γ is a regular epimorphism iff α is so;*
- *if γ and β are regular epimorphisms, then α is so.*
- *if γ is a regular epimorphism, then the induced map $\underline{f}\colon K_\alpha \to K_\beta$ is so.*

Proof. See Proposition 7, Proposition 8, Corollary 9 and Corollary 10 in [8]. □

Lemma 1.1.19. *Let $\mathbb{A}$ be semi-abelian and consider a morphism of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "f"] \arrow[d, "\alpha"'] & B \arrow[r, two heads, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, tail, "f'"'] & B' \arrow[r, two heads, "g'"'] & C' \arrow[r] & 0
\end{tikzcd} \tag{1.3}$$

If γ is an isomorphism, then $\binom{f'}{\beta}$ is a regular epimorphism.

Proof. Given the REM-factorisation

$$\begin{tikzcd}
A' + B \arrow[rr, "\binom{f'}{\beta}"] \arrow[dr, two heads, "\binom{e_1}{e_2}"'] & & B' \\
& I \arrow[ur, tail, "i"'] &
\end{tikzcd}$$

we can construct the following diagram

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, "\alpha"'] & B \arrow[r, two heads, "g"] \arrow[d, "e_2"'] & C \arrow[d, dotted, "\gamma_1"] \arrow[dd, bend left, "\gamma"] \\
A' \arrow[r, tail, "e_1"] \arrow[d, equal] & I \arrow[r, two heads, "c_{e_1}"] \arrow[d, tail, "i"'] & C_{e_1} \arrow[d, dotted, "\gamma_2"] \\
A' \arrow[r, tail, "f'"'] & B' \arrow[r, two heads, "g'"'] & C'
\end{tikzcd}$$

Notice that e_1 is a kernel since $i \circ e_1 = f'$ is a kernel and i is a monomorphism: this implies that e_1 is the kernel of its cokernel. Furthermore, since γ is an isomorphism, γ_2 is a split epimorphism, and hence a regular epimorphism. By applying Lemma 1.1.18 to the lower squares we deduce that i is a regular epimorphism since both $1_{A'}$ and γ_2 are so. This means that i is an isomorphism and hence $\binom{f'}{\beta}$ is a regular epimorphism. $\square$

The following result appears in [9] in a slightly more general context given by regular protomodular categories.

Lemma 1.1.20 (Theorem 4.1 in [9])**.** *Let $\mathbb{A}$ be a semi-abelian category and consider the following diagram*

$$\begin{tikzcd}
A \arrow[r, two heads, "p"] \arrow[d, tail, "k"'] & B \arrow[d, "m"] \\
C \arrow[r, two heads, "q"'] & D
\end{tikzcd}$$

with p, q regular epimorphisms, m a monomorphism and k a normal monomorphism. Then also m is normal. $\square$

Definition 1.1.21. Let $\mathbb{A}$ be a semi-abelian category and consider an object X with two subobjects $A \overset{a}{\hookrightarrow} X$ and $B \overset{b}{\hookrightarrow} X$. We define the *join* $A \vee B$ as the subobject of X defined through the REM-factorisation of the map

$$\begin{tikzcd} A+B \arrow[rr, "\binom{a}{b}"] \arrow[dr, two heads] & & X \\ & A \vee B \arrow[ur, tail, "a\vee b"'] & \end{tikzcd}$$

Remark 1.1.22. Notice that the inclusions

$$\begin{tikzcd} A \arrow[rr, "j_A"] \arrow[dr, tail, "i_A"'] & & A \vee B \\ & A+B \arrow[ur, two heads] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "j_B"] \arrow[dr, tail, "i_B"'] & & A \vee B \\ & A+B \arrow[ur, two heads] & \end{tikzcd}$$

are actually monomorphisms since postcomposing with $a \vee b$ we obtain respectively the monomorphisms a and b. Furthermore if both a and b are normal monomorphisms, then so is $a \vee b$: in particular if $a = k_f$ and $b = k_g$, then $a \vee b$ is the kernel of the diagonal of the pushout of f along g. For the proof of this fact and for more details see Proposition 2.7 in [40].

As a particular case we have the following lemma.

Lemma 1.1.23 (Section 2.4 in [40]). *Consider two regular epimorphisms and their pushout.*

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] \arrow[dr, dotted, two heads, "h"] & B \arrow[d, two heads, "f^*(g)"] \\ C \arrow[r, two heads, "g^*(f)"'] & D \end{tikzcd}$$

Then the kernel of the diagonal is given by the join of the kernels of the two regular epimorphisms, that is $K_h \cong K_f \vee K_g$. □

1.1.2 Regular Pushouts

Definition 1.1.24 ([9]). A *regular pushout* is a commutative square of regular epimorphisms such that the comparison map to the induced pullback square is a regular epimorphism as well.

Lemma 1.1.25 ([9]). *Let $\mathbb{A}$ be a regular category. Then every regular pushout is a pushout.*

Proof. See Definition 2.2 (and what comes after it) in [9]. □

Remark 1.1.26. In a semi-abelian category a commutative square of regular epimorphisms is a regular pushout if and only if it is a pushout. This actually holds in any regular exact Mal'cev category (see Theorem 5.7 in [17]), but if $\mathbb{A}$ is semi-abelian there is a simpler proof which goes as follows.

Consider two regular epimorphisms f and g with the same domain: in order to build their pushout consider the following diagram

$$\begin{tikzcd}
K_f \arrow[r, tail, "k_f"] \arrow[d, two heads, "e"'] & A \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & B \arrow[d, dotted, two heads, "\tilde{g}"] \\
I \arrow[r, tail, "i"'] & C \arrow[r, two heads, "c_i"'] & C_i
\end{tikzcd}$$

where e and i are defined as the REM-factorisation of $g \circ k_f$ and c_i is the cokernel of i. Since f is a regular epimorphism, it is the cokernel of its kernel, and this gives us the dotted map on the right. In particular Lemma 1.1.20 tells us that i is a normal monomorphism, and hence we have a morphism of short exact sequences. Since the map on the left is a regular epimorphism, the square on the right is a pushout. Due to this explicit construction of the pushout we are then able to show that the square on the right is actually a regular pushout. Take the pullback of c_i along $\tilde{g}$ and consider the induced diagram

$$\begin{tikzcd}
K_f \arrow[r, tail, "k_f"] \arrow[d, two heads, "e"'] & A \arrow[r, two heads, "f"] \arrow[d, dotted, "\phi"'] & B \arrow[d, equal] \\
K_{p_1} \arrow[r, tail, "k_{p_1}"] \arrow[d, equal] & P \arrow[r, two heads, "p_1"] \arrow[d, two heads, "p_2"'] & B \arrow[d, two heads, "\tilde{g}"] \\
I \arrow[r, tail, "i"'] & C \arrow[r, two heads, "c_i"'] & C_i
\end{tikzcd}$$

It suffices to use Lemma 1.1.18 to obtain that ϕ is a regular epimorphism, which is the thesis.

Lemma 1.1.27 ([9]). *Consider a regular pushout in a regular category and take the kernels of two of its parallel morphisms*

$$\begin{tikzcd}
K_f \arrow[r, tail, "k_f"] \arrow[d, dotted, "k"'] & A \arrow[r, two heads, "f"] \arrow[d, two heads, "\alpha"'] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[d, two heads, "\beta"] \\
K_{f'} \arrow[r, tail, "k_{f'}"'] & A' \arrow[r, two heads, "f'"'] & B'
\end{tikzcd}$$

Then the induced map k is again a regular epimorphism.

Proof. Consider the pullback of f' and β and the induced comparison map which is a regular epimorphism.

$$\begin{tikzcd}
K_f \arrow[r, tail, "k_f"] \arrow[d, dotted, "k_1"'] \arrow[dr, phantom, "(*_1)"] & A \arrow[r, two heads, "f"] \arrow[d, two heads, "\langle \alpha, f\rangle"] & B \arrow[d, equal] \\
K_{p_2} \arrow[r, tail, "k_{p_2}"] \arrow[d, dotted, "k_2"'] & A' \times_{B'} B \arrow[r, two heads, "p_2"] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "(*_2)"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "\beta"] \\
K_{f'} \arrow[r, tail, "k_{f'}"'] & A' \arrow[r, two heads, "f'"'] & B'
\end{tikzcd}$$

Since $(*_1)$ is a pullback, k_1 is a regular epimorphism as well. Furthermore, since $(*_2)$ is a pullback, k_2 is an isomorphism and the composition $k = k_2 \circ k_1$ is again a regular epimorphism. □

1.1.3 The bifunctor $\flat$

From now on we will consider $\mathbb{A}$ to be a semi-abelian category.

Definition 1.1.28 ([5]). The bifunctor $\flat\colon \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is defined on objects as the kernel

$$\begin{tikzcd}
A\flat B \arrow[r, tail, "k_{A,B}"] & A + B \arrow[r, "\binom{1}{0}"] & A
\end{tikzcd}$$

Using the universal property of kernels together with the functoriality of coproducts, its behaviour on arrows is determined by

$$\begin{tikzcd}
A\flat B \arrow[r, tail, "k_{A,B}"] \arrow[d, dotted, "f\flat g"'] & A + B \arrow[r, "\left(\begin{smallmatrix}1_A\\0\end{smallmatrix}\right)"] \arrow[d, "f+g"'] & A \arrow[d, "f"] \\
A'\flat B' \arrow[r, tail, "k_{A',B'}"'] & A' + B' \arrow[r, "\left(\begin{smallmatrix}1_{A'}\\0\end{smallmatrix}\right)"'] & A'
\end{tikzcd}$$

Example 1.1.29. In the category **Grp** the coproduct $A+B$ is the group freely generated by the disjoint union of A and B, modulo the relations that hold in A or in B. This means that an element in $A+B$ can be represented as a word obtained by juxtaposition of elements in A and in B. Then it is easy to deduce that $A\flat B$ is the subgroup of $A+B$ whose elements are represented by the words of the form $a_1b_1 \cdots a_nb_n$ such that $a_1 \cdots a_n = 1 \in A$. Furthermore, it can be shown that each word in $A\flat B$ can be written as a juxtaposition of formal conjugations, that is

$$A\flat B = \langle aba^{-1} \mid a \in A, b \in B\rangle.$$

The following example expresses the idea of the proof, which easily generalises to any word in $A\flat B$.

$$\begin{aligned}
a_1b_1a_2b_2a_3b_3 &= (a_1b_1a_1^{-1})(a_1a_2b_2a_2^{-1}a_1^{-1})(a_1a_2a_3)b_3 \\
&= (a_1b_1a_1^{-1})(a_1a_2b_2(a_1a_2)^{-1})1(1b_31^{-1})
\end{aligned}$$

There are two natural transformations that will be useful in the following chapters: we are going to define them and to explain what is their role.

Definition 1.1.30. If we consider the second projection functor $\pi_2\colon \mathbb{A}\times\mathbb{A}\to\mathbb{A}$, we can define a natural transformation

$$\eta\colon \pi_2 \stackrel{\cdot}{\longrightarrow} (-\flat-).$$

For $A, B \in \mathbb{A}$ we define the morphism $\eta^A_B\colon B \to A\flat B$ using the universal property of the kernel $A\flat B$

$$\begin{tikzcd} B \arrow[d, dotted, "\eta^A_B"'] \arrow[dr, "i_B"] & & \\ A\flat B \arrow[r, tail, "k_{A,B}"] & A+B \arrow[r, "\binom{1}{0}"] & A \end{tikzcd}$$

since $\binom{1}{0}\circ i_B = 0$.

Remark 1.1.31. The naturality of η follows from the universal property of kernels and says that for each pair of morphisms (f,g), the square

$$\begin{tikzcd} B \arrow[r, "\eta^A_B"] \arrow[d, "g"'] & A\flat B \arrow[d, "f\flat g"] \\ B' \arrow[r, "\eta^{A'}_{B'}"'] & A'\flat B' \end{tikzcd}$$

commutes. This can be shown by postcomposing with the monomorphism $k_{A',B'}$ obtaining the equalities

$$\begin{aligned} k_{A',B'}\circ\eta^{A'}_{B'}\circ g &= i_{B'}\circ g \\ &= (f+g)\circ i_B \\ &= (f+g)\circ k_{A,B}\circ\eta^A_B \\ &= k_{A',B'}\circ f\flat g\circ\eta^A_B. \end{aligned}$$

Definition 1.1.32. If we define the functor $-\flat(-\flat-)\colon \mathbb{A}\times\mathbb{A}\to\mathbb{A}$ which sends the pair (A,B) into $A\flat(A\flat B)$ we have another natural transformation

$$\mu\colon\ -\flat(-\flat-) \stackrel{\cdot}{\longrightarrow} (-\flat-).$$

Its component $\mu^A_B\colon A\flat(A\flat B)\to A\flat B$ is induced from the following diagram

$$\begin{tikzcd} A\flat(A\flat B) \arrow[r, tail, "k_{A,A\flat B}"] \arrow[d, dotted, "\mu^A_B"'] & A+(A\flat B) \arrow[r, "\binom{1}{0}"] \arrow[d, "\binom{i_A}{k_{A,B}}"] & A \arrow[d, equal] \\ A\flat B \arrow[r, tail, "k_{A,B}"'] & A+B \arrow[r, "\binom{1}{0}"'] & A \end{tikzcd}$$

since the right-hand square commutes.

Remark 1.1.33. The naturality of μ follows from the universal property of kernels and says that for each pair of morphisms (f, g), the square

$$\begin{tikzcd}
A\flat(A\flat B) \arrow[r, "\mu^A_B"] \arrow[d, "f\flat(f\flat g)"'] & A\flat B \arrow[d, "f\flat g"] \\
A'\flat(A'\flat B') \arrow[r, "\mu^{A'}_{B'}"'] & A'\flat B'
\end{tikzcd}$$

commutes. This can be shown again by postcomposing with the monomorphism $k_{A',B'}$ obtaining the equalities

$$\begin{aligned}
k_{A',B'} \circ \mu^{A'}_{B'} \circ (f\flat(f\flat g)) &= \begin{pmatrix} i_{A'} \\ k_{A',B'} \end{pmatrix} \circ k_{A',A'\flat B'} \circ (f\flat(f\flat g)) \\
&= \begin{pmatrix} i_{A'} \\ k_{A',B'} \end{pmatrix} \circ (f + (f\flat g)) \circ k_{A,A\flat B} \\
&= (f + g) \circ \begin{pmatrix} i_{A} \\ k_{A,B} \end{pmatrix} \circ k_{A,A\flat B} \\
&= (f + g) \circ k_{A,B} \circ \mu^A_B \\
&= k_{A',B'} \circ (f\flat g) \circ \mu^A_B.
\end{aligned}$$

Corollary 1.1.34 ([5])**.** *For any fixed object $A \in \mathbb{A}$ the triple $(A\flat(-), \eta^A, \mu^A)$ is a monad.* ☐

Lemma 1.1.35. *In a semi-abelian category, consider regular epimorphisms $\alpha\colon A \to A'$ and $\beta\colon B \to B'$. Then both $\alpha + \beta$ and $\alpha\flat\beta$ are regular epimorphisms as well.*

Proof. The first statement is easily shown checking that, if $\alpha = coeq(x_1, x_2)$ and $\beta = coeq(y_1, y_2)$, then $\alpha + \beta = coeq(x_1 + y_1, x_2 + y_2)$. As for the second statement, we build the diagram

$$\begin{tikzcd}
A\flat B \arrow[r, tail, "k_{A,B}"] \arrow[d, "\alpha\flat\beta"'] & A + B \arrow[r, two heads, "\binom{1_A}{0}"] \arrow[d, two heads, "\alpha+\beta"] & A \arrow[d, two heads, "\alpha"] \\
A'\flat B' \arrow[r, tail, "k_{A',B'}"'] & A' + B' \arrow[r, two heads, "\binom{1_{A'}}{0}"'] & A'.
\end{tikzcd}$$

Thanks to Lemma 1.1.27 and Remark 1.1.26 it suffices to show that the right-hand square is a pushout in order to obtain that $\alpha\flat\beta$ is a regular epimorphism as well. This is easy to do by direct calculation. ☐

1.1.4 The cosmash product $\diamond$

Definition 1.1.36 ([65, 16, 51])**.** Given two objects A and B in $\mathbb{A}$, consider the map

$$\Sigma_{A,B} := \begin{pmatrix} \langle 1, 0 \rangle \\ \langle 0, 1 \rangle \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \colon A + B \longrightarrow A \times B$$

Since $\mathbb{A}$ is semi-abelian, the morphism $\Sigma_{A,B}$ is a regular epimorphism. By taking its kernel we find the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \diamond B \arrow[r, tail, "h_{A,B}"] & A + B \arrow[r, "\Sigma_{A,B}"] & A \times B \arrow[r] & 0 \end{tikzcd}$$

where $A \diamond B$ is called *cosmash product* of A and B.

Remark 1.1.37. Notice that the inclusion of $A \diamond B$ into $A + B$ factors through $A \flat B$ due to the fact that the latter is the kernel of $\binom{1_A}{0} \colon A + B \to A$. Moreover we have another split short exact sequence involving the cosmash product, that is

$$\begin{tikzcd} 0 \arrow[r] & A \diamond B \arrow[r, tail, "i_{A,B}"] & A \flat B \arrow[r, shift left, "\tau^A_B"] & B \arrow[l, shift left, "\eta^A_B"] \arrow[r] & 0 \end{tikzcd}$$

where $\tau^A_B := \binom{0}{1} \circ k_{A,B}$ is the trivial action of A on B. This can be seen by constructing the 3×3 diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \diamond B \arrow[r, tail, "i_{A,B}"] \arrow[d, tail, "i_{B,A}"'] \arrow[dr, dotted, tail, "h_{A,B}"] & A \flat B \arrow[r, two heads, "\tau^A_B"] \arrow[d, tail, "k_{A,B}"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \flat A \arrow[r, tail, "k_{B,A}"'] \arrow[d, two heads, "\tau^B_A"'] & A + B \arrow[r, two heads, "\binom{0}{1}"] \arrow[d, two heads, "\binom{1}{0}"'] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, equal] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

from the bottom-right square by taking kernels, and then by noticing that the top-left object is the kernel of the comparison morphism from $A + B$ to the pullback induced by the lower-right square: since this morphism is precisely $\Sigma_{A,B}$, its kernel is $A \diamond B$.

Moreover the upper left square is a pullback and hence $A \diamond B$ can be seen as the intersection of the subobjects $A \flat B$ and $B \flat A$ of $A + B$. Furthermore, being $\mathbb{A}$ a protomodular category, each split short exact sequence leads to a regular epimorphism which covers the object in the middle with the sum of the two adjacent ones. In this particular case we obtain the regular epimorphism

$$\begin{tikzcd} (A \diamond B) + B \arrow[r, two heads, "\binom{i_{A,B}}{\eta^A_B}"] & A \flat B. \end{tikzcd}$$

Lemma 1.1.38. *Let X be an object in a semi-abelian category $\mathbb{A}$. Then the functor $-\flat X \colon \mathbb{A} \to \mathbb{A}$ preserves coequalisers of reflexive graphs.*

Proof. Consider a reflexive graph (see Definition 1.3.1) with its coequaliser

$$\begin{tikzcd} A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \arrow[r, two heads, "q"] & Q \end{tikzcd}$$

and the induced diagram

$$\begin{tikzcd} A \diamond X \arrow[r, shift left, "d\diamond 1_X"] \arrow[r, shift right, "c\diamond 1_X"'] \arrow[d, tail, "i_{A,X}"'] & B \diamond X \arrow[r, two heads, "q\diamond 1_X"] \arrow[d, tail, "i_{B,X}"'] & Q \diamond X \arrow[d, tail, "i_{Q,X}"] \\ A\flat X \arrow[r, shift left, "d\flat 1_X"] \arrow[r, shift right, "c\flat 1_X"'] \arrow[d, two heads, "\tau^A_X"'] & B\flat X \arrow[r, two heads, "q\flat 1_X"] \arrow[d, two heads, "\tau^B_X"'] & Q\flat X \arrow[d, two heads, "\tau^Q_X"] \\ X \arrow[r, shift left, "1_X"] \arrow[r, shift right, "1_X"'] & X \arrow[r, two heads, "1_X"'] & X \end{tikzcd}$$

By using Corollary 2.27 in [50] we know that $q\diamond 1_X$ is again the coequaliser of $d\diamond 1_X$ and $c\diamond 1_X$. We already know that $q\flat 1_X$ is a regular epimorphism by Lemma 1.1.35 and that $(q\flat 1_X)\circ(d\flat 1_X) = (q\flat 1_X)\circ(c\flat 1_X)$, so it remains to show the universal property. First of all, by examining the squares on the right, one can see that they form a morphism of short exact sequences, and being 1_X an isomorphism, we conclude that the top square is a pullback. This implies that it is also a pushout: indeed if we take kernels horizontally we obtain an induced isomorphism between them and this in turn implies that it is a pushout. Now suppose that there exists a map $z\colon B\flat X \to Z$ such that $z\circ(d\flat 1_X) = z\circ(c\flat 1_X)$. This in turn implies that $z\circ i^A_X \circ (d\diamond 1_X) = z\circ i^A_X \circ (c\diamond 1_X)$ and hence there exists a unique map $\phi\colon Q\diamond X \to Z$ such that $\phi\circ(q\diamond 1_X) = z\circ i^A_X$. Now by the universal property of the pushout we obtain the thesis. $\square$

1.1.5 The ternary cosmash product

Following [52, 16], in [51] Hartl and Van der Linden define the n-ary version of the cosmash product. We are interested in the ternary one and in some relations between this and the binary one.

Definition 1.1.39 ([51])**.** Given three objects A, B and C in $\mathbb{A}$, consider the map

$$\Sigma_{A,B,C} = \begin{pmatrix} i_A & i_A & 0 \\ i_B & 0 & i_B \\ 0 & i_C & i_C \end{pmatrix} \colon A+B+C \longrightarrow (A+B)\times(A+C)\times(B+C)$$

Its kernel is denoted as

$$\begin{tikzcd} A\diamond B\diamond C \arrow[r, tail, "h_{A,B,C}"] & A+B+C \end{tikzcd}$$

and it is called *ternary cosmash product* of A, B and C. It is trivial to notice that, as it happens for the binary cosmash product, the ternary one does not depend (up to isomorphism) on the order of the objects.

In [51] the authors define folding operations linking cosmash product of different arities: for our purposes we only need to recall one of them.

Definition 1.1.40. Given two objects A and B we can construct a map

$$S_{2,1}^{A,B}\colon A \diamond A \diamond B \to A \diamond B$$

through the diagram

$$\begin{tikzcd}
A \diamond A \diamond B \arrow[r, tail, "h_{A,A,B}"] \arrow[d, dotted, "S_{2,1}^{A,B}"'] & A + A + B \arrow[r, "\Sigma_{A,A,B}"] \arrow[d, "\binom{1_A}{1_A}+1_B"] & (A+A) \times (A+B) \times (A+B) \arrow[d, "\binom{1_A}{1_A}\times\left(\binom{0}{1_B}\circ\pi_i\right)"] \\
A \diamond B \arrow[r, tail, "h_{A,B}"'] & A + B \arrow[r, "\Sigma_{A,B}"'] & A \times B
\end{tikzcd}$$

Finally we need a map that links objects of the form $(A+B)\flat C$ to the corresponding ternary cosmash product $A \diamond B \diamond C$.

Definition 1.1.41. Consider the object $(A+B)\flat C$ and define the map $j_{A,B,C}$ as in the diagram

$$\begin{tikzcd}
A \diamond B \diamond C \arrow[d, dotted, "j_{A,B,C}"'] \arrow[dr, tail, "h_{A,B,C}"] & & \\
(A+B)\flat C \arrow[r, tail, "k_{(A+B),C}"'] & A + B + C \arrow[r, "\binom{1_{A+B}}{0}"] \arrow[dr, "\Sigma_{A,B,C}"'] & A + B \\
& & (A+B) \times (A+C) \times (B+C) \arrow[u, "\pi_1"']
\end{tikzcd}$$

In particular if $A = B$ we have the commutative diagram

$$\begin{tikzcd}
A \diamond A \diamond C \arrow[r, "j_{A,A,C}"] \arrow[d, "S_{2,1}^{A,C}"'] \arrow[rr, tail, bend left, "h_{A,A,B}"] & (A+A)\flat C \arrow[r, tail, "k_{(A+A),C}"] \arrow[d, "\binom{1_A}{1_A}\flat 1_C"] & A + A + C \arrow[d, "\binom{1_A}{1_A}+1_C"] \\
A \diamond C \arrow[r, tail, "i_{A,C}"'] \arrow[rr, tail, bend right, "h_{A,B}"'] & A\flat C \arrow[r, tail, "k_{A,C}"'] & A + C
\end{tikzcd}$$

Lemma 1.1.42. *It is possible to cover the object $(A+B)\flat C$ with the three components $(A \diamond B \diamond C)$, $(A\flat C)$ and $(B\flat C)$.*

Proof. By Lemma 2.12 in [51] we know that there is a regular epimorphism of the form

$$(A \diamond B \diamond C) + (A \diamond C) + (B \diamond C) \xrightarrow{\;e\;} (A+B) \diamond C$$

Using Remark 1.1.37 we are able to construct the square

$$\begin{tikzcd}
(A \diamond B \diamond C) + (A \diamond C) + (B \diamond C) + C + C \arrow[r, two heads, "1+\binom{i_{A,C}}{\eta^A_C}+\binom{i_{B,C}}{\eta^B_C}"] \arrow[d, two heads, "e+\binom{1_C}{1_C}"'] & (A \diamond B \diamond C) + (A\flat C) + (B\flat C) \arrow[d, two heads, "\begin{pmatrix} j_{A,B,C} \\ i_A\flat 1_C \\ i_B\flat 1_C \end{pmatrix}"] \\
((A+B) \diamond C) + C \arrow[r, two heads, "\binom{i_{(A+B),C}}{\eta^{A+B}_C}"'] & (A+B)\flat C
\end{tikzcd}$$

from which we can see that the vertical map on the right is a regular epimorphism. $\square$

1.1.6 Some notions in commutator theory

Definition 1.1.43 ([52, 65])**.** Given two subobjects (L, l) and (M, m) of X, we define their *Higgins commutator* as the image of the map $\binom{l}{m} \circ h_{L,M}$, that is the subobject of X given by the following factorisation

$$\begin{tikzcd}
L \diamond M \arrow[r, tail, "h_{L,M}"] \arrow[d, two heads] & L + M \arrow[d, "\binom{l}{m}"] \\
{[L, M]^{\mathcal{H}}_X} \arrow[r, tail] & X.
\end{tikzcd}$$

Proposition 1.1.44 (Theorem 5.3 in [65])**.** *Let $\mathbb{A}$ be a semi-abelian category and $K \hookrightarrow X$ a monomorphism. Then K is a normal subobject of X iff $[K, X]^{\mathcal{H}}_X$ is a subobject of K.* $\square$

Lemma 1.1.45. *Consider a diagram of the form*

$$\begin{tikzcd}
A \arrow[r, tail, "a"] \arrow[d, "\alpha"'] & X \arrow[d, "\gamma"] & B \arrow[l, tail, "b"'] \arrow[d, "\beta"] \\
A \arrow[r, tail, "a'"'] & X & B \arrow[l, tail, "b'"]
\end{tikzcd}$$

Then there exists a unique induced morphism

$$[\alpha, \beta]^{\mathcal{H}}_\gamma \colon [A, B]^{\mathcal{H}}_X \to [A', B']^{\mathcal{H}}_{X'}$$

Moreover, if γ is a monomorphism, then also $[\alpha, \beta]^{\mathcal{H}}_\gamma$ is a monomorphism. If furthermore α and β are regular epimorphisms, then $[\alpha, \beta]^{\mathcal{H}}_\gamma$ is an isomorphism.

Proof. The map $[\alpha, \beta]^{\mathcal{H}}_\gamma$ is given by the universal properties of the cokernels involved in the definition of the REM-factorisation. When γ is a monomorphism, then the lower face in the cube

$$\begin{tikzcd}
A \diamond B \arrow[rr, tail, "h_{A,B}"] \arrow[dr, "\alpha\diamond\beta"] \arrow[dd, two heads] & & A+B \arrow[dr, "\alpha+\beta"] \arrow[dd, "\binom{a}{b}"] & \\
& A' \diamond B' \arrow[rr, tail, "h_{A',B'}"] \arrow[dd, two heads] & & A'+B' \arrow[dd, "\binom{a'}{b'}"] \\
{[A,B]^{\mathcal{H}}_X} \arrow[rr, hook] \arrow[dr, "{[\alpha,\beta]^{\mathcal{H}}_\gamma}"'] & & X \arrow[dr, hook, "\gamma"] & \\
& {[A',B']^{\mathcal{H}}_{X'}} \arrow[rr, hook] & & X'
\end{tikzcd}$$

shows us that also $[\alpha,\beta]^{\mathcal{H}}_\gamma$ is so. Now, if α and β are regular epimorphisms, they are isomorphisms since from γ being a monomorphism we can deduce that they are monomorphisms as well. Hence $\alpha \diamond \beta$ is an isomorphism and from the left face of the cube we deduce that $[\alpha,\beta]^{\mathcal{H}}_\gamma$ is also a regular epimorphism. $\square$

Definition 1.1.46 ([53])**.** Given a coterminal pair

$$K \xrightarrow{k} A \xleftarrow{l} L$$

we say that k and l *Huq-commute* if there exists a (necessarily unique) map ϕ such that the diagram

$$\begin{tikzcd}
& K \arrow[dl, "\langle 1,0\rangle"'] \arrow[dr, "k"] & \\
K \times L \arrow[rr, dotted, "\phi"] & & A \\
& L \arrow[ul, "\langle 0,1\rangle"] \arrow[ur, "l"'] &
\end{tikzcd} \tag{1.4}$$

commutes.

We are mainly interested in the case in which both K and L are (normal) subobjects of A.

Definition 1.1.47 ([53])**.** Given a pair of subobjects

$$K \overset{k}{\rightarrowtail} A \overset{l}{\leftarrowtail} L$$

we define the *Huq commutator* of (K,k) and (L,l), denoted by $[K,L]^{\mathcal{Q}}_A$ as follow

$$\begin{tikzcd}
& K+L \arrow[r, two heads, "\Sigma_{K,L}"] \arrow[d, "\binom{k}{l}"'] & K \times L \arrow[d, "\phi"] \\
{[K,L]^{\mathcal{Q}}_A} \arrow[r, tail, "k_q"'] & A \arrow[r, two heads, "q"'] \arrow[ur, phantom, "\ulcorner"] & Q
\end{tikzcd} \tag{1.5}$$

where the square on the right is a pushout. Since $\Sigma_{K,L}$ is a regular epimorphism, so is q (the pushout of a regular epimorphism is again a regular epimorphism): in particular this implies that $q = c_{k_q} = c_i$ (since $\mathbb{A}$ is both pointed and protomodular).

Remark 1.1.48. It is trivial to observe that two coterminal morphisms Huq-commute iff their Huq commutator is trivial: indeed a map ϕ as in (1.4) exists iff q as in (1.5) is an isomorphism, and since it is always a regular epimorphism, this is equivalent to q being a monomorphism and hence to its kernel being 0.

Remark 1.1.49. Notice that the Huq commutator is the normalisation of the Higgins one. In order to see this, consider the diagram

$$\begin{tikzcd}
K \diamond L \arrow[r, "h_{K,L}", tail] \arrow[d, two heads] & K+L \arrow[r, "\Sigma_{K,L}", two heads] \arrow[d, "\binom{k}{l}"'] & K \times L \arrow[d, "\phi"] \\
{[K,L]^{\mathcal{H}}_A} \arrow[r, tail, "m"'] \arrow[d, dotted, two heads] & A \arrow[r, two heads, "c_m"'] & Q \\
{[K,L]^{\mathcal{Q}}_A} \arrow[ur, tail, "k_{c_m}"'] & &
\end{tikzcd} \tag{1.6}$$

Since $\Sigma_{K,L}$ is the cokernel of $h_{K,L}$ and since the square on the right is a pushout, we have that c_m is the cokernel of m. Hence, being the Huq commutator the kernel of the cokernel of m, it is the normalisation of $[K,L]^{\mathcal{H}}_A$.

Moreover if K and L cover A, that is if $\binom{k}{l}$ is a regular epimorphism, by Lemma 1.1.20 we obtain that m is already a normal monomorphism, and therefore it is the kernel of its cokernel: this means that

$$[K,L]^{\mathcal{H}}_A \cong [K,L]^{\mathcal{Q}}_A.$$

Remark 1.1.50. Looking at (1.6) it is easy to see that $[K,L]^{\mathcal{H}}_A = 0$ iff $[K,L]^{\mathcal{Q}}_A = 0$: indeed if the first one vanishes, then its cokernel is the identity on A and being the Huq commutator the kernel of this map, it is again trivial; the other implication is given by the fact that the Higgins commutator is a subobject of the Huq commutator.

Proposition 1.1.51 (Proposition 5.1.2 in [27]). *The construction of the Huq commutator is functorial. This means that if we have a commutative diagram as on the left*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "f"'] & A \arrow[d, "h"] & L \arrow[l, "l"'] \arrow[d, "g"] & & {[K,L]^{\mathcal{Q}}_A} \arrow[d, "{[f,g]^{\mathcal{Q}}_h}"] \\
K' \arrow[r, "k'"'] & A' & L' \arrow[l, "l'"] & \longmapsto & {[K',L']^{\mathcal{Q}}_{A'}}
\end{tikzcd}$$

we can construct a map as on the right. □

Remark 1.1.52. Notice that by definition of the induced map on the Huq commutators we have that the following square commutes

$$\begin{tikzcd}
{[K,L]^{\mathcal{Q}}_A} \arrow[r, tail] \arrow[d, "{[f,g]^{\mathcal{Q}}_h}"'] & A \arrow[d, "h"] \\
{[K',L']^{\mathcal{Q}}_{A'}} \arrow[r, tail] & A'
\end{tikzcd}$$

Lemma 1.1.53 (Lemma 5.1.3 in [27]). *If K and L are normal subobjects of A, then $[K,L]^{\mathcal{Q}}_A$ is a normal subobject of both K and L (and hence of $K \wedge L$).* □

Proposition 1.1.54 (Lemma 5.1.5 in [27]). *Consider a pair of subobjects*

$$K \overset{k}{\rightarrowtail} A \overset{l}{\leftarrowtail} L$$

and a regular epimorphism $h\colon A \to A'$. Construct the morphism of coterminal pairs

$$\begin{tikzcd}
K \arrow[r, tail, "k"] \arrow[d, two heads, "h_K"'] & A \arrow[d, two heads, "h"] & L \arrow[l, tail, "l"'] \arrow[d, two heads, "h_L"] \\
K' \arrow[r, tail, "k'"'] & A' & L' \arrow[l, tail, "l'"]
\end{tikzcd}$$

induced by the REM-factorisation. Then

$$[K',L']^{\mathcal{Q}}_{A'} = [h(K),h(L)]^{\mathcal{Q}}_{h(A)} \cong h([K,L]^{\mathcal{Q}}_A).$$

that is the morphism $[h_K,h_L]^{\mathcal{Q}}_h$ is a regular epimorphism. □

Proposition 1.1.55 (Proposition 5.1.6 in [27]). *If K and L are normal subobjects of A, then the commutator*

$$\left[\frac{K}{[K,L]^{\mathcal{Q}}_A}, \frac{L}{[K,L]^{\mathcal{Q}}_A}\right]^{\mathcal{Q}}_{\frac{A}{[K,L]^{\mathcal{Q}}_A}}$$

vanishes. □

Definition 1.1.56 ([71, 74]). Given two equivalence relations R and S on the same object A, depicted as

$$\begin{tikzcd}
R \arrow[r, shift left=2, "r_0"] \arrow[r, shift right=2, "r_1"'] & A \arrow[l, "\delta_R" description] \arrow[r, "\delta_S" description] & S \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "s_1"]
\end{tikzcd}$$

consider the pullback

$$\begin{tikzcd}
R \times_A S \arrow[r] \arrow[d] & S \arrow[d, "s_0"] \\
R \arrow[r, "r_1"'] & A
\end{tikzcd}$$

We say that R *and* S *commute in the sense of Smith* (or *Smith-commute*) if there exists a (necessarily unique) map θ such that the following diagram commutes

$$\begin{tikzcd} & R \arrow[dl, "\langle 1,\delta_S \circ r_1\rangle"'] \arrow[dr, "r_0"] & \\ R \times_A S \arrow[rr, dotted, "\theta"] & & A \\ & S \arrow[ul, "\langle \delta_R \circ s_0,1\rangle"] \arrow[ur, "s_1"'] & \end{tikzcd} \tag{1.7}$$

There is a correspondent definition of *Smith-commutator* between two equivalence relations on the same object A: this commutator $[R,S]^{\mathcal{S}}_A$ is again an equivalence relation on A and it can easily be proved that $[R,S]^{\mathcal{S}}_A = \Delta_A$ (the discrete relation on A) iff R and S Smith-commute. For more details on this see [72, 71].

Definition 1.1.57 ([67])**.** We say that in $\mathbb{A}$ the *Smith-is-Huq* condition (from now on denoted simply with SH) holds if two effective equivalence relations commute in the sense of Smith as soon as their normalisations commute in the sense of Huq.

Definition 1.1.58. Let $\mathbb{A}$ be a semi-abelian category. We say that $A \in \mathbb{A}$ is an *abelian object* if it carries the structure of an internal abelian group. In particular the subcategory $\mathbf{Ab}(\mathbb{A})$ is defined as the category of abelian objects and morphisms between them: notice that all morphisms between abelian objects automatically preserve the additional structure since we are in a semi-abelian category (for more details see Lemma 3.9 in [46]). An equivalent condition for A to be an abelian object is that $[A,A]^{\mathcal{H}}_A = 0$ (see Section 2.3 in [3]).

1.2 Points and actions

Throughout this section we consider $\mathbb{A}$ to be any category unless explicitly stated otherwise: in particular we will require $\mathbb{A}$ to be semi-abelian only when dealing with the equivalence between internal actions and points.

1.2.1 The categories $\mathbf{Pt}(\mathbb{A})$ and $\mathbf{Act}(\mathbb{A})$

Definition 1.2.1. A *point* in $\mathbb{A}$ is a split epimorphism p with a chosen splitting s, that is

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

with $p \circ s = 1_B$. A morphism of points is given by a pair of vertical maps (f,g) such that the left-pointing and the right-pointing squares in

$$\begin{tikzcd} A \arrow[r, shift left, "p"] \arrow[d, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, "g"] \\ A' \arrow[r, shift left, "p'"] & B' \arrow[l, shift left, "s'"] \end{tikzcd}$$

commute. The category of points in $\mathbb{A}$ and morphisms between them is denoted with $\mathbf{Pt}(\mathbb{A})$.

Lemma 1.2.2. *Consider a morphism of points of the form*

$$\begin{tikzcd}
A_0 \arrow[r, shift left, "p_0"] \arrow[d, "f"'] & B_0 \arrow[l, shift left, "s_0"] \arrow[d, "g"] \\
A_1 \arrow[r, shift left, "p_1"] & B_1 \arrow[l, shift left, "s_1"]
\end{tikzcd}$$

If f and g are epimorphisms then the right pointing square is a pushout. Dually, if f and g are monomorphisms, then the left pointing square is a pullback.

Proof. We only prove the first result since the second one is obtained by taking the dual proof. Consider (Z, α, β)

$$\begin{tikzcd}
A_0 \arrow[r, shift left, "p_0"] \arrow[d, "f"'] & B_0 \arrow[l, shift left, "s_0"] \arrow[d, "g"] \arrow[ddr, bend left, "\beta"] & \\
A_1 \arrow[r, shift left, "p_1"] \arrow[drr, bend right, "\alpha"'] & B_1 \arrow[l, shift left, "s_1"] \arrow[dr, dotted, "\phi"] & \\
 & & Z
\end{tikzcd}$$

such that $\alpha \circ f = \beta \circ p_0$. We want to construct a unique $\phi \colon B_1 \to Z$ such that $\phi \circ g = \beta$ and $\phi \circ p_1 = \alpha$. The uniqueness is given by the fact that g is an epimorphism, whereas to show existence we define $\phi = \alpha \circ s_1$ and we show that

$$\begin{cases} \alpha \circ s_1 \circ g = \beta \\ \alpha \circ s_1 \circ p_1 = \alpha \end{cases}$$

holds. In particular since both p_0 and f are epimorphisms, it suffices to show

$$\begin{cases} \alpha \circ s_1 \circ g \circ p_0 = \beta \circ p_0 \\ \alpha \circ s_1 \circ p_1 \circ f = \alpha \circ f \end{cases}$$

The first one is given by

$$\begin{aligned} \alpha \circ s_1 \circ g \circ p_0 &= \alpha \circ f \circ s_0 \circ p_0 \\ &= \beta \circ p_0 \circ s_0 \circ p_0 = \beta \circ p_0 \end{aligned}$$

whereas the second one is given by

$$\begin{aligned} \alpha \circ s_1 \circ p_1 \circ f &= \alpha \circ s_1 \circ g \circ p_0 \\ &= \alpha \circ f \circ s_0 \circ p_0 = \beta \circ p_0 \circ s_0 \circ p_0 \\ &= \beta \circ p_0 = \alpha \circ f. \end{aligned}$$

□

Corollary 1.2.3. *Consider a point in* $\mathbf{Pt}(\mathbb{A})$*, that is a commutative diagram of the form*

$$\begin{tikzcd} A_1 \arrow[r, shift left, "p_1"] \arrow[d, shift left, "p_A"] & B_1 \arrow[l, shift left, "s_1"] \arrow[d, shift left, "p_B"] \\ A_0 \arrow[r, shift left, "p_0"] \arrow[u, shift left, "s_A"] & B_0 \arrow[l, shift left, "s_0"] \arrow[u, shift left, "s_B"] \end{tikzcd} \tag{1.8}$$

in which every pair of parallel arrows is a morphism of points. Then the right-down pointing square of split epimorphisms is a pushout. Dually the left-up pointing square of split monomorphisms is a pullback. □

Having described the category of points, we now shift to internal actions, whose category is equivalent to the former whenever the base category $\mathbb{A}$ is semi-abelian.

Definition 1.2.4 ([5])**.** An *internal action* (or simply *action*) in $\mathbb{A}$ is a triple (A, X, ξ) with $\xi\colon A\flat X \to X$ a map in $\mathbb{A}$ such that (X, ξ) is an algebra for the monad $A\flat(-)\colon \mathbb{A} \to \mathbb{A}$. A morphism of actions from (A, X, ξ) to (A', X', ξ') is given by a pair (f, g) of maps in $\mathbb{A}$, with $f\colon A \to A'$ and $g\colon X \to X'$, such that the following diagram commutes:

$$\begin{tikzcd} A\flat X \arrow[r, "f\flat g"] \arrow[d, "\xi"'] & A'\flat X' \arrow[d, "\xi'"] \\ X \arrow[r, "g"'] & X' \end{tikzcd}$$

The category of actions and morphisms between them is denoted by $\mathbf{Act}(\mathbb{A})$.

Example 1.2.5 ([5]). If we fix $\mathbb{A} = \mathbf{Grp}$ we find that internal actions coincide with the usual notion of group action. Indeed due to Example 1.1.29, in order to define such an internal action $\xi\colon A\flat X \to X$ it suffices to define it only on elements of the form axa^{-1} since they generate the whole subgroup $A\flat X$; now an internal action ξ corresponds to the group action $\psi\colon A \times X \to X$ given by $\psi(a, x) := \xi(axa^{-1})$; viceversa starting from a group action ψ we define $\xi\colon A\flat X \to X$ on the generators by $\xi(axa^{-1}) := \psi(a, x)$. It is easy to show that these are actions in the corresponding sense (the fact that ξ is a morphism and the axioms for it to be an internal action amounts to requiring the group action axioms for the function ψ) and that the correspondence just depicted is a bijection between internal and group actions.

Remark 1.2.6. Every time we have an action $\xi\colon A\flat X \to X$ we can construct the corresponding *action core* ${}^{\diamond}\xi\colon A \diamond X \to X$ as the composition of ξ and $i_{A,X}\colon A \diamond X \to A\flat X$ (this is firstly defined in [51] and then studied also in [49]). Furthermore we obtain that the image of this action core is given by $Im({}^{\diamond}\xi) = [A, X]^{\mathcal{H}}_{X \rtimes_{\xi} A}$ (see Proposition 4.2.9 for more details).

Remark 1.2.7 ([11]). Whenever the base category $\mathbb{A}$ is semi-abelian we have an equivalence of categories $\mathbf{Pt}(\mathbb{A}) \simeq \mathbf{Act}(\mathbb{A})$. This sends a split epimorphism

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

into the action (B, K_p, ξ) where ξ is the unique morphism making the following diagram commute

$$\begin{tikzcd} B\flat K_p \arrow[r, "k_{B,K_p}"] \arrow[d, dotted, "\xi"'] & B + K_p \arrow[r, "\binom{1_B}{0}"] \arrow[d, "\binom{s}{k_p}"] & B \arrow[d, equal] \\ K_p \arrow[r, tail, "k_p"'] & A \arrow[r, "p"'] & B \end{tikzcd}$$

that is the map induced by the universal properties of kernels.

Viceversa, it sends an action (A, X, ξ) into the split epimorphism

$$\begin{tikzcd} X \rtimes_\xi A \arrow[r, shift left, "\pi_\xi"] & A \arrow[l, shift left, "i_\xi"] \end{tikzcd}$$

where $X \rtimes_\xi A$ is defined as the coequaliser

$$\begin{tikzcd} A\flat X \arrow[r, shift left, "i_X \circ \xi"] \arrow[r, shift right, "k_{A,X}"'] & A + X \arrow[r, "\sigma_\xi"] & X \rtimes_\xi A, \end{tikzcd}$$

the map π_ξ is the unique map such that

$$\begin{tikzcd} A + X \arrow[r, "\sigma_\xi"] \arrow[dr, "\pi_{A,X}"'] & X \rtimes_\xi A \arrow[d, dotted, "\pi_\xi"] \\ & A \end{tikzcd}$$

commutes, and finally $i_\xi = \sigma_\xi \circ i_A$. We will also denote $X \rtimes_\xi A$ just with $X \rtimes A$ if there is no risk of confusion regarding the action involved, and we will sometimes use the notation $\pi_\xi = \left\langle\begin{smallmatrix}1_A\\0\end{smallmatrix}\right|$ since $\pi_{A,X} = \left(\begin{smallmatrix}1_A\\0\end{smallmatrix}\right)$. Notice that the map

$$k := \sigma_\xi \circ i_X : X \to X \rtimes_\xi A$$

is the kernel of π_ξ: it's trivial to see that $\pi_\xi \circ k = 0$ whereas for the universal property we have some work to do. Consider the following diagram

$$\begin{tikzcd} A\flat X \arrow[r, tail, "k_{A,X}"] \arrow[d, two heads, "\xi"'] & A + X \arrow[r, shift left, "\binom{1}{0}"] \arrow[d, two heads, "\sigma_\xi"] & A \arrow[l, shift left, "i_A"] \arrow[d, equal] \\ X \arrow[r, "k"'] & X \rtimes_\xi A \arrow[r, shift left, "\pi_\xi"] & A \arrow[l, shift left, "i_\xi"] \end{tikzcd}$$

Notice that the square on the left commutes by the definition of σ_ξ. The fact that k is a monomorphism is shown in Proposition 31 in [66]. Thus it is a normal monomorphism by Lemma 1.1.20 and hence to show that $k = k_{\pi_\xi}$ it suffices to prove that π_ξ is the cokernel of k. This is done as follows, through the diagram

$$\begin{tikzcd} X \arrow[r, "k"] & X \rtimes_\xi A \arrow[r, "\pi_\xi"] \arrow[dr, "z"'] & A \arrow[d, dotted, "z \circ i_\xi"] \\ & & Z \end{tikzcd}$$

with $z \circ k = 0$. We want to show that $z \circ i_\xi \circ \pi_\xi = z$ (the uniqueness comes from π_ξ being an epimorphism), but since σ_ξ is an epimorphism, this amounts to showing that $z \circ i_\xi \circ \pi_\xi \circ \sigma_\xi = z \circ \sigma_\xi$. We can use the fact that A and X cover $A + X$ to decompose this condition into the system

$$\begin{cases} z \circ i_\xi \circ \pi_\xi \circ \sigma_\xi \circ i_A = z \circ \sigma_\xi \circ i_A \\ z \circ i_\xi \circ \pi_\xi \circ \sigma_\xi \circ i_X = z \circ \sigma_\xi \circ i_X \end{cases}$$

which is trivially satisfied.

Remark 1.2.8. Notice that by the definition of the semidirect product, it is easy to show that the diagram

$$\begin{tikzcd} A \flat X \arrow[r, tail, "k_{A,X}"] \arrow[d, "\xi"'] & A + X \arrow[d, "\sigma_\xi"] \\ X \arrow[r, "k_{\pi_\xi}"'] & X \rtimes_\xi A \end{tikzcd}$$

is a pushout. Thanks to this commutativity we can show that also the square

$$\begin{tikzcd} A \flat X \arrow[r, "\xi"] \arrow[d, "i_\xi \flat k_{\pi_\xi}"'] & X \arrow[d, "k_{\pi_\xi}"] \\ (X \rtimes_\xi A) \flat (X \rtimes_\xi A) \arrow[r, "\chi_{(X \rtimes_\xi A)}"'] & X \rtimes_\xi A \end{tikzcd}$$

commutes, which means that "computing an action" is the same as "computing the conjugation in the induced semidirect product".

Example 1.2.9. Consider the trivial action (A, X, τ) given by

$$\tau = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ k_{A,X} \colon A \flat X \to X.$$

Then we have that

$$(X \rtimes_\tau A, \sigma_\tau) \cong Coeq(i_X \circ (\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ k_{A,X}), k_{A,X}).$$

Both $\binom{1}{0}$ and $\binom{0}{1}$ coequalise the two maps, so the first guess (also reasoning on the trivial action in **Grp**) would be that

$$Coeq(i_X \circ \begin{pmatrix}0\\1\end{pmatrix} \circ k_{A,X}, k_{A,X}) \cong (A \times X, \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rangle).$$

In order to prove this, since it is quite difficult to show directly the universal property, we use once again the equivalence $\mathbf{Pt}(\mathbb{A}) \simeq \mathbf{Act}(\mathbb{A})$. In particular we claim that the desired point is given by

$$\begin{tikzcd} A \times X \arrow[r, shift left, "\pi_A"] & A \arrow[l, shift left, "\langle 1,0 \rangle"] \end{tikzcd}$$

and hence it suffices to show that $\tau = \binom{0}{1} \circ k_{A,X}$ makes the following diagram commute

$$\begin{tikzcd} A\flat X \arrow[r, "k_{A,X}"] \arrow[d, "\tau"'] & A + X \arrow[d, "\binom{\langle 1,0\rangle}{\langle 0,1\rangle}"] \\ X \arrow[r, "\langle 0,1 \rangle"'] & A \times X \end{tikzcd}$$

and this is done by direct and easy calculations.

Example 1.2.10. Consider the conjugation action (A, A, χ_A) given by

$$\chi_A = \begin{pmatrix}1\\1\end{pmatrix} \circ k_{A,A} \colon A\flat A \to A.$$

Then we have that

$$(A \rtimes_{\chi_A} A, \sigma_{\chi_A}) \cong Coeq(i_2 \circ (\begin{pmatrix}1\\1\end{pmatrix} \circ k_{A,A}), k_{A,A}).$$

Both $\binom{1}{0}$ and $\binom{1}{1}$ coequalise the two maps, so the first guess (again thinking about the case of **Grp**) would be that

$$Coeq(i_2 \circ \begin{pmatrix}1\\1\end{pmatrix} \circ k_{A,A}, k_{A,A}) \cong (A \times A, \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \rangle).$$

In order to prove this, we use the same strategy of the previous example.

In particular we claim that the desired point is given by

$$\begin{tikzcd} A \times A \arrow[r, shift left, "\pi_1"] & A \arrow[l, shift left, "\langle 1,1 \rangle"] \end{tikzcd}$$

and hence it suffices to show that $\chi_A = \binom{1}{1} \circ k_{A,A}$ makes the following diagram commute

$$\begin{tikzcd} A\flat A \arrow[r, "k_{A,A}"] \arrow[d, "\chi_A"'] & A + A \arrow[d, "\binom{\langle 1,1\rangle}{\langle 0,1\rangle}"] \\ A \arrow[r, "\langle 0,1 \rangle"'] & A \times A \end{tikzcd}$$

and this is done by direct and easy calculations.

Remark 1.2.11. Notice that if $(B, X, \xi \colon B\flat X \to X)$ is an action and $f \colon A \to B$ is any map, then also $(A, X, \xi \circ (f\flat 1_X) \colon A\flat X \to X)$ is an action. Indeed we can obtain the commutativity of the diagrams required by the action axioms by using the naturality of η and μ and the axioms for ξ to be an action:

$$
\begin{tikzcd}
X \arrow[r, "\eta^A_X"] \arrow[d, equal] & A\flat X \arrow[d, "f\flat 1_X"] \\
X \arrow[r, "\eta^A_X"] \arrow[dr, equal] & B\flat X \arrow[d, "\xi"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
A\flat(A\flat X) \arrow[rr, "\mu^A_X"] \arrow[d, "1_A\flat(f\flat 1_X)"'] \arrow[dr, "f\flat(f\flat 1_X)"] & & A\flat X \arrow[d, "f\flat 1_X"] \\
A\flat(B\flat X) \arrow[r, "f\flat 1_{B\flat X}"] \arrow[d, "1_A\flat\xi"'] \arrow[dr, "f\flat\xi"] & B\flat(B\flat X) \arrow[r, "\mu^B_X"] \arrow[d, "1_B\flat\xi"] & B\flat X \arrow[d, "\xi"] \\
A\flat X \arrow[r, "f\flat 1_X"'] & B\flat X \arrow[r, "\xi"'] & X
\end{tikzcd}
$$

The action $\xi \circ (f\flat 1_X)$ is often called *pullback action* of ξ along f and the reason is the following. Consider the diagram

$$
\begin{tikzcd}
X \arrow[r, tail, "k_{\pi_{\xi'}}"] \arrow[d, equal] & X \rtimes_{\xi'} A \arrow[r, shift left, "\pi_{\xi'}"] \arrow[d, "1_X \rtimes f"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[l, shift left, "\sigma_{\xi'}"] \arrow[d, "f"] \\
X \arrow[r, tail, "k_{\pi_\xi}"'] & X \rtimes_\xi B \arrow[r, shift left, "\pi_\xi"] & B \arrow[l, shift left, "\sigma_\xi"]
\end{tikzcd}
$$

where the bottom row is the point associated to ξ, whereas the first row is obtained taking the pullback of π_ξ along f. The action ξ' is called pullback action and it is easy to see that this coincides with $\xi \circ (f\flat 1_X)$, indeed we have the commutative diagram

$$
\begin{tikzcd}
A\flat X \arrow[rr, tail, "k_{A,X}"] \arrow[dr, "f\flat 1"] \arrow[dd, "\xi'"'] & & A + X \arrow[rr, two heads, "\binom{1}{0}"] \arrow[dr, "f+1"] \arrow[dd, "\binom{\sigma_{\xi'}}{k_{\pi_{\xi'}}}"'] & & A \arrow[dr, "f"] \arrow[dd, equal] & \\
 & B\flat X \arrow[rr, tail, "k_{B,X}"] \arrow[dd, "\xi"] & & B + X \arrow[rr, two heads, "\binom{1}{0}"] \arrow[dd, "\binom{\sigma_\xi}{k_{\pi_\xi}}"'] & & B \arrow[dd, equal] \\
X \arrow[rr, tail, "k_{\pi_{\xi'}}"'] \arrow[dr, equal] & & X \rtimes_{\xi'} A \arrow[rr, two heads, "\pi_{\xi'}"'] \arrow[dr, "1\rtimes f"'] & & A \arrow[dr, "f"] & \\
 & X \arrow[rr, tail, "k_{\pi_\xi}"'] & & X \rtimes_\xi B \arrow[rr, two heads, "\pi_\xi"'] & & B
\end{tikzcd}
$$

1.2.2 The categories $\mathbf{Pt}_L(\mathbb{A})$ and $\mathbf{Act}_L(\mathbb{A})$

In the following we will also need the categories of points and actions over a fixed object L and some basic results about them: we think that all of them are well-known results, but we recall some of the proofs for completeness.

Definition 1.2.12. A *point over* L (or L-*point*) in $\mathbb{A}$ for a fixed $L \in \mathbb{A}$ is a split epimorphism with codomain L and with a chosen splitting, that is

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \end{tikzcd}$$

with $p \circ s = 1_L$. A morphism of points over L is given by a vertical map such that the two squares

$$\begin{tikzcd} A \arrow[r, shift left, "p"] \arrow[d, "\alpha"'] & L \arrow[l, shift left, "s"] \arrow[d, equal] \\ A' \arrow[r, shift left, "p'"] & L \arrow[l, shift left, "s'"] \end{tikzcd}$$

commute. The category of points over L is denoted with $\mathbf{Pt}_L(\mathbb{A})$.

