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Published
DOI:10.1093/logcom/exz006

Uniform Interpolation for Propositional and Modal Team Logics

Giovanna D'Agostino
Department of Math., Comp. Science, and Physics,
University of Udine, Via delle Scienze 206, 33100 Udine, Italy
giovanna.dagostino@uniud.it

Abstract

In this paper we discuss the property of uniform interpolation in Propositional and Modal Team Logics

1 Introduction

Interpolation is a desirable property for a logic. In very general terms it states that if a formula G is a consequence of a formula F, then only the common language between the two formulas is important, because G is also a consequence of a formula in the common language. Uniform interpolation is stronger than Craig interpolation, in the sense that the interpolant between F and G only depends on F and on the common language between F and G, but not on G itself: in a logic enjoying uniform interpolation, given a sublanguage L of the language of F, if G_1, G_2 are two consequences of F having L as common language with F, then both G_1, G_2 will be logical conseuqnce of the uniform interpolant of F w.r.t. L.

Although uniform interpolation is a stronger property than Craig interpolation, it is, in some way, more stable. Suppose we have two logics, L_1, L_2 where L_1 is more expressive than L_2, and enjoys Craig interpolation. Hence, if ϕ, ψ are L_2 formulas with $\phi \models \psi$, then ϕ, ψ have an interpolant in L_1, which, however, could be a formula which is not equivalent to a formula in L_2. On the other hand, we will prove that $\mathcal{FPTL}$ enjoys uniform interpolation and we that the uniform interpolant of a $\mathcal{PDEP}$ formula is equivalent to a $\mathcal{PDEP}$ formula. More generally, if L_1, L_2 are as above, the logic L_1 enjoys uniform interpolation, and L_2 has some nice semantical characterization inside L_1, we can try to prove that the L_1 uniform interpolant of a formula in L_2 is (equivalent to) an L_2-formulas.

As we shall see, this strategy proved to be quite fruitful for team logics.

2 Propositional and Modal Team Logics

In the sequel *Prop* denotes a nonempty set of propositional letters. A team X over *Prop* is a set of valuations, where a valuation s is a function $s : Prop \to \{0,1\}$.

Team formulas are built from literals $p, \neg p$ (for $p \in Prop$) and the constant $\perp$ using conjunction $\wedge$ and the team disjunction $\otimes$, and are interpreted on teams as follows:

$$
\begin{array}{lcl}
X \models \perp & \Leftrightarrow & X = \emptyset \\
X \models p_i & \Leftrightarrow & s(p_i) = 1 \ \text{ for all } \ s \in X \\
X \models \neg p_i & \Leftrightarrow & s(p_i) = 0 \ \text{ for all } \ s \in X \\
X \models \phi_1 \wedge \phi_2 & \Leftrightarrow & X \models \phi_1 \text{ and } \ X \models \phi_2 \\
X \models \phi_1 \otimes \phi_1 & \Leftrightarrow & \exists X_1, X_2 \ \ X = X_1 \cup X_2, \ X_1 \models \phi_1, \ X_2 \models \phi_2.
\end{array}
$$

We use the constant $\top$ to denote the formula $p \otimes \neg p$, for a propositional variable p in the language; notice that $\top$ is true in any team. Moreover, we consider also the non empty disjunction $\circledast$, classical disjunction $\vee$, and a constant for non-emptiness NE:

$$
\begin{array}{lcl}
X \models \phi_1 \circledast \phi_2 & \Leftrightarrow & X = \emptyset \text{ or } \exists X_1 \neq \emptyset, X_2 \neq \emptyset, \ X = X_1 \cup X_2, \\
 & & X_1 \models \phi_1, \ X_2 \models \phi_2 \\
X \models \phi \vee \psi & \Leftrightarrow & X \models \phi \text{ or } X \models \psi \\
X \models NE & \Leftrightarrow & X \neq \emptyset.
\end{array}
$$

Next, we add the modal operators:

Definition 2.1 *The formulas of Modal Logic $\mathcal{ML}$ are defined by*

$$\alpha := \quad p \mid \neg p \mid \perp \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \otimes \alpha_2 \mid \Diamond\alpha_1 \mid \Diamond\alpha_2$$

where $p \in Prop$.

To interpret the modality operators $\Box\phi, \Diamond\phi$ we enrich the semantics by considering teams X as subsets of the set of words of a Kripke model, defined, as usual, as a tuple $M = (W, R, V)$, where W is a non empty set, $R \subseteq W \times W$ is the *accessibility* relation, and $V : W \to Pow(Prop)$.

We use the following notation for $X, Y \subseteq W$:

$$R(X) := \{s \in W \ : \ \exists t \in X \ sRt\};$$

$$XRY \Leftrightarrow \forall x \in X \exists y \in Y \ xRy \wedge \forall y \in Y \exists x \in X \ xRy.$$

Definition 2.2 *(Modal Team Semantics)* *If $M = (W, R, V)$ is a Kripke model and $X \subseteq W$ is a team in M then the semantics of $\mathcal{ML}$ is defined as follows*

$$\begin{array}{lcl}
(M,X) \models p & \Leftrightarrow & s \in V(p), \ \textit{for all} \ \ s \in X \\
(M,X) \models \neg p & \Leftrightarrow & s \notin V(p), \ \textit{for all} \ \ s \in X \\
(M,X) \models \bot & \Leftrightarrow & X = \emptyset \\
(M,X) \models \alpha_1 \wedge \alpha_2 & \Leftrightarrow & (M,X) \models \alpha_1 \ \textit{and} \ \ (M,X) \models \alpha_2 \\
(M,X) \models \alpha_1 \otimes \alpha_1 & \Leftrightarrow & \exists X_1, X_2 \ \ (M,X_1) \models \alpha_1, \ (M,X_2) \models \alpha_2 \\
 & & \ \ \textit{and} \ \ X = X_1 \cup X_2 \\
(M,X) \models \Diamond\alpha & \Leftrightarrow & \exists Y \ XRY \ \textit{and} \ \ (M,Y) \models \alpha \\
(M,X) \models \Box\alpha & \Leftrightarrow & (M, R(X)) \models \alpha
\end{array}$$

If the team X is a singleton, then the semantics of $\mathcal{ML}$ formulas coincides with the classical modal semantics on Kripke model, with $\otimes$ behaving as a standard disjunction. If $\phi \in \mathcal{ML}$ we call *singleton semantics* the usual semantics of modal logic, where formulas are interpreted over pointed Kripke models (M, w) and not over teams (M, X).

2.1 Dependence, Inclusion, Independence

Team semantics allow us to consider various notions of dependence between data. To this end, new 'atoms' are added to the basic framework discussed above (both in the propositional and in the modal case). In this paper we consider dependence atoms $=\!(\overline{\alpha}, \gamma)$, inclusions atoms $\overline{\alpha} \subseteq \overline{\alpha}'$, independence atoms $\overline{\alpha} \bot \overline{\beta}$, where $\overline{\alpha}, \overline{\alpha}', \overline{\beta}$ are sequences of formulas in $\mathcal{ML}$ (with α, α' of the same lenght), and $\gamma \in \mathcal{ML}$.

