On a Babuška Paradox for Polyharmonic Operators: Spectral Stability and Boundary Homogenization for Intermediate Problems

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On a Babuška paradox for polyharmonic operators: spectral stability and boundary homogenization for intermediate problems

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Abstract

We analyse the spectral convergence of high order elliptic differential operators subject to singular domain perturbations and homogeneous boundary conditions of intermediate type. We identify sharp assumptions on the domain perturbations improving, in the case of polyharmonic operators of higher order, conditions known to be sharp in the case of fourth order operators. The optimality is proved by analysing in detail a boundary homogenization problem, which provides a smooth version of a polyharmonic Babuška paradox.

1 Introduction

A recurrent topic in the Analysis of Partial Differential Equations, in Spectral Theory, and their applications is the study of the variation of the solutions to elliptic boundary value problems on domains subject to boundary perturbation, with contributions rooting back in the works of Courant and Hilbert [27], and Keldysh [37]. The mathematical interest in this type of problems is also given by the possible appearance of an unexpected asymptotic behaviour of the solutions, which can be understood as a spectral instability phenomenon. Probably the most famous example in elasticity theory is the celebrated Babuška paradox which concerns the approximation of a thin hinged circular plate by means of an invading sequence of convex polygons. This problem was considered by Babuška in [10] and was further discussed by Maz'ya and Nazarov in [38] where among various results they present a variant of the Babuška paradox consisting in the approximation a thin hinged circular plate by means of an invading sequence of non-convex, indented polygons (see [33, § 1.4] for a recent discussion on this subject and for more details concerning the related results of Sapondžhyan [44]). We find convenient to briefly recall the formulation of the paradox.

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Given a circle Ω in $\mathbb{R}^2$ and a datum $f \in L^2(\Omega)$, consider the following boundary value problem

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \frac{\partial^2 u}{\partial n^2} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in the unknown real-valued function u. Note that here and in the sequel, boundary value problems will be understood in the weak sense. Thus, problem (1.1) consists in finding $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that

$$\int_\Omega D^2 u : D^2 \varphi \, dx = \int_\Omega f \varphi \, dx, \ \text{ for all } \varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

where $D^2 u : D^2 \varphi = \sum_{i,j=1}^N u_{x_i x_j} \varphi_{x_i x_j}$ is the Frobenius product of the two Hessian matrices of u and φ. In the theory of elastic plates, u represents the deflection of a *hinged* thin plate with midplane Ω and normal load f.

Define inside Ω an invading sequence of indented polygons Ω_n obtained by modifying an inscribed convex polygon with n vertexes p_j^n, $j = 1, \dots, n$, and replacing its contour line in a neighbourhood of each p_j^n by a V-shaped line as in Figure[1]. The small curvilinear triangles appearing have height equal to h_j^n and base of length η_j^n, while the length of the nearby chord (the side of the polygon) is denoted by ζ_j^n. Consider now the same boundary value problem in Ω_n

$$\begin{cases} \Delta^2 u_n = f, & \text{in } \Omega_n, \\ u_n = 0, & \text{on } \partial\Omega_n, \\ \frac{\partial^2 u_n}{\partial n^2} = 0, & \text{on } \partial\Omega_n, \end{cases} \tag{1.2}$$

in the unknown $u_n \in W^{2,2}(\Omega_n) \cap W_0^{1,2}(\Omega_n)$. The paradox lies in the fact that if

$$\max_{1 \le j \le n} \frac{|\zeta_j^n|}{|\eta_j^n|} = O(1), \quad \max_{1 \le j \le n} \frac{|\eta_j^n|}{|h_j^n|^{2/3}} = o(1),$$

as $n \longrightarrow \infty$, then the solution $u_n \in W^{2,2}(\Omega_n) \cap W_0^{1,2}(\Omega_n)$ of (1.2) does not converge to the solution u of (1.1), but to the solution v of the boundary value problem

$$\begin{cases} \Delta^2 v = f, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here v represents the deflection of a *clamped* thin plate. Note that it is possible to choose $|\zeta_j^n| = 0$ for all j and n in order to obtain the wild looking set Ω_n in Figure 2.

In [7, 8] the authors considered a smooth version of this paradox. Given a sufficiently regular bounded domain W in $\mathbb{R}^{N-1}$, $N \geq 2$, they define a family of domains $(\Omega_\epsilon)_{0<\epsilon<\epsilon_0}$ by setting

$$\Omega = W \times (-1, 0), \ \ \Omega_\epsilon = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, -1 < x_N < \epsilon^\alpha b(\bar{x}/\epsilon)\} \tag{1.4}$$

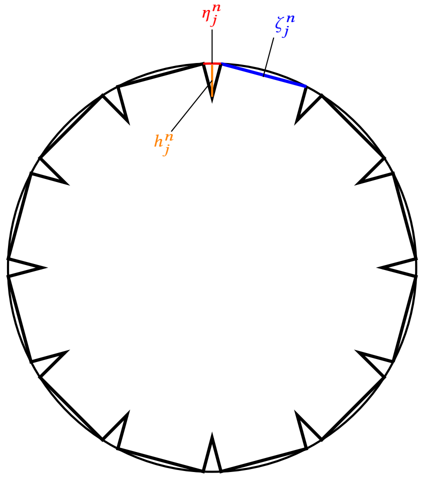


Figure 1: Indented polygon

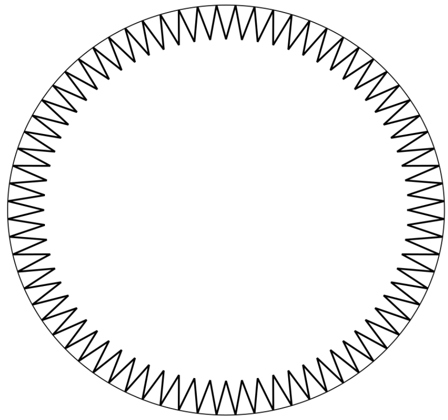


Figure 2: Degenerate indented polygon

where $\bar{x} = (x_1, \ldots, x_{N-1})$, and b is a non-constant, smooth, positive, periodic function of period $Y = [-1/2, 1/2]^{N-1}$. The geometry of this perturbation is described in figure [3] below.

By comparing Figure 3(a) and Figure 2, one realizes that the perturbations look similar locally at the boundary. This analogy goes further if we define $h_j^n = \epsilon^\alpha$ and $\eta_j^n = \epsilon$, with $\epsilon = 1/n$. Indeed, in [8] it was proved that if

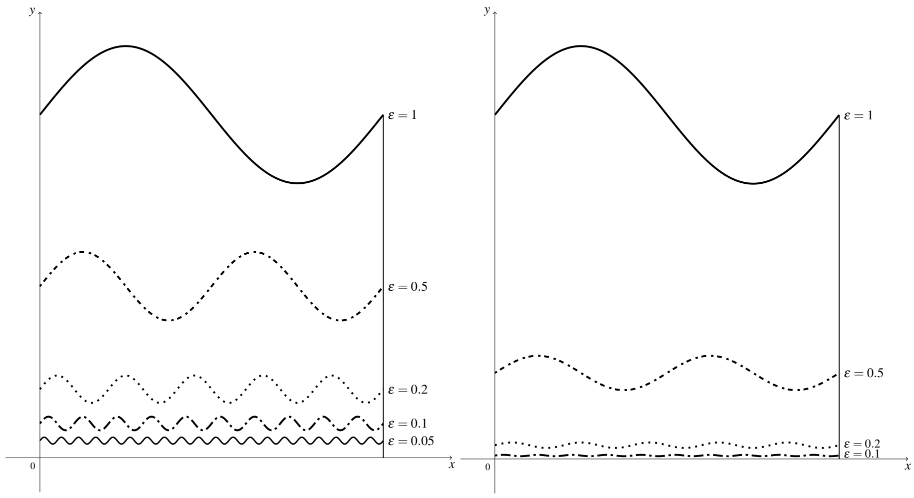
$$\frac{|\eta_j^n|}{|h_j^n|^{2/3}} = \frac{\epsilon}{\epsilon^{2/3\alpha}} = o(1),$$

as $\epsilon \longrightarrow 0$, that is if $\alpha < 3/2$, then the same Babuška-type paradox appears. Moreover, it was also proved that if $\alpha > 3/2$ then no Babuška paradox appears and there is spectral stability. The threshold $\alpha = 3/2$ is then critical and represents a typical case of study for homogenization theory: in fact, it was proved in [8] that the limiting problem contains a 'strange term' which could be interpreted as a 'strange curvature'.

It is then natural to wonder whether Babuška-type paradoxes may be detected in the case of polyharmonic operators $(-\Delta)^m$, $m > 2$ subject to intermediate boundary conditions. The answer is not as straightforward as it may appear, and it is necessary to clarify first what are the possible boundary conditions for those operators. Indeed, there exists a whole family of boundary value problems depending on a parameter $k = 0, 1 \ldots, m$, the weak formulation of which reads as follows: given a bounded domain (i.e., a connected open set) Ω in $\mathbb{R}^N$ with sufficiently smooth boundary, $m \in \mathbb{N}$, and $f \in L^2(\Omega)$, find $u \in W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$ such that

$$\int_\Omega D^m u : D^m \varphi dx + \int_\Omega u\varphi dx = \int_\Omega f\varphi dx, \quad \forall \varphi \in W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega). \tag{1.5}$$

Here we denote by $W^{m,2}(\Omega)$ the standard Sobolev space of functions in $L^2(\Omega)$ with weak derivatives up to order m in $L^2(\Omega)$ and by $W_0^{k,2}(\Omega)$ the closure in $W^{k,2}(\Omega)$ of the C^∞-functions with compact support in Ω. Note that for $k = m$ one obtains the Dirichlet



(a) $\alpha = 1$ (b) $\alpha = 2$

Figure 3: Oscillations of the upper boundary of Ω_ϵ as $\epsilon \longrightarrow 0$, depending on α.

problem

$$\begin{cases} (-\Delta)^m u + u = f, & \text{in } \Omega, \\ \frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial\Omega, \qquad \text{for all } 0 \le l \le m-1, \end{cases} \tag{1.6}$$

while for $k = m - 1$ one gets the significantly different problem

$$\begin{cases} (-\Delta)^m u + u = f, & \text{in } \Omega, \\ \frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial\Omega, \qquad \text{for all } 0 \le l \le m-2. \\ \frac{\partial^m u}{\partial n^m} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Finally, for $k = 0$ one gets the problem with natural boundary conditions, also known as Neumann problem, and this explains why problem (1.7) is called intermediate. Actually, in this paper we refer to problem (1.7) as to the *strong intermediate problem* to emphasise the fact that (1.7) is the intermediate problem with the largest k and to distinguish it from the other cases where $0 < k < m - 1$ which are called here *weak intermediate problems*. According to these considerations, one is led to ask the following:

Question: *Are there Babuška-type paradoxes for polyharmonic operators* $(-\Delta)^m$, $m > 2$ *satisfying intermediate boundary conditions, and which are the natural assumptions which prevent the appearance of this paradox?*

We are able to answer to this question in the geometric setting given by (1.4). Since when $m = 2$ problem (1.7) coincides with the hinged plate (1.1), the Babuška paradox will be discussed for polyharmonic operators with strong intermediate boundary conditions (in short, (*SIBC*)), being the natural higher order version of the intermediate boundary conditions for the biharmonic operator.

Let us describe one of the two main results of this paper. Let Ω_ϵ and Ω be as in (1.4),

$V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W_0^{m-1,2}(\Omega_\epsilon)$. For every $\epsilon > 0$, let $u_\epsilon \in V(\Omega_\epsilon)$ be the solution of

$$\int_{\Omega_\epsilon} D^m u_\epsilon : D^m \varphi + u_\epsilon \varphi \, dx = \int_{\Omega_\epsilon} f \varphi \, dx, \quad \text{for all } \varphi \in V(\Omega_\epsilon). \tag{1.8}$$

Recall that this is the weak formulation of the Poisson problem for $(-\Delta)^m + \mathbb{I}$ with $(SIBC)$. For $u \in W^{m,2}(\Omega)$, define $T_\epsilon u = u \circ \Phi_\epsilon$ where Φ_ϵ is a smooth diffeomorphism mapping Ω_ϵ into Ω that coincides with the identity on a large part K_ϵ of Ω, with $|\Omega \setminus K_\epsilon| \to 0$ as $\epsilon \to 0$, see (3.5). Let u be such that $\|u_\epsilon - T_\epsilon u\|_{L^2(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$.

Theorem 7 states that the limit u solves different differential problems according to the values of the parameter α. More precisely, we have the following trichotomy:

(i) *(Stability)* If $\alpha > 3/2$, then u solves (1.8) in Ω, that is, u satisfies $(-\Delta)^m u + u = f$ in Ω and $(SIBC)$ on $\partial\Omega$;

(ii) *(Degeneration)* If $\alpha < 3/2$, then u satisfies $(-\Delta)^m u + u = f$ in Ω, with Dirichlet boundary conditions on $W \times \{0\}$, that is

$$\frac{\partial^l u}{\partial n^l} = 0, \quad \text{for all } 0 \le l \le m-1,$$

and $(SIBC)$ on the rest of the boundary of Ω;

(iii) *(Strange term)* If $\alpha = 3/2$, then u satisfies $(-\Delta)^m u + u = f$ in Ω with the following boundary conditions on $W \times \{0\}$

$$\begin{cases} D^l u = 0, & \text{for all } 0 \le l \le m-2, \\ \frac{\partial^m u}{\partial n^m} + K \frac{\partial^{m-1} u}{\partial n^{m-1}} = 0, & \end{cases}$$

and $(SIBC)$ on the rest of the boundary of Ω. Here K is a certain positive constant that can be characterized as the energy of a suitable m-harmonic function in $Y \times (-\infty, 0)$.

It follows that if $\alpha < 3/2$ a polyharmonic Babuška paradox appears. It is interesting to observe that the critical value $3/2$ is the same for all the polyharmonic operators with $(SIBC)$.

The techniques used to prove Theorem 7 vary drastically depending on the case $(i) - (iii)$ considered. Theorem 7(i) is a consequence of Theorem 2, which is the second main result of the paper and provides a general stability criterion for self-adjoint elliptic differential operators of order $2m$ with non-constant coefficients and compact resolvents (or, more precisely, for their realization in the space $W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$, $0 < k < m$) on varying domains featuring a fast oscillating boundary.

Theorem 2 is an improvement of a previous result (see [8, Lemma 6.2]) and can be summarized and simplified in the following way. Let Ω and Ω_ϵ be bounded domains in $\mathbb{R}^N$ defined as follows:

$$\Omega = \{(\bar{x}, x_N) \in W \times (a, b) : \bar{x} \in W, a < g(\bar{x}) < b\},$$
$$\Omega_\epsilon = \{(\bar{x}, x_N) \in W \times (a, b) : \bar{x} \in W, a < g_\epsilon(\bar{x}) < b\},$$

where $W \subset \mathbb{R}^{n-1}$ is as above, $a+\rho < g, g_\epsilon < b-\rho$, $a, b \in \mathbb{R}$, and $g, g_\epsilon \in C^m(\overline{W})$. If $\|g-g_\epsilon\|_\infty$ converges to zero as ϵ goes to zero and, for all $|\beta| = m$, $\|D^\beta(g-g_\epsilon)\|_\infty$ converges to zero or diverges to infinity with a suitable rate expressed in terms of a power of $\|g - g_\epsilon\|_\infty$, then the spectrum of the realization of a self-adjoint elliptic differential operator in $W^{m,2}(\Omega_\epsilon) \cap W_0^{k,2}(\Omega_\epsilon)$, $1 \le k \le m-1$ is stable as $\epsilon \to 0$. We note that [8, Lemma 6.2] is sharp in the case $m = 2$ and $k = 1$. In Theorem 2 we allow a rate of convergence or divergence for $\|D^\beta(g - g_\epsilon)\|_\infty$ which is much better when $k > 1$. For example, going back to Theorem 7(i), we note the following fact: upon considering profile functions g_ϵ of the type $g_\epsilon(\bar{x}) = \epsilon^\alpha b(\frac{\bar{x}}{\epsilon})$, where b is a non-constant periodic function, we could apply [8, Lemma 6.2] to the polyharmonic problem in a straightofrward way; however, this would only guarantee the spectral stability for $\alpha > m - 1/2$. Our improved stability Theorem2 guarantees the spectral stability for the better range $\alpha > m - k + 1/2$.

The proof of Theorem 7(ii) is based on a consequence of a degeneration argument that was introduced in [21], and which was already exploited in [8].

The reader may wonder if it is possible to push the arguments contained in the proof of Theorem 7 in order to discuss the general case of weak intermediate problems for polyharmonic operators. The main issue is that the degeneration argument in Theorem 7(ii) is restricted to the case of $(SIBC)$. Hence, a detailed analysis of the various possible situations seems to us much more involved and almost prohibitive for arbitrary values of m and k. We mention that the case $m = 3$, $k = 1$ will be the object of a forthcoming paper and we refer to [30] for a number of results in this direction.

We remark that our main results, in particular Theorem 2 and Theorem 7, are based on the notion of $\mathcal{E}$-convergence in the sense of Vainikko [46] which is related to Stummel's discrete convergence and to Anselone and Palmer's collective compactness, see [45] and [2] respectively. For a recent survey on these topics and further generalisations, we refer to [11].

Finally, we mention that, in the case of second-order operators, counterexamples to the spectral stability with respect to domain perturbation are well-known, see for example the classical [27, Chp. VI, 2.6]. Related problems for the Neumann Laplace operator and for the Schrödinger operator with Neumann boundary conditions have been considered in [6, 22] and [3] respectively. Regarding higher order elliptic operators on variable domains, several contributions can be found in [4, 12, 13, 14, 17, 16, 18, 32]. In particular, for a possible approach to these topics via asymptotic analysis, we refer to the articles [19, 26, 34] and to the monographs [39, 40]. We refer also to the monograph [33] and the articles [31, 43] where polyharmonic operators are considered. For a wider discussion about perturbation theory for linear operators we mention the monographs [35, 36, 41].

This paper is organised as follows. Section 2 is devoted to preliminaries and notation, in particular to the definition of the class of operators and open sets under consideration. Section 3 contains a general discussion concerning the spectral stability of elliptic operators, and the proof of Theorem 2 and its corollaries, see in particular Theorem 4. In Section 4 we prove a Polyharmonic Green Formula which is used in the sequel and has its own interest. Section 5 is devoted to the analysis of strong intermediate boundary conditions and to the proof of Theorem 7. In the Appendix we prove a technical lemma used in the proof of Theorem 7(iii).

