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# Treating the Gibbs phenomenon in barycentric rational interpolation and approximation via the S-Gibbs algorithm

J.-P. Berrut\*, S. De Marchi\*\*, G. Elefante\*, F. Marchetti\*\*\*

\**Département de Mathématiques, Université de Fribourg, Switzerland;*

\*\**Dipartimento di Matematica “Tullio Levi-Civita”, Università di Padova, Italy;*

\*\*\**Dipartimento di Salute della Donna e del Bambino, Università di Padova, Italy*

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## Abstract

In this work, we extend the so-called mapped bases or *fake nodes* approach to the barycentric rational interpolation of Floater-Hormann and to AAA approximants. More precisely, we focus on the reconstruction of discontinuous functions by the *S-Gibbs* algorithm introduced in [? ]. Numerical tests show that it yields an accurate approximation of discontinuous functions.

*Keywords:* Barycentric rational interpolation, Gibbs phenomenon, Floater-Hormann interpolant, AAA algorithm, fake nodes  
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## 1. Introduction

2 In the seminal paper [? ], Floater and Hormann (FH) have introduced a  
3 family of linear barycentric rational interpolants, which contains the first Berrut  
4 interpolant [? ? ]. FH interpolants have shown good approximation properties  
5 for smooth functions, in particular using equidistant nodes. Because of their  
6 high accuracy, these interpolants have been applied in several frameworks, such  
7 as for solving Volterra integral equations [? ], or as collocation methods for  
8 nonlinear parabolic partial differential equations [? ]. Among other favourable  
9 properties, the Lebesgue constant grows logarithmically with the number of  
10 nodes [? ]. Other instances in which linear barycentric rational interpolation  
11 is extremely efficient, actually exponentially convergent, is the trigonometric  
12 interpolant presented in [? ] and used with conformally mapped equispaced  
13 points [? ? ], and its special case on the interval, Berrut’s second interpolant [?  
14 ] between conformally mapped Chebyshev points [? ]. The Berrut interpolants

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*Email addresses:* [jean-paul.berrut@unifr.ch](mailto:jean-paul.berrut@unifr.ch) (J.-P. Berrut\*), [demarchi@math.unipd.it](mailto:demarchi@math.unipd.it) (S. De Marchi\*\*), [giacomo.elefante@unifr.ch](mailto:giacomo.elefante@unifr.ch) (G. Elefante\*), [francesco.marchetti.1@phd.unipd.it](mailto:francesco.marchetti.1@phd.unipd.it) (F. Marchetti\*\*\*)

15 enjoy a Lebesgue constant that grows logarithmically for a wide class of nodes  
 16 as well [? ? ].

17 In [? ], the *Adaptive Antoulas-Anderson* (AAA) greedy algorithm for com-  
 18 puting a barycentric rational approximant has been presented. This recent  
 19 method leads to impressively well-conditioned bases, which can be used in var-  
 20 ious fields, such as in computing conformal maps, or in rational minimax ap-  
 21 proximations (see [? ? ]). Note that a similar approach has been considered in  
 22 [? ] for kernel-based interpolation.

23 The FH interpolants and the approximants obtained by the AAA algorithm  
 24 suffer from the well-known *Gibbs phenomenon*, when the underlying function  
 25 presents jump discontinuities. For a general overview of that phenomenon,  
 26 interested readers may refer to [? ? ].

27 In [? ], a new interpolation procedure has been suggested in the frame-  
 28 work of univariate polynomial interpolation to reduce the Gibbs phenomenon.  
 29 The method essentially maps the polynomial basis, in which the interpolant is  
 30 expressed, by a suitable map  $S$ , or equivalently uses the so-called *fake nodes*,  
 31 without resampling the underlying function. This led to the *S-Gibbs* algorithm,  
 32 which basically constructs the map  $S$  that is then used for eliminating the Gibbs  
 33 effect (see *S-Gibbs* [? , Algorithm 2]).

34 In this work, we propose an extension of the fake nodes approach (cf. [? ]) to  
 35 the framework of barycentric rational interpolation, focusing on FH interpolants  
 36 and the AAA algorithm.

## 37 2. Barycentric polynomial interpolation

38 Let  $\mathcal{X}_n := \{x_i : i = 0, \dots, n\}$  be a set of  $n+1$  distinct nodes in  $I = [a, b] \subset \mathbb{R}$ ,  
 39 increasingly ordered from  $x_0 = a$  to  $x_n = b$ . We consider the problem of  
 40 interpolating a function  $f : I \rightarrow \mathbb{R}$  given the set of samples  $\mathcal{F}_n := \{f_i = f(x_i) :$   
 41  $i = 0, \dots, n\}$ .