Definition 2.3 *[4, 5, 3, 2]*
The formulas of Modal Dependence Logic $\mathcal{MDL}$ are defined by

$$\phi := \quad p \mid \neg p \mid \bot \mid \phi_1 \wedge \phi_2 \mid \phi_1 \otimes \phi_2 \mid =\!(\overline{\alpha}, \gamma) \mid \Diamond\phi \mid \Box\phi$$

where $p \in Prop$, $\overline{\alpha} = \alpha_1, \ldots, \alpha_h$, and α_i, γ are formulas in $\mathcal{ML}$.

The formulas of Modal Inclusion Logic $\mathcal{MINC}$ are defined by

$$\phi := \quad p \mid \neg p \mid \bot \mid \phi_1 \wedge \phi_2 \mid \phi_1 \otimes \phi_2 \mid \overline{\alpha} \subseteq \overline{\alpha}' \mid \Diamond\phi \mid \Box\phi$$

where $p \in Prop$, $\overline{\alpha} = \alpha_1, \ldots, \alpha_h$, $\overline{\alpha}' = \alpha'_1, \ldots, \alpha_h$ and $\alpha_i, \alpha'_j \in \mathcal{ML}$.

The formulas of Modal Independence Logic $\mathcal{MIND}$ are defined by

$$\phi := \quad p \mid \neg p \mid \bot \mid \phi_1 \wedge \phi_2 \mid \phi_1 \otimes \phi_2 \mid \overline{\alpha} \bot \overline{\beta} \mid \Diamond\phi \mid \Box\phi$$

where $p \in Prop$, $\overline{\alpha} = \alpha_1, \ldots, \alpha_h$, $\overline{\beta} = \beta_1, \ldots, \beta_k$ *and* $\alpha_i, \beta_j \in \mathcal{ML}$.

Notice that the new atoms $=\!(\overline{\alpha}, \gamma)$, $\overline{\alpha} \subseteq \overline{\alpha}'$, $\overline{\alpha} \perp \overline{\beta}$ are defined only on $\mathcal{ML}$ formula and cannot be nested. By convention, we use letters $\alpha, \beta, \ldots$ to denote $\mathcal{ML}$ formulas, and letters $\phi, \psi, \ldots$ to denote formulas in $\mathcal{MDL}, \mathcal{MINC}, \mathcal{MIND}$.

To give a semantics to the new atoms, given a Kripke model $M = (W, R, V)$ and $s \in W$, we consider the valuation functions M_s on $\mathcal{ML}$-formulas defined as follows:

$$M_s(\alpha) = \begin{cases} 1, \text{ if } M, s \models \alpha; \\ 0, \text{ if } M, s \not\models \alpha, \end{cases}$$

where $M, s \models \alpha$ denote the usual (singeton) semantics for modal formulas. Moreover, if $\overline{\alpha} = \alpha_1, \ldots, \alpha_h$ is a sequence of formulas in $\mathcal{ML}$ we define $M_s(\overline{\alpha}) = (M_s(\alpha_1), \ldots, M_s(\alpha_h))$.

Definition 2.4 *The semantics of the new atoms is defined over a Kripke team model* (M, X) *as follows:*

$$(M, X) \models \ =\!(\overline{\alpha}, \beta) \ \Leftrightarrow$$
$$\forall s, s' \in X \ (M_s(\overline{\alpha}) = M_{s'}(\overline{\alpha}) \Rightarrow M_s(\beta) = M_{s'}(\beta));$$
$$(M, X) \models \ \overline{\alpha} \subseteq \overline{\alpha}' \ \Leftrightarrow$$
$$\forall s \in X \ \exists s' \in X \quad M_s(\overline{\alpha}) = M_{s'}(\overline{\alpha}');$$
$$(M, X) \models \ \overline{\alpha} \perp \overline{\beta} \ \Leftrightarrow$$
$$\forall s \forall s' \in X \ \exists s''(M_{s''}(\overline{\alpha}) = M_s(\overline{\alpha}) \wedge M_{s''}(\overline{\beta}) = M_{s'}(\overline{\beta})).$$

In correspondence to any Modal Team Logic we also consider its propositional variant, which has the same syntax except for the absence of the modal operators. Hence, we consider Propositional Dependence Logic, Propositional Inclusion Logic, and Propositional Independence Logic. In the propositional case, the new atoms are evaluated over a set X of valuations $s : Prop \to \{0, 1\}$, as expected: e.g.

$$X \models \ =\!(\overline{\alpha}, \beta) \Leftrightarrow \qquad \forall s, s' \in X \ (s(\overline{\alpha}) = s'(\overline{\alpha}) \Rightarrow s(\beta) = s'(\beta)).$$

In the following table we list all team logics we will consider in this paper:

LOGIC	ATOMS	CONN.	MOD. OPER.
Classical Prop.Team Logic $\mathcal{CPL}$	$p_i, \neg p_i, \bot$	$\wedge, \otimes$	–
Prop. Dependence Logic $\mathcal{PDEP}$	$p_i, \neg p_i, \bot, =(\overline{\alpha}, \beta)$	$\wedge, \otimes$	–
Prop. Inclusion Logic $\mathcal{PINC}$	$p_i, \neg p_i, \bot, \overline{\alpha} \subseteq \overline{\beta}$	$\wedge, \otimes$	–
Prop. Independence Logic $\mathcal{PIND}$	$p_i, \neg p_i, \bot, \overline{\alpha} \bot \overline{\beta}$	$\wedge, \otimes$	–
Full Prop.Team Logic $\mathcal{FPTL}$	$p_i, \neg p_i, \bot, NE$	$\wedge, \otimes, \vee$	–
Modal Team Logic $\mathcal{ML}$	$p_i, \neg p_i, \bot$	$\wedge, \otimes$	$\Box$, $\Diamond$
Modal Dependence Logic $\mathcal{MDEP}$	$p_i, \neg p_i, \bot, =(\overline{\alpha}, \beta)$	$\wedge, \otimes$	$\Box$, $\Diamond$
Modal Inclusion Logic $\mathcal{MINC}$	$p_i, \neg p_i, \bot, \overline{\alpha} \subseteq \overline{\beta}$	$\wedge, \otimes$	$\Box$, $\Diamond$
Modal Independence Logic $\mathcal{MIND}$	$p_i, \neg p_i, \bot, \overline{\alpha} \bot \overline{\beta}$	$\wedge, \otimes$	$\Box$, $\Diamond$
Full Modal Team Logic $\mathcal{FMTL}$	$p_i, \neg p_i, \bot, NE$	$\wedge, \otimes, \vee$	$\Box$, $\Diamond$

Table 1: A list of (Modal) Team Logics

When L is one of the logic above, we denote by $(L, \models)$ the pair consisting of the logic and its consequence relation, defined as usual.

3 Uniform Interpolation in the Propositional Team Context

If ϕ is a formula of a team logic we denote by $\mathcal{L}(\phi) \subseteq Prop$ the finite set of proposition from which ϕ is constructed.

Let ϕ and ψ be two formulas in a team logic $(L, \models)$ such that $\phi \models \psi$. Then θ is an *interpolant* of ϕ, ψ iff:

1. $\phi \models \theta$ and $\theta \models \psi$;
2. $\mathcal{L}(\theta) \subseteq \mathcal{L}(\phi) \cap \mathcal{L}(\psi)$. $\Box$

In words: if $\phi \models \psi$, an interpolant of ϕ, ψ is a formula in the common language of ϕ and ψ which sits in between ϕ and ψ.