Definition 1.2.13. An L-*action* in $\mathbb{A}$ with fixed $L \in \mathbb{A}$ is an action (L, X, ξ) in which the acting object is L. A morphism of L-actions from (L, X, ξ) to (L, X', ξ') is given by a map of actions in which the first component is the identity over L, that is an $f\colon X \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} L\flat X \arrow[r, "1_L\flat f"] \arrow[d, "\xi"'] & L\flat X' \arrow[d, "\xi'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

The category of L-actions and morphisms between them is denoted by $\mathbf{Act}_L(\mathbb{A})$.

Remark 1.2.14. As for the general case, when $\mathbb{A}$ is a semi-abelian category we have an equivalence of categories $\mathbf{Pt}_L(\mathbb{A}) \simeq \mathbf{Act}_L(\mathbb{A})$ which is simply the restriction of the previous equivalence to these subcategories. Therefore also in this case we will often switch from one formalism to the other if there is no risk of confusion.

Definition 1.2.15. Given two L-actions $\xi\colon L\flat X \to X$ and $\xi'\colon L\flat X' \to X'$ and a morphism $f\colon X \to X'$ in $\mathbb{A}$, we say that f is *equivariant with respect to* ξ *and* ξ' if it forms with the identity over L a map of L-actions. This corresponds to $(1, f)$ being a morphism between the corresponding L-points, as can be seen through the diagram

$$\begin{tikzcd}
L\flat X \arrow[rr, tail, "k_{L,X}"] \arrow[dr, "1\flat f"] \arrow[dd, "\xi"'] & & L+X \arrow[rr, shift left, two heads, "\binom{1}{0}"] \arrow[dr, "1+f"] \arrow[dd, "\binom{\sigma_\xi}{k_{\pi_\xi}}"' near start] & & L \arrow[ll, shift left] \arrow[dr, equal] \arrow[dd, equal] & \\
& L\flat X' \arrow[rr, tail, "k_{L,X'}"' near start] \arrow[dd, "\xi'" near start] & & L+X' \arrow[rr, shift left, two heads, "\binom{1}{0}"] \arrow[dd, "\binom{\sigma_{\xi'}}{k_{\pi_{\xi'}}}"' near start] & & L \arrow[ll, shift left] \arrow[dd, equal] \\
X \arrow[rr, tail, "k_{\pi_\xi}" near start] \arrow[dr, "f"'] & & X \rtimes_\xi L \arrow[rr, shift left, two heads, "\pi_\xi" near end] \arrow[dr, "f \rtimes 1"] & & L \arrow[ll, shift left] \arrow[dr, equal] & \\
& X' \arrow[rr, tail, "k_{\pi_{\xi'}}"'] & & X' \rtimes_{\xi'} L \arrow[rr, shift left, two heads, "\pi_{\xi'}"] & & L \arrow[ll, shift left]
\end{tikzcd}$$

Lemma 1.2.16. *A map in the category* $\mathbf{Pt}_L(\mathbb{A})$ *is a regular epimorphism if and only if the morphism between the domains is a regular epimorphism in* $\mathbb{A}$*.*

Proof. First of all we are going to show how coequalisers are computed in $\mathbf{Pt}_L(\mathbb{A})$.

Consider two parallel morphism in $\mathbf{Pt}_L(\mathbb{A})$ and the point over L induced by the coequaliser of the morphisms between the domains

$$\begin{tikzcd}
A \arrow[r, shift left, "p"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & L \arrow[l, shift left, "s"] \arrow[d, equal] \\
A' \arrow[r, shift left, "p'"] \arrow[d, two heads, "c_{f,g}"'] & L \arrow[l, shift left, "s'"] \arrow[d, equal] \\
C_{f,g} \arrow[r, shift left, "\overline{p}"] & L \arrow[l, shift left, "\overline{s}"]
\end{tikzcd}$$

Here the map $\overline{p}$ is induced by the universal property of $C_{f,g}$ since $p' \circ f = p = p' \circ g$, whereas the map $\overline{s}$ is defined as the composition $c_{f,g} \circ s'$: it is trivial to see that $\overline{p} \circ \overline{s} = \overline{p} \circ c_{f,g} \circ s' = p' \circ s' = 1_L$. Now suppose that there is another point coequalising the two morphisms of points

$$\begin{tikzcd}
A \arrow[r, shift left, "p"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & L \arrow[l, shift left, "s"] \arrow[d, equal] \\
A' \arrow[r, shift left, "p'"] \arrow[d, "h"'] & L \arrow[l, shift left, "s'"] \arrow[d, equal] \\
Z \arrow[r, shift left, "\tilde{p}"] & L \arrow[l, shift left, "\tilde{s}"]
\end{tikzcd}$$

Obviously we find a map $\phi\colon C_{f,g} \to Z$ such that $h = \phi \circ c_{f,g}$ since h coequalises f and g. The only thing that remains to be proved is that this ϕ is a morphism of points over L, which corresponds to the commutativity of the left-pointing and right-pointing squares in

$$\begin{tikzcd} C_{f,g} \arrow[r, shift left, "\overline{p}"] \arrow[d, "\phi"'] & L \arrow[l, shift left, "\overline{s}"] \arrow[d, equal] \\ Z \arrow[r, shift left, "\tilde{p}"] & L \arrow[l, shift left, "\tilde{s}"] \end{tikzcd}$$

But we have that

$$\tilde{p} \circ \phi \circ c_{f,g} = \tilde{p} \circ h = p' = \overline{p} \circ c_{f,g}$$

which means $\tilde{p} \circ \phi = \overline{p}$ since $c_{f,g}$ is an epimorphism, and finally $\tilde{s} = h \circ s' = \phi \circ c_{f,g} \circ s' = \phi \circ \overline{s}$.

This means that in order to take the coequaliser of two morphisms of L-points it suffices to compute the coequaliser of the morphisms between the domains and then take the induced structure. In other words we just showed that if a morphism is a regular epimorphism in $\mathbf{Pt}_L(\mathbb{A})$, then the morphism between the domain is a regular epimorphism in $\mathbb{A}$.

Let us show the converse: suppose that we have a morphism of points in which the first component is the coequaliser of two maps in $\mathbb{A}$

$$\begin{tikzcd} A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \\ A' \arrow[r, shift left, "p'"] \arrow[d, two heads, "c_{f,g}"'] & L \arrow[l, shift left, "s'"] \arrow[d, equal] \\ C_{f,g} \arrow[r, shift left, "\overline{p}"] & L \arrow[l, shift left, "\overline{s}"] \end{tikzcd}$$

We want to construct two morphisms of points such that the lower square is their coequaliser. The first step is to observe that $c_{f,g}$ is also the coequaliser of its kernel pair

$$\begin{tikzcd} Kp(c_{f,g}) \arrow[r, shift left, dotted, "p"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & L \arrow[l, shift left, dotted, "s"] \arrow[d, equal] \\ A' \arrow[r, shift left, "p'"] \arrow[d, two heads, "c_{f,g}"'] & L \arrow[l, shift left, "s'"] \arrow[d, equal] \\ C_{f,g} \arrow[r, shift left, "\overline{p}"] & L \arrow[l, shift left, "\overline{s}"] \end{tikzcd}$$

then we construct the structure of point over L as follows: first we define $p := p' \circ r_0 = p' \circ r_1$ and then we define s using the universal property of the kernel pair (which is a pullback)

$$\begin{tikzcd}
L \arrow[drr, bend left, "s'"] \arrow[dr, dotted, "s"] \arrow[ddr, bend right, "s'"'] & & \\
& Kp(c_{f,g}) \arrow[r, "r_0"] \arrow[d, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "c_{f,g}"] \\
& A' \arrow[r, "c_{f,g}"'] & C_{f,g}
\end{tikzcd}$$

obtaining $p \circ s = p' \circ r_0 \circ s = p' \circ s' = 1_L$. □

Lemma 1.2.17. *A square in the category* $\mathbf{Pt}_L(\mathbb{A})$ *is a pushout if and only if the square between the domains is a pushout in* $\mathbb{A}$*. The same holds for pullbacks. This means that pushouts and pullbacks can be computed in the base category using only the domains and then inducing the additional structure.*

Proof. First of all we want to show that the diagram

$$\begin{tikzcd}
A \arrow[rr, shift left, two heads, "p_A"] \arrow[dr, "g"] \arrow[dd, "f"'] & & L \arrow[ll, shift left, "s_A"] \arrow[dr, equal] \arrow[dd, equal] & \\
& C \arrow[rr, shift left, two heads, "p_C"] \arrow[dd, "g_*(f)" near start] & & L \arrow[ll, shift left, "s_C"] \arrow[dd, equal] \\
B \arrow[rr, shift left, two heads, "p_B"] \arrow[dr, two heads, "f_*(g)"'] & & L \arrow[ll, shift left, "s_B"] \arrow[dr, equal] & \\
& B +_A C \arrow[rr, shift left, two heads, "\overline{p}"] & & L \arrow[ll, shift left, "\overline{s}"]
\end{tikzcd} \tag{1.9}$$

is a pushout of L-points. So consider a point

$$\begin{tikzcd}
Z \arrow[r, shift left, "\tilde{p}"] & L \arrow[l, shift left, "\tilde{s}"]
\end{tikzcd}$$

with morphism in $\mathbf{Pt}_L(\mathbb{A})$ given by $\beta\colon B \to Z$ and $\gamma\colon C \to Z$ such that the cube

$$\begin{tikzcd}
A \arrow[rr, shift left, two heads, "p_A"] \arrow[dr, "g"] \arrow[dd, "f"'] & & L \arrow[ll, shift left, "s_A"] \arrow[dr, equal] \arrow[dd, equal] & \\
& C \arrow[rr, shift left, two heads, "p_C"] \arrow[dd, "\gamma"' near start] & & L \arrow[ll, shift left, "s_C"] \arrow[dd, equal] \\
B \arrow[rr, shift left, two heads, "p_B" near end] \arrow[dr, "\beta"'] & & L \arrow[ll, shift left, "s_B" near start] \arrow[dr, equal] & \\
& Z \arrow[rr, shift left, two heads, "\tilde{p}"] & & L \arrow[ll, shift left, "\tilde{s}"]
\end{tikzcd}$$

commutes. By the universal property of $B +_A C$ we have that there exists a unique arrow $\phi\colon B +_A C \to Z$ such that $\phi \circ f_*(g) = \beta$ and $\phi \circ g_*(f) = \gamma$. It remains to show that ϕ is a morphism of L-points, that is the commutativity of the squares

$$\begin{tikzcd}
B +_A C \arrow[r, shift left, "\overline{p}"] \arrow[d, "\phi"'] & L \arrow[l, shift left, "\overline{s}"] \arrow[d, equal] \\
Z \arrow[r, shift left, "\tilde{p}"] & L \arrow[l, shift left, "\tilde{s}"]
\end{tikzcd}$$

As far as it concerns the left-pointing square, we have

$$\tilde{s} = \beta \circ s_B = \phi \circ f_*(g) \circ s_B = \phi \circ \overline{s}$$

whereas for the right-pointing one it suffices to use the universal property of the pushout as depicted in

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "g_*(f)"] \arrow[ddr, bend left, "p_C"] & \\
B \arrow[r, "f_*(g)"'] \arrow[drr, bend right, "p_B"'] & B +_A C \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted] & \\
& & L
\end{tikzcd}$$

and the fact that both $\overline{p}$ and $\tilde{p} \circ \phi$ satisfy the property of the dotted map and hence they have to coincide. This means that the diagram (1.9) is a pushout in $\mathbf{Pt}_L(\mathbb{A})$.

In order to show that the same holds for pullbacks it suffices to show that the diagram

$$
\begin{tikzcd}
A \times_C B \arrow[rr, shift left, two heads, "\overline{p}"] \arrow[dr, "f_*(g)"] \arrow[dd, "g_*(f)"'] & & L \arrow[ll, shift left, "\overline{s}"] \arrow[dr, equal] \arrow[dd, equal] & \\
& A \arrow[rr, shift left, two heads, "p_A"] \arrow[dd, "f"'] & & L \arrow[ll, shift left, "s_A"] \arrow[dd, equal] \\
B \arrow[rr, shift left, two heads, "p_B"] \arrow[dr, two heads, "f_*(g)"'] & & L \arrow[ll, shift left, "s_B"] \arrow[dr, equal] & \\
& C \arrow[rr, shift left, two heads, "p_C"] & & L \arrow[ll, shift left, "s_C"]
\end{tikzcd}
\tag{1.10}
$$

is a pullback in $\mathbf{Pt}_L(\mathbb{A})$: in order to see this, one can simply repeat the same reasoning that we used for pushouts. □

As a consequence we have a simple way to compute kernels in $\mathbf{Pt}_L(\mathbb{A})$.

Corollary 1.2.18. *Consider a morphism of L-points*

$$
\begin{tikzcd}
A \arrow[r, shift left, two heads, "p_A"] \arrow[d, "f"'] & L \arrow[l, shift left, "s_A"] \arrow[d, equal] \\
B \arrow[r, shift left, two heads, "p_B"] & L \arrow[l, shift left, "s_B"]
\end{tikzcd}
$$

Then its kernel is the L-point induced by the pullback diagram

$$
\begin{tikzcd}
A \times_B L \arrow[rr, shift left, dotted, two heads, "p_{A\times_B L}"] \arrow[dr] \arrow[dd] & & L \arrow[ll, shift left, dotted, "s_{A\times_B L}"] \arrow[dr, equal] \arrow[dd, equal] & \\
& A \arrow[rr, shift left, two heads, "p_A"] \arrow[dd, "f"'] & & L \arrow[ll, shift left, "s_A"] \arrow[dd, equal] \\
L \arrow[rr, shift left, "1_L"] \arrow[dr, "s_B"'] & & L \arrow[ll, shift left, "1_L"] \arrow[dr, equal] & \\
& B \arrow[rr, shift left, two heads, "p_B"] & & L \arrow[ll, shift left, "s_B"]
\end{tikzcd}
$$

in $\mathbf{Pt}_L(\mathbb{A})$. □

1.3 Internal graphs, groupoids and categories

Each notion mentioned in the title of this section is well-known. We will recall some basic definitions and results, explicitly stating the properties that the category $\mathbb{A}$ has to satisfy: for the sake of simplicity one can always consider $\mathbb{A}$ to be semi-abelian with (SH), since this is the most stringent requirement within this section.

1.3.1 The category $\mathbf{RG}(\mathbb{A})$

Definition 1.3.1. A *reflexive graph* (C_1, C_0, d, c, e) in $\mathbb{A}$ is given by a diagram

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

such that

$$\begin{tikzcd} & C_0 \arrow[dl, equal] \arrow[d, "e"] \arrow[dr, equal] & \\ C_0 & C_1 \arrow[l, "d"] \arrow[r, "c"'] & C_0 \end{tikzcd} \tag{1.11}$$

commutes.

A morphism of reflexive graphs from (C_1, C_0, d, c, e) to (C_1', C_0', d', c', e') is a pair $(f_1 \colon C_1 \to C_1', f_0 \colon C_0 \to C_0')$ such that

$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[d, "f_0"'] & C_1 \arrow[d, "f_1"] \\ C_0' \arrow[r, "e'"'] & C_1' \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "d"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"] \\ C_1' \arrow[r, "d'"'] & C_0' \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "c"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"] \\ C_1' \arrow[r, "c'"'] & C_0' \end{tikzcd}$$

commute. This completes the definition of the category $\mathbf{RG}(\mathbb{A})$.

Lemma 1.3.2 (Proposition 3.9 in [39])**.** *Let $\mathbb{A}$ be a pointed protomodular category and consider a reflexive graph with its normalisation*

$$\begin{tikzcd} K_d \arrow[r, tail, "k_d"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

Then $C_{c \circ k_d} \cong C_{(d,c)}$. □

1.3.2 The category $\mathbf{RMG}(\mathbb{A})$

Definition 1.3.3. A *reflexive multiplicative graph* (C_1, C_0, d, c, e, m) in $\mathbb{A}$ is given by a diagram

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

such that (C_1, C_0, d, c, e) is a reflexive graph and such that the multiplication m makes the following diagram commute

$$\begin{tikzcd} C_1 \arrow[r, "\langle ec, 1_{C_1}\rangle"] \arrow[dr, equal] & C_1 \times_{C_0} C_1 \arrow[d, "m"] & C_1 \arrow[l, "\langle 1_{C_1}, ed\rangle"'] \arrow[dl, equal] \\ & C_1 & \end{tikzcd} \tag{1.12}$$

A morphism of reflexive multiplicative graph from (C_1, C_0, d, c, e, m) to $(C_1', C_0', d', c', e', m')$ is morphism (f_1, f_0) of the underlying reflexive graphs such that

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] \arrow[d, "f_1 \times_{C_0} f_1"'] & C_1 \arrow[d, "f_1"] \\ C_1' \times_{C_0'} C_1' \arrow[r, "m'"'] & C_1' \end{tikzcd}$$

This completes the definition of the category $\mathbf{RMG}(\mathbb{A})$.

1.3.3 The category $\mathbf{Cat}(\mathbb{A})$ of internal categories

Definition 1.3.4. Let $\mathbb{A}$ be a category with pullbacks. An *internal category* $C = (C_1, C_0, d, c, e, m)$ in $\mathbb{A}$ is a reflexive multiplicative graph in $\mathbb{A}$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

such that the following additional diagrams commute:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] \arrow[d, "\pi_2"'] & C_1 \arrow[d, "d"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "1_{C_1} \times_{C_0} m"] \arrow[d, "m \times_{C_0} 1_{C_1}"'] & C_1 \times_{C_0} C_1 \arrow[d, "m"] \\ C_1 \times_{C_0} C_1 \arrow[r, "m"'] & C_1 \end{tikzcd}$$

Remark 1.3.5. Sometimes the condition (1.12) is stated in an equivalent way by asking that the following diagram commutes

$$\begin{tikzcd} C_0 \times_{C_0} C_1 \arrow[r, "e \times_{C_0} 1_{C_1}"] \arrow[dr, "\pi_2'"'] & C_1 \times_{C_0} C_1 \arrow[d, "m"] & C_1 \times_{C_0} C_0 \arrow[l, "1_{C_1} \times_{C_0} e"'] \arrow[dl, "\pi_1'"] \\ & C_1 & \end{tikzcd}$$

In order to see that the two are equivalent one has to use the diagram

$$\begin{tikzcd}
C_1 \arrow[dr, dotted, "\langle 1,d\rangle"] \arrow[drrr, bend left, "d"] \arrow[ddd, equal, bend right] & & & \\
& C_1 \times_{C_0} C_0 \arrow[rr, "\pi_2'"] \arrow[dd, "\pi_1'"'] \arrow[dr, dotted, "1\times_{C_0} e"] & & C_0 \arrow[d, "e"] \\
& & C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] \\
& C_1 \arrow[r, equal] & C_1 \arrow[r, "d"'] & C_0
\end{tikzcd}$$

to deduce that $\langle 1, ed\rangle = (1 \times_{C_0} e)\langle 1, d\rangle$ and from this equality it is easy to show that

$$m(1 \times_{C_0} e) = \pi_1' \quad \Longleftrightarrow \quad m\langle 1, ed\rangle = 1_{C_1}.$$

Inverting the roles of c and d leads to the equivalence of the other two equalities.

Definition 1.3.6. Let C and D be internal categories in $\mathbb{A}$, then an *internal functor* $f\colon C \to D$ is given by a morphism (f_1, f_0) of the underlying reflexive multiplicative graphs.

This completes the definition of the category $Cat(\mathbb{A})$ of internal categories and functors in $\mathbb{A}$.

1.3.4 The category $\mathbf{Grpd}(\mathbb{A})$ of internal groupoids

Definition 1.3.7. A *groupoid* in $\mathbb{A}$ is an internal category endowed with an additional map

$$C_1 \xrightarrow{i} C_1$$

such that the following diagrams commute

$$\begin{tikzcd}
& C_1 \arrow[dl, "d"'] \arrow[d, "i"] \arrow[dr, "c"] & \\
C_0 & C_1 \arrow[l, "c"] \arrow[r, "d"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "\langle 1,i\rangle"] \arrow[d, "d"'] & C_1 \times_{C_0} C_1 \arrow[d, "m"] \\
C_0 \arrow[r, "e"'] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "\langle i,1\rangle"] \arrow[d, "c"'] & C_1 \times_{C_0} C_1 \arrow[d, "m"] \\
C_0 \arrow[r, "e"'] & C_1
\end{tikzcd}$$

Notice that being a groupoid is a property that an internal category may have of not: it is not an additional structure because i is uniquely determined whenever it exists.

Definition 1.3.8. The category $\mathbf{Grpd}(\mathbb{A})$ of internal groupoid in $\mathbb{A}$ is given by the full subcategory of $\mathbf{Cat}(\mathbb{A})$ with groupoids as objects.

1.3.5 Comparisons between these categories

Let us start by noticing that there is a chain of forgetful functors given by

$$\mathbf{Grpd}(\mathbb{A}) \xhookrightarrow{U} \mathbf{Cat}(\mathbb{A}) \xhookrightarrow{V} \mathbf{RMG}(\mathbb{A}) \xhookrightarrow{W} \mathbf{RG}(\mathbb{A})$$

In general both U and V are full and faithful, whereas W is only faithful. If the base category is Mal'tsev we have the following

Theorem 1.3.9 ([19]). *Let $\mathbb{A}$ be a Mal'tsev category. Then*

1) W is full;

2) U and V are isomorphisms;

3) Any internal reflexive graph admits at most one structure of reflexive multiplicative graph.

Proof. See Proposition 2.1 and Theorem 2.2 in [19]. □

It is possible to define a functor $F\colon \mathbf{RG}(\mathbb{A}) \to \mathbf{RMG}(\mathbb{A})$, which is left adjoint to the inclusion W. In order to construct the functor F we need to use the following results.

Lemma 1.3.10 ([72]). *Let $\mathbb{A}$ be a semi-abelian category (it suffices Mal'cev and exact). Given a reflexive graph (C_1, C_0, d, c, e), it admits a (unique) internal groupoid structure if and only if $[Kp(d), Kp(c)]^{\mathcal{S}}_{C_1} = \Delta_{C_1}$ (that is iff $Kp(d)$ and $Kp(c)$ Smith-commute).*

Proof. See Proposition 1.8 and Corollary 1.9 in [72]. □

Now we restrict ourselves to categories in which the SH condition holds, and this gives us a better way to state the previous result.

Lemma 1.3.11 ([67]). *Let $\mathbb{A}$ be a semi-abelian category with SH. Given a reflexive graph (C_1, C_0, d, c, e), it admits a (unique) internal groupoid structure if and only if $[K_d, K_c]^{\mathcal{Q}}_{C_1} = 0$.*

Proof. We use SH to go from the description in terms of kernel pairs to the one in terms of kernels, that is to show that

$$[Kp(d), Kp(c)]^{\mathcal{S}}_{C_1} = \Delta_{C_1} \qquad \Longleftrightarrow \qquad [K_d, K_c]^{\mathcal{Q}}_{C_1} = 0. \qquad □$$

Construction 1.3.12. Consider $C \in \mathbf{RG}(\mathbb{A})$ given by

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

We want to construct $F(C) \in \mathbf{RMG}(\mathbb{A}) \simeq \mathbf{Grpd}(\mathbb{A})$. Let us denote it as (C_1', C_0', d', c', e') and let us define it as follows

- $C'_0 := C_0$;
- $C'_1 := Q = \frac{C_1}{[K_d, K_c]^Q_{C_1}}$;
- the map d' is constructed using the universal property of the cokernel Q as follows

$$\begin{tikzcd}
{[K_d, K_c]^Q_{C_1}} \arrow[r, tail, "i"] \arrow[d, tail, "j_d"'] & C_1 \arrow[r, two heads, "q"] \arrow[dr, "d"'] & Q \arrow[d, dotted, "d'"] \\
K_d \arrow[ur, tail, "k_d"'] & & C_0
\end{tikzcd}$$

- similarly the map c' is constructed using the universal property of the cokernel Q as follows

$$\begin{tikzcd}
{[K_d, K_c]^Q_{C_1}} \arrow[r, tail, "i"] \arrow[d, tail, "j_c"'] & C_1 \arrow[r, two heads, "q"] \arrow[dr, "c"'] & Q \arrow[d, dotted, "c'"] \\
K_c \arrow[ur, tail, "k_c"'] & & C_0
\end{tikzcd}$$

- finally the map e' is defined as $e' = qe$.

It is trivial to verify that (1.11) is satisfied and therefore this is again a reflexive graph. In order to show that there exists a multiplicative structure on $F(C)$ it suffices to use Lemma 1.3.11. But from the diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & {[K_d, K_c]^Q_{C_1}} \arrow[r, tail] \arrow[d, equal] & K_d \arrow[r, two heads, "\overline{q}"] \arrow[d, tail] & K_{d'} \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & {[K_d, K_c]^Q_{C_1}} \arrow[r, tail, "i"] \arrow[d, two heads] & C_1 \arrow[r, two heads, "q"] \arrow[d, two heads, "d"'] & C'_1 \arrow[r] \arrow[d, two heads, "d'"] & 0 \\
0 \arrow[r] & 0 \arrow[r, tail] \arrow[d] & C_0 \arrow[r, equal] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

(notice that $\overline{q}$ is a regular epimorphism thanks to Lemma 1.1.18) and from the similar one involving c instead of d we can deduce that

$$K_{d'} \cong \frac{K_d}{[K_d, K_c]^Q_{C_1}} \qquad\qquad K_{c'} \cong \frac{K_c}{[K_d, K_c]^Q_{C_1}}$$

Therefore we have

$$[K_{d'}, K_{c'}]^{\mathcal{Q}}_{C_1'} \cong \left[\frac{K_d}{[K_d, K_c]^{\mathcal{Q}}_{C_1}}, \frac{K_c}{[K_d, K_c]^{\mathcal{Q}}_{C_1}} \right]^{\mathcal{Q}}_{\frac{C_1}{[K_d, K_c]^{\mathcal{Q}}_{C_1}}}$$

and this is 0 thanks to Proposition 1.1.55. Hence $F(C) \in \mathbf{RMG}(\mathbb{A})$.

Now consider a morphism of reflexive graphs

$$(f_1, f_0)\colon (B_1, B_0, d_B, c_B, e_B) \to (C_1, C_0, d_C, c_C, e_C),$$

in order to define the morphism $F((f_1, f_0)) = (f_1', f_0')$ (which is automatically a morphisms of reflexive multiplicative graphs since W is full and faithful) we notice that f induces a morphism of coterminal pairs

$$\begin{tikzcd}
K_{d_B} \arrow[r, tail, "k_{d_B}"] \arrow[d, dotted] & B_1 \arrow[d, "f_1"] & K_{c_B} \arrow[l, tail, "k_{c_B}"'] \arrow[d, dotted] \\
K_{d_C} \arrow[r, tail, "k_{d_C}"'] & C_1 & K_{c_C} \arrow[l, tail, "k_{c_C}"]
\end{tikzcd}$$

which, thanks to Proposition 1.1.51, gives us a map

$$[K_{d_B}, K_{c_B}]^{\mathcal{Q}}_{B_1} \longrightarrow [K_{d_C}, K_{c_C}]^{\mathcal{Q}}_{C_1}.$$

This, in turn, by the universal property of cokernels gives us the dotted map

$$\begin{tikzcd}
{[K_{d_B}, K_{c_B}]^{\mathcal{Q}}_{B_1}} \arrow[r, tail, "i_B"] \arrow[d] & B_1 \arrow[r, two heads, "q_B"] \arrow[d, "f_1"] & B_1' \arrow[d, dotted, "f_1'"] \\
{[K_{d_C}, K_{c_C}]^{\mathcal{Q}}_{C_1}} \arrow[r, tail, "i_C"] & C_1 \arrow[r, two heads, "q_C"] & C_1'
\end{tikzcd}$$

Finally by putting $f_0' = f_0$ it can be easily shown that (f_1', f_0') is a morphism of reflexive graphs.

Proposition 1.3.13. *The functor $F\colon \mathbf{RG}(\mathbb{A}) \to \mathbf{RMG}(\mathbb{A})$ is left adjoint to the inclusion $W\colon \mathbf{RMG}(\mathbb{A}) \to \mathbf{RG}(\mathbb{A})$. This means that we have an adjunction $F \dashv W$ and consequently also an adjunction $(U^{-1} \circ V^{-1} \circ F) \dashv (W \circ V \circ U)$ between internal groupoids and reflexive graphs.*

Proof. First of all notice that if we take a reflexive multiplicative graph $C = (C_1, C_0, d, c, e, m) \in \mathbf{RMG}(\mathbb{A})$ and we take $W(C)$, since it obviously admits a reflexive multiplicative structure, we have that $[K_d, K_c]^{\mathcal{Q}}_{C_1} = 0$ and therefore $F(W(C)) \cong C$.
Let us define the unit of the adjunction and show it's universal property. Consider

$C = (C_1, C_0, d, c, e) \in \mathbf{RG}(\mathbb{A})$ and take $WF(C) = (C_1', C_0, d', c', e')$. The map $(q, 1_{C_0})$ makes the following triple square commute

$$\begin{tikzcd} C_1 \arrow[r, "q"] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] & C_1' \arrow[d, shift right=2, "c'"'] \arrow[d, shift left=2, "d'"] \\ C_0 \arrow[u, "e" description] \arrow[r, equal] & C_0 \arrow[u, "e'" description] \end{tikzcd}$$

and therefore it is a morphism of reflexive graphs. We denote it as

$$\eta_C \coloneqq (q, 1_{C_0}) \colon C \to WF(C)$$

and our aim is to prove that this is the unit of the adjunction. Notice that $\eta \colon 1_{\mathbf{RG}(\mathbb{A})} \to WF$ is a natural transformation: indeed by definition of $F(f)$ we have that the following naturality square in $\mathbf{RG}(\mathbb{A})$ commutes

$$\begin{tikzcd} X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & WF(X) \arrow[d, "WF(f)"] \\ Y \arrow[r, two heads, "\eta_Y"'] & WF(Y) \end{tikzcd}$$

Now consider a map $f \colon X \to W(Z)$, we want to show that there exists a unique $\hat{f} \colon F(X) \to Z$ such that

$$\begin{tikzcd} X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & WF(X) \arrow[dl, dotted, "W(\hat{f})"] \\ W(Z) & \end{tikzcd} \tag{1.13}$$

commutes, that is $W(\hat{f}) \circ \eta_X = f$.

This $\hat{f}$ is given by $W^{-1}(\eta_{W(Z)}^{-1} \circ WF(f))$: indeed $\eta_{W(Z)}$ is the isomorphism induced by quotienting $W(Z)$ for the subobject 0. From the naturality square

$$\begin{tikzcd} X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & WF(X) \arrow[d, "WF(f)"] \\ W(Z) \arrow[r, equal, "\sim", "\eta_{W(Z)}"'] & WFW(Z) \end{tikzcd}$$

we deduce the commutativity of (1.13). Finally this is the unique choice for $\hat{f}$ since W is full and faithful and η_X is an epimorphism. $\square$

1.4 Pre-crossed modules and crossed modules

In this section we recall the concepts of internal pre-crossed modules and internal crossed modules (first defined by Janelidze in [55] in the context of semi-abelian categories), which are respectively equivalent to reflexive graphs and to groupoids.

1.4.1 The category $\mathbf{PreXMod}(\mathbb{A})$

Definition 1.4.1 ([55, 64])**.** An *internal pre-crossed module* $(X \xrightarrow{\partial} A, \xi)$ is given by an internal action (A, X, ξ) with a morphism $\partial\colon X \to A$ such that the following diagram commutes

$$\begin{tikzcd} A\flat X \arrow[r, "\xi"] \arrow[d, "1_A\flat\partial"'] & X \arrow[d, "\partial"] \\ A\flat A \arrow[r, "\chi_A"'] & A \end{tikzcd} \tag{1.14}$$

Example 1.4.2. One of the easiest examples of internal pre-crossed module (actually an internal crossed module) is given by $(A \xrightarrow{1_A} A, \chi_A)$ for which the previous commutativity is trivial.

Definition 1.4.3. Consider two internal pre-crossed modules $(X \xrightarrow{\partial} A, \xi)$ and $(X' \xrightarrow{\partial'} A', \xi')$. A morphism of internal pre-crossed modules is given by a pair (g, α) with $\alpha\colon A \to A'$ and $g\colon X \to X'$ such that the following diagrams commute:

$$\begin{tikzcd} A\flat X \arrow[r, "\alpha\flat g"] \arrow[d, "\xi"'] & A'\flat X' \arrow[d, "\xi'"] \\ X \arrow[r, "g"'] & X' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\partial"'] & X' \arrow[d, "\partial'"] \\ A \arrow[r, "\alpha"'] & A' \end{tikzcd}$$

The first of the two diagrams says that the pair (f, g) is equivariant with respect to the actions ξ and ξ' (i.e. (α, g) is a morphism of actions).

Remark 1.4.4. In particular from this definition one can deduce that each pre-crossed module $(X \xrightarrow{\delta} A, \xi)$ gives rise to a morphism of internal pre-crossed modules

$$(X \xrightarrow{\partial} A, \xi) \xrightarrow{(\partial, 1_A)} (A \xrightarrow{1_A} A, \chi_A)$$

Remark 1.4.5. The first definition of internal pre-crossed module was given in [55] and it was stated in a slightly different way: an *internal pre-crossed module* $(X \xrightarrow{\partial} A, \xi)$ is given by an internal action (A, X, ξ) with a morphism $\partial\colon X \to A$ such that the following diagram commutes

$$\begin{tikzcd} A\flat X \arrow[r, "k_{A,X}"] \arrow[d, "\xi"'] & A + X \arrow[d, "\binom{1_A}{\partial}"] \\ X \arrow[r, "\partial"'] & A \end{tikzcd} \tag{1.15}$$

The two definitions are equivalent, indeed it suffices to consider the following diagram

$$\begin{tikzcd}
A\flat X \arrow[r, tail, "k_{A,X}"] \arrow[d, "1_A\flat\partial"'] & A+X \arrow[r, "\binom{1_A}{0}"] \arrow[d, "1_A+\partial"] & A \arrow[d, equal] \\
A\flat A \arrow[r, tail, "k_{A,A}"'] & A+A \arrow[r, "\binom{1_A}{0}"'] & A
\end{tikzcd}$$

to go from diagram (1.14) to diagram (1.15) since

$$\begin{aligned}
\chi_A \circ (1_A\flat\partial) &= \begin{pmatrix}1_A\\1_A\end{pmatrix} \circ k_{A,A} \circ (1_A\flat\partial)\\
&= \begin{pmatrix}1_A\\1_A\end{pmatrix} \circ (1_A+\partial) \circ k_{A,X}\\
&= \begin{pmatrix}1_A\\\partial\end{pmatrix} \circ k_{A,X}.
\end{aligned}$$

Being more similar to the group one we will always use the first definition from now on.

1.4.2 The category XMod($\mathbb{A}$)

Let us recall the definition of internal crossed module in a semi-abelian category $\mathbb{A}$ that satisfies the (SH) condition.

Definition 1.4.6 ([55, 64, 51])**.** An *internal crossed module* (in a semi-abelian category $\mathbb{A}$ with SH), is given by $(X \xrightarrow{\partial} A, \xi)$ where $\partial\colon X \to A$ is a morphism in $\mathbb{A}$ and $\xi\colon A\flat X \to X$ is an internal action such that the following diagram commutes

$$\begin{tikzcd}
X\flat X \arrow[r, "\chi_X"] \arrow[d, "\partial\flat 1_X"'] & M \arrow[d, equal] \\
A\flat X \arrow[r, "\xi"] \arrow[d, "1_A\flat\partial"'] & X \arrow[d, "\partial"] \\
A\flat A \arrow[r, "\chi_A"'] & A
\end{tikzcd}$$

Remark 1.4.7. In particular an internal crossed module in $\mathbb{A}$ is an internal pre-crossed module $(X \xrightarrow{\partial} A, \xi)$ satisfying the so called *Peiffer condition*, which is the commutativity of the following diagram

$$\begin{tikzcd}
X\flat X \arrow[r, "\chi_X"] \arrow[d, "\partial\flat 1_X"'] & X \arrow[d, equal] \\
A\flat X \arrow[r, "\xi"'] & X
\end{tikzcd} \tag{1.16}$$

A morphism of internal crossed modules is just a morphism of the underlying internal pre-crossed modules, that is $\mathbf{XMod}(\mathbb{A})$ is the full subcategory of $\mathbf{PreXMod}(\mathbb{A})$ whose objects are internal crossed modules.

Construction 1.4.8. By using the correspondence between $\mathbf{Pt}(\mathbb{A})$ and $\mathbf{Act}(\mathbb{A})$ we can map each internal pre-crossed module into a particular reflexive graph, precisely given by a diagram of the form

$$\begin{tikzcd} X \arrow[r, tail, "k_d"] & X \rtimes_\xi A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "e" description] \end{tikzcd}$$

where $ce = 1_A = de$. Precisely this is given as follows:

- First we obtain $X \rtimes_\xi A$, and the maps d, e and k_d by computing the point associated to the action ξ; recall from Remark 1.2.7 that $X \rtimes_\xi A$ is defined as the coequaliser

$$\begin{tikzcd} A\flat X \arrow[r, shift left, "i_X\circ\xi"] \arrow[r, shift right, "k_{A,X}"'] & A+X \arrow[r, two heads, "\sigma_\xi"] \arrow[dr, "\binom{1}{0}"] & X \rtimes_\xi A \arrow[d, dotted, "d"] \\ & A \arrow[u, "i_A"] \arrow[r, equal] & A \end{tikzcd}$$

  and that the point

$$\begin{tikzcd} X \arrow[r, tail, "k_d"] & X \rtimes_\xi A \arrow[r, shift left, "d"] & A \arrow[l, shift left, "e"] \end{tikzcd}$$

  is given by $e = \sigma_\xi \circ i_A$, by $d \circ \sigma_\xi = \binom{1}{0}$, that is $d = \left\langle {}^{1_A}_{0} \right|$, and by $k_d = \sigma_\xi \circ i_X$.

- Similarly we define the map c, so that $c \circ \sigma_\xi = \binom{1}{\partial}$, that is $c = \left\langle {}^{1_A}_{\partial} \right|$, using the diagram

$$\begin{tikzcd} A\flat X \arrow[r, shift left, "i_X\circ\xi"] \arrow[r, shift right, "k_{A,X}"'] & A+X \arrow[r, two heads, "\sigma_\xi"] \arrow[dr, "\binom{1}{\partial}"] & X \rtimes_\xi A \arrow[d, dotted, "c"] \\ & X \arrow[u, "i_X"] \arrow[r, "\partial"'] & A \end{tikzcd} \tag{1.17}$$

  Notice that $\binom{1}{\partial} \circ (i_X \circ \xi) = \binom{1}{\partial} \circ k_{A,X}$ due to the fact that (A, X, ξ, ∂) is a crossed module. Finally we deduce that $c \circ k = \partial$ and that $ce = 1_A$.

From now on we will often use this formalism instead of the one given by the action and by the crossed module conditions. In particular from a pre-crossed module (A, X, ξ, ∂) we will construct a reflexive graph given by the point $(A, X \rtimes_\xi A, e, d)$ endowed with additional maps c and k such that $ce = 1_A = de$, $k = k_d$ and $ck = \delta$.

Example 1.4.9. Consider the pre-crossed module $(X \xrightarrow{0} A, \tau^A_X)$ given by the trivial action. Then the diagram (1.17) becomes (see Example 1.2.9)

$$\begin{tikzcd}
A\flat X \arrow[r, shift left, "i_2\circ\tau^A_X"] \arrow[r, shift right, "k_{A,X}"'] & A+X \arrow[r, two heads, "\langle\binom{1}{0},\binom{0}{1}\rangle"] \arrow[dr, "\binom{1}{0}"] & A\times X \arrow[d, dotted, "\pi_A"] \\
& X \arrow[u, "i_X"] \arrow[r, "0"'] & A
\end{tikzcd}$$

So $d = \pi_A$, $c = \pi_A$, $e = \langle 1, 0\rangle$ and $k = \langle 0, 1\rangle$. This means that the reflexive graph associated to the trivial pre-crossed module is given by

$$\begin{tikzcd}
X \arrow[r, tail, "\langle 0,1\rangle"] & A\times X \arrow[r, shift left=2, "\pi_A"] \arrow[r, shift right=2, "\pi_A"'] & A \arrow[l, "\langle 1,0\rangle" description]
\end{tikzcd}$$

Furthermore we are able to show that if $(X \xrightarrow{0} A, \tau^A_X)$ is a crossed module, then X is an abelian object. In order to prove this, we use the equivalent condition $[X,X]^{\mathcal{H}}_X = 0$ and the definition of this commutator through the diagram

$$\begin{tikzcd}
X\diamond X \arrow[r, "h_{X,X}"] \arrow[d, two heads] & X+X \arrow[d, "\binom{1}{1}"] \\
{[X,X]^{\mathcal{H}}_X} \arrow[r, tail] & X
\end{tikzcd}$$

It suffices to show that $\binom{1}{1} \circ h_{X,X} = 0$ but this is given by the Peiffer condition through the equalities

$$\begin{aligned}
\begin{pmatrix}1\\1\end{pmatrix} \circ k_{X,X} \circ i_{X,X} &= \chi_X \circ i_{X,X} = \tau^A_X \circ (0\flat 1) \circ i_{X,X} \\
&= \tau^A_X \circ i_{A,X} \circ (0 \diamond 1) \\
&= \begin{pmatrix}0\\1\end{pmatrix} \circ k_{A,X} \circ i_{A,X} \circ (0\diamond 1) \\
&= \pi_2 \circ \Sigma_{A,X} \circ h_{A,X} \circ (0\diamond 1) \\
&= \pi_2 \circ 0 \circ (0 \diamond 1) = 0.
\end{aligned}$$

Example 1.4.10. Consider the crossed module $(A \xrightarrow{1_A} A, \chi_A)$ given by the conjugation action. Then the diagram (1.17) becomes (see Example 1.2.10)

$$\begin{tikzcd}
A\flat A \arrow[r, shift left, "i_2\circ\chi_A"] \arrow[r, shift right, "k_{A,A}"'] & A+A \arrow[r, two heads, "\langle\binom{1}{0},\binom{1}{1}\rangle"] \arrow[dr, "\binom{1}{1}"] & A\times A \arrow[d, dotted, "\pi_2"] \\
& A \arrow[u, "i_2"] \arrow[r, equal] & A
\end{tikzcd}$$

So $d = \pi_1$, $c = \pi_2$, $e = \langle 1,1\rangle$ and $k = \langle 0,1\rangle$. This means that the reflexive graph associated to the conjugation crossed module is given by

$$\begin{tikzcd} A \arrow[r, tail, "\langle 0,1\rangle"] & A\times A \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & A \arrow[l, "\langle 1,1\rangle" description] \end{tikzcd}$$

Proposition 1.4.11. *Given a crossed module $(X \xrightarrow{\partial} A, \xi)$ we have that $[K_\partial, X]^{\mathcal{H}}_X = 0$.*

Proof. Looking at the diagram

$$\begin{tikzcd} K_\partial \diamond X \arrow[r, tail, "h_{K_\partial,X}"] \arrow[d, two heads] & K_\partial + X \arrow[d, "\binom{k_\partial}{1}"] \\ {[K_\partial, X]^{\mathcal{H}}_X} \arrow[r, tail] & M \end{tikzcd}$$

defining $[K_\partial, X]^{\mathcal{H}}_X$ we just want to show that the equality $\binom{k_\partial}{1} \circ k_{K_\partial,X} \circ i_{K_\partial,X} = 0$. In order to do this we use the equality $\binom{k_\partial}{1} \circ k_{K_\partial,X} = \xi \circ (0\flat 1)$ given by the diagram

$$\begin{tikzcd} K_\partial\flat X \arrow[r, tail, "k_{K_\partial,X}"] \arrow[d, "k_\partial\flat 1"'] & K_\partial + X \arrow[r, "\binom{k_\partial}{1}"] \arrow[d, "k_\partial+1"] & X \arrow[d, equal] \\ X\flat X \arrow[r, tail, "k_{X,X}"'] \arrow[d, "\partial\flat 1"'] & X + X \arrow[r, "\binom{1}{1}"'] & X \arrow[d, equal] \\ A\flat X \arrow[rr, "\xi"'] & & X \end{tikzcd}$$

where the two top squares commute trivially whereas the lower one is given by the Peiffer condition. Furthermore we have that $0\flat 1 = \eta^A_X \circ \tau^{K_\partial}_X$: to show this, it suffices to postcompone with the monomorphism $k_{A,X}$. Now it remains to compute the following chain of equalities

$$\begin{aligned} \binom{k_\partial}{1} \circ k_{K_\partial,X} \circ i_{K_\partial,X} &= \xi \circ (0\flat 1) \circ i_{K_\partial,X} \\ &= \xi \circ \eta^A_X \circ \tau^{K_\partial}_X \circ i_{K_\partial,X} \\ &= \tau^{K_\partial}_X \circ i_{K_\partial,X} = 0. \end{aligned}$$

□

Lemma 1.4.12 ([8])**.** *Consider the following diagram*

$$\begin{tikzcd} A \arrow[r, tail, "k_p"] \arrow[d, "\alpha"'] & B \arrow[r, shift left, "p"] \arrow[d, "\beta"] & C \arrow[l, shift left, "s"] \arrow[d, two heads, "\gamma"] \\ A' \arrow[r, tail, "k_{p'}"'] & B' \arrow[r, shift left, "p'"] & C' \arrow[l, shift left, "s'"] \end{tikzcd}$$

with $p \circ s = 1_C$, $p' \circ s' = 1_{C'}$ and γ being a regular epimorphism. Then α is a regular epimorphism iff β is so.

Proof. If α is a regular epimorphism, then β is so by Lemma 1.1.18. Instead if β is a regular epimorphism, the right-pointing square on the right is a pushout by Lemma 1.2.2. Consequently it is a regular pushout by Remark 1.1.26. Finally by applying Lemma 1.1.27 we obtain that α is a regular epimorphism as well. $\square$

Lemma 1.4.13. *Consider a morphism of internal crossed modules*

$$(X \xrightarrow{\partial} A, \xi) \xrightarrow{(f,\alpha)} (X' \xrightarrow{\partial'} A', \xi').$$

Then (f, α) is a regular epimorphism in $\mathbf{XMod}(\mathbb{A})$ if and only f and α are regular epimorphisms in $\mathbb{A}$.

Proof. In the category $\mathbf{RG}(\mathbb{A})$ of reflexive graphs in $\mathbb{A}$, coequalisers are computed pointwise, and due to Theorem 3.1 and Lemma 3.1 in [44] this implies that also in $\mathbf{Cat}(\mathbb{A})$ the coequalisers are computed pointwise. This means that a morphism

$$(A, A_0, d, c, e, m) \xrightarrow{(\alpha,\alpha_0)} (A', A_0', d', c', e', m')$$

is the coequaliser of (g, g_0) and (h, h_0) in $\mathbf{Cat}(\mathbb{A})$ if and only if α is the coequaliser $c_{g,h}$ and if α_0 is the coequaliser c_{g_0,h_0}. Using the equivalence of categories $\mathbf{XMod}(\mathbb{A}) \simeq \mathbf{Cat}(\mathbb{A})$ and the diagram

$$\begin{tikzcd}
M \arrow[r, triangle, "k_d"] \arrow[d, "f"'] & A_0 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "\alpha_0"] & A \arrow[l, "e" description] \arrow[d, "\alpha"] \\
M' \arrow[r, triangle, "k_{d'}"'] & A_0' \arrow[r, shift left=2, "d'"] \arrow[r, shift right=2, "c'"'] & A' \arrow[l, "e'" description]
\end{tikzcd}$$

we conclude that (f, α) is a regular epimorphism in $\mathbf{XMod}(\mathbb{A})$ iff both α and α_0 are regular epimorphisms in $\mathbb{A}$. Now it suffices to apply Lemma 1.4.12 to conclude the proof. $\square$

Another category that we will deal with, is denoted by $\mathbf{XMod}_L(\mathbb{A})$ for a fixed object $L \in \mathbb{A}$: it is the subcategory of $\mathbf{XMod}(\mathbb{A})$ whose objects are the internal crossed modules of the form $(M \xrightarrow{\partial} L, \xi)$ and whose maps between $(M \xrightarrow{\partial} L, \xi)$ and $(M' \xrightarrow{\partial'} L, \xi')$ are the morphisms of internal crossed modules of the form $(M \xrightarrow{g} M', L \xrightarrow{1_L} L)$.

Remark 1.4.14. The category $\mathbf{XMod}_L(\mathbb{A})$ is not semi-abelian only because there is no zero object, but it is quasi-pointed (and sequentiable). Indeed we have that $(0 \to L, \tau_0^L)$ is the initial object, that $(L \xrightarrow{1_L} L, \chi_L)$ is the terminal object and that the morphism

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & L \arrow[dl, equal] \\
& L &
\end{tikzcd}$$

is a monomorphism of internal crossed modules.

Lemma 1.4.15. *Consider a morphism of L-crossed modules*

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f,1_L)} (M' \xrightarrow{\partial'} L, \xi').$$

Then $(f, 1_L)$ is a monomorphism in $\mathbf{XMod}_L(\mathbb{A})$ if and only f is a monomorphism in $\mathbb{A}$.

Proof.

($\Leftarrow$) Suppose that $(\alpha, 1_L)$ and $(\beta, 1_L)$ are two morphisms of L-crossed modules such that $(f, 1_L) \circ (\alpha, 1_L) = (f, 1_L) \circ (\beta, 1_L)$: we have $f \circ \alpha = f \circ \beta$ and since f is a monomorphism in $\mathbb{A}$ we deduce $\alpha = \beta$, that is $(\alpha, 1_L) = (\beta, 1_L)$.

($\Rightarrow$) Suppose $(f, 1_L)$ is a monomorphism in $\mathbf{XMod}_L(\mathbb{A})$: we construct the kernel pair $(Kp(f), r_0, r_1)$ of f in $\mathbb{A}$ and we want to show that $r_0 = r_1$ in order to show that f is a monomorphism. We start by defining a L-crossed module structure around $Kp(f)$ by using the diagrams

$$\begin{tikzcd}
L\flat Kp(f) \arrow[rr, "1\flat r_1"] \arrow[dd, "1\flat r_0"'] \arrow[dr, dotted, "\tilde{\xi}"] & & L\flat M \arrow[d, "\xi"] \\
& Kp(f) \arrow[r, "r_1"] \arrow[d, "r_0"'] & M \arrow[d, "f"] \\
L\flat M \arrow[r, "\xi"'] & M \arrow[r, "f"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
Kp(f) \arrow[r, dotted, "\tilde{\partial}"] \arrow[d, "r_i"'] & L \arrow[d, equal] \\
M \arrow[r, "\partial"'] & L
\end{tikzcd}$$

It is easy to show that $\tilde{\xi}$ is actually an action, that $(Kp(f) \xrightarrow{\tilde{\partial}} L, \tilde{\xi})$ is an L-crossed module and that the maps $(r_i, 1_L)$ are morphisms of L-crossed modules. Now we use the fact that $(f, 1_L)$ is a monomorphism in $\mathbf{XMod}_L(\mathbb{A})$: therefore from the equality $(f, 1_L) \circ (r_0, 1_L) = (f, 1_L) \circ (r_1, 1_L)$ we can deduce that $(r_0, 1_L) = (r_1, 1_L)$, that is $r_0 = r_1$, which in turn is equivalent to f being a monomorphism in $\mathbb{A}$. $\square$

Remark 1.4.16. Consider a morphism of L-crossed modules

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f,1_L)} (M' \xrightarrow{\partial'} L, \xi').$$

The kernel of this morphism is given by

$$(K_f \xrightarrow{0} L, \underline{\xi}) \xrightarrow{(k_f,1_L)} (M \xrightarrow{\partial} L, \xi)$$

where the action $\underline{\xi}$ is induced by the universal property of K_f as shown in the following diagram

$$\begin{tikzcd}
L\flat K_f \arrow[r, "1_L\flat k_f"] \arrow[d, dotted, "\underline{\xi}"'] & L\flat M \arrow[r, "1_L\flat f"] \arrow[d, "\xi"'] & L\flat M' \arrow[d, "\xi'"] \\
K_f \arrow[r, tail, "k_f"'] & M \arrow[r, "f"'] & M'
\end{tikzcd}$$

The fact that this is an L-crossed module is trivial to check.

Chapter 2

Compatible actions in semi-abelian categories

The concept of a pair of compatible actions was first introduced in the category of groups by Brown and Loday, in relation to their work on the non-abelian tensor product of groups [15]. Later, in [37], the definition was adapted to the context of Lie algebras, where it was further studied in [59]. Since then, several other particular instances of compatible actions have been defined, in various settings: see for example [43, 21, 20]. The aim of this chapter is to provide a general definition in semi-abelian categories (in the sense of [58]), in a way that extends these as special cases. In particular this will give us the tools to develop a unified theory, in such a way that computing the non-abelian tensor product of compatible actions is the same as computing the non-abelian tensor product of internal crossed modules. This process generalises the diverse particular notions of non-abelian tensor product that appear in the literature so far.

With this idea in mind, we first examine the cases of groups and Lie algebras from a new perspective, aiming to use a diagrammatic and internal approach whenever this is possible. To do so, we take advantage of the equivalence between group actions (resp. Lie algebra actions) in the usual sense and internal actions (introduced in [11, 5]) in the category **Grp** (resp. $\mathbf{Lie}_R$), as well as the equivalence (see [55]) between crossed modules of groups (resp. crossed modules of Lie algebras) and internal crossed modules in **Grp** (resp. $\mathbf{Lie}_R$). Thus we may separate properties which are specific for groups and Lie algebras from those that are purely categorical.

The conditions which we single out in the categories **Grp** and $\mathbf{Lie}_R$ in terms of the internal action formalism become our general definition of "a pair of compatible actions". This definition makes sense as soon as the surrounding category is semi-abelian. However, we shall always assume the (SH) condition to hold as well: this is a relatively mild condition which excludes loops, for instance, but includes all categories of groups with operations; see [67, 29]. This simplifies our work, since under (SH) internal crossed modules allow an easier description [55, 67].

Our main tool is an extension, to the semi-abelian context, of the *Peiffer product* $M \bowtie N$ of two objects M and N acting on each other (via an action ξ_M^N of N on M

and an action ξ_N^M of M on N). A notion of Peiffer product has already been introduced in [30], in the special case of a pair of internal precrossed modules over a common base object. Ours, however, is a different approach, and a priori the two notions do not coincide. Our definition is a direct generalisation of the group and Lie algebra versions of the Peiffer product, which were originally introduced respectively in [75] and in [37]. It is well defined as soon as the two objects M and N act on each other, whereas for the definition in [30] they also need to satisfy some compatibility conditions. Moreover, when the actions ξ_M^N and ξ_N^M are compatible, the Peiffer product $M \bowtie N$ is endowed with internal crossed module structures $(M \xrightarrow{l_M} M \bowtie N, \xi_M^{M\bowtie N})$ and $(N \xrightarrow{l_N} M \bowtie N, \xi_N^{M\bowtie N})$.

We use this as an ingredient in the generalisation of a result, stated in [15] for groups and in [59] for Lie algebras, to any semi-abelian category that satisfies the condition (SH). We show namely that two objects M and N act on each other compatibly if and only if there exists a third object L with two internal crossed module structures $(M \xrightarrow{\mu} L, \xi_M^L)$ and $(N \xrightarrow{\nu} L, \xi_N^L)$. Amongst other things, this allows us to deduce that our definition of compatibility for pairs of internal actions restricts to the classical definitions for groups and Lie algebras. Another consequence of this result is that the non-abelian tensor product introduced in Chapter 3 has two natural interpretations: either as a tensor product of compatible internal actions, or equivalently as a tensor product of crossed modules over a common base object.

Finally, we study the Peiffer product via its universal properties. We also prove that, under the additional hypothesis of algebraic coherence [29], our definition of Peiffer product coincides with the one given in [30].

The chapter is organised as follows:

- In Section 2.1 we examine the concept of a pair of compatible actions in the category of groups. First we consider the definition of compatibility given in [15] and we translate it into its diagrammatic form. Then we construct the Peiffer product as a coequaliser and we prove that it coincides with the definition already known for the case of groups. In Proposition 2.1.10 we prove a result stated in [15], namely that two groups M and N act on each other compatibly if and only if there exists a third group L with two crossed module structures $(M \xrightarrow{\mu} L, \xi_M^L)$ and $(N \xrightarrow{\nu} L, \xi_N^L)$.

- In order to deal with the Lie algebra case we open Section 2.2 with a quick recap of some specific tools that we are going to use in the rest of the section. We then show the link between the notions of compatible actions for groups and for Lie algebras, supporting the idea of a possible generalisation to semi-abelian categories. We show that two crossed modules with a common codomain in $\mathbf{Lie}_R$ induce compatible actions and, in order to prove the converse, we use the Lie algebra version of the internal construction of the Peiffer product introduced in the previous section, endowing it with crossed module structures. Lastly, we prove that the coproduct in $\mathbf{XMod}_L(\mathbf{Lie}_R)$ can be obtained through the Peiffer product and we draw some consequences of this result.