42 It is well-known (see e.g. [? ]) that it is possible to write the unique  
 43 interpolating polynomial  $P_n[f]$  of degree at most  $n$  of  $f$  at  $\mathcal{X}_n$  for any  $x \in I$  in  
 44 the second barycentric form

$$P_n[f](x) = \frac{\sum_{i=0}^n \frac{\lambda_i}{x-x_i} f_i}{\sum_{i=0}^n \frac{\lambda_i}{x-x_i}}, \quad (1)$$

45 where  $\lambda_i = \prod_{j \neq i} \frac{1}{x_i - x_j}$  are the so-called *weights*. This expression is one of  
 46 the most stable formulas for evaluating  $P_n[f]$  (see [? ]). If the weights  $\lambda_i$  are  
 47 changed to other nonzero weights, say  $w_i$ , then the corresponding barycentric  
 48 rational function

$$r_n[f](x) = \frac{\sum_{i=0}^n \frac{w_i}{x-x_i} f_i}{\sum_{i=0}^n \frac{w_i}{x-x_i}} \quad (2)$$

49 still satisfies the interpolation conditions  $r_n[f](x_i) = f_i$ ,  $i = 0, \dots, n$ . For more  
 50 details about barycentric rational interpolation, we refer to [? ].

51 *2.1. The Floater-Hormann family*

Let  $n \in \mathbb{N}$ ,  $d \in \{0, \dots, n\}$ . Let  $p_i$ ,  $i = 0, \dots, n-d$  denote the unique polynomial interpolant of degree at most  $d$  interpolating the  $d+1$  points  $(x_k, f_k)$ ,  $k = i, \dots, i+d$ . One can write the FH rational interpolant as

$$R_{n,d}[f](x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \text{ where } \lambda_i(x) = \frac{(-1)^i}{(x-x_i) \cdots (x-x_{i+d})},$$

52 which interpolates  $f$  at the set of nodes  $\mathcal{X}_n$ . It has been proved in [?] that  
 53  $R_{n,d}[f]$  has no real poles and that it reduces to the unique interpolating poly-  
 54 nomial of degree at most  $n$  when  $d = n$ .

One can derive the barycentric form of this family of interpolants as well. Indeed, with considering the sets  $J_i = \{k \in \{0, 1, \dots, n-d\} : i-d \leq k \leq i\}$ , one has

$$R_{n,d}[f](x) = \frac{\sum_{i=0}^n \frac{w_i}{x-x_i} f_i}{\sum_{i=0}^n \frac{w_i}{x-x_i}}, \text{ where } w_i = (-1)^{i-d} \sum_{k \in J_i} \prod_{\substack{j=k \\ j \neq i}}^{j+d} \frac{1}{|x_i - x_j|}.$$

55 *2.2. The AAA algorithm*

Let us consider a set of points  $\mathcal{X}_N$  with a large value of  $N$  and a function  $f$ . The *AAA algorithm* [?] is a greedy technique that in the step  $m \geq 0$  considers the set  $\mathcal{X}^{(m)} = \mathcal{X}_N \setminus \{x_0, \dots, x_m\}$  and constructs the interpolant

$$r_m[f](x) = \frac{\sum_{i=0}^m \frac{w_i}{x-x_i} f_i}{\sum_{j=0}^m \frac{w_j}{x-x_j}} = \frac{n(x)}{d(x)},$$

by solving the discrete least squares problem

$$\min \|fd - n\|_{\mathcal{X}^{(m)}} \quad \|\mathbf{w}\|_2 = 1,$$

56 for the unknown vector  $\mathbf{w} = (w_0, \dots, w_m)$ , where  $\|\cdot\|_{\mathcal{X}^{(m)}}$  is the discrete 2-norm  
 57 over  $\mathcal{X}^{(m)}$ . The subsequent data site  $x_{m+1} \in \mathcal{X}^{(m)}$  is chosen by maximizing the  
 58 residual  $|f(x) - n(x)/d(x)|$  with respect to  $x \in \mathcal{X}^{(m)}$ .

59 **3. Mapped bases and fake nodes in barycentric rational interpolation**

60 Here we investigate the extension of the interpolation method presented in  
 61 [?] to the *Floater-Hormann* interpolants and to the approximants produced  
 62 via the *AAA algorithm*.