Definition 3.1 *Given a formula ϕ and a language $\mathcal{L}' \subseteq \mathcal{L}(\phi)$, the* uniform interpolant *of ϕ with respect to $\mathcal{L}'$ is a formula θ such that:*

1. *$\phi \models \theta$;*
2. *Whenever $\phi \models \psi$ and $\mathcal{L}(\phi) \cap \mathcal{L}(\psi) \subseteq \mathcal{L}'$ then $\theta \models \psi$.*
3. *$\mathcal{L}(\theta) \subseteq \mathcal{L}'$.* $\Box$

When we say that a logic has (uniform) interpolation we mean that we can always find a (uniform) interpolant when the appropriate conditions are satisfied. Clearly, if a logic has uniform interpolation, it also enjoys Craig interpolation. For if $\phi \models \psi$, simply choose $\mathcal{L}' = \mathcal{L}(\phi) \cap \mathcal{L}(\psi)$. The interpolant between ϕ and ψ is then the uniform interpolant of ϕ relative to $\mathcal{L}'$. This explains why we call this formula a *uniform* interpolant: no information is needed about the formula ψ except which non-logical symbols it has in common with ϕ.

Before proving uniform interpolation in the modal team context, we recall the easy proof of uniform interpolation for Classical Propositional Logic, and show that it cannot be applied to the propositional team context, except for the case of $\mathcal{CPL}$ (which is the simpler logic with team semantics we shall consider).

In Classical Propositional (singleton) Logic[1] it is well know (and easy to prove) that the formula $\phi[p|\top] \vee \phi[p|\bot]$ is a uniform interpolant for ϕ with respect to $\mathcal{L}(\phi) \setminus \{p\}$. Moreover, we can iterate this construction to obtain a uniform interpolant with respect to any subset of $\mathcal{L}(\phi)$. This immediately implies that Classical Propositional Team Logic $\mathcal{CPL}$ enjoys uniform interpolation, the uniform interpolant of a propositional formula $\phi(p)$ with respect to $\mathcal{L}(\phi) \setminus \{p\}$ being again $\phi[p|\top] \otimes \phi[p|\bot]$.

Note that all formulas in Classical Propositional Team Logic are downward closed, union closed, and local, where a formula ϕ is:

Definition 3.2 *(see [1])*

1. *local, if* $X \models \phi$ *and* $Y =_{L(\phi)} X$ *implies* $Y \models \phi$,

$$(\textit{where } \; Y =_{\mathcal{L}(\phi)} X \;\Leftrightarrow\; \begin{array}{l} \forall s \in Y \exists s' \in X\, s(p) = s'(p), \forall p \in \mathcal{L}(\phi) \\ \forall s \in X \exists s' \in Y\, s(p) = s'(p), \forall p \in \mathcal{L}(\phi)); \end{array}$$

2. *downward closed, if* $X \models \phi$ *and* $Y \subseteq X$ *implies* $Y \models \phi$;
3. *union closed, if* $X_1 \models \phi$ *and* $X_2 \models \phi$ *implies* $X_1 \cup X_2 \models \phi$.

We next show how these properties play a separate role in order to ensure that the formula $\phi[p|\top] \otimes \phi[p|\bot]$ is a a uniform interpolant for a formul ϕ in $\mathcal{CPL}$, with respect to $\mathcal{L}(\phi) \setminus \{p\}$. To this end, we have first to recall some lemma on substitutions in a team context.

Given a team X, we define

$$X[p|\top] := \{s[p|\top] : s \in X\}, \; X[p|\bot] := \{s[p|\bot] : s \in X\},$$

where

$$s[p|\top](q) = \begin{cases} s(q) & \text{if } \; q \neq p \\ 1 & \text{if } \; q = p \end{cases} \qquad s[p|\bot](q) = \begin{cases} s(q) & \text{if } \; q \neq p \\ 0 & \text{if } \; q = p \end{cases}$$

[1] that is: propositional logic with the usual semantics, which coincides with the team semantics ristrected to singleton teams

In general, team logics do not have a good notion of substitution, unless we restrict to classical substitutions. In particular, if we define $\phi[p|\top], \phi[p|\bot]$ by induction, as usual:

$$\phi[p|\top] := \begin{cases} \top, & \text{if } \phi = p; \\ \bot, & \text{if } \phi = \neg p \text{ or } \phi = \bot; \\ \phi_1[p|\top] \circ \phi_1[p|\top], & \text{if } \phi = \phi_1 \circ \phi_2, \circ \in \{\otimes, \vee, \wedge\}; \\ \overline{\alpha}[p|\top] \subseteq \beta[p|\top] & \text{if } \phi = \overline{\alpha} \subseteq \beta; \\ \vdots \end{cases}$$

$$\phi[p|\bot] := \begin{cases} \bot, & \text{if } \phi = p; \\ \top, & \text{if } \phi = \neg p; \\ \phi_1[p|\bot] \circ \phi_1[p|\bot], & \text{if } \phi = \phi_1 \circ \phi_2, \circ \in \{\otimes, \vee, \wedge\}; \\ \overline{\alpha}[p|\bot] \subseteq \beta[p|bot] & \text{if } \phi = \overline{\alpha} \subseteq \beta; \\ \vdots \end{cases}$$

then the syntactic substitution reflects on the semantics side as follows (see [1]):

Lemma 3.1 *If ϕ is a propositional team formula and X is a team then*

$$X[p|\top] \models \phi \Leftrightarrow X \models \phi[p|\top]; \quad X[p|\bot] \models \phi \Leftrightarrow X \models \phi[p|\bot].$$

In particular, if $X \models p$, then $X \models \phi \Leftrightarrow X \models \phi[p|\top]$.
Similarly, if $X \models \neg p$, then $X \models \phi \Leftrightarrow X \models \phi[p|\bot]$.

Lemma 3.2 *If ϕ is downward closed then $\phi \models \phi[p|\top] \otimes \phi[p|\bot]$.*

Proof Suppose $X \models \phi$. Consider $X_0 = \{s \in X : s(p) = 0\}$ and $X_1 = \{s \in X : s(p) = 1\}$. By downward closure, $X_0 \models \phi$ and hence $X_0 \models \phi[p|\bot]$ by the observation above. Similarly, $X_1 \models \phi[p|\top]$, and $X \models \phi[p|\top] \otimes \phi[p|\bot]$ follows. □

However, if ϕ is not downward closed, the previous lemma does not hold, as the following example shows.