- Section 2.3 contains this chapter's main results. We work in the context of a semi-abelian category $\mathbb{A}$ that satisfies the (SH) condition. We express the definition of compatibility in this general context and show in Proposition 2.3.3 that whenever we have a pair of internal crossed modules over a common base object, they induce a pair of compatible internal actions. Then we construct the Peiffer product of two internal actions in three distinct ways: first we imitate what happens in the case of groups, constructing the Peiffer product for each pair of objects acting on each other. In Proposition 2.3.5 we prove that this is the same as taking the pushout of the two semi-direct products. Then we give a more specific definition that requires the actions to be compatible. Finally, we show in Proposition 2.3.8 that, if the compatibility conditions are satisfied, then the two definitions coincide.

We prove in Proposition 2.3.9 that whenever the actions are compatible, their Peiffer product is automatically endowed with internal crossed module structures $(M \xrightarrow{l_M} M \bowtie N, \xi_M^{M \bowtie N})$ and $(N \xrightarrow{l_N} M \bowtie N, \xi_N^{M \bowtie N})$. This leads to Theorem 2.3.10, which is a generalisation to semi-abelian categories of Proposition 2.1.10, proven for groups in Section 2.1: two objects M and N act on each other compatibly if and only if there exists a third object L with two internal crossed module structures $(M \xrightarrow{\mu} L, \xi_M^L)$ and $(N \xrightarrow{\nu} L, \xi_N^L)$. Via this result we obtain Corollary 2.3.11 and Corollary 2.3.12 as confirmations of the equivalence between our general definition of compatibility and the specific ones in the cases of groups and Lie algebras.

We conclude the chapter with a study of the Peiffer product via its universal properties. Here we also prove that, under the additional hypothesis of algebraic coherence [29], our definition of the Peiffer product coincides with the one given in [30]. Via results in [30], this further implies that under an additional condition called (UA), the actions induced by two L-crossed module structures have a Peiffer product which is again an L-crossed module; furthermore, it is the coproduct in $\mathbf{XMod}_L(\mathbb{A})$ of the given L-crossed modules. This generalises Proposition 2.2.26 in Section 2.2.

2.1 Compatible actions of groups

Definition 2.1.1. Consider two groups M and N acting on each other via

$$\xi_N^M : M \flat N \to N \qquad\qquad \xi_M^N : N \flat M \to M$$

and on themselves by conjugation. We are able to define induced actions ξ_M^{M+N} and ξ_N^{M+N} of the coproduct $M+N$ on M and on N, that is such that the following diagrams commute:

$$\begin{tikzcd} M\flat N \arrow[r, "i_M\flat 1_N"] \arrow[dr, "\xi_N^M"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\ & N \end{tikzcd} \qquad \begin{tikzcd} N\flat M \arrow[r, "i_N\flat 1_M"] \arrow[dr, "\xi_M^N"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\ & M \end{tikzcd}$$

$$\begin{tikzcd} N\flat N \arrow[r, "i_N\flat 1_N"] \arrow[dr, "\chi_N"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\ & N \end{tikzcd} \qquad \begin{tikzcd} M\flat M \arrow[r, "i_M\flat 1_M"] \arrow[dr, "\chi_M"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\ & M \end{tikzcd} \tag{2.1}$$

This is done simply by defining the action $\xi_M^{M+N}\colon (M+N)\flat M \to M$ on the generators $s\overline{m}s^{-1}$ with $\overline{m} \in M$ and $s \in M+N$, inductively on the length of s:

$$\xi_M^{M+N}(s\overline{m}s^{-1}) := \begin{cases} \overline{m} & \text{if } s = \epsilon, \\ \xi_M^{M+N}(s'\xi_M^N(n\overline{m}n^{-1})s'^{-1}) & \text{if } \exists n \in N \mid s = s'n, \\ \xi_M^{M+N}(s'\chi_M(m\overline{m}m^{-1})s'^{-1}) & \text{if } \exists m \in M \mid s = s'm. \end{cases} \tag{2.2}$$

and similarly for ξ_N^{M+N}.

Remark 2.1.2. In particular we have that the following always hold

$$^{(^n m)}m' = (^n m)m'(^n m)^{-1} = {}^n(m(^{n^{-1}}m')m^{-1}) = {}^{nmn^{-1}}m', \tag{2.3}$$

$$^{(^m n)}n' = (^m n)n'(^m n)^{-1} = {}^m(n(^{m^{-1}}n')n^{-1}) = {}^{mnm^{-1}}n', \tag{2.4}$$

where the right-hand side of each equality is given by the induced action of the coproduct. This is given diagrammatically by the commutativity of the squares

$$\begin{tikzcd} (N\flat M)\flat M \arrow[r, "k_{N,M}\flat 1_M"] \arrow[d, "\xi_M^N\flat 1_M"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\ M\flat M \arrow[r, "\chi_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} (M\flat N)\flat N \arrow[r, "k_{M,N}\flat 1_N"] \arrow[d, "\xi_N^M\flat 1_N"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\ N\flat N \arrow[r, "\chi_N"'] & N \end{tikzcd} \tag{2.5}$$

Definition 2.1.3. Two actions are said to be *compatible* if also the following equalities hold for each $m, m' \in M$ and $n, n' \in N$

$$^{(^m n)}m' = {}^{mnm^{-1}}m', \qquad\qquad ^{(^n m)}n' = {}^{nmn^{-1}}n'. \tag{2.6}$$

If once again we examine these equalities from a diagrammatic point of view, we obtain that they are equivalent to the commutativity of

$$\begin{tikzcd} (M\flat N)\flat M \arrow[r, "k_{M,N}\flat 1_M"] \arrow[d, "\xi_N^M\flat 1_M"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\ N\flat M \arrow[r, "\xi_M^N"'] & M \end{tikzcd} \qquad \begin{tikzcd} (N\flat M)\flat N \arrow[r, "k_{N,M}\flat 1_N"] \arrow[d, "\xi_M^N\flat 1_N"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\ M\flat N \arrow[r, "\xi_N^M"'] & N \end{tikzcd} \tag{2.7}$$

A second look on these four equalities leads us to the following remark.

Remark 2.1.4. The meaning of the equations (2.3) and (2.4) is that for each $m \in M$ and $n \in N$

- $({}^{n}m)nm^{-1}n^{-1}$ acts trivially on M,
- $({}^{m}n)mn^{-1}m^{-1}$ acts trivially on N,

whereas the meaning of equations (2.6) is that for each $m \in M$ and $n \in N$

- $({}^{n}m)nm^{-1}n^{-1}$ acts trivially on N,
- $({}^{m}n)mn^{-1}m^{-1}$ acts trivially on M.

If we define $K \leqslant M + N$ to be the normal closure of the subgroup generated by elements of the form $({}^{n}m)nm^{-1}n^{-1}$ or $({}^{m}n)mn^{-1}m^{-1}$, we have that K acts trivially on both M and N if and only if the two actions are compatible.

The previous remark leads to the following definition given in [42].

Definition 2.1.5. Given a pair of compatible actions as above, we define their *Peiffer product* $M \bowtie N$ of M and N as the quotient

$$K \rightarrowtail M + N \overset{q_K}{\twoheadrightarrow} M \bowtie N =: \tfrac{M+N}{K}$$

Remark 2.1.6. Notice that the map q_K and the Peiffer product $M \bowtie N$ can also be defined in the following equivalent way, as the coequaliser in the diagram

$$(N\flat M) + (M\flat N) \underset{\xi_M^N + \xi_N^M}{\overset{\binom{k_{N,M}}{k_{M,N}}}{\rightrightarrows}} M + N \xrightarrow{\quad q \quad} M \bowtie N \tag{2.8}$$

In order to show why this definition is equivalent to the previous one, consider the map q_K given by the first definition. It is easy to show that

$$\begin{cases} q_K \circ i_M \circ \xi_M^N = q_K \circ k_{N,M} \\ q_K \circ i_N \circ \xi_N^M = q_K \circ k_{M,N} \end{cases}$$

since this is exactly what taking the quotient by K means. But this is the same as saying

$$\begin{cases} q_K \circ (\xi_M^N + \xi_N^M) \circ i_{N\flat M} = q_K \circ k_{N,M} \\ q_K \circ (\xi_M^N + \xi_N^M) \circ i_{M\flat N} = q_K \circ k_{M,N} \end{cases}$$

which in turn is

$$q_K \circ (\xi_M^N + \xi_N^M) = q_K \circ \begin{pmatrix} k_{N,M} \\ k_{M,N} \end{pmatrix}.$$

The universal property of the coequaliser is given by the universal property of the quotient by K in a straightforward way.

Since K acts trivially on both M and N we can define induced actions $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ of $M \bowtie N$ on M and N, that is such that the following diagrams commute

$$\begin{tikzcd}
(M+N)\flat M \arrow[r, "q\flat 1_M"] \arrow[dr, "\xi_M^{M+N}"'] & (M\bowtie N)\flat M \arrow[d, "\xi_M^{M\bowtie N}"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
(M+N)\flat N \arrow[r, "q\flat 1_N"] \arrow[dr, "\xi_N^{M+N}"'] & (M\bowtie N)\flat N \arrow[d, "\xi_N^{M\bowtie N}"] \\
& M
\end{tikzcd} \tag{2.9}$$

We can describe these actions of the Peiffer product through its universal property, but in order to do this, we need Lemma 1.1.38 and a preliminar remark.

Remark 2.1.7. Note that the two composites

$$\begin{tikzcd}
M+N \arrow[r, "\eta_M^N+\eta_N^M"] & (N\flat M)+(M\flat N) \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}"] \arrow[r, shift right, "\xi_M^N+\xi_N^M"'] & M+N
\end{tikzcd}$$

are equal to 1_{M+N}: one is obvious and the other one is clear once we draw the diagram involved. Hence we have that

$$\begin{tikzcd}
(N\flat M)+(M\flat N) \arrow[r, bend left, "\binom{k_{N,M}}{k_{M,N}}"] \arrow[r, bend right, "\xi_M^N+\xi_N^M"'] & M+N \arrow[l, "\eta_M^N+\eta_N^M"']
\end{tikzcd}$$

is a reflexive graph.

Lemma 1.1.38 implies that $q\flat 1_M$ is the coequaliser of $\binom{k_{N,M}}{k_{M,N}}\flat 1_M$ and $(\xi_M^N+\xi_N^M)\flat 1_M$ and that $q\flat 1_N$ is the coequaliser of $\binom{k_{N,M}}{k_{M,N}}\flat 1_N$ and $(\xi_M^N+\xi_N^M)\flat 1_N$. We want to use these universal properties to define induced actions $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ of $M\bowtie N$ on M and N as in the following diagram

$$\begin{tikzcd}
((N\flat M)+(M\flat N))\flat M \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}\flat 1_M"] \arrow[r, shift right, "(\xi_M^N+\xi_N^M)\flat 1_M"'] & (M+N)\flat M \arrow[r, "q\flat 1_M"] \arrow[dr, "\xi_M^{M+N}"'] & (M\bowtie N)\flat M \arrow[d, dotted, "\xi_M^{M\bowtie N}"] \\
& & M
\end{tikzcd}$$

$$\begin{tikzcd}
((N\flat M)+(M\flat N))\flat N \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}\flat 1_N"] \arrow[r, shift right, "(\xi_M^N+\xi_N^M)\flat 1_N"'] & (M+N)\flat N \arrow[r, "q\flat 1_N"] \arrow[dr, "\xi_N^{M+N}"'] & (M\bowtie N)\flat N \arrow[d, dotted, "\xi_N^{M\bowtie N}"] \\
& & N
\end{tikzcd}$$

In order to do this, we need the following result.

Proposition 2.1.8. *The action ξ_M^{M+N} coequalises $\binom{k_{N,M}}{k_{M,N}}\flat 1_M$ and $(\xi_M^N + \xi_N^M)\flat 1_M$. Similarly, the action ξ_N^{M+N} coequalises $\binom{k_{N,M}}{k_{M,N}}\flat 1_N$ and $(\xi_M^N + \xi_N^M)\flat 1_N$.*

Proof. Consider a generator $s\overline{m}s^{-1}$ of $((N\flat M)+(M\flat N))\flat M$ and write s as juxtapposition of generators of $N\flat M$ and $M\flat N$, that is $s = s_1 \cdots s_k$ with $s_j = n_j m_j n_j^{-1} \in N\flat M$ or $s_j = m_j n_j m_j^{-1} \in M\flat N$. We are going to prove the equality

$$\xi_M^{M+N}\left(\left(\begin{pmatrix} k_{N,M} \\ k_{M,N} \end{pmatrix}\flat 1_M\right)(s\overline{m}s^{-1})\right) = \xi_M^{M+N}\left(\left(\left(\xi_M^N + \xi_N^M\right)\flat 1_M\right)(s\overline{m}s^{-1})\right)$$

by induction on k. First of all notice that this is equivalent to the equality

$$\xi_M^{M+N}\left(s\overline{m}s^{-1}\right) = \xi_M^{M+N}\left(\epsilon(s)\overline{m}\epsilon(s)^{-1}\right) \tag{2.10}$$

where $\epsilon(s) := \left(\xi_M^N + \xi_N^M\right)(s) \in M + N$. In order to prove it when s is the empty word, it suffices to notice that also $\epsilon(s)$ is the empty word. Now suppose we proved (2.10) for each word whose decomposition involves at most $k-1$ generators of $N\flat M$ and $M\flat N$, consider $s = s_1 \cdots s_k$ and denote $s' = s_1 \cdots s_{k-1}$: we have the chain of equalities

$$\begin{aligned}
\xi_M^{M+N}\left(s\overline{m}s^{-1}\right) &= \xi_M^{M+N}\left(s' s_k \overline{m} s_k^{-1} s'^{-1}\right) \\
&= \xi_M^{M+N}\left(s' \left({}^{s_k}\overline{m}\right) s'^{-1}\right) \\
&= \xi_M^{M+N}\left(s' \left({}^{\epsilon(s_k)}\overline{m}\right) s'^{-1}\right) \\
&= \xi_M^{M+N}\left(\epsilon(s') \left({}^{\epsilon(s_k)}\overline{m}\right) \epsilon(s')^{-1}\right) \\
&= \xi_M^{M+N}\left(\epsilon(s')\epsilon(s_k)\overline{m}\epsilon(s_k)^{-1}\epsilon(s')^{-1}\right) \\
&= \xi_M^{M+N}\left(\epsilon(s)\overline{m}\epsilon(s)^{-1}\right)
\end{aligned}$$

where

$$\epsilon(s_k) = \begin{cases} {}^{n_k}m_k & \text{if } s_k = n_k m_k n_k^{-1} \in N\flat M, \\ {}^{m_k}n_k & \text{if } s_k = m_k n_k m_k^{-1} \in M\flat N. \end{cases}$$

Finally we apply the same reasoning to ξ_N^{M+N}. □

Proposition 2.1.9. *We have two crossed module structures*

$$(M \xrightarrow{l_M} M \bowtie N, \xi_M^{M\bowtie N}) \qquad\qquad (N \xrightarrow{l_N} M \bowtie N, \xi_N^{M\bowtie N})$$

where the actions of the Peiffer product are induced as above and the maps l_M and l_N are defined through the compositions

$$\begin{tikzcd}
M \arrow[dr, hook, "i_M"] \arrow[ddr, bend right, "l_M"'] & & N \arrow[dl, hook', "i_N"'] \arrow[ddl, bend left, "l_N"] \\
& M+N \arrow[d, two heads, "q"] & \\
& M \bowtie N &
\end{tikzcd} \tag{2.11}$$

Proof. We will show the thesis only for $\xi_M^{M\bowtie N}$ since the proof in the other case uses the same strategy. We need to show the commutativity of the following squares

$$\begin{tikzcd}
M\flat M \arrow[r, "\chi_M"] \arrow[d, "l_M\flat 1_M"'] & M \arrow[d, equal] \\
(M \bowtie N)\flat M \arrow[r, "\xi_M^{M\bowtie N}"] \arrow[d, "1_{M\bowtie N}\flat l_M"'] & M \arrow[d, "l_M"] \\
(M \bowtie N)\flat(M \bowtie N) \arrow[r, "\chi_{M\bowtie N}"'] & (M \bowtie N)
\end{tikzcd}$$

For what regards the commutativity of the upper square we have the following chain of equalities

$$\begin{aligned}
\xi_M^{M\bowtie N} \circ (l_M\flat 1_M) &= \xi_M^{M\bowtie N} \circ (q\flat 1_M) \circ (i_M\flat 1_M) \\
&= \xi_M^{M+N} \circ (i_M\flat 1_M) \\
&= \chi_M
\end{aligned}$$

given by commutativity of diagrams (2.5) and (2.9).

For what regards the lower square, it can be shown to be commutative by direct calculations, using the explicit definition of the coproduct action given in (2.2). First of all we can precompose with the regular epimorphism $q\flat 1_M$: this shows that the required commutativity is equivalent to the equation

$$q \circ \chi_{M+N} \circ (1\flat i_M) = q \circ i_M \circ \xi_M^{M+N}. \tag{2.12}$$

Now we can take a word $s \in M + N$, an element $\overline{m} \in M$ and prove by induction on the length of s that the generator $s\overline{m}s^{-1} \in (M + N)\flat M$ is sent through the two maps in (2.12) to

$$q\left({}^{s}\overline{m}\right) = q(s\overline{m}s^{-1}). \tag{2.13}$$

Let us first show this equality for s with length 0, that is the empty word: we have that ${}^{s}\overline{m} = \overline{m} = s\overline{m}s^{-1}$ and hence (2.13). For the inductive step we are going to use the equality $q\left({}^{n}\overline{m}\right) = q(n\overline{m}n^{-1})$ coming from the definition of the Peiffer product. Suppose that (2.13) holds for words s with length $l(s) < k$. Given s with length k we can write it as $s = xs'$ with $x = m \in M$ or $x = n \in N$ and $l(s') = k - 1$: now we have the chain of equalities

$$\begin{aligned}
q\left({}^{s}\overline{m}\right) = q\left({}^{xs'}\overline{m}\right) &= q\left({}^{x}\left({}^{s'}\overline{m}\right)\right) \\
&= q\left(x\left({}^{s'}\overline{m}\right)x^{-1}\right) = q(x)q\left({}^{s'}\overline{m}\right)q\left(x^{-1}\right) \\
&= q(x)q\left(s'\overline{m}s'^{-1}\right)q\left(x^{-1}\right) \\
&= q\left(xs'\overline{m}s'^{-1}x^{-1}\right) \\
&= q\left(s\overline{m}s^{-1}\right).
\end{aligned}$$

Hence $(M \xrightarrow{l_M} M \bowtie N, \xi_M^{M\bowtie N})$ and $(N \xrightarrow{l_N} M \bowtie N, \xi_N^{M\bowtie N})$ are crossed modules. $\square$

Furthermore we know that the actions ξ_N^M and ξ_M^N are in turn induced by $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ through the maps l_M and l_N, that is

$$\begin{tikzcd} M\flat N \arrow[r, "l_M\flat 1_N"] \arrow[dr, "\xi_N^M"'] & (M\bowtie N)\flat N \arrow[d, "\xi_N^{M\bowtie N}"] \\ & N \end{tikzcd} \qquad \begin{tikzcd} N\flat M \arrow[r, "l_N\flat 1_M"] \arrow[dr, "\xi_M^N"'] & (M\bowtie N)\flat M \arrow[d, "\xi_M^{M\bowtie N}"] \\ & M \end{tikzcd}$$

commute. This can be proved by using the commutativity of diagrams (2.1), (2.9) and (2.11).

Proposition 2.1.10 (Remark 2.16 in [15]). *Two actions as above are compatible if and only if there exists a group L with two crossed module structures $(M \xrightarrow{\mu} L, \psi_M)$ and $(N \xrightarrow{\nu} L, \psi_N)$ such that the action of M on N and the action of N on M are induced from L and its actions.*

Proof. ($\Leftarrow$) Let us first show that the actions $\xi_N^M := \psi_N \circ (\mu\flat 1_N)$ and $\xi_M^N := \psi_M \circ (\nu\flat 1_M)$ are compatible. To see that they are actually actions it suffices to use Remark 1.2.11. In order to show (2.6) (we will show only one of the two equalities since the proof of the other follows the same steps) we are going to use the commutative diagrams induced from the crossed module structures involving L, that is

$$\begin{tikzcd} M\flat M \arrow[r, "\chi_M"] \arrow[d, "\mu\flat 1_M"'] \arrow[dr, phantom, "*_1"] & M \arrow[d, equal] \\ L\flat M \arrow[r, "\psi_M"] \arrow[d, "1_L\flat\mu"'] \arrow[dr, phantom, "*_2"] & M \arrow[d, "\mu"] \\ L\flat L \arrow[r, "\chi_L"'] & L \end{tikzcd} \qquad \begin{tikzcd} N\flat N \arrow[r, "\chi_N"] \arrow[d, "\nu\flat 1_N"'] \arrow[dr, phantom, "*_3"] & N \arrow[d, equal] \\ L\flat N \arrow[r, "\psi_N"] \arrow[d, "1_L\flat\nu"'] \arrow[dr, phantom, "*_4"] & N \arrow[d, "\nu"] \\ L\flat L \arrow[r, "\chi_L"'] & L \end{tikzcd}$$

Therefore we have the following chain of equalities

$$\begin{aligned} {}^{({}^m n)}m' &= {}^{\nu({}^{\mu(m)}n)}m' = {}^{\mu(m)\nu(n)\mu(m^{-1})}m' \\ &= {}^{\mu(m)\nu(n)}\left({}^{\mu(m^{-1})}m'\right) = {}^{\mu(m)\nu(n)}\left({}^{m^{-1}}m'\right) \\ &= {}^{\mu(m)}\left({}^{\nu(n)}\left({}^{m^{-1}}m'\right)\right) = {}^{\mu(m)}\left({}^{n}\left({}^{m^{-1}}m'\right)\right) \\ &= {}^{\mu(m)}\left({}^{nm^{-1}}m'\right) = {}^{m}\left({}^{nm^{-1}}m'\right) \\ &= {}^{mnm^{-1}}m' \end{aligned}$$

($\Rightarrow$) This implication is given by Proposition 2.1.9. $\square$

2.2 Compatible actions of Lie algebras

2.2.1 Preliminaries for Lie algebras

We start by recalling some well-known facts that we are going to use in the following. In the meantime we use this subsection to fix some notation.

Definition 2.2.1. Let R be a commutative ring and let M be an R-module. We say that M is a *Lie algebra over* R if it is endowed with a binary operation

$$[-,-]\colon M \times M \to M$$

called *Lie bracket*, such that the following conditions hold:

1) $[ax+by,z] = a[x,z]+b[y,z]$ and $[x,ay+bz] = a[x,y]+b[x,z]$ (R-bilinearity);

2) $[x,x]=0$ and $[x,y]+[y,x]=0$ (alternating);

3) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ (Jacobi identity).

Remark 2.2.2. We recall that the above definition is redundant: notice that the two conditions in 1) are equivalent under the condition 2), so it suffices to check just one of them. Moreover, $[x,x]=0$ always implies $[x,y]+[y,x]=0$, and the converse is true whenever the multiplication by 2 is injective in M (that is, M is 2-torsion free). Furthermore, the equation $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ is equivalent to $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ thanks to 2).

Definition 2.2.3. Let M and L be R-Lie algebras. A morphism of R-Lie algebras $f\colon M\to L$ is a morphism of R-modules such that

$$f([x,y]) = [f(x),f(y)].$$

This defines the category $\mathbf{Lie}_R$ of R-Lie algebras and R-Lie algebra morphisms.

Remark 2.2.4. There is an obvious forgetful functor $U\colon \mathbf{Lie}_R \to \mathbf{Set}$ and it has a left adjoint $F\colon \mathbf{Set}\to\mathbf{Lie}_R$: this functor builds the free R-Lie algebra on a given set X by means of the following well-known procedure.

i) First of all we build the free magma on X, denoted $Mag(X)$, writing $[-,-]\colon Mag(X)\times Mag(X)\to Mag(X)$ for the binary operation: this means that an element of $Mag(X)$ is given by a word with square brackets, as for instance "$[[x_1,[x_2,x_3]],x_4]$".

ii) Then we take the free R-module on it $R[Mag(X)]$ and we extend the product by defining
$$\left[\sum_{i=0}^{n} r_i x_i, \sum_{j=0}^{m} s_j y_j\right] = \sum_{i=0}^{n}\sum_{j=0}^{m} r_i s_j [x_i,y_j].$$
This product gives to $R[Mag(X)]$ the structure of a R-algebra.

iii) Finally consider the ideal I generated by the symbols

- $[x, x]$,
- $[x, y] + [y, x]$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$,

with $x, y, z \in X$ and define $F(X) := R[Mag(X)]/I$.

Remark 2.2.5. Let M and N be two R-Lie algebras. Their coproduct $M + N$ is the R-Lie algebra given by $F(U(M) \sqcup U(N))/J$ where J is the ideal generated by the identities coming separately from M and from N: this means that it is a quotient of the free algebra on the disjoint union of the underlying sets of the two algebras.

Definition 2.2.6. Given a word $s \in M + N$, we say that it is *well nested* if it is a simple bracket—$[x_1, x_2]$ where x_1, $x_2 \in M \cup N$—or if it is obtained by taking the bracket of an element with a well-nested word. Equivalently this means that s does not contain a bracket between two brackets. The *height* of a well nested word is simply the number of pair of brackets appearing in it. Given a word $s \in M + N$, any simple bracket $[x_1, x_2]$ is contained in a maximal well nested subword of s and we say that the *relative height* of x_1 and of x_2 in s is the height of this subword.

Since we couldn't find a clear reference for the following lemma, we prove it here, even if we think it is a well-known result.

Lemma 2.2.7. *Every element in $M + N$ can be written as a linear combination of elements of the form*

$$[x_k, [x_{k-1}, [\dots, [x_3, [x_2, x_1]] \cdots]]] \tag{2.14}$$

with $x_i \in M$ or $x_i \in N$.

Proof. Consider a word s which has n pairs of brackets and apply the following algorithm:

1) Choose a subword t of s which is well nested: this always exists, because we can take one of the innermost (and hence simple) brackets.

2) If $t = s$ go to 3). Otherwise t is contained in a subword of the form

$$[t, [w_1, w_2]] \qquad \text{or} \qquad [[w_1, w_2], t].$$

with w_1 and w_2 subwords of s. Use the Jacobi identity to break $[t, [w_1, w_2]]$ into $[w_1, [t, w_2]] + [[t, w_1], w_2]$ (and similarly in the other case). Now s can be seen as the sum of the two words in which we substituted $[t, [w_1, w_2]]$ with the two summands resulted from the application of the Jacobi identity. For each of these words repeat the step 2) choosing them as new s and the maximal well nested word containing the old t as new t.

3) Since s is now well nested it suffices to apply the alternating property until all the brackets have a simple element on the left. This has only the effect of possibly changing the sign in front of the word.

The reason why this algorithm works is simply because at each application of 2) we obtain one of the following:

i) the relative height of t increases by at least 1: this will eventually lead to the relative height reaching n, which means that the word in question is well nested;

ii) the complexity of the bracket near t decreases: in one application it goes from $[w_1, w_2]$ to both w_1 and w_2 which individually contains less brackets than $[w_1, w_2]$. This will eventually lead to w_1 or w_2 being a single element and hence to i) at the next iteration. □

Remark 2.2.8. Notice that for each word $s \in M + N$ and for each letter x in it, we can decompose s as a linear combination of words of the form (2.14) in such a way that each word in the decomposition has $x_1 = x$. This is possible because, by using the Jacobi identity, we can first decompose s as a linear combination of words in which x appears in a simple bracket. Then we can use the algorithm described in Lemma 2.2.7 choosing as starting t the simple bracket containing x.

Remark 2.2.9. By examining the general definition of the functor $\flat$ and of its monad structure when restricted to $\mathbf{Lie}_R$, we see that an element of the R-Lie algebra $P\flat M$ is an element of $P+M$ such that each of its monomials contains an element from M: indeed the arrow $\binom{1}{0}$ takes a linear combination of "words" and sends it to the linear combination of "words" obtained by substituting every element from M with 0 (therefore only monomials with an element in M go to zero). Notice that $(P\flat M, k_{P,M}) = Ker(Coker(i_M : M \to P + M))$ and therefore $P\flat M$ is the ideal generated by M in $P + M$.

In particular $\eta^P : 1_{\mathbf{Lie}_R} \to P\flat(-)$ is given by

$$\eta^P_M : M \to P\flat M : m \mapsto m$$

and $\mu^P : P\flat(P\flat(-)) \to P\flat(-)$ has components

$$\mu^P_M : P\flat(P\flat M) \to P\flat M$$

which map the two different brackets in $P\flat(P\flat M)$ to the one bracket in $P\flat M$.

Furthermore if $f : A \to B$ is a morphism, then $P\flat(f) = 1_P\flat f : P\flat A \to P\flat B$ is given by sending each linear combination of words in $P\flat A$ into the one obtained by substituting every element $a \in A$ with its image $f(a) \in B$.

2.2.2 Compatible actions of Lie algebras

We start by recalling the equivalent definitions of action and internal action in $\mathbf{Lie}_R$.

Definition 2.2.10. Let M and P be R-Lie algebras. An *action of P on M* is given by a R-bilinear map $\psi : P \times M \to M$ with $(p, m) \mapsto {}^p m = \psi(p, m)$, such that for each $p, p' \in P$ and $m, m' \in M$ we have

- ${}^{[p,p']}m = {}^p({}^{p'}m) - {}^{p'}({}^p m)$ and

- ${}^{p}[m, m'] = [{}^{p}m, m'] + [m, {}^{p}m']$

Remark 2.2.11. Recalling Definition 1.2.4 we see that an internal action is a morphism of R-Lie algebras $\xi \colon P\flat M \to M$ such that

$$\xi(m) = m, \qquad\qquad \xi(\mu^P_M(s)) = \xi((1_P\flat\xi)(s))$$

for all $m \in M$ and $s \in P\flat(P\flat M)$. For example if $s = \{p, [m, p']\}$, then $\mu^P_M(s) = [p, [m, p']]$ and $(1_P\flat\xi)(s) = [p, \xi([m, p'])]$, so we want that

$$\xi([p, [m, p']]) = \xi([p, \xi([m, p'])]).$$

This means that the image of the action on a complicated word can be obtained by taking the image of the most internal bracket and iterating this process until there are no brackets left. We will call this property *decomposability.*

Remark 2.2.12. It is easy to notice that there is an equivalence between actions and internal actions. In particular this correspondence sends an internal action $\xi \colon P\flat M \to M$ to the action $\psi \colon P \times M \to M$ defined via $\psi(p, m) := \xi([p, m])$, and conversely it sends an action $\psi \colon P \times M \to M$ to the internal action $\xi \colon P\flat M \to M$ defined via

$$\begin{cases} \xi(m) := m, \\ \xi([p, m]) := \psi(p, m). \end{cases}$$

The behavior of ξ on more complex elements is uniquely determined by the hypothesis of decomposability. From now one we are going to use actions or internal actions equivalently, depending on which is the more convenient approach in each specific case.

Example 2.2.13. Given an R-Lie algebra M, the conjugation action $\chi_M \colon M\flat M \to M$ corresponds to the map $M \times M \to M$ given by $(m, m') \mapsto [m, m']$.

Definition 2.2.14. Consider an action $\xi^N_M \colon N\flat M \to M$ and the conjugation $\chi_M \colon M\flat M \to M$. We can always construct an action $\xi^{M+N}_M \colon (M + N)\flat M \to M$ of the coproduct $M + N$ on M such that it extends both ξ^N_M and χ_M. It is defined via

- $\overline{m} \longmapsto \overline{m}$,
- $[m, \overline{m}] \longmapsto [m, \overline{m}]$,
- $[n, \overline{m}] \longmapsto \xi^N_M([n, \overline{m}])$.

where $\overline{m} \in M$ and $m, n \in M + N$. Notice that the images of those three types of elements are fixed by the fact that ξ^{M+N}_M is an action and by the fact that it extends both the conjugation of M and the action ξ^N_M. Furthermore, ξ^{M+N}_M is uniquely determined by these requirements since we can easily deduce its behavior on more complex elements by using the Jacobi identity and the decomposability of the action ξ^{M+N}_M. For example we can show that

$$\xi^{M+N}_M([[n, m], \overline{m}]) = [\xi^N_M([n, m]), \overline{m}]$$

by the following chain of equalities

$$\begin{aligned}
\xi_M^{M+N}([[n,m],\overline{m}]) &= \xi_M^{M+N}(-[[m,\overline{m}],n]-[[\overline{m},n],m]) \\
&= \xi_M^{M+N}([n,[m,\overline{m}]]-[m,[n,\overline{m}]]) \\
&= \xi_M^{M+N}([n,[m,\overline{m}]])-\xi_M^{M+N}([m,[n,\overline{m}]]) \\
&= \xi_M^{M+N}([n,\xi_M^{M+N}([m,\overline{m}])])-\xi_M^{M+N}([m,\xi_M^{M+N}([n,\overline{m}])]) \\
&= \xi_M^{N}([n,\chi_M([m,\overline{m}])])-\chi_M([m,\xi_M^{N}([n,\overline{m}])]) \\
&= \xi_M^{N}([n,[m,\overline{m}]])-[m,\xi_M^{N}([n,\overline{m}])] \\
&= [\xi_M^{N}([n,m]),\overline{m}]+[m,\xi_M^{N}([n,\overline{m}])]-[m,\xi_M^{N}([n,\overline{m}])] \\
&= [\xi_M^{N}([n,m]),\overline{m}]
\end{aligned}$$

Definition 2.2.15. Given two R-Lie algebras M and N, we say that two actions

$$\psi_N^M\colon M\times N\to N \qquad\qquad \psi_M^N\colon N\times M\to M$$

are *compatible* (see [37]) if the following equations hold

$$\begin{cases} {}^{({}^m n)}m' = [m', {}^n m]\,, \\ {}^{({}^n m)}n' = [n', {}^m n]\,. \end{cases} \tag{2.15}$$

Remark 2.2.16. The link between this definition and the compatibility condition in the case of groups is given by the following general idea: *the element ${}^m n$ (resp. ${}^n m$) has to act as the formal conjugation of m and n in the coproduct would do.* In particular in **Grp** this amounts to requiring the equalities

$$\begin{cases} {}^{({}^n m)}n' = {}^{(nmn^{-1})}n', \\ {}^{({}^m n)}m' = {}^{(mnm^{-1})}m', \end{cases} \tag{2.16}$$

(see [15] for further details) whose internal translation is given by

$$\begin{cases} \xi_N^M\left(\xi_M^N(x)n'\xi_M^N(x)^{-1}\right) = \xi_N^{M+N}\left(xn'x^{-1}\right), \\ \xi_M^N\left(\xi_N^M(y)m'\xi_N^M(y)^{-1}\right) = \xi_M^{M+N}\left(ym'y^{-1}\right), \end{cases}$$

with $x = nmn^{-1}$ and $y = mnm^{-1}$. Notice that these can also be seen as the commutativity of the diagrams

$$\begin{tikzcd}
(N\flat M)\flat N \arrow[r, "k_{N,M}\flat 1_N"] \arrow[d, "\xi_M^N\flat 1_N"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\
M\flat N \arrow[r, "\xi_N^M"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
(M\flat N)\flat M \arrow[r, "k_{M,N}\flat 1_M"] \arrow[d, "\xi_N^M\flat 1_M"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\
N\flat M \arrow[r, "\xi_M^N"'] & M
\end{tikzcd} \tag{2.17}$$

Besides (2.16), we should also require the equalities

$$\begin{cases} {}^{({}^n m)}m' = {}^{(nmn^{-1})}m', \\ {}^{({}^m n)}n' = {}^{(mnm^{-1})}n', \end{cases}$$

or their internal version

$$\begin{cases} \chi_M\left(\xi_M^N(x)m'\xi_M^N(x)^{-1}\right) = \xi_M^{M+N}\left(xn'x^{-1}\right), \\ \chi_N\left(\xi_N^M(y)n'\xi_N^M(y)^{-1}\right) = \xi_N^{M+N}\left(yn'y^{-1}\right), \end{cases}$$

coming from the commutativity of the diagrams

$$\begin{tikzcd} (N\flat M)\flat M \arrow[r, "k_{N,M}\flat 1_M"] \arrow[d, "\xi_M^N\flat 1_M"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\ M\flat M \arrow[r, "\chi_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} (M\flat N)\flat N \arrow[r, "k_{M,N}\flat 1_N"] \arrow[d, "\xi_N^M\flat 1_N"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\ N\flat N \arrow[r, "\chi_N"'] & N \end{tikzcd}$$

However, as one can easily check, these always hold for every pair of actions.

The same idea applied in $\mathbf{Lie}_R$ leads to the equations

$$\begin{cases} {}^{({}^n m)}n' = {}^{[n,m]}n', \\ {}^{({}^m n)}m' = {}^{[m,n]}m', \end{cases}$$

whose internal version is given by the system

$$\begin{cases} \xi_N^M\left(\left[\xi_M^N\left([n,m]\right), n'\right]\right) = \xi_N^{M+N}\left(\left[[n,m], n'\right]\right), \\ \xi_M^N\left(\left[\xi_N^M\left([m,n]\right), m'\right]\right) = \xi_M^{M+N}\left(\left[[m,n], m'\right]\right), \end{cases}$$

or again by the commutativity of (2.17). By using the decomposability of the coproduct actions one can show that these requirements are the same as (2.15) in Definition 2.2.15: indeed we have the chains of equalities

$$\begin{aligned} {}^{[n,m]}n' &= \xi_N^{M+N}\left(\left[[n,m], n'\right]\right) = \left[\xi_N^M([n,m]), n'\right] = \left[n', \xi_N^M([m,n])\right] = \left[n', {}^m n\right], \\ {}^{[m,n]}m' &= \xi_M^{M+N}\left(\left[[m,n], m'\right]\right) = \left[\xi_M^N([m,n]), m'\right] = \left[m', \xi_M^N([n,m])\right] = \left[m', {}^n m\right]. \end{aligned}$$

Furthermore, in the case of $\mathbf{Lie}_R$ the other two equations

$$\begin{cases} {}^{({}^n m)}m' = {}^{[n,m]}m', \\ {}^{({}^m n)}n' = {}^{[m,n]}n', \end{cases}$$

are automatically satisfied: indeed by looking at their internal version

$$\begin{cases} \left[\xi_M^N\left([n,m]\right), m'\right] = \xi_M^{M+N}\left(\left[[n,m], m'\right]\right), \\ \left[\xi_N^M\left([m,n]\right), n'\right] = \xi_N^{M+N}\left(\left[[m,n], n'\right]\right), \end{cases}$$

one can see that they are precisely a consequence of the decomposability of the coproduct actions shown in Definition 2.2.14.

Definition 2.2.17. A *crossed module of R-Lie algebras* is given by (M, P, ∂, ψ) where M and P are R-Lie algebras, $\partial\colon M \to P$ is a morphism between them, and $\psi\colon P \times M \to M$ is an action such that the diagram

$$\begin{tikzcd} M \times M \arrow[r, "\chi_M"] \arrow[d, "\partial \times 1_M"'] & M \arrow[d, equal] \\ P \times M \arrow[r, "\psi"] \arrow[d, "1_P \times \partial"'] & M \arrow[d, "\partial"] \\ P \times P \arrow[r, "\chi_P"'] & P \end{tikzcd}$$

commutes. That is, such that $[m, m'] = {}^{\partial(m)}m'$ and $\partial({}^{p}m) = [p, \partial(m)]$.

Again by using the equivalence between actions and internal actions one can see that crossed modules of R-Lie algebras are the same as internal crossed modules in $\mathbf{Lie}_R$ according to Definition 1.4.6 (see [67] for further details).

Proposition 2.2.18. *Let M and N be R-Lie algebras. Consider two crossed module structures $(M \xrightarrow{\mu} P, \psi_M)$ and $(N \xrightarrow{\nu} P, \psi_N)$*

$$\begin{tikzcd} & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

and construct two induced actions ψ_N^M and ψ_M^N as follows:

$$\begin{tikzcd} M \times N \arrow[rr, "\psi_N^M"] \arrow[dr, "\mu \times 1_N"'] & & N \\ & P \times N \arrow[ur, "\psi_M"'] & \end{tikzcd} \qquad \begin{tikzcd} N \times M \arrow[rr, "\psi_M^N"] \arrow[dr, "\nu \times 1_M"'] & & M \\ & P \times M \arrow[ur, "\psi_N"'] & \end{tikzcd}$$

These two actions are compatible.

Proof. We need to prove the equation ${}^{({}^{n}m)}n' = [n', {}^{m}n]$ by using the crossed module conditions $[m, m'] = {}^{\mu(m)}m'$ and $\mu({}^{p}m) = [p, \mu(m)]$, and $[n, n'] = {}^{\nu(n)}n'$ and $\nu({}^{p}n) = [p, \nu(n)]$. We have the chain of equalities

$$\begin{aligned} {}^{({}^{n}m)}n' &= {}^{({}^{\nu(n)}m)}n' = {}^{\mu({}^{\nu(n)}m)}n' = {}^{[\nu(n), \mu(m)]}n' \\ &= {}^{-[\mu(m), \nu(n)]}n' = {}^{-\nu({}^{\mu(m)}n)}n' = {}^{\nu(-{}^{\mu(m)}n)}n' \\ &= [-{}^{\mu(m)}n, n'] = [n', {}^{\mu(m)}n] = [n', {}^{m}n]. \end{aligned}$$

For the second equation, the reasoning is the same. $\square$

Imitating what has been done in the case of groups in [75, 42], we are able to define the Peiffer product of two Lie algebras acting on each other (this was firstly defined in [59]).

Definition 2.2.19. Given two Lie algebras M and N acting on each other, consider their coproduct $M+N$ and its ideal K, generated by the elements

$$({}^{n}m) - [n,m] \qquad \text{and} \qquad ({}^{m}n) - [m,n]\,,$$

for $m \in M$ and $n \in N$. We define the *Peiffer product* $M \bowtie N$ *of* M *and* N as the quotient

$$K \rightarrowtriangle M+N \xrightarrow{q_K} \frac{M+N}{K} =: M \bowtie N.$$

By repeating what we did in Remark 2.1.6 we can see that $M \bowtie N$ is the following coequaliser

$$\begin{tikzcd} (N\flat M)+(M\flat N) \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}"] \arrow[r, shift right, "\xi_M^N+\xi_N^M"'] & M+N \arrow[r, two heads, "q"] & M \bowtie N. \end{tikzcd}$$

Since K acts trivially on both M and N, we can define induced actions $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ of $M \bowtie N$ on M and N, that is such that the following diagrams commute

$$\begin{tikzcd} (M+N)\flat M \arrow[r, "q\flat 1_M"] \arrow[dr, "\xi_M^{M+N}"'] & (M\bowtie N)\flat M \arrow[d, "\xi_M^{M\bowtie N}"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} (M+N)\flat N \arrow[r, "q\flat 1_N"] \arrow[dr, "\xi_N^{M+N}"'] & (M\bowtie N)\flat N \arrow[d, "\xi_N^{M\bowtie N}"] \\ & M \end{tikzcd} \tag{2.18}$$

We can describe these actions of the Peiffer product through its universal property, but in order to do this, we need Lemma 1.1.38 and the Lie algebra version of Remark 2.1.7.

Lemma 1.1.38 implies that $q\flat 1_M$ is the coequaliser of $\binom{k_{N,M}}{k_{M,N}}\flat 1_M$ and $(\xi_M^N+\xi_N^M)\flat 1_M$ and that $q\flat 1_N$ is the coequaliser of $\binom{k_{N,M}}{k_{M,N}}\flat 1_N$ and $(\xi_M^N+\xi_N^M)\flat 1_N$. We want to use these universal properties to define induced actions $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ of $M \bowtie N$ on M and N as in the next two diagrams

$$\begin{tikzcd} ((N\flat M)+(M\flat N))\flat M \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}\flat 1_M"] \arrow[r, shift right, "(\xi_M^N+\xi_N^M)\flat 1_M"'] & (M+N)\flat M \arrow[r, "q\flat 1_M"] \arrow[dr, "\xi_M^{M+N}"'] & (M\bowtie N)\flat M \arrow[d, dotted, "\xi_M^{M\bowtie N}"] \\ & & M \end{tikzcd}$$

$$\begin{tikzcd} ((N\flat M)+(M\flat N))\flat N \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}\flat 1_N"] \arrow[r, shift right, "(\xi_M^N+\xi_N^M)\flat 1_N"'] & (M+N)\flat N \arrow[r, "q\flat 1_N"] \arrow[dr, "\xi_N^{M+N}"'] & (M\bowtie N)\flat N \arrow[d, dotted, "\xi_N^{M\bowtie N}"] \\ & & N \end{tikzcd}$$

In order to do so, we need the following result.

Proposition 2.2.20. *The action* ξ_M^{M+N} *coequalises* $\binom{k_{N,M}}{k_{M,N}}\flat 1_M$ *and* $(\xi_M^N + \xi_N^M)\flat 1_M$. *Similarly, the action* ξ_N^{M+N} *coequalises* $\binom{k_{N,M}}{k_{M,N}}\flat 1_N$ *and* $(\xi_M^N + \xi_N^M)\flat 1_N$.

Proof. We need to show that

$$\xi_M^{M+N}\left(\left(\left(\begin{pmatrix} k_{N,M} \\ k_{M,N} \end{pmatrix}\flat 1_M\right)(s)\right) = \xi_M^{M+N}\left(\left(\left(\xi_M^N + \xi_N^M\right)\flat 1_M\right)(s)\right) \tag{2.19}$$

holds for each element $s \in ((N\flat M) + (M\flat N))\flat M$. By Lemma 2.2.7 and Remark 2.2.8, it suffices to check this for the generators of the form $s = [x_k, [\ldots, [x_1, \overline{m}]\cdots]]$ with $x_i \in N\flat M$ or $x_i \in M\flat N$ and $\overline{m} \in M$. This means that to prove (2.19) it suffices to show the equality

$$\xi_M^{M+N}\left([x_k, [\ldots, [x_1, \overline{m}]\cdots]]\right) = \xi_M^{M+N}\left([\epsilon(x_k), [\ldots, [\epsilon(x_1), \overline{m}]\cdots]]\right) \tag{2.20}$$

where

$$\epsilon(x_i) = \begin{cases} \xi_N^M(x_i) & \text{if } x_i \in M\flat N, \\ \xi_M^N(x_i) & \text{if } x_i \in N\flat M. \end{cases}$$

In order to see this, we can use the decomposability of the action ξ_M^{M+N} on both sides of (2.20) obtaining that the one on the left becomes

$$\xi_M^{M+N}\left([x_k, \xi_M^{M+N}\left([\ldots, \xi_M^{M+N}\left([x_1, \overline{m}]\right)\cdots]\right)]\right)$$

whereas the one on the right becomes

$$\xi_M^{M+N}\left([\epsilon(x_k), \xi_M^{M+N}\left([\ldots, \xi_M^{M+N}\left([\epsilon(x_1), \overline{m}]\right)\cdots]\right)]\right).$$

This means that it suffices to show

$$\xi_M^{M+N}\left([x, \overline{m}]\right) = \xi_M^{M+N}\left([\epsilon(x), \overline{m}]\right)$$

for $x \in M\flat N$ or $x \in N\flat M$, but this is given again by decomposability of ξ_M^{M+N}.

Finally, we repeat the whole reasoning with ξ_N^{M+N}. □

Proposition 2.2.21. *We have two crossed module structures*

$$(M \xrightarrow{l_M} M \bowtie N, \xi_M^{M\bowtie N}) \qquad\qquad (N \xrightarrow{l_N} M \bowtie N, \xi_N^{M\bowtie N})$$

where the actions of the Peiffer product are induced as above and the morphisms l_M *and* l_N *are defined through the compositions*

$$\begin{tikzcd} M \arrow[dr, tail, "i_M"] \arrow[ddr, bend right, "l_M"'] & & N \arrow[dl, tail, "i_N"'] \arrow[ddl, bend left, "l_N"] \\ & M + N \arrow[d, two heads, "q"] & \\ & M \bowtie N & \end{tikzcd} \tag{2.21}$$

Proof. We will prove the claim only for $\xi_M^{M \bowtie N}$, since the proof in the other case uses the same strategy. We need to show the commutativity of the following squares

$$\begin{tikzcd}
M \flat M \arrow[r, "\chi_M"] \arrow[d, "l_M \flat 1_M"'] & M \arrow[d, equal] \\
(M \bowtie N) \flat M \arrow[r, "\xi_M^{M \bowtie N}"] \arrow[d, "1_{M \bowtie N} \flat l_M"'] & M \arrow[d, "l_M"] \\
(M \bowtie N) \flat (M \bowtie N) \arrow[r, "\chi_{M \bowtie N}"'] & (M \bowtie N)
\end{tikzcd}$$

For what concerns the commutativity of the upper square, we have the chain of equalities

$$\begin{aligned}
\xi_M^{M \bowtie N} \circ (l_M \flat 1_M) &= \xi_M^{M \bowtie N} \circ (q_K \flat 1_M) \circ (i_M \flat 1_M) \\
&= \xi_M^{M+N} \circ (i_M \flat 1_M) \\
&= \chi_M
\end{aligned}$$

given by the definition of the coproduct action and of the Peiffer product action.

As for the lower square, we can precompose with the regular epimorphism $q \flat 1_M$: this shows that the required commutativity is equivalent to the equation

$$q \circ \chi_{M+N} \circ (1 \flat i_M) = q \circ i_M \circ \xi_M^{M+N}.$$

Consider a generator $[s_k, [\ldots, [s_1, m] \cdots]] \in (M + N) \flat M$ with $m \in M$ and $s_j \in M + N$ or $s_j \in M$ (see Lemma 2.2.7 and Remark 2.2.8): we want to show that

$$q\left(\xi_M^{M+N}\left([s_k, [\ldots, [s_1, m] \cdots]]\right)\right) = q\left([s_k, [\ldots, [s_1, m] \cdots]]\right). \tag{2.22}$$

We are going to prove this by induction on k:

- If $k = 0$ we trivially have
$$q\left(\xi_M^{M+N}(m)\right) = q(m);$$

- Suppose that (2.22) holds for $j < k$. Then by using the decomposability of ξ_M^{M+N} and the equality
$$q([s, m]) = q(k_{N,M}([s, m])) = q\left(\xi_M^N([s, m])\right)$$
induced from the definition of the Peiffer product as coequaliser, we have the chain

of equalities

$$\begin{aligned}
q\left(\xi_M^{M+N}([s_k,[\ldots,[s_1,m]\cdots]])\right) &= q\left(\xi_M^{M+N}\left(\left[s_k,\left[\ldots,\xi_M^{M+N}([s_1,m])\cdots\right]\right]\right)\right)\\
&= q\left(\left[s_k,\left[\ldots,\xi_M^{M+N}([s_1,m])\cdots\right]\right]\right)\\
&= q\left(\left[s_k,\left[\ldots,\xi_M^{N}([s_1,m])\cdots\right]\right]\right)\\
&= \left[q(s_k),\left[\ldots,q\left(\xi_M^{N}([s_1,m])\right)\cdots\right]\right]\\
&= \left[q(s_k),\left[\ldots,q([s_1,m])\cdots\right]\right]\\
&= q\left([s_k,[\ldots,[s_1,m]\cdots]]\right).
\end{aligned}$$

Notice that the induction hypothesis is used for the equality on the second line, considering $\xi_M^{M+N}([s_1,m])$ as $m' \in M$. □

Furthermore we know that the actions ξ_N^M and ξ_M^N are in turn induced by $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ through the morphisms l_M and l_N, that is

$$\begin{tikzcd}
M\flat N \arrow[r, "l_M\flat 1_N"] \arrow[dr, "\xi_N^M"'] & (M\bowtie N)\flat N \arrow[d, "\xi_N^{M\bowtie N}"] \\
& N
\end{tikzcd}
\qquad
\begin{tikzcd}
N\flat M \arrow[r, "l_N\flat 1_M"] \arrow[dr, "\xi_M^N"'] & (M\bowtie N)\flat M \arrow[d, "\xi_M^{M\bowtie N}"] \\
& M
\end{tikzcd}$$

commute. This can be proved by using the definition of the coproduct actions and the commutativity of diagrams (2.18) and (2.21).

Putting together Proposition 2.2.18 and Proposition 2.2.21, we find the following characterisation of compatible actions.

Theorem 2.2.22. *Consider two Lie algebras M and N acting on each other. These actions are compatible if and only if there exists a Lie algebra L and two crossed module structures $(M \xrightarrow{\mu} L, \psi_M)$ and $(N \xrightarrow{\nu} L, \psi_N)$ such that the action of M on N and the action of N on M are induced from L and its actions, through μ and ν.* □

2.2.3 The Peiffer product as a coproduct

As an additional result we want to show that the coproduct in $\mathbf{XMod}_L(\mathbf{Lie}_R)$ can be obtained through the Peiffer product: this coproduct has already been characterised in a different way in [24] by using semi-direct products instead of the Peiffer product, but this approach generalises the one used for $\mathbf{XMod}_L(\mathbf{Grp})$ in [13]. Consequently, we also obtain that the Peiffer product defined above (and hence the one from [59]) coincides with the one defined in [30] when restricted to $\mathbf{Lie}_R$.

Definition 2.2.23. Given a pair of actions of L respectively on M and on N, we can define an action of L on the coproduct $M+N$ by imposing the equalities

$${}^l s := \begin{cases} {}^l m & \text{if } s = m \in M \\ {}^l n & \text{if } s = n \in N \\ \left[{}^l s_1, s_2\right] + \left[s_1, {}^l s_2\right], & \text{if } s = [s_1,s_2] \in M+N \end{cases}$$

and by extending the definition by linearity. In order to see that this is well defined it suffices to use Lemma 2.2.7 and induction on the length of $s \in M + N$.

Proposition 2.2.24. *The action ψ_{M+N} restricts to an action on K. Consequently it induces an action $\psi_{M \bowtie N}$ of L on the quotient $M \bowtie N$.*

Proof. Let us show that ${}^{l}k$ lies in K (that is $q\left({}^{l}k\right) = 0$) as soon as $k \in K$. In order to do this, it suffices to prove it for the generators

$$({}^{n}m) - [n, m] \qquad \text{and} \qquad ({}^{m}n) - [m, n]\,,$$

We prove it for the first one since the reasoning can be repeated for the other one:

$$\begin{aligned}
q\left({}^{l}\left({}^{n}m - [n,m]\right)\right) &= q\left({}^{l}\left({}^{n}m\right)\right) - q\left({}^{l}\left([n,m]\right)\right)\\
&= q\left({}^{[l,\nu(n)]}m\right) + q\left({}^{\nu(n)}\left({}^{l}m\right)\right) - q\left(\left[{}^{l}n, m\right]\right) - q\left(\left[n, {}^{l}m\right]\right)\\
&= q\left({}^{\nu({}^{l}n)}m\right) + q\left(\left[n, {}^{l}m\right]\right) - q\left(\left[{}^{l}n, m\right]\right) - q\left(\left[n, {}^{l}m\right]\right)\\
&= q\left({}^{({}^{l}n)}m\right) - q\left(\left[{}^{l}n, m\right]\right)\\
&= q\left({}^{({}^{l}n)}m - \left[{}^{l}n, m\right]\right) = 0
\end{aligned}$$

For the second part of the claim it suffices to apply Theorem 5.5 in [69] and use the fact that, as shown in [64], $\mathbf{Lie}_R$ is a strongly protomodular category in the sense of [7]. □

Proposition 2.2.25. *If in the previous situation the actions on M and N are part of crossed module structures $(M \xrightarrow{\mu} L, \psi_M)$ and $(N \xrightarrow{\nu} L, \psi_N)$, then also the induced action on the Peiffer product is part of a crossed module structure*

$$(M \bowtie N \xrightarrow{\left|\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right|} L, \psi_{M \bowtie N}).$$

Proof. Since $q\colon M + N \to M \bowtie N$ is an epimorphism, it suffices to show that for each $s, s' \in M + N$ and for each $l \in L$ the equalities

$$q\left(\begin{pmatrix}\mu\\ \nu\end{pmatrix}\left({}^{l}s\right)\right) = q\left(\left[l, \begin{pmatrix}\mu\\ \nu\end{pmatrix}(s)\right]\right) \qquad\qquad q\left({}^{\left(\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right)(s')}(s)\right) = q\left([s', s]\right)$$

hold.