2 Preliminaries and notation

In the sequel, we will use the following basic notation:

- $\mathbb{N}$ denotes the set of positive integers. Moreover, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
- Given a normed space X, $\mathcal{L}(X)$ is the space of bounded linear operators on X;
- If not otherwise specified, $m \in \mathbb{N}$ will always be greater or equal to 2;
- Ω, Ω_ϵ, $\epsilon_0 \geq \epsilon > 0$ will always denote bounded domains (i.e., open connected open sets in $\mathbb{R}^N$);
- The standard Sobolev spaces with summability order 2 and smoothness order m are denoted by $W_0^{m,2}(\Omega)$ and $W^{m,2}(\Omega)$.
- The notation $V(\Omega)$, $V(\Omega_\epsilon)$ will often be used for subspaces of $W^{m,2}(\Omega)$ (resp. $W^{m,2}(\Omega_\epsilon)$), containing $W_0^{m,2}(\Omega)$ (resp. $W_0^{m,2}(\Omega_\epsilon)$).

2.1 Classes of operators

Let M be the number of multiindices $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ with length $|\alpha| = \alpha_1 + \dots + \alpha_N = m$. For all $\alpha, \beta \in \mathbb{N}_0^N$ such that $|\alpha| = |\beta| = m$, let $A_{\alpha\beta}$ be bounded measurable real-valued functions defined on $\mathbb{R}^N$ satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq 0, \tag{2.1}$$

for all $x \in \mathbb{R}^N$, $(\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$. For all open subsets Ω of $\mathbb{R}^N$ we define

$$Q_\Omega(u,v) = \sum_{|\alpha|=|\beta|=m} \int_\Omega A_{\alpha\beta} D^\alpha u D^\beta v \, dx + \int_\Omega uv \, dx, \tag{2.2}$$

for all $u, v \in W^{m,2}(\Omega)$ and we set $Q_\Omega(u) = Q_\Omega(u,u)$. Note that by (2.1) Q_Ω is a positive quadratic form, densely defined in the Hilbert space $L^2(\Omega)$. Hence, $Q_\Omega(\cdot,\cdot)$ defines a scalar product in $W^{m,2}(\Omega)$.

Let $V(\Omega)$ be a linear subspace of $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$. By standard Spectral Theory, if $V(\Omega)$ is complete with respect to the norm $Q_\Omega^{1/2}$, then there exists a uniquely determined non-negative self-adjoint operator $H_{V(\Omega)}$ such that $\mathscr{D}(H_{V(\Omega)}^{1/2}) = V(\Omega)$ and

$$Q_\Omega(u,v) = \left(H_{V(\Omega)}^{1/2}u, H_{V(\Omega)}^{1/2}v\right)_{L^2(\Omega)}, \qquad \text{for all } u, v \in V(\Omega). \tag{2.3}$$

By [29, Lemma 4.4.1] it follows that the domain $\mathscr{D}(H_{V(\Omega)})$ of $H_{V(\Omega)}$ is the subset of $W^{m,2}(\Omega)$ containing all the functions $u \in V(\Omega)$ for which there exists $f \in L^2(\Omega)$ such that

$$Q_\Omega(u,v) = (f,v)_{L^2(\Omega)}, \quad \text{for all } v \in V(\Omega), \tag{2.4}$$

in which case $H_{V(\Omega)}u = f$. If u is a smooth function satisfying identity (2.4) and the coefficients $A_{\alpha\beta}$ are smooth, by integration by parts it is immediate to verify that (2.4) is the weak formulation of problem $Lu = f$ in Ω, where L is the operator defined by

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha (A_{\alpha\beta} D^\beta u) + u,$$

and the unknown u is subject to suitable boundary conditions depending on the choice of $V(\Omega)$.

If the embedding $V(\Omega) \subset L^2(\Omega)$ is compact, then the operator $H_{V(\Omega)}$ has compact resolvent. Consequently, its spectrum is discrete, and it consists of a sequence of isolated eigenvalues $\lambda_n[V(\Omega)]$ of finite multiplicity diverging to $+\infty$. By [29, Theorem 4.5.3] the eigenvalues $\lambda_n[V(\Omega)]$ are determined by the following Min-Max principle:

$$\lambda_n[V(\Omega)] = \min_{\substack{E \subset V(\Omega) \\ \dim E = n}} \max_{\substack{u \in E \\ u \neq 0}} \frac{Q_\Omega(u)}{\|u\|^2_{L^2(\Omega)}},$$

for all $n \geq 1$. Furthermore, there exists an orthonormal basis in $L^2(\Omega)$ of eigenfunctions $\varphi_n[V(\Omega)]$ associated with the eigenvalues $\lambda_n[V(\Omega)]$.
We remark that in our assumptions there exist two positive constants $c, C \in \mathbb{R}$ independent of u such that

$$c\|u\|_{W^{m,2}(\Omega)} \leq Q_\Omega^{1/2}(u) \leq C\|u\|_{W^{m,2}(\Omega)},$$

which means that the two norms $Q_\Omega^{1/2}$ and $\|\cdot\|_{W^{m,2}(\Omega)}$ are equivalent on $V(\Omega)$. Note that in general the constant c may depend on Ω. However, if the coefficients $A_{\alpha\beta}$ satisfy the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \theta \sum_{|\alpha|=m} |\xi_\alpha|^2, \tag{2.5}$$

for all $x \in \mathbb{R}^N$, $(\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$ and for some $\theta > 0$, then c can be chosen independent of Ω.

2.2 Classes of open sets

We recall the following definition from [16, Definition 2.4] where for any given set $V \in \mathbb{R}^N$ and $\delta > 0$, V_δ is the set $\{x \in \mathbb{R}^N : d(x, \partial\Omega) > \delta\}$, and by a cuboid we mean any rotation of a rectangular parallelepiped in $\mathbb{R}^N$.

Definition 1. *Let $\rho > 0$, $s, s' \in \mathbb{N}$ with $s' < s$. Let also $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$. We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$. Moreover, we consider the family of all open sets $\Omega \subset \mathbb{R}^N$ satisfying the following:*
i) $\Omega \subset \cup_{j=1}^s (V_j)_\rho$ and $(V_j)_\rho \cap \Omega \neq \emptyset$
ii) $V_j \cap \partial\Omega \neq \emptyset$ for $j = 1, \ldots, s'$ and $V_j \cap \partial\Omega = \emptyset$ for $s' < j \leq s$

iii) for $j = 1, \dots, s$ we have

$$r_j(V_j) = \{x \in \mathbb{R}^N : a_{ij} < x_i < b_{ij}, i = 1, \dots, N\}, \quad j = 1, \dots, s$$
$$r_j(V_j \cap \Omega) = \{x \in \mathbb{R}^N : a_{Nj} < x_N < g_j(\bar{x})\}, \qquad j = 1, \dots, s'$$

where $\bar{x} = (x_1, \dots, x_{N-1})$, $W_j = \{x \in \mathbb{R}^{N-1} : a_{ij} < x_i < b_{ij}, i = 1, \dots, N-1\}$ and $g_j \in C^{k,\gamma}(W_j)$ for $j = 1, \dots, s'$, with $k \in \mathbb{N}_0$ and $0 \le \gamma \le 1$. Moreover, for $j = 1, \dots, s'$ we have $a_{Nj} + \rho \le g_j(\bar{x}) \le b_{Nj} - \rho$, for all $\bar{x} \in W_j$.

We say that an open set Ω is of class $C^{k,\gamma}_M(\mathcal{A})$ if all the functions g_j, $j = 1, \dots, s'$ defined above are of class $C^{k,\gamma}(W_j)$ and $\|g_j\|_{C^{k,\gamma}(W_j)} \le M$. We say that an open set Ω is of class $C^{k,\gamma}(\mathcal{A})$ if it is of class $C^{k,\gamma}_M(\mathcal{A})$ for some $M > 0$. Also, we say that an open set Ω is of class $C^{k,\gamma}$ if it is of class $C^{k,\gamma}_M(\mathcal{A})$ for some atlas $\mathcal{A}$ and some $M > 0$. Finally, we denote by C^k the class $C^{k,0}$ for $k \in \mathbb{N} \cup \{0\}$.

It is important to note that if Ω is a C^0 bounded open set then the Sobolev space $W^{m,2}(\Omega)$ (and consequently all the spaces $W^{m,2}(\Omega) \cap W^{k,2}_0(\Omega)$, $1 \le k \le m$) is compactly embedded in $L^2(\Omega)$, see e.g., Burenkov [15]. Moreover, by using a common atlas as in Definition 1, it is possible to define a distance.

Definition 2. (Atlas distance) *Let $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ be an atlas in $\mathbb{R}^N$. For all $\Omega_1, \Omega_2 \in C^m(\mathcal{A})$ and for all $h = 0, \dots, m$ we set*

$$d^{(h)}_{\mathcal{A}}(\Omega_1, \Omega_2) = \max_{j=1,\dots,s'} \sup_{0 \le |\beta| \le h} \sup_{(\bar{x}, x_N) \in r_j(V_j)} \left| D^\beta g_{1j}(\bar{x}) - D^\beta g_{2j}(\bar{x}) \right|,$$

where g_{1j}, g_{2j} respectively, are the functions describing the boundaries of Ω_1, Ω_2 respectively, as in Definition 1. Moreover, we set $d_{\mathcal{A}} = d^{(0)}_{\mathcal{A}}$ and we call $d_{\mathcal{A}}$ 'atlas distance'.

2.3 Formulae for higher order derivatives of composite functions

We recall here few well-known multidimensional formulae for the derivatives of composite functions. We will use the following notation: by $\mathcal{P}(A)$ we denote the set of all subsets of a given finite non-empty set A and by Part(A) we denote the set of all possible partitions of A. Namely, $\pi \in$ Part(A) is a set the elements of which are pairwise disjoint subsets of A whose union is A. Given $n \in \mathbb{N}$, we often write Part(n) in place of Part($\{1, \dots, n\}$) and $\mathcal{P}(n)$ in place of $\mathcal{P}(\{1, \dots, n\})$. Moreover we use the symbol $|A|$ to denote the cardinality of A; hence, for example $|\pi|$ with $\pi \in$ Part(A) is the number of subsets of A in the partition π. Let Ω be an open set in $\mathbb{R}^N$. If I is an open set in $\mathbb{R}$ and f is a C^n-function from I to $\mathbb{R}$ and Φ is a C^n function from Ω to I, then the Faà di Bruno formula reads

$$\frac{\partial^n f(\Phi(x))}{\partial x_{i_1} \cdots \partial x_{i_n}} = \sum_{\pi \in \mathrm{Part}(n)} f^{(|\pi|)}(\Phi(x)) \prod_{S \in \pi} \frac{\partial^{|S|} \Phi(x)}{\prod_{j \in S} \partial x_{i_j}}. \tag{2.6}$$

Moreover, the Leibnitz formula for the derivatives of the product of two functions u, v of class $C^n(\Omega)$ can can be written as follows

$$\frac{\partial^n (uv)}{\partial x_{i_1} \cdots \partial x_{i_n}} = \sum_{S \in \mathcal{P}(n)} \frac{\partial^{|S|} u}{\prod_{j \in S} \partial x_{i_j}} \frac{\partial^{(n-|S|)} v}{\prod_{j \notin S} \partial x_{i_j}}, \tag{2.7}$$

where $j \notin S$ means that j lies in the complementary of S in $\{1, \dots, n\}$. We recall that in general, if Φ is a C^n function from an open subset U of $\mathbb{R}^N$ to an open subset V of $\mathbb{R}^r$, and f is a function in $W^{n,1}_{loc}(V)$ then the Faà di Bruno formula reads

$$\frac{\partial^n f(\Phi(x))}{\partial x_{i_1} \cdots \partial x_{i_n}} = \sum_{\pi \in \mathrm{Part}(n)} \sum_{j_1, \dots, j_{|\pi|} \in \{1, \dots, r\}} \frac{\partial^{|\pi|} f}{\prod_{k=1}^{|\pi|} \partial x_{j_k}}(\Phi(x)) \prod_{k=1}^{|\pi|} \frac{\partial^{|S_k|} \Phi^{(j_k)}}{\prod_{l \in S_k} \partial x_{i_l}} \tag{2.8}$$

3 Higher order operators on domains with perturbed boundaries

Let $m \in \mathbb{N}$, $m \geq 2$ and let $\epsilon > 0$. Let $V(\Omega), V(\Omega_\epsilon)$ be subspaces of $W^{m,2}(\Omega)$, $W^{m,2}(\Omega_\epsilon)$ respectively, containing $W_0^{m,2}(\Omega)$, $W_0^{m,2}(\Omega_\epsilon)$ respectively. Moreover, let $H_{V(\Omega)}$, $H_{V(\Omega_\epsilon)}$, Q_Ω, Q_{Ω_ϵ} be as in (2.3). A fundamental part of our analysis will be based on the following:

Definition 3. *([8, Definition 3.1]). Given open sets Ω_ϵ, $\epsilon > 0$ and $\Omega \in \mathbb{R}^N$ with corresponding elliptic operators $H_{V(\Omega_\epsilon)}$, $H_{V(\Omega)}$ defined on Ω_ϵ, Ω respectively, we say that condition (C) is satisfied if there exists open sets $K_\epsilon \subset \Omega \cap \Omega_\epsilon$ such that*

$$\lim_{\epsilon \to 0} |\Omega \setminus K_\epsilon| = 0, \tag{3.1}$$

and the following conditions are satisfied:
(C1) If $v_\epsilon \in V(\Omega_\epsilon)$ and $\sup_{\epsilon>0} Q_{\Omega_\epsilon}(v_\epsilon) < \infty$ then $\lim_{\epsilon \to 0} \|v_\epsilon\|_{L^2(\Omega_\epsilon \setminus K_\epsilon)} = 0$.
(C2) For each $\epsilon > 0$ there exists an operator T_ϵ from $V(\Omega)$ to $V(\Omega_\epsilon)$ such that for all fixed $\varphi \in V(\Omega)$

- *(i)* $\lim_{\epsilon \to 0} Q_{K_\epsilon}(T_\epsilon \varphi - \varphi) = 0$;
- *(ii)* $\lim_{\epsilon \to 0} Q_{\Omega_\epsilon \setminus K_\epsilon}(T_\epsilon \varphi) = 0$;
- *(iii)* $\lim_{\epsilon \to 0} \|T_\epsilon \varphi\|_{L^2(\Omega_\epsilon)} < \infty$.

(C3) For each $\epsilon > 0$ there exists an operator E_ϵ from $V(\Omega_\epsilon)$ to $W^{m,2}(\Omega)$ such that the set $E_\epsilon(V(\Omega_\epsilon))$ is compactly embedded in $L^2(\Omega)$ and such that

- *(i) If $v_\epsilon \in V(\Omega_\epsilon)$ is a sequence such that $\sup_{\epsilon>0} Q_{V(\Omega_\epsilon)}(v_\epsilon) < \infty$, then $\lim_{\epsilon \to 0} Q_{K_\epsilon}(E_\epsilon v_\epsilon - v_\epsilon) = 0$;*
- *(ii)*
$$\sup_{\epsilon>0} \sup_{v \in V(\Omega_\epsilon) \setminus \{0\}} \frac{\|E_\epsilon v\|_{W^{m,2}(\Omega)}}{Q^{1/2}_{\Omega_\epsilon}(v)} < \infty;$$
- *(iii) If $v_\epsilon \in V(\Omega_\epsilon)$ for all $\epsilon > 0$, $\sup_{\epsilon>0} Q_{\Omega_\epsilon}(v_\epsilon) < \infty$ and there exists $v \in L^2(\Omega)$ such that, up to a subsequence, we have $\|E_\epsilon v_\epsilon - v\|_{L^2(\Omega)} \to 0$, then $v \in V(\Omega)$.*

It is proved in [8, Theorem 3.5] that Condition (C) guarantees the spectral convergence of the operators $H_{V(\Omega_\epsilon)}$ to the operator $H_{V(\Omega)}$ as $\epsilon \to 0$.
The convergence of the operators is understood in the sense of the compact convergence, as defined in [46]. Let us briefly recall the setting. Let $\mathcal{E}$ be the *extension-by-zero operator*, mapping any given real-valued function u defined on some subset A of $\mathbb{R}^N$, to the function $\mathcal{E}u$ such that $\mathcal{E}u = u$ a.e. in A and $\mathcal{E}u = 0$ a.e. in $\mathbb{R}^N \setminus A$. By using $\mathcal{E}$ we can map functions in $L^2(\Omega)$ to the space $L^2(\Omega_\epsilon)$, for every $\epsilon > 0$, so that $\mathcal{E}$ defines a *"connecting system"* between $L^2(\Omega)$ and the family of spaces $(L^2(\Omega_\epsilon))_{\epsilon>0}$. We then say that:

- $v_\epsilon \in L^2(\Omega_\epsilon)$ *$\mathcal{E}$-converges* to $v \in L^2(\Omega)$ if $\|v_\epsilon - \mathcal{E}v\|_{L^2(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$;
- a family of bounded linear operators $B_\epsilon \in \mathcal{L}(L^2(\Omega_\epsilon))$ *$\mathcal{E}\mathcal{E}$- converges* to $B \in \mathcal{L}(L^2(\Omega))$ if $B_\epsilon v_\epsilon$ $\mathcal{E}$-converges to Bv whenever v_ϵ $\mathcal{E}$-converges to v;
- a family of bounded, compact linear operators $B_\epsilon \in \mathcal{L}(L^2(\Omega_\epsilon))$ is said to *$\mathcal{E}$-compact converges* to $B \in \mathcal{L}(L^2(\Omega))$ if B_ϵ $\mathcal{E}\mathcal{E}$-converges to B and for any family of functions $v_\epsilon \in L^2(\Omega_\epsilon)$ with $\|v_\epsilon\|_{L^2(\Omega_\epsilon)} \le 1$ there exists a subsequence, denoted by v_ϵ again, and a function $w \in L^2(\Omega)$ such that $B_\epsilon v_\epsilon$ $\mathcal{E}$-converges to w.