Example 3.1 *If $\phi := (p \wedge q) \circledast (\neg p \wedge q)$, then $\phi[p|\top] \otimes \phi[p|\bot]$ is not a logical consequence of ϕ. In particular, $\phi[p|\top] \otimes \phi[p|\bot]$ is not a uniform interpolant of ϕ.*

Proof $\phi[p|\top] = (\top \wedge q) \circledast (\bot \wedge q)$, and $\phi[p|\bot] = (\bot \wedge q) \circledast (\top \wedge q)$; both formulas are easily seen to be true only for the empty team, hence $X \models \phi[p|\top] \otimes \phi[p|\bot]$ implies $X = \emptyset$. On the other hand, ϕ is satisfied by the non empty team $X = \{s_1, s_2\}$ with $s_1(p) = s_1(q) = s_2(q) = 1$, and $s_2(p) = 0$. Hence, $\phi[p|\top] \otimes \phi[p|\bot]$ is not a logical consequence of ϕ. □

Lemma 3.3 *Suppose ϕ is union closed and ψ is a local formula such that $\phi \models \psi$ and $p \notin \mathcal{L}(\psi)$. Then*

$$\phi[p|\top] \otimes \phi[p|\bot] \models \psi$$

Proof Suppose $X \models \phi[p|\top] \otimes \phi[p|\bot]$. Then there are X_1, X_2 such that $X = X_1 \cup X_2$ and $X_1 \models \phi[p|\top]$, $X_2 \models \phi[p|\bot]$. By Lemma 3.1 we obtain $X_1[p|\top] \models \phi$, and $X_2[p|\bot] \models \phi$, and, by union closure, $X_1[p|\top] \cup X_2[p|\bot] \models \phi$. From $\phi \models \psi$ we obtain that $X_1[p|\top] \cup X_2[p|\bot] \models \psi$, and hence $X \models \psi$, since ψ is local and

$$X =_{L(\psi)\setminus\{p\}} (X_1[p|\top] \cup X_2[p|\bot]).$$

□

If ϕ is not union closed, the previous lemma does not hold, as the following example shows.

Example 3.2 *Consider the formula*

$$\phi(p, q) := \ =\!(p, q) \wedge =\!(p)$$

We have:

$$\phi[p|\top] \equiv \ =\!(\top, q) \wedge =\!(\top) \equiv \ =\!(q) \equiv \ \phi[p|\bot]$$

Hence, $\phi[p|\top] \otimes \phi[p|\bot] \equiv =\!(q) \otimes =\!(q) \equiv \top$. On the other hand, it is clear that $\phi \models =\!(q)$, although $\top \not\models =\!(q)$. It follows that the formula $\phi[p|\top] \otimes \phi[p|\bot]$ is not a uniform interpolant for ϕ with respect to $\mathcal{L}(\phi) \setminus \{p\}$.

Hence, if we consider a propositional team logic which is not downward closed or not union closed, we cannot prove uniform interpolation using the formula $\phi[p|\top] \otimes \phi[p|\bot]$. On the other hand, one can easily check that in Example (3.2) the formula $=\!(q)$ is the correct uniform interpolant in any local logic containing ϕ: one can easily verify that $\phi \models =\!(q)$; moreover, suppose $\phi \models \psi$ with $p \notin \mathcal{L}(\psi)$, and $X \models =\!(q)$; then if $Y := X[p|\top]$ we have $Y \models \phi$ and hence $Y \models \psi$; but then $X \models \psi$ because $Y =_{\mathcal{L}(\phi)} X$ and ψ is local.

As we shall see, at least for Propositional Dependence Logic and for Propositional Inclusion Logic we can prove uniform interpolation. Similarly, uniform interpolation for Modal Team Logic can be easily proved from uniform interpolation of standard modal logic, but this easy proof cannot be used for other team modal logics, where we have to use other means.

4 Expressiveness of Team Logics

Given a formula ϕ in a propositional logic with team semantics we denote by $||\phi||$ the class of team models of ϕ:

$$||\phi|| = \{X : X \models \phi\}.$$

If L is a fragment of Full Propositional Team Logic $\mathcal{FPTL}$, we denote by $Team_L$ the class of team models of formulas in L:

$$Team_L = \{||\phi|| : \phi \in L\}$$

We say that a team propositional logic L is expressively complete for a class of team properties $\mathcal{X}$ if,

$$\mathcal{X} = Team_L,$$

that is: for every formula $\phi \in L$, the set of team satisfying ϕ belongs to $\mathcal{X}$ and, moreover, every team property belonging to $\mathcal{X}$ coincides with the set of team satisfying a formula in L. We have:

Lemma 4.1 *[1]*

1. *$\mathcal{CPL}$ is expressively complete for the class of non empty, downward and union closed team properties;*
2. *$\mathcal{PDEP}$ is expressively complete for the class of non empty, downward closed team properties;*
3. *$\mathcal{PINC}$ is expressively complete for the class of non empty, union closed team properties;*
4. *$\mathcal{FPT}$ is expressively complete for the class of all team properties.*

In order to state an analogous lemma for modal team logics, we consider modal team properties as sets of modal team models (M, X), and define:

Definition 4.1 *A team property $\mathcal{C}$ is:*

1. downward closed: $(M, X) \in \mathcal{C}$ *and* $Y \subseteq X$ *implies* $(M, Y) \in \mathcal{C}$*;*
2. union closed: $(M, X_i) \in \mathcal{C}$ *implies* $(M, \bigcup_i X_i) \in \mathcal{C}$.

Moreover, to state the expressiveness results, we need the notion of team (bounded) bisimulation, a generalization of the usual notion of (bounded) bisimulation:

Definition 4.2 *If $M = (W, R, V), M' = (W', R', V')$ are Kripke models, $(w, w') \in M \times M'$, and $k \in \mathbb{N}$, we say that $(M, w), (M', w')$ are k-bisimilar (notation: $(M, w) \rightleftharpoons^k (M', w')$), iff for all $i \leq k$ there is $B_i \subseteq M \times M'$ such that $(w, w') \in B_k$, and for all $(v, v') \in B_{i+1}$ it holds:*

1. $V(v) = V(v')$;
2. if vRu there exists u' such that $v'R'u'$ and $(u, u') \in B_i$;
3. if $v'Ru'$ there exists u such that vRu and $(u, u') \in B_i$.

We say that $(M, w), (M', w')$ are bisimilar (notation: $(M, w) \rightleftharpoons (M', w')$), iff there exists $B \subseteq M \times M'$ such that $(w, w') \in B$, and for all $(v, v') \in B$ it holds:

1. $V(v) = V(v')$;
2. if vRu there exists u' such that $v'R'u'$ and $(u, u') \in B$;
3. if $v'Ru'$ there exists u such that vRu and $(u, u') \in B$.

We shall also consider bisimulations and bounded bisimulation where condition 1. above is resticted to a subset $\mathcal{P}$ of propositions, that is, we require that $V(v) \cap \mathcal{P} = V(v') \cap \mathcal{P}$. Two models $(M, w), (N, v)$ which are bisimilar (k-bisimilar) w.r.t. the propositions in $\mathcal{P}$ are denoted by $(M, w) \rightleftharpoons_{\mathcal{P}} (M', w')$, $((M, w) \rightleftharpoons^k_{\mathcal{P}} (M', w')$, respectively).

The notion of bisimulation is extended to team models as follows.

Definition 4.3 *Let $\mathcal{P}$ be a set of propositional variables. The team models $(M, X), (N, Y)$ are $\mathcal{P}$-bisimilar if*

$$\forall x \in X\ \exists y \in Y (M, x) \rightleftharpoons_{\mathcal{P}} (N, y) \text{ and } \forall y \in Y\ \exists x \in X (M, x) \rightleftharpoons_{\mathcal{P}} (N, y).$$

Team bisimilar models are denoted by $(M, X) \rightleftharpoons_{\mathcal{P}} (N, Y)$.