We are going to show them only in the case in which $s = [m, n]$ and $s' = [m', n']$, but the reasoning easily generalises to give the induction step needed for a complete proof by induction on the complexity of s and s'. Notice that we already have the equalities

$$\begin{aligned}
\binom{\mu}{\nu}\left({}^{l}[m,n]\right) &= \binom{\mu}{\nu}\left(\left[{}^{l}m,n\right]+\left[m,{}^{l}n\right]\right)\\
&= \left[\mu\left({}^{l}m\right),\nu(n)\right]+\left[\mu(m),\nu\left({}^{l}n\right)\right]\\
&= \left[\left[l,\mu(m)\right],\nu(n)\right]+\left[\mu(m),\left[l,\nu(n)\right]\right]\\
&= \left[l,\left[\mu(m),\nu(n)\right]\right]\\
&= \left[l,\binom{\mu}{\nu}\left([m,n]\right)\right],
\end{aligned}$$

hence by applying q to both sides we obtain the first equation. As for the second one we have

$$\begin{aligned}
q\left({}^{\binom{\mu}{\nu}([m',n'])}([m,n])\right) &= q\left({}^{\mu(m')}\left({}^{\nu(n')}[m,n]\right)-{}^{\nu(n')}\left({}^{\mu(m')}[m,n]\right)\right)\\
&= q\left(\left[m,\left[{}^{m'}n',n\right]\right]-\left[n,\left[m,{}^{n'}m'\right]\right]\right)\\
&= \left[q\left(m\right),\left[q\left({}^{m'}n'\right),q\left(n\right)\right]\right]-\left[q\left(n\right),\left[q\left(m\right),q\left({}^{n'}m'\right)\right]\right]\\
&= \left[q\left(m\right),\left[q([m',n']),q\left(n\right)\right]\right]-\left[q\left(n\right),\left[q\left(m\right),q([n',m'])\right]\right]\\
&= q\left(\left[[m',n'],[m,n]\right]\right).
\end{aligned}$$

□

Proposition 2.2.26. *Given a pair of L-crossed modules*

$$(M \xrightarrow{\mu} L, \psi_M) \qquad \text{and} \qquad (N \xrightarrow{\nu} L, \psi_N),$$

their coproduct in $\mathbf{XMod}_L(\mathbf{Lie}_R)$ is given by $(M \bowtie N \xrightarrow{\left|\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right|} L, \psi_{M\bowtie N})$.

Proof. Suppose we have a crossed module $(Z \xrightarrow{z} L, \psi_Z)$ with two morphisms $(z_M, 1_L)$ and $(z_N, 1_L)$ as in the following diagram

$$\begin{tikzcd}
(M \xrightarrow{\mu} L, \psi_M) \arrow[dr, "(l_M{,}1_L)"] \arrow[ddr, "(z_M{,}1_L)"'] & & (N \xrightarrow{\nu} L, \psi_N) \arrow[dl, "(l_N{,}1_L)"'] \arrow[ddl, "(z_N{,}1_L)"] \\
& (M \bowtie N \xrightarrow{\left|\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right|} L, \psi_{M\bowtie N}) \arrow[d, dotted, "(\left|\begin{smallmatrix}z_M\\ z_N\end{smallmatrix}\right|{,}1_L)"] & \\
& (Z \xrightarrow{z} L, \psi_Z) &
\end{tikzcd}$$

We want to construct the dotted morphism of crossed modules such that the two triangles commute. The first step is constructing the arrow $\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|$ through the diagram

$$\begin{tikzcd}
(N\flat M)+(M\flat N) \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}"] \arrow[r, shift right, "\xi^N_M+\xi^M_N"'] & M+N \arrow[r, two heads, "q"] \arrow[dr, "\binom{z_M}{z_N}"'] & M\bowtie N \arrow[d, dotted, "\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"] \\
& & Z
\end{tikzcd}$$

In order to do so we need to show that $\binom{z_M}{z_N}$ coequalises the arrows on the left. This is done by using the Peiffer condition for $(Z \xrightarrow{z} L, \psi_Z)$ and the fact that $(z_M, 1_L)$ and $(z_N, 1_L)$ are morphisms of crossed modules:

$$\begin{aligned}
\begin{pmatrix} z_M \\ z_N \end{pmatrix} \circ (\xi^N_M + \xi^M_N) &= \begin{pmatrix} z_M \circ \xi^N_M \\ z_N \circ \xi^M_N \end{pmatrix} = \begin{pmatrix} \psi_Z \circ (\nu\flat z_M) \\ \psi_Z \circ (\mu\flat z_N) \end{pmatrix} \\
&= \psi_Z \circ \begin{pmatrix} \nu\flat z_M \\ \mu\flat z_N \end{pmatrix} = \psi_Z \circ \begin{pmatrix} (z\flat 1)\circ(z_N\flat z_M) \\ (z\flat 1)\circ(z_M\flat z_N) \end{pmatrix} \\
&= \psi_Z \circ (z\flat 1) \circ \begin{pmatrix} z_N\flat z_M \\ z_M\flat z_N \end{pmatrix} = \chi_Z \circ \begin{pmatrix} z_N\flat z_M \\ z_M\flat z_N \end{pmatrix} \\
&= \begin{pmatrix} \binom{z_M}{z_N} \circ k_{N,M} \\ \binom{z_M}{z_N} \circ k_{M,N} \end{pmatrix} = \begin{pmatrix} z_M \\ z_N \end{pmatrix} \circ \begin{pmatrix} k_{N,M} \\ k_{M,N} \end{pmatrix}.
\end{aligned}$$

Finally we need to show the commutativity of the diagrams

$$\begin{tikzcd}
L\flat(M\bowtie N) \arrow[r, "1\flat\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"] \arrow[d, "\psi_{M\bowtie N}"'] & L\flat Z \arrow[d, "\psi_Z"] \\
M\bowtie N \arrow[r, "\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
M\bowtie N \arrow[r, "\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"] \arrow[d, "\left|\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right|"'] & Z \arrow[d, "z"] \\
L \arrow[r, equal] & L
\end{tikzcd}$$

To obtain the second one it suffices to precompose with the epimorphism q

$$\begin{vmatrix} \mu \\ \nu \end{vmatrix} \circ q = \begin{pmatrix} \mu \\ \nu \end{pmatrix} = z \circ \begin{pmatrix} z_M \\ z_N \end{pmatrix} = \begin{vmatrix} z_M \\ z_N \end{vmatrix} \circ q$$

whereas for the first one, we need to use the fact that $\mathbf{Lie}_R$ is an algebraically coherent category, and hence $1\flat l_M$ and $1\flat l_M$ are jointly strongly epimorphic, since l_M and l_N are so (see Theorem 3.18 in [29] for further details). This means that in order to prove the claim, we only need to check the commutativity of the outer rectangles

$$\begin{tikzcd}
L\flat M \arrow[r, "1\flat l_M"] \arrow[d, "\psi_M"'] & L\flat(M\bowtie N) \arrow[r, "1\flat\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"] \arrow[d, "\psi_{M\bowtie N}"'] & L\flat Z \arrow[d, "\psi_Z"] \\
M \arrow[r, "l_M"'] & M\bowtie N \arrow[r, "\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
L\flat N \arrow[r, "1\flat l_N"] \arrow[d, "\psi_N"'] & L\flat(M\bowtie N) \arrow[r, "1\flat\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"] \arrow[d, "\psi_{M\bowtie N}"'] & L\flat Z \arrow[d, "\psi_Z"] \\
N \arrow[r, "l_N"'] & M\bowtie N \arrow[r, "\left|\begin{smallmatrix} z_M \\ z_N \end{smallmatrix}\right|"'] & Z
\end{tikzcd}$$

which is given by hypothesis. □

2.3 Compatible actions in semi-abelian categories

From now on we will consider $\mathbb{A}$ to be a semi-abelian category in which the condition SH holds.

We are going to give a definition of compatible internal actions which comes from Definition 2.1.1 and Definition 2.1.3, with some differences that we will explain in the rest of this section. Notice that, as explained in Remark 2.3.2, for a pair of actions, being compatible is a property, and not additional structure.

Definition 2.3.1. Consider two objects M, $N \in \mathbb{A}$ which act on each other and on themselves by conjugation and denote the actions as

$$\chi_M \colon M\flat M \to M \qquad\qquad \chi_N \colon N\flat N \to N$$
$$\xi_N^M \colon M\flat N \to N \qquad\qquad \xi_M^N \colon N\flat M \to M.$$

We say that the actions ξ_N^M and ξ_M^N are compatible if there exist two actions

$$\xi_N^{M+N} \colon (M+N)\flat N \to N \qquad\qquad \xi_M^{M+N} \colon (M+N)\flat M \to M$$

"induced" from ξ_N^M, ξ_M^N and the conjugations, that is such that

$$\begin{tikzcd}
N\flat M \arrow[r, "i_N\flat 1_M"] \arrow[dr, "\xi_M^N"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
M\flat N \arrow[r, "i_M\flat 1_N"] \arrow[dr, "\xi_N^M"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\
& N
\end{tikzcd}$$

$$\begin{tikzcd}
M\flat M \arrow[r, "i_M\flat 1_M"] \arrow[dr, "\chi_M"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
N\flat N \arrow[r, "i_N\flat 1_N"] \arrow[dr, "\chi_N"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\
& N
\end{tikzcd}
\tag{CA0}$$

$$\begin{tikzcd}
M\diamond N\diamond M \arrow[r, "j_{M,N,M}"] \arrow[d, "S_{1,2}^{N,M}"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\
N\diamond M \arrow[r, "\diamond\xi_M^N"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M\diamond N\diamond N \arrow[r, "j_{M,N,N}"] \arrow[d, "S_{1,2}^{M,N}"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\
M\diamond N \arrow[r, "\diamond\xi_N^M"'] & N
\end{tikzcd}$$

with the additional property that the following diagrams commute

$$\begin{tikzcd}
((N\flat M)+(M\flat N))\flat M \arrow[r, "(\xi_M^N+\xi_N^M)\flat 1_M"] \arrow[d, "\binom{k_{N,M}}{k_{M,N}}\flat 1_M"'] & (M+N)\flat M \arrow[d, "\xi_M^{M+N}"] \\
(M+N)\flat M \arrow[r, "\xi_M^{M+N}"'] & M
\end{tikzcd} \tag{CA.M}$$

$$\begin{tikzcd}
((N\flat M)+(M\flat N))\flat N \arrow[r, "(\xi_M^N+\xi_N^M)\flat 1_N"] \arrow[d, "\binom{k_{N,M}}{k_{M,N}}\flat 1_N"'] & (M+N)\flat N \arrow[d, "\xi_N^{M+N}"] \\
(M+N)\flat N \arrow[r, "\xi_N^{M+N}"'] & N
\end{tikzcd} \tag{CA.N}$$

This definition obviously implies the one given in the case of groups, but we will see later (Corollary 2.3.11) that in **Grp** the two definitions coincide. The difference between these two definitions is threefold.

- First of all, in the case of groups we know that the actions ξ_M^{M+N} and ξ_N^{M+N} of the coproduct are given (uniquely) for free as soon as we have the actions ξ_M^N and ξ_N^M of the single components (together with the conjugation actions). The conditions on the base category required to obtain such a construction for free are still not well characterised.

- The fact that the two squares in (CA0) involving the ternary cosmash products are commutative for free in **Grp** and $\mathbf{Lie}_R$. These commutativities are a key requirement to have the uniqueness of the coproduct action once we fix its components, but right now it is not clear to us what are the conditions that the category $\mathbb{A}$ must satisfy for the commutativity of these squares to be implied by the other four triangles in (CA0).

- Similarly, we see a difference between diagrams (CA.M) and (CA.N), and their group version given by (2.7). In particular the former two can be decomposed into the latter, together with (2.5) and with additional conditions involving higher order cosmash products and bemolle: also this aspect still needs further investigation.

Remark 2.3.2. Notice that in the situation of the previous definition, the coproduct actions ξ_M^{M+N} and ξ_N^{M+N} are uniquely determined by the commutativities of (CA0) due to Lemma 1.1.42.

Proposition 2.3.3. *Given a pair of coterminal crossed modules*

$$(M \xrightarrow{\mu} L, \psi_M) \qquad\qquad (N \xrightarrow{\nu} L, \psi_N)$$

we can define actions ξ_N^M and ξ_M^N through the diagrams

$$\begin{tikzcd} M\flat N \arrow[rr, "\xi_N^M"] \arrow[dr, "\mu\flat 1_N"'] & & N \\ & L\flat N \arrow[ur, "\psi_N"'] & \end{tikzcd} \qquad \begin{tikzcd} N\flat M \arrow[rr, "\xi_M^N"] \arrow[dr, "\nu\flat 1_M"'] & & M \\ & L\flat M \arrow[ur, "\psi_M"'] & \end{tikzcd}$$

These actions are then compatible in the sense of Definition 2.3.1.

Proof. First of all, notice that ξ_N^M and ξ_M^N are actually actions due to Remark 1.2.11. Now, in order to show that they are compatible, we need to define the coproduct actions

$$\xi_N^{M+N} \colon (M+N)\flat N \to N \qquad\qquad \xi_M^{M+N} \colon (M+N)\flat M \to M$$

such that diagrams (CA0), (CA.M) and (CA.N) commute. These are defined as the compositions

$$\begin{tikzcd} (M+N)\flat M \arrow[rr, "\xi_M^{M+N}"] \arrow[dr, "\binom{\mu}{\nu}\flat 1_M"'] & & M \\ & L\flat M \arrow[ur, "\psi_M"'] & \end{tikzcd} \qquad \begin{tikzcd} (M+N)\flat N \arrow[rr, "\xi_N^{M+N}"] \arrow[dr, "\binom{\mu}{\nu}\flat 1_N"'] & & N \\ & L\flat N \arrow[ur, "\psi_N"'] & \end{tikzcd}$$

Once again the fact that they are actions is given by Remark 1.2.11. In order to show that the four triangles in (CA0) commute, we simply compute the following

$$\begin{aligned}
\xi_M^{M+N} \circ (i_M \flat 1_M) &= \psi_M \circ \left(\begin{pmatrix}\mu\\ \nu\end{pmatrix}\flat 1_M\right) \circ (i_M\flat 1_M) = \psi_M \circ (\mu\flat 1_M) = \chi_M, \\
\xi_M^{M+N} \circ (i_N \flat 1_M) &= \psi_M \circ \left(\begin{pmatrix}\mu\\ \nu\end{pmatrix}\flat 1_M\right) \circ (i_N\flat 1_M) = \psi_M \circ (\nu\flat 1_M) = \xi_M^N, \\
\xi_N^{M+N} \circ (i_M \flat 1_N) &= \psi_N \circ \left(\begin{pmatrix}\mu\\ \nu\end{pmatrix}\flat 1_N\right) \circ (i_M\flat 1_N) = \psi_N \circ (\mu\flat 1_N) = \xi_N^M, \\
\xi_N^{M+N} \circ (i_N \flat 1_N) &= \psi_N \circ \left(\begin{pmatrix}\mu\\ \nu\end{pmatrix}\flat 1_N\right) \circ (i_N\flat 1_N) = \psi_N \circ (\nu\flat 1_N) = \chi_N,
\end{aligned}$$

by using the crossed module conditions. For what regards the first square in (CA0), we use the diagrams

$$\begin{tikzcd} M\diamond N\diamond M \arrow[r, "j_{M,N,M}"] \arrow[d, "\mu\diamond\nu\diamond 1_M"'] & (M+N)\flat M \arrow[d, "(\mu+\nu)\flat 1_M"] \\ L\diamond L\diamond M \arrow[r, "j_{L,L,M}"] \arrow[d, "S_{2,1}^{L,M}"'] & (L+L)\flat M \arrow[d, "\binom{1_L}{1_L}\flat 1_M"] \\ L\diamond M \arrow[r, "i_{L,M}"'] & L\flat M \end{tikzcd} \qquad \begin{tikzcd} M\diamond N\diamond M \arrow[r, "S_{1,2}^{N,M}"] \arrow[d, "1_M\diamond\nu\diamond 1_M"'] & N\diamond M \arrow[r, "\diamond\xi_M^N"] \arrow[d, "\nu\diamond 1_M"'] & M \arrow[d, equal] \\ M\diamond L\diamond M \arrow[r, "S_{1,2}^{L,M}"'] \arrow[d, "\mu\diamond 1_L\diamond 1_M"'] & L\diamond M \arrow[r, "\diamond\psi_M"'] & M \arrow[d, equal] \\ L\diamond L\diamond M \arrow[r, "S_{2,1}^{L,M}"'] & L\diamond M \arrow[r, "\diamond\psi_M"'] & M \end{tikzcd}$$

induced by naturality and by the crossed module conditions (see Theorem 5.6 in [51]), to obtain the chain of equalities

$$\begin{aligned}
\xi_M^{M+N} \circ j_{M,N,M} &= \psi_M \circ \left(\begin{pmatrix} \mu \\ \nu \end{pmatrix} \flat 1_M \right) \circ j_{M,N,M} \\
&= \psi_M \circ i_{L,M} \circ S_{2,1}^{L,M} \circ \mu \diamond \nu \diamond 1_M \\
&= {}^{\diamond}\xi_M^N \circ S_{1,2}^{N,M}
\end{aligned}$$

With the same reasoning we can show the commutativity of the other square in (CA0). Finally we need to show the commutativity of (CA.M), that is the fact that ξ_M^{M+N} coequalises the maps

$$((N \flat M) + (M \flat N)) \flat M \underset{(\xi_M^N + \xi_N^M)\flat 1_M}{\overset{\binom{k_{N,M}}{k_{M,N}}\flat 1_M}{\rightrightarrows}} (M+N)\flat M$$

but we have the chain of equalities

$$\begin{aligned}
\xi_M^{M+N} \circ (\xi_M^N + \xi_N^M)\flat 1_M &= \psi_M \circ \begin{pmatrix} \mu \\ \nu \end{pmatrix} \flat 1_M \circ \left(\left((\psi_M \circ \nu \flat 1_M) + (\psi_N \circ \mu \flat 1_N) \right) \flat 1_M \right) \\
&= \psi_M \circ \left(\left(\begin{pmatrix} \mu \\ \nu \end{pmatrix} \circ ((\psi_M \circ \nu \flat 1_M) + (\psi_N \circ \mu \flat 1_N)) \right) \flat 1_M \right) \\
&= \psi_M \circ \begin{pmatrix} \mu \circ \psi_M \circ \nu \flat 1_M \\ \nu \circ \psi_N \circ \mu \flat 1_N \end{pmatrix} \flat 1_M \\
&= \psi_M \circ \begin{pmatrix} \chi_L \circ \nu \flat \mu \\ \chi_L \circ \mu \flat \nu \end{pmatrix} \flat 1_M \\
&= \psi_M \circ \begin{pmatrix} \binom{\mu}{\nu} \circ k_{N,M} \\ \binom{\mu}{\nu} \circ k_{M,N} \end{pmatrix} \flat 1_M \\
&= \psi_M \circ \left(\begin{pmatrix} \mu \\ \nu \end{pmatrix} \flat 1_M \right) \circ \begin{pmatrix} k_{N,M} \\ k_{M,N} \end{pmatrix} \flat 1_M \\
&= \xi_M^{M+N} \circ \left(\begin{pmatrix} k_{N,M} \\ k_{M,N} \end{pmatrix} \flat 1_M \right).
\end{aligned}$$

With the same reasoning we can show that (CA.N) commutes. □

Let us consider the construction of the Peiffer product given in (2.8) as a definition in the general case.

Definition 2.3.4. Given two objects M and N acting on each other with actions ξ_M^N and ξ_N^M, we define their Peiffer product $M \bowtie N$ as the following coequaliser

$$(N \flat M) + (M \flat N) \underset{\xi_M^N + \xi_N^M}{\overset{\binom{k_{N,M}}{k_{M,N}}}{\rightrightarrows}} M + N \xrightarrow{q} M \bowtie N \tag{2.23}$$

An equivalent definition of the Peiffer product of two actions can be given through the following proposition, which characterises it as the pushout of the two semi-direct products induced by the two actions.

Proposition 2.3.5. *Given a pair of actions $\xi_N^M\colon M\flat N \to N$ and $\xi_M^N\colon N\flat M \to M$ we can obtain the Peiffer product $M \bowtie N$ as the pushout*

$$\begin{tikzcd}
M+N \arrow[r, two heads, "\sigma_{\xi_M^N}"] \arrow[d, two heads, "\sigma_{\xi_N^M}"'] \arrow[dr, two heads, "q"] & M \rtimes N \arrow[d, two heads, "q_{M\rtimes N}"] \\
N \rtimes M \arrow[r, two heads, "q_{N\rtimes M}"'] & M \bowtie N \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{2.24}$$

of the two semi-direct products.

Proof. Recall that the semi-direct products are defined as the coequalisers

$$N\flat M \underset{k_{N,M}}{\overset{i_M\circ\xi_M^N}{\rightrightarrows}} M+N \xrightarrow{\sigma_{\xi_M^N}} M \rtimes N,$$

$$M\flat N \underset{k_{M,N}}{\overset{i_N\circ\xi_N^M}{\rightrightarrows}} M+N \xrightarrow{\sigma_{\xi_N^M}} N \rtimes M,$$

By definition we know that q coequalises each of these pairs of maps, and hence we obtain the unique regular epimorphisms $q_{N\rtimes M}$ and $q_{M\rtimes N}$ making the triangles

$$\begin{tikzcd}
M+N \arrow[r, two heads, "\sigma_{\xi_M^N}"] \arrow[dr, two heads, "q"'] & M \rtimes N \arrow[d, two heads, "q_{M\rtimes N}"] \\
& M \bowtie N
\end{tikzcd}
\qquad
\begin{tikzcd}
M+N \arrow[r, two heads, "\sigma_{\xi_N^M}"] \arrow[dr, two heads, "q"'] & N \rtimes M \arrow[d, two heads, "q_{N\rtimes M}"] \\
& M \bowtie N
\end{tikzcd}$$

commute. Now in order to prove that (2.24) is a pushout, suppose there exist $f\colon M\rtimes N \to Z$ and $g\colon N \rtimes M \to Z$ such that $\gamma := f\circ\sigma_{\xi_M^N} = g\circ\sigma_{\xi_N^M}$. It suffices to show that γ coequalises the maps defining q:

$$\begin{aligned}
\gamma\circ\begin{pmatrix}k_{N,M}\\ k_{M,N}\end{pmatrix} = \begin{pmatrix}\gamma\circ k_{N,M}\\ \gamma\circ k_{M,N}\end{pmatrix} &= \begin{pmatrix}f\circ\sigma_{\xi_M^N}\circ k_{N,M}\\ g\circ\sigma_{\xi_N^M}\circ k_{M,N}\end{pmatrix}\\
&= \begin{pmatrix}f\circ\sigma_{\xi_M^N}\circ i_M\circ\xi_M^N\\ g\circ\sigma_{\xi_N^M}\circ i_N\circ\xi_N^M\end{pmatrix} = \begin{pmatrix}\gamma\circ i_M\circ\xi_M^N\\ \gamma\circ i_N\circ\xi_N^M\end{pmatrix}\\
&= \gamma\circ(\xi_M^N+\xi_N^M).
\end{aligned}$$

This gives us a unique map $\gamma'\colon M\bowtie N \to Z$ such that $\gamma'\circ q_{M\rtimes N} = f$ and $\gamma'\circ q_{N\rtimes M} = g$ because $\sigma_{\xi_M^N}$ and $\sigma_{\xi_N^M}$ are epimorphisms. $\square$

The idea behind the Peiffer product $M \bowtie N$ is that it should be the universal object acting on M and N with two crossed modules structures, as soon as these two objects act on each other compatibly. In particular, if we are in the situation of two compatible actions, we have induced coproduct actions whose precrossed module conditions

$$\begin{tikzcd}
(M+N)\flat M \arrow[r, "\xi_M^{M+N}"] \arrow[d, "1_{M+N}\flat i_M"'] & M \arrow[d, "i_M"] & (M+N)\flat N \arrow[r, "\xi_N^{M+N}"] \arrow[d, "1_{M+N}\flat i_N"'] & N \arrow[d, "i_N"] \\
(M+N)\flat(M+N) \arrow[r, "\chi_{M+N}"'] & M+N & (M+N)\flat(M+N) \arrow[r, "\chi_{M+N}"'] & M+N
\end{tikzcd} \tag{2.25}$$

are not satisfied (whereas the Peiffer conditions already hold for free).

Hence we want to do two things: we want to define actions of the Peiffer product on M and N induced from the coproduct actions, and then we want to show that the postcomposition with the quotient defining the Peiffer product makes the previous square commute, obtaining two crossed module structure.

Again by using Lemma 1.1.38 and Remark 2.1.7 we deduce that in order to define the actions $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ of the Peiffer product as in

$$\begin{tikzcd}
((N\flat M)+(M\flat N))\flat M \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}\flat 1_M"] \arrow[r, shift right, "(\xi_M^N+\xi_N^M)\flat 1_M"'] & (M+N)\flat M \arrow[r, "q\flat 1_M"] \arrow[dr, "\xi_M^{M+N}"'] & (M\bowtie N)\flat M \arrow[d, dotted, "\xi_M^{M\bowtie N}"] \\
 & & M
\end{tikzcd} \tag{2.26}$$

$$\begin{tikzcd}
((N\flat M)+(M\flat N))\flat N \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}\flat 1_N"] \arrow[r, shift right, "(\xi_M^N+\xi_N^M)\flat 1_N"'] & (M+N)\flat N \arrow[r, "q\flat 1_N"] \arrow[dr, "\xi_N^{M+N}"'] & (M\bowtie N)\flat N \arrow[d, dotted, "\xi_N^{M\bowtie N}"] \\
 & & N
\end{tikzcd} \tag{2.27}$$

it suffices to show that ξ_M^{M+N} coequalises the parallel maps in (2.26) and that ξ_N^{M+N} coequalises the parallel maps in (2.27). But these are precisely given by (CA.M) and by (CA.N).

Now we have the desired actions of the Peiffer product, but in order to obtain the crossed module structures we need to show that postcomposing with the quotient q makes the diagrams (2.25) commute. In fact we are going to prove more than this: the Peiffer product is the coequaliser of those maps.

Definition 2.3.6. Given a pair of compatible actions ξ_M^N and ξ_N^M, we define the *strong Peiffer product* $M \bowtie_S N$ as the coequaliser in the diagram

$$\begin{tikzcd}
((M+N)\flat M)+((M+N)\flat N) \arrow[r, shift left, "\chi_{M+N}\binom{1_{M+N}\flat i_M}{1_{M+N}\flat i_N}"] \arrow[r, shift right, "\xi_M^{M+N}+\xi_N^{M+N}"'] & M+N \arrow[r, two heads, "q_S"] & M\bowtie_S N
\end{tikzcd} \tag{2.28}$$

Remark 2.3.7. It is important to notice that in principle there is a huge difference between the coequaliser in (2.23) and the one in (2.28):

- the latter makes sense only if the two actions are already compatible (otherwise the existence of the coproduct actions is not guaranteed) and it is directly asking that the Peiffer product coequalises the compositions in (2.25);

- the former makes sense even when the two actions are not compatible and it is obtained following the ideas from the particular compatibility conditions in the case of **Grp** through Remark 2.1.4 and Remark 2.1.6.

This means that taking (2.23) as a definition of $M \bowtie N$, we would not immediately have that the Peiffer product is the universal way to coequalises the compositions in (2.25). Obviously if we precompose the maps in (2.28) with $(i_N \flat 1_M) + (i_M \flat 1_N)$, we see that q_S coequalises also the maps defining q

$$q_S \circ (\xi_M^N + \xi_N^M) = q_S \circ \begin{pmatrix} k_{N,M} \\ k_{M,N} \end{pmatrix}$$

but for the converse we need the following proposition.

Proposition 2.3.8. *Consider two actions ξ_N^M and ξ_M^N which are compatible in the sense of Definition 2.3.1. Then the two coequalisers* (2.23) *and* (2.28) *are isomorphic, so that the Peiffer product $M \bowtie N$ coincides with the strong Peiffer product $M \bowtie_S N$.*

Proof. In order to show the isomorphism between the two Peiffer products it suffices to show that q coequalises the maps defining q_S: since the converse already holds due to Remark 2.3.7, we obtain the thesis by the universal properties of the coequalisers. Recalling Lemma 1.1.42 we just need to show that q coequalises the two compositions in

$$\begin{tikzcd}
(M\flat M)+(N\flat M)+(M\diamond N\diamond M)+(M\flat N)+(N\flat N)+(M\diamond N\diamond N) \arrow[d, "\begin{pmatrix} i_M\flat 1_M \\ i_N\flat 1_M \\ j_{M,N,M} \end{pmatrix} + \begin{pmatrix} i_M\flat 1_N \\ i_N\flat 1_N \\ j_{M,N,N} \end{pmatrix}"'] & \\
((M+N)\flat M) + ((M+N)\flat N) \arrow[r, shift left, "\chi_{M+N}\begin{pmatrix} 1_{M+N}\flat i_M \\ 1_{M+N}\flat i_N \end{pmatrix}"] \arrow[r, shift right, "\xi_M^{M+N}+\xi_N^{M+N}"'] & M+N
\end{tikzcd}$$

By the universal property of the coproduct we can consider each component separately and since the last three are similar to the first three (it suffices to change the role of M and N), we are going to examine only the first three.

- Precomposing with the inclusion of $M\flat M$, we obtain

$$\begin{aligned}
q \circ \chi_{M+N} \begin{pmatrix} 1_{M+N}\flat i_M \\ 1_{M+N}\flat i_N \end{pmatrix} \circ i_1 \circ (i_M \flat 1_M) &= q \circ \chi_{M+N} \circ (i_M \flat i_M) \\
&= q \circ i_M \circ \chi_M \\
&= q \circ i_M \circ \xi_M^{M+N} \circ (i_M \flat 1_M) \\
&= q \circ \left(\xi_M^{M+N} + \xi_N^{M+N} \right) \circ i_1 \circ (i_M \flat 1_M).
\end{aligned}$$

- Precomposing with the inclusion of $N\flat M$, and using the definition of q we obtain

$$\begin{aligned} q \circ \chi_{M+N} \begin{pmatrix} 1_{M+N}\flat i_M \\ 1_{M+N}\flat i_N \end{pmatrix} \circ i_1 \circ (i_N\flat 1_M) &= q \circ \chi_{M+N} \circ (i_N\flat i_M) \\ &= q \circ k_{N,M} \\ &= q \circ i_M \circ \xi_M^N \\ &= q \circ i_M \circ \xi_M^{M+N} \circ (i_N\flat 1_M) \\ &= q \circ \left(\xi_M^{M+N} + \xi_N^{M+N}\right) \circ i_1 \circ (i_N\flat 1_M). \end{aligned}$$

- Precomposing with the inclusion of $M \diamond N \diamond M$, we obtain

$$\begin{aligned} q \circ \chi_{M+N} \begin{pmatrix} 1_{M+N}\flat i_M \\ 1_{M+N}\flat i_N \end{pmatrix} \circ i_1 \circ j_{M,N,M} &= q \circ \chi_{M+N} \circ (1_{M+N}\flat i_M) \circ j_{M,N,M} \\ &= q \circ h_{N,M} \circ S_{1,2}^{N,M} \\ &= q \circ k_{N,M} \circ i_{N,M} \circ S_{1,2}^{N,M} \\ &= q \circ k_{N,M} \circ i_{N,M} \circ S_{1,2}^{N,M} \\ &= q \circ i_M \circ \xi_M^N \circ i_{N,M} \circ S_{1,2}^{N,M} \\ &= q \circ i_M \circ \xi_M^{M+N} \circ j_{M,N,M} \\ &= q \circ \left(\xi_M^{M+N} + \xi_N^{M+N}\right) \circ i_1 \circ j_{M,N,M}. \quad \square \end{aligned}$$

This means that $M \bowtie_S N \cong M \bowtie N$ and that q is the universal map making (2.25) commute through postcomposition.

Our aim now is to show that $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ are indeed actions and that they give to $M \bowtie N$ two crossed module structures over M and N.

Proposition 2.3.9. *The maps $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ are internal actions and we have two crossed module structures*

$$(M \xrightarrow{l_M} M \bowtie N, \xi_M^{M\bowtie N}) \qquad\qquad (N \xrightarrow{l_N} M \bowtie N, \xi_N^{M\bowtie N})$$

where the maps l_M and l_N are defined through the compositions

$$\begin{tikzcd} M \arrow[dr, tail, "i_M"] \arrow[ddr, bend right, "l_M"'] & & N \arrow[dl, tail, "i_N"'] \arrow[ddl, bend left, "l_N"] \\ & M+N \arrow[d, two heads, "q"] & \\ & M \bowtie N & \end{tikzcd}$$

Furthermore the compatible actions induced by these crossed module structures as in Proposition 2.3.3 coincide with actions ξ_M^N and ξ_N^M.

Proof. We are going to prove the thesis only for $\xi_M^{M\bowtie N}$ and l_M since the reasoning can be repeated for $\xi_N^{M\bowtie N}$ and l_N.

In order to see that $\xi_M^{M\bowtie N}$ is automatically an action it suffices to follow these steps:

- from diagram

$$\begin{tikzcd}
M \arrow[d, "\eta_M^{M+N}"'] \arrow[dr, tail, "i_M"] & \\
(M+N)\flat M \arrow[r, "k_{M+N,M}"] \arrow[d, "q\flat 1_M"'] & (M+N)+M \arrow[d, "q+1_M"] \\
(M\bowtie N)\flat M \arrow[r, "k_{M\bowtie N,M}"] & (M\bowtie N)+M
\end{tikzcd}$$

  we can see that $\eta_M^{M\bowtie N} = (q\flat 1_M)\circ \eta_M^{M+N}$ and consequently the first axiom

$$\xi_M^{M\bowtie N}\circ \eta_M^{M\bowtie N} = \xi_M^{M\bowtie N}\circ (q\flat 1_M)\circ \eta_M^{M+N} = \xi_M^{M+N}\circ \eta_M^{M+N} = 1_M;$$

- by using the fact that $q\flat(q\flat 1_M)$ is a regular epimorphism (due to Lemma 1.1.35) we can show the second axiom

$$\xi_M^{M\bowtie N}\circ \mu_M^{M\bowtie N} = \xi_M^{M\bowtie N}\circ (1_{M\bowtie N}\flat \xi_M^{M\bowtie N})$$

  through the commutativity of the outer rectangle in

$$\begin{tikzcd}
(M+N)\flat((M+N)\flat M) \arrow[r, "\mu_M^{M+N}"] \arrow[d, two heads, "q\flat(q\flat 1_M)"'] & (M+N)\flat M \arrow[d, two heads, "q\flat 1_M"] \\
(M\bowtie N)\flat((M\bowtie N)\flat M) \arrow[r, "\xi_M^{M\bowtie N}"] \arrow[d, "1_{M\bowtie N}\flat\xi_M^{M\bowtie N}"'] & (M\bowtie N)\flat M \arrow[d, "\xi_M^{M\bowtie N}"] \\
(M\bowtie N)\flat M \arrow[r, "\xi_M^{M\bowtie N}"'] & M
\end{tikzcd}$$

  given by the second axiom for the action ξ_M^{M+N}.

It remains to prove that $(M \xrightarrow{l_M} M\bowtie N, \xi_M^{M\bowtie N})$ is indeed a crossed module. We need to show the commutativity of the following squares

$$\begin{tikzcd}
M\flat M \arrow[r, "\chi_M"] \arrow[d, "l_M\flat 1_M"'] & M \arrow[d, equal] \\
(M\bowtie N)\flat M \arrow[r, "\xi_M^{M\bowtie N}"] \arrow[d, "1_{M\bowtie N}\flat l_M"'] & M \arrow[d, "l_M"] \\
(M\bowtie N)\flat(M\bowtie N) \arrow[r, "\chi_{M\bowtie N}"'] & (M\bowtie N)
\end{tikzcd}$$

For what regards the commutativity of the upper square we have the following chain of equalities

$$\begin{aligned}
\xi_M^{M\bowtie N} \circ (l_M \flat 1_M) &= \xi_M^{M\bowtie N} \circ (q\flat 1_M) \circ (i_M \flat 1_M)\\
&= \xi_M^{M+N} \circ (i_M \flat 1_M)\\
&= \chi_M
\end{aligned}$$

In order to show the commutativity of the lower square, consider the following diagram

$$\begin{tikzcd}
(M+N)\flat M \arrow[d, twoheadrightarrow, "q\flat 1_M"'] \arrow[dr, bend left, "\xi_M^{M+N}"] & \\
(M\bowtie N)\flat M \arrow[r, "\xi_M^{M\bowtie N}"] \arrow[d, "1_{M\bowtie N}\flat l_M"'] & M \arrow[d, "l_M"] \\
(M\bowtie N)\flat(M\bowtie N) \arrow[r, "\chi_{M\bowtie N}"'] & (M\bowtie N)
\end{tikzcd}$$

and since $q\flat 1_M$ is a regular epimorphism, it suffices to show that this last one commutes. To deduce this, we decompose it as

$$\begin{tikzcd}
(M+N)\flat M \arrow[r, "\xi_M^{M+N}"] \arrow[d, "1_{M+N}\flat i_M"'] & M \arrow[d, "i_M"] \\
(M+N)\flat(M+N) \arrow[r, "\chi_{M+N}"'] \arrow[d, "q\flat q"'] & M+N \arrow[d, "q"] \\
(M\bowtie N)\flat(M\bowtie N) \arrow[r, "\chi_{M\bowtie N}"'] & M\bowtie N
\end{tikzcd}$$

It is trivial to check that the lower square commutes and thanks to this, by using Proposition 2.3.8, we obtain that the whole rectangle commutes.

Finally we know that the actions ξ_N^M and ξ_M^N are in turn induced by $\xi_M^{M\bowtie N}$ and $\xi_N^{M\bowtie N}$ through the maps l_M and l_N, that is

$$\begin{tikzcd}
M\flat N \arrow[r, "l_M\flat 1_N"] \arrow[dr, "\xi_N^M"'] & (M\bowtie N)\flat N \arrow[d, "\xi_N^{M\bowtie N}"] \\
& N
\end{tikzcd}
\qquad
\begin{tikzcd}
N\flat M \arrow[r, "l_N\flat 1_M"] \arrow[dr, "\xi_M^N"'] & (M\bowtie N)\flat M \arrow[d, "\xi_M^{M\bowtie N}"] \\
& M
\end{tikzcd}$$

commute. This can be proved by using the definition of l_M and l_N and the commutativity of diagrams (CA0), (2.26) and (2.27). □

Combining Proposition 2.3.3 and Proposition 2.3.9 we obtain the following characterisation of compatible actions.

Theorem 2.3.10. *Two actions ξ_N^M and ξ_M^N are compatible if and only if there exists an object L and two crossed module structures*

$$(M \xrightarrow{\mu} L, \psi_M) \qquad\qquad (N \xrightarrow{\nu} L, \psi_N)$$

which induce the given actions via the commutative triangles

$$\begin{tikzcd} M\flat N \arrow[rr, "\xi_N^M"] \arrow[dr, "\mu\flat 1_N"'] & & N \\ & L\flat N \arrow[ur, "\psi_N"'] & \end{tikzcd} \qquad \begin{tikzcd} N\flat M \arrow[rr, "\xi_M^N"] \arrow[dr, "\nu\flat 1_M"'] & & M \\ & L\flat M \arrow[ur, "\psi_M"'] & \end{tikzcd}$$

Consequently we obtain that our definition of compatible internal actions is indeed a generalisation of the specific ones in the group and Lie algebra cases.

Corollary 2.3.11. *In* **Grp** *Definition 2.3.1 coincides with Definition 2.1.3.*

Proof. This is a corollary of Theorem 2.3.10 and Proposition 2.1.10. □

Corollary 2.3.12. *The definition of compatible actions of Lie algebras given in [37] coincides with Definition 2.3.1 restricted to the category* $\mathbf{Lie}_R$.

Proof. This is a corollary of Theorem 2.3.10 and Theorem 2.2.22. □

2.3.1 Universal properties of the Peiffer product

The Peiffer product $M \bowtie N$ is the universal way to associate a coterminal pair of crossed modules to a pair of compatible actions.

Proposition 2.3.13. *Consider a pair of compatible actions ξ_N^M and ξ_M^N and all the pairs of coterminal crossed modules inducing them. The pair given by the Peiffer product is the universal one, in the sense that for each other pair of crossed module*

$$(M \xrightarrow{\mu} L, \psi_M) \qquad\qquad (N \xrightarrow{\nu} L, \psi_N)$$

inducing ξ_N^M and ξ_M^N there exists a unique morphism $\left|\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right| \colon M \bowtie N \to L$ making the diagram

$$\begin{tikzcd} & M \arrow[d, "l_M"'] \arrow[ddr, bend left, "\mu"] & \\ N \arrow[r, "l_N"] \arrow[drr, bend right, "\nu"'] & M \bowtie N \arrow[dr, dotted, "\left|\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right|"] & \\ & & L \end{tikzcd}$$

commute.

Proof. It suffices to show that $\binom{\mu}{\nu}\colon M+N \to L$ coequalises the two maps defining $M \bowtie N$. Indeed that would give us a unique map $\left|\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right|$ such that

$$\begin{tikzcd} M+N \arrow[r, "q"] \arrow[dr, "\binom{\mu}{\nu}"'] & M \bowtie N \arrow[d, dotted, "\left|\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right|"] \\ & L \end{tikzcd}$$

and then by precomposing with the inclusion we would get

$$\mu = \begin{pmatrix}\mu\\ \nu\end{pmatrix} \circ i_M = \begin{vmatrix}\mu\\ \nu\end{vmatrix} \circ q \circ i_M = \begin{vmatrix}\mu\\ \nu\end{vmatrix} \circ l_M,$$
$$\nu = \begin{pmatrix}\mu\\ \nu\end{pmatrix} \circ i_N = \begin{vmatrix}\mu\\ \nu\end{vmatrix} \circ q \circ i_N = \begin{vmatrix}\mu\\ \nu\end{vmatrix} \circ l_N.$$

Therefore we have to show that the two compositions

$$\begin{tikzcd} (N\flat M)+(M\flat N) \arrow[r, shift left, "\binom{k_{N,M}}{k_{M,N}}"] \arrow[r, shift right, "\xi^N_M+\xi^M_N"'] & M+N \arrow[r, "\binom{\mu}{\nu}"] & L \end{tikzcd}$$

are equal and this is obtained through the chain of equalities

$$\begin{pmatrix}\mu\\ \nu\end{pmatrix} \circ (\xi^N_M + \xi^M_N) = \begin{pmatrix}\mu\circ\psi_M\circ\nu\flat 1_M\\ \nu\circ\psi_N\circ\mu\flat 1_N\end{pmatrix} = \begin{pmatrix}\binom{\mu}{\nu}\circ k_{N,M}\\ \binom{\mu}{\nu}\circ k_{M,N}\end{pmatrix} = \begin{pmatrix}\mu\\ \nu\end{pmatrix} \circ \begin{pmatrix}k_{N,M}\\ k_{M,N}\end{pmatrix}. \qquad \square$$

Lemma 2.3.14. *Consider two different pairs of coterminal crossed modules*

$$\begin{tikzcd} & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[d, "\mu'"] \\ N \arrow[r, "\nu'"'] & L' \end{tikzcd}$$

such that they induce the same actions between M and N, that is such that the following diagrams commute

$$\begin{tikzcd} N\flat M \arrow[r, "\nu\flat 1_M"] \arrow[d, "\nu'\flat 1_M"'] \arrow[dr, "\xi^N_M"] & L\flat M \arrow[d, "\psi_M"] \\ L'\flat M \arrow[r, "\psi'_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} M\flat N \arrow[r, "\mu\flat 1_N"] \arrow[d, "\mu'\flat 1_N"'] \arrow[dr, "\xi^M_N"] & L\flat N \arrow[d, "\psi_N"] \\ L'\flat N \arrow[r, "\psi'_N"'] & N \end{tikzcd}$$

Then they induce the same Peiffer product $M \bowtie N$.

Proof. The induced actions ξ^{M+N}_M and ξ'^{M+N}_M (resp. ξ^{M+N}_N and ξ'^{M+N}_N) coincide when restricted to $M\flat M$, $N\flat M$ and $M\diamond N\diamond M$ (resp. $M\flat N$, $N\flat N$ and $M\diamond N\diamond N$), therefore it suffices to use Remark 2.3.2 to obtain that $\xi^{M+N}_M = \xi'^{M+N}_M$ (resp. $\xi^{M+N}_N = \xi'^{M+N}_N$). As a consequence they induce the same Peiffer product and the same crossed module structures. $\square$

Remark 2.3.15. We know from Proposition 3.2 in [30] that, as soon as $(M \xrightarrow{\mu} L, \psi_M)$ and $(N \xrightarrow{\nu} L, \psi_N)$ are (pre)crossed modules, we have induced actions of L on $M \rtimes N$ and $N \rtimes M$ with corresponding (pre)crossed module structures. In general this is not true for $M \bowtie N$, but if $\mathbb{A}$ is algebraically coherent, by Proposition 4.1 and Proposition 4.3 in [30], and by Proposition 2.3.5 we obtain that our definition of Peiffer product coincides with the one given by Cigoli, Mantovani and Metere: consequently $M \bowtie N$ is endowed with a precrossed module structure $(M \bowtie N \xrightarrow{\left|\substack{\mu \\ \nu}\right|} L, \psi_{M \bowtie N})$ as soon as M and N are so. Finally when $\mathbb{A}$ satisfies also the condition (UA), Theorem 5.2 in [30] tells us that the Peiffer product precrossed module is actually a crossed module as soon as M and N are so, and it is the coproduct of $(M \xrightarrow{\mu} L, \psi_M)$ and $(N \xrightarrow{\nu} L, \psi_N)$ in $\mathbf{XMod}_L(\mathbb{A})$.

Remark 2.3.16. We do not know whether L acts on $M \bowtie N$ when $\mathbb{A}$ is not algebraically coherent. If so, it is also not clear to us whether this action defines a precrossed module structure.

Chapter 3

Non-abelian tensor product

The aim of this chapter is to explain how, in the context of a semi-abelian category, the concept of an *internal crossed square* may be used to set up an intrinsic approach to the *non-abelian tensor product*. Both concepts were originally introduced for groups by Brown and Loday in [15] and for Lie algebras by Ellis in [37].

A *crossed square* (of groups) is a *two-dimensional crossed module*, in the following precise sense. The internal groupoid construction may be repeated, obtaining the category $\mathbf{Grpd}^2(\mathbf{Grp}) = \mathbf{Grpd}(\mathbf{Grpd}(\mathbf{Grp}))$ of internal *double* groupoids in $\mathbf{Grp}$. Given such an internal double groupoid

$$\begin{tikzcd}[row sep=huge, column sep=huge]
Z \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift right=2, "c_L"'] \arrow[d, shift left=2, "d_L"] & X \arrow[l, "e_U" description] \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
Y \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] \arrow[u, "e_L" description] & L \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd}$$

viewed as a diagram in $\mathbf{Grp}$ (in which the composition maps are omitted), we may take the normalisation functor vertically and horizontally to obtain a commutative square

$$\begin{tikzcd}
P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & L.
\end{tikzcd}$$

The given double groupoid structure naturally induces actions of L on M, P and N (and consequently also actions of M on P and N, and of N on P and M) that satisfy some properties. One may now ask, whether it is possible to equip a given commutative square of group homomorphisms with suitable actions (and, possibly, additional maps), in such a way that an internal double groupoid may be recovered, thus extending the equivalence $\mathbf{XMod} \simeq \mathbf{Grpd}(\mathbf{Grp})$ in order to capture double groupoids in $\mathbf{Grp}$ as commutative squares equipped with extra structure. The concept of a crossed square [15] answers this question, and does indeed give rise to a category equivalence $\mathbf{XSqr} \simeq \mathbf{Grpd}^2(\mathbf{Grp})$.

Internal crossed squares answer the same question, now asked for a different, general base category $\mathbb{A}$, which we take to be semi-abelian. The work of Janelidze [55] provides an explicit description of internal crossed modules in $\mathbb{A}$, together with an equivalence of categories $\mathbf{XMod}(\mathbb{A}) \simeq \mathbf{Grpd}(\mathbb{A})$ which reduces to the well-known equivalence when $\mathbb{A} = \mathbf{Grp}$. Since the category of internal crossed modules in a semi-abelian category is again semi-abelian, this construction may be repeated, and thus we see that $\mathbf{XMod}^2(\mathbb{A}) \simeq \mathbf{Grpd}^2(\mathbb{A})$. We may now write $\mathbf{XSqr}(\mathbb{A}) := \mathbf{XMod}^2(\mathbb{A})$ and say that a *crossed square in* $\mathbb{A}$ is an internal crossed module of internal crossed modules in $\mathbb{A}$. Indeed, any such double internal crossed module has an underlying commutative square in $\mathbb{A}$, which the crossed module structures equip with suitable internal actions in such a way that an internal double groupoid may be recovered. The internal action structure is, however, far from being transparent, and thus merits further explicitation.

Yet, we shall see that even this tentative and very abstract general definition is concrete enough to serve as a basis for an intrinsic approach to the *non-abelian tensor product*. Originally this tensor product (of two groups M and N acting on each other compatibly) was defined in [15] via a presentation in terms of generators and relations. In Chapter 2 we investigated how to extend the concept of a pair of compatible actions to the semi-abelian setting, showing that such a pair of compatible actions is equivalent to the datum of a third object L and two internal crossed module structures $\mu\colon M \to L$ and $\nu\colon N \to L$. According to Proposition 2.15 in [15], given two L-crossed modules μ and ν, and a crossed square of groups

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

the crossed module

$$\mu p_M = \nu p_N \colon P \to L$$

happens to be the tensor product of M and N (with respect to the actions of M and N on each other, induced by the crossed module structures of $\mu\colon M \to L$ and $\nu\colon N \to L$) if and only if the crossed square is the initial object in the category of all crossed squares over the given crossed modules μ and ν. This property of course determines the tensor product, and it may actually be taken as a definition.

Concretely this means that in a semi-abelian category (satisfying (SH)), the non-abelian tensor product of two objects acting compatibly on one another may be constructed as follows.

1. Consider the internal L-crossed modules $\mu\colon M \to L$ and $\nu\colon N \to L$ corresponding to the given actions.

2. Use the equivalence $\mathbf{XMod}(\mathbb{A}) \simeq \mathbf{Grpd}(\mathbb{A})$ to transform these into internal groupoids

$$\begin{tikzcd}
 & M \rtimes L \arrow[d, shift right=2, "c_M"'] \arrow[d, shift left=2, "d_M"] \\
N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[u, "e_M"'] \arrow[l, "e_N"]
\end{tikzcd}$$

3. Take the pushout of e_N and e_M to find the double reflexive graph

$$\begin{tikzcd}
Q \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift right=2, "c_L"'] \arrow[d, shift left=2, "d_L"] & M \rtimes L \arrow[l, "e_U"'] \arrow[d, shift right=2, "c_M"'] \arrow[d, shift left=2, "d_M"] \\
N \rtimes L \arrow[u, "e_L"'] \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[u, "e_M"'] \arrow[l, "e_N"]
\end{tikzcd}$$

4. This double reflexive graph is not yet a double groupoid; it may be reflected into $\mathbf{Grpd}^2(\mathbb{A})$ by taking the quotient of Q by the join of commutators

$$[K_{c_L}, K_{d_L}] \vee [K_{c_U}, K_{d_U}].$$

5. The resulting internal double groupoid normalises to a crossed square

$$\begin{tikzcd}
M \otimes N \arrow[r, "\pi_M^{M\otimes N}"] \arrow[d, "\pi_N^{M\otimes N}"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & L,
\end{tikzcd}$$

whose structure involves a crossed module $M \otimes N \to L$. By definition, this is the non-abelian tensor product of M and N with respect to the given pair of compatible actions.

By known properties of the non-abelian tensor product for groups and Lie algebras, this reduces to the classical definitions in those cases (Proposition 3.2.3 and Proposition 3.2.17).

This chapter is devoted to exploring some basic properties of the definition, and showing that in some cases, the tensor product may be used to give an explicit description of an object of $\mathbf{XSqr}(\mathbb{A})$ as a square

$$\begin{tikzcd}
P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & L
\end{tikzcd}$$

in $\mathbb{A}$ equipped with suitable actions and a morphism $h\colon M \otimes N \to P$. This extends the explicit descriptions for groups and monoids to the general setting. It is, however, not yet clear to us whether this description is always valid (see Section 3.3).

The chapter is organised as follows:

- In Section 3.1.1 we recall the notions of double reflexive graphs and internal double groupoids; we construct the reflection into $\mathbf{Grpd}^2(\mathbb{A})$ of a particular type of double reflexive graphs that we will encounter in the following; then we give a quick recap on the basic theory of crossed squares.
- Section 3.2 is devoted to the non-abelian tensor product: we explain in detail how the tools from the previous section may be used to obtain an intrinsic approach to the non-abelian tensor product in any semi-abelian category satisfying (SH); we show that this approach coincides with the already existing ones in simpler cases like **Grp** and $\mathbf{Lie}_R$.
- In Section 3.3 we give a (partial) description of internal crossed squares in terms of the non-abelian tensor product, by introducing the new definition of weak crossed square and showing some conditions under which the two notions coincide.

3.1 Two-dimensional background

In this section and in the following ones we will always assume $\mathbb{A}$ to be a semi-abelian category with the (SH)condition, even if for most of the results this is not strictly necessary.

3.1.1 Double groupoids and double reflexive graphs

We recall the categories of double groupoids and double reflexive graphs, and describe how one is embedded into the other as a reflective subcategory.

Definition 3.1.1. A *double reflexive graph in* $\mathbb{A}$ is a reflexive graph in $\mathbf{RG}(\mathbb{A})$. This means that the category $\mathbf{RG}^2(\mathbb{A})$ is defined as $\mathbf{RG}(\mathbf{RG}(\mathbb{A}))$.

Lemma 3.1.2. *Every double reflexive graph can be depicted as a diagram of the form*

$$\begin{tikzcd}
A_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "c_1"'] \arrow[d, shift right=2, "c_A"'] \arrow[d, shift left=2, "d_A"] & B_1 \arrow[l, "e_1" description] \arrow[d, shift right=2, "c_B"'] \arrow[d, shift left=2, "d_B"] \\
A_0 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "c_0"'] \arrow[u, "e_A" description] & B_0 \arrow[l, "e_0" description] \arrow[u, "e_B" description]
\end{tikzcd} \tag{3.1}$$

in which every pair of adjacent vertices form a reflexive graph.

Proof. In order to prove this it suffices to explicit the definition. Consider two reflexive graphs in $\mathbb{A}$

$$\mathcal{A} := (A_1, A_0, d_A, c_A, e_A) \qquad\qquad \mathcal{B} := (B_1, B_0, d_B, c_B, e_B)$$

and the double reflexive graph

$$\mathcal{C} := (\mathcal{B}, \mathcal{A}, d, c, e) = (\mathcal{B}, \mathcal{A}, (d_1, d_0), (c_1, c_0), (e_1, e_0)).$$

By writing diagrammatically the commutativities that it has to satisfy, we easily see that they are the same conditions for the diagram (3.1) to commute. □

Definition 3.1.3. A *double reflexive multiplicative graph in* $\mathbb{A}$ is a reflexive multiplicative graph in $\mathbf{RMG}(\mathbb{A})$. This means that the category $\mathbf{RMG}^2(\mathbb{A})$ is defined as $\mathbf{RMG}(\mathbf{RMG}(\mathbb{A}))$.

Lemma 3.1.4. *A double reflexive multiplicative graph can be depicted as a diagram of the form* (3.1) *in which every pair of adjacent vertices form a reflexive multiplicative graph (that is it is endowed with a multiplication).*

Proof. By using the previous result, we say that a double reflexive multiplicative graph is given by a diagram of the form (3.1) endowed with multiplications

$$A_1 \times_{A_0} A_1 \xrightarrow{m_A} A_1 \qquad\qquad B_1 \times_{B_0} B_1 \xrightarrow{m_B} B_1$$

and with a multiplication $(m_0, m_1)\colon \mathcal{A} \times_{\mathcal{B}} \mathcal{A} \to \mathcal{A}$ in $\mathbf{RMG}(\mathbb{A})$ given by

$$A_1 \times_{B_1} A_1 \xrightarrow{m_1} A_1 \qquad\qquad A_0 \times_{B_0} A_0 \xrightarrow{m_0} A_0$$

It is easy to show that also these two maps satisfy the properties of multiplication for a reflexive multiplicative graph: this means that in the diagram (3.1) not only are the vertical reflexive graphs multiplicative, but also the horizontal ones. □

Definition 3.1.5. An *internal double groupoid in* $\mathbb{A}$ is an internal groupoid in $\mathbf{Grpd}(\mathbb{A})$. This means that the category $\mathbf{Grpd}^2(\mathbb{A})$ is defined as $\mathbf{Grpd}(\mathbf{Grpd}(\mathbb{A}))$.