We refer to [8, Section 2.2], for further information on this type of convergence. Importantly, in our assumptions on the operators $H_{V(\Omega_\epsilon)}$, $H_{V(\Omega)}$, the compact convergence of the resolvent operators is a sufficient condition for the spectral convergence. In particular, we have the following

Theorem 1. *Let Ω_ϵ, $\epsilon > 0$ and Ω be open sets in $\mathbb{R}^N$. Let $H_{V(\Omega_\epsilon)}$, $H_{V(\Omega)}$ be operators with compact resolvents, associated with $V(\Omega_\epsilon)$, $V(\Omega)$, respectively, as in* (2.3), *such that condition (C) is satisfied. Let λ_k, λ_k^ϵ be the k-th eigenvalue of $H_{V(\Omega)}$, $H_{V(\Omega_\epsilon)}$, respectively. Then $H^{-1}_{V(\Omega_\epsilon)}$ $\mathcal{E}$-compact converges to $H^{-1}_{V(\Omega)}$ as $\epsilon \to 0$. Moreover,*

(i) $\lambda_n^\epsilon \to \lambda_n$ *as* $\epsilon \to 0$, *for all* $n \in \mathbb{N}$.

(ii) If $\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+h-1}$ is an eigenvalue of multiplicity h and $\varphi_n^\epsilon, \varphi_{n+1}^\epsilon, \dots, \varphi_{n+h-1}^\epsilon$ is an orthonormal set in $L^2(\Omega_\epsilon)$ of eigenfunctions associated with the corresponding eigenvalues $\lambda_n^\epsilon, \lambda_{n+1}^\epsilon, \dots, \lambda_{n+h-1}^\epsilon$, then there exists an orthonormal set $\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+h-1}$ in $L^2(\Omega)$ of eigenfunctions associated with the eigenvalues $(\lambda_{n+t-1})_{t=1}^h$ such that, possibly passing to a suitable subsequence, φ_{n+i-1}^ϵ $\mathcal{E}$-converges to φ_{n+i-1} as $\epsilon \to 0$ for all $i = 1, \dots, h$.

(iii) If $\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+h-1}$ is an eigenvalue of multiplicity h and $\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+h-1}$ is an orthonormal set $L^2(\Omega)$ of eigenfunctions associated with $(\lambda_{n+t-1})_{t=1}^h$ then for every $\epsilon > 0$ there exists an orthonormal set in $L^2(\Omega_\epsilon)$ of eigenfunctions $\varphi_n^\epsilon, \varphi_{n+1}^\epsilon, \dots, \varphi_{n+h-1}^\epsilon$ associated with the corresponding eigenvalues $\lambda_n^\epsilon, \lambda_{n+1}^\epsilon, \dots, \lambda_{n+h-1}^\epsilon$ such that φ_{n+i-1}^ϵ $\mathcal{E}$-converges to φ_{n+i-1} as $\epsilon \to 0$ for all $i = 1, \dots, h$.

When the claims $(i) - (ii) - (iii)$ of the previous theorem are verified, we say that $H_{V(\Omega_\epsilon)}$ *spectrally converges* to $H_{V(\Omega)}$ as $\epsilon \to 0$.

3.1 An explicit condition for the spectral stability

We consider now the following geometric setting:

(G1) There exists a cuboid V of the form $W \times (a, b)$, where $W \subset \mathbb{R}^{N-1}$ is an open, connected and bounded set of class C^m, and $g, g_\epsilon \in C^m(\overline{W})$ such that

$$\Omega \cap V = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g(\bar{x})\}, \tag{3.2}$$

$$\Omega_\epsilon \cap V = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g_\epsilon(\bar{x})\}. \tag{3.3}$$

Assume that $\Omega \setminus V = \Omega_\epsilon \setminus V$ for all $\epsilon > 0$.

It is convenient to set $\Omega_0 = \Omega$. According to Def. 1, if $\Omega_\epsilon \in C^m(\mathcal{A})$ for all $\epsilon \geq 0$, then we can assume **(G1)** without loss of generality. For all $\epsilon \geq 0$, let us consider the quadratic forms Q_{Ω_ϵ} on Ω_ϵ defined as in (2.2), where the coefficients $A_{\alpha\beta}$ are independent of $\epsilon \geq 0$ and satisfy the uniform ellipticity condition (2.5). Then we consider the non-negative self-adjoint operators $H_{V(\Omega_\epsilon)}$ defined by (2.3) with $V(\Omega)$ replaced by $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W_0^{k,2}(\Omega_\epsilon)$ for some $1 \leq k < m$. Since Ω_ϵ is of class C^m, $V(\Omega_\epsilon)$ is compactly embedded in $L^2(\Omega_\epsilon)$ hence $H_{V(\Omega_\epsilon)}$ has compact resolvent.

We now state our first result, concerning an explicit condition sufficient to guarantee the spectral convergence of the operators $H_{V(\Omega_\epsilon)}$. This theorem is a generalisation of [8, Lemma 6.2].

Theorem 2. *Let Ω_ϵ, $\epsilon \geq 0$ satisfy assumption **(G1)**. Suppose that for some $k \in \mathbb{N}$, with $1 \leq k < m$, $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W_0^{k,2}(\Omega_\epsilon)$ for all $\epsilon \geq 0$. If for all $\epsilon > 0$ there exists $\kappa_\epsilon > 0$ such that*

(i) $\kappa_\epsilon > \|g_\epsilon - g\|_\infty, \quad \forall \epsilon > 0, \quad \lim_{\epsilon \to 0} \kappa_\epsilon = 0,$

(ii) $\lim_{\epsilon \to 0} \frac{\|D^\beta (g_\epsilon - g)\|_\infty}{\kappa_\epsilon^{m-|\beta|-k+1/2}} = 0, \ \forall \beta \in \mathbb{N}_0^N$ *with* $|\beta| \leq m$,

then $H_{V(\Omega_\epsilon)}^{-1}$ $\mathcal{E}$-compact converges to $H_{V(\Omega)}^{-1}$ as $\epsilon \to 0$. In particular, $H_{V(\Omega_\epsilon)}$ spectrally converges to $H_{V(\Omega)}$ as $\epsilon \to 0$

Proof. We first observe that the last statement is a direct consequence of Theorem 1. The case $k = 1$ is proved in [8, Lemma 6.2]. Thus, we suppose $k > 1$. It is possible to assume directly that $\Omega = \Omega \cap V$ and $\Omega_\epsilon = \Omega_\epsilon \cap V$ as in (3.2) and (3.3) respectively. Define $k_\epsilon = M\kappa_\epsilon$ for a suitable constant $M > 2m$. Let $\tilde{g}_\epsilon = g_\epsilon - k_\epsilon$ and

$$K_\epsilon = \{(\bar{x}, x_N) \in W \times]a, b[: a < x_N < \tilde{g}_\epsilon(\bar{x})\}.$$

Note that with this definition of K_ϵ (3.1) is satisfied. By the standard one dimensional estimate

$$\|f\|_{L^\infty(a,b)} \leq C\|f\|_{W^{1,2}(a,b)}, \tag{3.4}$$

and Tonelli Theorem it follows that condition (C1) is satisfied.
We now define a suitable family of diffeomorphisms $\Phi_\epsilon : \overline{\Omega}_\epsilon \to \overline{\Omega}$ by setting

$$\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N)),$$

for all $(\bar{x}, x_N) \in \overline{\Omega}_\epsilon$, where

$$h_\epsilon(\bar{x}, x_N) = \begin{cases} 0, & \text{if } a \le x_N \le \tilde{g}_\epsilon(\bar{x}), \\ (g_\epsilon(\bar{x}) - g(\bar{x}))\left(\frac{x_N - \tilde{g}_\epsilon(\bar{x})}{g_\epsilon(\bar{x}) - \tilde{g}_\epsilon(\bar{x})}\right)^{m+1} & \text{if } \tilde{g}_\epsilon(\bar{x}) < x_N \le g_\epsilon(\bar{x}). \end{cases}$$

Then consider the map T_ϵ from $V(\Omega)$ to $V(\Omega_\epsilon)$ defined by

$$T_\epsilon \varphi = \varphi \circ \Phi_\epsilon, \tag{3.5}$$

for all $\varphi \in V(\Omega)$. One can check that T_ϵ is well-defined and that condition (C2)(i) is satisfied. We now want to prove that conditions (C2)(ii), (iii) are satisfied. We need to estimate the derivatives of $\varphi \circ \Phi_\epsilon$. Here we can improve the estimate given in [8, Lemma 6.2] by taking advantage of the decay of $D^\gamma \varphi$ in a neighbourhood of $\partial\Omega$, for $|\gamma| \le k-1$. We divide the proof in two steps.

Step 1. We aim at proving a decay estimates for the L^2-norms of the derivatives of φ near the boundary, namely estimate (3.12). First, note that

$$\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon) = \Omega \setminus K_\epsilon = \{(\bar{x}, x_N) \in \Omega : \bar{x} \in W,\ g_\epsilon(\bar{x}) - k_\epsilon \le x_N \le g(\bar{x})\},$$

for any $\epsilon > 0$. Fix $x \in \Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)$ and $\beta \in \mathbb{N}_0^N$, $|\beta| \le k-1$. Suppose for the moment $\varphi \in C^m(\overline{\Omega})$. By the Taylor's formula with remainder in integral form, we get that

$$D^\beta \varphi(x) = \sum_{l=0}^{k-1-|\beta|} \frac{1}{l!} \frac{\partial^l (D^\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} (x_N - g(\bar{x}))^l + R(\beta, x),$$

where

$$R(\beta, x) := \frac{(x_N - g(\bar{x}))^{k-|\beta|}}{(k-|\beta|-1)!} \int_0^1 (1-t)^{k-1-|\beta|} \frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}} D^\beta \varphi(\bar{x}, g(\bar{x}) + t(x_N - g(\bar{x}))\, \mathrm{d}t\,.$$

Note that $-2k_\epsilon \le g_\epsilon(\bar{x}) - g(\bar{x}) - k_\epsilon \le x_N - g(\bar{x}) \le 0$. By Jensen's inequality,

$$|R(\beta, x)|^2 \le (2k_\epsilon)^{2(k-|\beta|)} \int_0^1 \left| \frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}} D^\beta \varphi(\bar{x}, g(\bar{x}) + t(x_N - g(\bar{x})) \right|^2 \mathrm{d}t. \tag{3.6}$$

An integration in the variable x_N in (3.6) and inequality (3.4) applied to the interval $(a, g(\bar{x}))$ yield

$$\int_{g_\epsilon(\bar{x}) - k_\epsilon}^{g(\bar{x})} |R(\beta, x)|^2\, \mathrm{d}x_N \le C k_\epsilon^{2(k-|\beta|)+1} \left\| \frac{\partial^{k-|\beta|+1}}{\partial x_N^{k-|\beta|+1}} D^\beta \varphi(\bar{x}, \cdot) \right\|^2_{W^{2,2}(a, g(\bar{x}))} \tag{3.7}$$

By integrating both sides of (3.7) with respect to $\bar{x} \in W$, we finally get

$$\int_{\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)} |R(\beta, x)|^2\, \mathrm{d}x \le C k_\epsilon^{2(k-|\beta|)+1} \|\varphi\|^2_{W^{m,2}(\Omega)}, \tag{3.8}$$

for sufficiently small ϵ, for all $|\beta| \le k-1$. Thus, by (3.1) we get

$$\begin{aligned}\int_{\Phi_\epsilon(\Omega_\epsilon\setminus K_\epsilon)} |D^\beta\varphi(x)|^2\,\mathrm{d}x &\le Ck_\epsilon^{2(k-|\beta|)+1}\|\varphi\|^2_{W^{m,2}(\Omega)}\\ &+ C\int_W\int_{g_\epsilon(\bar{x})-k_\epsilon}^{g(\bar{x})}\left|\sum_{l=0}^{k-1-|\beta|}\frac{\partial^l(D^\beta\varphi(\bar{x},g(\bar{x}))}{\partial x_N^l}\right|^2|x_N-g(\bar{x})|^{2l}\mathrm{d}\bar{x}\,\mathrm{d}x_N\,,\end{aligned} \tag{3.9}$$

for all sufficiently small ϵ, and $|\beta| \le k-1$. We now estimate the last integral in the right-hand side of (3.9) in the following way

$$\begin{aligned}&\sum_{l=0}^{k-1-|\beta|}\int_W\int_{g_\epsilon(\bar{x})-k_\epsilon}^{g(\bar{x})}\left|\frac{\partial^l(D^\beta\varphi(\bar{x},g(\bar{x}))}{\partial x_N^l}\right|^2|x_N-g(\bar{x})|^{2l}\mathrm{d}\bar{x}\,\mathrm{d}x_N\\ &\le \sum_{l=0}^{k-1-|\beta|}k_\epsilon^{2l+1}\int_W\left|\frac{\partial^l(D^\beta\varphi(\bar{x},g(\bar{x}))}{\partial x_N^l}\right|^2\mathrm{d}\bar{x}\\ &= \sum_{l=0}^{k-1-|\beta|}Ck_\epsilon^{2l+1}\left\|\frac{\partial^l(D^\beta\varphi)}{\partial x_N^l}\right\|^2_{L^2(\Gamma)},\end{aligned} \tag{3.10}$$

where $\Gamma := \{(\bar{x}, g(\bar{x})) : \ \bar{x} \in W\}$. Thus, by (3.9), (3.10) we obtain

$$\begin{aligned}&\int_{\Phi_\epsilon(\Omega_\epsilon\setminus K_\epsilon)} |D^\beta\varphi(x)|^2\,\mathrm{d}x\\ &\le \sum_{l=0}^{k-1-|\beta|}Ck_\epsilon^{2l+1}\left\|\frac{\partial^l(D^\beta\varphi)}{\partial x_N^l}\right\|^2_{L^2(\Gamma)} + Ck_\epsilon^{2(k-|\beta|)+1}\|\varphi\|^2_{W^{m,2}(\Omega)}.\end{aligned} \tag{3.11}$$

Inequality (3.11) holds for smooth functions. If $\varphi \in W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$, then we can choose a sequence $(\psi_n)_{n\ge1} \subset C^\infty(\overline{\Omega})$ such that $\psi_n \longrightarrow \varphi$ in $W^{m,2}(\Omega)$ (this is possible because $\partial\Omega$ is Lipschitz continuous). We then use (3.11) for ψ_n, and we pass to the limit as $n \longrightarrow \infty$ by using the continuity of the trace operator and standard estimates on the intermediate derivatives of Sobolev functions (see e.g., [15, §4.4]). We deduce that

$$\int_{\Phi_\epsilon(\Omega_\epsilon\setminus K_\epsilon)} |D^\beta\varphi(x)|^2\,\mathrm{d}x \le Ck_\epsilon^{2(k-|\beta|)+1}\|\varphi\|^2_{W^{m,2}(\Omega)}, \tag{3.12}$$

for all sufficiently small ϵ. Actually, inequality (3.12) holds also for $|\beta| = k$ (possibly modifying the constant in the right hand side). Indeed, $D^\beta\varphi \in W^{2,2}(\Omega)$, for any $|\beta| = k$, hence by standard boundedness of Sobolev functions on almost all vertical lines (see (3.4)) we find that

$$\int_W\int_{g_\epsilon-k_\epsilon}^{g(\bar{x})}|D^\beta\varphi(x)|^2\,\mathrm{d}x_N\,\mathrm{d}\bar{x} \le 2k_\epsilon\int_W\|D^\beta\varphi(\bar{x},\cdot)\|^2_\infty\mathrm{d}\bar{x} \le 2Ck_\epsilon\|\varphi\|^2_{W^{m,2}(\Omega)}.$$

This concludes Step 1.

Step 2. We claim that Condition (C2)(ii) holds. Let $\varphi \in V(\Omega)$ and let α be a fixed multiindex such that $|\alpha| = m$. We write

$$D^\alpha \varphi(\Phi_\epsilon(x)) = \sum_{1\le|\beta|\le m} D^\beta \varphi(\Phi_\epsilon(x)) p^\alpha_{m,\beta}(\Phi_\epsilon)(x), \tag{3.13}$$

where $p^\alpha_{m,\beta}(\Phi_\epsilon)$ is a homogeneous polynomial of degree $|\beta|$ in derivatives of Φ_ϵ of order not exceeding $m - |\beta| + 1$. Note that the polynomial $p^\alpha_{m,\beta}(\Phi_\epsilon)$ appearing in (3.13) is the sum of several terms Θ in the following form

$$\Theta = D^{k_1}\left(\delta_{j_1,N} - \frac{\partial h_\epsilon}{\partial x_{j_1}}\right)\cdots D^{k_n}\left(\delta_{j_n,N} - \frac{\partial h_\epsilon}{\partial x_{j_n}}\right)\frac{\partial \Phi^{(i_{n+1})}}{\partial x_{i_{n+1}}}\cdots\frac{\partial \Phi^{(i_{|\beta|})}}{\partial x_{i_{|\beta|}}},$$

where[1] $1 \le n \le |\beta|$, $1 \le j_i \le N$ for all $i = 1, \dots, n$, $i_{n+1}, \dots, i_{|\beta|}$ are in $\{1, \dots, N-1\}$, and $k_1, \dots, k_n$ are multiindexes satisfying $|k_1| + \cdots + |k_n| = m - |\beta|$. Moreover, Θ is a sum of terms of the type $D^{L_1}h_\epsilon \cdots D^{L_l}h_\epsilon$, for all $1 \le l \le n$, for suitable multiindexes $L_1, \dots, L_l$ satisfying

$$|L_1| + \cdots + |L_l| = m - |\beta| + l. \tag{3.14}$$

Now by [8, Inequality (6.7)] and hypothesis (iii) we have

$$\begin{aligned}
&\|D^{L_1}h_\epsilon \cdots D^{L_l}h_\epsilon\|_\infty \\
&\le C\left(\sum_{|\gamma_1|\le|L_1|}\frac{\|D^{\gamma_1}(g_\epsilon - g)\|_\infty}{\kappa_\epsilon^{|L_1|-|\gamma_1|}}\right)\cdots\left(\sum_{|\gamma_l|\le|L_l|}\frac{\|D^{\gamma_l}(g_\epsilon - g)\|_\infty}{\kappa_\epsilon^{|L_l|-|\gamma_l|}}\right)\\
&\le o(1)\left(\sum_{|\gamma_1|\le|L_1|}\frac{\kappa_\epsilon^{m-|\gamma_1|-k+1/2}}{\kappa_\epsilon^{|L_1|-|\gamma_1|}}\right)\cdots\left(\sum_{|\gamma_l|\le|L_l|}\frac{\kappa_\epsilon^{m-|\gamma_l|-k+1/2}}{\kappa_\epsilon^{|L_l|-|\gamma_l|}}\right)\\
&\le o(1)\kappa_\epsilon^{l(m-k+1/2)-\sum_i|L_i|} = o(1)\kappa_\epsilon^{l(m-k+1/2)-\sum_i|L_i|-|\beta|+k+1/2}\cdot\kappa_\epsilon^{|\beta|-k-1/2}\\
&\le o(1)\kappa_\epsilon^{|\beta|-k-1/2}
\end{aligned}$$

where the last inequality holds provided that

$$l(m-k+1/2) - \sum_i |L_i| - |\beta| + k + 1/2 \ge 0.$$

By (3.14), we have to check that $l(m-k+1/2) - (m - |\beta| + l) - |\beta| + k + 1/2 \ge 0$, which is verified if and only if $l(m-k-1/2) \ge m-k-1/2$, and this holds true because $m-k-1/2 > 0$ and $l \ge 1$. Hence we have proved that

$$\|p^\alpha_{m,\beta}(\Phi_\epsilon)\|_\infty \le o(1)\,\kappa_\epsilon^{|\beta|-k-1/2}. \tag{3.15}$$

[1]Here it is understood that for $|\beta| = 1$ the terms $\frac{\partial\Phi^{(j_{n+1})}}{\partial x_{i_{n+1}}}\cdots\frac{\partial\Phi^{(j_{|\beta|})}}{\partial x_{i_{|\beta|}}}$ are not present; recall that $m \ge 2$.