Remark 4.1 *By considering the maximal bisimulation between models, if $(M, X) \rightleftharpoons_{\mathcal{P}} (N, Y)$ we may suppose w.l.o.g. that there exists a $\mathcal{P}$-bisimulation B between M and N such that*

$$\forall x \in X\ \exists y \in Y (x, y) \in B \text{ and } \forall y \in Y\ \exists x \in X (x, y) \in B.$$

One can easily prove that all formulas ϕ in the logics listed on table 1. are invariant under bisimulation, that is,

$$(M, X) \models \phi \text{ and } (M, X) \rightleftharpoons_{\mathcal{L}(\phi)} (N, Y) \text{ implies } (N, Y) \models \phi.$$

Similarly, if $md(\phi)$, the modal depth of ϕ, is defined as the maximal number of nested modal operators in ϕ, we have:

$$(M, X) \models \phi \text{ and } (M, X) \rightleftharpoons^k_{\mathcal{L}(\phi)} (N, Y) \text{ implies } (N, Y) \models \phi.$$

The expressiveness results for modal team logics are based on the following definition:

Definition 4.4

- *A class $\mathcal{K}$ of team models is* bisimulation invariant *if*

$$(M, X) \in \mathcal{K} \quad \textit{and } (M, X) \rightleftharpoons (N, Y) \quad \textit{implies } (N, Y) \in \mathcal{K}.$$

- *A class $\mathcal{K}$ of team models is* first order definable *if there exists a first order formula $\phi(V)$ with a monadic variable V in the language $\{R, =, P_1, \ldots, P_n, \ldots\}$, where R is a binary relational symbol representing the accessibilty relation and $P_1, \ldots, P_n, \ldots$ are unary relational symbol representing the propositions, such that, for all team model (M, X) it holds:*

$$(M, X) \in \mathcal{K} \Leftrightarrow M, V := X \models \phi(V),$$

where $M, V := X$ on the right M is considered as a first order model for the language $\{R, =, P_1, \ldots, P_n, \ldots\}$, and we interpret the monadic variable V by the set X.

Given a formula ϕ in a modal logic with team semantics we denote by $||\phi||$ the class of team models of ϕ:

$$||\phi|| = \{(M, X) : (M, X) \models \phi\}.$$

As in the propositional case, we say that a team modal logic L is expressively complete for a class of team properties $\mathcal{X}$ if

$$\mathcal{X} = Team_L,$$

where $Team_L = \{||\phi|| : \phi \in L\}$. We have:

Theorem 4.1 *[2, 3, 4]*

1. *$\mathcal{ML}$ is expressively complete for the class of all first order definable, non empty, downward and union closed, bisimulation invariant team properties;*
2. *$\mathcal{MDEP}$ is expressively complete for the class of all first order definable, non empty, downward closed, bisimulation invariant team properties;*
3. *$\mathcal{MINC}$ is expressively complete for the class of all first order definable, non empty, union closed, bisimulation invariant team properties;*
4. *Full Modal Team Logic $\mathcal{FMTL}$ is expressively complete for the class of all first order definable, bisimulation invariant team properties.*

5 Amalgamation

To prove uniform interpolation we need the notion of amalgamation, defined below.

Lemma 5.1 *Let $\mathcal{P}, \mathcal{Q}$ are sets of propositions and let B be a $\mathcal{P} \cap \mathcal{Q}$-bisimulation between M, N with $B \neq \emptyset$. The B-*amalgamation *K of M, N, is a Kripke model over the propositions $\mathcal{P} \cup \mathcal{Q}$ defined as follows:*
-The domain of K is the relation B.
- The accessibility relation R^K is given by

$$(m,n)R^K(m',n') \ \Leftrightarrow \ mR^M m' \quad \text{and } nR^N n'.$$

For all propositional variables $r \in \mathcal{P} \cup \mathcal{Q}$ and $(m,n) \in B$ we have

$$(m,n) \models r \ \Leftrightarrow \ \begin{cases} (M,m) \models r, \ \text{if} \ \ r \in \mathcal{P} \\ (N,n) \models r, \ \text{if} \ \ r \in \mathcal{Q}. \end{cases}$$

Then the projection over the first component is a $\mathcal{P}$-bisimulation between K and M, while the projection over the second component is a $\mathcal{Q}$-bisimulation between K and N.

We next show that the amalgamation property of Kripke models extends easily to team models:

Lemma 5.2 *If $\mathcal{P}, \mathcal{Q}$ are sets of propositions and (M, X), (N, Y) are team models such that*

$$(M,X) \rightleftharpoons_{\mathcal{P}\cap\mathcal{Q}} (N,Y) \tag{1}$$

then there is a team model (K, Z) such that

$$(M,X) \rightleftharpoons_{\mathcal{P}} (K,Z) \rightleftharpoons_{\mathcal{Q}} (N,Y)$$

Proof Let B be the bisimulation witnessing $(M,X) \rightleftharpoons_{\mathcal{P}\cap\mathcal{Q}} (N,Y)$ as in Remark 4.1; consider the B-amalgamation K of M, N as defined in 5.1.

We define $Z = (X \times Y) \cap B$; if $x \in X$ then, since B is a $\mathcal{P} \cap \mathcal{Q}$-bisimulation between the team models (M, X) and (N, Y), there exists $y \in Y$ such that $(x, y) \in B$. Hence, the pair (x, y) belongs to Z, and the projection over the first component is a witness fort a $\mathcal{P}$-team bisimulation between (K, Z) and (M, X). Similarly the projection over the first component is a witness fort a $\mathcal{Q}$-team bisimulation between (K, Z) and (N, Y). □

6 Bisimulation Quantifiers and Uniform Interpolation in Modal Team Logic

Given a logic L with modal team semantics such that all formulas are bisimulation invariant, we extend its syntax by means of the *existential bisimulation quantifier*, $\tilde{\exists} p\ \phi$, obtaining the logic $\tilde{\exists} L$. E.g. a formula ϕ in $\tilde{\exists}\mathcal{ML}$ is defined by:

$$\phi := p \mid \neg p \mid \bot \mid \phi_1 \wedge \phi_2 \mid \phi_1 \otimes \phi_2 \mid \Diamond\phi \mid \Box\phi \mid \tilde{\exists} p\ \phi.$$

The semantics of $\tilde{\exists} p\ \phi$ is defined as follows: for any team model (M, X) over a set of proposition $\mathcal{P}$ and for any $p \in \mathcal{P}$ it holds:

$$(M, X) \models \tilde{\exists} p\ \phi \Leftrightarrow \tag{2}$$

$$\exists (M', X') \rightleftharpoons_{Free(\phi)\setminus\{p\}} (M, X) \text{ and } (M', X') \models \phi,$$

where the set of free variables $Free(\phi)$ of a formula $\psi \in \tilde{\exists} L$ are defined inductively as expected, stipulating that $Free(\tilde{\exists} p\ \phi) = Free(\phi) \setminus \{p\}$. One can easily prove that all formulas in $\tilde{\exists} L$ are bisimulation invariant:

Lemma 6.1 *Suppose all formula of L are bisimulation invariant, $\phi \in \tilde{\exists} L$, $(M, X) \models \phi$, and $(M, X) \rightleftharpoons_{Free(\phi)} (N, Y)$ then $(N, Y) \models \phi$.*