Let  $S : I \rightarrow \mathbb{R}$  be a mapping that we assume injective. We construct the “new” interpolant  $r_n^S : I \rightarrow \mathbb{R}$  at the nodes  $\mathcal{X}_n$  and function values  $\mathcal{F}_n$  as

$$r_n^S[f](x) := \frac{\sum_{i=0}^n \frac{w_i}{S(x)-S(x_i)} f_i}{\sum_{i=0}^n \frac{w_i}{S(x)-S(x_i)}}.$$

63 As discussed in [?, p. 3] in the polynomial interpolation setting, we can see  
 64  $r_n^S[f]$  from two different perspectives.

65 First, since the interpolant  $r_n[f]$  defined in (2) admits a cardinal basis form  
 66  $r_n[f](x) = \sum_{j=0}^n f_j b_j(x)$ , where  $b_j(x) = \frac{\frac{w_j}{x-x_j}}{\sum_{i=0}^n \frac{w_i}{x-x_i}}$  is the  $j$ -th basis function, in  
 67 the same spirit, we can write  $r_n^S[f]$  in the mapped cardinal basis form  $r_n^S[f](x) =$   
 68  $\sum_{i=0}^n f_i b_i^S(x)$ , where  $b_j^S(x) = \frac{\frac{w_j}{S(x)-S(x_j)}}{\sum_{i=0}^n \frac{w_i}{S(x)-S(x_i)}}$  is the  $j$ -th mapped basis function.

69 Using the  $S$  mapping approach, a more stable interpolant may arise. We  
 70 present an upper bound on the  $S$ -Lebesgue constant which involves the classical  
 71 Lebesgue constant.

**Theorem 1.** Let  $\Lambda_n(\mathcal{X}_n) = \max_{x \in I} \sum_{j=0}^n |b_j(x)|$  and  $\Lambda_n^S(\mathcal{X}_n) := \max_{x \in I} \sum_{j=0}^n |b_j^S(x)|$  be  
 the classical and the  $S$ -Lebesgue constants, respectively. We then have

$$\Lambda_n^S(\mathcal{X}_n) \leq C \Lambda_n(\mathcal{X}_n),$$

where  $C = \frac{\max_k A^k}{\min_k A_k}$  with

$$A^k = \max_{x \in I} \prod_{\substack{l=0 \\ l \neq k}}^n \left| \frac{S(x) - S(x_l)}{x - x_l} \right|, \quad A_k = \min_{x \in I} \prod_{\substack{l=0 \\ l \neq k}}^n \left| \frac{S(x) - S(x_l)}{x - x_l} \right|.$$

*Proof.* We bound each basis function  $b_j^S$  in terms of  $b_j$  for all  $x \in I$ . We compute

$$\begin{aligned} |b_j^S(x)| &= \left| \frac{\frac{w_j}{S(x)-S(x_j)}}{\sum_{i=0}^n \frac{w_i}{S(x)-S(x_i)}} \right| = \left| \frac{\frac{w_j}{S(x)-S(x_j)} \prod_{l=0}^n (S(x) - S(x_l))}{\sum_{i=0}^n \frac{w_i}{S(x)-S(x_i)} \prod_{l=0}^n (S(x) - S(x_l))} \right| \\ &= \left| \frac{w_j \prod_{\substack{l=0 \\ l \neq j}}^n (S(x) - S(x_l))}{\sum_{i=0}^n w_i \prod_{\substack{l=0 \\ l \neq i}}^n (S(x) - S(x_l))} \right| \\ &= \left| \frac{w_j \prod_{\substack{l=0 \\ l \neq j}}^n (S(x) - S(x_l)) \left( \frac{\prod_{\substack{m=0 \\ m \neq j}}^n x - x_m}{\prod_{\substack{m=0 \\ m \neq j}}^n x - x_m} \right)}{\sum_{i=0}^n w_i \prod_{\substack{l=0 \\ l \neq i}}^n (S(x) - S(x_l)) \left( \frac{\prod_{\substack{m=0 \\ m \neq i}}^n x - x_m}{\prod_{\substack{m=0 \\ m \neq i}}^n x - x_m} \right)} \right| \\ &= \left| \frac{\frac{w_j}{x-x_j} \prod_{\substack{l=0 \\ l \neq j}}^n \frac{S(x)-S(x_l)}{x-x_l}}{\sum_{i=0}^n \frac{w_i}{x-x_i} \prod_{\substack{l=0 \\ l \neq i}}^n \frac{S(x)-S(x_l)}{x-x_l}} \right| \\ &\leq \frac{\max_k A^k}{\min_k A_k} \left| \frac{\frac{w_j}{x-x_j}}{\sum_{i=0}^n \frac{w_i}{x-x_i}} \right| = \frac{\max_k A^k}{\min_k A_k} |b_j(x)| = C |b_j(x)|. \end{aligned}$$