By inequalities (3.12) and (3.15), we deduce that

$$
\begin{aligned}
Q_{\Omega_\epsilon\setminus K_\epsilon}(T_\epsilon\varphi) &\le \int_{\Omega_\epsilon\setminus K_\epsilon}|\varphi(\Phi_\epsilon)|^2\,dx + C\sum_{|\alpha|=m}\int_{\Omega_\epsilon\setminus K_\epsilon}|D^\alpha\varphi(\Phi_\epsilon)|^2\,dx \\
&\le C\int_{\Phi_\epsilon(\Omega_\epsilon\setminus K_\epsilon)}|\varphi|^2\,dx + C\sum_{\substack{|\alpha|=m\\ 1\le|\beta|\le k}}\|p^\alpha_{m,\beta}(\Phi_\epsilon)\|^2_\infty\int_{\Omega_\epsilon\setminus K_\epsilon}|D^\beta\varphi(\Phi_\epsilon(x))|^2\,dx \\
&+ C\sum_{\substack{|\alpha|=m\\ k<|\beta|\le m}}\|p^\alpha_{m,\beta}(\Phi_\epsilon)\|^2_\infty\int_{\Omega_\epsilon\setminus K_\epsilon}|D^\beta\varphi(\Phi_\epsilon(x))|^2\,dx \\
&\le C\|\varphi\|^2_{L^2(\Omega\setminus K_\epsilon)} + o(1)\kappa_\epsilon^{2(|\beta|-k-1/2)}\kappa_\epsilon^{2(k-|\beta|)+1} + o(1)\|\varphi\|^2_{W^{m,2}(\Omega\setminus K_\epsilon)},
\end{aligned} \tag{3.16}
$$

for all $\epsilon > 0$ sufficiently small. Since the right-hand side of (3.16) vanishes as $\epsilon \to 0$ we conclude that condition $(C2)(ii)$ is satisfied.
It remains to prove condition (C3). To prove that conditions (C3)(i), (C3)(ii) are satisfied it is sufficient to set $E_\epsilon u = (\mathrm{Ext}_{\Omega_\epsilon} u)|_\Omega$ for all $u \in V(\Omega_\epsilon)$, where $\mathrm{Ext}_{\Omega_\epsilon}$ is the standard Sobolev extension operator mapping $W^{m,2}(\Omega_\epsilon)$ to $W^{m,2}(\mathbb{R}^N)$. Finally, in order to prove condition (C3)(iii) it is sufficient to prove that the weak limit v of the uniformly bounded sequence v_ϵ (appearing in the statement of condition (C3)(iii)) lies in $W^{k,2}_0(\Omega)$. This is easily achieved by considering the extension-by-zero of the functions v_ϵ outside Ω_ϵ, passing to the limit and recalling that the limit set Ω has Lipschitz boundary. □

Theorem 2 can be actually applied to open sets Ω in the atlas class $C^m(\mathcal{A})$ by requiring that the assumptions of Lemma 2 are satisfied by all the profile functions g_j describing their boundaries. Then we can prove the following

Theorem 3. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$, $M > 0$, $m \in \mathbb{N}$, $m \ge 2$. For all $\epsilon \ge 0$, let $\Omega_\epsilon \in C^m_M(\mathcal{A})$. Let $k \in \mathbb{N}$ with $1 \le k < m$ and define, for all $\epsilon \ge 0$, $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W^{k,2}_0(\Omega_\epsilon)$. If*

$$\lim_{\epsilon\to 0} d^{(m-k)}_{\mathcal{A}}(\Omega_\epsilon, \Omega) = 0,$$

then condition **(C)** *is satisfied, hence $H^{-1}_{V(\Omega_\epsilon)}$ $\mathcal{E}$-compact converges to $H^{-1}_{V(\Omega)}$ as $\epsilon \to 0$.*

Proof. By using a standard partition of unity argument, it suffices to prove that the assumptions of Theorem 2 are satisfied by all the profile functions $g_{j,\epsilon}$, g_j describing the boundaries of Ω_ϵ, Ω, respectively, and this follows by choosing $\kappa_\epsilon = (d^{(m-k)}_{\mathcal{A}}(\Omega_\epsilon, \Omega))^{\frac{1}{m}}$. □

In order to prove that the assumptions of Lemma 2 are sharp, we now consider a the following geometric setting:
(G2) Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Let $b \in C^\infty(\overline{W})$ a positive, non-constant periodic function, with periodicity cell given by $Y =]-1/2, 1/2[^{N-1}$. Let us set

$$g_\epsilon(\bar{x}) = \epsilon^\alpha b\left(\frac{\bar{x}}{\epsilon}\right), \qquad g(\bar{x}) = 0,$$

for all $\bar{x} \in W$. For simplicity, we set $g_0 = g$ and for all $\epsilon \geq 0$ we consider the open sets

$$\Omega_\epsilon = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, \ -1 < x_N < g_\epsilon(\bar{x})\}$$

Then we have the following

Theorem 4. *Let Ω_ϵ, $\epsilon \geq 0$ be as in* **(G2)** *and let $k \in \mathbb{N}$ satisfy $1 \leq k \leq m-1$. Let $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W_0^{k,2}(\Omega_\epsilon)$ for all $\epsilon \geq 0$. If $\alpha > m - k + \frac{1}{2}$, then $H^{-1}_{V(\Omega_\epsilon)}$ $\mathcal{E}$-compact converges to $H^{-1}_{V(\Omega)}$ as $\epsilon \to 0$.*

Proof. We aim at applying Theorem 2 with $\kappa_\epsilon = \epsilon^{\alpha\theta}\|b\|_\infty$, for some $\theta \in (0,1)$ to be specified. By the classical Gagliardo-Nirenberg interpolation inequality

$$\|D^\beta f\|_\infty \leq C\Big(\sum_{|\alpha|=m} \|D^\alpha f\|_\infty\Big)^{|\beta|/m} \|f\|_\infty^{1-|\beta|/m},$$

for all $f \in W^{m,\infty}(\Omega)$ (see e.g., [42, p.125]), in order to verify condition (iii) in Theorem 2 it is sufficient to verify it for $|\beta| = 0$ and $|\beta| = m$ (see also [8, Proposition 6.17]). When $|\beta| = 0$ we have

$$\lim_{\epsilon\to 0} \frac{\|g_\epsilon - g\|_\infty}{\kappa_\epsilon^{m-k+1/2}} = c \lim_{\epsilon\to 0} \frac{\epsilon^\alpha}{\epsilon^{\alpha\theta(m-k+1/2)}} = c \lim_{\epsilon\to 0} \epsilon^{\alpha(1-\theta(m-k-1/2))},$$

where c is a constant depending only on $\|b\|_\infty$. The right-hand side clearly tends to 0 as soon as $\theta < \frac{1}{m-k+1/2}$.
When $|\beta| = m$, we must check that $\lim_{\epsilon\to 0} \frac{D^\beta g_\epsilon}{\kappa_\epsilon^{-k+1/2}} = 0$. Note that

$$\left\| \frac{D^\beta g_\epsilon}{\kappa_\epsilon^{-k+1/2}} \right\|_\infty = c \frac{\epsilon^{\alpha - m}}{\epsilon^{\alpha\theta(-k+1/2)}} = \epsilon^{\alpha(1-\theta(-k+1/2))-m},$$

and the right hand side tends to zero if and only if

$$\alpha\Big(1 + \theta\Big(k - \frac{1}{2}\Big)\Big) - m > 0. \tag{3.17}$$

By letting $\theta \to \frac{1}{m-k+1/2}$ in (3.17) we obtain that inequality (3.17) is satisfied when $\alpha > m - k + 1/2$, true by assumption. By Lemma 2 we deduce the validity of Theorem 4. □

Remark 1. *When $k = m-1$, Theorem 4 states that if $\alpha > \frac{3}{2}$, $H^{-1}_{V(\Omega_\epsilon)} \xrightarrow{C} H^{-1}_{V(\Omega)}$ as $\epsilon \to 0$, independently on $m \geq 2$. Actually, it is possible to prove that $\alpha = 3/2$ in this case is the critical exponent, in the sense that when $\alpha \leq 3/2$ the operator $H^{-1}_{V(\Omega_\epsilon)}$ does not converge to $H^{-1}_{V(\Omega)}$. We refer to Theorem 7 for a complete discussion about the spectral convergence of $H_{V(\Omega_\epsilon)}$ depending on α.*

4 A polyharmonic Green formula

In this section we provide a formula which turns out to be useful in recognising the possible natural boundary conditions for polyharmonic operators of any order. Let us begin by stating an easy integration-by-parts formula.

Proposition 1. *Let Ω be a bounded domain of class $C^{0,1}$ in $\mathbb{R}^N$. Let $m \in \mathbb{N}$ and let $f \in C^{m+1}(\overline{\Omega})$, $\varphi \in C^m(\overline{\Omega})$. Then*

$$\int_\Omega D^m f : D^m \varphi \, \mathrm{d}x = -\int_\Omega D^{m-1}(\Delta f) : D^{m-1}\varphi \, \mathrm{d}x + \int_{\partial\Omega} D^m f : (n \otimes D^{m-1}\varphi) \, \mathrm{d}S, \tag{4.1}$$

where the symbol $:$ stands for the Frobenius product, n is the unit outer normal to $\partial\Omega$, and $\otimes$ is the tensor product, defined by $(n \otimes D^{m-1}\varphi)_{i,j_1,\cdots,j_{m-1}} = n_i \frac{\partial^{m-1}\varphi}{\partial x_{j_1}\cdots\partial x_{j_{m-1}}}$ for all $i, j_1, \cdots, j_{m-1} \in \{1, \cdots, N\}$.

Proof. The proof is a simple integration by parts. Indeed, dropping the summation symbols we get

$$\begin{aligned}
&\int_\Omega D^m f : D^m \varphi \, \mathrm{d}x = \int_\Omega \frac{\partial^m f}{\partial x_{j_1}\cdots\partial x_{j_m}} \frac{\partial^m \varphi}{\partial x_{j_1}\cdots\partial x_{j_m}} \, \mathrm{d}x \\
&= -\int_\Omega \frac{\partial^{m+1} f}{\partial x_{j_1}^2\cdots\partial x_{j_m}} \frac{\partial^{m-1}\varphi}{\partial x_{j_2}\cdots\partial x_{j_m}} \, \mathrm{d}x + \int_{\partial\Omega} \frac{\partial^m f}{\partial x_{j_1}\cdots\partial x_{j_m}} \frac{\partial^{m-1}\varphi}{\partial x_{j_2}\cdots\partial x_{j_m}} n_{j_1} \, \mathrm{d}S \\
&= -\int_\Omega D^{m-1}(\Delta f) : D^{m-1}\varphi \, \mathrm{d}x + \int_{\partial\Omega} (D^m f) : (n \otimes D^{m-1}\varphi) \, \mathrm{d}S.
\end{aligned}$$

□

By applying m times the integration by parts argument used in the proof of formula (4.1), we deduce the validity of the following

Corollary 1. *Let $m \in \mathbb{N}$. Let $f \in C^{2m}(\overline{\Omega})$, $\varphi \in C^m(\overline{\Omega})$.*

$$\int_\Omega D^m f : D^m \varphi \, \mathrm{d}x = (-1)^m \int_\Omega \Delta^m f \varphi \, \mathrm{d}x + \sum_{k=0}^{m-1} (-1)^k \int_{\partial\Omega} (D^{m-k}(\Delta^k f)) : (n \otimes D^{m-k-1}\varphi) \, \mathrm{d}S. \tag{4.2}$$

Theorem 5 (Polyharmonic Green Formula - Flat case)**.** *Let H be the half-space $H = \{(\bar{x}, x_N) \in \mathbb{R}^N : x_N < 0\}$. Let $m \in \mathbb{N}$. Let $f \in C^{2m}(\overline{H})$, $\varphi \in C^m(\overline{H})$ with compact support in $\overline{H}$. Then,*

$$\int_H D^m f : D^m \varphi \, \mathrm{d}x = (-1)^m \int_H \Delta^m f \varphi \, \mathrm{d}x + \sum_{t=0}^{m-1} \int_{\mathbb{R}^{N-1}} B_t(f) \frac{\partial^t \varphi}{\partial x_N^t} \mathrm{d}\bar{x}, \tag{4.3}$$

where $B_t : C^{2m}(\partial H) \to C^{t+1}(\partial H)$ *is defined by*

$$B_t(f) = \sum_{l=t}^{m-1} (-1)^{m-t-1} \binom{l}{t} \Delta_{N-1}^{l-t} \left(\frac{\partial^{t+1}}{\partial x_N^{t+1}} (\Delta^{m-l-1} f) \right), \tag{4.4}$$

and Δ_{N-1} *is the Laplace operator in the first* $N-1$ *variables.*

Proof. Let $r = m - k - 1$. First note that we can write

$$\int_{\mathbb{R}^{N-1}} \left(D^r \left(\Delta^k \left(\frac{\partial f}{\partial x_N} \right) \right) \right) : D^r \varphi \, \mathrm{d}\bar{x} \\ = \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \left(D_{\bar{x}}^{r-t} \left(\Delta^k \left(\frac{\partial^{t+1} f}{\partial x_N^{t+1}} \right) \right) \right) : \left(D_{\bar{x}}^{r-t} \left(\frac{\partial^t \varphi}{\partial x_N^t} \right) \right) d\bar{x}. \tag{4.5}$$

Then, by using (4.5) in the last integral in the right-hand side of (4.2) we get the following as boundary term

$$\sum_{k=0}^{m-1} (-1)^k \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} D_{\bar{x}}^{r-t} \left(\frac{\partial^{t+1} (\Delta^k f)}{\partial x_N^{t+1}} \right) : D_{\bar{x}}^{r-t} \left(\frac{\partial^t \varphi}{\partial x_N^t} \right) \mathrm{d}\bar{x}. \tag{4.6}$$

By dropping the summation symbols, the integrand in (4.6) becomes

$$\int_{\mathbb{R}^{N-1}} \frac{\partial^{r-t}}{\partial x_{i_1} \cdots \partial x_{i_{r-t}}} \left(\frac{\partial^{t+1} (\Delta^k f)}{\partial x_N^{t+1}} \right) \frac{\partial^{r-t}}{\partial x_{i_1} \cdots \partial x_{i_{r-t}}} \left(\frac{\partial^t \varphi}{\partial x_N^t} \right) \mathrm{d}\bar{x}, \tag{4.7}$$

where the indexes i_j run on the first $N-1$ coordinates. By integrating by parts $r-t$ times in $i_1, \dots, i_{r-t}$ in (4.7) we deduce that (4.6) equals

$$\sum_{k=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \frac{\partial^{2(r-t)}}{\partial^2 x_{i_1} \cdots \partial^2 x_{i_{r-t}}} \left(\frac{\partial^{t+1} (\Delta^k f)}{\partial x_N^{t+1}} \right) \frac{\partial^t \varphi}{\partial x_N^t} \, \mathrm{d}\bar{x},$$

where we have no other boundary terms because φ has compact support. We rewrite the last expression as

$$\sum_{k=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \Delta_{N-1}^{r-t} \left(\frac{\partial^{t+1} (\Delta^k f)}{\partial x_N^{t+1}} \right) \frac{\partial^t \varphi}{\partial x_N^t} \mathrm{d}\bar{x}. \tag{4.8}$$

We now apply the change of summation index $r = m - k - 1$ in the first sum of (4.8). We deduce that (4.8) equals

$$\sum_{r=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \Delta_{N-1}^{r-t} \left(\frac{\partial^{t+1} (\Delta^{m-r-1} f)}{\partial x_N^{t+1}} \right) \frac{\partial^t \varphi}{\partial x_N^t} \mathrm{d}\bar{x}. \tag{4.9}$$

By exchanging the two sums in (4.9) we get (4.3). □

Remark 2. *If $m = 2$, then* (4.3) *reads*

$$\int_H D^2 f : D^2\varphi \,\mathrm{d}x = \int_H \Delta^2 f\varphi \,\mathrm{d}x + \int_{\mathbb{R}^{N-1}} \frac{\partial^2 f}{\partial x_N^2}\frac{\partial\varphi}{\partial x_N}\,\mathrm{d}\bar{x} \\ -\int_{\mathbb{R}^{N-1}} \left(\Delta_{N-1}\left(\frac{\partial f}{\partial x_N}\right) + \Delta\left(\frac{\partial f}{\partial x_N}\right)\right)\varphi \,\mathrm{d}\bar{x},$$

which is consistent with the formula provided in [8, Lemma 8.56]. Indeed, if the domain is a hyperplane, the boundary integral $\int_{\partial H}(\mathrm{div}_{\partial H}(D^2 f\cdot n)_{\partial\Omega})\,\varphi\,\mathrm{d}S$ appearing in [8, Lemma 8.56] coincides with $\int_{\mathbb{R}^{N-1}}\Delta_{N-1}(\frac{\partial f}{\partial x_N})\,\varphi\,\mathrm{d}\bar{x}$.