Corollary 3.1.6. *Double groupoids are diagrams of the form* (3.1) *in which each reflexive graph has an internal groupoid structure.*

Proof. Being $\mathbb{A}$ a Mal'cev category, $\mathbf{RMG}(\mathbb{A})$ is Mal'cev as well (see [44] for more details), and since the isomorphism $\mathbf{RMG}(\mathbb{C}) \simeq \mathbf{Grpd}(\mathbb{C})$ holds for each Mal'cev category $\mathbb{C}$, we have

$$\begin{aligned}\mathbf{Grpd}^2(\mathbb{A}) &\simeq \mathbf{Grpd}(\mathbf{Grpd}(\mathbb{A})) \simeq \mathbf{Grpd}(\mathbf{RMG}(\mathbb{A})) \\ &\simeq \mathbf{RMG}(\mathbf{RMG}(\mathbb{A})) \simeq \mathbf{RMG}^2(\mathbb{A}).\end{aligned}$$

This means that a double groupoid is a double reflexive multiplicative graph, which by the previous lemma is just a square of reflexive multiplicative graphs, which in turn is a square of internal groupoids. □

3.1.2 Double groupoids induced by particular double reflexive graphs

Consider a double reflexive graph

$$
\begin{tikzcd}
A \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift right=2, "c_L"'] \arrow[d, shift left=2, "d_L"] & B \arrow[l, "e_U" description] \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
C \arrow[u, "e_L" description] \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] & D \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd}
\tag{3.2}
$$

such that the reflexive graph on the right and the one at the bottom already admit a multiplicative structure, that is such that

$$[K_{d_R}, K_{c_R}]^{\mathcal{Q}}_{B} = 0 \qquad\qquad [K_{d_D}, K_{c_D}]^{\mathcal{Q}}_{C} = 0$$

We want to construct an induced double reflexive multiplicative graph by means of the following construction.

Construction 3.1.7. Consider the diagram

$$
\begin{tikzcd}
 & & S \arrow[dl, tail, "j_S"'] \arrow[dd, tail, "i_S"] & \\
 & S \vee T \arrow[dr, tail, "i"] & & \\
T \arrow[ur, tail, "j_T"] \arrow[rr, tail, "i_T"'] & & A \arrow[r, two heads, "q_T"] \arrow[d, two heads, "q_S"'] \arrow[dr, two heads, "q"] & \frac{A}{T} \arrow[d, two heads, "\tilde{q_S}"] \\
 & & \frac{A}{S} \arrow[r, two heads, "\tilde{q_T}"'] & \frac{A}{S\vee T}
\end{tikzcd}
\tag{3.3}
$$

where S and T are two normal subobjects of A such that

$$i_S = k_{q_S}, \qquad\qquad i_T = k_{q_T},$$
$$q_S = c_{i_S}, \qquad\qquad q_T = c_{i_T}.$$

Recall from Definition 1.1.21 and Remark 1.1.22 that $S \vee T$ is a normal subobject of A with $i = k_q$, where $\frac{A}{S\vee T}$ is defined as the pushout of q_T along q_S, and q is its diagonal: this immediately implies that $q = c_i$.

Now we are going to apply this result to the particular situation depicted in (3.2) with

$$S := [K_{d_L}, K_{c_L}] \qquad\qquad T := [K_{d_U}, K_{c_U}]$$

So let us consider the special double reflexive graph depicted in (3.2) and using Construction 1.3.12 we define the maps d'_U, c'_U, e'_U and d'_L, c'_L, e'_L as shown in the following diagram

$$
\begin{tikzcd}
A \arrow[rrr, shift left=2, "d_U"] \arrow[rrr, shift right=2, "c_U"'] \arrow[ddd, shift right=2, "c_L"'] \arrow[ddd, shift left=2, "d_L"] \arrow[dr, equal] & & & B \arrow[lll, "e_U" description] \arrow[d, equal] \\
& A \arrow[r, two heads, "q_T"] \arrow[d, two heads, "q_S"'] \arrow[dr, "q"] & \frac{A}{T} \arrow[r, shift left=2, "d'_U"] \arrow[r, shift right=2, "c'_U"'] \arrow[d, two heads, "\tilde{q_S}"] \arrow[dr, phantom, "(*_1)"] & B \arrow[l, "e'_U" description] \arrow[d, equal] \\
& \frac{A}{S} \arrow[r, two heads, "\tilde{q_T}"'] \arrow[d, shift right=2, "c'_L"'] \arrow[d, shift left=2, "d'_L"] \arrow[dr, phantom, "(*_2)"] & \frac{A}{S\vee T} \arrow[r, dotted, shift left=2, "d''_U"] \arrow[r, dotted, shift right=2, "c''_U"'] \arrow[d, dotted, shift right=2, "c''_L"'] \arrow[d, dotted, shift left=2, "d''_L"] \arrow[dr, phantom, "(*_3)"] & B \arrow[l, dotted, "e''_U" description] \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
C \arrow[r, equal] \arrow[uuu, "e_L" description] & C \arrow[r, equal] \arrow[u, "e'_L" description] & C \arrow[u, dotted, "e''_L" description] \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] & D \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd}
\tag{3.4}
$$

We want to define the dotted maps so that each square of parallel arrows in (3.4) commutes. This means that the following must hold:

$$
\begin{cases} d''_U \tilde{q_S} = d'_U, \\ c''_U \tilde{q_S} = c'_U, \\ e''_U = \tilde{q_S} e'_U, \end{cases} \qquad \begin{cases} d''_L \tilde{q_T} = d'_L, \\ c''_L \tilde{q_T} = c'_L, \\ e''_L = \tilde{q_T} e'_L. \end{cases} \tag{3.5}
$$

Indeed notice that if (3.5) holds, then not only do the three squares at $(*_1)$ and the three at $(*_2)$ commute, but also the nine at $(*_3)$ (in order to show this it suffices to use the fact that q is an epimorphism). Furthermore we immediately have that $(\frac{A}{S\vee T}, B, d''_U, c''_U, e''_U)$ and $(\frac{A}{S\vee T}, C, d''_L, c''_L, e''_L)$ are reflexive graphs.

We can define e''_U and e''_L using (3.5), therefore it remains to show the existence of the other four maps satisfying (3.5) and that both the reflexive graphs just constructed have a multiplicative structure, that is

$$
[K_{d''_U}, K_{c''_U}]^{\mathcal{Q}}_{\frac{A}{S\vee T}} = 0 \qquad [K_{d''_L}, K_{c''_L}]^{\mathcal{Q}}_{\frac{A}{S\vee T}} = 0 \tag{3.6}
$$

Let us define for example d''_U (the other three maps are defined using the same strategy). In order to have the existence of such a map, we will take the following steps:

- we will show that $d_U \circ i_S = 0$,
- this will give us a map $\phi \colon \frac{A}{S} \to B$ such that $\phi \circ q_S = d_U = d'_U \circ q_T$,

- finally the universal property of the pushout will give us the map d''_U.

The only point that we need to treat more in detail is the first one since the others are straightforward.

Consider the commutative diagrams induced by taking kernels vertically

$$
\begin{tikzcd}
K_{d_L} \arrow[r, dotted] \arrow[d, tail] & K_{d_R} \arrow[d, tail] \\
A \arrow[r, "d_U"] \arrow[d, "d_L"'] & B \arrow[d, "d_R"] \\
C \arrow[r, "d_D"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
K_{c_L} \arrow[r, dotted] \arrow[d, tail] & K_{c_R} \arrow[d, tail] \\
A \arrow[r, "d_U"] \arrow[d, "c_L"'] & B \arrow[d, "c_R"] \\
C \arrow[r, "d_D"'] & D
\end{tikzcd}
$$

The dotted maps give us a morphism of coterminal pairs

$$
\begin{tikzcd}
K_{d_L} \arrow[r, tail] \arrow[d, dotted] & A \arrow[d, "d_U"] & K_{c_L} \arrow[l, tail] \arrow[d, dotted] \\
K_{d_R} \arrow[r, tail] & B & K_{c_R} \arrow[l, tail]
\end{tikzcd}
$$

which by Remark 1.1.52 implies the commutativity of

$$
\begin{tikzcd}
{[K_{d_L}, K_{c_L}]^{\mathcal{Q}}_A} \arrow[r, tail, "i_S"] \arrow[d] & A \arrow[d, "d_U"] \\
{[K_{d_R}, K_{c_R}]^{\mathcal{Q}}_B} \arrow[r, tail] & B
\end{tikzcd}
$$

Finally it suffices to use that (B, D, d_R, c_R, e_R) already admits a multiplicative structure: hence $[K_{d_R}, K_{c_R}]^{\mathcal{Q}}_B \cong 0$ and $d_U \circ i_S = 0$.

The last step of this construction is the proof of the equalities (3.6): we prove only the first one since the strategy for the second one is the same. Consider the diagram of exact sequences

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_{\tilde{q_S}} \arrow[r, tail] \arrow[d, equal] & K_{d'_U} \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "(**)"] & K_{d''_U} \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & K_{\tilde{q_S}} \arrow[r, tail] \arrow[d, two heads] & \frac{A}{T} \arrow[r, two heads, "\tilde{q_S}"] \arrow[d, two heads, "d'_U"'] & \frac{A}{S \vee T} \arrow[r] \arrow[d, two heads, "d''_U"] & 0 \\
0 \arrow[r] & 0 \arrow[r, tail] \arrow[d] & B \arrow[r, equal] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Once again the fact that we have a regular epimorphism on the first row is given by Lemma 1.1.18: hence the REM-factorisation of $\tilde{q_S} \circ k_{d'_U}$ is the one given by the square $(**)$. In the exact same way it is possible to describe the REM-factorisation of $\tilde{q_S} \circ k_{c'_U}$.

Now we can apply Proposition 1.1.54 to the diagram

$$\begin{tikzcd}
K_{d'_U} \arrow[r, tail] \arrow[d, two heads] & \frac{A}{T} \arrow[d, two heads, "\tilde{q_S}"] & K_{c'_U} \arrow[l, tail] \arrow[d, two heads] \\
K_{d''_U} \arrow[r, tail] & \frac{A}{S \vee T} & K_{c''_U} \arrow[l, tail, "l'"]
\end{tikzcd}$$

obtaining that the induced morphism

$$[K_{d'_U}, K_{c'_U}]^{\mathcal{Q}}_{\frac{A}{T}} \twoheadrightarrow [K_{d''_U}, K_{c''_U}]^{\mathcal{Q}}_{\frac{A}{S \vee T}}$$

is a regular epimorphism. But we know that

$$[K_{d'_U}, K_{c'_U}]^{\mathcal{Q}}_{\frac{A}{T}} = 0$$

by construction (see Construction 1.3.12) therefore we deduce that

$$[K_{d''_U}, K_{c''_U}]^{\mathcal{Q}}_{\frac{A}{S \vee T}} = 0$$

This means that $(\frac{A}{S \vee T}, B, d''_U, c''_U, e''_U)$ admits a (unique) multiplicative structure. The same reasoning works for $(\frac{A}{S \vee T}, C, d''_L, c''_L, e''_L)$ giving us that the square $(*_3)$ in (3.4) is a double reflexive multiplicative graph.

Proposition 3.1.8. *Consider a special double reflexive graph as in* (3.2), *then the morphism of double reflexive graphs*

$$\begin{tikzcd}
A \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift right=2, "c_L"'] \arrow[d, shift left=2, "d_L"] & B \arrow[l, "e_U"' description] \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
C \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] \arrow[u, "e_L" description] & D \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd}
\xrightarrow{\begin{pmatrix} q & 1_B \\ 1_C & 1_D \end{pmatrix}}
\begin{tikzcd}
\frac{A}{S \vee T} \arrow[r, shift left=2, "d''_U"] \arrow[r, shift right=2, "c''_U"'] \arrow[d, shift right=2, "c''_L"'] \arrow[d, shift left=2, "d''_L"] & B \arrow[l, "e''_U"' description] \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
C \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] \arrow[u, "e''_L" description] & D \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd} \tag{3.7}$$

constructed in Construction 3.1.7 coincides with the unit of the adjunction between double groupoids and double reflexive graphs induced by the adjunction between groupoids and reflexive graphs (see Construction 1.3.12 and Proposition 1.3.13).

Proof. Consider another morphism of double reflexive graphs

$$
\begin{tikzcd}
A \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift right=2, "c_L"'] \arrow[d, shift left=2, "d_L"] & B \arrow[l, "e_U" description] \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
C \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] \arrow[u, "e_L" description] & D \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd}
\xrightarrow{\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)}
\begin{tikzcd}
A' \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] \arrow[d, shift right=2, "c_W"'] \arrow[d, shift left=2, "d_W"] & B' \arrow[l, "e_N" description] \arrow[d, shift right=2, "c_E"'] \arrow[d, shift left=2, "d_E"] \\
C' \arrow[r, shift left=2, "d_S"] \arrow[r, shift right=2, "c_S"'] \arrow[u, "e_W" description] & D' \arrow[l, "e_S" description] \arrow[u, "e_E" description]
\end{tikzcd}
$$

in which the codomain is a double groupoid. We want to define a map $\phi\colon \frac{A}{S \vee T} \to A'$ such that $\phi \circ q = \alpha$ and in order to do this, consider the diagrams

$$
\begin{tikzcd}
A \arrow[rr, shift left=2, "d_U"] \arrow[rr, shift right=2, "c_U"'] \arrow[dr, two heads, "q_T"] \arrow[dd, "\alpha"'] & & B \arrow[ll, "e_U" description] \arrow[d, equal] \\
& \frac{A}{T} \arrow[r, shift left=2, "d'_U"] \arrow[r, shift right=2, "c'_U"'] \arrow[dl, dotted, "\phi_T"] & B \arrow[l, "e'_U" description] \arrow[d, "\beta"] \\
A' \arrow[rr, shift left=2, "d_N"] \arrow[rr, shift right=2, "c_N"'] & & B' \arrow[ll, "e_N" description]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, shift left=2, "d_L"] \arrow[rr, shift right=2, "c_L"'] \arrow[dr, two heads, "q_S"] \arrow[dd, "\alpha"'] & & C \arrow[ll, "e_L" description] \arrow[d, equal] \\
& \frac{A}{S} \arrow[r, shift left=2, "d'_L"] \arrow[r, shift right=2, "c'_L"'] \arrow[dl, dotted, "\phi_S"] & C \arrow[l, "e'_L" description] \arrow[d, "\gamma"] \\
A' \arrow[rr, shift left=2, "d_W"] \arrow[rr, shift right=2, "c_W"'] & & C' \arrow[ll, "e_W" description]
\end{tikzcd}
$$

with the same notation as in (3.4). Here the dotted maps are defined through the universal property of the unit of the adjunction between $\mathbf{RG}(\mathbb{A})$ and $\mathbf{Grpd}(\mathbb{A})$ (see Proposition 1.3.13). Now we can simply define ϕ by using the universal property of the pushout

$$
\begin{tikzcd}
A \arrow[r, two heads, "q_T"] \arrow[d, two heads, "q_S"'] \arrow[dr, two heads, "q"] & \frac{A}{T} \arrow[d, two heads, "\tilde{q_S}"] \arrow[ddr, bend left, "\phi_T"] & \\
\frac{A}{S} \arrow[r, two heads, "\tilde{q_T}"'] \arrow[drr, bend right, "\phi_S"'] & \frac{A}{S \vee T} \arrow[dr, dotted, "\phi"] & \\
& & A'
\end{tikzcd}
$$

Now it is trivial to see that $\left(\begin{smallmatrix}\phi & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ is a morphism of double groupoids and that it is the only one such that

$$
\begin{pmatrix}\phi & \beta \\ \gamma & \delta\end{pmatrix} \circ \begin{pmatrix} q & 1_B \\ 1_C & 1_D \end{pmatrix} = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}. \qquad \square
$$

Corollary 3.1.9. *Consider two groupoids of the form*

$$
\begin{tikzcd}
& B \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
C \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] & D \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd}
\tag{3.8}
$$

and construct the following span

$$
\begin{tikzcd}
D \arrow[rr, shift left=2, "1"] \arrow[rr, shift right=2, "1"'] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & & D \arrow[ll, "1" description] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & & B \arrow[rr, shift left=2, "1"] \arrow[rr, shift right=2, "1"'] \arrow[dd, shift right=2, "c_R"'] \arrow[dd, shift left=2, "d_R"] & & B \arrow[ll, "1" description] \arrow[dd, shift right=2, "c_R"'] \arrow[dd, shift left=2, "d_R"] \\
& & {} \arrow[rr, "\left(\begin{smallmatrix} e_R & e_R \\ 1_D & 1_D \end{smallmatrix}\right)"] & & {} & & \\
D \arrow[uu, "1" description] \arrow[rr, shift left=2, "1"] \arrow[rr, shift right=2, "1"'] & {} \arrow[dd, "\left(\begin{smallmatrix} e_D & 1_D \\ e_D & 1_D \end{smallmatrix}\right)"'] & D \arrow[ll, "1" description] \arrow[uu, "1" description] & & D \arrow[uu, "e_R" description] \arrow[rr, shift left=2, "1"] \arrow[rr, shift right=2, "1"'] & & D \arrow[ll, "1" description] \arrow[uu, "e_R" description] \\
& & & & & & \\
C \arrow[rr, shift left=2, "d_D"] \arrow[rr, shift right=2, "c_D"'] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & {} & D \arrow[ll, "e_D" description] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & & & & \\
& & & & & & \\
C \arrow[uu, "1" description] \arrow[rr, shift left=2, "d_D"] \arrow[rr, shift right=2, "c_D"'] & & D \arrow[ll, "e_D" description] \arrow[uu, "1" description] & & & &
\end{tikzcd}
\tag{3.9}
$$

in $\mathbf{Grpd}^2(\mathbb{A})$*. Then in order to obtain its pushout in* $\mathbf{Grpd}^2(\mathbb{A})$*, we can see it as a diagram in* $\mathbf{RG}^2(\mathbb{A})$ *(through the inclusion functor* $J\colon \mathbf{Grpd}^2(\mathbb{A}) \hookrightarrow \mathbf{RG}^2(\mathbb{A})$*), take its pushout in* $\mathbf{RG}^2(\mathbb{A})$ *and then apply Construction 3.1.7 to the double reflexive graph that we obtained.*

Proof. By computing the pushout in $\mathbf{RG}^2(\mathbb{A})$ (which is done componentwise) we obtain a special double reflexive graph of the form (3.2). Consequently, by Proposition 3.1.8 we deduce that applying Construction 3.1.7 is the same as applying the reflector $R\colon \mathbf{RG}^2(\mathbb{A}) \to \mathbf{Grpd}^2(\mathbb{A})$. But being the reflector a left adjoint, it preserves colimits: in particular it sends the pushout in $\mathbf{RG}^2(\mathbb{A})$ to a pushout in $\mathbf{Grpd}^2(\mathbb{A})$, that is the thesis. $\square$

3.1.3 Crossed squares

Definition 3.1.10 ([48, 61, 15])**.** A *crossed square (of groups)* is given by a commutative square

$$
\begin{tikzcd}
P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & L
\end{tikzcd}
$$

in **Grp**, together with actions of L on M, N and P (and hence actions of M on P and N via μ, and of N on M and P via ν) and a function (not a group morphism!) $h\colon M \times N \to P$ such that the following axioms hold:

0) $h(mm', n) = {}^{m}h(m', n)h(m, n)$ and $h(m, nn') = h(m, n)^{n}h(m, n')$;

i) the maps p_M and p_N preserve the actions of L, furthermore with the given actions $(M \xrightarrow{\mu} L)$, $(N \xrightarrow{\nu} L)$ and $(P \xrightarrow{\mu\circ p_M=\nu\circ p_N} L)$ are crossed modules;

ii) $p_M(h(m,n)) = m^n m^{-1}$ and $p_N(h(m,n)) = {}^m n n^{-1}$;

iii) $h(p_M(p),n) = p^n p^{-1}$ and $h(m,p_N(p)) = {}^m p p^{-1}$;

iv) ${}^l h(m,n) = h({}^l m, {}^l n)$;

for all $l \in L$, $m,m' \in M$, $n,n' \in N$ and $p \in P$.

A map of crossed squares is given by four group morphisms which are compatible with the actions and with the map h. Crossed squares and morphisms between them form the category $\mathbf{XSqr}(\mathbf{Grp})$.

Definition 3.1.11 ([15]). Given a pair of L-crossed modules $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$ in $\mathbf{Grp}$ we have an action ξ_N^M of M on N induced via μ and an action ξ_M^N of N on M induced via ν. We say that a map $h\colon M \times N \to P$ is a *crossed pairing* if the following hold for each $m,m' \in M$ and $n,n' \in N$

- $h(mm',n) = h({}^m m', {}^m n)h(m,n)$,
- $h(m,nn') = h(m,n)h({}^n m, {}^n n')$.

Remark 3.1.12. Notice that if we have a crossed square, then the map $h\colon M \times N \to P$ is actually a crossed pairing. Indeed by using *iv*) and the fact that the actions involved are induced from the actions of L we can show the equivalence between condition 0) and h being a crossed pairing, through the equalities

$$\begin{aligned}
{}^m h(m',n) &= {}^{\mu(m)}h(m',n) = h({}^{\mu(m)}m', {}^{\mu(m)}n) = h({}^m m', {}^m n),\\
{}^n h(m,n') &= {}^{\nu(n)}h(m,n') = h({}^{\nu(n)}m, {}^{\nu(n)}n') = h({}^n m, {}^n n').
\end{aligned}$$

In Proposition 5.2 in [61] and in Theorem 18 of [73], it is proved that Definition 3.1.10 is equivalent to the one of a cat^2-group. By using the fact that crossed modules can be equivalently described as groupoids or as cat^1-groups, that is by using the equivalences of categories

$$\mathbf{cat}^1\text{-}\mathbf{Grp} \simeq \mathbf{Grpd}(\mathbb{A}) \simeq \mathbf{XMod}(\mathbb{A})$$

one can then show that any crossed square can be depicted as an internal crossed module in the category of crossed modules of groups. This means that we have the equivalences

$$\mathbf{XSqr}(\mathbf{Grp}) \simeq \mathbf{cat}^2\text{-}\mathbf{Grp} \simeq \mathbf{Grpd}^2(\mathbb{A}) \simeq \mathbf{XMod}(\mathbf{XMod}(\mathbb{A}))$$

In particular the functor from $\mathbf{Grpd}^2(\mathbf{Grp})$ to $\mathbf{XSqr}(\mathbf{Grp})$ is given by normalisation.

In general internal crossed square don't have an explicit description as in Definition 3.1.10, but, following the idea of the previous chain of equivalences, they are directly defined as follows.

Definition 3.1.13. An *internal crossed square in* $\mathbb{A}$ is an internal crossed module in $\mathbf{XMod}(\mathbb{A})$. This means that the category $\mathbf{XSqr}(\mathbb{A})$ is defined as $\mathbf{XMod}(\mathbf{XMod}(\mathbb{A}))$.

This means that an internal crossed square is simply a square (endowed with additional structure) obtained by normalising a double groupoid.

Lemma 3.1.14. *An internal crossed square in* $\mathbb{A}$ *is the normalisation of a double groupoid in* $\mathbb{A}$*, that is the outer square in the diagram*

$$\begin{tikzcd}
P \arrow[r, tail, "k_{d_T}"] \arrow[d, tail, "k_{d_W}"'] \arrow[dr, phantom, "\lrcorner", very near start] & K_{d_L} \arrow[r, shift left=2, "d_T"] \arrow[r, shift right=2, "c_T"'] \arrow[d, tail, "k_{d_L}"] & K_{d_R} \arrow[l, "e_T" description] \arrow[d, tail, "k_{d_R}"] \\
K_{d_U} \arrow[r, tail, "k_{d_U}"'] \arrow[d, shift right=2, "c_W"'] \arrow[d, shift left=2, "d_W"] & A \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift right=2, "c_L"'] \arrow[d, shift left=2, "d_L"] & B \arrow[l, "e_U" description] \arrow[d, shift right=2, "c_R"'] \arrow[d, shift left=2, "d_R"] \\
K_{d_D} \arrow[r, tail, "k_{d_D}"'] \arrow[u, "e_W" description] & C \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] \arrow[u, "e_L" description] & D \arrow[l, "e_D" description] \arrow[u, "e_R" description]
\end{tikzcd} \tag{3.10}$$

obtained taking kernels of the domain morphisms and the induced maps. Similarly a morphism of internal crossed squares is the (unique) normalisation of a morphism of double groupoids.

Proof. It suffices to use the equivalence $\mathbf{XMod}(\mathbb{A}) \simeq \mathbf{Grpd}(\mathbb{A})$ given by normalisation and denormalisation to obtain

$$\mathbf{XSqr}(\mathbb{A}) = \mathbf{XMod}(\mathbf{XMod}(\mathbb{A})) \simeq \mathbf{Grpd}^2(\mathbb{A}). \qquad \square$$

One can easily see that there is a lack of a definition of internal crossed square that follows the lines of Definition 3.1.10, that is one that describes explicitly these objects as squares of crossed modules with additional structure satisfying some axioms. We will do a step in this direction with the definition of *weak crossed squares* (see Section 3.3).

Referring to the diagram (3.10) we will denote with j the diagonal of the upper left square, with (D, A, c, d, e) the reflexive graph structure induced diagonally in the lower right square and with λ the composition $c \circ j$.

Remark 3.1.15. Given a double groupoid as in (3.10) we can define an action of D on P in the following different ways:

- First of all we can define it as the dotted arrow in the diagram

$$\begin{tikzcd}
D \flat P \arrow[r, "e \flat k"] \arrow[d, dotted, "\xi"'] & A \flat A \arrow[r, "\langle d_U, d_L\rangle \flat \langle d_U, d_L\rangle"] \arrow[d, "\chi_A"] & (B \times C) \flat (B \times C) \arrow[d, "\chi_{(B\times C)}"] \\
P \arrow[r, tail, "k"'] & A \arrow[r, "\langle d_U, d_L\rangle"'] & B \times C
\end{tikzcd} \tag{3.11}$$

where $k = k_{d_L} \circ k_{d_T} = k_{d_U} \circ k_{d_W}$ is the kernel of $\langle d_U, d_L \rangle$;

- either we induce it through the diagram

$$\begin{tikzcd}
D\flat P \arrow[r, "e_R \flat k_{d_W}"] \arrow[d, dotted, "\xi"'] & B\flat K_{d_U} \arrow[r, "d_R \flat d_W"] \arrow[d, "\psi_U"] & D\flat K_{d_D} \arrow[d, "\psi_D"] \\
P \arrow[r, tail, "k_{d_W}"'] & K_{d_U} \arrow[r, "d_W"'] & K_{d_D}
\end{tikzcd} \tag{3.12}$$

- or simmetrically we can also use

$$\begin{tikzcd}
D\flat P \arrow[r, "e_D \flat k_{d_T}"] \arrow[d, dotted, "\xi"'] & C\flat K_{d_L} \arrow[r, "d_D \flat d_T"] \arrow[d, "\psi_L"] & D\flat K_{d_R} \arrow[d, "\psi_R"] \\
P \arrow[r, tail, "k_{d_T}"'] & K_{d_L} \arrow[r, "d_T"'] & K_{d_R}
\end{tikzcd} \tag{3.13}$$

Notice that these three actions are uniquely determined by the universal property of the kernels and that they are actually the same: indeed it suffices to show that if such a ξ makes one of the previous diagrams commute, then also the other two are satisfied. This is easily shown by the diagrams

$$\begin{tikzcd}
D\flat P \arrow[r, "e_R \flat k_{d_W}"] \arrow[d, dotted, "\xi"'] & B\flat K_{d_U} \arrow[r, "e_U \flat k_{d_U}"] \arrow[d, "\psi_U"] & A\flat A \arrow[d, "\chi_A"] \\
P \arrow[r, tail, "k_{d_W}"'] & K_{d_U} \arrow[r, tail, "k_{d_U}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
D\flat P \arrow[r, "e_D \flat k_{d_T}"] \arrow[d, dotted, "\xi"'] & C\flat K_{d_L} \arrow[r, "e_D \flat k_{d_L}"] \arrow[d, "\psi_L"] & A\flat A \arrow[d, "\chi_A"] \\
P \arrow[r, tail, "k_{d_T}"'] & K_{d_L} \arrow[r, tail, "k_{d_L}"'] & A
\end{tikzcd}$$

In each rectangle the rightmost square commutes by Remark 1.2.8, therefore the leftmost square in each rectangle commutes if and only if the corresponding rectangle commutes (also because k_{d_U} and k_{d_L} are monomorphisms), but both the rectangles are the same as the left square in (3.11). Hence the three definitions are the same.

Remark 3.1.16. We can also define an action of D on P in the following way. Consider the diagram

$$\begin{tikzcd}
P \arrow[r, tail, "k_{d_T}"] \arrow[d, tail, "k_{d_W}"'] & K_{d_L} \arrow[r, equal] \arrow[d] & K_{d_L} \arrow[r, equal] \arrow[dd] & K_{d_L} \arrow[ddd, tail, "k_{d_L}"] \\
K_{d_U} \arrow[r] \arrow[d, equal] & K_{d_L} + K_{d_U} \arrow[dr, two heads] & & \\
K_{d_U} \arrow[rr] \arrow[d, equal] & & K_{d_L} \vee K_{d_U} \arrow[dr, tail, "k_{d_L} \vee k_{d_U}"] & \\
K_{d_U} \arrow[rrr, tail, "k_{d_U}"'] & & & A
\end{tikzcd}$$

and denote with l the map going from P to $K_{d_L} \vee K_{d_U}$. Consider the diagonal point defined through diagram (3.10)

$$\begin{tikzcd} K_{d_L} \vee K_{d_U} \arrow[r, tail, "k_d"] & A \arrow[r, shift left, "d"] & D \arrow[l, shift left, "e"] \end{tikzcd}$$

Notice that $K_{d_L} \vee K_{d_U}$ is the kernel of d (and that $k_d = k_{d_L} \vee k_{d_U}$) because of Lemma 1.1.23 and Corollary 1.2.3. Now since $k_d \circ l$ is a normal monomorphism, through Lemma 2.6 in [28], we can construct the diagram

$$\begin{tikzcd} P \arrow[r, tail, dotted, "k_{\hat{d}}"] \arrow[d, tail, "l"'] & \hat{A} \arrow[r, shift left, dotted, "\hat{d}"] \arrow[d, dotted, tail, "l \rtimes 1"'] & D \arrow[l, shift left, dotted, "\hat{e}"] \arrow[d, equal] \\ K_{d_L} \vee K_{d_U} \arrow[r, tail, "k_d"'] & A \arrow[r, shift left, "d"] & D \arrow[l, shift left, "e"] \end{tikzcd} \tag{3.14}$$

which gives us an action of D on P through the equivalence between points and actions.

Lemma 3.1.17. *The actions defined in Remark 3.1.15 and in Remark 3.1.16 are the same action ξ.*

Proof. In order to show this, it suffices to prove that the equivalence $\mathbf{Act}(\mathbb{A}) \simeq \mathbf{Pt}(\mathbb{A})$ sends the point constructed in (3.14) into the action ξ uniquely defined through the commutativity of (3.11). To do this consider the diagram

$$\begin{tikzcd} D\flat P \arrow[r, "k_{D,P}"] \arrow[d, dotted, tail] & D+P \arrow[r, shift left, "\binom{1}{0}"] \arrow[d, "\binom{\hat{e}}{k_{\hat{d}}}"'] & D \arrow[l, shift left, "i_D"] \arrow[d, equal] \\ P \arrow[r, "k_{\hat{d}}"] & \hat{A} \arrow[r, shift left, "\hat{d}"] & D \arrow[l, shift left, "\hat{e}"] \end{tikzcd}$$

and let us prove that $k_{\hat{d}} \circ \xi = \binom{\hat{e}}{k_{\hat{d}}} \circ k_{D,P}$. The map $l \rtimes 1$ is a monomorphism since l is so, therefore the thesis become

$$(l \rtimes 1) \circ k_{\hat{d}} \circ \xi = (l \rtimes 1) \circ \begin{pmatrix} \hat{e} \\ k_{\hat{d}} \end{pmatrix} \circ k_{D,P}.$$

The lefthand side is equal to $k \circ \xi$ which in turn (by definition of ξ) is $\chi_A \circ (e\flat k)$, whereas for the righthand side we have the following chain of equalities

$$\begin{aligned} (l \rtimes 1) \circ \begin{pmatrix} \hat{e} \\ k_{\hat{d}} \end{pmatrix} \circ k_{D,P} &= \begin{pmatrix} (l \rtimes 1) \circ \hat{e} \\ (l \rtimes 1) \circ k_{\hat{d}} \end{pmatrix} \circ k_{D,P} \\ &= \begin{pmatrix} e \\ k \end{pmatrix} \circ k_{D,P} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ (e+k) \circ k_{D,P} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} k_{A,A} \circ (e\flat k) = \chi_A \circ (e\flat k). \end{aligned}$$

□

Proposition 3.1.18. *In the previous situation we have that* $(P \xrightarrow{\lambda} D, \xi)$ *is a crossed module.*

Proof. Notice that if we define $\hat{c} := c \circ (l \rtimes 1)$ we have that $\hat{c} \circ k_{\hat{d}} = c \circ j = \lambda$. Therefore it suffices to show that the first row in (3.14) is actually a groupoid once it is endowed with $\hat{c}$ as second leg. In order to prove this, by using Remark 1.1.50 and Lemma 1.3.11 we only need to show that $[P, K_{\hat{c}}] = 0$ since $P = K_{\hat{d}}$. But $K_{\hat{c}} \hookrightarrow K_c$ implies $[P, K_{\hat{c}}] \hookrightarrow [P, K_c]$, hence it suffices to show that $[P, K_c] = 0$. We have the following chain of monomorphisms

$$
\begin{aligned}
[P, K_c] &\cong [P, K_{c_U} \vee K_{c_L}] \\
&\cong [P, K_{c_U}] \vee [P, K_{c_L}] \vee [P, K_{c_U}, K_{c_L}] \\
&\subseteq [K_{d_U}, K_{c_U}] \vee [K_{d_L}, K_{c_L}] \vee [K_{c_U}, K_{c_U}, A] \\
&\cong 0 \vee 0 \vee 0 \cong 0
\end{aligned}
$$

where the first isomorphism is given by Lemma 1.1.23, the second one is given by the join decomposition formula described in Proposition 2.22 in [51] and the last ones are due to Theorem 5.2 again in [51]. □

Proposition 3.1.19. *Given a morphism of double groupoids*

$$
\begin{tikzcd}
A \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] & B \arrow[l] \arrow[d, shift left=2] \arrow[d, shift right=2] \\
C \arrow[u] \arrow[r, shift left=2] \arrow[r, shift right=2] & D \arrow[l] \arrow[u]
\end{tikzcd}
\xrightarrow{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}
\begin{tikzcd}
A' \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] & B' \arrow[l] \arrow[d, shift left=2] \arrow[d, shift right=2] \\
C' \arrow[u] \arrow[r, shift left=2] \arrow[r, shift right=2] & D' \arrow[l] \arrow[u]
\end{tikzcd}
$$

consider the unique morphism of internal crossed squares induced between their normalisations, and denote $\rho \colon P \to P'$ *the upper-left component. Then*

$$(P \xrightarrow{\lambda} D, \xi) \xrightarrow{(\rho, \delta)} (P' \xrightarrow{\lambda'} D', \xi')$$

is a morphism of internal crossed module.

Proof. We want to show the commutativity of the diagrams

$$
\begin{tikzcd}
P \arrow[r, "\lambda"] \arrow[d, "\rho"'] & D \arrow[d, "\delta"] \\
P' \arrow[r, "\lambda'"'] & D'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
D \flat P \arrow[r, "\xi"] \arrow[d, "\delta \flat \rho"'] & P \arrow[d, "\rho"] \\
D' \flat P' \arrow[r, "\xi'"'] & P'
\end{tikzcd}
$$

The first one is obvious by construction of the map ρ. For the second one we need to use one of the explicit constructions for the actions ξ and ξ', in particular the one depicted in (3.11). From this we construct the cube

$$
\begin{tikzcd}
D\flat P \arrow[rr, "e\flat k"] \arrow[dr, "\delta\flat\rho"] \arrow[dd, "\xi"'] & & A\flat A \arrow[dr, "\alpha\flat\alpha"] \arrow[dd, "\chi_A"' near start] & \\
& D'\flat P' \arrow[rr, "e'\flat k'" near start] \arrow[dd, "\xi'"' near start] & & A'\flat A' \arrow[dd, "\chi_{A'}"] \\
P \arrow[rr, tail, "k" near start] \arrow[dr, "\rho"'] & & A \arrow[dr, "\alpha"] & \\
& P' \arrow[rr, tail, "k'"'] & & A'
\end{tikzcd}
$$

We want to show that the face on the left commutes, but since we already know that every other face commutes, it suffice to postcompose with the monomorphism k' to obtain

$$
\begin{aligned}
k' \circ \xi' \circ (\delta\flat\rho) &= \chi_{A'} \circ (e'\flat k') \circ (\delta\flat\rho) \\
&= \chi_{A'} \circ (\alpha\flat\alpha) \circ (e\flat k) \\
&= \alpha \circ \chi_A \circ (e\flat k) \\
&= \alpha \circ k \circ \xi \\
&= k' \circ \rho \circ \xi.
\end{aligned}
$$

□

3.2 Construction of the Non-Abelian Tensor Product

3.2.1 Groups case

First of all, let us examine what happens in the category **Grp**: the aim of this subsection is to show how to construct the non-abelian tensor product of two coterminal crossed modules of groups, without passing through set-theoretical constructions.

Consider two groups M and N acting on each other via $\xi_N^M \colon M\flat N \to N$ and $\xi_M^N \colon N\flat M \to M$ and denote with ${}^m n$ the action of $m \in M$ on $n \in N$ and with ${}^n m$ the action of $n \in N$ on $m \in M$.

Definition 3.2.1. Given two groups M and N acting on each other (and on themselves by conjugation) we define their *non-abelian tensor product* $M \otimes N$ as the group generated by the symbols $m \otimes n$ with $m \in M$ and $n \in N$ with the relations

- $(mm') \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n)$,
- $m \otimes (nn') = (m \otimes n)({}^n m \otimes {}^n n')$,

for all $m, m' \in M$ and $n, n' \in N$.

Even if it is possible to give a definition of the non-abelian tensor product even when the two actions are not compatible, the main results of [15] that we are interested in, always assume compatibility. Hence from now on we will do the same, dealing only with non-abelian tensor products of two compatible actions.

Remark 3.2.2. Notice that by Chapter 2 we can equivalently use pairs of compatible actions or pair of coterminal crossed modules, both in the group case and in the semi-abelian one. Therefore we choose to use the latter from now on.

In order to describe the alternative construction of the non-abelian tensor product in the semi-abelian context we need to recall the following result from [15].

Proposition 3.2.3 (Proposition 2.15 in [15])**.** *Let $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$ be crossed modules, so that M and N act on both M and N via P. Then there is a crossed square*

$$\begin{tikzcd} M \otimes N \arrow[r, "\pi_M"] \arrow[d, "\pi_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

where $\pi_M(m \otimes n) = m^n m^{-1}$, $\pi_N(m \otimes n) = {}^m n n^{-1}$ and $h(m,n) = m \otimes n$. This crossed square is universal in the sense that it satisfies the following two equivalent conditions:

(1) If

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

is another crossed square (with the same μ and ν), then there is a unique morphism

$$\begin{tikzcd} M \otimes N \arrow[r, "\pi_M"] \arrow[d, "\pi_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd} \xrightarrow{\begin{pmatrix} \phi & 1_M \\ 1_N & 1_L \end{pmatrix}} \begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

of crossed squares, which is the identity on M, N and L.

(2) The diagram of inclusions of crossed squares

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[d] & & & 0 \arrow[r] \arrow[d] & M \arrow[d, "\mu"] \\
0 \arrow[r] & L & {} \arrow[r, "\left(\begin{smallmatrix}1_0 & 0 \\ 1_0 & 1_L\end{smallmatrix}\right)"] & {} & 0 \arrow[r] & L \\
& {} \arrow[d, "\left(\begin{smallmatrix}1_0 & 1_0 \\ 0 & 1_L\end{smallmatrix}\right)"'] & {} \arrow[r, phantom, "\ulcorner"] & {} & {} \arrow[d, "\left(\begin{smallmatrix}0 & 1_M \\ 0 & 1_L\end{smallmatrix}\right)"] & \\
& {} & & & {} & \\
0 \arrow[r] \arrow[d] & 0 \arrow[d] & & & M \otimes N \arrow[r, "\pi_M^{M\otimes N}"] \arrow[d, "\pi_N^{M\otimes N}"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & L & {} \arrow[r, "\left(\begin{smallmatrix}0 & 0 \\ 1_N & 1_L\end{smallmatrix}\right)"'] & {} & N \arrow[r, "\nu"'] & L
\end{tikzcd}
\tag{3.15}
$$

is a pushout in **XSqr**(**Grp**). □

We can reinterpret this result as a way to construct the non-abelian tensor product $M \otimes N$ as the upper-left group in the pushout crossed square: this process does not involve generators and relations and hence completely avoids the use of set-theoretical tools. In order to generalise this construction to $\mathbf{XSqr}(\mathbb{A})$ we need what we have done in the previous section, that is a description of how to compute pushouts of the previous kind in the category of $\mathbf{Grpd}^2(\mathbb{A})$ (since it is equivalent to $\mathbf{XSqr}(\mathbb{A})$).

3.2.2 Construction in semi-abelian categories

Imitating in a semi-abelian category $\mathbb{A}$ what we've done so far for groups we have the following construction.

Construction 3.2.4. Consider two L-crossed modules $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$ and their induced internal groupoid structures

$$
\begin{tikzcd}
& & M \arrow[d, tail, "k_M"] \\
& & M \rtimes L \arrow[d, shift right=3, "c_M"'] \arrow[d, shift left=3, "d_M"] \\
N \arrow[r, tail, "k_N"] & N \rtimes L \arrow[r, shift left=3, "d_N"] \arrow[r, shift right=3, "c_N"'] & L \arrow[l, "e_N"'] \arrow[u, "e_M"]
\end{tikzcd}
\tag{3.16}
$$

We construct the span (3.9) in $\mathbf{Grpd}^2(\mathbb{A})$ and in order to compute its pushout we use Lemma 3.1.9. This means that we see it as a diagram in $\mathbf{RG}^2(\mathbb{A})$ and we compute the pushout

$$\begin{tikzcd}[column sep=huge, row sep=huge]
Q \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift right=2, "c_L"'] \arrow[d, shift left=2, "d_L"] \arrow[dr, phantom, very near start, "\ulcorner"] & M \rtimes L \arrow[l, "e_U" description] \arrow[d, shift right=2, "c_M"'] \arrow[d, shift left=2, "d_M"] \\
N \rtimes L \arrow[u, "e_L" description] \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[l, "e_N" description] \arrow[u, "e_M" description]
\end{tikzcd} \tag{3.17}$$

in $\mathbf{RG}^2(\mathbb{A})$, which is given by the pointwise pushout of the previous diagram: this means that

$$Q := (M \rtimes L) +_L (N \rtimes L)$$

is the pushout of e_M along e_N and that the maps d_U, c_U, d_L and c_L are defined as follows, using the universal property of the pushout:

$$d_U := \left\langle \begin{matrix} 1_{M \rtimes L} \\ e_M \circ d_N \end{matrix} \right\rangle \qquad d_L := \left\langle \begin{matrix} e_N \circ d_M \\ 1_{N \rtimes L} \end{matrix} \right\rangle$$

$$c_U := \left\langle \begin{matrix} 1_{M \rtimes L} \\ e_M \circ c_N \end{matrix} \right\rangle \qquad c_L := \left\langle \begin{matrix} e_N \circ c_M \\ 1_{N \rtimes L} \end{matrix} \right\rangle$$

Finally Corollary 3.1.9 tells us that by applying Construction 3.1.7 to (3.17) we obtain the desired pushout

$$\begin{tikzcd}[column sep=huge, row sep=huge]
Q_{M \otimes N} \arrow[r, shift left=2, "\overline{d_U}"] \arrow[r, shift right=2, "\overline{c_U}"'] \arrow[d, shift right=2, "\overline{c_L}"'] \arrow[d, shift left=2, "\overline{d_L}"] & M \rtimes L \arrow[l, "\overline{e_U}" description] \arrow[d, shift right=2, "c_M"'] \arrow[d, shift left=2, "d_M"] \\
N \rtimes L \arrow[u, "\overline{e_L}" description] \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[l, "e_N" description] \arrow[u, "e_M" description]
\end{tikzcd} \tag{3.18}$$

of (3.9) in $\mathbf{Grpd}^2(\mathbb{A})$ (the notation for the maps here is a bit different: we use an overline instead of double apices). Here $Q_{M \otimes N}$ is given by

$$Q_{M \otimes N} := \frac{(M \rtimes L) +_L (N \rtimes L)}{[K_{d_L}, K_{c_L}] \vee [K_{d_U}, K_{c_U}]}$$

By normalising this double groupoid (that is computing the kernels of the "domain" morphisms and of the induced maps), we go back from $\mathbf{Grpd}^2(\mathbb{A})$ to $\mathbf{XSqr}(\mathbb{A})$ obtaining the crossed square

$$
\begin{tikzcd}
M \otimes N \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & K_{\overline{d_L}} \arrow[r, shift left=2] \arrow[d, tail] & M \arrow[l] \arrow[l, shift left=2, leftarrow] \arrow[d, tail] \\
K_{\overline{d_U}} \arrow[r, tail] \arrow[d, shift right=2] \arrow[d, shift left=2] & Q_{M\otimes N} \arrow[r, shift left=2, "\overline{d_U}"] \arrow[r, shift right=2, "\overline{c_U}"'] \arrow[d, shift right=2, "\overline{c_L}"'] \arrow[d, shift left=2, "\overline{d_L}"] & M \rtimes L \arrow[l, "\overline{e_U}" description] \arrow[d, shift right=2, "c_M"'] \arrow[d, shift left=2, "d_M"] \\
N \arrow[r, tail] \arrow[u] & N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] \arrow[u, "\overline{e_L}" description] & L \arrow[l, "e_N" description] \arrow[u, "e_M" description]
\end{tikzcd}
\tag{3.19}
$$

Using the equivalence $\mathbf{XSqr}(\mathbb{A}) \simeq \mathbf{Grpd}^2(\mathbb{A})$ we now have that this crossed square is the pushout in $\mathbf{XSqr}(\mathbb{A})$ depicted in (3.15).

Definition 3.2.5. Given a pair of L-crossed modules $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$ we define their *non-abelian tensor product* $M \otimes N$ as the top left object in the crossed square

$$
\begin{tikzcd}
M \otimes N \arrow[r, "\pi_M^{M\otimes N}"] \arrow[d, "\pi_N^{M\otimes N}"'] \arrow[dr, "\lambda" description] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & L
\end{tikzcd}
$$

constructed above.

Corollary 3.2.6. *The non-abelian tensor product $M \otimes N$ has an L-crossed module structure, namely $(M \otimes N \xrightarrow{\lambda} L, \xi)$ where the action ξ is defined as in Remark 3.1.15.*

Proof. This is just an application of Proposition 3.1.18. $\square$

Notice that the previous corollary implies that the non-abelian tensor product is a "binary operation" on the objects

$$- \otimes - \colon \mathbf{XMod}_L(\mathbb{A}) \times \mathbf{XMod}_L(\mathbb{A}) \to \mathbf{XMod}_L(\mathbb{A}).$$

This is obviously commutative, up to isomorphism, by construction; but it is not associative (see [37]).

Proposition 3.2.7. *Consider two L-crossed modules*

$$(M \xrightarrow{\mu} L, \xi_M^L), \qquad (N \xrightarrow{\nu} L, \xi_N^L),$$

two L'-crossed modules

$$(M' \xrightarrow{\mu'} L', \xi'^{L'}_{M'}), \qquad (N' \xrightarrow{\nu'} L', \xi'^{L'}_{N'}),$$

and two morphisms of internal crossed modules

$$(M \xrightarrow{\mu} L, \xi^L_M) \xrightarrow{(f,l)} (M' \xrightarrow{\mu'} L', \xi'^{L'}_{M'})$$
$$(N \xrightarrow{\nu} L, \xi^L_N) \xrightarrow{(g,l)} (N' \xrightarrow{\nu'} L', \xi'^{L'}_{N'})$$

Then there exists a unique morphism $f \otimes g\colon M \otimes N \to M' \otimes N'$ *such that* $\left(\begin{smallmatrix} f\otimes g & f \\ g & l \end{smallmatrix}\right)$ *is a morphism of internal crossed squares.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
M \otimes N \arrow[rr, tail] \arrow[dd, tail] \arrow[dr, dotted, "f\otimes g"] & & (M\otimes N) \rtimes M \arrow[rr, shift left=1.5] \arrow[rr, leftarrow] \arrow[rr, shift right=1.5] \arrow[dd, tail] \arrow[dr, dotted, "(f\otimes g)\rtimes f"] & & M \arrow[dd, tail] \arrow[dr, "f"] & \\
& M' \otimes N' \arrow[rr, tail] \arrow[dd, tail] & & (M'\otimes N') \rtimes M' \arrow[rr, shift left=1.5] \arrow[rr, leftarrow] \arrow[rr, shift right=1.5] \arrow[dd, tail] & & M' \arrow[dd, tail] \\
(M\otimes N) \rtimes N \arrow[rr, tail] \arrow[dd, shift right=1.5] \arrow[dd, leftarrow] \arrow[dd, shift left=1.5] \arrow[dr, dotted, "(f\otimes g)\rtimes g"] & & Q_{M\otimes N} \arrow[rr, shift left=1.5] \arrow[rr, leftarrow] \arrow[rr, shift right=1.5] \arrow[dd, shift right=1.5] \arrow[dd, leftarrow] \arrow[dd, shift left=1.5] \arrow[dr, dotted, "\phi"] & & M \rtimes L \arrow[dd, shift right=1.5] \arrow[dd, leftarrow] \arrow[dd, shift left=1.5] \arrow[dr, "f\rtimes l"] & \\
& (M'\otimes N') \rtimes N' \arrow[rr, tail] \arrow[dd, shift right=1.5] \arrow[dd, leftarrow] \arrow[dd, shift left=1.5] & & Q_{M'\otimes N'} \arrow[rr, shift left=1.5] \arrow[rr, leftarrow] \arrow[rr, shift right=1.5] \arrow[dd, shift right=1.5] \arrow[dd, leftarrow] \arrow[dd, shift left=1.5] & & M' \rtimes L' \arrow[dd, shift right=1.5] \arrow[dd, leftarrow] \arrow[dd, shift left=1.5] \\
N \arrow[rr, tail] \arrow[dr, "g"] & & N \rtimes L \arrow[rr, shift left=1.5] \arrow[rr, leftarrow] \arrow[rr, shift right=1.5] \arrow[dr, "g\rtimes l"] & & L \arrow[dr, "l"] & \\
& N' \arrow[rr, tail] & & N' \rtimes L \arrow[rr, shift left=1.5] \arrow[rr, leftarrow] \arrow[rr, shift right=1.5] & & L
\end{tikzcd}$$

Here ϕ is uniquely determined by the universal property of

$$\begin{tikzcd}
Q_{M\otimes N} \arrow[r, shift left=1.5] \arrow[r, leftarrow] \arrow[r, shift right=1.5] \arrow[d, shift right=1.5] \arrow[d, leftarrow] \arrow[d, shift left=1.5] & M \rtimes L \arrow[d, shift right=1.5] \arrow[d, leftarrow] \arrow[d, shift left=1.5] \\
N \rtimes L \arrow[r, shift left=1.5] \arrow[r, leftarrow] \arrow[r, shift right=1.5] & L
\end{tikzcd}$$

being defined as a pushout in $\mathbf{Grpd}^2(\mathbb{A})$: in particular ϕ is the only morphism which makes $\left(\begin{smallmatrix} \phi & f\rtimes l \\ g\rtimes l & l \end{smallmatrix}\right)$ a morphism of double groupoids. Since the other dotted maps are uniquely induced by taking kernels, $f \otimes g$ is automatically the unique morphism such that $\left(\begin{smallmatrix} f\otimes g & f \\ g & l \end{smallmatrix}\right)$ is a morphism of internal crossed squares. □

Corollary 3.2.8. *In the situation depicted in the previous proposition*

$$(M \otimes N \xrightarrow{\lambda} L, \xi) \xrightarrow{(f\otimes g, l)} (M' \otimes N' \xrightarrow{\lambda'} L', \xi')$$

is a morphism of internal crossed modules, and consequently the non-abelian tensor product

$$- \otimes - \colon \mathbf{XMod}_L(\mathbb{A}) \times \mathbf{XMod}_L(\mathbb{A}) \to \mathbf{XMod}_L(\mathbb{A}).$$

is a bifunctor.

Proof. The first result is just an application of Proposition 3.1.19 to the morphism of internal crossed square $\left(\begin{smallmatrix} f\otimes g & f \\ g & l \end{smallmatrix}\right)$. The second part is simply a particular case in which $l = 1_L$. □

Example 3.2.9. Consider the two crossed modules

$$(N \xrightarrow{\nu} L, \xi^L_N) \qquad\qquad (0 \xrightarrow{0} L, \tau^L_0).$$

Let us compute their non-abelian tensor product. First of all we need to explicit their internal groupoid structure: to do this, we follow Construction 1.4.8 and Example 1.4.9 obtaining

$$\begin{tikzcd}
 & & 0 \arrow[d, tail, "0"] \\
 & & L \arrow[d, shift right=2, "1"'] \arrow[d, shift left=2, "1"] \\
N \arrow[r, tail, "k_N"] & N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[l, "e_N" description] \arrow[u, "1" description]
\end{tikzcd}$$

But now the double groupoid given by Construction 3.1.7 is clearly

$$\begin{tikzcd}
N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] \arrow[d, shift right=2, "1"'] \arrow[d, shift left=2, "1"] & L \arrow[l, "e_N" description] \arrow[d, shift right=2, "1"'] \arrow[d, shift left=2, "1"] \\
N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] \arrow[u, "1" description] & L \arrow[l, "e_N" description] \arrow[u, "1" description]
\end{tikzcd}$$

which means that $0 \otimes N \cong 0$.

At this point one would hope that taking the non-abelian tensor product of an L-crossed module with the conjugation crossed module on L would give us back the first L-crossed module: this would mean that the conjugation L-crossed module is precisely the neutral element for the non-abelian tensor product. Unfortunately this happens to be false even in **Grp** and even when the starting L-crossed modules are both given by the conjugation on L. A simple counterexample is given by the Klein group: if we take L

to be the Klein group endowed with the conjugation action, then the non-abelian tensor square is

$$L \otimes L = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 = L$$

For this example, other computations of the non-abelian tensor product in **Grp** and for further details look on [14] and [68].

Proposition 3.2.10. *Consider an internal crossed square of the form*

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

Then there exists a unique ϕ such that the following diagram commutes

$$\begin{tikzcd} M \otimes N \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, "\pi_M^{M\otimes N}"] \arrow[ddr, bend right, "\pi_N^{M\otimes N}"'] & & \\ & P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ & N \arrow[r, "\nu"'] & L \end{tikzcd}$$

making $\begin{pmatrix} \phi & 1_M \\ 1_N & 1_L \end{pmatrix}$ a morphism of crossed squares.

Proof. We first shift to the double-groupoid setting and construct the following diagram

$$\begin{tikzcd}[column sep=small]
M \otimes N \arrow[rr, tail] \arrow[dd, tail] \arrow[dr, dotted, "\phi"] & & (M \otimes N) \rtimes M \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, tail] \arrow[dr, dotted, "\phi \rtimes 1_M"] & & M \arrow[ll] \arrow[dd, tail] \arrow[dr, equal] & \\
& P \arrow[rr, tail] \arrow[dd, tail] & & P \rtimes M \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, tail] & & M \arrow[ll] \arrow[dd, tail] \\
(M \otimes N) \rtimes N \arrow[rr, tail] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr, dotted, "\phi \rtimes 1_N"] & & Q_{M \otimes N} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr, dotted, "\phi_0"] & & M \rtimes L \arrow[ll] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr, equal] & \\
& P \rtimes N \arrow[rr, tail] \arrow[dd, shift left] \arrow[dd, shift right] & & R \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, shift left] \arrow[dd, shift right] & & M \rtimes L \arrow[ll] \arrow[dd, shift left] \arrow[dd, shift right] \\
N \arrow[rr, tail] \arrow[uu] \arrow[dr, equal] & & N \rtimes L \arrow[rr, shift left] \arrow[rr, shift right] \arrow[uu] \arrow[dr, equal] & & L \arrow[ll] \arrow[uu] \arrow[dr, equal] & \\
& N \arrow[rr, tail] \arrow[uu] & & N \rtimes L \arrow[rr, shift left] \arrow[rr, shift right] \arrow[uu] & & L \arrow[ll] \arrow[uu]
\end{tikzcd} \tag{3.20}$$

Here ϕ_0 is induced by the fact that the double groupoid involving $Q_{M\otimes N}$ is defined as a pushout in $\mathbf{Grpd}^2(\mathbb{A})$, whereas the maps $\phi \rtimes 1_M$ and $\phi \rtimes 1_N$ are the maps induced between the kernels and finally ϕ is given by the front square in the upper-left cube being a pullback. The fact that ϕ is the unique map making $\left(\begin{smallmatrix}\phi & 1_M \\ 1_N & 1_L\end{smallmatrix}\right)$ a morphism of crossed squares comes from the fact that ϕ_0 is the unique map such that $\left(\begin{smallmatrix}\phi_0 & 1_{M \rtimes L} \\ 1_{N \rtimes L} & 1_L\end{smallmatrix}\right)$ is a morphism of double groupoids. $\square$

3.2.3 The Lie algebras case

The aim of this subsection is to show that the non-abelian tensor product of Lie algebras defined in [37] coincides with the general definition of non-abelian tensor product when $\mathbb{A} = \mathbf{Lie}_R$. In order to do that we need to recall some definitions and results regarding the Lie algebra case from [37, 23].