72

□

Second, equivalently to the above mapped basis perspective, we can discuss the construction of the interpolant  $r_n^S[f]$  via the so-called fake nodes approach. Let  $\tilde{r}_n[g]$  be the barycentric interpolant as in (2) that interpolates, at the set of fake nodes  $S(\mathcal{X}_n)$ , the function  $g : S(I) \rightarrow \mathbb{R}$ , making use of the same functional values  $\mathcal{F}_n$ , that is

$$g|_{S(\mathcal{X}_n)} = f|_{\mathcal{X}_n}.$$

73 Observe that  $r_n^S[f](x) = \tilde{r}_n[g](S(x))$  for every  $x \in I$ . Hence, we may also build  
 74  $r_n^S[f]$  upon a standard barycentric interpolation process, thereby providing a  
 75 more intuitive interpretation of the method.

76 The choice of the mapping  $S$  is crucial for the accuracy of the proposed  
 77 interpolant  $r_n^S[f]$ . Here, we assume that  $f$  presents jump discontinuities and  
 78 we adopt the so-called *S-Gibbs* Algorithm (SGA) [?] to construct an effective  
 79 mapping  $S$ .

#### 80 4. Numerical Examples

81 In this section, we test the fake nodes approach with SGA in the framework  
 82 of FH interpolants and the AAA algorithm for approximation. We fix  $k = 10$   
 83 in the SGA. As observed in [?], also in this setting the choice of the shifting  
 84 parameter is non-critical as long as it is taken "sufficiently large".

In  $I = [-5, 5]$  we consider the discontinuous functions

$$f_1(x) = \begin{cases} e^{\frac{1}{x+5.5}}, & -5 \leq x < -3 \\ \cos(3x), & -3 \leq x < 2 \\ -\frac{x^3}{30} + 2, & 2 \leq x \leq 5. \end{cases} \quad f_2(x) = \begin{cases} \log(-\sin(x/2)), & -5 \leq x < -2.5 \\ \tan(x/2), & -2.5 \leq x < 2 \\ \arctan(e^{-\frac{1}{x-5.1}}), & 2 \leq x \leq 5. \end{cases}$$

85 We evaluate the constructed interpolants on a set of 5000 equispaced evaluation  
 86 points  $\Xi = \{\bar{x}_i = -5 + \frac{i}{1000} : i = 0, \dots, 5000\}$  and compute the Relative  
 87 Maximum Absolute Error  $\text{RMAE} = \max_i \frac{|r_n(\bar{x}_i) - f(\bar{x}_i)|}{|f(\bar{x}_i)|}$  and the same for  $r_n^S$ .

#### 88 The FH interpolants

89 Here, we take various sets of equispaced nodes  $\mathcal{X}_n = \{-5 + \frac{5i}{n} : i = 0, \dots, n\}$ ,  
 90 varying the size of  $n$ . The results are displayed in Figure 1. We observe that  
 91 the proposed reconstruction via the fake nodes approach by far outperforms the  
 92 standard technique.

93 Figure 2 displays a comparison between the direct application of the FH  
 94 interpolant and the one modified by the SGA.

#### 95 The AAA algorithm

96 As the starting set for the AAA algorithm, we consider 10000 nodes randomly  
 97 uniformly distributed in  $I$ , which we denote by  $\mathcal{X}_{rand}$ .