Theorem 6. *Let Ω be a bounded domain of $\mathbb{R}^N$ of class $C^{0,1}$, $m \in \mathbb{N}$, $m \geq 2$. Let $f \in W^{2m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$ and $\varphi \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$. Then*

$$\int_\Omega D^m f : D^m\varphi \,dx = (-1)^m\int_\Omega \Delta^m f\varphi dx + \int_{\partial\Omega}\frac{\partial^m f}{\partial n^m}\frac{\partial^{m-1}\varphi}{\partial n^{m-1}}\,dS. \tag{4.10}$$

Proof. By (4.2) it is easy to see that

$$\int_\Omega D^m f : D^m\varphi \,dx = (-1)^m\int_\Omega \Delta^m f\varphi \,dx + \int_{\partial\Omega} D^m f : (n\otimes D^{m-1}\varphi)\,dS, \tag{4.11}$$

for all $\varphi \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$, since $D^l\varphi = 0$ on $\partial\Omega$ for all $l \leq m-2$. We note that $D^m f : (n\otimes D^{m-1}\varphi) = (n^T D^m f) : D^{m-1}\varphi$. Moreover we claim that $D^{m-1}\varphi = \frac{\partial^{m-1}\varphi}{\partial n^{m-1}}\bigotimes_{i=1}^{m-1} n$ on $\partial\Omega$ and we prove it by induction. If $m = 2$ the claim is a direct consequence of the gradient decomposition $\nabla|_{\partial\Omega} = \nabla_{\partial\Omega} + \frac{\partial}{\partial n}n$. Now we assume that $m > 2$ and that the claim holds for $m-1$. Then, by using the fact that $D^{m-2}\varphi|_{\partial\Omega} = 0$, for all $\varphi \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$, we get

$$D^{m-1}\varphi|_{\partial\Omega} = D(D^{m-2}\varphi)|_{\partial\Omega} = \left(D\left(\frac{\partial^{m-2}\varphi}{\partial n^{m-2}}\bigotimes_{i=1}^{m-2} n\right)n\right)\otimes n = \frac{\partial^{m-1}\varphi}{\partial n^{m-1}}\bigotimes_{i=1}^{m-1} n,$$

for all $\varphi \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$. This proves the claim. Then we can rewrite (4.11) as

$$\int_\Omega D^m f : D^m\varphi \,dx = (-1)^m\int_\Omega \Delta^m f\varphi \,dx + \int_{\partial\Omega}\frac{\partial^{m-1}\varphi}{\partial n^{m-1}}(n^T D^m f) : \left(\bigotimes_{i=1}^{m-1} n\right)dS, \tag{4.12}$$

and since $(n^T D^m f) : \left(\bigotimes_{i=1}^{m-1} n\right) = D^m f : \left(\bigotimes_{i=1}^{m} n\right) = \frac{\partial^m f}{\partial n^m}$ we deduce (4.10). □

5 Polyharmonic operators with strong intermediate boundary conditions

Let Ω_ϵ, $\epsilon \geq 0$ be as in **(G2)**. Consider the polyharmonic operators $(-\Delta)^m + \mathbb{I}$ subject to strong intermediate boundary conditions, corresponding to the energy space

$V(\Omega_\epsilon) := W^{m,2}(\Omega_\epsilon) \cap W_0^{m-1,2}(\Omega_\epsilon)$. More precisely, let $H_{\Omega_\epsilon,S}$ be the non-negative self-adjoint operator such that

$$(H_{\Omega_\epsilon,S}u, v)_{L^2(\Omega_\epsilon)} = (H^{1/2}_{\Omega_\epsilon,S}u,\, H^{1/2}_{\Omega_\epsilon,S}v)_{L^2(\Omega_\epsilon)} = Q_{\Omega_\epsilon}(u,v), \tag{5.1}$$

for all functions $u, v \in W^{m,2}(\Omega_\epsilon)\cap W_0^{m-1,2}(\Omega_\epsilon)$, where $Q_{\Omega_\epsilon}(u,v) := \int_{\Omega_\epsilon} D^m u : D^m v + uv\, dx$, is the quadratic form canonically associated with $H_{\Omega_\epsilon,S}$. As it is explained in Section 2 the equation $H_{\Omega_\epsilon,S}u = f$ with datum $f \in L^2(\Omega_\epsilon)$, corresponds exactly to the weak Poisson problem (1.8).

Let $H_{\Omega,D}$ be the polyharmonic operator satisfying strong intermediate boundary conditions on $\partial\Omega \setminus \overline{W}$ and Dirichlet boundary conditions on W, whose associated boundary value problem reads

$$\begin{cases} (-\Delta)^m u + u = f, & \text{in } \Omega_\epsilon,\\ \frac{\partial^l u}{\partial n^l} = 0, & \text{on } W, \text{ for all } 0 \le l \le m-1,\\ \frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial\Omega_\epsilon \setminus \overline{W}, \text{ for all } 0 \le l \le m-2,\\ \frac{\partial^m u}{\partial n^m} = 0, & \text{on } \partial\Omega_\epsilon \setminus \overline{W}. \end{cases} \tag{5.2}$$

Note that we are identifying W with $W \times \{0\}$. Then the following theorem holds.

Theorem 7. *Let $m \in \mathbb{N}$, $m \ge 2$, Ω_ϵ as in* **(G2)**, *H_{Ω_ϵ} as in* (5.1), *for all $\epsilon > 0$. Then the following statements hold true.*

(i) [Spectral stability] *If $\alpha > 3/2$, then $H^{-1}_{\Omega_\epsilon,S} \xrightarrow{C} H^{-1}_{\Omega,S}$ as $\epsilon \to 0$.*

(ii) [Instability] *If $\alpha < 3/2$, then $H^{-1}_{\Omega_\epsilon,S} \xrightarrow{C} H^{-1}_{\Omega,D}$ as $\epsilon \to 0$, where $H_{\Omega,D}$ is defined in* (5.2).

(iii) [Strange term] *If $\alpha = 3/2$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{C} \hat{H}^{-1}_{\Omega}$ as $\epsilon \to 0$, where $\hat{H}_\Omega$ is the operator $(-\Delta)^m + \mathbb{I}$ with strong intermediate boundary conditions on $\partial\Omega\setminus\overline{W}$ and the following boundary conditions on W: $D^l u = 0$, for all $l \le m-2$, $\partial^m_{x_N} u + K\partial^{m-1}_{x_N} u = 0$, where the factor K is given by*

$$K = \int_{Y\times(-\infty,0)} |D^m V|^2\, \mathrm{d}y = -\int_Y \left(\frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}} + (m-1)\Delta_{N-1}\left(\frac{\partial^{m-1} V}{\partial x_N^{m-1}}\right)\right) b(\bar{y})\mathrm{d}\bar{y},$$

and the function V is Y-periodic in the variable $\bar{y}$ and satisfies the following microscopic problem

$$\begin{cases} (-\Delta)^m V = 0, & \text{in } Y \times (-\infty, 0),\\ \frac{\partial^l V}{\partial n^l}(\bar{y},0) = 0, & \text{on } Y, \text{ for all } 0 \le l \le m-3,\\ \frac{\partial^{m-2} V}{\partial y_N^{m-2}}(\bar{y},0) = b(\bar{y}), & \text{on } Y,\\ \frac{\partial^m V}{\partial y_N^m}(\bar{y},0) = 0, & \text{on } Y. \end{cases}$$

Proof. Statement (i) is a straightforward application of Theorem 4 with $k = m - 1$. To prove (ii) we check that Condition (C) in Definition 3 is satisfied with $V(\Omega) = W^{m,2}_{0,W}(\Omega) \cap W^{m-1,2}_0(\Omega)$, and $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W^{m-1,2}_0(\Omega_\epsilon)$. Here $W^{m,2}_{0,W}(\Omega)$ is the closure in $W^{m,2}(\Omega)$ of the space of functions vanishing in a neighborhood of W. Let $K_\epsilon = \Omega$ for all $\epsilon > 0$. Then we see immediately that condition (3.1) and condition (C1) are satisfied. We define now T_ϵ as the extension by zero operator from $W^{m,2}_{0,W}(\Omega)$ to $W^{m,2}(W \times (-1, +\infty))$ and E_ϵ as the restriction operator to Ω. With these definitions it is not difficult to prove that conditions (C2) and (C3)(i),(ii) are satisfied. It remains to prove that condition (C3)(iii) holds. Let $v_\epsilon \in W^{m,2}(\Omega_\epsilon) \cap W^{m-1,2}_0(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)} \le C$ for all $\epsilon > 0$. Possibly passing to a subsequence there exists a function $v \in W^{m-1,2}(\Omega)$ such that $v_\epsilon|_\Omega \rightharpoonup v$ in $W^{m,2}(\Omega)$ and $v_\epsilon|_\Omega \to v$ in $W^{m-1,2}(\Omega)$. By considering the sequence of functions $T_\epsilon(v_\epsilon|\Omega)$ it is not difficult to prove that $v \in W^{m-1,2}_0(\Omega)$. It remains to check that $\frac{\partial^{m-1} v}{\partial x_N^{m-1}} = 0$ on $W \times \{0\}$. This is proven exactly as in [8, Theorem 7.3] by applying Lemma 4.3 from [20] to the vector field V^i_ϵ defined by

$$V^i_\epsilon = \left(0, \cdots, 0, -\frac{\partial^{m-1} v_\epsilon}{\partial x_N^{m-1}}, 0, \cdots, 0, \frac{\partial^{m-1} v_\epsilon}{\partial x_N^{m-2} \partial x_i}\right),$$

for all $i = 1, \dots, N-1$, where the only non-zero entries are the i-th and the N-th ones. We remark that it is possible to apply Lemma 4.3 from [20] because by Theorem 4 the critical threshold for all the polyharmonics operator with strong intermediate boundary conditions is $\alpha = 3/2$, which coincides with the critical value in [20]. We then deduce that $\frac{\partial^{m-1} v(\bar{x}, 0)}{\partial x_N^{m-1}} \frac{\partial b(\bar{y})}{\partial y_i} = 0$, a.e. $W \times Y$. Since b is a non-constant smooth function we must have $\frac{\partial^{m-1} v(\bar{x}, 0)}{\partial x_N^{m-1}} = 0$ a.e. on W. This concludes the proof of condition $(C3)(iii)$.

We provide a proof of (iii) in Sections 5.1 and 5.2. □

Remark 3. *We take the chance to point out a misprint in [5, Theorem 1, (ii)] where the condition $\partial^m_{x_N} u + K\partial^{m-1}_{x_N} u = 0$ in our Theorem 7 (iii) above, appears for $m = 3$ with $-K$ instead of $+K$ as it should be.*

5.1 Critical case - Macroscopic problem.

In this section we prove Theore 7 (iii). Let us define a diffeomorphism Φ_ϵ from Ω_ϵ to Ω by

$$\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N)), \quad \text{for all } x = (\bar{x}, x_N) \in \Omega_\epsilon,$$

where h_ϵ is defined by

$$h_\epsilon(\bar{x}, x_N) = \begin{cases} 0, & \text{if } -1 \le x_N \le -\epsilon, \\ g_\epsilon(\bar{x}) \left(\frac{x_N + \epsilon}{g_\epsilon(\bar{x}) + \epsilon}\right)^{m+1}, & \text{if } -\epsilon \le x_N \le g_\epsilon(\bar{x}). \end{cases}$$

By standard calculus one can prove the following

Lemma 1. *The map Φ_ϵ is a diffeomorphism of class C^m and there exists a constant $c > 0$ independent of ϵ such that $|h_\epsilon| \le c\epsilon^\alpha$ and $\left|D^l h_\epsilon\right| \le c\epsilon^{\alpha - l}$, for all $l = 1, \dots, m$, $\epsilon > 0$ sufficiently small.*

As in [8, Section 8.1], we introduce the pullback operator T_ϵ from $L^2(\Omega)$ to $L^2(\Omega_\epsilon)$ given by $T_\epsilon u = u \circ \Phi_\epsilon$ for all $u \in L^2(\Omega)$.

In order to proceed we find convenient to recall some notation and results in homogenization theory regarding the unfolding operator. We refer to [1, 23, 24, 28] for the proof of the main properties of the operator, and we mention that recent developments can be found in the article [9].

For any $k \in \mathbb{Z}^{N-1}$ and $\epsilon > 0$ we define

$$\begin{cases} C_\epsilon^k = \epsilon k + \epsilon Y, \\ I_{W,\epsilon} = \{k \in \mathbb{Z}^{N-1} : C_\epsilon^k \subset W\}, \\ \widehat{W}_\epsilon = \bigcup_{k \in I_{W,\epsilon}} C_\epsilon^k. \end{cases} \tag{5.3}$$

Then we give the following

Definition 4. *Let u be a real-valued function defined in Ω. For any $\epsilon > 0$ sufficiently small the unfolding $\hat{u}$ of u is the real-valued function defined on $\widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0)$ by*

$$\hat{u}(\bar{x}, \bar{y}, y_N) = u\left(\epsilon\left[\frac{\bar{x}}{\epsilon}\right] + \epsilon\bar{y}, \epsilon y_N\right),$$

for almost all $(\bar{x}, \bar{y}, y_N)) \in \widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0)$, where $\left[\frac{\bar{x}}{\epsilon}\right]$ denotes the integer part of the vector $\bar{x}\epsilon^{-1}$ with respect to Y, i.e., $[\bar{x}\epsilon^{-1}] = k$ if and only if $\bar{x} \in C_\epsilon^k$.

The following lemma will be often used in the sequel. For a proof we refer to [25, Proposition 2.5(i)].

Lemma 2. *Let $a \in [-1, 0[$ be fixed. Then*

$$\int_{\widehat{W}_\epsilon \times (a,0)} u(x) dx = \epsilon \int_{\widehat{W}_\epsilon \times Y \times (a/\epsilon, 0)} \hat{u}(\bar{x}, y) d\bar{x} dy \tag{5.4}$$

for all $u \in L^1(\Omega)$ and $\epsilon > 0$ sufficiently small. Moreover

$$\int_{\widehat{W}_\epsilon \times (a,0)} \left|\frac{\partial^l u(x)}{\partial x_{i_1} \cdots \partial x_{i_l}}\right|^2 dx = \epsilon^{1-2l} \int_{\widehat{W}_\epsilon \times Y \times (a/\epsilon, 0)} \left|\frac{\partial^l \hat{u}}{\partial y_{i_1} \cdots \partial y_{i_l}}(\bar{x}, y) d\bar{x}\right|^2 dy,$$

for all $l \le m$, $u \in W^{m,2}(\Omega)$ and $\epsilon > 0$ sufficiently small.

Let $W^{m,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y \times (-\infty, 0))$ be the subspace of $W^{m,2}_{\mathrm{loc}}(\mathbb{R}^{N-1} \times (-\infty, 0))$ containing Y-periodic functions in the first $(N-1)$ variables $\bar{y}$. We then define $W^{m,2}_{loc}(Y \times (-\infty, 0))$ to be the space of functions in $W^{m,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y \times (-\infty, 0))$ restricted to $Y \times (-\infty, 0)$. Finally we set

$$w^{m,2}_{\mathrm{Per}_Y}(Y \times (-\infty, 0)) := \big\{u \in W^{m,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y \times (-\infty, 0)) \\ : \|D^\gamma u\|_{L^2(Y \times (-\infty,0))} < \infty, \forall |\gamma| = m\big\}. \tag{5.5}$$

For any $d < 0$, let $\mathcal{P}^l_{hom,y}(Y \times (d,0))$ be the space of homogeneous polynomials of degree at most l restricted to the domain $(Y\times(d,0))$. Let $\epsilon > 0$ be fixed. We define the projectors P_i from $L^2(\widehat{W}_\epsilon, W^{m,2}(Y \times (-1/\epsilon, 0)))$ to $L^2(\widehat{W}_\epsilon, \mathcal{P}^i_{hom,y}(-1/\epsilon, 0))$ by setting

$$P_i(\psi) = \sum_{|\eta|=i} \int_Y D^\eta \psi(\bar{x}, \bar{\zeta}, 0) d\bar{\zeta} \frac{y^\eta}{\eta!}$$

for all $i = 0, \dots, m-1$. We now set $Q_{m-1} = P_{m-1}$, $Q_{m-2} = P_{m-2}(\mathbb{I} - Q_{m-1})$, etc., up to $Q_0 = P_0\big(\mathbb{I} - \sum_{j=1}^{m-1} Q_j\big)$. Note that Q_{m-j}, $j = 1, \dots, m$ is a projection on the space of homogeneous polynomials of degree $m-j$, with the property that $Q_{m-k}(p) = 0$ for all polynomials p of degree $m-k$ with $k \neq j$. We finally set

$$\mathcal{P} = Q_0 + Q_1 + \cdots + Q_{m-1}, \tag{5.6}$$

which is a projector on the space of polynomials in y of degree at most $m-1$. Note that $D^\beta_y \mathcal{P}(\psi)(\bar{x}, \bar{y}, 0) = \int_Y D^\beta_y \psi(\bar{x}, \bar{y}, 0) d\bar{y}$ for all $|\beta| = 0, \dots, m-1$. In particular, it follows that $\int_Y (D^\beta_y \psi(\bar{x}, \bar{y}, 0) - D^\beta_y \mathcal{P}(\psi)(\bar{x}, \bar{y}, 0)) d\bar{y} = 0$ for almost all $\bar{x}$ in $\widehat{W}_\epsilon$, for all $|\beta| = 0, \dots, m-1$.

Lemma 3. *Let $m \in \mathbb{N}$, $m \geq 2$ be fixed. The following statements hold:*

(i) Let $v_\epsilon \in W^{m,2}(\Omega)$ with $\|\hat{v}_\epsilon\|_{W^{m,2}(\Omega)} \leq M$, for all $\epsilon > 0$. Let V_ϵ be defined by

$$V_\epsilon(\bar{x}, y) = \hat{v}_\epsilon(\bar{x}, y) - \mathcal{P}(v_\epsilon)(\bar{x}, y),$$

for $(\bar{x}, y) \in \widehat{W_\epsilon} \times Y \times (-1/\epsilon, 0)$, where $\mathcal{P}$ is defined by (5.6) . Then there exists a function $\hat{v} \in L^2(W, w^{m,2}_{\mathrm{Per}_Y}(Y \times (-\infty, 0)))$ such that, possibly passing to a subsequence, for every $d < 0$

(a) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{m-1/2}} \rightharpoonup D^\gamma_y \hat{v}$ in $L^2(W \times Y \times (d, 0))$ as $\epsilon \to 0$, for any $\gamma \in \mathbb{N}^N_0$, $|\gamma| \leq m-1$.