Moreover, existential bisimulation quantifiers in $\tilde{\exists} L$ are related to uniform interpolants as follows:

Lemma 6.2 *Consider a modal team logic L, invariant under bisimulation, and let $\tilde{\exists} L$ be its existential bisimulation extension. If ϕ is a formula of $\tilde{\exists} L$ then $\tilde{\exists} p\ \phi$ is a uniform interpolant for the formula ϕ in $\tilde{\exists} L$, with respect to $Free(\phi) \setminus \{p\}$.*

Proof It is clear that $\phi \models \tilde{\exists} p\ \phi$ and, by definition, $Free(\tilde{\exists} p\ \phi) = Free(\phi) \setminus \{p\}$.
Suppose $\phi \models \psi$ with $\psi \in \tilde{\exists} L$ and $Free(\phi) \cap Free(\psi) \subseteq Free(\phi) \setminus \{p\}$, that is, $p \notin Free(\psi)$. We prove that $\tilde{\exists} p\ \phi \models \psi$. If $(M, X) \models \tilde{\exists} p\ \phi$ then there exists (M', X') such that

$$(M', X') \rightleftharpoons_{Free(\phi)\setminus\{p\}} (M, X)$$

and $(M', X') \models \phi$. Since $Free(\phi) \cap Free(\psi) \subseteq Free(\phi) \setminus \{p\}$, by the amalgamation property proved in Lemma 5.2, there exists a team model (N, Y) such that

$$(M', X') \rightleftharpoons_{Free(\phi)} (N, Y) \rightleftharpoons_{Free(\psi)} (M, X).$$

Since $(M', X') \models \phi$, the first bisimulation implies $(N, Y) \models \phi$. Then, from $\phi \models \psi$ we obtain $(N, Y) \models \psi$, and from $(N, Y) \rightleftharpoons_{Free(\psi)} (M, X)$ we finally have $(M, X) \models \psi$. $\square$

Lemma 6.2 allows to use, in the Modal Team context, the well known strategy that consists on proving uniform interpolation in a logic L by showing that, for any formula $\phi \in L$ there is a formula $\theta \in L$ which is equivalent to $\tilde{\exists} p\ \theta$. Our first task is to use this strategy to prove uniform interpolation for Full Modal Team Logic $\mathcal{FMTL}$. Notice that we cannot apply Theorem 4.1 directly, proving that for all $\phi \in \mathcal{FMTL}$ the formula $\tilde{\exists} p\ \phi \in \mathcal{FMTL}$, because, although we proved that the property expressed by $\tilde{\exists} p\ \phi$ is bisimulation invariant, we do not know whether it is an FO-property.

In the following we prove that the existential bisimulation quantifier commutes with both disjunctions $\otimes$, $\vee$, and with the non-emptyness atom NE. First we prove that the singleton semantics of the bisimulation quantifier over classical modal formulas is equivalent to its team semantics.

Lemma 6.3 *Suppose $\theta, \phi \in \mathcal{ML}$, and θ behaves as $\tilde{\exists} p\ \phi$ w.r.t. singleton semantics, that is, for all Kripke models (M, w) it holds*

$$(M, w) \models \theta \Leftrightarrow \exists (N, v) \rightleftharpoons_{\mathcal{L}(\phi)\setminus\{p\}} (M, w) \ \textit{and} \ \ (N, v) \models \phi;$$

then θ is equivalent to $\tilde{\exists} p\ \phi$ in the modal team semantics, that is, for all team models (M, X) it holds

$$(M, X) \models \theta \Leftrightarrow \exists (N, Y) \rightleftharpoons_{\mathcal{L}(\phi)\setminus\{p\}} (M, X) \ \textit{and} \ \ (N, Y) \models \phi;$$

Proof If $(M, X) \models \theta$ then, since $\theta \in \mathcal{ML}$, for all $w \in X$ we have $(M, w) \models \theta$; hence, for all $w \in X$ there exists (N_w, v_w) such that

$$(N_w, v_w) \rightleftharpoons_{\mathcal{L}(\phi)\setminus\{p\}} (M, w) \text{ and } \ (N_w, v_w) \models \phi.$$

Consider the disjoint union N of all the N_w, for $w \in X$, and the team $Y = \{v_w : w \in X\}$. Since ϕ is a classical modal formula, we have $(N, Y) \models \phi$ and $(N, Y) \rightleftharpoons_{\mathcal{L}(\phi)\setminus\{p\}} (M, X)$, so that $(M, X) \models \tilde{\exists} p\ \phi$. Hence $\tilde{\exists} p \phi$ is a logical consequence of θ in the modal team semantics.

Vice versa, if $(M, X) \models \tilde{\exists} p\ \phi$ we prove that $(M, w) \models \theta$, for all $w \in X$. Let (N, Y) be such that $(N, Y) \models \phi$ and $(N, Y) \rightleftharpoons_{\mathcal{L}(\phi)\setminus\{p\}} (M, X)$. Since ϕ is a classical modal formula, $(N, y) \models \phi$, for all $y \in Y$. If $w \in X$ then there exists $v \in Y$ such that $(N, v) \rightleftharpoons_{\mathcal{L}(\phi)\setminus\{p\}} (M, w)$ and hence, by hypothesis $(M, w) \models \theta$.

Finally, from $(M, w) \models \theta$, for all $w \in X$, it follows $(M, X) \models \theta$, since θ is a classical modal formula. □

Corollary 6.1 *If $\phi \in \mathcal{ML}$ then $\tilde{\exists} p\ \phi \in \mathcal{ML}$.*

Proof The corollary follows from the previous Lemma and the closure of Classical Modal Logic under the existential bisimulation quantifier w.r.t. singleton semantics: for any formula ϕ of classical modal logic, there exists a formula θ such that for all Kripke models (M, w) it holds

$$(M, w) \models \theta \Leftrightarrow \exists (N, v) \rightleftharpoons_{\mathcal{L}(\phi)\setminus\{p\}} (M, w) \text{ and } (N, v) \models \phi$$

(for a proof see [7]). $\Box$

Finally, we prove a lemma stating that in the team semantics of $\tilde{\exists} p\ \phi$ we can substitute $Free(\phi)$ by any set of propositions containing the free variable of ϕ.

Lemma 6.4 *If ϕ is a formula of $\mathcal{MTL}$, (M, X) is a Kripke model over a set Prop of propositional variables containing the free variables of ϕ, and $(M, X) \models \tilde{\exists} p\ \phi$, then there exists a model (K, Z) such that:*

$$(K, Z) \rightleftharpoons_{Prop\setminus\{p\}} (M, X) \textit{ and } (K, Z) \models \phi.$$

Proof If $(M, X) \models \tilde{\exists} p\ \phi$ then by definition there exists a model (N, Y) such that:

$$(N, Y) \rightleftharpoons_{Free(\phi)\setminus\{p\}} (M, X) \text{ and } (N, Y) \models \phi$$

Since $Free(\phi) \setminus \{p\} = Free(\phi) \cap (Prop \setminus \{p\})$, by Lemma 5.2 there exists a team model (K, Z) with

$$(N, Y) \rightleftharpoons_{Free(\phi)} (K, Z) \text{ and } (M, X) \rightleftharpoons_{Prop\setminus\{p\}} (K, Z)$$

Then, since $(N, Y) \models \phi$, using the first bisimulation we obtain $(K, Z) \models \phi$, and the lemma follows. $\Box$

Lemma 6.5 *If $\phi_1, \phi_2 \in \mathcal{MTL}$ then*

$$\tilde{\exists} p\ (\phi_1 \wedge NE) \equiv (\tilde{\exists} p\ \phi_1) \wedge NE$$

$$\tilde{\exists} p\ (\phi_1 \vee \phi_2) \equiv \tilde{\exists} p\ \phi_1 \vee \tilde{\exists} p\ \phi_2$$

$$\tilde{\exists} p\ (\phi_1 \otimes \phi_2) \equiv \tilde{\exists} p\ \phi_1 \otimes \tilde{\exists} p\ \phi_2$$

Proof The first equivalence holds because if the team of a team model in not empty, so is any team of a bisimilar team model.