From now on we will assume that M and N are two Lie algebras with crossed module structures on a common Lie algebra L, since in Chapter 2 we proved that this is the same as having two compatible actions of Lie algebras.

Definition 3.2.11 ([37])**.** Given two R-Lie algebras M and N acting on each other, their non-abelian tensor product $M \otimes_{Lie} N$ is the Lie algebra generated by the symbols $m \otimes n$ with $m \in M$ and $n \in N$, subject to the relations:

i) $(\lambda m) \otimes n = \lambda(m \otimes n) = m \otimes (\lambda n)$,

ii) $(m + m') \otimes n = m \otimes n + m' \otimes n$ and $m \otimes (n + n') = m \otimes n + m \otimes n'$,

iii) $[m, m'] \otimes n = m \otimes ({}^{m'}n) - m' \otimes ({}^{m}n)$ and $m \otimes [n, n'] = ({}^{n'}m) \otimes n - ({}^{n}m) \otimes n'$,

iv) $[m \otimes n, m' \otimes n'] = -({}^{n}m) \otimes ({}^{m'}n')$,

for all $\lambda \in R$, $m, m' \in M$ and $n, n' \in N$.

Definition 3.2.12 ([37])**.** Given two R-Lie algebras M and N acting on each other, and a third Lie Algebra P, we say that a bilinear function $h\colon M \times N \to P$ is a *Lie paring* if

i) $h([m, m'], n) = h(m, {}^{m'}n) - h(m', {}^{m}n)$,

ii) $h(m, [n, n']) = h({}^{n'}m, n) - h({}^{n}m, n')$,

iii) $h({}^{n}m, {}^{m'}n') = -[h(m, n), h(m', n')]$,

for all $m, m' \in M$ and $n, n' \in N$. The Lie pairing h is said to be *universal* if for any other Lie pairing $h'\colon M \times N \to P'$ there exists a unique Lie homomorphism $\phi\colon P \to P'$ that makes the following triangle commute

$$\begin{tikzcd} M \times N \arrow[r, "h"] \arrow[dr, "h'"'] & P \arrow[d, "\phi"] \\ & P' \end{tikzcd}$$

Proposition 3.2.13 (Proposition 1 in [37]). *Given two R-Lie algebras M and N acting on each other, the mapping*

$$\begin{aligned} h\colon M \times N &\to M \otimes_{Lie} N \\ (m,n) &\mapsto m \otimes n \end{aligned}$$

is a universal Lie pairing. Hence the non-abelian tensor product $M \otimes_{Lie} N$ of two Lie algebras acting on each other is uniquely characterised (up to isomorphism) as the codomain of their universal Lie pairing. □

Definition 3.2.14 ([36, 23]). A *crossed square* in $\mathbf{Lie}_R$ is a commutative square of Lie algebras

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

endowed with Lie actions of L on P, M and N (and hence Lie actions of M on N and L via μ, and of N on M and L via ν) and a function $h\colon M \times N \to P$ such that

0) h is bilinear and such that $h([m,m'],n) = {}^{m}h(m',n) - {}^{m'}h(m,n)$ and $h(m,[n,n']) = {}^{n}h(m,n') - {}^{n'}h(m,n)$;

i) the maps p_M and p_N preserve the actions of L, furthermore with the given actions $(M \xrightarrow{\mu} L, \xi_M)$, $(N \xrightarrow{\nu} L, \xi_N)$ and $(P \xrightarrow{\mu\circ p_M = \nu\circ p_N} L, \xi_P)$ are crossed modules;

ii) $p_M(h(m,n)) = -{}^{n}m$ and $p_N(h(m,n)) = {}^{m}n$;

iii) $h(p_M(p),n) = -{}^{n}p$ and $h(m, p_N(p)) = {}^{m}p$;

iv) ${}^{l}h(m,n) = h({}^{l}m,n) + h(m,{}^{l}n)$;

for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

Lemma 3.2.15 (Theorem 30 in [23]). *Lie crossed squares, as just defined, coincide with internal crossed squares in the category* $\mathbf{Lie}_R$. □

Lemma 3.2.16 ([59]). *For a pair of crossed modules $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$, the square*

$$\begin{tikzcd} M \otimes_{Lie} N \arrow[r, "\rho_M"] \arrow[d, "\rho_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

in $\mathbf{Lie}_R$, with ρ_M and ρ_N defined via

$$\rho_M(m \otimes n) = -{}^{n}m \qquad\qquad \rho_N(m \otimes n) = {}^{m}n$$

endowed with

- *the actions* ξ_M, ξ_N,
- *the action of* L *on* $M \otimes_{Lie} N$ *given by*
$$ {}^{l}(m \otimes n) := \left({}^{l}m\right) \otimes n + m \otimes \left({}^{l}n\right), \tag{3.21}$$
- *the map* $h\colon M \times N \to M \otimes N$ *defined in Proposition 3.2.13,*

is a crossed square (according to Definition 3.2.14).

Proof. Let us check conditions 0) - iv) in Definition 3.2.14.

i) The first step is to show that the maps ρ_M preserves the actions of L. This amounts to showing the equality
$$ {}^{l}(\rho_M(m \otimes n)) = \rho_M\left({}^{l}(m \otimes n)\right)$$
We prove this by using the fact that ξ_M is an action of Lie algebras, obtaining the chain of equalities
$$\begin{aligned} \rho_M\left({}^{l}(m \otimes n)\right) &= \rho_M\left(\left({}^{l}m\right) \otimes n\right) + \rho_M\left(m \otimes \left({}^{l}n\right)\right) \\ &= -{}^{n}({}^{l}m) - {}^{({}^{l}n)}m \\ &= -{}^{\nu(n)}({}^{l}m) - {}^{\nu({}^{l}n)}m \\ &= -{}^{\nu(n)}({}^{l}m) - {}^{[l,\nu(n)]}m \\ &= -{}^{l}({}^{\nu(n)}m) \\ &= {}^{l}(-{}^{n}m) \\ &= {}^{l}(\rho_M(m \otimes n)). \end{aligned}$$
The same reasoning works for ρ_N. Now it remains to show that the diagonal $\lambda := \mu \circ \rho_M = \nu \circ \rho_N$ is a crossed module once endowed with the action (3.21): in order to do that we use the compatibility conditions as stated in [37], the crossed module conditions for $(M \xrightarrow{\mu} L, \xi_M)$ and $(N \xrightarrow{\nu} L, \xi_N)$, and the equation $\lambda(m \otimes n) = [\mu(m), \nu(n)]$ obtained through
$$\begin{aligned} \lambda(m \otimes n) &= \mu(\rho_M(m \otimes n)) = \mu\left(-{}^{n}m\right) \\ &= -\mu\left({}^{\nu(n)}m\right) = -[\nu(n), \mu(m)] \\ &= [\mu(m), \nu(n)]. \end{aligned}$$
In particular we obtain the following chains of equalities
$$\begin{aligned} \lambda\left({}^{l}(m \otimes n)\right) &= \lambda\left(\left({}^{l}m\right) \otimes n\right) + \lambda\left(m \otimes \left({}^{l}n\right)\right) \\ &= \left[\mu\left({}^{l}m\right), \nu(n)\right] + \left[\mu(m), \nu\left({}^{l}n\right)\right] \\ &= [[l, \mu(m)], \nu(n)] + [\mu(m), [l, \nu(n)]] \\ &= [l, [\mu(m), \nu(n)]] \\ &= [l, \lambda(\mu(m) \otimes \nu(n))], \end{aligned}$$

$$\begin{aligned}
{}^{\lambda(m'\otimes n')}(m\otimes n) &= \left({}^{\lambda(m'\otimes n')}m\right)\otimes n + m\otimes\left({}^{\lambda(m'\otimes n')}n\right)\\
&= \left[-{}^{n'}m', m\right]\otimes n + m\otimes\left[{}^{m'}n', n\right]\\
&= (-{}^{n'}m')\otimes({}^{m}n) - m\otimes\left({}^{\left({}^{n'}m'\right)}n\right) + m\otimes\left[{}^{m'}n', n\right]\\
&= (-{}^{n'}m')\otimes({}^{m}n) + m\otimes\left(\left[{}^{m'}n', n\right] - {}^{\left({}^{n'}m'\right)}n\right)\\
&= (-{}^{n'}m')\otimes({}^{m}n)\\
&= [m'\otimes n', m\otimes n].
\end{aligned}$$

0) The map h is bilinear by construction and by using the definition of the tensor product $M\otimes_{Lie} N$ we have the following chain of equalities

$$\begin{aligned}
{}^{m}h(m',n) - {}^{m'}h(m,n) &= [m,m']\otimes n + m'\otimes({}^{m}n) - [m',m]\otimes n - m\otimes\left({}^{m'}n\right)\\
&= [m,m']\otimes n\\
&= h([m,m'],n)
\end{aligned}$$

and similarly for the second equality required.

ii) $\rho_M(h(m,n)) = -{}^{n}m$ and $\rho_N(h(m,n)) = {}^{m}n$ by definition of ρ_M and ρ_N.

iii) Again by definition of $M\otimes_{Lie} N$ we have

$$\begin{aligned}
h(\rho_M(m\otimes n), n') &= h(-{}^{n}m, n') = -\left({}^{n}m\right)\otimes n'\\
&= -\left({}^{n'}m\right)\otimes n - m\otimes[n',n]\\
&= -{}^{n}(m\otimes n)
\end{aligned}$$

and similarly for the second equality.

iv) ${}^{l}h(m,n) = {}^{l}(m\otimes n) = ({}^{l}m\otimes n) + (m\otimes{}^{l}n) = h({}^{l}m,n) + h(m,{}^{l}n)$. □

Proposition 3.2.17. *When $\mathbb{A} = \mathbf{Lie}_R$, the non-abelian tensor product $M\otimes N$ as described in Definition 3.2.5 coincides with the tensor product of Lie algebras $M\otimes_{Lie} N$ defined Definition 3.2.11.*

Proof. By Proposition 3.2.13 it suffices to show that the general version of $M\otimes N$ has the same universal property as $M\otimes_{Lie} N$.

The first step is to construct a Lie pairing from $M\times N$ to $M\otimes N$. In order to do this, consider diagram (3.19) and denote with j_M and j_N the diagonal inclusions of M and N in $Q_{M\otimes N}$. We are going to define a function h from $M\times N$ to $Q_{M\otimes N}$, we are going to show that it factors through $M\otimes N$ as $h\colon M\times N\to M\otimes N$ and then we are going to prove that it is a universal Lie pairing.

Since we are in $\mathbf{Lie}_R$ we can define h directly on the elements by imposing $h(m,n) := [j_M(m), j_N(n)]$. To prove that it factors through $M\otimes N$ it suffices to show that $\overline{d_U}\circ h =$

$0 = \overline{d_L} \circ h$ (the rest is trivial since $M \otimes N$ is the pullback of the kernels $K_{\overline{d_U}}$ and $K_{\overline{d_L}}$), and this is done through the equalities

$$\begin{aligned}\overline{d_U} \circ h(m,n) &= \overline{d_U}\left([j_M(m), j_N(n)]\right) = \left[\overline{d_U}(j_M(m)), \overline{d_U}(j_N(n))\right] \\ &= \left[\overline{d_U}(j_M(m)), 0\right] = 0,\end{aligned}$$

$$\begin{aligned}\overline{d_L} \circ h(m,n) &= \overline{d_L}\left([j_M(m), j_N(n)]\right) = \left[\overline{d_L}(j_M(m)), \overline{d_L}(j_N(n))\right] \\ &= \left[0, \overline{d_L}(j_N(n))\right] = 0.\end{aligned}$$

Now we have shown that the image of h lies in $M \otimes N$ obtaining $h\colon M \times N \to M \otimes N$. Let us prove that it is a Lie pairing according to Definition 3.2.12:

- for $i)$ we have the following chain of equalities

$$\begin{aligned}h([m,m'],n) &= [j_M([m,m']), j_N(n)] = [[j_M(m), j_M(m')], j_N(n)] \\ &= -[[j_M(m'), j_N(n)], j_M(m)] - [[j_N(n), j_M(m)], j_M(m')] \\ &= [j_M(m), [j_M(m'), j_N(n)]] - [j_M(m'), [j_M(m), j_N(n)]] \\ &= [j_M(m), j_N({}^{m'}n)] - [j_M(m'), j_N({}^{m}n)] \\ &= h(m, {}^{m'}n) - h(m', {}^{m}n)\end{aligned}$$

  and a similar one shows $ii)$;

- for $iii)$ we have

$$\begin{aligned}h({}^{n}m, {}^{m'}n') &= \left[j_M({}^{n}m), j_N({}^{m'}n')\right] \\ &= \left[[j_N(n), j_M(m)], [j_M(m'), j_N(n')]\right] \\ &= -\left[[j_M(m), j_N(n)], [j_M(m'), j_N(n')]\right] \\ &= -\left[h(m,n), h(m',n')\right].\end{aligned}$$

It remains to prove that h is a universal Lie pairing. In order to do that, we take a universal Lie pairing $\tilde{h}$ (we know that it exists by Proposition 3.2.13) and we show that there exists a morphism ϕ such that

$$\begin{tikzcd} M \times N \arrow[r, "h"] \arrow[dr, "\tilde{h}"'] & M \otimes N \arrow[d, "\phi"] \\ & M \otimes_{Lie} N \end{tikzcd} \tag{3.22}$$

This would imply that h is automatically universal and hence by uniqueness (up to isomorphism) of the universal Lie pairing we would obtain that $M \otimes N \cong M \otimes_{Lie} N$.

Let us show that such a ϕ exist. The first step is to use Lemma 3.2.16 which shows that the non-abelian tensor product $M \otimes_{Lie} N$ induces a crossed square of Lie algebras

$$\begin{tikzcd} M \otimes_{Lie} N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & L \end{tikzcd}$$

according to Definition 3.2.14. By Lemma 3.2.15 we know that in $\mathbf{Lie}_R$ Definition 3.1.13 coincides with Definition 3.2.14 and hence we can use the universal property of $M \otimes N$, which gives us a map $\phi\colon M \otimes N \to M \otimes_{Lie} N$ such that (3.22) commutes. □

Consequently from now on can denote ρ_M and ρ_N respectively as $\pi_M^{M\otimes N}$ and $\pi_N^{M\otimes N}$.

3.3 Towards crossed squares through the non-abelian tensor product

The aim of this section is to generalise the explicit description of crossed squares of groups (given in Definition 3.1.10) and Lie algebras (given in Definition 3.2.14) to the semi-abelian case (with SH), without passing through the double groupoid formalism. In order to obtain this, we are going to use the construction of the non-abelian tensor product, first in the categories **Grp** and $\mathbf{Lie}_R$, and then in $\mathbb{A}$. We call this object *weak crossed square*, and we prove that it is the same as a crossed square as soon as we are in **Grp** or $\mathbf{Lie}_R$. We then show that the canonical definition of crossed square in the semi-abelian context implies the one of weak crossed square. The converse is still an open question: the aim would be to find suitable conditions under which the two definitions coincide. This would mean that an "explicit" definition of crossed square is possible in such a general setting.

The idea behind this internalisation is given by a bijection introduced in [15] (see Definition 2.2 and following): the authors say that, given a pair of compatible group actions, to each crossed pairing $h\colon M \times N \to P$ it corresponds a group homomorphism $h^*\colon M \otimes N \to P$ defined by extending $h^*(m \otimes n) = h(m, n)$ (from now on we will drop this notation and we will write h for both these maps since there is no risk of confusion).

Using this hint as the basis of our reasoning we can give the following definition.

Definition 3.3.1. A *weak crossed square in* $\mathbb{A}$ is given by a commutative square

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] \arrow[dr, dotted, "\lambda"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

in $\mathbb{A}$, together with internal actions

$$\xi_M^L\colon L\flat M \to M \qquad \xi_N^L\colon L\flat N \to N \qquad \xi_P^L\colon L\flat P \to P$$

and a morphism $h\colon M \otimes N \to P$ such that the following axioms hold:

i') the maps p_M and p_N are equivariant with respect to the L-actions, that is the squares

$$\begin{tikzcd} L\flat P \arrow[r, "\xi_P^L"] \arrow[d, "1\flat p_M"'] & P \arrow[d, "p_M"] \\ L\flat M \arrow[r, "\xi_M^L"'] & M \end{tikzcd} \qquad \begin{tikzcd} L\flat P \arrow[r, "\xi_P^L"] \arrow[d, "1\flat p_N"'] & P \arrow[d, "p_N"] \\ L\flat N \arrow[r, "\xi_N^L"'] & N \end{tikzcd}$$

commute, and furthermore $(M \xrightarrow{\mu} L, \xi_M^L)$, $(N \xrightarrow{\nu} L, \xi_N^L)$ and $(P \xrightarrow{\lambda} L, \xi_P^L)$ are L-crossed modules.

ii') the diagram

$$\begin{tikzcd} & M \otimes N \arrow[dl, "\pi_N^{M\otimes N}"'] \arrow[d, "h"] \arrow[dr, "\pi_M^{M\otimes N}"] & \\ N & P \arrow[l, "p_N"] \arrow[r, "p_M"'] & M \end{tikzcd}$$

commutes;

iii') the diagram

$$\begin{tikzcd} P \otimes N \arrow[r, "p_M\otimes 1_N"] \arrow[dr, "\pi_P^{P\otimes N}"'] & M \otimes N \arrow[d, "h"] & M \otimes P \arrow[l, "1_M\otimes p_N"'] \arrow[dl, "\pi_P^{M\otimes P}"] \\ & P & \end{tikzcd}$$

commutes;

iv') the map h is equivariant with respect to the action $\xi_{M\otimes N}^L \colon L\flat(M \otimes N) \to M \otimes N$ (induced as in Remark 3.1.15), that is the square

$$\begin{tikzcd} L\flat(M \otimes N) \arrow[r, "\xi_{M\otimes N}^L"] \arrow[d, "1_L\flat h"'] & M \otimes N \arrow[d, "h"] \\ L\flat P \arrow[r, "\xi_P^L"'] & P \end{tikzcd}$$

commutes.

A morphism of weak crossed squares of the form

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd} \xrightarrow{\left(\begin{smallmatrix} p & f \\ g & l \end{smallmatrix}\right)} \begin{tikzcd} P' \arrow[r, "p'_{M'}"] \arrow[d, "p'_{N'}"'] & M' \arrow[d, "\mu'"] \\ N' \arrow[r, "\nu'"'] & L' \end{tikzcd}$$

is given by a quadruple of morphisms

$$
\begin{aligned}
p\colon P \to P' &\qquad& f\colon M \to M' \\
g\colon N \to N' &\qquad& l\colon L \to L'
\end{aligned}
$$

such that the cube

$$
\begin{tikzcd}
P \arrow[rr] \arrow[dr] \arrow[dd] & & M \arrow[dr] \arrow[dd] & \\
& P' \arrow[rr] \arrow[dd] & & M' \arrow[dd] \\
N \arrow[rr] \arrow[dr] & & L \arrow[dr] & \\
& N' \arrow[rr] & & L'
\end{tikzcd}
$$

commutes and the h-maps are respected, that is the square

$$
\begin{tikzcd}
M \otimes N \arrow[r, "f \otimes g"] \arrow[d, "h"'] & M' \otimes N' \arrow[d, "h'"] \\
P \arrow[r, "p"'] & P'
\end{tikzcd}
$$

commutes as well.

Remark 3.3.2. Notice that from the three L-actions ξ^L_M, ξ^L_N and ξ^L_P we are able to construct the actions ξ^M_P, ξ^N_P, ξ^M_N and ξ^N_M through the diagrams

$$
\begin{tikzcd}
M\flat P \arrow[rr, "\xi^M_P"] \arrow[dr, "\mu\flat 1"'] & & P \\
& L\flat P \arrow[ur, "\xi^L_P"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
N\flat P \arrow[rr, "\xi^N_P"] \arrow[dr, "\nu\flat 1"'] & & P \\
& L\flat P \arrow[ur, "\xi^L_P"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
M\flat N \arrow[rr, "\xi^M_N"] \arrow[dr, "\mu\flat 1"'] & & N \\
& L\flat N \arrow[ur, "\xi^L_N"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
N\flat M \arrow[rr, "\xi^N_M"] \arrow[dr, "\nu\flat 1"'] & & M \\
& L\flat M \arrow[ur, "\xi^L_M"'] &
\end{tikzcd}
$$

and condition $i)$ implies that also $(P \xrightarrow{p_M} M, \xi^M_P)$ and $(P \xrightarrow{p_N} N, \xi^N_P)$ are crossed modules. This is just an application of the following lemma.

Lemma 3.3.3. *Let $\mathbb{A}$ be a semi-abelian category with SH. Consider a triangle*

$$
\begin{tikzcd}
P \arrow[r, "p"] \arrow[dr, "\lambda"'] & M \arrow[d, "\mu"] \\
& L
\end{tikzcd}
$$

with internal crossed module structures $(M \xrightarrow{\mu} L, \xi_M^L)$ *and* $(P \xrightarrow{\lambda} L, \xi_P^L)$*, and the induced action* $\xi_P^M := \xi_P^L \circ (\mu \flat 1_P)$*. If* p *is equivariant with respect to the* L*-actions, that is if the square*

$$\begin{tikzcd} L\flat P \arrow[r, "\xi_P^L"] \arrow[d, "1\flat p"'] & P \arrow[d, "p"] \\ L\flat M \arrow[r, "\xi_M^L"'] & M \end{tikzcd}$$

commutes, then also $(P \xrightarrow{p} M, \xi_P^M)$ *is an internal crossed module.*

Proof. We need to show the commutativity of the diagram

$$\begin{tikzcd} P\flat P \arrow[r, "\chi_P"] \arrow[d, "p\flat 1"'] & P \arrow[d, equal] \\ M\flat P \arrow[r, "\xi_P^M"] \arrow[d, "1\flat p"'] & P \arrow[d, "p"] \\ M\flat M \arrow[r, "\chi_M"'] & M \end{tikzcd}$$

For the upper square we have the chain of equalities

$$\xi_P^M \circ (p\flat 1) = \xi_P^L \circ (\mu\flat 1) \circ (p\flat 1) = \xi_P^L \circ (\lambda\flat 1) = \chi_P,$$

whereas for the lower one we have

$$\begin{aligned} p \circ \xi_P^M &= p \circ \xi_P^L \circ (\mu\flat 1) = \xi_M^L \circ (1\flat p) \circ (\mu\flat 1) \\ &= \xi_M^L \circ (\mu\flat 1) \circ (1\flat p) = \chi_M \circ (1\flat p). \end{aligned}$$

□

Proposition 3.3.4. *If* $\mathbb{A} = \mathbf{Grp}$*, then weak crossed squares are the same as internal crossed squares, that is the group version of Definition 3.3.1 coincides with Definition 3.1.10.*

Proof. As explained in [15], given a crossed pairing $h\colon M \times N \to P$ (see Remark 3.1.12), we can decompose it as

$$\begin{tikzcd} M \times N \arrow[rr, "h"] \arrow[dr, "-\otimes-"'] & & P \\ & M \otimes N \arrow[ur, "h^*"'] & \end{tikzcd}$$

where the first map, which sends (m, n) to $m \otimes n$, is called *universal crossed pairing*, whereas h^* is a morphism of groups. Viceversa, we can associate a crossed pairing $h^* \circ (- \otimes -)$ to every morphism of groups $h^*\colon M \otimes N \to P$. This means that having a crossed pairing amounts to having a morphism going out of the non-abelian tensor product (for the sake of simplicity we are going to name both of them as h).

Notice that $i')$ is only the internal reformulation of $i)$, and hence they are clearly equivalent. Let us prove that $ii) \iff ii')$. The only non-trivial step is given by the explicit description

$$\begin{cases} \pi_M^{M\otimes N}(m\otimes n) = m^n m^{-1}, \\ \pi_N^{M\otimes N}(m\otimes n) = {}^m n n^{-1}, \end{cases}$$

for the projection maps: for further details see Proposition 2.3 (b) in [15]. Using these equations together with

$$\begin{cases} p_M(h(m\otimes n)) = p_M(h(m,n)), \\ p_N(h(m\otimes n)) = p_N(h(m,n)), \end{cases}$$

we obtain the desired equivalence.

Similarly, in order to show $iii) \iff iii')$, we use the equations

$$\begin{cases} \pi_P^{P\otimes N}(p\otimes n) = p^n p^{-1}, \\ \pi_P^{M\otimes P}(m\otimes p) = {}^m p p^{-1}, \end{cases} \qquad \begin{cases} h(p_M(p)\otimes n) = h(p_M(p),n), \\ h(m\otimes p_N(p)) = h(m,p_N(p)). \end{cases}$$

We've already explained that, whenever $iv)$ holds, $0)$ is equivalent to the requirement that $h\colon M\times N\to P$ is a crossed pairing and that this is in turn equivalent to having a morphism $h\colon M\otimes N\to P$.

Finally, to show that $iv) \iff iv')$, we first take the action $\xi^L_{M\otimes N}$ as defined in Remark 3.1.15: in the particular case of groups it can be described through the equation

$$^l(m\otimes n) = (^l m)\otimes(^l n)$$

(for more details about this action see Proposition 2.3 (a) in [15]). Then, to obtain the thesis, we use the equations

$$^l h(m\otimes n) = {}^l h(m,n), \qquad h((^l m)\otimes(^l n)) = h(^l m, {}^l n). \qquad \square$$

Remark 3.3.5. Consider a crossed square of groups as in Definition 3.1.10: according to Proposition 3.2.10 we have a unique morphism $\phi\colon M\otimes N\to P$ such that $\left(\begin{smallmatrix}\phi & 1_M \\ 1_N & 1_L\end{smallmatrix}\right)$ is a morphism of crossed squares. In particular this map ϕ is the same as the map $h\colon M\otimes N\to P$ induced by the crossed pairing $h\colon M\times N\to P$. To see this, it suffices to show that $\left(\begin{smallmatrix}h & 1_M \\ 1_N & 1_L\end{smallmatrix}\right)$ is again a morphism of crossed squares: following the description of morphisms given in Definition 3.3.1, this amounts to proving that h makes the outer cube in (3.20) commute as well as the diagram

$$\begin{tikzcd} M\otimes N \arrow[r, equal] \arrow[d, equal] & M\otimes N \arrow[d, "h"] \\ M\otimes N \arrow[r, "h"'] & P \end{tikzcd}$$

The latter is trivial due to the fact $1_{M\otimes N}\colon M\otimes N\to M\otimes N$ is the h-map associated to the crossed square defining the non-abelian tensor product, and the former is given by condition $ii')$.

From the last remark we can deduce that, if there is a way to show the equivalence between the notion of weak crossed square and the one of internal crossed square, then the morphism $h\colon M \otimes N \to P$ has to be the one given by Proposition 3.2.10.

Proposition 3.3.6. *If $\mathbb{A} = \mathbf{Lie}_R$, then weak crossed squares are the same as internal crossed squares, that is the Lie algebra version of Definition 3.3.1 coincides with Definition 3.2.14.*

Proof. Let us consider conditions $0) - iv)$ as in Definition 3.2.14 and conditions $i') - iv')$ as in Definition 3.3.1.

As follows from Proposition 3.2.13, having a function $h\colon M \times N \to P$ such that 0) holds (that is a Lie pairing) is the same as having a morphism $h^*\colon M \otimes N \to P$ (from now on denoted again with h).

Notice that $i')$ is only the internal reformulation of $i)$, and hence they are clearly equivalent. The equivalence $ii) \iff ii')$ is given by the equivalence between the systems

$$\begin{cases} \pi_M^{M\otimes N}(m \otimes n) = p_M(h(m \otimes n)), \\ \pi_N^{M\otimes N}(m \otimes n) = p_N(h(m \otimes n)), \end{cases} \qquad \begin{cases} -{}^n m = p_M(h(m, n)), \\ {}^m n = p_N(h(m, n)), \end{cases}$$

which in turn is obtained via the explicit description of the maps $\pi_M^{M\otimes N}$ and $\pi_M^{M\otimes N}$ in the Lie algebra case. Similarly, in order to show $iii') \iff iii)$, we use the equivalence between the systems

$$\begin{cases} \pi_P^{P\otimes N}(p \otimes n) = h((p_N \otimes 1_N)(p \otimes n)), \\ \pi_P^{M\otimes P}(m \otimes p) = h((1_M \otimes p_M)(m \otimes p)), \end{cases} \qquad \begin{cases} h(p_M(p), n) = -{}^n p, \\ h(m, p_N(p)) = {}^m p. \end{cases}$$

Finally, to show that $iv) \iff iv')$, we first take the action $\xi^L_{M\otimes N}$ as defined in Remark 3.1.15: in the particular case of Lie algebras it can be described through the equation

$${}^l(m \otimes n) = ({}^l m) \otimes n + m \otimes ({}^l n).$$

Then we use the equivalent equalities

$$\begin{gathered} h\left({}^l(m \otimes n)\right) = {}^l(h(m \otimes n)), \\ \Updownarrow \\ h\left(({}^l m) \otimes n + m \otimes ({}^l n)\right) = {}^l(h(m \otimes n)), \\ \Updownarrow \\ h\left(({}^l m) \otimes n\right) + h\left(m \otimes ({}^l n)\right) = {}^l(h(m \otimes n)), \\ \Updownarrow \\ h\left({}^l m, n\right) + h\left(m, {}^l n\right) = {}^l(h(m, n)). \end{gathered}$$

□

Proposition 3.3.7. *An internal crossed square is automatically a weak crossed square, that is Definition 3.1.13 implies Definition 3.3.1.*

Proof. Consider an internal crossed square with respect to the implicit definition

$$
\begin{tikzcd}[column sep=large, row sep=large]
P \arrow[r, tail, "k_{d_T}"] \arrow[d, tail, "k_{d_W}"'] \arrow[dr, dotted, "k"] & P \rtimes M \arrow[r, shift left=2, "d_T"] \arrow[r, shift right=2, "c_T"'] \arrow[d, tail, "k_{d_L}"] & M \arrow[l, "e_T"'] \arrow[d, tail, "k_{d_R}"] \\
P \rtimes N \arrow[r, tail, "k_{d_U}"'] \arrow[d, shift left=2, "d_W"] \arrow[d, shift right=2, "c_W"'] & Q_P \arrow[r, shift left=2, "d_U"] \arrow[r, shift right=2, "c_U"'] \arrow[d, shift left=2, "d_L"] \arrow[d, shift right=2, "c_L"'] \arrow[dr, dotted, shift left=2, "d"] \arrow[dr, dotted, shift right=2, "c"'] & M \rtimes L \arrow[l, "e_U"'] \arrow[d, shift left=2, "d_R"] \arrow[d, shift right=2, "c_R"'] \\
N \arrow[u, "e_W"] \arrow[r, tail, "k_{d_D}"'] & N \rtimes L \arrow[u, "e_L"] \arrow[r, shift left=2, "d_D"] \arrow[r, shift right=2, "c_D"'] & L \arrow[l, "e_D"'] \arrow[u, "e_R"] \arrow[ul, dotted, "e"]
\end{tikzcd}
\tag{3.23}
$$

Let us start by fixing the basic ingredients. We define the maps $p_M := c_T \circ k_{d_T}$, $p_N := c_W \circ k_{d_W}$ and $\lambda := c \circ k$. The actions ξ^L_M and ξ^L_N are already given, whereas ξ^L_P and $\xi^L_{M \otimes N}$ are constructed as in Remark 3.1.16 and $h\colon M \otimes N \to P$ is given by Proposition 3.2.10. Now we are ready to show the properties of these objects.

As far as it concerns $i')$, we already know by hypothesis that $(M \xrightarrow{\mu} L, \xi^L_M)$ and $(N \xrightarrow{\nu} L, \xi^L_N)$ are crossed modules. The fact that also $(P \xrightarrow{\lambda} L, \xi^L_P)$ is so, is given by Proposition 3.1.18. It remains to show that $p_M\colon P \to M$ is equivariant with respect to these actions (for p_N the reasoning is totally similar). Consider the diagram

$$
\begin{tikzcd}
P \arrow[r, tail, "k_{\hat{d}}"] \arrow[d, tail, "l"'] \arrow[dr, "k"] \arrow[dd, bend right=60, "p_M"'] & \widehat{Q_P} \arrow[r, shift left=1, "\hat{d}"] \arrow[d, "l \rtimes 1"] & L \arrow[l, shift left=1, "\hat{e}"] \arrow[d, equal] \\
K_d \arrow[r, tail, "k_d"] \arrow[d, dotted, "\phi"'] & Q_P \arrow[r, shift left=1, "d"] \arrow[d, "d_U"'] & L \arrow[l, shift left=1, "e"] \arrow[d, equal] \\
M \arrow[r, tail, "k_{d_R}"'] & M \rtimes L \arrow[r, shift left=1, "d_R"] & L \arrow[l, shift left=1, "e_R"]
\end{tikzcd}
$$

where the two top squares are the ones defining the action ξ^L_P, whereas the dotted map is induced by the fact that M is the kernel of d_R. In order to show that $(p_M, 1)$ is a map of points from the top row to the bottom one (and hence an equivariant map), it suffices to show that $p_M = \phi \circ l$, since each square commutes: this is done using the chain of equalities

$$k_{d_R} \circ \phi \circ l = d_U \circ k_d \circ l = d_U \circ k = k_{d_R} \circ p_M$$

and the fact that k_{d_R} is a monomorphism.

Condition $ii')$ is already given by definition of the map h.

In order to show $iii')$ it suffices to prove that

$$\begin{tikzcd}
M\otimes P \arrow[r, "\pi_M^{M\otimes P}"] \arrow[d, "\pi_P^{M\otimes P}"'] & M \arrow[d, "\mu"] \arrow[r, shift left, "\left(\begin{smallmatrix}\pi_P^{M\otimes P} & 1\\ p_N & 1\end{smallmatrix}\right)"] \arrow[r, shift right, "\left(\begin{smallmatrix}h\circ(1\otimes p_N) & 1\\ p_N & 1\end{smallmatrix}\right)"'] & P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\
P \arrow[r, "\lambda"'] & L & N \arrow[r, "\nu"'] & L
\end{tikzcd}$$

are both morphisms of crossed squares, so that the thesis follows from the universal property of $M\otimes P$ (and similarly for $P\otimes N$). The map $\pi_P^{M\otimes P}$ clearly satisfies the universal property depicted in Proposition 3.2.10 and therefore it induces the morphism of crossed squares on the top. As far as it concerns the second one, it is easy to see that it is obtained as the composition

$$\begin{pmatrix} h\circ(1\otimes p_N) & 1 \\ p_N & 1 \end{pmatrix} = \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1\otimes p_N & 1 \\ p_N & 1 \end{pmatrix}$$

The first one is a morphism of crossed squares by definition of $1\otimes p_N$, whereas the second one is so by definition of h (and by Remark 3.3.5).

It remains to show that $iv')$ holds and to do so, consider the following diagram:

$$\begin{tikzcd}
L\flat(M\otimes N) \arrow[rr, tail] \arrow[dr, "1\flat h"] \arrow[dd, "\xi^L_{M\otimes N}"'] & & L+(M\otimes N) \arrow[rr, shift left] \arrow[dr, "1+h"] \arrow[dd] & & L \arrow[ll, shift left] \arrow[dr, equal] \arrow[dd, equal] & \\
& L\flat P \arrow[rr, tail] \arrow[dd, "\xi^L_P"] & & L+P \arrow[rr, shift left] \arrow[dd] & & L \arrow[ll, shift left] \arrow[dd, equal] \\
M\otimes N \arrow[rr, tail] \arrow[dr, "h"] \arrow[dd, tail] & & \widehat{Q_{M\otimes N}} \arrow[rr, shift left] \arrow[dr, "\hat{\phi}"] \arrow[dd] & & L \arrow[ll, shift left] \arrow[dr, equal] \arrow[dd, equal] & \\
& P \arrow[rr, tail] \arrow[dd, tail] & & \widehat{Q_P} \arrow[rr, shift left] \arrow[dd] & & L \arrow[ll, shift left] \arrow[dd, equal] \\
A_{M\otimes N} \arrow[rr, tail] \arrow[dr, "\underline{\phi}"] & & Q_{M\otimes N} \arrow[rr, shift left] \arrow[dr, "\phi"] & & L \arrow[ll, shift left] \arrow[dr, equal] & \\
& A_P \arrow[rr, tail] & & Q_P \arrow[rr, shift left] & & L \arrow[ll, shift left]
\end{tikzcd}$$

We want to show that the top square in the left face commutes. Notice that by definition of ϕ and $\underline{\phi}$ we already know that the squares on the bottom face commute, and similarly, by definition of h the bottom square on the left face commutes. The two lower cubes then commute by construction of $\widehat{Q_{M\otimes N}}$, $\widehat{Q_P}$ and $\hat{\phi}$ (see Lemma 2.6 in [28]). This means that $(h,1)$ is a morphism between the lifted points and therefore h is equivariant. $\square$

It remains an open question whether the converse of Proposition 3.3.7 holds; a stronger condition on the base category $\mathbb{A}$ might be necessary for this to be the case. We have a partially positive answer in the situation where h happens to be a regular epimorphism: such a weak crossed square is always a crossed square, as soon as in the induced diagram

$$\begin{tikzcd}[column sep=small]
M\otimes N \arrow[rr, tail] \arrow[dr, "h"] \arrow[dd, tail] & & (M\otimes N)\rtimes M \arrow[rr, shift left=2] \arrow[rr] \arrow[dr, "h\rtimes 1_M"] \arrow[dd, tail] & & M \arrow[ll, shift left=2] \arrow[dr, equal] \arrow[dd, tail] & \\
& P \arrow[rr, tail] \arrow[dd, tail] & & P\rtimes M \arrow[rr, shift left=2] \arrow[rr] \arrow[dd, dotted, tail] & & M \arrow[ll, shift left=2] \arrow[dd, tail] \\
(M\otimes N)\rtimes N \arrow[rr, tail] \arrow[dr, "h\rtimes 1_N"] \arrow[dd, shift left=2] \arrow[dd] & & Q_{M\otimes N} \arrow[rr, shift left=2] \arrow[rr] \arrow[dr, dotted, "\tilde{h}"] \arrow[dd, shift left=2] \arrow[dd] & & M\rtimes L \arrow[ll, shift left=2] \arrow[dr, equal] \arrow[dd, shift left=2] \arrow[dd] & \\
& P\rtimes N \arrow[rr, dotted, tail] \arrow[dd, shift left=2] \arrow[dd] & & Q' \arrow[rr, dotted, shift left=2] \arrow[rr, dotted] \arrow[dd, dotted, shift left=2] \arrow[dd, dotted] & & M\rtimes L \arrow[ll, dotted, shift left=2] \arrow[dd, shift left=2] \arrow[dd] \\
N \arrow[rr, tail] \arrow[dr, equal] \arrow[uu, shift left=2] & & N\rtimes L \arrow[rr, shift left=2] \arrow[rr] \arrow[dr, equal] \arrow[uu, shift left=2] & & L \arrow[ll, shift left=2] \arrow[dr, equal] \arrow[uu, shift left=2] & \\
& N \arrow[rr, tail] \arrow[uu, shift left=2] & & N\rtimes L \arrow[rr, shift left=2] \arrow[rr] \arrow[uu, dotted, shift left=2] & & L \arrow[ll, shift left=2] \arrow[uu, shift left=2]
\end{tikzcd} \tag{3.24}$$

the kernel of h is normal in $Q_{M\otimes N}$.

In this situation, we can construct the object Q' and the dotted arrows in (3.24) in such a way that the double reflexive graph in the front face is an internal double groupoid. We may then use the idea contained in the following remark.

Remark 3.3.8. Suppose for the moment that the front face in (3.24) is already an internal crossed square. Then both squares in the diagram

$$\begin{tikzcd}
M\otimes N \arrow[r] \arrow[d, two heads, "h"'] \arrow[dr, phantom, "(*_1)"] & (M\otimes N)\rtimes M \arrow[r] \arrow[d, two heads, "h\rtimes 1_M"] \arrow[dr, phantom, "(*_2)"] & Q_{M\otimes N} \arrow[d, two heads, "\tilde{h}"] \\
P \arrow[r] & P\rtimes M \arrow[r] & Q'
\end{tikzcd} \tag{3.25}$$

are pullbacks and hence the outer rectangle is so. This implies that $K_h \cong K_{\tilde{h}}$, but since $\tilde{h}$ is a regular epimorphism if and only if so is h (by applying the Lemma 1.1.18 twice), it is the cokernel of its kernel: this means that Q' can be described as the cokernel of the inclusion of K_h into $Q_{M\otimes N}$. Furthermore, this inclusion is normal.

Proposition 3.3.9. *In a semi-abelian category that satisfies* (SH)*, a weak crossed square where h is a regular epimorphism is also an internal crossed square—that is, Definition 3.3.1 implies Definition 3.1.13 in that case—as soon as in the induced diagram* (3.24)*, the kernel of h is normal in $Q_{M\otimes N}$.*

Proof. By using the idea in the previous remark we define Q' as the cokernel of $\gamma \circ k_h$, where γ is the composition depicted in the first row of (3.25). In particular we obtain that

$$\begin{tikzcd} M \otimes N \arrow[r, "\gamma"] \arrow[d, "h"'] & Q_{M\otimes N} \arrow[d, "\tilde{h}"] \\ P \arrow[r, "\gamma'"'] & Q' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{3.26}$$

is a pushout. Since Q' is the cokernel of $\gamma \circ k_h$, from $d_U \circ \gamma = 0 = d_L \circ \gamma$ we find unique morphisms

$$d'_U \colon Q' \to M \rtimes L \qquad\qquad d'_L \colon Q' \to N \rtimes L$$

such that $d'_U \circ \tilde{h} = d_U$ and $d'_L \circ \tilde{h} = d_L$. Similarly, by using the universal property of the pushout (3.26) we obtain unique morphisms

$$c'_U \colon Q' \to M \rtimes L \qquad\qquad c'_L \colon Q' \to N \rtimes L$$

such that $c'_U \circ \tilde{h} = c_U$ and $c'_L \circ \tilde{h} = c_L$. Then we define $e'_U := \tilde{h} \circ e_U$ and $e'_L := \tilde{h} \circ e_L$. With these data we already have that $(Q', M \rtimes L, d'_U, c'_U, e'_U)$ and $(Q', N \rtimes L, d'_L, c'_L, e'_L)$ are reflexive graphs. Since they are quotients of groupoids, they are groupoids as well. In particular the square of groupoids involving them is a double groupoid: this can be shown by proving the commutativity of each of the nine squares by using the fact that $\tilde{h}$ is a regular epimorphism.

It remains to define two morphisms

$$\alpha \colon M \rtimes L \to Q' \qquad\qquad \beta \colon N \rtimes L \to Q'$$

making (3.24) commute and to show that $\alpha = k_{d'_L}$ and $\beta = k_{d'_U}$. We are going to construct only α since a symmetric strategy works also for β. Let us first of all notice that the square

$$\begin{tikzcd} M + (M \otimes N) \arrow[r, "\binom{e_U \circ k^L_M}{\gamma}"] \arrow[d, "1+h"'] & Q_{M\otimes N} \arrow[d, "\tilde{h}"] \\ M + P \arrow[r, "\binom{e'_U \circ k^L_M}{\gamma'}"'] & Q' \end{tikzcd} \tag{3.27}$$

is commutative due to the commutativity of the two components

$$\begin{tikzcd} M \arrow[r, "k^L_M"] \arrow[d, equal] & M \rtimes L \arrow[r, "e_U"] \arrow[d, equal] & Q_{M\otimes N} \arrow[d, "\tilde{h}"] \\ M \arrow[r, "k^L_M"'] & M \rtimes L \arrow[r, "e'_U"] & Q' \end{tikzcd} \qquad \begin{tikzcd} M \otimes N \arrow[r, "\gamma"] \arrow[d, "h"'] & Q_{M\otimes N} \arrow[d, "\tilde{h}"] \\ P \arrow[r, "\gamma'"'] & Q' \end{tikzcd}$$

Also the triangle

$$
\begin{tikzcd}
M+(M\otimes N) \arrow[rr, "\sigma_{\xi^M_{M\otimes N}}"] \arrow[dr, "\left(\begin{smallmatrix} e_U\circ k^L_M \\ \gamma \end{smallmatrix}\right)"'] & & (M\otimes N)\rtimes M \arrow[dl, "k_{d_U}"] \\
& Q_{M\otimes N} &
\end{tikzcd}
\tag{3.28}
$$

commutes, due to the commutativity of the two components

$$
\begin{tikzcd}
M \arrow[r, "e_T"] \arrow[d, "k^L_M"'] & (M\otimes N)\rtimes M \arrow[d, "k_{d_L}"] \\
M\rtimes L \arrow[r, "e_U"'] & Q_{M\otimes N}
\end{tikzcd}
\qquad
\begin{tikzcd}
M\otimes N \arrow[r, "k_{d_T}"] \arrow[d, "k_{d_W}"'] \arrow[dr, "\gamma"] & (M\otimes N)\rtimes M \arrow[d, "k_{d_L}"] \\
(M\otimes N)\rtimes N \arrow[r, "k_{d_U}"'] & Q_{M\otimes N}
\end{tikzcd}
$$

Now we can use the definition of the semidirect product $P\rtimes M$ as a coequaliser to obtain the dotted arrow α from the commutative diagram of solid arrows

$$
\begin{tikzcd}
M\flat(M\otimes N) \arrow[r, shift left, "k_{M,M\otimes N}"] \arrow[r, shift right, "i_{M\otimes N}\circ\xi^M_{M\otimes N}"'] \arrow[dd, two heads, "1\flat h"'] & M+(M\otimes N) \arrow[rr, "\sigma_{\xi^M_{M\otimes N}}"] \arrow[dr, "\left(\begin{smallmatrix} e_U\circ k^L_M \\ \gamma \end{smallmatrix}\right)"'] \arrow[dd, two heads, "1+h"'] & & (M\otimes N)\rtimes M \arrow[dl, "k_{d_U}"] \arrow[dd, "h\rtimes 1"] \\
& & Q_{M\otimes N} \arrow[dd, "\tilde{h}"] & \\
M\flat P \arrow[r, shift left, "k_{M,P}"] \arrow[r, shift right, "i_P\circ\xi^M_P"'] & M+P \arrow[rr, "\sigma_{\xi^M_P}"] \arrow[dr, "\left(\begin{smallmatrix} e'_U\circ k^L_M \\ \gamma' \end{smallmatrix}\right)"'] & & P\rtimes M \arrow[dl, dotted, "\alpha"] \\
& & Q' &
\end{tikzcd}
$$

In particular we need to show that $\left(\begin{smallmatrix} e'_U\circ k^L_M \\ \gamma' \end{smallmatrix}\right)$ coequalises $k_{M,P}$ and $i_P\circ\xi^M_P$: this is done by precomposing with the regular epimorphism $1\flat h$ and by using the commutativity of (3.27) and (3.28). In a similar way we build also $\beta\colon P\rtimes N\to Q'$. Let us now show that every square in (3.24) involving α and β commutes:

- We already know that the square

$$
\begin{tikzcd}
(M\otimes N)\rtimes M \arrow[r, "k_{d_L}"] \arrow[d, "h\rtimes 1"'] & Q_{M\otimes N} \arrow[d, "\tilde{h}"] \\
P\rtimes M \arrow[r, "\alpha"'] & Q'
\end{tikzcd}
$$

  commutes by construction and similarly for the one involving β;

- The square

$$\begin{tikzcd}
P \arrow[r, "k_P^M"] \arrow[d, "k_P^N"'] \arrow[dr, "\gamma'"] & P \rtimes M \arrow[d, "\alpha"] \\
P \rtimes N \arrow[r, "\beta"'] & Q'
\end{tikzcd}$$

  commutes because both the compositions are equal to γ' by construction;

- Finally we need to show that the two right-pointing squares and the left-pointing square in

$$\begin{tikzcd}
P \rtimes M \arrow[r, shift left=2, "d_P^M"] \arrow[r, shift right=2, "c_P^M"'] \arrow[d, "\alpha"'] & M \arrow[l, "e_P^M"] \arrow[d, "k_M^L"] \\
Q' \arrow[r, shift left=2, "d'_U"] \arrow[r, shift right=2, "c'_U"'] & M \rtimes L \arrow[l, "e'_U"]
\end{tikzcd}$$

  commute. For the left-pointing one we have the chain of equalities

$$\alpha \circ e_P^M = \alpha \circ \sigma_{\xi_P^M} \circ i_P = \alpha \circ \sigma_{\xi_P^M} \circ i_P = \begin{pmatrix} e'_U \circ k_M^L \\ \gamma' \end{pmatrix} \circ i_P = e'_U \circ k_M^L,$$

  whereas for the right-pointing ones we need to precompose with the regular epimorphism $\sigma_{\xi_P^M}$ obtaining

$$\begin{aligned} d'_U \circ \alpha \circ \sigma_{\xi_P^M} &= \begin{pmatrix} d'_U \circ e'_U \circ k_M^L \\ d'_U \circ \gamma' \end{pmatrix} = \begin{pmatrix} k_M^L \\ 0 \end{pmatrix} = k_M^L \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= k_M^L \circ d_P^M \circ \sigma_{\xi_P^M}, \end{aligned}$$

  and

$$\begin{aligned} c'_U \circ \alpha \circ \sigma_{\xi_P^M} &= \begin{pmatrix} c'_U \circ e'_U \circ k_M^L \\ c'_U \circ \gamma' \end{pmatrix} = \begin{pmatrix} k_M^L \\ k_M^L \circ p_M \end{pmatrix} = k_M^L \begin{pmatrix} 1 \\ p_M \end{pmatrix} \\ &= k_M^L \circ c_P^M \circ \sigma_{\xi_P^M}. \end{aligned}$$

  Finally we can repeat the same argument for the corresponding squares involving β.

It remains to prove that $\alpha = k_{d_L}$ (and similarly that $\beta = k_{d_U}$): to do this, we first show that d_L is the cokernel of α and then that α is a normal monomorphism, which implies the thesis. The first step is easily done by directly showing the universal property of the cokernel through the diagram

$$\begin{tikzcd}
(M \otimes N) \rtimes M \arrow[r, "k_{d_L}"] \arrow[d, "h \rtimes 1"'] & Q_{M \otimes N} \arrow[r, "d_L"] \arrow[d, "\tilde{h}"'] & N \rtimes L \arrow[d, equal] \\
P \rtimes M \arrow[r, "\alpha"'] & Q' \arrow[r, "d'_L"'] & N \rtimes L
\end{tikzcd}$$

and the universal property of the cokernel d_L.

The kernel K of h is normal in $Q_{M\otimes N}$ if and only if (3.26) is a pullback. Let us show that then α is a monomorphism. Since $\mathbb{A}$ is protomodular, pullbacks reflect monomorphisms; since k_{d_L} is a monomorphism, so is α. Furthermore it is normal as a direct image of a normal monomorphism, which implies our claim. $\square$

Chapter 4

Universal central extensions through the non-abelian tensor product

The aim of this chapter is to show a direct application of the non-abelian tensor product construction by studying a result on universal central extensions of crossed modules due of Brown-Loday ([15], in the case of groups) and Edalatzadeh ([35], in the case of Lie algebras). We prove, namely, that a crossed module over a fixed base object is perfect (in an appropriate sense) if and only if it admits a universal central extension. We first follow an ad-hoc approach, extending it to the context of semi-abelian categories by using a general version of the non-abelian tensor product of Brown and Loday. We then provide two interpretations from the perspective of categorical Galois theory. A first one follows the line of Edalatzadeh [35] in the context of quasi-pointed categories. This allows to capture centrality, but we couldn't find a natural way to treat perfectness in this setting. We then switch to the pointed context where the theory developed by Casas and Van der Linden can be used. In this simpler environment we find a convenient interpretation both of centrality and of perfectness.

The chapter is oganised as follows:

- In Section 4.1 we recall well-known results on central extension theory in the context of semi-abelian categories following [56, 25].

- Section 4.2 is devoted to the construction of the Birkhoff subcategory $\mathbf{TrivAct}_L(\mathbb{A})$ of $\mathbf{Act}_L(\mathbb{A})$ and to the study of the so-called *coinvariance commutator*: we show some of its useful properties and we prove that it is isomorphic to the Higgins commutator.

- In Section 4.3 we generalise, with an ad-hoc approach, the definitions of central extensions and perfect objects from the category of L-crossed modules of groups and Lie algebras to the categories of internal L-crossed modules in any semi-abelian category with (SH). We then show in Theorem 4.3.9 that an internal crossed module admits a universal central extension if and only if it is perfect.

- By using the coinvariance reflector defined in Section 4.2, in Section 4.4 we construct a Birkhoff subcategory of $\mathbf{XMod}_L(\mathbb{A})$ which generalises the approach adopted by Edalatzadeh [35]: in this way we are able to show that the previous ad-hoc definitions of central extensions and perfect objects coincides with the more general version of the definitions given in [35].

- Finally in Section 4.5 we describe a more natural approach, through which we are able to reinterpret what we have done so far, in the theory of central extension in the semi-abelian context. This means that we can construct a Birkhoff subcategory of $\mathbf{XMod}(\mathbb{A})$ in such a way that the definitions of central extensions of crossed modules and of perfect crossed modules induced by the standard theory from [25] coincide with the former definitions given in the previous sections.

4.1 Extensions and central extensions in semi-abelian categories

We recall some basic definitions and results of categorical Galois theory [4, 54, 56, 57].

Definition 4.1.1 ([56])**.** Let $\mathbb{C}$ be an exact category and $\mathbb{X}$ a subcategory of $\mathbb{C}$. We say that $\mathbb{X}$ is a *Birkhoff subcategory of* $\mathbb{C}$ if the following hold:

(a) $\mathbb{X}$ is a full and reflective subcategory of $\mathbb{C}$,

(b) $\mathbb{X}$ is closed under subobjects in $\mathbb{C}$ and

(c) $\mathbb{X}$ is closed under quotients (i.e., regular epimorphisms) in $\mathbb{C}$.

We usually denote the left adjoint as $I\colon \mathbb{C} \to \mathbb{X}$ and, when we do not omit it, the right adjoint as $J\colon \mathbb{X} \to \mathbb{C}$. The largest Birkhoff subcategory of $\mathbb{C}$ is obviously $\mathbb{C}$ itself, whereas the smallest one is given by $Sub(1)$ where 1 denotes the terminal object. When $\mathbb{C}$ is a variety, being a Birkhoff subcategory is the same as being a subvariety.

Lemma 4.1.2 ([56])**.** *A reflective subcategory $\mathbb{X}$ of an exact category $\mathbb{C}$ is a Birkhoff subcategory if and only if for each regular epimorphism $f\colon A \to B$, the naturality square*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & JI(A) \arrow[d, "JI(f)"] \\ B \arrow[r, "\eta_B"'] & JI(B) \end{tikzcd}$$

is a regular pushout. In particular $\mathbb{X}$ is closed under subobjects if η_A is a regular epimorphism for each $A \in \mathbb{C}$, and it is closed under regular quotients when the above square is a regular pushout. □

Since we shall always be dealing with categories that are Mal'tsev, for each Birkhoff subcategory there is a Galois theory à la Janelidze (see [54, 56, 4]). We recall the main definitions.

Definition 4.1.3 ([56])**.** We denote with $(\mathbb{C} \downarrow B)$ or with $Ext_{\mathbb{C}}(B)$ the *category of extensions of* B, which is the full subcategory of $\mathbb{C}/B$ whose objects are the regular epimorphisms having B as codomain; notice that a morphism in $Ext_{\mathbb{C}}(B)$ is any triangle in $\mathbb{C}$ from a regular epimorphism to another regular epimorphism with the same codomain B.

Definition 4.1.4 ([56])**.** Given a Birkhoff subcategory $\mathbb{X} \hookrightarrow \mathbb{C}$ we say that an extension $f\colon A \to B$ is an $\mathbb{X}$*-trivial extension (of* B*)* when the naturality square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & JI(A) \arrow[d, "JI(f)"] \\ B \arrow[r, "\eta_B"'] & JI(B) \end{tikzcd}$$

is a pullback. We will denote with $Triv_{\mathbb{C}}^{\mathbb{X}}(B)$ the full subcategory of $Ext_{\mathbb{C}}(B)$ whose objects are the $\mathbb{X}$-trivial extensions of B.