(b) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{m-1/2}} \rightharpoonup D^\gamma_y \hat{v}$ in $L^2(W \times Y \times (-\infty, 0))$ as $\epsilon \to 0$, for any $\gamma \in \mathbb{N}^N_0$, $|\gamma| = m$,

where it is understood that the functions $V_\epsilon, D^\gamma_y V_\epsilon$ are extended by zero to the whole of $W \times Y \times (-\infty, 0)$ outside their natural domain of definition $\widehat{W_\epsilon} \times Y \times (-1/\epsilon, 0)$.

(ii) If $\psi \in W^{1,2}(\Omega)$, then $\lim_{\epsilon\to 0} \widehat{(T_\epsilon \psi)_{|\Omega}} = \psi(\bar{x}, 0)$ in $L^2(W \times Y \times (-1, 0))$.

Proof. The proof follows as in the proof [8, Lemma 8.9] by noting that $\mathcal{P}$ is a projector on the space of polynomials of degree at most $m-1$, so that a Poincaré-Wirtinger-type inequality still holds. □

Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\mathbb{R}^N)$ as $\epsilon \to 0$, with the understanding that the functions are extended by zero outside their natural domains. Let $v_\epsilon \in V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W^{m-1,2}_0(\Omega_\epsilon)$ be such that for all $\epsilon > 0$ small enough

$$H_{\Omega_\epsilon, S} v_\epsilon = f_\epsilon. \tag{5.7}$$

Then $\|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)} \leq M$ for all $\epsilon > 0$ sufficiently small, hence, possibly passing to a subsequence there exists $v \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$ such that $v_\epsilon \rightharpoonup v$ in $W^{m,2}(\Omega)$ and $v_\epsilon \to v$ in $L^2(\mathbb{R}^N)$.

Let $\varphi \in V(\Omega) = W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega)$ be fixed. Since $T_\epsilon \varphi \in V(\Omega_\epsilon)$, by (5.7) we have

$$\int_{\Omega_\epsilon} D^m v_\epsilon : D^m T_\epsilon \varphi \, dx + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon \varphi \, dx = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \varphi \, dx, \tag{5.8}$$

and passing to the limit as $\epsilon \to 0$ we get $\int_{\Omega_\epsilon} v_\epsilon T_\epsilon \varphi \, dx \to \int_\Omega v \varphi \, dx$ and $\int_{\Omega_\epsilon} f_\epsilon T_\epsilon \varphi \, dx \to \int_\Omega f \varphi \, dx$.

Now consider the first integral in the right hand-side of (5.8). Set $K_\epsilon = W \times (-1, -\epsilon)$. By splitting the integral in three terms corresponding to $\Omega_\epsilon \setminus \Omega$, $\Omega \setminus K_\epsilon$ and K_ϵ and by arguing as in [8, Section 8.3] one can show that $\int_{K_\epsilon} D^m v_\epsilon : D^m \varphi \, dx \to \int_\Omega D^m v : D^m \varphi \, dx$ and $\int_{\Omega_\epsilon \setminus \Omega} D^m v_\epsilon : D^m T_\epsilon \varphi \, dx \to 0$, as $\epsilon \to 0$. Let us define Q_ϵ by

$$Q_\epsilon = \widehat{W_\epsilon} \times (-\epsilon, 0).$$

We split again the remaining integral in two summands as follows:

$$\int_{\Omega_\epsilon \setminus K_\epsilon} D^m v_\epsilon : D^m T_\epsilon \varphi \, dx = \int_{\Omega_\epsilon \setminus (K_\epsilon \cup Q_\epsilon)} D^m v_\epsilon : D^m T_\epsilon \varphi \, dx + \int_{Q_\epsilon} D^m v_\epsilon : D^m T_\epsilon \varphi \, dx. \tag{5.9}$$

As in [8, Section 8.3], $\int_{\Omega_\epsilon \setminus (K_\epsilon \cup Q_\epsilon)} D^m v_\epsilon : D^m T_\epsilon \varphi \, dx \to 0$, as $\epsilon \to 0$. It remains to analyse the limit as $\epsilon \to 0$ of the last summand in the right-hand side of (5.9). To do so, we also need the following lemma in the proof of which we use notation and rules of calculus recalled in Section 2.

Lemma 4. *Let $l \in \mathbb{N}$, $l \le m$, and let $i_1, \dots, i_l \in \{1, \dots, N\}$. The functions $\hat{h}_\epsilon(\bar{x}, y)$, $\widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y)$ defined for $y \in Y \times (-1, 0)$, are independent of $\bar{x}$. Moreover, $\|\hat{h}_\epsilon\|_{L^\infty} = O(\epsilon^{3/2})$, $\left\| \widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y) \right\|_{L^\infty} = O(\epsilon^{3/2 - l})$ as $\epsilon \to 0$, and if $l \ge 2$ we have $\epsilon^{l-3/2} \widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y) \to \frac{\partial^l (b(\bar{y})(y_N + 1)^{m+1})}{\partial y_{i_1} \cdots \partial y_{i_l}}$ as $\epsilon \to 0$, uniformly in $y \in Y \times (-1, 0)$.*

Proof. First, note that the part of the statement involving the asymptotic behaviour of $\widehat{h}_\epsilon$ as $\epsilon \to 0$ follows directly from Lemma 1 and Definition 4. Assume now that $l \ge 2$. By applying formula (2.7) we have that

$$\widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y) = \sum_{S \in \mathcal{P}(l)} \frac{\epsilon^\alpha}{\epsilon^{|S|}} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j \in S} \partial y_{i_j}} \widehat{\frac{\partial^{l-|S|}}{\prod_{j \notin S} \partial x_{i_j}}} \left(\frac{x_N + \epsilon}{g_\epsilon(\bar{x}) + \epsilon} \right)^{m+1} \tag{5.10}$$

Standard Calculus computations based on Formulas (2.6) and (2.7) give

$$\begin{aligned} \widehat{\frac{\partial^{l-|S|}}{\prod_{j \notin S} \partial x_{i_j}}} \left(\frac{x_N + \epsilon}{g_\epsilon(\bar{x}) + \epsilon} \right)^{m+1} &= C(|S|)\, \epsilon^{-l+|S|} \frac{(y_N + 1)^{m+1-l+|S|}}{(\epsilon^{\alpha-1} b(\bar{y}) + 1)^{m+1}} \prod_{j \notin S} \delta_{i_j N} \\ &+ \sum_{\substack{\Lambda \in \mathcal{P}(S^C) \\ \Lambda \neq \emptyset}} \sum_{\pi \in \mathrm{Part}(\Lambda)} \epsilon^{\alpha|\pi| - |\pi| - l + |S|} (-1)^{|\pi|} \frac{(m + |\pi|)!}{m!} \frac{(m+1)!}{(m + 1 - l + |S| + |\Lambda|)!} \\ &\cdot \frac{(y_N + 1)^{m+1-l+|S|+|\Lambda|}}{(\epsilon^{\alpha-1} b(\bar{y}) + 1)^{m+1+|\pi|}} \prod_{k \in (S^C \setminus \Lambda)} \delta_{i_k N} \prod_{B \in \pi} \frac{\partial^{|B|} b(\bar{y})}{\prod_{l \in B} \partial y_{i_l}}. \end{aligned} \tag{5.11}$$

where $C(|S|) = \frac{(m+1)!}{(m+1-l+|S|)!}$. By (5.10) and (5.11) we deduce that

$$
\begin{aligned}
&\epsilon^{l-\alpha} \widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y) \\
&= \epsilon^{l-\alpha} \sum_{S \in \mathcal{P}(l)} \epsilon^{\alpha-|S|} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j \in S} \partial y_{i_j}} C(|S|) \epsilon^{-l+|S|} \frac{(y_N+1)^{m+1-l+|S|}}{(\epsilon^{\alpha-1} b(\bar{y})+1)^{m+1}} \prod_{j \notin S} \delta_{i_j N} \\
&+ \epsilon^{l-\alpha} \sum_{S \in \mathcal{P}(l)} \epsilon^{\alpha-|S|} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j \in S} \partial y_{i_j}} \sum_{\substack{\Lambda \in \mathcal{P}(S^C) \\ \Lambda \neq \emptyset}} \sum_{\pi \in \mathrm{Part}(\Lambda)} \epsilon^{|\Lambda|-|\pi|-l+|S|} (-1)^{|\pi|} \frac{(m+|\pi|)!}{m!} \\
&\cdot C(|S \cup \Lambda|) \frac{(y_N+1)^{m+1-l+|S|+|\Lambda|}}{(\epsilon^{\alpha-1}+1)^{m+1+|\pi|}} \prod_{k \in (S^C \setminus \Lambda)} \delta_{i_k N} \prod_{B \in \pi} \epsilon^{\alpha-|B|} \frac{\partial^{|B|} b(\bar{y})}{\prod_{l \in B} \partial y_{i_l}}.
\end{aligned}
\tag{5.12}
$$

It is possible to prove by direct computation that all the summands appearing in the second line in the right-hand side of (5.12) are vanishing as $\epsilon \to 0$. By letting $\epsilon \to 0$ in (5.12) we see that

$$
\begin{aligned}
\lim_{\epsilon \to 0} \epsilon^{l-\alpha} \widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y) &= \sum_{S \in \mathcal{P}(l)} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j \in S} \partial y_{i_j}} C(|S|) (y_N+1)^{m+1-l+|S|} \prod_{j \notin S} \delta_{i_j N} \\
&= \frac{\partial^l}{\partial y_{i_1} \cdots \partial y_{i_l}} \big(b(\bar{y})(y_N+1)^{m+1} \big),
\end{aligned}
$$

concluding the proof. □

Finally, we are ready to prove the following

Proposition 2. *Let $v_\epsilon \in V(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)} \leq M$ for all $\epsilon > 0$. Let $\widetilde{Y} = Y \times (-1, 0)$ and $g(y) = b(\bar{y})(1+y_N)^{m+1}$ for all $y \in \widetilde{Y}$. Moreover, let $\hat{v} \in L^2(W, w^{m,2}_{Per_Y}(Y \times (-\infty, 0)))$ be as in Lemma 3. Then*

$$
\begin{aligned}
&\int_{Q_\epsilon} D^m v_\epsilon : D^m(T_\epsilon \varphi)\,\mathrm{d}x \to \\
&\qquad - \sum_{l=1}^{m-1} \binom{m}{l+1} \int_W \int_{\widetilde{Y}} \frac{y_N^{l-1}}{(l-1)!} D_y^{l+1} \left(\frac{\partial^{m-l-1} \hat{v}(\bar{x}, y)}{\partial y_N^{m-l-1}} \right) : D_y^{l+1} g(y)\,\mathrm{d}y\, \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x}, 0) \mathrm{d}\bar{x},
\end{aligned}
$$

for all $\varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega)$, as $\epsilon \to 0$.

Proof. We set

$$
\begin{aligned}
P_1(t) &= \{\pi = (S_1, \dots, S_t) \in \mathrm{Part}(\{1, \dots, m\}) : \exists!\, S_k \text{ with } |S_k| > 1\}, \\
P_2(t) &= \{\pi \in \mathrm{Part}(\{1, \dots, m\}) : |\pi| = t, \pi \notin P_1(t)\}.
\end{aligned}
$$

We note that in the definition of $P_1(t)$ we may assume without loss of generality that the only element S_k with cardinality strictly bigger than 1 is S_1. In the sequel, we always assume that a given partition π of cardinality t is represented by $\pi = \{S_1, \dots, S_t\}$. In the following calculations, we use the index notation and we drop the summation symbols $\sum_{j_1,\dots,j_{|\pi|}=1}^N$ and $\sum_{i_1,\cdots,i_m=1}^N$. With the help of (2.8) we compute

$$\begin{aligned}
&\int_{Q_\epsilon} D^m v_\epsilon : D^m(T_\epsilon\varphi)\,\mathrm{d}x = \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}} \frac{\partial^m(\varphi\circ\Phi_\epsilon)}{\partial x_{i_1}\cdots\partial x_{i_m}}\,\mathrm{d}x \\
&= \sum_{\substack{\pi\in\mathrm{Part}(\{1,\dots,m\})\\ \pi=\{S_1,\dots,S_{|\pi|}\}}} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}} \frac{\partial^{|\pi|}\varphi}{\prod_{k=1}^{|\pi|}\partial x_{j_k}}(\Phi_\epsilon(x)) \prod_{k=1}^{|\pi|} \frac{\partial^{|S_k|}\Phi_\epsilon^{(j_k)}}{\prod_{l\in S_k}\partial x_{i_l}}\mathrm{d}x \\
&= \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}} \frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\Phi_\epsilon(x)) \frac{\partial\Phi_\epsilon^{(j_1)}}{\partial x_{i_1}}\cdots\frac{\partial\Phi_\epsilon^{(j_m)}}{\partial x_{i_m}}\mathrm{d}x, \qquad (5.13)\\
&+ \sum_{t=1}^{m-1}\sum_{\pi\in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}} \frac{\partial^t\varphi}{\prod_{k=1}^{t}\partial x_{j_k}}(\Phi_\epsilon(x)) \prod_{k=1}^{t} \frac{\partial^{|S_k|}\Phi_\epsilon^{(j_k)}}{\prod_{l\in S_k}\partial x_{i_l}}\mathrm{d}x \\
&+ \sum_{t=2}^{m-2} F_t(v_\epsilon,\varphi,\Phi_\epsilon),
\end{aligned}$$

where $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ is defined by

$$F_t(v_\epsilon,\varphi,\Phi_\epsilon) = \sum_{\pi\in P_2(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}} \frac{\partial^t\varphi}{\prod_{k=1}^{t}\partial x_{j_k}} \prod_{k=1}^{t} \frac{\partial^{|S_k|}\Phi_\epsilon^{(j_k)}}{\prod_{l\in S_k}\partial x_{i_l}} dx.$$

We consider separately the three summands in the right hand side of (5.13). Let us remark for future use that

$$\frac{\partial\Phi_\epsilon^{(k)}}{\partial x_i} = \begin{cases}\delta_{ki}, & \text{if } k\neq N,\\ \delta_{Ni}-\frac{\partial h_\epsilon}{\partial x_i}, & \text{if } k=N,\end{cases} \qquad \frac{\partial^l\Phi_\epsilon^{(k)}}{\partial x_{i_1}\cdots\partial x_{i_l}} = \begin{cases}0, & \text{if } k\neq N,\\ -\frac{\partial^l h_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_l}}, & \text{if } k=N.\end{cases}$$

for all $2\le l\le m$. Consider now the first term in the right hand side of (5.13). We unfold it by taking into account (5.4) in order to obtain

$$\begin{aligned}
&\left|\epsilon\int_{\hat W_\epsilon}\int_{\widetilde Y} \widehat{\frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}} \frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat\Phi_\epsilon(y)) \widehat{\frac{\partial\Phi_\epsilon^{(j_1)}}{\partial x_{i_1}}}\cdots\widehat{\frac{\partial\Phi_\epsilon^{(j_m)}}{\partial x_{i_m}}}\,dyd\bar x\right| \\
&= \epsilon^{-2m+1}\left|\int_{\hat W_\epsilon}\int_{\widetilde Y} \frac{\partial^m \hat v_\epsilon}{\partial y_{i_1}\cdots\partial y_{i_m}} \frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat\Phi_\epsilon(y)) \frac{\partial\widehat\Phi_\epsilon^{(j_1)}}{\partial y_{i_1}}\cdots\frac{\partial\widehat\Phi_\epsilon^{(j_m)}}{\partial y_{i_m}}\,dyd\bar x\right| \\
&\le C\epsilon^{-m+1}\epsilon^{m-1/2}\int_{\hat W_\epsilon}\int_{\widetilde Y}\left|\epsilon^{-m+1/2}\frac{\partial^m \hat v_\epsilon}{\partial y_{i_1}\cdots\partial y_{i_m}} \frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat\Phi_\epsilon(y))\right|dyd\bar x \\
&\le C\epsilon^{1/2}\left\|\epsilon^{-m+1/2}\frac{\partial^m \hat v_\epsilon}{\partial y_{i_1}\cdots\partial y_{i_m}}\right\|_{L^2(\hat W_\epsilon\times\widetilde Y)}\left\|\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat\Phi_\epsilon(y))\right\|_{L^2(\hat W_\epsilon\times\widetilde Y)} \\
&\le C\epsilon^{1/2}\left\|\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat\Phi_\epsilon(y))\right\|_{L^2(\hat W_\epsilon\times\widetilde Y)} \le C\left\|\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}\right\|_{L^2(\Phi_\epsilon(Q_\epsilon))},
\end{aligned}$$

which vanishes as $\epsilon \longrightarrow 0$. In the first inequality we have used the fact that $\left|\frac{\partial \hat{\Phi}_\epsilon^{(k)}}{\partial y_i}\right| \leq C\epsilon$, for sufficiently small $\epsilon > 0$. Let now $1 \leq t \leq m-1$ be fixed and consider

$$\begin{aligned}&\sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\prod_{k=1}^t \partial x_{j_k}}(\Phi_\epsilon(x)) \prod_{k=1}^t \frac{\partial^{|S_k|} \Phi_\epsilon^{(j_k)}}{\prod_{l \in S_k} \partial x_{i_l}} \mathrm{d}x \\ &\quad = \sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\prod_{k=1}^t \partial x_{j_k}}(\Phi_\epsilon(x)) \frac{\partial^{m-t+1} \Phi_\epsilon^{(j_1)}}{\prod_{l \in S_1} \partial x_{i_l}} \frac{\partial \Phi_\epsilon^{(j_2)}}{\partial x_{i_{S_2}}} \cdots \frac{\partial \Phi_\epsilon^{(j_t)}}{\partial x_{i_{S_t}}} \mathrm{d}x, \quad (5.14)\end{aligned}$$

where to shorten the notation we have identified $S_2, \ldots, S_t$ with the only element they contain. Note that if $j_1 \neq N$ then the integral in (5.14) is zero. Thus, without loss of generality we set $j_1 = N$. Note that we have $\frac{\partial \Phi_\epsilon^{(N)}}{\partial x_{i_t}} = \delta_{N i_t} + \frac{\partial h_\epsilon}{\partial x_{i_t}}$ and $\left|\frac{\partial h_\epsilon}{\partial x_{i_t}}\right| \leq C\epsilon^{1/2}$ as $\epsilon \longrightarrow 0$. In order to simplify the expressions we will not write down the higher order terms in ϵ. Hence, by setting $j_1 = N$ in (5.14) we deduce that the lower order terms in (5.14) are given by