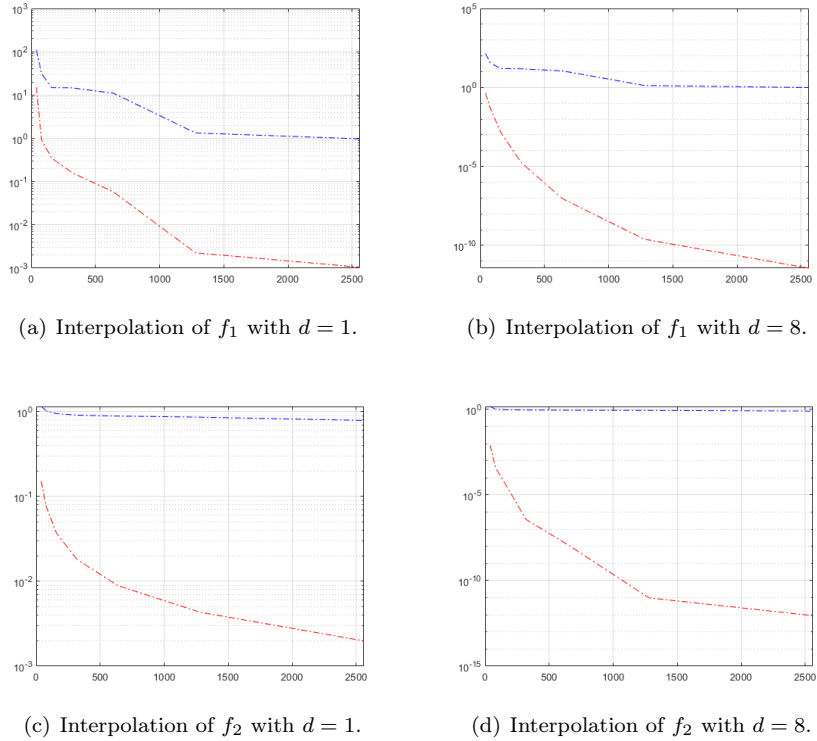


Figure 1: The RMAE for  $f_1$  and  $f_2$  when one doubles the number of nodes from 40 to 2560. In blue, the standard interpolant  $R_{n,d}$ . In red, the proposed interpolant  $R_{n,d}^s$ .

98 Looking at Table 1, we observe that using the AAA algorithm with starting  
 99 set  $S(\mathcal{X}_{rand})$  (indicated in the Table as AAA<sup>S</sup>), that is, constructing the ap-  
 100 proximants via the fake nodes approach, does not suffer from the effects of the  
 101 Gibbs phenomenon. For both approximants we fix the maximum degree to 20  
 102 and to 40 (by default 100 in the algorithm).

$f$	$m_{max}$	AAA	AAA <sup>S</sup>
$f_1$	20	1.5674e+02	8.5189e-05
	40	1.4308e+00	2.9550e-09
$f_2$	20	3.6034e+02	2.2066e-07
	40	1.4656e+00	6.3485e-11

Table 1: RMAE for AAA and AAA<sup>S</sup> approximants

## 103 5. Conclusions

104 This work introduces an extension of the fake nodes approach to barycentric  
 105 rational approximation, in particular to the family of FH interpolants and to

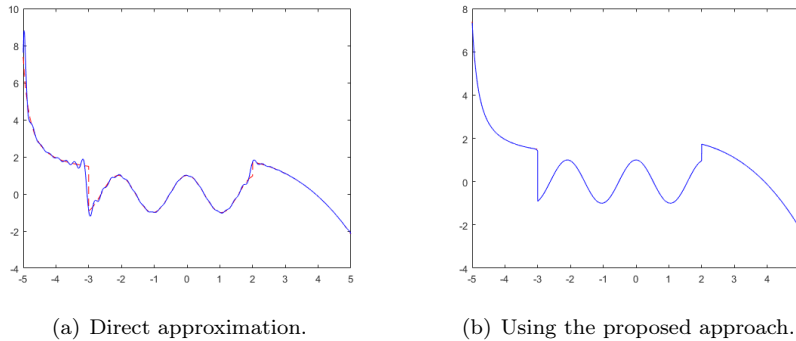


Figure 2: Interpolation of the function  $f_1$  using 80 nodes and  $d = 8$  in the FH interpolant. In red the function and in blue the interpolant.

106 the AAA algorithm for approximation, focusing on the treatment of the Gibbs  
 107 phenomenon via the S-Gibbs algorithm. The results show that the proposed  
 108 reconstructions outperform their classical versions, as they are not affected by  
 109 distortions and oscillations.