Definition 4.1.5 ([56])**.** Given a Birkhoff subcategory $\mathbb{X} \hookrightarrow \mathbb{C}$ we say that an extension $f\colon A \to B$ is an $\mathbb{X}$*-central extension (of* B*)* when there exists an extension $g\colon C \to B$ such that the pullback $g^*(f)$

$$\begin{tikzcd} A \times_B C \arrow[r, "g^*(f)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

of f along g is a $\mathbb{X}$-trivial extension. We will denote by $Centr_{\mathbb{C}}^{\mathbb{X}}(B)$ the full subcategory of $Ext_{\mathbb{C}}(B)$ whose objects are the $\mathbb{X}$-central extensions of B. We have the chain of inclusions

$$Triv_{\mathbb{C}}^{\mathbb{X}}(B) \subseteq Centr_{\mathbb{C}}^{\mathbb{X}}(B) \subseteq Ext_{\mathbb{C}}(B).$$

Remark 4.1.6. In the general setting of [56] an $\mathbb{X}$*-normal extension* is defined as an extension $f\colon A \to B$ such that one of the projections r_0, r_1 in the kernel pair

$$\begin{tikzcd} Kp(f) \arrow[r, "r_0"] \arrow[d, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a $\mathbb{X}$-trivial extension. Of course this is a stronger notion than the one of $\mathbb{X}$-central extension, but in our context the two coincide, as stated in the following.

Lemma 4.1.7 ([56])**.** *In the context of an exact protomodular category $\mathbb{C}$, an extension is $\mathbb{X}$-central if and only if it is $\mathbb{X}$-normal. Furthermore a split epimorphism is a $\mathbb{X}$-central extension if and only if it is $\mathbb{X}$-trivial.*

Proof. Proposition 4.7 in [56] tells us that the two claims are equivalent and Theorem 4.8 again in [56] proves that they hold in every Goursat category. Exactness and protomodularity imply the Mal'cev property which is stronger than the Goursat property. □

The following definitions are borrowed from [25], where the theory of universal central extensions is explored in detail.

Definition 4.1.8 ([25])**.** We say that an extension $u\colon U \to B$ is a *universal $\mathbb{X}$-central extension of B* if it is an initial object in $\mathit{Centr}^{\mathbb{X}}_{\mathbb{C}}(B)$.

Definition 4.1.9 ([25])**.** Given a Birkhoff subcategory $\mathbb{X}$ of a pointed exact category $\mathbb{C}$, we say that an object $A \in \mathbb{C}$ is *$\mathbb{X}$-perfect* whenever its reflection $I(A)$ is the zero object $0 \in \mathbb{X}$.

Example 4.1.10. A key example of a Birkhoff subcategory is the subcategory of abelian objects in any semi-abelian category, which are those objects that admit an internal abelian group structure. For instance, abelian groups in the category of all groups, or vector spaces equipped with a trivial (zero) multiplication in the category of Lie algebras over any field. It is clear that $\mathbf{Ab}(\mathbb{A})$ is an abelian category, but it is also a Birkhoff subcategory of $\mathbb{A}$: indeed it is a full reflective subcategory of $\mathbb{A}$, closed under subobjects and regular quotients.

This means that we have a definition of $\mathbf{Ab}(\mathbb{A})$-central extensions, also called *categorically central extensions* in contrast with *algebraically central extensions*: the former ones are given through Definition 4.1.5, whereas the latter ones arise naturally from commutator theory (see [45, 57] for further details). They are the extensions $f\colon A \to B$ whose kernel pair congruence $Kp(f)$ is contained in the center of A, or equivalently commutes with ∇_A: this means $[Kp(f), \nabla_A]^{\mathcal{S}}_A = \Delta_A$, where ∇_A and Δ_A are the largest and the smallest congruences on A.

Lemma 4.1.11 ([47, 65])**.** *Lef $f\colon A \to B$ be a regular epimorphism. It is an $\mathbf{Ab}(\mathbb{A})$-central extension iff $[K_f, A]^{\mathcal{H}}_A = 0$.*

Proof. From Theorem 6.1 in [46] we know that f is a categorically central extension iff it is a algebraically central extension, that is iff $[Kp(f), \nabla_A]^{\mathcal{S}}_A = \Delta_A$. But since $\nabla_A = Kp(t_A)$ (with $t_A\colon A \to 0$ the only map from A to the terminal object), we know that

$$[Kp(f), \nabla_A]^{\mathcal{S}}_A = \Delta_A \qquad \Longleftrightarrow \qquad [K_f, K_{t_A}]^{\mathcal{Q}}_A = 0.$$

Notice that for this equivalence to hold we need not SH: indeed it holds whenever the two subobjects of A cover it (see Proposition 4.6 in [40]) and we obviously have $K_f \vee K_{t_A} = A$ since $K_{t_A} = A$. This also implies that their Huq commutator coincides with their Higgins commutator, obtaining the equivalent condition $[K_f, A]^{\mathcal{H}}_A = 0$. □

4.2 Trivial actions and coinvariants

In this section we work out a less trivial example of a Galois structure, which later on will be useful for us: we study the so-called *coinvariants reflector* from internal actions to trivial actions. This is a categorical conceptualisation of a classical construction, well known in group cohomology (see [12]). Throughout, we let L be a fixed object in a semi-abelian category $\mathbb{A}$.

Let us start by defining a suitable Birkhoff subcategory of $\mathbf{Act}_L(\mathbb{A})$: the subcategory $\mathbf{TrivAct}_L(\mathbb{A})$ of trivial L-actions.

Definition 4.2.1. Consider an L-action expressed as a point with a chosen kernel

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "k_p"] & X \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \arrow[r] & 0. \end{tikzcd}$$

We say that it is a *trivial action* when there exists an isomorphism of split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "k_p"] \arrow[d, equal] & X \arrow[r, shift left, "p"] \arrow[d, "\phi"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "\langle 1_M{,}0\rangle"'] & M \times L \arrow[r, shift left, "\pi_2"] & L \arrow[l, shift left, "\langle 0{,}1_L\rangle"] \arrow[r] & 0 \end{tikzcd}$$

with the splitting induced by the product. The category $\mathbf{TrivAct}_L(\mathbb{A})$ of trivial L-actions is the full subcategory of $\mathbf{Act}_L(\mathbb{A})$ whose objects are trivial L-actions.

Construction 4.2.2. We wish to construct a functor $I\colon \mathbf{Act}_L(\mathbb{A}) \to \mathbf{TrivAct}_L(\mathbb{A})$ which is left adjoint to the inclusion functor $J\colon \mathbf{TrivAct}_L(\mathbb{A}) \to \mathbf{Act}_L(\mathbb{A})$. Given a split epimorphism, the first step is to construct another split epimorphism using the cokernel C_s of s. That is, taking the pushout of s along the zero morphism:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "k_p"] \arrow[d, "c_s \circ k_p"'] & X \arrow[r, shift left, two heads, "p"] \arrow[d, two heads, "c_s"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_s \arrow[r, equal] & C_s \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r] & 0. \end{tikzcd} \tag{4.1}$$

Then we take the pullback of the morphisms with codomain 0, thus obtaining the product

$$\begin{tikzcd} 0 \arrow[r] & C_s \arrow[r, tail, "\langle 1_{C_s}{,}0\rangle"] \arrow[d, equal] & C_s \times L \arrow[r, shift left, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[l, shift left, "\langle 0{,}1_L\rangle"] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_s \arrow[r, equal] & C_s \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r] & 0. \end{tikzcd}$$

This trivial L-action in the upper sequence is called the *objects of coinvariants* of the given action, and it is the image through I of the action we began with. This gives us the morphism

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, "c_s\circ k_p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, shift left, two heads, "p"] \arrow[d, "\langle c_s,p\rangle"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_s \arrow[r, tail, "\langle 1_{C_s},0\rangle"'] & C_s\times L \arrow[r, shift left, two heads, "\pi_2"] & L \arrow[l, shift left, "\langle 0,1_L\rangle"] \arrow[r] & 0
\end{tikzcd}$$

which happens to be the unit $\eta\colon Id_{\mathbf{Act}_L(\mathbb{A})} \to HI(-)$ of the adjunction. When there is no risk of confusion, we will denote the component of η depicted in the diagram by η_M.

The following definition follows the pattern of [39, 40]: the kernel of the unit of a Birkhoff reflector is viewed as a commutator, relative to this reflector.

Definition 4.2.3. With the notation of the previous construction we take the kernel of the unit η_M

$$\begin{tikzcd}
0 \arrow[r] & \llbracket L,M\rrbracket \arrow[r, equal] \arrow[d, tail, "k_{(c_s\circ k_p)}"'] & \llbracket L,M\rrbracket \arrow[r, shift left, two heads] \arrow[d, tail, "k_{\langle c_s,p\rangle}"] & 0 \arrow[l, shift left] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, "c_s\circ k_p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, shift left, two heads, "p"] \arrow[d, "\langle c_s,p\rangle"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_s \arrow[r, tail, "\langle 1_{C_s},0\rangle"'] & C_s\times L \arrow[r, shift left, two heads, "\pi_2"] & L \arrow[l, shift left, "\langle 0,1_L\rangle"] \arrow[r] & 0
\end{tikzcd}$$

and we define the *coinvariants commutator* (the reason behind the commutator notation will become clear in the following sections) $\llbracket L,M\rrbracket$ as the top left kernel. Notice that the equality in the first row comes from the fact that the lower left hand square is a pullback. Therefore we could equivalently define $\llbracket L,M\rrbracket$ as the kernel of $c_s\circ k_p$, computed in $\mathbb{A}$.

Remark 4.2.4. The mapping which sends an L-point as above to $\llbracket L,M\rrbracket$ is functorial, indeed for each morphism of L-points as in (4.6) (not necessarily a regular epimorphism) we have a unique map $\llbracket 1,f\rrbracket\colon \llbracket L,M\rrbracket \to \llbracket L,M'\rrbracket$ induced by the universal properties of the kernels and cokernels involved.

Remark 4.2.5. By construction we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, "c_s\circ k_p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, shift left, two heads, "p"] \arrow[d, "\langle c_s,p\rangle"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_s \arrow[r, tail, "\langle 1_{C_s},0\rangle"'] \arrow[d, equal] & C_s\times L \arrow[r, shift left, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] & L \arrow[l, shift left, "\langle 0,1_L\rangle"] \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & C_s \arrow[r, equal] & C_s \arrow[r, shift left, two heads] & 0 \arrow[l, shift left] \arrow[r] & 0
\end{tikzcd}$$

where the rightmost vertical composite rectangle

$$\begin{tikzcd} X \arrow[r, two heads, "p"] \arrow[d, two heads, "c_s"'] & L \arrow[d, two heads, "0"] \\ C_s \arrow[r, two heads, "0"'] \arrow[ur, phantom, "\ulcorner" very near end] & 0 \end{tikzcd}$$

is a pushout of regular epimorphisms, that is a regular pushout: indeed the universal property can be shown directly by using the fact that $p \circ s = 1_L$ and that c_s is the cokernel of s. Since $\langle c_s, p\rangle$ is the comparison morphism to the induced pullback, it is automatically a regular epimorphism. By Lemma 1.2.16, this is equivalent to η_M being a regular epimorphism of L-points. Furthermore, since the top left square is a pullback, also $c_s \circ k_p$ is a regular epimorphism.

Proposition 4.2.6. $\mathbf{TrivAct}_L(\mathbb{A})$ *is a Birkhoff subcategory of* $\mathbf{Act}_L(\mathbb{A})$.

Proof. According to Lemma 4.1.2 we need to prove three things:

(a) $\mathbf{TrivAct}_L(\mathbb{A})$ is a reflective subcategory of $\mathbf{Act}_L(\mathbb{A})$, that is the inclusion functor has a left adjoint;

(b) for each L-action

$$\begin{tikzcd} X \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \end{tikzcd}$$

the unit η_X is a regular epimorphism of L-actions;

(c) for each regular epimorphism of L-actions

$$\begin{tikzcd} X \arrow[r, shift left, "p"] \arrow[d, two heads, "f"'] & L \arrow[l, shift left, "s"] \arrow[d, equal] \\ X' \arrow[r, shift left, "p'"] & L \arrow[l, shift left, "s'"] \end{tikzcd}$$

its associated naturality square is a regular pushout of L-actions.

For (a) it suffices to show the universal property of η: consider a morphism of L-actions with a trivial L-action as codomain

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, "g \circ k_p"'] & X \arrow[r, shift left, "p"] \arrow[d, "\langle g,p\rangle"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r, tail, "\langle 1,0\rangle"'] & M' \times L \arrow[r, shift left, "\pi_2"] & L \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[r] & 0 \end{tikzcd} \tag{4.2}$$

We know that

$$g \circ s = \pi_1 \circ \langle g, p\rangle \circ s = \pi_1 \circ \langle 0, 1\rangle = 0$$

and therefore there exists a unique map $\phi\colon C_s \to M'$ such that

$$\begin{tikzcd}
X \arrow[r, "c_s"] \arrow[dr, "g"'] & C_s \arrow[d, dotted, "\phi"] \\
& M'
\end{tikzcd}$$

commutes. But this means that we can decompose in a unique way the morphism (4.2) as

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, "c_s \circ k_p"'] & X \arrow[r, shift left, "p"] \arrow[d, "\langle c_s, p\rangle"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_s \arrow[r, tail, "\langle 1,0\rangle"'] \arrow[d, "\phi"'] & C_s \times L \arrow[r, shift left, "\pi_2"] \arrow[d, "\phi \times 1_L"] & L \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r, tail, "\langle 1,0\rangle"'] & M' \times L \arrow[r, shift left, "\pi_2"] & L \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[r] & 0
\end{tikzcd}$$

Step (b) is already proved in Remark 4.2.5. It remains to show that (c) holds, and by Lemma 1.2.17 (and Lemma 1.2.16) this amounts to showing that for each regular epimorphism $f\colon X \to X'$ the left face of the cube

$$\begin{tikzcd}
X \arrow[rr, shift left, two heads, "p"] \arrow[dr, two heads, "f"] \arrow[dd, two heads, "\langle c_s, p\rangle"'] & & L \arrow[ll, shift left, "s"] \arrow[dr, equal] \arrow[dd, equal] & \\
& X' \arrow[rr, shift left, two heads, "p'"] \arrow[dd, two heads, "\langle c_{s'}, p'\rangle"' near start] & & L \arrow[ll, shift left, "s'"] \arrow[dd, equal] \\
C_s \times L \arrow[rr, shift left, two heads, "\pi_2"] \arrow[dr, two heads, "c \times 1_L"'] & & L \arrow[ll, shift left, "\langle 0,1\rangle"] \arrow[dr, equal] & \\
& C_{s'} \times L \arrow[rr, shift left, two heads, "\pi_2"] & & L \arrow[ll, shift left, "\langle 0,1\rangle"]
\end{tikzcd}$$

is a regular pushout in $\mathbb{A}$. By definition it suffices to show that the comparison morphism ϕ

$$\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[dr, dotted, "\phi"] \arrow[ddr, bend right, "\langle c_s, p\rangle"'] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "\langle c_{s'}, p'\rangle"] \\
& C_s \times L \arrow[r, "c \times 1_L"'] & C_{s'} \times L
\end{tikzcd} \tag{4.3}$$

is a regular epimorphism.

In order to prove this, we need two steps: the first one is to show an equivalent description of the pullback P, that is the fact that it can be obtained also as the pullback

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "c_{s'}"] \\ C_s \arrow[r, "c"'] & C_{s'} \end{tikzcd} \tag{4.4}$$

since this can be decomposed as the two pullbacks

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\ C_s \times L \arrow[r, "c\times 1_L"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_{s'} \times L \arrow[d, "\pi_1"] \\ C_s \arrow[r, "c"'] & C_{s'} \end{tikzcd}$$

So in order to show that ϕ is a regular epimorphism, we can use the diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, "f"] \arrow[ddr, two heads, bend right, "c_s"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & X' \arrow[d, two heads, "c_{s'}"] \\ & C_s \arrow[r, two heads, "c"'] & C_{s'} \end{tikzcd} \tag{4.5}$$

The second step consists into showing that the outer square in (4.5) is a regular pushout and since the four maps are already regular epimorphisms it amounts to showing that it has the desired universal property.

$$\begin{tikzcd} L \arrow[r, equal] \arrow[d, "s"'] & L \arrow[d, "s'"] & \\ X \arrow[r, two heads, "f"] \arrow[d, two heads, "c_s"'] \arrow[dr, phantom, "(**)"] & X' \arrow[d, two heads, "c_{s'}"] \arrow[ddr, bend left, "u"] & \\ C_s \arrow[r, two heads, "c"'] \arrow[drr, bend right, "v"'] & C_{s'} \arrow[dr, dotted, "w"] & \\ & & Z \end{tikzcd}$$

We have that

$$u \circ s' = u \circ f \circ s = v \circ c_s \circ s = 0$$

and therefore by the universal property of $C_{s'}$ there exists a unique $w\colon C_{s'} \to Z$ such that $w \circ c_{s'} = u$. The commutativity of the other triangle is given by the fact that c_s is an epimorphism.

Consequently $(**)$ is a regular pushout and the induced comparison morphism ϕ in (4.5) is a regular epimorphism. This in turn implies that the outer square in (4.3) is a regular pushout, that is the thesis. □

Remark 4.2.7. According to Definition 4.1.9 we have that an L-action

$$0 \longrightarrow M \rightarrowtail X \rightleftarrows L \longrightarrow 0$$

is $\mathbf{TrivAct}_L(\mathbb{A})$-perfect iff its image through the reflector $\mathbf{TrivAct}_L(\mathbb{A}) \to \mathbf{Act}_L(\mathbb{A})$ is the zero L-action, that is the point

$$0 \longrightarrow 0 \rightarrowtail L \underset{1_L}{\overset{1_L}{\rightleftarrows}} L \longrightarrow 0.$$

This in turn is equivalent to the equality of subobjects $[\![L, M]\!] = M$ since a map is zero iff its kernel is an isomorphism. Hence an L-action is perfect iff $[\![L, M]\!] = M$.

The following is a special case of a result in [39].

Lemma 4.2.8. *Consider a regular epimorphism in* $\mathbf{Pt}_L(\mathbb{A})$

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, two heads, "f"'] & X \arrow[r, shift left, "p"] \arrow[d, two heads, "g"'] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r, tail, "k_{p'}"'] & X' \arrow[r, shift left, "p'"] & L \arrow[l, shift left, "s'"] \arrow[r] & 0
\end{tikzcd} \tag{4.6}$$

then the induced map $[\![1, f]\!]\colon [\![L, M]\!] \to [\![L, M']\!]$ *is again a regular epimorphism.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
{[\![L,M]\!]} \arrow[dr, "{[\![1,f]\!]}"] \arrow[dd, tail, "k_{(c_s\circ k_p)}"'] & & & & & \\
& {[\![L,M']\!]} \arrow[dd, tail, "k_{(c_{s'}\circ k_{p'})}"] & & & & \\
M \arrow[rr, tail, "k_p"] \arrow[dr, two heads, "f"'] \arrow[dd, two heads, "c_s\circ k_p"'] & & X \arrow[rr, shift left, two heads, "p"] \arrow[dr, two heads, "g"] \arrow[dd, "{\langle c_s,p\rangle}"'] & & L \arrow[ll, shift left, "s"] \arrow[dr, equal] \arrow[dd, equal] & \\
& M' \arrow[rr, tail, "k_{p'}"] \arrow[dd, two heads, "c_{s'}\circ k_{p'}"] & & X' \arrow[rr, shift left, two heads, "p'"] \arrow[dd, "{\langle c_{s'},p'\rangle}"'] & & L \arrow[ll, shift left, "s'"] \arrow[dd, equal] \\
C_s \arrow[rr, tail, "{\langle 1,0\rangle}"] \arrow[dr, two heads, "\overline{f}"'] & & C_s \times L \arrow[rr, shift left, two heads, "\pi_2"] \arrow[dr, two heads, "\overline{f}\times 1_L"'] & & L \arrow[ll, shift left, "{\langle 0,1\rangle}"] \arrow[dr, equal] & \\
& C_{s'} \arrow[rr, tail, "{\langle 1,0\rangle}"'] & & C_{s'} \times L \arrow[rr, shift left, two heads, "\pi_2"] & & L \arrow[ll, shift left, "{\langle 0,1\rangle}"]
\end{tikzcd}
\tag{4.7}
$$

and consider the leftmost face of the cube on the left, denoted $(*)$ from now on: the strategy is to prove that $(*)$ is a regular pushout since the map induced between the kernels of a regular pushout is again a regular epimorphism.

So the first thing to do is to notice that every map in $(*)$ is a regular epimorphism (the only one we need to say something about is $\overline{f}$, but this is a regular epimorphism from the fact that it is the second map in a composition which is regular epimorphism).

Now to show the universal property of the pushout we use Lemma 1.1.25 and hence it suffices to show that the comparison map ϕ_1 (see (4.8)) induced by the pullback P_1 is a regular epimorphism. In order to prove this we can use the fact that ϕ_2 is a regular epimorphism: this is because the middle square is a regular pushout due to *iii*) in Proposition 4.2.6.

$$
\begin{tikzcd}
M \arrow[rr, tail, "k_p"] \arrow[dr, dotted, two heads, "\phi_1"] \arrow[ddr, bend left, "f"] \arrow[dddr, bend right, "c_s\circ k_p"'] & & X \arrow[rr, shift left, two heads, "p"] \arrow[dr, dotted, two heads, "\phi_2"] \arrow[ddr, bend left, "g"] \arrow[dddr, bend right, "\langle c_s{,}p\rangle"'] & & L \arrow[ll, shift left, "s"] \arrow[dr, equal] \arrow[ddrr, bend left, equal] \arrow[dddr, bend right, equal] & & \\
& P_1 \arrow[rr, tail, "\overline{k}"] \arrow[dr] \arrow[dd] & & P_2 \arrow[rr, shift left, two heads, "\overline{p}"] \arrow[dr] \arrow[dd] & & L \arrow[ll, shift left, "\overline{s}"] \arrow[dr, equal] \arrow[dd, equal] & \\
& & M' \arrow[rr, tail, "k_{p'}"] \arrow[dd, "c_{s'}\circ k_{p'}"] & & X' \arrow[rr, shift left, two heads, "p'"] \arrow[dd, "\langle c_{s'}{,}p'\rangle"] & & L \arrow[ll, shift left, "s'"] \arrow[dd, equal] \\
& C_s \arrow[rr, tail, "\langle 1{,}0\rangle"] \arrow[dr, "\overline{f}"'] & & C_s\times L \arrow[rr, shift left, two heads, "\pi_2"] \arrow[dr, "\overline{f}\times 1_L"'] & & L \arrow[ll, shift left, "\langle 0{,}1\rangle"] \arrow[dr, equal] & \\
& & C_{s'} \arrow[rr, tail, "\langle 1{,}0\rangle"'] & & C_{s'}\times L \arrow[rr, shift left, two heads, "\pi_2"] & & L \arrow[ll, shift left, "\langle 0{,}1\rangle"]
\end{tikzcd}
\tag{4.8}
$$

But since the square

$$
\begin{tikzcd}
M \arrow[r, "k_p"] \arrow[d, "\phi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "\phi_2"] \\
P_1 \arrow[r, "\overline{k}"'] & P_2
\end{tikzcd}
$$

is a pullback, also ϕ_1 is a regular epimorphism and hence the square $(*)$ is a regular pushout.

Finally it suffices to use Lemma 1.1.27 to obtain that $[\![1, f]\!]$ is a regular epimorphism. $\square$

Proposition 4.2.9. *Given an L-point*

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, tail, "k"] & X \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \arrow[r] & 0.
\end{tikzcd}
\tag{4.9}
$$

its coinvariance commutator $[\![L, M]\!]$, seen as a subobject of X, coincides with the Higgins commutator $[L, M]^H_X$ of L and M.

Proof. Consider the action $\xi\colon L\flat M \to M$ associated to the point in (4.9) and the induced "action core" $\xi^{\diamond}\colon L \diamond M \to M$. We have a morphism of split short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & L\flat M \arrow[r, tail, "k_{L,M}"] \arrow[d, "\xi"'] & L+M \arrow[r, shift left, "\binom{1}{0}"] \arrow[d, "\binom{s}{k}"] & L \arrow[l, shift left, "i_L"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, tail, "k"'] & X \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}
$$

Precomposing the first square with the map $i_{L,M}$ we obtain the commutative triangle of solid arrows

$$\begin{tikzcd}
0 \arrow[r] & L \diamond M \arrow[r, "k_{L,M}\circ i_{L,M}", triangle] \arrow[dr, "k\circ\xi^{\diamond}"'] & L+M \arrow[r, "\Sigma_{L,M}"] \arrow[d, "\binom{s}{k}"] & L\times M \arrow[r] \arrow[dl, dotted, "\phi_{s,k}"] & 0 \\
 & & X & &
\end{tikzcd}$$

Here we can see that $k \circ \xi^{\diamond} = 0$ iff s and k cooperate: first of all s and k cooperate iff there exists a map $\phi_{s,k}$ such that the triangle on the right commutes; if this map exists we deduce that

$$\begin{aligned}
k \circ \xi^{\diamond} &= \begin{pmatrix} s \\ k \end{pmatrix} \circ k_{L,M} \circ i_{L,M} \\
&= \phi_{s,k} \circ \Sigma_{L,M} \circ k_{L,M} \circ i_{L,M} = 0
\end{aligned}$$

On the other hand if $k \circ \xi^{\diamond} = 0$ then $\binom{s}{k} \circ k_{L,M} \circ i_{L,M} = 0$ and by the universal property of the cokernel $\Sigma_{L,M}$, such a $\phi_{s,k}$ exists.

Notice that the existence of $\phi_{s,k}$ is equivalent to the triviality of the action ξ, as shown by applying the Short Five Lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\langle 1,0\rangle", triangle] \arrow[d, equal] & M \times L \arrow[r, shift left, "\pi_L"] \arrow[d, dotted, "\phi_{s,k}"] & L \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "k_p"', triangle] & X \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

We then have that $k \circ \xi^{\diamond} = 0$ if and only if its image is 0, and its image is precisely $[L, M]^{\mathcal{H}}_X$. Hence we have shown that $[L, M]^{\mathcal{H}}_X = 0$ iff the action ξ is trivial. Resuming what we have so far: $[L, M]^{\mathcal{H}}_X$ is a normal subobject of X (since $X = L \vee M$), contained in M (we always have $[L, M]^{\mathcal{H}}_X \hookrightarrow [X, M]^{\mathcal{H}}_X$ and being k normal implies $[X, M]^{\mathcal{H}}_X \hookrightarrow M$), such that $[L, M]^{\mathcal{H}}_X = 0$ iff the action ξ is trivial.

But the coinvariance commutator share all these properties, therefore the two satisfy the same universal property and hence they are isomorphic. $\square$

Remark 4.2.10. In order to justify the last sentence in the previous proof it suffices to think of constructing two functors starting from the two commutators: in both cases, dividing out the commutator (which is a normal subobject) determines an adjunction to a RE-reflective subcategory defined by the condition that this commutator vanishes. But both commutators are zero at the same time, so they describe the same reflective subcategory. Therefore the two adjunctions are the same and consequently the units are the same. Finally their kernels are the same.

Lemma 4.2.11. *Given an internal crossed module* $(M \xrightarrow{\partial} L, \xi)$ *the object* $\frac{M}{[L,M]^{\mathcal{H}}_X}$ *is abelian.*

Proof. First of all notice that the Higgins commutator $[L, M]^{\mathcal{H}}_X$ can always be obtained as the image of the action core: indeed we already know that it is the image of $k \circ \xi^{\diamond}$ but since the second map is a monomorphism, the image of $\xi^{\diamond}$ is the same. Then by using the Peiffer condition and precomposing with $i_{M,M}$ we obtain the commutative diagram

$$\begin{tikzcd}
M \diamond M \arrow[r, "\chi^{\diamond}_M"] \arrow[d, "\partial \diamond 1"'] & M \arrow[d, equal] \\
L \diamond M \arrow[r, "\xi^{\diamond}"'] & M
\end{tikzcd}$$

Now by computing the REM-factorisation we find

$$\begin{tikzcd}
M \diamond M \arrow[r, two heads] \arrow[d, "\partial \diamond 1"'] & {[M, M]^{\mathcal{H}}_M} \arrow[r, tail] \arrow[d, dotted, "\phi"] & M \arrow[d, equal] \\
L \diamond M \arrow[r, two heads] & {[L, M]^{\mathcal{H}}_X} \arrow[r, tail] & M
\end{tikzcd}$$

Finally by taking cokernels of the horizontal maps in the square on the right we obtain

$$\begin{tikzcd}
{[M, M]^{\mathcal{H}}_M} \arrow[r, tail] \arrow[d, "\phi"] & M \arrow[r, two heads] \arrow[d, equal] & \frac{M}{[M,M]^{\mathcal{H}}_M} \arrow[d, dotted, two heads] \\
{[L, M]^{\mathcal{H}}_X} \arrow[r, tail] & M \arrow[r, two heads] & \frac{M}{[L,M]^{\mathcal{H}}_X}
\end{tikzcd}$$

Since $\frac{M}{[M,M]^{\mathcal{H}}_M}$ is an abelian object and since $\frac{M}{[L,M]^{\mathcal{H}}_X}$ is obtained as its quotient, it is abelian as well. $\square$

Remark 4.2.12. Notice that, if we consider the diagram in Definition 4.2.3, since $c_s \circ k_p$ is a regular epimorphism (see Remark 4.2.5), it is the cokernel of its kernel. This means that $C_s \cong \frac{M}{[L,M]^{\mathcal{H}}_X}$.

4.3 Central extensions of crossed modules, ad-hoc approach

We let $\mathbb{A}$ be a semi-abelian category. Imitating what has been done for groups and Lie algebras, we give the following definitions. Later on we will justify them from a Galois theory perspective.

Definition 4.3.1. Let L be an object of $\mathbb{A}$. A *central extension* in $\mathbf{XMod}_L(\mathbb{A})$ is a regular epimorphism of crossed modules

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f, 1_L)} (M' \xrightarrow{\partial'} L, \xi')$$

where for the kernel K_f of f we have that $[L, K_f]^{\mathcal{H}}_X = 0$.

Remark 4.3.2. Notice that this means that the kernel $K_f \xrightarrow{0} L$ of $(f, 1_L)$ has a trivial L-action: indeed by Proposition 4.2.9, we have $[\![L, K_f]\!] = [L, K_f]^{\mathcal{H}}_X$.

Definition 4.3.3. Let L be an object of $\mathbb{A}$. A *perfect object* in $\mathbf{XMod}_L(\mathbb{A})$ is an L-crossed module $(M \xrightarrow{\partial} L, \xi)$ such that $[L, M]^{\mathcal{H}}_X = M$.

Lemma 4.3.4. *An L-crossed module $(M \xrightarrow{\partial} L, \xi)$ is perfect if and only if in the corresponding internal groupoid*

$$\begin{tikzcd} X \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & L \arrow[l, "e" description] \end{tikzcd}$$

the normal closure $\overline{L}$ of e is all of X.

Proof. Consider the diagram

$$\begin{tikzcd} {[\![L, M]\!]} \arrow[r, tail] \arrow[d, tail] & \overline{L} \arrow[r, shift left, "d \circ k_{c_e}"] \arrow[d, tail, "k_{c_e}"] & L \arrow[l, shift left, "\overline{e}"] \arrow[d, equal] \\ M \arrow[r, "k_d", triangle] \arrow[d, two heads, "c_e \circ k_d"'] & X \arrow[r, shift left, "d"] \arrow[d, two heads, "c_e"] & L \arrow[l, shift left, "e"] \arrow[d, two heads] \\ C_e \arrow[r, equal] & C_e \arrow[r, shift left] & 0 \arrow[l, shift left] \end{tikzcd}$$

whose rows are split short exact sequences and whose columns are short exact sequences. If $[\![L, M]\!] = M$ then by the Short Five Lemma applied to the first two rows, we obtain that $\overline{L} = X$. The converse holds because the upper left square is a pullback. The result now follows from Proposition 4.2.9. □

Proposition 4.3.5. *Given a crossed module $(M \xrightarrow{\partial} L, \xi)$ we can construct the crossed square*

$$\begin{tikzcd} L \otimes M \arrow[r, "\delta_M"] \arrow[d] & M \arrow[d, "\partial"] \\ L \arrow[r, equal] & L \end{tikzcd}$$

by taking the non-abelian tensor product. Then δ_M is a regular epimorphism iff $(M \xrightarrow{\partial} L, \xi)$ is perfect.

Proof. Let us recall the construction of the non-abelian tensor product. First of all we denormalise both $(M \xrightarrow{\partial} L, \xi)$ and $(L \xrightarrow{1_L} L, \chi_L)$ and we take the pushout of the two split monomorphisms e and Δ_L obtaining the square of reflexive graphs

$$
\begin{tikzcd}
P \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] \arrow[d, shift right=2] \arrow[d, shift left=2] & M \rtimes L \arrow[l] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] \\
L \times L \arrow[u] \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & L \arrow[l, "\Delta_L" description] \arrow[u, "e" description]
\end{tikzcd}
$$

Then we take a quotient of P in order to obtain a double groupoid

$$
\begin{tikzcd}
P \arrow[rr, shift left=2, "p_1"] \arrow[rr, shift right=2, "p_2"'] \arrow[dr] \arrow[dd, shift right=2] \arrow[dd, shift left=2] & & M \rtimes L \arrow[ll] \arrow[d, equal] \\
 & \tilde{P} \arrow[r, shift left=2, "\tilde{p_1}"] \arrow[r, shift right=2, "\tilde{p_2}"'] \arrow[d, shift right=2] \arrow[d, shift left=2] & M \rtimes L \arrow[l] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] \\
L \times L \arrow[uu] \arrow[r, equal] & L \times L \arrow[u] \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & L \arrow[l, "\Delta_L" description] \arrow[u, "e" description]
\end{tikzcd}
$$

and finally we normalise back the whole double groupoid obtaining

$$
\begin{tikzcd}
L \otimes M \arrow[r, tail] \arrow[d, tail] & (L \otimes M) \rtimes M \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, tail] & M \arrow[l] \arrow[d, tail] \\
(L \otimes M) \rtimes L \arrow[r, tail, "k_{\tilde{p_1}}"] \arrow[d, shift right=2] \arrow[d, shift left=2] & \tilde{P} \arrow[r, shift left=2, "\tilde{p_1}"] \arrow[r, shift right=2, "\tilde{p_2}"'] \arrow[d, shift right=2] \arrow[d, shift left=2] & M \rtimes L \arrow[l] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] \\
L \arrow[u] \arrow[r, tail] & L \times L \arrow[u] \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & L \arrow[l, "\Delta_L" description] \arrow[u, "e" description]
\end{tikzcd}
$$

Now we are ready to prove the result. First of all, by applying Lemma 1.1.18 to the diagram

$$
\begin{tikzcd}
L \otimes M \arrow[r, "\delta_M"] \arrow[d, tail] & M \arrow[d, tail] \\
(L \otimes M) \rtimes L \arrow[r, two heads, "\delta_M \rtimes 1_L"] \arrow[d, shift right=2, "c_\otimes"'] \arrow[d, shift left=2, "d_\otimes"] & M \rtimes L \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] \\
L \arrow[u, "e_\otimes"] \arrow[r, equal] & L \arrow[u, "e"]
\end{tikzcd}
\tag{4.10}
$$

we deduce that δ_M is a regular epimorphism iff $\delta_M \rtimes 1_L$ is so. Then using the diagram

$$
\begin{tikzcd}
K_{p_1} \arrow[r, tail, "k_{p_1}"] \arrow[d, dotted, two heads] \arrow[rr, bend left, "\delta_P"] & P \arrow[r, "p_2"] \arrow[d, two heads] & M \rtimes L \arrow[d, equal] \\
K_{\tilde{p_1}} \arrow[r, tail, "k_{\tilde{p_1}}"'] \arrow[rr, bend right, "\delta_M \rtimes 1_L"'] & \tilde{P} \arrow[r, "\tilde{p_2}"'] & M \rtimes L
\end{tikzcd}
$$

we see that $\delta_M \rtimes 1_L$ is a regular epimorphism if and only if δ_P is so.

Being δ_P a proper map (see Lemma 1.1.20), it is a regular epimorphism iff it has trivial cokernel.

On the other hand, Lemma 1.3.2 says that for any reflexive graph the cokernel of the normalisation is the same as the coequaliser of the two split epimorphisms: this implies that the first one is trivial iff the second one is. Let us draw the picture involving the desired coequaliser Q

$$
\begin{tikzcd}
P \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] \arrow[d, dotted, shift right=2] \arrow[d, dotted, shift left=2] & M \rtimes L \arrow[l, "s" description] \arrow[r, two heads, "q"] \arrow[d, dotted, shift right=2, "c"'] \arrow[d, dotted, shift left=2, "d"] & Q \arrow[d, dotted, shift right=2] \arrow[d, dotted, shift left=2] \\
L \times L \arrow[u, "e'"] \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & L \arrow[u, "e"] \arrow[l, "\Delta_L" description] \arrow[r, two heads] & 0 \arrow[u]
\end{tikzcd}
$$

Here the second row involves also the coequaliser of π_1 and π_2 which is 0 (because it is the cokernel of the normalisation 1_L).

Let us prove by hand that q is the cokernel of e. Consider $\gamma\colon M \rtimes L \to Z$ such that $\gamma \circ e = 0$: in order to have the unique map $\phi\colon Q \to Z$ such that $\phi \circ q = \gamma$ it suffices to show that $\gamma \circ p_1 = \gamma \circ p_2$ since q is the coequaliser of p_1 and p_2. For that we use the fact that P is the pushout of e and Δ_L, and hence that (s, e') is a jointly epimorphic pair:

we have the equalities

$$\begin{cases} \gamma \circ p_1 \circ s = f = \gamma \circ p_2 \circ s \\ \gamma \circ p_1 \circ e' = \gamma \circ e \circ \pi_1 = 0 = \gamma \circ e \circ \pi_2 = \gamma \circ p_2 \circ e' \end{cases}$$

and so $\gamma \circ p_1 = \gamma \circ p_2$. This means that $Q = C_e$.

Finally by Lemma 4.3.4 we know that $C_e = 0$ iff $(M \xrightarrow{\partial} L, \xi)$ is perfect, and this gives us the thesis. $\square$

Proposition 4.3.6. *Consider a regular epimorphism of crossed module*

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f, 1_L)} (M' \xrightarrow{\partial'} L, \xi').$$

Then f considered as a morphism in $\mathbb{A}$ is a central extension (with respect to $\mathbf{Ab}(\mathbb{A})$).

Proof. Since $(M \xrightarrow{\partial} L, \xi)$ is a crossed module we have that $[K_\partial, M]^{\mathcal{H}}_M = 0$. From the commutativity of the triangle

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\partial"'] & M' \arrow[d, "\partial'"] \\ & L \end{tikzcd}$$

we can construct a monomorphism $K_f \hookrightarrow K_\partial$ which in turn induces a monomorphism $[K_f, M]^{\mathcal{H}}_M \hookrightarrow [K_\partial, M]^{\mathcal{H}}_M$ (since the Higgins commutator is monotone). This means that $[K_f, M]^{\mathcal{H}}_M = 0$ and therefore by Lemma 4.1.11 we obtain that f is central as a morphism in $\mathbb{A}$ with respect to $\mathbf{Ab}(\mathbb{A})$. $\square$

Proposition 4.3.7. *In the situation of Proposition 4.3.5, if $(M \xrightarrow{\partial} L, \xi)$ is perfect, then the map $(\delta_M, 1_L)$ is a central extension of L-crossed modules.*

Proof. We know from the proof of Proposition 4.3.5 that $\delta_M \rtimes 1_L$ is a regular epimorphism. Since it is also the differential of a crossed module (coming from a double groupoid square), it is a central extension in $\mathbb{A}$ with respect to the Birkhoff subcategory $\mathbf{Ab}(\mathbb{A})$.

Notice that, being the upper square in (4.10) a pullback, we have an equality between kernels $K_{\delta_M} = K_{\delta_M \rtimes 1}$. Consider the following diagram

$$\begin{tikzcd} K_{\delta_M} \arrow[r, tail] \arrow[d, equal] & K_{\delta_M} \rtimes L \arrow[r, two heads, shift left, "\overline{d}"] \arrow[d, "h"'] & L \arrow[l, shift left, "\overline{e}"] \arrow[d, "e"] \\ K_{\delta_M} \arrow[r, tail] & (L \otimes M) \rtimes L \arrow[r, two heads, "\delta_M \rtimes 1"'] & M \rtimes L \end{tikzcd} \tag{4.11}$$

where the square on the right is a pullback according to Corollary 1.2.18. The right inverse $\overline{e}$ of $\overline{d}$ is induced by the diagram

$$\begin{tikzcd}
L \arrow[drr, bend left, equal] \arrow[dr, dotted, "\overline{e}"] \arrow[ddr, bend right, "e_{\otimes}"'] & & \\
& K_{\delta_M} \rtimes L \arrow[r, two heads, "\overline{d}"] \arrow[d, "h"'] & L \arrow[d, "e"] \\
& (L \otimes M) \rtimes L \arrow[r, two heads, "\delta_M \rtimes 1"'] & M \rtimes L
\end{tikzcd}$$

The first row in (4.11) is the L-action which is part of the L-crossed module structure of the kernel of $(\delta_M, 1_L)$. In order to conclude the proof, we need to show that this action is trivial, so that $[L, K_{\delta_M}] = [\![L, K_{\delta_M}]\!] = 0$ as in Remark 4.3.2. The map $\overline{d}$ is obviously a split extension, so it suffices to show that it is central with respect to $\mathbf{Ab}(\mathbb{A})$: but we already know that $\delta_M \rtimes 1$ is so and because $\overline{d}$ is its pullback it is central as well. $\square$

Proposition 4.3.8. *Suppose that* $\mathbb{A}$ *satisfies the* Smith is Huq *condition. Then any central extension of* L*-crossed modules*

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f, 1_L)} (M' \xrightarrow{\partial'} L, \xi')$$

induces a crossed square

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\partial"'] & M' \arrow[d, "\partial'"] \\
L \arrow[r, equal] & L.
\end{tikzcd}$$

Proof. The first step is to prove that

$$[M \rtimes L, K_{f \rtimes 1}] = 0.$$

In order to do that we use the decomposition formula given in [67]

$$\begin{aligned}
[M \rtimes L, K_{f \rtimes 1}]^{\mathcal{H}}_{M \rtimes L} &= [M, K_{f \rtimes 1}]^{\mathcal{H}}_{M \rtimes L} \vee [L, K_{f \rtimes 1}]^{\mathcal{H}}_{M \rtimes L} \vee [M, L, K_{f \rtimes 1}]^{\mathcal{H}}_{M \rtimes L} \\
&= [M, K_{f \rtimes 1}]^{\mathcal{H}}_{M} \vee [L, K_{f \rtimes 1}]^{\mathcal{H}}_{K_f \rtimes L} \vee [M, L, K_{f \rtimes 1}]^{\mathcal{H}}_{M \rtimes L}
\end{aligned}$$

and we show that each component is trivial:

- notice that $K_f = K_{f \rtimes 1}$ since f is the pullback of $f \rtimes 1$;
- since $(f, 1_L)$ is a central extension, we know that $[L, K_f] = 0$;
- from Proposition 4.3.6 it follows that f is a central extension with respect to $\mathbf{Ab}(\mathbb{A})$, and therefore $[M, K_f]^{\mathcal{H}}_M = [M, K_f]^{\mathcal{H}}_{M \rtimes L} = 0$;

- since both K_f and M are normal subobjects of $M \rtimes L$, the *Smith is Huq* condition implies that $[K_f, M, M \rtimes L]^{\mathcal{H}}_{M\rtimes L} = 0$, which in turn implies $[M, L, K_f]^{\mathcal{H}}_{M\rtimes L} = 0$ since this is a subobject of the previous one.

Now consider the extension $f \rtimes 1$ (it is a regular epimorphism because f is so): since $[M \rtimes L, K_{f\rtimes 1}]^{\mathcal{H}}_{M\rtimes L} = 0$ we deduce that $f \rtimes 1$ is a central extension with respect to $\mathbf{Ab}(\mathbb{A})$ and therefore it is the differential of a crossed module.

We now use the fact that in $\mathbf{Grpd}(\mathbb{A})$ the central extensions (with respect to $\mathbf{Ab}(\mathbf{Grpd}(\mathbb{A}))$) are computed pointwise, that is they are couples of central extensions in $\mathbb{A}$ (with respect to $\mathbf{Ab}(\mathbb{A})$): this is shown in Proposition 4.1 of [10]. Since both $f \rtimes 1_L$ and 1_L are central with respect to $\mathbf{Ab}(\mathbb{A})$, the lower square in the diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, tail, "k_d"'] & M' \arrow[d, tail, "k_{d'}"] \\
M \rtimes L \arrow[r, two heads, "f\rtimes 1_L"] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] & M' \rtimes L \arrow[d, shift right=2, "c'"'] \arrow[d, shift left=2, "d'"] \\
L \arrow[u, "e" description] \arrow[r, equal] & L \arrow[u, "e'" description]
\end{tikzcd}$$

is a central extension in $\mathbf{Grpd}(\mathbb{A})$ (with respect to $\mathbf{Ab}(\mathbf{Grpd}(\mathbb{A}))$) and therefore it is the differential of an internal crossed module in $\mathbf{Grpd}(\mathbb{A})$. This means that its denormalisation is a double groupoid and therefore the square we are interested in is an internal crossed square. $\square$

Theorem 4.3.9. *An L-crossed module is perfect iff it admits an universal central extension.*

We split the proof in the two following implications.

Theorem 4.3.10. *Every perfect L-crossed module has a universal central extension.*

Proof. Let $(M \xrightarrow{\partial} L, \xi)$ be a perfect L-crossed module and consider the crossed square

$$\begin{tikzcd}
L \otimes M \arrow[r, two heads, "\delta_M"] \arrow[d, "\partial^{\otimes}"'] & M \arrow[d, "\partial"] \\
L \arrow[r, equal] & L
\end{tikzcd}$$

By Proposition 4.3.7 $(\delta_M, 1_L)$ is a central extension of L-crossed modules. Now we want to show that this central extension is universal, that is it is initial among central extensions of $(M \xrightarrow{\partial} L, \xi)$. So consider another central extension

$$(\tilde{M} \xrightarrow{\tilde{\partial}} L, \tilde{\xi}) \xrightarrow{(f,1_L)} (M \xrightarrow{\partial} L, \xi)$$

Due to Proposition 4.3.8 we know that also

$$\begin{tikzcd} \tilde{M} \arrow[r, two heads, "f"] \arrow[d, "\tilde{\partial}"'] & M \arrow[d, "\partial"] \\ L \arrow[r, equal] & L \end{tikzcd}$$

is a crossed square. Now it suffices to use the universal property of the non-abelian tensor product (see Proposition 3.2.10) to conclude that there exists a unique map $\phi\colon L \otimes M \to \tilde{M}$ such that the diagram

$$\begin{tikzcd} L \otimes M \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, two heads, "\delta_M"] \arrow[ddr, bend right, "\partial^{\otimes}"'] & & \\ & \tilde{M} \arrow[r, two heads, "f"] \arrow[d, "\tilde{\partial}"'] & M \arrow[d, "\partial"] \\ & L \arrow[r, equal] & L \end{tikzcd}$$

commutes. This implies that $(\delta_M, 1_L)$ is initial as central extension of L-crossed modules. □

Also the converse of this result holds.

Theorem 4.3.11. *Every object that admits a universal central extension is perfect.*

Proof. Consider an L-crossed module $(M' \xrightarrow{\partial'} L, \xi')$ and an abelian object $A \in \mathbf{Ab}(\mathbb{A})$: the fact that A is abelian can be seen as $(A \xrightarrow{0} L, \tau^L_A)$ being an L-crossed module. We can construct the crossed module $(A \times M' \xrightarrow{\partial' \circ \pi_{M'}} L, \xi_{A \times M'})$ where the action $\xi_{A \times M'}$ is induced by the universal property of the product as shown in the diagram

$$\begin{tikzcd} & L\flat(A \times M') \arrow[dl, "1\flat\pi_A"'] \arrow[dr, "1\flat\pi_{M'}"] \arrow[dd, dotted, "\xi_{A \times M'}"] & \\ L\flat A \arrow[dd, "\tau^L_A"'] & & L\flat M' \arrow[dd, "\xi'"] \\ & A \times M' \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_{M'}"] & \\ A & & M' \end{tikzcd} \tag{4.12}$$

In order to see that this is an L-action it suffices to use the naturality diagrams for η and μ and the fact that both τ^L_A and ξ' are L-actions. Similarly to see that this gives rise to an L-crossed module it suffices to use that both $(M' \xrightarrow{\partial'} L, \xi')$ and $(A \xrightarrow{0} L, \tau^L_A)$ are so.

Now consider the triangle

$$\begin{tikzcd}
A \times M' \arrow[rr, "\pi_{M'}"] \arrow[dr, "\partial' \circ \pi_{M'}"'] & & M' \arrow[dl, "\partial'"] \\
& L &
\end{tikzcd}$$

This is a morphism of L-crossed module (due to (4.12)) which is a regular epimorphism: we want to follow Remark 4.3.2 and show that it is a central extension by proving that its kernel has a trivial L-action. But its kernel is simply $(A \xrightarrow{0} L, \tau^L_A)$: to see this is suffices to use the description of kernels in $\mathbf{XMod}_L(\mathbb{A})$, to notice that $A = K_{\pi_{M'}}$ in the base category $\mathbb{A}$ and to show the commutativity of the square on the left in the following diagram

$$\begin{tikzcd}
L\flat A \arrow[r, "1\flat\langle 1,0\rangle"] \arrow[d, "\tau^L_A"'] & L\flat(A \times M') \arrow[r, "1\flat\pi_{M'}"] \arrow[d, "\xi_{A\times M'}"'] & L\flat M' \arrow[d, "\xi'"'] \\
A \arrow[r, tail, "\langle 1,0\rangle"'] & A \times M' \arrow[r, "\pi_{M'}"'] & M'
\end{tikzcd}$$

Therefore since its action is trivial, $(\pi_{M'}, 1_L)$ is a central extension.

Now suppose that

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f,1_L)} (M' \xrightarrow{\partial'} L, \xi')$$

is a universal central extension of L-crossed modules. By definition we have a unique map from this extension to the previous one

$$\begin{tikzcd}
(M \xrightarrow{\partial} L, \xi) \arrow[rr, "(f,1_L)"] \arrow[dr, dotted, "(\langle g,f\rangle,1_L)"'] & & (M' \xrightarrow{\partial'} L, \xi') \\
& (A \times M' \xrightarrow{\partial'\circ\pi_{M'}} L, \xi_{A\times M'}) \arrow[ur, "(\pi_{M'},1_L)"'] &
\end{tikzcd}$$

Let us focus on this induced map: the way we wrote it comes from the fact that it makes the previous diagram commute, but what can we say about $g\colon M \to A$? It is the unique map that makes $(\langle g, f\rangle, 1_L)$ a morphism of L-crossed modules, that is such that the following squares commute

$$\begin{tikzcd}
M \arrow[r, "\partial"] \arrow[d, "\langle g,f\rangle"'] & L \arrow[d, equal] \\
A \times M' \arrow[r, "\partial'\circ\pi_{M'}"'] & L
\end{tikzcd}
\qquad
\begin{tikzcd}
L\flat M \arrow[r, "1\flat\langle g,f\rangle"] \arrow[d, "\xi"'] & L\flat(A \times M') \arrow[d, "\xi_{A\times M'}"] \\
M \arrow[r, "\langle g,f\rangle"'] & A \times M'
\end{tikzcd}$$

The first one trivially commutes for each choice of g, whereas the second one does so iff

$$\begin{tikzcd} L\flat M \arrow[r, "1\flat g"] \arrow[d, "\xi"'] & L\flat A \arrow[d, "\tau^L_A"] \\ M \arrow[r, "g"'] & A \end{tikzcd} \tag{4.13}$$

commutes. Now, since

$$\begin{tikzcd} L \diamond M \arrow[r, tail, "i_{L,M}"] & L\flat M \arrow[r, shift left, "\tau^L_M"] & M \arrow[l, shift left, "\eta^L_M"] \end{tikzcd}$$

is a split short exact sequence, we have that the map $\binom{\eta^L_M}{\tau^L_M} \colon (L \diamond M) + M \to L\flat M$ is an epimorphism. This implies that the commutativity of (4.13) is equivalent to the commutativity of the same diagram precomposed with $\binom{\eta^L_M}{\tau^L_M}$. This in turn amounts to having that $g \circ \xi \circ i_{L,M} = 0$, that is $g([L, M]^{\mathcal{H}}_X) = 0$.

Now fix $A = \frac{M}{[L,M]^{\mathcal{H}}_X}$ (it is abelian by Lemma 4.2.11) in order to deduce that $[L, M]^{\mathcal{H}}_X = M$. Notice that both the quotient $g = q \colon M \twoheadrightarrow \frac{M}{[L,M]^{\mathcal{H}}_X}$ and the zero map $g = 0$ satisfy the condition $g([L, M]^{\mathcal{H}}_X) = 0$, we conclude that $q = 0$, that is $[L, M]^{\mathcal{H}}_X = M$. This means that $(M \xrightarrow{\partial} L, \xi)$ is perfect and consequently $(M' \xrightarrow{\partial'} L, \xi')$ is perfect too, since it is a quotient of a perfect. $\square$

4.4 Galois theory interpretation, quasi-pointed setting

The aim here is to use the coinvariants reflector to construct a Birkhoff subcategory of $\mathbf{XMod}_L(\mathbb{A})$ with respect to which we find the "right" class of central extensions of L-crossed modules.

Definition 4.4.1. Consider an internal crossed module $(M \xrightarrow{\partial} L, \xi)$ with the kernel K_∂ and the commutator $[\![L, M]\!]$ induced by ξ as in Definition 4.2.3.

We say that an internal crossed module *has an acyclic action* when $K_\partial \cap [\![L, M]\!] = 0$. Here the intersection is the subobject of M defined via the pullback

$$\begin{tikzcd} K_\partial \cap [\![L, M]\!] \arrow[r, tail] \arrow[d, tail] \arrow[dr, dotted, "i"] & {[\![L, M]\!]} \arrow[d, tail, "k_{(c_s \circ k_p)}"] \\ K_\partial \arrow[r, tail, "k_\partial"'] & M \end{tikzcd}$$

Notice that since i is the diagonal of the pullback of a kernel along another kernel, it is itself a kernel.

The idea behind this definition is that the action has no cycles (elements of K_∂) in its image.

We will denote $\mathbf{AAXMod}_L(\mathbb{A})$ the full subcategory of $\mathbf{XMod}_L(\mathbb{A})$ whose objects are the crossed modules with an acyclic action.

Construction 4.4.2. We will now define the functor $F\colon \mathbf{XMod}_L(\mathbb{A}) \to \mathbf{AAXMod}_L(\mathbb{A})$ which is left adjoint to the inclusion functor J. Given an internal L-crossed module $(M \xrightarrow{\partial} L, \xi)$, we start by defininig the subcrossed module $(K_\partial \cap \llbracket L, M \rrbracket) \xrightarrow{0} 0, \tau^0_{K_\partial \cap \llbracket L,M \rrbracket})$ where $\tau^0_{K_\partial \cap \llbracket L,M \rrbracket}$ is the trivial action: this is a crossed module, indeed in the diagram

$$\begin{tikzcd}
(K_\partial \cap \llbracket L, M \rrbracket) \flat (K_\partial \cap \llbracket L, M \rrbracket) \arrow[r, "\chi_{(K_\partial \cap \llbracket L,M \rrbracket)}"] \arrow[d, "0\flat 1"'] & (K_\partial \cap \llbracket L, M \rrbracket) \arrow[d, equal] \\
0\flat (K_\partial \cap \llbracket L, M \rrbracket) \arrow[r, "\tau^0_{(K_\partial \cap \llbracket L,M \rrbracket)}"] \arrow[d, "1\flat 0"'] & (K_\partial \cap \llbracket L, M \rrbracket) \arrow[d, "0"] \\
0\flat 0 \arrow[r, "\chi_0"'] & 0
\end{tikzcd}$$

the lower square trivially commutes (each composition is 0), whereas for the upper one it suffices to use the fact that $\tau^0_{(K_\partial \cap \llbracket L,M \rrbracket)}$ is an isomorphism with inverse given by $\eta^0_{(K_\partial \cap \llbracket L,M \rrbracket)}$ and the commutativity of the diagram

$$\begin{tikzcd}
0\flat(K_\partial \cap \llbracket L, M \rrbracket) \arrow[d, "\tau^0_{(K_\partial \cap \llbracket L,M \rrbracket)}"] \arrow[dd, equal, bend right=60] & \\
(K_\partial \cap \llbracket L, M \rrbracket) \arrow[r, "i"] \arrow[d, "\eta^0_{(K_\partial \cap \llbracket L,M \rrbracket)}"] & M \arrow[d, "\eta^L_M"'] \arrow[dd, equal, bend left=60] \\
0\flat(K_\partial \cap \llbracket L, M \rrbracket) \arrow[r, "0\flat i"'] & L\flat M \arrow[d, "\xi"'] \\
 & M
\end{tikzcd} \tag{4.14}$$

in order to obtain

$$\begin{aligned}
i \circ \chi_{(K_\partial \cap \llbracket L,M \rrbracket)} &= \chi_M \circ (i\flat i) = \xi \circ (\partial\flat 1) \circ (i\flat i) = \xi \circ (0\flat i) \circ (0\flat 1) \\
&= i \circ \tau^0_{(K_\partial \cap \llbracket L,M \rrbracket)} \circ (0\flat 1)
\end{aligned}$$

which is the thesis since i is a monomorphism.