$$\begin{aligned}&\sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\partial x_N \partial x_{j_{S_2}} \cdots \partial x_{j_{S_t}}}(\Phi_\epsilon) \frac{\partial^{m-t+1} \Phi_\epsilon^{(N)}}{\prod_{l \in S_1} \partial x_{i_l}} \delta_{i_{S_2} j_2} \cdots \delta_{i_{S_t} j_N} \mathrm{d}x \\ &= \sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}}(\Phi_\epsilon) \frac{\partial^m v_\epsilon}{\prod_{l \in S_1} \partial x_{i_l} \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}} \frac{\partial^{m-t+1} \Phi_\epsilon^{(N)}}{\prod_{l \in S_1} \partial x_{i_l}} \mathrm{d}x \qquad (5.15) \\ &= \binom{m}{t-1} \int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}}(\Phi_\epsilon) \frac{\partial^m v_\epsilon}{\prod_{l \in S_1} \partial x_{i_l} \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}} \frac{\partial^{m-t+1} \Phi_\epsilon^{(N)}}{\prod_{l \in S_1} \partial x_{i_l}} \mathrm{d}x,\end{aligned}$$

where in the last equality in (5.15) we have used the fact that each of the summands

$$\int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}}(\Phi_\epsilon) \frac{\partial^m v_\epsilon}{\prod_{l \in S_1} \partial x_{i_l} \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}} \frac{\partial^{m-t+1} \Phi_\epsilon^{(N)}}{\prod_{l \in S_1} \partial x_{i_l}} \mathrm{d}x$$

equals

$$\int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}}(\Phi_\epsilon) D^{m-t+1} \left(\frac{\partial^{t-1} v_\epsilon}{\partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}} \right) : D^{m-t+1} \Phi_\epsilon^{(N)} \mathrm{d}x.$$

and in particular they do not depend on the choice of π (note that the cardinality of $P_1(t)$ equals $\binom{m}{t-1}$). By unfolding the right-hand side of (5.15) and using the fact that $m-t+1 \geq 2$ we have that

$$\begin{aligned}&\binom{m}{t-1} \epsilon \int_{\hat{W}_\epsilon} \int_{\widetilde{Y}} \frac{\partial^t \varphi}{\partial x_N \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}}(\hat{\Phi}_\epsilon(y)) \frac{\widehat{\partial^m v_\epsilon}}{\prod_{l \in S_1} \partial x_{i_l} \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}} \frac{\widehat{\partial^{m-t+1} \Phi_\epsilon^{(N)}}}{\prod_{l \in S_1} \partial x_{i_l}} dy d\bar{x} \\ &= -\binom{m}{t+1} \frac{\epsilon}{\epsilon^m} \int_{\hat{W}_\epsilon} \int_{\widetilde{Y}} \frac{\partial^m \hat{v}_\epsilon}{\prod_{l \in S_1} \partial y_{i_l} \partial y_{i_{S_2}} \cdots \partial y_{i_{S_t}}} \frac{\partial^t \varphi}{\partial x_N \partial x_{i_{S_2}} \cdots \partial x_{i_{S_t}}}(\hat{\Phi}_\epsilon(y)) \frac{\widehat{\partial^{m-t+1} h_\epsilon}}{\prod_{l \in S_1} \partial x_{i_l}} dy d\bar{x}.\end{aligned} \qquad (5.16)$$

It is easy to see that the final expression appearing in the right-hand side of (5.16) can be written as

$$\begin{aligned}-\binom{m}{t+1} \int_{\hat{W}_\epsilon} \int_{\widetilde{Y}} &\left[\epsilon^{-m+1/2} \frac{\partial^m \hat{v}_\epsilon}{\prod_{l \in S_1} \partial y_{i_l} \partial y_{i_{S_2}} \cdots \partial y_{i_{S_t}}} \right] \\ &\cdot \left[\frac{1}{\epsilon^{m-t-1}} \frac{\partial^t \varphi}{\partial x_N \partial y_{i_{S_2}} \cdots \partial y_{i_{S_t}}}(\hat{\Phi}_\epsilon(y)) \right] \left[\epsilon^{m-t+1-3/2} \frac{\widehat{\partial^{m-t+1} h_\epsilon}}{\prod_{l \in S_1} \partial x_{i_l}} \right] dy d\bar{x}. \quad (5.17)\end{aligned}$$

Now
$$\epsilon^{-m+1/2}\frac{\partial^m \hat{v}_\epsilon}{\prod_{l\in S_1}\partial y_{i_l}\partial y_{i_{S_2}}\cdots\partial y_{i_{S_t}}}\to\frac{\partial^m \hat{v}}{\prod_{l\in S_1}\partial y_{i_l}\partial y_{i_{S_2}}\cdots\partial y_{i_{S_t}}},$$
weakly in $L^2(\widehat{W}_\epsilon\times Y\times(-1,0))$ as $\epsilon\to 0$, by Lemma 3, and
$$\epsilon^{m-t+1-3/2}\frac{\widehat{\partial^{m-t+1}h_\epsilon}}{\prod_{l\in S_1}\partial x_{i_l}}\to\frac{\partial^{m-t+1}(b(\bar{y})(1+y_N)^{m+1})}{\prod_{l\in S_1}\partial y_{i_l}},$$
in $L^\infty(\widehat{W}_\epsilon\times Y\times(-1,0))$ as $\epsilon\to 0$, by Lemma 4. Moreover, by Lemma 6 in the Appendix it follows that
$$\frac{1}{\epsilon^{m-t-1}}\frac{\partial^t\varphi}{\partial x_N^t}(\hat{\Phi}_\epsilon(y))\to\frac{y_N^{m-t-1}}{(m-t-1)!}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0),$$
and
$$\frac{1}{\epsilon^{m-t-1}}\frac{\partial^t\varphi}{\partial x_N\partial x_{i_{S_2}}\cdots\partial x_{i_{S_t}}}(\hat{\Phi}_\epsilon(y))\to 0,$$
strongly in $L^2(W\times Y\times(-1,0))$ as $\epsilon\to 0$, if at least one of the indexes $i_{S_2},\dots,i_{S_N}$ is not equal to N. Hence (5.17) tends to
$$-\binom{m}{t+1}\int_W\int_{Y\times(-1,0)}\frac{y_N^{m-t-1}}{(m-t-1)!}D_y^{m-t+1}\left(\frac{\partial^{t-1}\hat{v}}{\partial y_N^{t-1}}\right):D_y^{m-t+1}\Big(b(\bar{y})(1+y_N)^{m+1}\Big)dy\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)d\bar{x}.$$
By setting $m-t=l$ we recover the limiting expression in the statement. Then, in order to conclude the proof it is sufficient to prove that the integrals in $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ vanish as $\epsilon\to 0$. We will show this by comparing each integral appearing in the definition of $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ with the corresponding integral of the form (5.14), which is convergent as $\epsilon\to 0$, hence it is uniformly bounded in ϵ. Note that by Lemma 4
$$\frac{\partial^{m-t+1}\hat{\Phi}_\epsilon^{(j_1)}}{\prod_{l\in S_1}\partial y_{i_l}}\frac{\partial\hat{\Phi}_\epsilon^{(j_2)}}{\partial y_{i_{S_2}}}\cdots\frac{\partial\hat{\Phi}_\epsilon^{(j_t)}}{\partial y_{i_{S_t}}}=O(\epsilon^{3/2+t-1})=O(\epsilon^{1/2+t}),$$
for all $\pi\in P_1(t)$, whereas if we consider $\pi'=(S_1',\dots,S_t')\in P_2(t)$ with $|S_1'|=m-t<m-t+1$ there must exists S_k', $k>1$ with $|S_k'|=2$. Let us assume that $k=2$. Then we have
$$\frac{\partial^{m-t}\hat{\Phi}_\epsilon^{(j_1)}}{\prod_{l\in S_1'}\partial y_{i_l}}\frac{\partial^2\hat{\Phi}_\epsilon^{(j_2)}}{\prod_{l\in S_2'}\partial y_{i_l}}\frac{\partial\hat{\Phi}_\epsilon^{(j_3)}}{\partial y_{i_{S_3'}}}\cdots\frac{\partial\hat{\Phi}_\epsilon^{(j_t)}}{\partial y_{i_{S_t'}}}=O(\epsilon^{3/2+t}\epsilon^{3/2-2})=O(\epsilon^{1+t}),$$
and since $\epsilon^{1+t}=o\big(\epsilon^{1/2+t}\big)$ as $\epsilon\to 0$ and the integral (5.14) is bounded, we deduce that the integral in $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ involving
$$\frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}\frac{\partial^t\varphi}{\prod_{k=1}^t\partial x_{j_k}}\frac{\partial^{m-t}\hat{\Phi}_\epsilon^{(j_1)}}{\prod_{l\in S_1'}\partial y_{i_l}}\frac{\partial^2\hat{\Phi}_\epsilon^{(j_2)}}{\prod_{l\in S_2'}\partial y_{i_l}}\frac{\partial\hat{\Phi}_\epsilon^{(j_3)}}{\partial y_{i_{S_3'}}}\cdots\frac{\partial\hat{\Phi}_\epsilon^{(j_t)}}{\partial y_{i_{S_t'}}},$$
for all $\pi'\in P_2(t)$ defined above, vanishes as $\epsilon\to 0$. By arguing in a similar way for all the terms in $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ we deduce the validity of the statement. □

We summarise the previous discussion in the following

Theorem 8. *Let $f_\epsilon \in L^2(\Omega_\epsilon)$, $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\Omega)$. Let $g(y) = b(\bar{y})(1+y_N)^{m+1}$ for all $y \in Y \times (-1,0)$. Moreover, let us assume that $v_\epsilon \in W^{m,2}(\Omega_\epsilon) \cap W^{m-1,2}_0(\Omega_\epsilon)$ is the solution to $H_{\Omega_\epsilon,S} v_\epsilon = f_\epsilon$ for all $\epsilon > 0$. Then there exist $v \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$ and a function $\hat{v}$ in the space $L^2(W, w^{m,2}_{Per_Y}(Y \times (-\infty,0)))$ such that, possibly passing to a subsequence, $v_\epsilon \rightharpoonup v$ in $W^{m,2}(\Omega)$, $v_\epsilon \to v$ in $L^2(\mathbb{R}^N)$, and statements (a) and (b) in Lemma 3 hold. Moreover, the following integral equality holds*

$$-\sum_{l=1}^{m-1}\binom{m}{l+1}\int_W\int_{Y\times(-1,0)}\left[\frac{y_N^{l-1}}{(l-1)!}D_y^{l+1}\left(\frac{\partial^{m-l-1}\hat{v}(\bar{x},y)}{\partial y_N^{m-l-1}}\right):D_y^{l+1}g(y)\right]\mathrm{d}y\,\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x}$$
$$+\int_\Omega D^m v : D^m\varphi + u\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x. \quad (5.18)$$

for all $\varphi \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$.

Notation. We will use the following notation:

$$q_Y(f,g) := \sum_{l=1}^{m-1}\binom{m}{l+1}\int_{Y\times(-1,0)}\left[\frac{y_N^{l-1}}{(l-1)!}D_y^{l+1}\left(\frac{\partial^{m-l-1}f(\bar{x},y)}{\partial y_N^{m-l-1}}\right):D_y^{l+1}g(y)\right]\mathrm{d}y$$

for all $f \in L^2(W, w^{m,2}_{Per_Y}(Y\times(-\infty,0)))$, $g \in C^m_{Per_Y}(Y\times(-1,0))$. We refer to

$$-\int_W q_Y(\hat{v},g)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x} \quad (5.19)$$

as the *strange term* appearing in the homogenization.

5.2 Critical case - Microscopic problem.

The aim of this section is to characterize the strange term (5.19) as the energy of a suitable polyharmonic function and in particular to conclude that it is different from zero. We will use periodically oscillating test functions matching the intrinsic ϵ-scaling of the problem.
Let then $\psi \in C^\infty(\overline{W}\times\overline{Y}\times]-\infty,0])$ be such that $\mathrm{supp}\,\psi \subset C\times\overline{Y}\times[d,0]$ for some compact set $C \subset W$ and for some $d \in (-\infty,0)$. Moreover, assume that $\psi(\bar{x},\bar{y},0) = D^l\psi(\bar{x},\bar{y},0) = 0$ for all $(\bar{x},\bar{y}) \in W\times Y$, for all $1 \le l \le m-2$. Let also ψ be Y-periodic in the variable $\bar{y}$. We set

$$\psi_\epsilon(x) = \epsilon^{m-\frac{1}{2}}\psi\Big(\bar{x},\frac{\bar{x}}{\epsilon},\frac{x_N}{\epsilon}\Big),$$

for all $\epsilon > 0$, $x \in W\times]-\infty,0]$. Then $T_\epsilon\psi_\epsilon \in V(\Omega_\epsilon)$ for all sufficiently small ϵ, hence we can use it as a test function in the weak formulation of the problem in Ω_ϵ, getting

$$\int_{\Omega_\epsilon} D^m v_\epsilon : D^m T_\epsilon\psi_\epsilon\,\mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x.$$

It is not difficult to prove that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x \to 0, \qquad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x \to 0 \quad (5.20)$$

as $\epsilon \longrightarrow 0$. By arguing as in [8, §8.4], it is also possible to prove that

$$\int_{\Omega_\epsilon \setminus \Omega} D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \,\mathrm{d}x \longrightarrow 0, \tag{5.21}$$

as $\epsilon \longrightarrow 0$. Moreover, a suitable modification of [8, Lemma 8.47] yields

$$\int_\Omega D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \,\mathrm{d}x \longrightarrow \int_{W\times Y\times(-\infty,0)} D_y^m \hat{v}(\bar{x}, y) : D_y^m \psi(\bar{x}, y)\,\mathrm{d}\bar{x}\mathrm{d}y. \tag{5.22}$$

Theorem 9. *Let $\hat{v} \in L^2(W, w^{m,2}_{Per_Y}(Y \times (\infty, 0)))$ be the function from Theorem 8. Then*

$$\int_{W\times Y\times(-\infty,0)} D_y^m \hat{v}(\bar{x}, y) : D_y^m \psi(\bar{x}, y)\ \mathrm{d}\bar{x}\,\mathrm{d}y = 0, \tag{5.23}$$

for all $\psi \in L^2(W, w^{m,2}_{Per_Y}(Y \times (\infty, 0)))$ such that $\psi(\bar{x}, \bar{y}, 0) = D_y^l \psi(\bar{x}, \bar{y}, 0) = 0$ for all $(\bar{x}, \bar{y}) \in W \times Y$, for all $1 \le l \le m-2$. Moreover, for any $j = 1, \dots, N-1$, we have

$$\frac{\partial^{m-1}\hat{v}}{\partial y_j \partial y_N^{m-2}}(\bar{x}, \bar{y}, 0) = -\frac{\partial b}{\partial y_j}(\bar{y})\frac{\partial^{m-1} v}{\partial x_N^{m-1}}(\bar{x}, 0), \qquad \text{on } W \times Y, \tag{5.24}$$

and

$$\frac{\partial^{m-1}\hat{v}}{\partial y_{i_1} \cdots \partial y_{i_{m-1}}}(\bar{x}, \bar{y}, 0) = 0, \qquad \text{on } W \times Y, \tag{5.25}$$

for all $i_1, \dots, i_{m-1} = 1, \dots, N-1$.

Proof. The first part of the statement follows from (5.20), (5.21) and (5.22) by arguing as in [8, Theorem 8.53]. In order to prove formulas (5.24) and (5.25) we note that, since $D^{m-2} v_\epsilon(\bar{x}, g_\epsilon(\bar{x})) = 0$ for all $\bar{x} \in W$, we have

$$\frac{\partial^{m-2} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}}}(\bar{x}, g_\epsilon(\bar{x})) = 0, \qquad \text{for all } i_1, \dots, i_{m-2} = 1, \dots, N, \bar{x} \in W.$$

Differentiating with respect to x_j, $j \in \{1, \dots, N-1\}$ yields

$$\frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_j}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_j} = 0,$$

for all $\bar{x} \in W$. Hence, by setting

$$V_\epsilon^j = \left(0, \dots, 0, -\frac{\partial^{m-1} v_\epsilon}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-2}}}, 0, \dots, 0, \frac{\partial^{m-1} v_\epsilon}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_{m-2}}}\right),$$

for all $i_1, \dots, i_{m-2} = 1, \dots, N$, $j = 1, \dots, N-1$, where the only non-zero entries are the j-th and the N-th, we obtain that $V_\epsilon^j \cdot n_\epsilon = 0$, on Γ_ϵ, where n_ϵ is the outer normal to $\Gamma_\epsilon \equiv \{(\bar{x}, g_\epsilon(\bar{x})) : \bar{x} \in W\}$. By using Lemma 3

$$\frac{\widehat{\frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_j}} - \int_Y \widehat{\frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_j}}(\bar{x}, \bar{y}, 0) d\bar{y}}{\sqrt{\epsilon}} \overset{\epsilon \to 0}{\rightharpoonup} \frac{\partial^{m-1}\hat{v}}{\partial y_{i_1} \cdots \partial y_{i_{m-2}} \partial y_j},$$

in $L^2(W \times Y \times]d, 0[)$ for any $d < 0$. This combined with [20, Lemma 4.3] (see also [8, Lemma 8.56]) yields

$$\frac{\partial^{m-1}\hat{v}}{\partial y_{i_1}\cdots\partial y_{i_{m-2}}\partial y_j}(\bar{x},\bar{y},0) = -\frac{\partial b}{\partial y_j}(\bar{y})\frac{\partial^{m-1}v}{\partial x_N\partial x_{i_1}\cdots\partial x_{i_{m-2}}}(\bar{x},0),$$

for all $(\bar{x},\bar{y}) \in W \times Y$, $i_1,\dots,i_{m-2} = 1,\dots,N$, $j = 1,\dots,N-1$. We deduce that since $v \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega)$, then $D^{m-2}\, v(\bar{x},0) = 0$ for all $x \in W$. This implies that all the derivatives $\frac{\partial^{m-1}v}{\partial x_N\partial x_{i_1}\cdots\partial x_{i_{m-2}}}(\bar{x},0)$, where one of the indexes i_k is different from N, are zero. This concludes the proof. □

Now we have the following

Lemma 5. *There exists $V \in w^{m,2}_{Per_Y}(Y \times (-\infty,0))$ satisfying the equation*

$$\int_{Y\times(-\infty,0)} D^mV : D^m\psi \;\mathrm{d}y = 0, \tag{5.26}$$

for all $\psi \in w^{m,2}_{Per_Y}(Y \times (-\infty,0))$ such that $D^l\psi(\bar{y},0) = 0$ on Y, for all $0 \le l \le m-2$, and the boundary conditions

$$\begin{cases} \frac{\partial^l V}{\partial y_N^l}(\bar{y},0) = 0, \text{ for all } l = 0,\dots,m-3, & \text{on } Y,\\ \frac{\partial^{m-2}V}{\partial y_N^{m-2}}(\bar{y},0) = b(\bar{y}), & \text{on } Y. \end{cases}$$