We prove the third equivalence, leaving the second one to the reader. Let (M, X) be a tem model. If $(M, X) \models \tilde{\exists} p\ (\phi_1 \otimes \phi_2)$ there exists a team model (N, Y) such that $(N, Y) \models \phi_1 \otimes \phi_2$ and

$$(N, Y) \rightleftharpoons_{Free(\phi_1 \otimes \phi_2)\setminus\{p\}} (M, X).$$

Then, there are Y_1, Y_2 with $Y = Y_1 \cup Y_2$ and $(N, Y_i) \models \phi_i$, for $i = 1, 2$. Let $X_i = \{x \in X : \exists y \in Y_i\ (N, y) \rightleftharpoons_{Free(\phi_1 \otimes \phi_2)\setminus\{p\}} (M, x)\}$. Then $X = X_1 \cup X_2$ and

$$(N, Y_i) \rightleftharpoons_{Free(\phi_1 \otimes \phi_2)\setminus\{p\}} (M, X_i).$$

Since $Free(\phi_1 \otimes \phi_2) \supseteq Free(\phi_i)$ we also have

$$(N, Y_i) \rightleftharpoons_{Free(\phi_i)\setminus\{p\}} (M, X_i),$$

and from $(N, Y_i) \models \phi_i$ it follows $(M, X_i) \models \ \exists p\ \phi_i$, for $i = 1, 2$. This implies $(M, X) \models \tilde{\exists} p\ \phi_1 \otimes \tilde{\exists} p\ \phi_2$.

Conversely, suppose $(M, X) \models \tilde{\exists} p\ \phi_1 \otimes \tilde{\exists} p\ \phi_2$. Then $X = X_1 \cup X_2$ with $(M, X_i) \models \tilde{\exists} p\ \phi_i$, for $i = 1, 2$. Using Lemma 6.4 we obtain models (K_i, Z_i) such that $(K_i, Z_i) \models \phi_i$ and

$$(K_i, Z_i) \rightleftharpoons_{Free(\phi_1 \otimes \phi_2)\setminus\{p\}} (M, X_i).$$

Define $(N, Z) := (K_1 \dot{\cup} K_2, Z_1 \dot{\cup} Z_2)$, where $\dot{\cup}$ denotes disjoint union. For all $y \in Z_i$ we have

$$(N, y) \rightleftharpoons_{Free(\phi_1 \otimes \phi_2)} (K_i, y),$$

hence

$$(N, Z_i) \rightleftharpoons_{Free(\phi_1 \otimes \phi_2)} (K_i, Z_i).$$

It follows that $(N, Z_i) \models \phi_i$, hence $(N, Z) \models \phi_1 \otimes \phi_2$ and

$$(N, Z) \rightleftharpoons_{Free(\phi_1 \otimes \phi_2)\setminus\{p\}} (M, X).$$

This implies $(M, X) \models \tilde{\exists} p\ (\phi_1 \otimes \phi_2)$. □

Using the previous lemmas we can now prove that $\tilde{\exists}\mathcal{MTL}$ is expressively equivalent to $\mathcal{MTL}$, but first we need to fix some notation and recall some well known result of characteristic formulas for modal logic. If (M, w) is a Kripke model and $\mathcal{P}$ is a finite set of propositions we denote by $\phi^{k,\mathcal{P}}_{M,w}$ the modal formula characterizing (M, w) modulo k-bisimulation w.r.t. the variables in $\mathcal{P}$, so that, for all Kripke models (M', w') it holds:

$$(M', w') \models \phi^{k,\mathcal{P}}_{M,w} \quad \Leftrightarrow \quad (M', w') \rightleftharpoons^k_{\mathcal{P}} (M, w).$$

We omit the reference to $\mathcal{P}$ if this set is clear from the context (for a definition of $\phi^k_{M,w}$ see e.g. [6]).

Theorem 6.1 *Elimination of bisimulation quantifiers in $\mathcal{MTL}$: for any $\phi \in \mathcal{MTL}$ and $p \in \mathcal{P}$ there exists a formula $\theta \in \mathcal{MTL}$ which is equivalent to $\tilde{\exists} p\ \phi$.*

Proof Given a team model (M, X), for all $w \in X$ consider the characteristic formulas $\chi^k_{(M,w)}$ with respect to $\mathcal{L}(\phi)$ and define

$$\chi^k_{(M,X)} := \bigotimes_{w \in X} (\chi^k_{(M,w)} \wedge NE).$$

Then, as proved in [4], for all models (N, Y) it holds:

$$(N, Y) \models \chi^k_{(M,X)} \;\Leftrightarrow\; (N, Y) \rightleftharpoons^k_{\mathcal{L}(\phi)} (M, X).$$

This implies that any formula $\phi \in \mathcal{MTL}$ of modal depth k is equivalent to a disjunction of formulas $\chi^k_{(M,X)}$:

$$\phi \equiv \bigvee_{(M,X) \models \phi} \chi^k_{(M,X)}.$$

By Lemma 6.5 we have

$$\tilde{\exists} p\ \phi \equiv \bigvee_{(M,X) \models \phi} \tilde{\exists} p\ \chi^k_{(M,X)} \equiv \bigotimes_{w \in X} (\tilde{\exists} p\ \chi^k_{(M,w)} \wedge NE).$$

By Corollary 6.1, the formulas $\tilde{\exists} p\ \chi^k_{(M,w)}$ are equivalent to modal formulas and the theorem follows.

Corollary 6.2 *The team logic $\mathcal{MTL}$ enjoys uniform interpolation.*

Proof This follows from the previous theorem and Lemma 6.2. □

6.1 Uniform Interpolation for Modal Team Fragments.

In this section we prove uniform interpolation for all fragments of $\mathcal{MTL}$ described in Section 2.1, with one notable exception: Modal Independence Logic and its propositional fragment.

If $\mathcal{C}$ is a team model property and $Q \subseteq Prop$, we denote by $\mathcal{C}_{\sim Q}$ the team model property of all team models which are in $\mathcal{C}$ modulo bisimulations forgetting the variables in Q:

$$\mathcal{C}_{\sim Q} = \{(N, Y) : \exists (M, X) \in \mathcal{C} \;\; (N, Y) \rightleftharpoons_{Prop \setminus Q} (M, X)\}.$$

A collection of team model properties is said to be *forgetting* if it is closed under the previous construction with respect to finite sets of propositions:

Definition 6.1 *A collection $\mathcal{T}$ of team model properties is said to be* forgetting *if for every finite $Q \subseteq Prop$ and $\mathcal{C} \in \mathcal{T}$ it holds:*

$$\mathcal{C}_{\sim Q} \in \mathcal{T}.$$

A fragment L of $\mathcal{MTL}$ is said to be forgetting if $Team_L$ is so.