Furthermore the inclusion between the two crossed modules is given by the arrow $(i, 0) \in \mathbf{XMod}(\mathbb{A})$:

$$\begin{tikzcd}
0\flat(K_\partial \cap \llbracket L, M \rrbracket) \arrow[r, "0\flat i"] \arrow[d, "\tau^0_{K_\partial \cap \llbracket L,M \rrbracket}"'] & L\flat M \arrow[d, "\xi"] \\
(K_\partial \cap \llbracket L, M \rrbracket) \arrow[r, "i"'] & M
\end{tikzcd} \qquad \begin{tikzcd}
(K_\partial \cap \llbracket L, M \rrbracket) \arrow[r, "i"] \arrow[d, "0"'] & M \arrow[d, "\partial"] \\
0 \arrow[r, "0"'] & L
\end{tikzcd}$$

here the commutativity of the second square is trivial whereas the first one is given by (4.14).

The image of $(M \xrightarrow{\partial} L, \xi)$ through F is given by the cokernel in $\mathbf{XMod}(\mathbb{A})$ of the previous inclusion, that is

$$\left(\frac{M}{K \cap \llbracket L, M \rrbracket} \xrightarrow{\overline{\partial}} L, \overline{\xi}\right)$$

where the action $\overline{\xi}$ is obtained as follows: first we pass to the category of points and we take the cokernel

$$\begin{tikzcd}
0 \arrow[r] & K_\partial \cap \llbracket L, M \rrbracket \arrow[r, equal] \arrow[d, tail, "i"'] & K_\partial \cap \llbracket L, M \rrbracket \arrow[r, shift left, "0"] \arrow[d, "k_p \circ i"'] & 0 \arrow[l, shift left, "0"] \arrow[r] \arrow[d, tail, "0"] & 0 \\
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, dotted] & X \arrow[r, shift left, "p"] \arrow[d, "c_{(k_p \circ i)}"] & L \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_{\overline{p}} \arrow[r, tail, "k_{\overline{p}}"'] & C_{(k_p \circ i)} \arrow[r, shift left, "\overline{p}"] & L \arrow[l, shift left, "\overline{s}"] \arrow[r] & 0
\end{tikzcd} \tag{4.15}$$

and then we go back to the associated action $\overline{\xi}$ given by the diagram

$$\begin{tikzcd}
L \flat K_{\overline{p}} \arrow[r, tail, "k_{L, k_{\overline{p}}}"] \arrow[d, dotted, "\overline{\xi}"'] & L + K_{\overline{p}} \arrow[r, two heads, "\binom{1_L}{0}"] \arrow[d, "\binom{\overline{s}}{k_{\overline{p}}}"] & L \arrow[d, equal] \\
K_{\overline{p}} \arrow[r, tail, "k_{\overline{p}}"'] & C_{(k_p \circ i)} \arrow[r, two heads, "\overline{p}"'] & L
\end{tikzcd}$$

The first thing we need, is to prove that $K_{\overline{p}} \cong C_i$ and to do so, we need three steps:

i) We show that the outer square

$$\begin{tikzcd}
K_\partial \cap \llbracket L, M \rrbracket \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket L, M \rrbracket \arrow[r, equal] \arrow[d, tail, "k_{(c_s \circ k_p)}"] & \llbracket L, M \rrbracket \arrow[dd] \\
K_\partial \arrow[r, tail, "k_\partial"'] \arrow[d, equal] & M \arrow[dr, tail, "k_p"] & \\
K_\partial \arrow[rr] & & X
\end{tikzcd}$$

is again a pullback by using the universal property of the smaller pullback square and the fact that k_p is a monomorphism. This means that if we show that the maps $k_p \circ k_\partial$ and $k_p \circ k_{(c_s \circ k_p)}$ are kernels, then $k_p \circ i$ is again a kernel, and in particular the kernel of its cokernel.

ii) We show that the maps $k_p \circ k_\partial$ and $k_p \circ k_{(c_s \circ k_p)}$ are kernels. To see this, it suffices to take the following kernels of split epimorphisms

$$\begin{tikzcd}
0 \arrow[r] & \llbracket L,M\rrbracket \arrow[r, equal] \arrow[d, tail, "k_{(c_s\circ k_p)}"'] & \llbracket L,M\rrbracket \arrow[r, shift left, "0"] \arrow[d, tail, "k_p\circ k_{(c_s\circ k_p)}"'] & 0 \arrow[r] \arrow[l, shift left, "0"] \arrow[d, tail, "0"] & 0 \\
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, "c_s\circ k_p"'] & X \arrow[r, shift left, "p"] \arrow[d, "\langle c_s,p\rangle"'] & L \arrow[r] \arrow[l, shift left, "s"] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_s \arrow[r, tail, "\langle 1,0\rangle"'] & C_s\times L \arrow[r, shift left, "\pi_2"] & L \arrow[r] \arrow[l, shift left, "\langle 0,1\rangle"] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & K_\partial \arrow[r, equal] \arrow[d, tail, "k_\partial"'] & K_\partial \arrow[r, shift left, "0"] \arrow[d, tail, "k_p\circ k_\partial"'] & 0 \arrow[r] \arrow[l, shift left, "0"] \arrow[d, tail, "0"] & 0 \\
0 \arrow[r] & M \arrow[r, tail, "k_p"] \arrow[d, "\partial"'] & X \arrow[r, shift left, "p"] \arrow[d, "z"'] & L \arrow[r] \arrow[l, shift left, "s"] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, tail, "\langle 1,0\rangle"'] & L\times L \arrow[r, shift left, "\pi_2"] & L \arrow[r] \arrow[l, shift left, "\langle 1,1\rangle"] & 0
\end{tikzcd}$$

where the map z in the second diagram is part of the structure of the unique morphism of points induced (through the equivalence $\mathbf{Act}(\mathbb{A}) \simeq \mathbf{Pt}(\mathbb{A})$) by the morphism of actions

$$\begin{tikzcd}
L\flat M \arrow[r, "1_L\flat\partial"] \arrow[d, "\xi"'] & L\flat L \arrow[d, "\chi_L"] \\
M \arrow[r, "\partial"'] & L
\end{tikzcd}$$

(this commutativity is given by the precrossed module condition for $(M \xrightarrow{\partial} L, \xi)$). In each diagram, the middle upper vertical arrow is a kernel by definition, and it is identical to the desired composition since the lower left square is a pullback.

iii) With *i*) and *ii*) we proved that the middle vertical sequence in (4.15) is exact since being $k_p \circ i$ a kernel, it is the kernel of its cokernel. But since also the rightmost vertical one and the three rows are exact, by the 3×3 Lemma (see [8]) we obtain that the leftmost vertical sequence is exact, that is $K_{\overline{p}} \cong C_i$.

At this point one would expect that the action $\overline{\xi}$ just defined makes the diagram

$$\begin{tikzcd}
0\flat(K_\partial\cap\llbracket L,M\rrbracket) \arrow[r, "0\flat i"] \arrow[d, "\tau^0_{K_\partial\cap\llbracket L,M\rrbracket}"'] & L\flat M \arrow[r, two heads, "1_L\flat c_i"] \arrow[d, "\xi"] \arrow[dr, phantom, "(*)"] & L\flat\frac{M}{K_\partial\cap\llbracket L,M\rrbracket} \arrow[d, "\overline{\xi}"] \\
(K_\partial\cap\llbracket L,M\rrbracket) \arrow[r, "i"'] & M \arrow[r, two heads, "c_i"'] & \frac{M}{K_\partial\cap\llbracket L,M\rrbracket}
\end{tikzcd}$$

commute and indeed this can be shown by using the diagram

$$\begin{tikzcd}
L\flat M \arrow[rr, tail] \arrow[dr, "1_L\flat c_i"] \arrow[dd, "\xi"'] & & L+M \arrow[rr, two heads, "\binom{1}{0}"] \arrow[dr, "1+c_i"] \arrow[dd, "\binom{s}{k_p}"'] & & L \arrow[dr, equal] \arrow[dd, equal] & \\
& L\flat\frac{M}{K_\partial\cap[\![L,M]\!]} \arrow[rr, tail] \arrow[dd, "\overline{\xi}"] & & L+\frac{M}{K_\partial\cap[\![L,M]\!]} \arrow[rr, two heads, "\binom{1}{0}"] \arrow[dd, "\binom{\overline{s}}{k_{\overline{p}}}"'] & & L \arrow[dd, equal] \\
M \arrow[rr, tail, "k_p"'] \arrow[dr, "c_i"'] & & X \arrow[rr, two heads, "p"'] \arrow[dr, "c_{(k_p\circ i)}"] & & L \arrow[dr, equal] & \\
& \frac{M}{K_\partial\cap[\![L,M]\!]} \arrow[rr, tail, "k_{\overline{p}}"'] & & C_{(k_p\circ i)} \arrow[rr, two heads, "\overline{p}"'] & & L
\end{tikzcd}$$

The map $\overline{\partial}$ is induced via the universal property of the cokernel of i

$$\begin{tikzcd}
(K_\partial\cap[\![L,M]\!]) \arrow[r, "i"] \arrow[d, "0"'] & M \arrow[r, two heads, "c_i"] \arrow[d, "\partial"] \arrow[dr, phantom, "(**)"] & \frac{M}{K_\partial\cap[\![L,M]\!]} \arrow[d, dotted, "\overline{\partial}"] \\
0 \arrow[r, "0"'] & L \arrow[r, equal] & L
\end{tikzcd}$$

The fact that it is an internal crossed module is easy to show: it suffices to use that $(M \xrightarrow{\partial} L, \xi)$ is an internal crossed module and that both $q\flat q$ and $1_L\flat q$ are (regular) epimorphisms (by Lemma 1.1.35). From the commutativity of $(*)$ and $(**)$ we conclude that

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(c_i, 1_L)} \left(\frac{M}{K_\partial\cap[\![L,M]\!]} \xrightarrow{\overline{\partial}} L, \overline{\xi}\right)$$

is a morphism of L-crossed modules. Furthermore it can easily be checked that this map has the universal property of the cokernel of $(i,0)$ in $\mathbf{XMod}(\mathbb{A})$.

We should also verify that the internal L-crossed module obtained through F is actually an action-acyclic one and this is done in what follows.

Let us depict the situation through the diagram

$$\begin{tikzcd}
M \arrow[rr, tail, "k_p"] \arrow[dr, "c_i"] \arrow[dd, "c_s\circ k_p"'] & & X \arrow[rr, two heads, shift left, "p"] \arrow[dr, "c_{(k_p\circ i)}"] \arrow[dd, "\langle c_s,p\rangle"'] & & L \arrow[ll, shift left, "s"] \arrow[dr, equal] \arrow[dd, equal] & \\
& \frac{M}{K_\partial\cap[\![L,M]\!]} \arrow[rr, tail, "k_{\overline{p}}"] \arrow[dd, "c_{\overline{s}}\circ k_{\overline{p}}"] & & X' \arrow[rr, two heads, shift left, "\overline{p}"] \arrow[dd, "\langle c_{\overline{s}},\overline{p}\rangle"'] & & L \arrow[ll, shift left, "\overline{s}"] \arrow[dd, equal] \\
C_s \arrow[rr, tail, "\langle 1,0\rangle"] \arrow[dr, "\overline{c_{(k_p\circ i)}}"'] & & C_s\times L \arrow[rr, two heads, shift left, "\pi_2"] \arrow[dr, "\overline{c_{(k_p\circ i)}}\times 1_L"] & & L \arrow[ll, shift left, "\langle 0,1\rangle"] \arrow[dr, equal] & \\
& C_{\overline{s}} \arrow[rr, tail, "\langle 1,0\rangle"'] & & C_{\overline{s}}\times L \arrow[rr, two heads, shift left, "\pi_2"] & & L \arrow[ll, shift left, "\langle 0,1\rangle"]
\end{tikzcd}$$

where the map $\overline{c_{(k_p \circ i)}}$ is defined through the universal property of cokernels

$$\begin{tikzcd}
L \arrow[r, "s"] \arrow[d, equal] & X \arrow[r, two heads, "c_s"] \arrow[d, two heads, "c_{(k_p \circ i)}"'] & C_s \arrow[d, dotted, two heads, "\overline{c_{(k_p \circ i)}}"] \\
L \arrow[r, "\overline{s}"'] & X' \arrow[r, two heads, "c_{\overline{s}}"'] & C_{\overline{s}}
\end{tikzcd}$$

Now by defining the maps h as follows

$$\begin{tikzcd}
K_\partial \arrow[r, tail, "k_\partial"] \arrow[d, dotted, two heads, "h"'] & M \arrow[r, "\partial"] \arrow[d, two heads, "c_i"] & L \arrow[d, equal] \\
K_{\overline{\partial}} \arrow[r, tail, "k_{\overline{\partial}}"'] & \frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \arrow[r, "\overline{\partial}"'] & L
\end{tikzcd}$$

we obtain the following commutative cube

$$\begin{tikzcd}
K_\partial \cap \llbracket L, M \rrbracket \arrow[rr, tail] \arrow[dr, "\phi"] \arrow[dd, tail] & & \llbracket L, M \rrbracket \arrow[dr, two heads, "\llbracket 1, c_i \rrbracket"] \arrow[dd, tail] & \\
& K_{\overline{\partial}} \cap \llbracket L, \frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \rrbracket \arrow[rr, tail] \arrow[dd, tail] & & \llbracket L, \frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \rrbracket \arrow[dd, tail] \\
K_\partial \arrow[rr, tail, "k_\partial"] \arrow[dr, two heads, "h"'] & & M \arrow[dr, two heads, "c_i"] & \\
& K_{\overline{\partial}} \arrow[rr, tail, "k_{\overline{\partial}}"'] & & \frac{M}{K_\partial \cap \llbracket L,M \rrbracket}
\end{tikzcd}$$

in which the front face and the back face are pullbacks and the map ϕ, induced by the universal property of the former, is the zero morphism: this is due to the fact that the diagonal of the pullback on the back face is exactly i. Also the lower face is a pullback and this implies that the upper one is so. Finally, since $\llbracket 1, c_i \rrbracket$ is a regular epimorphism and the upper face is a pullback, also ϕ is a regular epimorphism. But an epimorphism can be 0 only when its codomain is 0 itself, that is

$$K_{\overline{\partial}} \cap \llbracket L, \frac{M}{K_\partial \cap \llbracket L, M \rrbracket} \rrbracket = 0$$

which means that the image through F is always an action-acyclic L-crossed module.

Finally we will denote the previous cokernel map as

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{\eta_{(\partial,\xi)} = (c_i, 1_L)} \left(\frac{M}{K_\partial \cap \llbracket L, M \rrbracket} \xrightarrow{\overline{\partial}} L, \overline{\xi} \right).$$

We now want to construct an exact sequence involving this map in the category $\mathbf{XMod}_L(\mathbb{A})$, and in order to do so we take its kernel

$$\begin{tikzcd}
((K_\partial \cap [\![L,M]\!]) \xrightarrow{0} L, \underline{\xi}) \arrow[r] \arrow[d] & (M \xrightarrow{\partial} L, \xi) \arrow[d, two heads, "{(c_i,1_L)}"] \\
(0 \xrightarrow{0} L, \tau_0^L) \arrow[r, "{(0,1_L)}"'] & (\frac{M}{K_\partial \cap [\![L,M]\!]} \xrightarrow{\overline{\partial}} L, \overline{\xi})
\end{tikzcd}$$

where the action $\underline{\xi}$ is defined through the diagram

$$\begin{tikzcd}
L\flat(K_\partial \cap [\![L,M]\!]) \arrow[r, "1_L\flat i"] \arrow[d, dotted, "\underline{\xi}"'] & L\flat M \arrow[r, "1_L\flat c_i"] \arrow[d, "\xi"] & L\flat \frac{M}{K_\partial \cap [\![L,M]\!]} \arrow[d, "\overline{\xi}"] \\
(K_\partial \cap [\![L,M]\!]) \arrow[r, "i"'] & M \arrow[r, "c_i"'] & \frac{M}{K_\partial \cap [\![L,M]\!]}
\end{tikzcd}$$

We know that $(c_i, 1_L)$ is a regular epimorphism in $\mathbf{XMod}_L(\mathbb{A})$ since c_i is so in $\mathbb{A}$. Due to Proposition 2 in [8] we know that in a quasi-pointed protomodular category a map is a regular epimorphism if and only if it is the cokernel of its kernel, therefore $(c_i, 1_L)$ is the cokernel of $((i, 1_L))$ and we have that the sequence

$$((K_\partial \cap [\![L,M]\!]) \xrightarrow{0} L, \underline{\xi}) \rightarrowtail^{(i,1_L)} (M \xrightarrow{\partial} L, \xi) \xrightarrow{(c_i,1_L)}\!\!\!\!\!\rightarrow \left(\frac{M}{K_\partial \cap [\![L,M]\!]} \xrightarrow{\overline{\partial}} L, \overline{\xi}\right)$$

is exact.

Lemma 4.4.3. *The category* $\mathbf{AAXMod}_L(\mathbb{A})$ *is a Birkhoff subcategory of* $\mathbf{XMod}_L(\mathbb{A})$.

Proof. By using directly Definition 4.1.1 we need to prove three things:

i) $\mathbf{AAXMod}_L(\mathbb{A})$ is a reflective subcategory of $\mathbf{XMod}_L(\mathbb{A})$, that is the inclusion functor has a left adjoint;

ii) **AAXMod** is closed under subobjects;

iii) **AAXMod** is closed under quotients.

For *i)* it suffices to show the universal property of η: consider a morphism of L-crossed modules with an action-acyclic L-crossed module as codomain, that is

$$\left(M \xrightarrow{\partial} L, \xi\right) \xrightarrow{(f,1_L)} \left(M' \xrightarrow{\partial'} L, \xi'\right) \tag{4.16}$$

where $K_{\partial'} \cap \llbracket L, M' \rrbracket = 0$. We want to show that there exists a unique morphism of action-acyclic L-crossed module $(\overline{f}, 1_L)$ such that the triangle

$$
\begin{tikzcd}
\left(M \xrightarrow{\partial} L, \xi\right) \arrow[rr, "\eta=(c_i,1_L)"] \arrow[dr, "(f,1_L)"'] & & \left(\frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \xrightarrow{\overline{\partial}} L, \overline{\xi}\right) \arrow[dl, "(\overline{f},1_L)"] \\
& \left(M' \xrightarrow{\partial'} L, \xi'\right) &
\end{tikzcd}
\tag{4.17}
$$

commutes. We will define the map $\overline{f}$ using the universal property of c_i, therefore we need to show that $f \circ i = 0$. So consider the cube

$$
\begin{tikzcd}
K_\partial \cap \llbracket L, M \rrbracket \arrow[rr, tail] \arrow[dd, tail] \arrow[dr, "\phi"] & & \llbracket L, M \rrbracket \arrow[dr, two heads, "\llbracket 1,f \rrbracket"] \arrow[dd, tail] & \\
& K_{\partial'} \cap \llbracket L, M' \rrbracket \arrow[rr, tail] \arrow[dd, tail] & & \llbracket L, M' \rrbracket \arrow[dd, tail] \\
K_\partial \arrow[rr, tail, "k_\partial"] \arrow[dr, two heads] & & M \arrow[dr, two heads, "f"] & \\
& K_{\partial'} \arrow[rr, tail, "k_{\partial'}"'] & & M'
\end{tikzcd}
\tag{4.18}
$$

Since i is the diagonal of the square on the back, the composition $f \circ i$ coincides with ϕ followed by the diagonal on the front face: now it suffices to notice that $K_{\partial'} \cap \llbracket L, M' \rrbracket = 0$ and hence that composition is the zero map. Therefore we have a unique map $\overline{f}$ such that

$$
\begin{tikzcd}
M \arrow[rr, "c_i"] \arrow[dr, "f"'] & & \frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \arrow[dl, "\overline{f}"] \\
& M' &
\end{tikzcd}
\tag{4.19}
$$

It remains to prove that $(\overline{f}, 1_L)$ is a morphism of (action-acyclic) L-crossed module, that is the commutativity the diagrams

$$
\begin{tikzcd}
\frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \arrow[r, "\overline{\partial}"] \arrow[d, "\overline{f}"'] & L \arrow[d, equal] \\
M' \arrow[r, "\partial'"'] & L
\end{tikzcd}
\qquad
\begin{tikzcd}
L \flat \frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \arrow[r, "1_L \flat \overline{f}"] \arrow[d, "\overline{\xi}"'] & L \flat M' \arrow[d, "\xi'"] \\
\frac{M}{K_\partial \cap \llbracket L,M \rrbracket} \arrow[r, "\overline{f}"'] & M'
\end{tikzcd}
$$

For the first one it suffices to precompose with the regular epimorphism c_i obtaining

$$\partial' \circ \overline{f} \circ c_i = \partial' \circ f = \partial = \overline{\partial} \circ c_i.$$

For the second one we use the fact that if c_i is a regular epimorphism, then so is $1_L \flat c_i$: therefore we precompose with $1_L \flat c_i$ obtaining

$$\xi' \circ (1_L \flat \overline{f}) \circ (1_L \flat c_i) = \xi' \circ (1_L \flat f) = f \circ \xi = \overline{f} \circ c_i \circ \xi = \overline{f} \circ \overline{\xi} \circ (1_L \flat c_i)$$

and hence the thesis. Notice that the commutativity of (4.17) is equivalent to the commutativity of (4.19).

As for *ii*) is concerned, let us consider a morphism as in (4.16) which is also a monomorphism in $\mathbf{XMod}_L(\mathbb{A})$ (see Lemma 1.4.15) and the induced cube as in (4.18). Since we can obtain the morphism from $K_\partial \cap [\![L, M]\!]$ to M' as a composition of three monomorphisms, it is a monomorphism itself. But this can also be seen as ϕ followed by the diagonal of the front face: we then use the fact that if a composition is monic, its first component is monic as well, to obtain that ϕ is a monomorphism as well. Now, since the codomain $K_{\partial'} \cap [\![L, M']\!]$ is 0 by hypothesis, also the domain is 0, that is also M is action-acyclic.

For what regards *iii*) consider a morphism as in (4.16) which is also a quotient in $\mathbf{XMod}_L(\mathbb{A})$ (that is such that f is a regular epimorphism due to Lemma 1.4.13) and the induced cube as in (4.18). Following the reasoning at the end of Construction 4.4.2 we obtain that ϕ is a regular epimorphism, but since the domain is 0 by hypothesis, also the codomain is 0, that is also M' is action-acyclic too. $\square$

Protoadditive functors were introduced and studied in [38].

Theorem 4.4.4. *The reflector F is protoadditive: this means that is preserves split short exact sequences.*

Proof. The proof consists of the following steps:

1) Show that the functor that sends an L-crossed module $(M \xrightarrow{\partial} L, \xi)$ to $[\![L, M]\!]$ is protoadditive;

2) Show that this implies that the functor that sends $(M \xrightarrow{\partial} L, \xi)$ to $K_\partial \cap [\![L, M]\!]$ is protoadditive;

3) Use the 3×3 Lemma to conclude that F is protoadditive.

For what regards 1) the aim is to prove that any split short exact sequence of L-crossed modules

$$(K \xrightarrow{0} L, \underline{\xi}) \overset{(1,k)}{\rightarrowtail} (M \xrightarrow{\partial} L, \xi) \underset{(1,g)}{\overset{(1,f)}{\rightleftarrows}} (M' \xrightarrow{\partial'} L, \xi') \tag{4.20}$$

induces a split short exact sequence of coinvariance commutators

$$0 \longrightarrow [\![L, K]\!] \xrightarrow{[\![1,k]\!]} [\![L, M]\!] \underset{[\![1,g]\!]}{\overset{[\![1,f]\!]}{\rightleftarrows}} [\![L, M']\!] \longrightarrow 0.$$

Using Proposition 4.2.9 we can reason with Higgins commutators instead, showing that

$$\begin{tikzcd} 0 \arrow[r] & {[L,K]} \arrow[r, "{[1,k]}"] & {[L,M]} \arrow[r, shift left, "{[1,f]}"] & {[L,M']} \arrow[l, shift left, "{[1,g]}"] \arrow[r] & 0 \end{tikzcd}$$

is still exact. From the fact that

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, tail, "k"] & M \arrow[r, shift left, "f"] & M' \arrow[l, shift left, "g"] \arrow[r] & 0 \end{tikzcd}$$

is a split exact sequence in the base category, by using Proposition 2.24 in [51] we obtain that

$$\begin{tikzcd} 0 \arrow[r] & (L \diamond K \diamond M) \rtimes (L \diamond K) \arrow[r, tail] & L \diamond M \arrow[r, shift left, "1\diamond f"] & L \diamond M' \arrow[l, shift left, "1\diamond g"] \arrow[r] & 0 \end{tikzcd}$$

is a split exact sequence as well. We have the comparison arrows

$$\begin{tikzcd} 0 \arrow[r] & (L \diamond K \diamond M) \rtimes (L \diamond K) \arrow[r, tail] \arrow[d] & L \diamond M \arrow[r, shift left, "1\diamond f"] \arrow[d, "\xi^{\diamond}"'] & L \diamond M' \arrow[l, shift left, "1\diamond g"] \arrow[r] \arrow[d, "\xi'^{\diamond}"] & 0 \\ 0 \arrow[r] & K \arrow[r, tail, "k"'] & M \arrow[r, shift left, "f"] & M' \arrow[l, shift left, "g"] \arrow[r] & 0 \end{tikzcd}$$

whose images, from right to left, are $[L, M']$, $[L, M]$ and $[L, K, M] \vee [L, K]$. These images form a split short exact sequence: taking kernels to the left,

$$\begin{tikzcd} 0 \arrow[r] & (L \diamond K \diamond M) \rtimes (L \diamond K) \arrow[r, tail] \arrow[d] & L \diamond M \arrow[r, shift left, "1\diamond f"] \arrow[d, two heads] & L \diamond M' \arrow[l, shift left, "1\diamond g"] \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & K_{[1,f]} \arrow[r, tail] \arrow[d] & {[L,M]} \arrow[r, shift left, "{[1,f]}"] \arrow[d, hook] & {[L,M']} \arrow[l, shift left, "{[1,g]}"] \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & K \arrow[r, tail, "k"'] & M \arrow[r, shift left, "f"] & M' \arrow[l, shift left, "g"] \arrow[r] & 0 \end{tikzcd}$$

we see that the bottom left square is a pullback, since the bottom right vertical arrow is a monomorphism; likewise, the top right square is a regular pushout, so the top left vertical arrow is a regular epimorphism. It follows that $K_{[1,f]}$ is the image $[L, K, M] \vee [L, K]$ of the left vertical composite. Now since K is central in M (we use the underlying crossed module structures only here) we have $[L, K, M] \hookrightarrow [M, K, M] \hookrightarrow [K, M] = 0$. Hence $K_{[1,f]} = [L, K]$, which is the thesis.

For what regards 2) consider the following diagram

$$\begin{tikzcd}
K_0 \cap \llbracket L,K \rrbracket \arrow[rr, equal] \arrow[ddd, tail] \arrow[dr, "\tilde{k}"] & & \llbracket L,K \rrbracket \arrow[ddd, tail] \arrow[dr, tail, "{\llbracket 1,k \rrbracket}"] & & \\
& K_{\partial} \cap \llbracket L,M \rrbracket \arrow[rr, tail, "\underline{k_{\partial}}"] \arrow[ddd, tail] \arrow[dr, shift left, "\tilde{f}"] & & \llbracket L,M \rrbracket \arrow[ddd, tail] \arrow[dr, shift left, "{\llbracket 1,f \rrbracket}"] & \\
& & K_{\partial'} \cap \llbracket L,M' \rrbracket \arrow[rr, tail, "\underline{k_{\partial'}}"] \arrow[ddd, tail] \arrow[ul, shift left, "\tilde{g}"] & & \llbracket L,M' \rrbracket \arrow[ddd, tail] \arrow[ul, shift left, "{\llbracket 1,g \rrbracket}"] \\
K_0 \arrow[rr, equal] \arrow[dr] & & K \arrow[dr, tail, "k"] & & \\
& K_{\partial} \arrow[rr, tail, "k_{\partial}"] \arrow[dr, shift left] & & M \arrow[dr, shift left, "f"] & \\
& & K_{\partial'} \arrow[rr, tail, "k_{\partial'}"] \arrow[ul, shift left] & & M' \arrow[ul, shift left, "g"]
\end{tikzcd}$$

It is trivial to notice that $\llbracket 1,f \rrbracket \circ \llbracket 1,g \rrbracket = 1$. Then it remains to show that $\tilde{k} = k_{\tilde{f}}$. Suppose that $A \xrightarrow{\alpha} K_{\partial} \cap \llbracket L,M \rrbracket$ is such that $\tilde{f} \circ \alpha = 0$ then $0 = \underline{k_{\partial'}} \circ \tilde{f} \circ \alpha = \llbracket 1,f \rrbracket \circ \underline{k_{\partial}} \circ \alpha$ and since $\llbracket 1,k \rrbracket = k_{\llbracket 1,f \rrbracket}$ we have a unique $\gamma \colon A \to \llbracket L,K \rrbracket$ such that $\llbracket 1,k \rrbracket \circ \gamma = \underline{k_{\partial}} \circ \alpha$. Using that $\underline{k_{\partial}}$ is a monomorphism and the equality $\llbracket 1,k \rrbracket = \underline{k_{\partial}} \circ \tilde{k}$ we obtain $\tilde{k} \circ \gamma = \alpha$ which is the thesis.

Finally, in order to prove 3), consider the following diagram:

$$\begin{tikzcd}
(K_0 \cap \llbracket L,K \rrbracket \xrightarrow{0} L, \underline{\phi}) \arrow[r, tail, "{(\tilde{k},1_L)}"] \arrow[d, tail] & (K_{\partial} \cap \llbracket L,M \rrbracket \xrightarrow{0} L, \phi) \arrow[r, shift left, "{(\tilde{f},1_L)}"] \arrow[d, tail] & (K_{\partial'} \cap \llbracket L,M' \rrbracket \xrightarrow{0} L, \phi') \arrow[l, shift left, "{(\tilde{g},1_L)}"] \arrow[d, tail] \\
(K \xrightarrow{0} L, \underline{\xi}) \arrow[r, tail, "{(k,1_L)}"] \arrow[d, two heads] & (M \xrightarrow{\partial} L, \xi) \arrow[r, shift left, "{(f,1_L)}"] \arrow[d, two heads] & (M' \xrightarrow{\partial'} L, \xi') \arrow[l, shift left, "{(g,1_L)}"] \arrow[d, two heads] \\
F(K \xrightarrow{0} L, \underline{\xi}) \arrow[r, "{F(k,1_L)}"] & F(M \xrightarrow{\partial} L, \xi) \arrow[r, shift left, "{F(f,1_L)}"] & F(M' \xrightarrow{\partial'} L, \xi') \arrow[l, shift left, "{F(g,1_L)}"]
\end{tikzcd}$$

Each column is exact by definition of the functor F and the middle row is exact by hypothesis. From the description of kernels in Remark 1.4.16 and from the previous step, we deduce that the top row is exact as well. Now it suffices to use the 3×3-lemma to obtain that also the bottom row is exact. The fact that it is split is given by functoriality. □

Remark 4.4.5. If $\mathbb{A}$ is strongly protomodular we can give a simpler proof of the protoadditivity of the functor F by changing the way in which 1) is shown in Theorem 4.4.4.

This proof uses Proposition 4.3.6 as follows.

Consider a split short exact sequence of L-crossed module as in (4.20): by Proposition 4.3.6 we know that f is a central extension in $\mathbb{A}$ (with respect to $\mathbf{Ab}(\mathbb{A})$), but since it is also split, it is a trivial extension and hence a product projection. In particular g is a normal monomorphism, and being $\mathbb{A}$ a strongly protomodular category we obtain that $(g, 1_L)$ is a normal monomorphism of L-actions: since it is also split we that $(g, 1_L)$ is a product injection and $(f, 1_L)$ is a product projection in $\mathbf{Act}_L(\mathbb{A})$.

Notice that in the semi-abelian context, any RE-reflector R preserves products. Indeed consider a product split short exact sequence

$$\begin{tikzcd} X \arrow[r, triangle, "\langle 1,0\rangle"] & X \times Y \arrow[r, shift left, two heads, "\pi_Y"] & Y \arrow[l, shift left, "\langle 0,1\rangle"] \end{tikzcd}$$

The images $R(\pi_Y)$ and $R(\langle 0, 1\rangle)$ are again respectively a split monomorphism and split epimorphism. Since R is a left adjoint, it preserves cokernels and consequently we know that $R(\pi_Y)$ is the cokernel of $R(\langle 1, 0\rangle)$, therefore in order to prove that $R(X \times Y) \cong R(X) \times R(Y)$ it suffices to show that the split monomorphism $R(\langle 1, 0\rangle)$ is a normal monomorphism (so that it is the kernel of its cokernel). However, this follows from the assumption that the adjunction units are regular epimorphisms and from the fact that a direct image of a kernel along a regular epimorphism is again a kernel (see Lemma 1.1.20): here we also used the fact that $\langle 1, 0\rangle$ is a (split) monomorphism.

This means that F preserves products, so it sends the split short exact sequence (4.20) into a sequence which is again split exact. Finally by the 3×3 lemma we deduce that the induced sequence of coivariance commutators is split short exact as well.

Now, using Lemma 4.1.7 we can reformulate centrality as follows.

Lemma 4.4.6. *Consider an extension*

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f,1_L)} (M' \xrightarrow{\partial'} L, \xi').$$

It is central with respect to $\mathbf{AAXMod}_L(\mathbb{A})$ *if and only if its kernel*

$$\begin{tikzcd} K_{(f,1_L)} = (K_f \xrightarrow{0} L, \underline{\xi}) \arrow[r] \arrow[d] & (M \xrightarrow{\partial} L, \xi) \arrow[d, "(f,1_L)"] \\ (0 \xrightarrow{0} L, \tau_0^L) \arrow[r] & (M' \xrightarrow{\partial'} L, \xi') \end{tikzcd}$$

is a crossed module with an acyclic action.

Proof. The previous is a central extension iff one of the projections

$$(r_i, 1_L) \colon Kp((1_L, f)) \to (M \xrightarrow{\partial} L, \xi)$$

is a trivial extension. Since r_0 and r_1 are split epimorphisms we can construct the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{(f,1_L)} \arrow[r, tail] \arrow[d, two heads] & Kp((f,1_L)) \arrow[r, shift left, "{(r_i,1_L)}"] \arrow[d, two heads] & (M \xrightarrow{\partial} L, \xi) \arrow[l, shift left] \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & F(K_{(f,1_L)}) \arrow[r, tail] & F(Kp((f,1_L))) \arrow[r, shift left, "{F((r_i,1_L))}"] & F((M \xrightarrow{\partial} L, \xi)) \arrow[l, shift left] \arrow[r] & 0
\end{tikzcd} \tag{4.21}$$

where the vertical maps are the components of the unit. Notice that the first row is exact since $K_{(r_i,1_L)} = K_{(f,1_L)}$ as follows from the fact that both the square in the diagram

$$\begin{tikzcd}
K_{(r_i,1_L)} \arrow[r] \arrow[d] & Kp((f,1_L)) \arrow[r, "{(r_j,1_L)}"] \arrow[d, "{(r_i,1_L)}"'] & (M \xrightarrow{\partial} L, \xi) \arrow[d, "{(f,1_L)}"] \\
(0 \xrightarrow{0} L, \tau_0^L) \arrow[r, "{(0,1_L)}"'] & (M \xrightarrow{\partial} L, \xi) \arrow[r, "{(f,1_L)}"'] & (M' \xrightarrow{\partial'} L, \xi')
\end{tikzcd}$$

are pullbacks and therefore the outer rectangle is a pullback as well. Going back to (4.21), the second row is exact since F is protoadditive. Then by definition we have that $(r_i, 1_L)$ is a trivial extension iff the square on the right is a pullback, but this is true iff the vertical map on the left is an isomorphism (see Lemma 1.1.18). $\square$

Theorem 4.4.7. *An extension of L-crossed modules in $\mathbb{A}$ is central with respect to* $\mathbf{AAXMod}_L(\mathbb{A})$ *if and only if it is a central extension in the sense of Definition 4.3.1.*

Proof. Recall that an extension is central in the sense of Definition 4.3.1 if and only if $[\![L, K_f]\!] = 0$ (by Proposition 4.2.9. From the previous lemma we know that the extension $(f, 1_L)$ is central with respect to $\mathbf{AAXMod}_L(\mathbb{A})$ iff the unit $(K_f \xrightarrow{0} L, \underline{\xi}) \to F(K_f \xrightarrow{0} L, \underline{\xi})$ is an isomorphism.

- ($\Rightarrow$) If it is an isomorphism, its kernel $((K_0 \cap [\![L, K_f]\!]) \xrightarrow{0} L, \zeta)$ is the initial object $(0 \xrightarrow{0} L, \tau_0^L)$: this implies that $(K_0 \cap [\![L, K_f]\!]) = 0$. Now notice that K_0 is the kernel of the zero map $K_f \xrightarrow{0} L$, therefore we have $K_0 = K_f$ and also $(K_0 \cap [\![L, K_f]\!]) = [\![L, K_f]\!]$ since the pullback defining the intersection becomes

$$\begin{tikzcd}
K_0 \cap [\![L, K_f]\!] \arrow[r, equal] \arrow[d, tail] & {[\![L, K_f]\!]} \arrow[d, tail] \\
K_0 \arrow[r, equal] & K_f
\end{tikzcd}$$

  Putting together the two equations we obtain the thesis.

- ($\Leftarrow$) Conversely if $[\![L, K_f]\!] = 0$ we have that also $K_0 \cap [\![L, K_f]\!] = 0$ and therefore the kernel of the unit is the initial object. This in turn is equivalent to the unit being monic, but since it is already a regular epimorphism, it is an isomorphism. $\square$

We have to generalise Definition 4.1.9 to the quasi-pointed exact environment of $\mathbf{XMod}_L(\mathbb{A})$. There seems to be no single categorically sound approach to this; so we stick with the following ad-hoc interpretation:

Definition 4.4.8. Given an L-crossed module $(M \xrightarrow{\partial} L, \xi)$, we say that it is *perfect* (with respect to $\mathbf{AAXMod}_L(\mathbb{A})$) whenever its underlying action is perfect (with respect to $\mathbf{TrivAct}_L(\mathbb{A})$).

Remark 4.4.9. Recalling Remark 4.2.7 and Proposition 4.2.9 we obtain that an L-crossed module $(M \xrightarrow{\partial} L, \xi)$ is perfect iff $[\![L, M]\!] = [L, M]^{\mathcal{H}}_X = M$.

The aim of the next section is to make this more natural: we set up a Galois theory with respect to which both the central extensions and the perfect objects agree with those needed in Section 4.3.

4.5 Galois theory interpretation, pointed setting

Consider an internal crossed module $(M \xrightarrow{\partial} L, \xi)$. Lemma 4.2.11 tells us that $\frac{M}{[L,M]^{\mathcal{H}}_X}$ is an abelian object. Furthermore the association

$$\begin{aligned} F\colon \mathbf{XMod}(\mathbb{A}) &\longrightarrow \mathbf{Ab}(\mathbb{A}) \\ (M \xrightarrow{\partial} L, \xi) &\longmapsto \frac{M}{[L, M]^{\mathcal{H}}_X} \end{aligned}$$

is functorial. Indeed if we have a morphism of internal crossed modules of the form

$$(M \xrightarrow{\partial} L, \xi) \xrightarrow{(f,l)} (M' \xrightarrow{\partial'} L', \xi')$$

we can construct the cube

$$\begin{tikzcd}
L \diamond M \arrow[rr, tail, "i_{L,M}"] \arrow[dr, two heads] \arrow[dd, "l \diamond f"'] & & L\flat M \arrow[dr, "\xi"] \arrow[dd, "l\flat f"] & \\
& {[L, M]^{\mathcal{H}}_{M \rtimes L}} \arrow[rr, tail] \arrow[dd, dotted] & & M \arrow[dd, "f"] \\
L' \diamond M' \arrow[rr, tail, "i_{L',M'}"] \arrow[dr, two heads] & & L'\flat M' \arrow[dr, "\xi'"] & \\
& {[L', M']^{\mathcal{H}}_{M' \rtimes L'}} \arrow[rr, tail] & & M'
\end{tikzcd}$$

and consequently the map $\overline{f}$ through the diagram

$$\begin{tikzcd}
{[L,M]^{\mathcal{H}}_{M\rtimes L}} \arrow[r, tail] \arrow[d, "{[l,f]}"'] & M \arrow[r, two heads] \arrow[d, "f"] & \frac{M}{[L,M]^{\mathcal{H}}_{M\rtimes L}} \arrow[d, dotted, "\overline{f}"] \\
{[L',M']^{\mathcal{H}}_{M'\rtimes L'}} \arrow[r, tail] & M' \arrow[r, two heads] & \frac{M'}{[L',M']^{\mathcal{H}}_{M'\rtimes L'}}
\end{tikzcd}$$

The mapping F is clearly a functor and it has a right adjoint given by the inclusion of abelian objects as particular crossed modules. In order to define this functor $G\colon \mathbf{Ab}(\mathbb{A}) \to \mathbf{XMod}(\mathbb{A})$ we first need the following lemma.

Lemma 4.5.1. *If $A \in \mathbf{Ab}(\mathbb{A})$ then $(A \xrightarrow{0} 0, \tau^0_A)$ is an internal crossed module.*

Now the functor G is determined by

$$\begin{aligned} G\colon \mathbf{Ab}(\mathbb{A}) &\longrightarrow \mathbf{XMod}(\mathbb{A}) \\ A &\longmapsto (A \xrightarrow{0} 0, \tau^0_A). \end{aligned}$$

Notice that if $f\colon A \to B$ is a morphism in $\mathbf{Ab}(\mathbb{A})$ (that is a morphism in $\mathbb{A}$), then

$$(f,0)\colon (A \xrightarrow{0} 0, \tau^0_A) \to (B \xrightarrow{0} 0, \tau^0_B)$$

is a morphism of internal crossed module.

Proposition 4.5.2. *The functor G is right adjoint to F.*

Proof. To prove the thesis we will show the universal property of the unit.

Let us start by considering $G \circ F(M \xrightarrow{\partial} L, \xi)) = \left(\frac{M}{[L,M]^{\mathcal{H}}_X} \xrightarrow{0} 0, \tau\right)$ and constructing the unit

$$(q,0)\colon (M \xrightarrow{\partial} L, \xi) \longrightarrow \left(\frac{M}{[L,M]^{\mathcal{H}}_X} \xrightarrow{0} 0, \tau\right)$$

The first step is showing that this is a morphism of internal crossed modules, that is the commutativity of the diagrams

$$\begin{tikzcd}
M \arrow[r, "\partial"] \arrow[d, "q"'] & L \arrow[d] \\
\frac{M}{[L,M]^{\mathcal{H}}_X} \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
L\flat M \arrow[r, "\xi"] \arrow[d, "0\flat q"'] & M \arrow[d, "q"] \\
0\flat \frac{M}{[L,M]^{\mathcal{H}}_X} \arrow[r, "\tau"'] & \frac{M}{[L,M]^{\mathcal{H}}_X}
\end{tikzcd}$$

The first one is obvious, whereas the second one comes from the isomorphism $C_s \cong \frac{M}{[L,M]^{\mathcal{H}}_X}$ (see Remark 4.2.12) and from the fact that (4.1) is a morphism of points.

Now we want to show the universal property of the unit: given a morphism of internal crossed modules of the form

$$(f,0)\colon (M \xrightarrow{\partial} L, \xi) \longrightarrow (A \xrightarrow{0} 0, \tau^0_A)$$

we need to construct a unique $\overline{f}\colon \frac{M}{[L,M]^{\mathcal{H}}_X} \to A$ such that $(\overline{f},0)\circ(q,0) = (f,0)$. Consider the following diagram

$$\begin{tikzcd}
M \arrow[rr, tail, "k_p"] \arrow[dr, "f"] \arrow[dd, two heads, "q"'] & & M \rtimes L \arrow[rr, shift left, two heads, "p"] \arrow[dr, "f \rtimes 0"] \arrow[dd, two heads, "\langle c_s, p\rangle"'] & & L \arrow[ll, shift left, "s"] \arrow[dr] \arrow[dd, equal] & \\
& A \arrow[rr, equal] \arrow[dd, equal] & & A \arrow[rr, shift left, two heads] \arrow[dd, equal] & & 0 \arrow[ll, shift left] \arrow[dd, equal] \\
C_s \arrow[rr, tail, "\langle 1,0\rangle"] \arrow[dr, "\overline{f}"'] & & C_s \times L \arrow[rr, shift left, two heads, "\pi_2"] \arrow[dr, "\overline{f} \times 0"] & & L \arrow[ll, shift left, "\langle 0,1\rangle"] \arrow[dr] & \\
& A \arrow[rr, equal] & & A \arrow[rr, shift left, two heads] & & 0 \arrow[ll, shift left]
\end{tikzcd}$$

Here the map $\overline{f}$ is uniquely determined by the construction of the lower layer via the universal property of cokernels C_s and A. It is the unique map such that $\overline{f}\circ q = f$ and this immediately implies that it is the unique such that $(\overline{f},0)\circ(q,0) = (f,0)$. Finally, $(\overline{f},0)$ is a morphism of crossed modules by the construction of G. □

Proposition 4.5.3. *The category* $\mathbf{Ab}(\mathbb{A})$ *is a Birkhoff subcategory of* $\mathbf{XMod}(\mathbb{A})$.

Proof. We already know from the previous proposition that $\mathbf{Ab}(\mathbb{A})$ is a full reflective subcategory of $\mathbf{XMod}(\mathbb{A})$. By using Lemma 4.1.2 it suffices to show that the unit $(q,0)$ is a regular epimorphism in $\mathbf{XMod}(\mathbb{A})$ and that $\mathbf{Ab}(\mathbb{A})$ is closed under quotients in $\mathbf{XMod}(\mathbb{A})$. Lemma 1.4.13 tells us that $(q,0)$ is a regular epimorphism in $\mathbf{XMod}(\mathbb{A})$ if and only if both q and 0 are regular epimorphisms in $\mathbb{A}$, and this is clear by construction. Now consider an internal crossed module $(A \xrightarrow{0} 0, \tau^0_A)$ coming from the subcategory $\mathbf{Ab}(\mathbb{A})$, and a quotient of it in $\mathbf{XMod}(\mathbb{A})$

$$(A \xrightarrow{0} 0, \tau^0_A) \overset{(q_1,q_2)}{\twoheadrightarrow} (M \xrightarrow{\partial} L, \xi)$$

Again by Lemma 1.4.13 we know that both q_1 and q_2 are regular epimorphisms. Since q_2 has 0 as domain it has to be an isomorphism, giving us $L \cong 0$. From q_1 being a regular epimorphism we deduce that $M \in \mathbf{Ab}(\mathbb{A})$ (because it is a quotient of an abelian object) and also that $\partial = 0$ (from the equality $\partial \circ q_1 = q_2 \circ 0 = 0$). Finally the action ξ has to be trivial because it is the only possible action of the zero object. This means that

$$(M \xrightarrow{\partial} L, \xi) \cong (M \xrightarrow{0} 0, \tau^0_M) = G(M)$$

i.e. $\mathbf{Ab}(\mathbb{A})$ is closed under quotients in $\mathbf{XMod}(\mathbb{A})$. □

Remark 4.5.4. If $\mathbf{XMod}(\mathbb{A})$ has enough projectives then so does $\mathbb{A}$, since $\mathbb{A}$ is included as a Birkhoff subcategory and Birkhoff reflectors preserve the property of existence of enough projectives. (Indeed, any left adjoint whose right adjoint preserves regular epimorphisms does so.)

Proving the converse (that $\mathbf{XMod}(\mathbb{A})$ has enough projectives if so does $\mathbb{A}$) is more difficult. By general results on functor categories we know that if $\mathbb{A}$ has enough projectives then the category of reflexive graphs in $\mathbb{A}$ has enough projectives as well. The claim now follows from the same argument as above: $\mathbf{XMod}(\mathbb{A})$ is equivalent to a Birkhoff subcategory of the category of reflexive graphs in A.

Now we are able to apply Theorem 3.5 in [25] to obtain the following.

Corollary 4.5.5. *Suppose $\mathbb{A}$ is semi-abelian with enough projectives. An internal crossed module $(M \xrightarrow{\partial} L, \xi)$ of $\mathbb{A}$ is perfect (with respect to the Birkhoff subcategory $\mathbf{Ab}(\mathbb{A})$ of $\mathbf{XMod}(\mathbb{A})$) iff it admits a universal central extension (with respect to the Birkhoff subcategory $\mathbf{Ab}(\mathbb{A})$ of $\mathbf{XMod}(\mathbb{A})$).* □

We now have to explain that the central extensions and the perfect objects in this sense agree with the definitions above. Once this is clear, we find Theorem 4.3.9 as a consequence of Corollary 4.5.5—under the condition that enough projectives exist in $\mathbb{A}$. If $\mathbb{A}$ happens to not have enough projectives, then Theorem 4.3.9 remains valid, of course.

Proposition 4.5.6. *Given an extension of a crossed module $(M \xrightarrow{\partial} L, \xi)$, we have that it is a universal central extension with respect to the Birkhoff subcategory*

$$\mathbf{Ab}(\mathbb{A}) \rightleftarrows \mathbf{XMod}(\mathbb{A}), \tag{B1}$$

if and only if it is a universal central extension with respect to the Birkhoff subcategory

$$\mathbf{AAXMod}_L(\mathbb{A}) \rightleftarrows \mathbf{XMod}_L(\mathbb{A}). \tag{B2}$$

Lemma 4.5.7. *Consider an extension of crossed modules*

$$(M' \xrightarrow{\partial'} L', \xi') \xrightarrow{(f,l)} (M \xrightarrow{\partial} L, \xi) \tag{4.22}$$

which is central with respect to (B1)*. Then l is an isomorphism and (f, l) can be considered as an extension of L-crossed modules.*

Proof. Let us start by proving that a morphism as in (4.22) is a trivial extension with respect to (B1) iff the following hold

$$\begin{cases} l \text{ is an isomorphism,} \\ [f,l]\colon\ [M', L'] \to [M, L] \ \text{ is an isomorphism.} \end{cases}$$

By definition (f,l) is trivial with respect to (B1), if and only if the cube on the right

$$\begin{tikzcd}
{[M',L']} \arrow[rr, tail] \arrow[dr, "{[f,l]}"'] & & M' \arrow[rr, two heads] \arrow[dr, "f"] \arrow[dd, "\partial'"'] & & \frac{M'}{[M',L']} \arrow[dd] \arrow[dr] & \\
& {[M,L]} \arrow[rr, tail] & & M \arrow[rr, two heads] \arrow[dd, "\partial'"'] & & \frac{M}{[M,L]} \arrow[dd] \\
& & L' \arrow[rr] \arrow[dr, "l"'] & & 0 \arrow[dr, equal] & \\
& & & L \arrow[rr] & & 0
\end{tikzcd}$$

is a pullback in $\mathbf{XMod}(\mathbb{A})$. But since pullbacks are computed levelwise in $\mathbf{XMod}(\mathbb{A})$, this is the same as asking that both the top and the bottom faces are pullbacks in $\mathbb{A}$. Now it's trivial to see that the top face is a pullback iff $[f,l]$ is an isomorphism and that the bottom face is a pullback iff l is an isomorphism as well.

The next step amounts to showing that for any extension (4.22) which is central with respect to (B1), l is an isomorphism. In order to show this, recall that (f,l) is central if there exists another extension

$$(\widetilde{M} \xrightarrow{\tilde{\partial}} \widetilde{L}, \tilde{\xi}) \xrightarrow{(g,k)} (M \xrightarrow{\partial} L, \xi)$$

such that the pullback $(\overline{f}, \bar{l})$ of (f,l) along (g,k) is trivial. By looking at the pullback

$$\begin{tikzcd}
\overline{M} \arrow[rr] \arrow[dr] \arrow[dd, "\bar{\partial}"'] & & \widetilde{M} \arrow[dr] \arrow[dd, "\tilde{\partial}"] & \\
& M' \arrow[rr] \arrow[dd, "\partial'"'] & & M \arrow[dd, "\partial"] \\
\overline{L} \arrow[rr, "\bar{l}"] \arrow[dr] & & \widetilde{L} \arrow[dr] & \\
& L' \arrow[rr, "l"] & & L
\end{tikzcd}$$

and by using the equivalent condition for triviality proven above, we know that $\bar{l}$ is an isomorphism and hence l is an isomorphism too: this is given by the fact that l is a regular epimorphism by hypothesis (it is part of an extension) and a monomorphism because $\bar{l}$ is so and because pullbacks reflect monomorphisms in protomodular categories. $\square$

Proof of Proposition 4.5.6. We already know that in order for an extension to be central with respect to (B1), it has to have an isomorphism in the second component. Let us therefore fix an extension $(f, 1_L)$. Consider its kernel pair

$$\begin{tikzcd}
\overline{M} \arrow[rr, "r_0"] \arrow[dr, "r_1"] \arrow[dd, "\bar{\partial}"'] & & M' \arrow[dr, "f"] \arrow[dd, "\partial'" near start] & \\
& M' \arrow[rr, "f" near start] \arrow[dd, "\partial'" near start] & & M \arrow[dd, "\partial"] \\
L \arrow[rr, equal] \arrow[dr, equal] & & L \arrow[dr, equal] & \\
& L \arrow[rr, equal] & & L
\end{tikzcd}$$

and in particular one of the two projections $(r_0, 1_L)$. We will use the following chain of equivalent conditions to obtain the thesis:

1) $(f, 1)$ is central with respect to (B1),

2) $(r_0, 1)$ is trivial with respect to (B1),

3) $K_{[r_0,1]} = 0$,

4) $[K_{r_0}, L] = 0$,

5) $[K_f, L] = 0$,

6) $(f, 1)$ is central with respect to (B2).

The equivalence between 1) and 2) is given by the fact that the extension $(f, 1)$ is central with respect to (B1) iff it is normal with respect to (B1).

To show 2) $\iff$ 3) we use Lemma 4.5.7 and the fact that $[r_0, 1]$ is already a split epimorphism by construction (it is defined through a kernel pair): this means that it is an isomorphism iff its kernel $K_{[r_0,1]}$ is trivial.

Now consider the diagram

$$\begin{tikzcd}
{[K_{r_0}, L]} \arrow[r, tail, "k_{[r_0,1]}"] \arrow[d, tail] & {[\overline{M}, L]} \arrow[r, shift left, "{[r_0,1]}"] \arrow[d, tail] & {[M', L]} \arrow[l, shift left, "{[s_0,1]}"] \arrow[d, tail] \\
K_{r_0} \arrow[r, tail, "k_{r_0}"'] & \overline{M} \arrow[r, shift left, "r_0"] & M' \arrow[l, shift left, "s_0"]
\end{tikzcd} \tag{4.23}$$

The functor that sends an L-crossed module $(M \xrightarrow{\partial} L, \xi)$ to $[M, L]$ is protoadditive (see Theorem 4.4.4) and hence the first row in (4.23) is again a split short exact sequence: this means that $K_{[r_0,1]} \cong [K_{r_0}, L]$, that is 3) $\iff$ 4).

The equivalence between 4) and 5) is simply given by the vertical isomorphism on the left of the diagram

$$\begin{tikzcd} K_{r_0} \arrow[r, tail, "k_{r_0}"] \arrow[d, dotted] & \overline{M} \arrow[r, "r_0"] \arrow[d, "r_1"'] & M' \arrow[d, "f"] \\ K_f \arrow[r, tail, "k_f"'] & M' \arrow[r, "f"'] & M \end{tikzcd}$$

due to the fact that the square on the right is a pullback by construction.

The last step is given by Corollary 4.4.7.

Finally it is trivial to observe that a central extension is universal with respect to (B1) iff it is universal with respect to (B2). □

Proposition 4.5.8. *An L-crossed module $(M \xrightarrow{\partial} L, \xi)$ is perfect with respect to Definition 4.4.8 if and only if it is perfect with respect to the Birkhoff subcategory* $\mathbf{Ab}(\mathbb{A})$ *when seen as an object in* $\mathbf{XMod}(\mathbb{A})$.

Proof. Recalling Remark 4.4.9 we have that $(M \xrightarrow{\partial} L, \xi)$ is perfect with respect to Definition 4.4.8 if and only if $[L, M] = M$. But this is equivalent to the requirement $\frac{M}{[L,M]} = 0$, which in turn is the same as $F(M \xrightarrow{\partial} L, \xi) = 0$, that is perfectness with respect to Definition 4.1.9. □

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