The function V is unique up to the sum of a monomial in y_N of degree $m-1$ of the type ay_N^{m-1} with $a \in \mathbb{R}$. Moreover $V \in W^{2m,2}_{Per_Y}(Y \times (d,0))$ for any $d < 0$ and it satisfies the equation

$$(-\Delta)^m V = 0, \qquad \text{in } Y \times (d,0),$$

subject to the boundary conditions

$$\begin{cases} \frac{\partial^l V}{\partial n^l}(\bar{y},0) = 0, & \text{on } Y, \text{ for all } 0 \le l \le m-3,\\ \frac{\partial^{m-2}V}{\partial y_N^{m-2}}(\bar{y},0) = b(\bar{y}), & \text{on } Y,\\ \frac{\partial^m V}{\partial y_N^m}(\bar{y},0) = 0, & \text{on } Y. \end{cases}$$

Proof. Similar to the proof of [8, Lemma 8.60]. We just note that in order to deduce the classical formulation of problem (5.26) it is sufficient to choose test functions ψ as in the statement with bounded support in the y_N direction. By using the Polyharmonic Green Formula (4.3) we then deduce that

$$\int_{Y\times(-\infty,0)} D^mV : D^m\psi \,\mathrm{d}y = (-1)^m \int_{Y\times(-\infty,0)} \Delta^m V\psi \,\mathrm{d}y + \int_Y \frac{\partial^m V}{\partial y_N^m}\frac{\partial^{m-1}\psi}{\partial y_N^{m-1}}\, d\bar{y}.$$

By the arbitrariness of ψ it is then easy to conclude the proof. □

Theorem 10. *Let V be as in Lemma 5 and $g(y) = b(\bar{y})(1+y_N)^{m+1}$, for all $y \in Y \times (-1,0)$. Then*

$$q_Y(V,g) = \int_{Y\times(-\infty,0)} |D^mV|^2 \,\mathrm{d}y. \tag{5.27}$$

Furthermore

$$\int_{Y\times(-\infty,0)} |D^m V|^2 \,\mathrm{d}y = -\int_Y \left(\frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}} + (m-1)\Delta_{N-1}\left(\frac{\partial^{m-1}V}{\partial x_N^{m-1}}\right)\right) b(\bar{y})\,\mathrm{d}\bar{y}. \tag{5.28}$$

Proof. Let ϕ be the real-valued function defined on $Y\times]-\infty,0]$ by $\phi(y) = \frac{y_N^{m-2}}{(m-2)!}g(y)$ if $-1 \le y_N \le 0$ and $\phi(y) = 0$ if $y_N < -1$. Then $\phi \in W^{m,2}(Y\times(-\infty,0))$, $\frac{\partial^l \phi}{\partial y_N^l}(\bar{y},0) = 0$ for all $0 \le l \le m-3$, and

$$\frac{\partial^{m-2}\phi}{\partial y_N^{m-2}}(\bar{y},0) = b(\bar{y}), \qquad \text{for all } y \in Y. \tag{5.29}$$

Now note that the function $\psi = V - \phi$ is a suitable test-function in equation (5.26); by plugging it in (5.26) we deduce that $\int_{Y\times(-\infty,0)} |D^m V|^2\,\mathrm{d}y = \int_{Y\times(-1,0)} D^m V : D^m\phi\,\mathrm{d}y$. By the Leibnitz rule we have that

$$\int_{Y\times(-1,0)} D^m V : D^m \phi\,\mathrm{d}y$$
$$= \int_{Y\times(-1,0)} \frac{\partial^m V}{\partial x_{j_1}\cdots\partial x_{j_m}} \sum_{S\in\mathcal{P}(m)} \frac{1}{(m-2)!} \frac{\partial^{|S|} y_N^{m-2}}{\prod_{j\in S}\partial x_{i_j}} \frac{\partial^{(n-|S|)} g}{\prod_{j\notin S}\partial x_{i_j}}\, dy. \tag{5.30}$$

Using the obvious fact that

$$\frac{\partial^{m-k} y_N^{m-2}}{\partial x_{i_1}\cdots\partial x_{i_{m-k}}} = \begin{cases} 0, & \text{if } k = 0,1; \\ y_N^{k-2}\delta_{i_1 N}\cdots\delta_{i_{m-k}N}, & \text{for } k \ge 2. \end{cases}$$

we can rewrite the right-hand side of (5.30) as follows

$$\sum_{k=2}^m \binom{m}{k} \int_{Y\times(-1,0)} D^k\left(\frac{\partial^{m-k}V(y)}{\partial y_N^{m-k}}\right) : \left(\frac{y_N^{k-2}}{(k-2)!} D^k g(y)\right) dy$$
$$= \sum_{k=2}^m \binom{m}{k} \int_{Y\times(-1,0)} \frac{y_N^{k-2}}{(k-2)!} D^k\left(\frac{\partial^{m-k}V(y)}{\partial y_N^{m-k}}\right) : D^k g(y)\, dy,$$

which coincides with the left-hand side of (5.27) up to the change of summation index defined by $k = l+1$. Finally, (5.28) follows by applying the polyharmonic Green formula (4.3) on $\int_{Y\times(-1,0)} D^m V : D^m\phi\,\mathrm{d}y$. Indeed, we note that the boundary integrals on $\partial Y \times (-1,0)$ are zero, due to the periodicity of V and b. Moreover the boundary integral on $\partial Y \times \{-1\}$ is zero since ϕ vanishes there together with all its derivatives. Then, the only non-trivial boundary integral is supported on $Y\times\{0\}$. More precisely, we have

$$\int_{Y\times(-1,0)} D^m V : D^m\phi\,\mathrm{d}y = (-1)^m \int_{Y\times(-1,0)} \Delta^m V \phi\,\mathrm{d}y + \sum_{t=0}^{m-1} \int_Y B_t(V)(\bar{y},0) \frac{\partial^t \phi(\bar{y},0)}{\partial y_N^t}\,\mathrm{d}\bar{y}, \tag{5.31}$$

and by recalling that $\Delta^m V = 0$ in $Y\times(-1,0)$, $\frac{\partial^m V}{\partial y_N^m} = 0$ on $Y\times\{0\}$, $\frac{\partial^l \phi}{\partial y_N^l} = 0$ on $Y\times\{0\}$, for all $0 \le l \le m-3$ and by (5.29), we deduce that

$$\int_{Y\times(-1,0)} D^m V : D^m \phi\,\mathrm{d}y = \int_Y B_{m-2}(V)(\bar{y},0) b(\bar{y})\,\mathrm{d}\bar{y}$$

and by formula (4.3) $B_{m-2}(V)(\bar{y},0) = -\sum_{l=m-2}^{m-1}\binom{l}{m-2}\Delta_{N-1}^{l-m+2}(\frac{\partial^{m-1}}{\partial y_N^{m-1}}(\Delta^{m-l-1}V))$, from which we deduce (5.28). □

Theorem 11. *Let $m \in \mathbb{N}$, $m \geq 2$. Let V be as in Lemma 5. Let $v, \hat{v}$ be the functions defined in Theorem 8. Let also $g(y) = b(\bar{y})(1+y_N)^{m+1}$ for all $y \in Y \times (-1,0)$. Then*

$$\hat{v}(\bar{x},y) = -V(y)\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0) + a(x)y^{m-1},$$

for some $a(\bar{x}) \in L^2(W)$. Moreover, the strange term (5.19) *is given by*

$$\begin{aligned}
-\int_W q_Y(\hat{v},g)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x} &= \int_{Y\times(-\infty,0)}|D^mV|^2dy\int_W \frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x}\\
&= -\int_Y\left(\frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}} + (m-1)\Delta_{N-1}\left(\frac{\partial^{m-1}V}{\partial x_N^{m-1}}\right)\right)b(\bar{y})\mathrm{d}\bar{y}\int_W \frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x}.
\end{aligned}$$

Proof. The proof follows by Lemma 5 and Theorems 9, 10 and by observing that $-V(y)\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)$ satisfies problem (5.23) with the boundary conditions (5.24). □

We are now ready to conclude the proof of (iii) of Theorem 7.

Proof of Theorem 7(iii). Define $g(y) = b(\bar{y})(1+y_N)^{m+1}$ for all $y = (\bar{y}, y_N)$ in $Y \times (-1,0)$. The function v in Theorem 8 satisfies

$$\int_W q_Y(V,g)\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x} + \int_\Omega D^mv : D^m\varphi + u\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x. \tag{5.32}$$

for all $\varphi \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$. By Theorem 11 we can rewrite the first integral on the left-hand side of (5.32) as

$$\int_{Y\times(-\infty,0)}|D^mV|^2dy\int_W \frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\,\mathrm{d}\bar{x}$$

and by the Green Formula (4.10) for all $\varphi \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$

$$\int_\Omega D^mv : D^m\varphi\,dx = (-1)^m\int_\Omega \Delta^m v\varphi + \int_{\partial\Omega}\frac{\partial^m v}{\partial n^m}\frac{\partial^{m-1}\varphi}{\partial n^{m-1}}\,dS. \tag{5.33}$$

Hence, in the weak formulation of the limiting problem we find the following boundary integral

$$\int_W\left(\frac{\partial^m v}{\partial x_N^m}(\bar{x},0) + \left(\int_{Y\times(-\infty,0)}|D^mV|^2\,dy\right)\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\right)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\,\mathrm{d}\bar{x}, \tag{5.34}$$

for all $\varphi \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$. By (5.32), (5.33), (5.34) and the arbitrariness of φ we deduce the statement of Theorem 7, part (iii). □

6 Appendix

In this section we prove the following technical result used in the proof of Proposition 2.

Lemma 6. *Let $l, m \in \mathbb{N}$, $m \geq 2$, $1 \leq l \leq m-1$, $i_1, \dots, i_{m-l-1} \in \{1, \dots, N\}$. Then for all $\varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega)$ we have*

$$\frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\hat{\Phi}_\epsilon(y)) \to \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x}, 0),$$

in $L^2(W \times Y \times (-1, 0)$ as $\epsilon \to 0$ and if at least one of the indexes $i_1, \dots, i_{m-l-1}$ does not coincide with N we also have

$$\frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-l-1}}}(\hat{\Phi}_\epsilon(y)) \to 0$$

in $L^2(W \times Y \times (-1, 0)$ as $\epsilon \to 0$.

Proof. Note that for $l = 1$ the claim follows by Lemma 3. Then assume $l > 1$. Fix $\varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega) \cap C^\infty(\Omega)$. Then

$$\begin{aligned}
&\int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \left| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\hat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x}, 0) \right|^2 d\bar{x}dy \\
&= \int_{-1}^0 \sum_{k \in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_Y \left| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}\Big(\epsilon\Big[\frac{\bar{x}}{\epsilon}\Big] + \epsilon\bar{y}, \epsilon y_N - h_\epsilon\Big(\epsilon\Big[\frac{\bar{x}}{\epsilon}\Big] + \epsilon\bar{y}, \epsilon y_N\Big)\Big) \right. \\
&\qquad\qquad \left. - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x}, 0) \right|^2 d\bar{y}d\bar{x}dy_N \qquad (6.1) \\
&= \int_{-1}^0 \sum_{k \in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \left| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}\Big(\bar{z}, \epsilon y_N - h_\epsilon\Big(\bar{z}, \epsilon y_N\Big)\Big) \right. \\
&\qquad\qquad \left. - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x}, 0) \right|^2 d\bar{x}\frac{d\bar{z}}{\epsilon^{N-1}}dy_N.
\end{aligned}$$

Now, let $\bar{z} \in C_\epsilon^k$ be fixed. By expanding φ in Taylor's series with remainder in Lagrange form we deduce that

$$\frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}\big(\bar{z}, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N)\big) = \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) \frac{(\epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N))^{l-1}}{(l-1)!},$$

for some $\xi \in (0, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N))$. We then deduce that the term appearing in the right-hand side of (6.1) can be rewritten as

$$\int_{-1}^0 \sum_{k \in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \left| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) \frac{(\epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N))^{l-1}}{(l-1)!} \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x}, 0) \right|^2 d\bar{x}\frac{d\bar{z}}{\epsilon^{N-1}}dy_N. \qquad (6.2)$$

We then estimate (6.2) from above. Note that

$$\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\frac{(\epsilon y_N-h_\epsilon(\bar z,\epsilon y_N))^{l-1}}{(l-1)!}\frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&\le\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\left(\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right)\frac{y_N^{l-1}}{(l-1)!}\right.\\
&\quad\left.+\sum_{s=1}^{l-1}\binom{l-1}{s}\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)(\epsilon y_N)^{l-1-s}(-h_\epsilon(\bar z,\epsilon y_N))^s\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N
\end{aligned}\tag{6.3}$$

and the right-hand side of (6.3) is estimated from above by

$$\begin{aligned}
&\le C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&+C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&+C\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\left|\frac{1}{\epsilon^{l-1}}(\epsilon y_N)^{l-1-s}|h_\epsilon(\bar z,\epsilon y_N)|^s\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N.
\end{aligned}\tag{6.4}$$

Now we consider separately the three integrals on the right-hand side of (6.4). The first integral can be estimated in the following way

$$\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&=\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\int_0^\xi\frac{\partial^{m}\varphi}{\partial x_N^{m}}(\bar z,t)dt\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\le C\epsilon\left\|\frac{\partial^{m}\varphi}{\partial x_N^{m}}\right\|^2_{L^2(W\times(-c\epsilon,0))},
\end{aligned}\tag{6.5}$$

Now consider the second integral in (6.4). We have the following estimate

$$\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&=\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2\frac{|\bar z-\bar x|^N}{|\bar z-\bar x|^N}d\bar x\frac{d\bar z}{\epsilon^{N-1}}\\
&\le C\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)}{|\bar z-\bar x|^{N/2}}\right|^2\epsilon^N d\bar x\frac{d\bar z}{\epsilon^{N-1}}\\
&\le C\epsilon\left\|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right\|^2_{B_2^{1/2}(W)}\le C\epsilon\left\|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right\|^2_{W^{2,2}(\Omega)},
\end{aligned}\tag{6.6}$$

where we have used the classical Trace Theorem and the standard Besov space $B_2^{1/2}(W)$ of exponents $2, 1/2$. Finally we consider the third integral in (6.4), which is easily estimated by using Lemma 1 as follows:

$$
\begin{aligned}
&\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\left|\frac{1}{\epsilon^{l-1}}(\epsilon y_N)^{l-1-s}|h_\epsilon(\bar z,\epsilon y_N)|^s\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&\le C\epsilon^{N-1}\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\left(\frac{1}{\epsilon^{l-1}}(\epsilon)^{l-1-s}|C\epsilon^{3/2}|^s\right)^2\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&\le C\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\epsilon^s d\bar z dy_N\le C\epsilon\left\|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}\right\|^2_{W^{1,2}(\Omega)}.
\end{aligned}
\tag{6.7}
$$

By using (6.5), (6.6), (6.7) in (6.2) we deduce that

$$
\begin{aligned}
\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}&\left|\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\frac{(\epsilon y_N-h_\epsilon(\bar z,\epsilon y_N))^{l-1}}{(l-1)!}\right.\\
&\left.-\frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\le C\epsilon\|\varphi\|_{W^{m,2}(\Omega)}\longrightarrow 0,
\end{aligned}
\tag{6.8}
$$

as $\epsilon\longrightarrow 0$. This concludes the proof in the case of smooth functions.

Now, if $\varphi\in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$, by [15, Theorem 9, p.77] there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}\subset W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)\cap C^\infty(\Omega)$ such that

$$\varphi_n\longrightarrow\varphi,\qquad \text{in } W^{m,2}(\Omega_\epsilon),$$

as $n\longrightarrow\infty$ hence $\mathrm{Tr}_{\partial\Omega}D^\eta\varphi_n=\mathrm{Tr}_{\partial\Omega}D^\eta\varphi$ for all $|\eta|\le m-1$. Then

$$
\begin{aligned}
&\left\|\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat\Phi_\epsilon(y))-\frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right\|_{L^2(\widehat W_\epsilon\times Y\times(-1,0))}\\
&\le\left\|\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat\Phi_\epsilon(y))-\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\widehat\Phi_\epsilon(y))\right\|_{L^2(\widehat W_\epsilon\times Y\times(-1,0))}\\
&+\left\|\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\widehat\Phi_\epsilon(y))-\frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar x,0)\right\|_{L^2(\widehat W_\epsilon\times Y\times(-1,0))}\\
&+\left\|\frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar x,0)-\frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right\|_{L^2(\widehat W_\epsilon\times Y\times(-1,0))}.
\end{aligned}
\tag{6.9}
$$

By using Lemma 2, a Trace Theorem, Poincaré inequality and a typical diagonal argument, it is not difficult to see that right hand-side of (6.9) tends to zero as $\epsilon\longrightarrow 0$, concluding the proof of the first part of the statement.

The second part of the second statement can be proved as follows. By assumption, at least one of the indexes i_j it is different from N. This implies that the function $\frac{\partial^{m-l}\varphi}{\partial x_N\partial x_{i_1}\cdots\partial x_{i_{m-l-1}}}$ is not only in $W^{l,2}(\Omega)\cap W_0^{l-1,2}(\Omega)$ but also in $W_{0,W}^{l,2}(\Omega)$. Thus, formula

(5.4) and an iterated application of the Poincaré inequality in the x_N direction, $l-1$ times, yield

$$\left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-l-1}}}(\hat{\Phi}_\epsilon(y)) \right\|_{L^2(W\times Y\times(-1,0)} \leq C \left\| \frac{\partial^{m-1}\varphi}{\partial x_N^l \partial x_{i_1} \cdots \partial x_{i_{m-l-1}}}(\hat{\Phi}_\epsilon(y)) \right\|_{L^2(W\times Y\times(-1,0)}$$

which allows to conclude since the right-hand side of the previous inequality tends to zero as $\epsilon \to 0$ in virtue of Lemma 3(ii) and of the vanishing of the trace of $\frac{\partial^{m-1}\varphi}{\partial x_N^l \partial x_{i_1} \cdots \partial x_{i_{m-l-1}}}$ on W.

□

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