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 113

#### 114 References

- 115 [1] A. ABDI, J.-P. BERRUT, AND S. A. HOSSEINI, *The linear barycentric*  
 116 *rational method for a class of delay Volterra integro-differential equations*,  
 117 *J. Sci. Comput.*, 75 (2018), pp. 1757–1775.
- 118 [2] R. BALTENSPERGER, *Some results on linear rational trigonometric inter-*  
 119 *polation*, *Comput. Math. Appl.*, 43 (2002), pp. 737–746.
- 120 [3] R. BALTENSPERGER AND J.-P. BERRUT, *The linear rational collocation*  
 121 *method*, *J. Comput. Appl. Math.*, 134 (2001), pp. 243–258.
- 122 [4] J.-P. BERRUT, *The conditioning of a linear barycentric rational inter-*  
 123 *polant*. submitted.
- 124 [5] ———, *Rational functions for guaranteed and experimentally well-*  
 125 *conditioned global interpolation*, *Comput. Math. Appl.*, 15 (1988), pp. 1–16.
- 126 [6] J.-P. BERRUT AND G. ELEFANTE, *A periodic map for linear barycentric*  
 127 *rational trigonometric interpolation*. to appear in *Appl. Math. Comput.*



- 128 [7] J.-P. BERRUT AND L. N. TREFETHEN, *Barycentric Lagrange interpolation*, SIAM Rev., 46 (2004), pp. 501–517.  
129
- 130 [8] L. BOS, S. DE MARCHI, K. HORMANN, AND J. SIDON, *Bounding the*  
131 *Lebesgue constant for Berrut’s rational interpolant at general nodes*, J. Approx. Theory, 169 (2013), pp. 7–22.  
132
- 133 [9] M. D. BUHMANN, S. DE MARCHI, AND E. PERRACCHIONE, *Analysis of*  
134 *a new class of rational RBF expansions*, IMA J. Numer. Analysis, (2019).  
135 doi: 10.1093/imanum/drz015.
- 136 [10] S. DE MARCHI, W. ERB, AND F. MARCHETTI, *Spectral filtering for the*  
137 *reduction of the Gibbs phenomenon for polynomial approximation meth-*  
138 *ods on Lissajous curves with applications in MPI*, Dolomites Res. Notes  
139 Approx., 10 (2017), pp. 128–137.
- 140 [11] S. DE MARCHI, K. HORMANN, AND G. KLEIN, *Barycentric rational inter-*  
141 *polation at quasi-equidistant nodes*, Dolomites Res. Notes Approx., 5  
142 (2012), pp. 1–6.
- 143 [12] S. DE MARCHI, F. MARCHETTI, E. PERRACCHIONE, AND D. POGGIALI,  
144 *Polynomial interpolation via mapped bases without resampling*, J. Comput.  
145 Appl. Math., 364 (2020), pp. 112347, 12.
- 146 [13] S.-I. FILIP, Y. NAKATSUKASA, L. N. TREFETHEN, AND B. BECKER-  
147 MANN, *Rational minimax approximation via adaptive barycentric represen-*  
148 *tations*, SIAM J. Sci. Comput., 40 (2018), pp. A2427–A2455.
- 149 [14] M. S. FLOATER AND K. HORMANN, *Barycentric rational interpolation*  
150 *with no poles and high rates of approximation*, Numer. Math., 107 (2007),  
151 pp. 315–331.
- 152 [15] A. GOPAL AND L. N. TREFETHEN, *Representation of conformal maps by*  
153 *rational functions*, Numer. Math., 142 (2019), pp. 359–382.
- 154 [16] D. GOTTLIEB AND C.-W. SHU, *On the Gibbs phenomenon and its resolu-*  
155 *tion*, SIAM Rev., 39 (1997), pp. 644–668.
- 156 [17] N. J. HIGHAM, *The numerical stability of barycentric Lagrange interpola-*  
157 *tion*, IMA J. Numer. Anal., 24 (2004), pp. 547–556.
- 158 [18] K. HORMANN, *Barycentric interpolation*, in Approximation Theory XIV:  
159 San Antonio 2013, vol. 83 of Springer Proc. Math. Stat., Springer, Cham,  
160 2014, pp. 197–218.
- 161 [19] W.-H. LUO, T.-Z. HUANG, X.-M. GU, AND Y. LIU, *Barycentric rational*  
162 *collocation methods for a class of nonlinear parabolic partial differential*  
163 *equations*, Appl. Math. Lett., 68 (2017), pp. 13–19.
- 164 [20] Y. NAKATSUKASA, O. SÈTE, AND L. N. TREFETHEN, *The AAA algorithm*  
165 *for rational approximation*, SIAM J. Sci. Comput., 40 (2018), pp. A1494–  
166 A1522.