If L is a team modal logic, $\phi \in L$, and $Q = \{p_1, \ldots, p_n\}$ is a finite set of propositional variables, then using Lemma 6.4 we get:

$$||\tilde{\exists} p_1 \ldots \tilde{\exists} p_n \ \phi|| = ||\phi||_{\sim Q}$$

because

$$||\tilde{\exists} p_1 \ldots \tilde{\exists} p_n \ \phi|| =$$

$$= \{(N, Y) : \exists (M, X) \in ||\phi|| \text{ with } (N, Y) \rightleftharpoons_{\mathcal{L}(\phi) \setminus Q} (M, X)\} =$$

$$\{(N, Y) : \exists (M, X) \in ||\phi|| \text{ with } (N, Y) \rightleftharpoons_{Prop \setminus Q} (M, X)\} = ||\phi||_{\sim Q}.$$

This easily implies:

Lemma 6.6 *If L is a fragment of $\mathcal{MTL}$ then*

L is forgetting $\quad \Leftrightarrow \quad$ L enjoys uniform interpolation.

Proof ($\Rightarrow$) If $\phi \in L$ and $p_1, \ldots, p_n$ are propositional variables consider the set $Q = \{p_1, \ldots, p_n\}$. By the forgetting hypothesis, since $||\phi|| \in Team_L$ then $||\phi||_{\sim Q} = ||\tilde{\exists} p_1 \ldots \tilde{\exists} p_n \ \phi|| \in Team_L$. Hence, $\tilde{\exists} p_1 \ldots \tilde{\exists} p_n \ \phi$ is equivalent to a formula θ in L. By Lemma 6.2 we know that θ is a uniform interpolant for ϕ in $\tilde{\exists} L$ and, even more so, in L.

($\Leftarrow$) If $\phi \in L$ and $Q = \{p_1, \ldots, p_n\}$, then the class of team models $||\phi||_{\sim Q}$ is expressible by the formula $\tilde{\exists} p_1 \ldots \tilde{\exists} p_n \ \phi$ which, by Lemma 6.2, is a $\tilde{\exists} L$ uniform interpolant in of the L-formula ϕ w.r.t. $\mathcal{L}(\phi) \setminus Q$. Since L enjoys uniform interpolation, and uniform interpolants are unique modulo equivalence, we obtain that $\tilde{\exists} p_1 \ldots \tilde{\exists} p_n \ \phi$ is equivalent to an L-formula and hence $||\phi||_{\sim Q} \in Team_L$.

□

Lemma 6.7 *The following collections of team model properties are forgetting:*

1. *the class of bisimulation invariant, non empty, downward closed and union closed (also called: flat) properties;*
2. *the class of bisimulation invariant, non empty, downward closed properties;*
3. *the class of bisimulation invariant, non empty, union closed properties.*

Proof We prove the second and the third property, from which the first property follows.

1. Suppose $\mathcal{C}$ is a non empty, bisimulation invariant, downward closed set of teams, and Q is a finite set of propositions. We want to prove that

$$\mathcal{C}_{\sim Q} = \{(N,Y) : \exists (M,X) \in \mathcal{C} \ \ (N,Y) \rightleftharpoons_{Prop \setminus Q} (M,X)\}$$

   is non empty, bisimulation invariant and downward closed. The first two properties are easily verified and we leave them to the reader. As for the third property, suppose $(N,Y) \in \mathcal{C}_{\sim Q}$, and $Y' \subseteq Y$. Then there exists (M,X) with

$$(N,Y) \rightleftharpoons_{Prop \setminus Q} (M,X) \ \text{ and } \ (M,X) \in \mathcal{C}.$$

   For $s \in Y'$, let $s' \in X$ be such that $(N,s') \rightleftharpoons_{Prop \setminus Q} (M,s)$; let $X' \subseteq X$ be $X' = \{s' \ : \ s \in Y'\}$. By downward closure of $\mathcal{C}$, we have $(M,X') \in \mathcal{C}$; since

$$(N,Y') \rightleftharpoons_{Prop \setminus Q} (M,X')$$

   we obtain $(N,Y') \in \mathcal{C}_Q$.

2. If $\mathcal{C}$ is a non empty, bisimulation invariant, union closed set of teams over $Prop$ and Q is a finite set $Q \subseteq Prop$, we want to prove that

$$\mathcal{C}_{\sim Q} = \{(N,Y) : \exists (M,X) \in \mathcal{C} \ \ (N,Y) \rightleftharpoons_{Prop \setminus Q} (M,X)\}$$

   is non empty, bisimulation invariant, and union closed. The first two properties are easily verified and we leave them to the reader. As for the third property, suppose $(N,Y_1),(N,Y_2) \in \mathcal{C}_{\sim Q}$. Then there are teams $(M_1,X_1),(M_2,X_2) \in \mathcal{C}$ such that, for $i = 1,2$ it holds:

$$(N,Y_i) \rightleftharpoons_{Prop \setminus Q} (M_i,X_i) \ \text{ and } \ (M_i,X_i) \in \mathcal{C}.$$

   Let M be the disjoint union of M_1, M_2, and let X be the disjoint union of X_1, X_2. Then one can easily prove that, for $i = 1,2$ it holds:

$$(M,X_i) \rightleftharpoons_{Prop} (M_i,X_i), \ \text{ and } (M,X_i \cup X_2) \rightleftharpoons_{Prop \setminus Q} (N, Y_1 \cup Y_2).$$

   From the first two bisimulations we obtain $(M,X_i) \in \mathcal{C}$, which implies $(M, X_i \cup X_2)$ since $\mathcal{C}$ is closed under union. It follows that $(N, Y_1 \cup Y_2) \in \mathcal{C}_{\sim Q}$.

□

Using Lemma 6.7 and Lemma 6.6 we easily obtain:

Corollary 6.3 *The following fragment of $\mathcal{MTL}$ enjoys uniform interpolation:*

$$\mathcal{ML},\ \ \mathcal{MDEP},\ \ \mathcal{MINC}$$

Finally, we consider propositional fragments. Being complete for the class of all team properties, it is clear that $\mathcal{FPT}$ enjoys uniform interpolation. Then we can use the semantical characterization of the fragments $\mathcal{CPL}$, $\mathcal{PD}$, $\mathcal{PINC}$ to prove, as we did for modal fragments, that all these logics enjoy uniform interpolation.

Corollary 6.4 *The following fragments of $\mathcal{FPT}$ enjoys uniform interpolation:*

$$\mathcal{CPL},\ \ \mathcal{PD},\ \ \mathcal{PINC}$$

6.2 Conclusion and Open Questions

The method above allow us to prove uniform interpolation for propositional and modal team logics whose class of team models is *forgetting* (see Def. 6.1). However, to prove that the class of team models for a logic is forgetting, we used a good description of the class, given in Theorem 4.1. To our knowledge, this description is missing for indipendence logic, for which uniform interpolation remains an open question.

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