Contents

Chapter 1

A value for j-cooperative games: some theoretical aspects and applications

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1.1 Introduction

Shapley, in his seminal article [48], introduced a value for cooperative games which is uniquely characterized by some natural axioms, since then, the value has had a great impact both theoretical and practical. As conspicuous examples of theoretical works we mention [24, 27, 42], and [26, 43] which contain compilations of practical results in the cooperative and voting contexts.

The restriction of the Shapley value to simple games is known as the Shapley-Shubik power index [49]. As pointed out by Felsenthal and Machover [30] the index can be interpreted as measure of power as a payoff (e.g., when dividing a cake or a unit of a divisible object among players). The index as a measure of influence is more questionable because the axiom of efficiency has no interest, a remarkable characterization of the index without using efficiency can be found in [25]. In addition the valuable probabilistic model considered for the Shapley value in cooperative games loses its interest for simple games in which the players do not necessarily wish to vote in favor of the proposal in their turn of vote. As observed in [7] and in [28] an alternative model based on roll-calls also extends to the cooperative framework and it is the key for finding, as done in this paper, an explicit formula for a value on the class of multi-choice games (here denoted j-cooperative games for coherence) that respects the original model by Shapley. We do not call it 'Shapley value' because as explained below there are already many different values with such denomination, which can cause confusion. The idea of this value is based on the player gain capacity and on the blocking capacity in her turn to vote.

In the context of cooperative games players decide whether or not to cooperate and this is their only possible action. Several more general models have been considered with more than two actions for players. Just to recall some of them Bolger [13, 14, 15, 16] considered games with n players and r alternatives, not necessarily ordered or comparable among them, Amer et. al. [4, 5] considered games with multiple alternatives and called them r-games, closely related with Bolger's model. Bolger defines and axiomatically characterizes an extension of the Shapley value to games with alternatives, whereas the index due to Penrose [45], Banzhaf [6], Coleman [21] is extended by Amer et al. [4]. Nevertheless, all these values refer to the value of a player for a particular alternative.

Bicooperative games are introduced in [9]. In these games ordered pairs of disjoint coalitions of players are considered. Each such pair yields a partition of the set of all players in three groups. Players in the first coalition are in favor of the proposal, and players in the second coalition object to it. The remaining players are not convinced of its benefits, but they have no intention of objecting to it. The characteristic function can be interpreted as a positive maximal gain or as a negative minimal loss. A value of zero is assigned to the tripartition in which everybody is indifferent. Thus, the value zero plays a central position in the characteristic function of a bicooperative game and the game can be regarded as a balance between two opposite forces. A notion of the Shapley value in this context is provided in [11, 12].

Multi-choice games are considered by Hsiao and Raghavan [39, 40]. These authors consider games in which the actions of the players are ordered in the sense that, for every pair of different actions one action carries more weight than the other action. In their model they reserve an action for those who are not active at any level. Hsiao and Raghavan also define a (matricial) notion of the Shapley value in multi-choice games that depends on actions. Some variants of their value are proposed in [38, 47].

Extensions of simple games are mainly proposed in [29], for voting games including abstention as an intermediate input level, and in [33] where (j, k)-simple games are considered a class of games in which voters may choose any of j ordered levels of approval and k stands for the number of aggregated ordered results. The last work provides a notion of weighted game endorsed by characterizations of the property of trade-robustness. Other important notions as those of the desirability relation, transitivity, acyclicity, and hierarchies, are extended in this broader context in [35, 36, 44, 46, 51].

In this paper we propose a value that has nothing to do with those cited above, with all the ingredients for both j-cooperative games (a trivial more convenient adaptation of multi-choice games) and j-simple games (i.e., $(j, 2)$-simple games as defined in [33]). As shown below, the proposed value is consistent in both frameworks and it gives a numerical evaluation for each player independently of the input alternatives for players. The probabilistic model used to create this value is that of roll-calls, which shows to be the correct one for both j-cooperative games and j-simple games. This feature is opposed to the original probabilistic model used by Shapley [48, 50] and Shapley-Shubik [49] and which has been rightly criticized by several authors as highly artificial (see, for instance, [17] or [41]) when refering to simple games.

The rest of the chapter is organized as follows. In Section 14.2 some motivating examples are presented. Section 14.3 introduces some preliminaries and the contexts of j-cooperative games and j-simple games. A value with its explicit formula is proposed for the class of j-cooperative games in Section 14.4. Section 14.5 provides a probabilistic model as a justification of the value. Section 14.6 proves that the value for two input alternatives coincides, as expected, with the Shapley value. An alternative formula to compute the proposed value is given in Section 14.7. Section 14.8 proposes an axiomatic characterization following the seminal ideas of Shapley's axiomatization for his value; the main contribution lies on the introduction of a fifth axiom for unanimity games. After defining the meaning of constant sum game for j-cooperative games we compute their proposed value in Section 14.9. The method of generating functions for computing the value for weighted 3-simple games is shown in Section 14.10 and used to compute the value for the voting system of the United Nations Security Council and for a variant of it that avoids the veto-right of permanent nations. The examples are revisited in Section 14.11 and the value is computed for them. A brief Conclusion ends the paper in Section 14.12.

1.2 Some motivating examples

In this section, we present some examples to illustrate the versatility of the kind of games we consider. We start with a very simple example of a ternary voting game already considered by Felsenthal and Machover [30]. Then we continue with an example of economic nature, another of academic activities, the description of the United Nations Security Council voting system and also a new modified version for it that avoids the veto-right of the permanent members without harming these five nations too much. In describing these examples, we use some intuitive terminology which is concisely defined in next section. A value that captures the idea of Shapley's value for cooperative games is proposed in Section 14.4 for a more general context. Such value will be computed in Section 14.11 for all the examples described in the rest of this section.

Example 1.1 (A ternary voting game) *Consider Example 8.3.7, page 288 in [30]. The set of voters is* $N = \{a, b, c\}$ *and the bill is passed* if *voter* a *votes for it and at least one of the other two does not oppose it. From the* 27 *possible ways to vote for members in* N*, there are only* 8 *that pass the bill.*

Example 1.2 (A team of workers) *A team of three workers have to perform a task. All three can carry out their task at three different levels: full involvement, medium involvement and lack of involvement. Only one of them, called* a*, is qualified to operate a machine that is essential to achieve a satisfactory execution of the work to be done. The other two workers, called* b *and* c*, play a symmetrical role and also turn out to be indispensable together and a lack of involvement on the part of the two would be fatal for the execution of the task. Other combinations for these two workers with at least a medium involvement by worker* a *lead to more or less satisfactory results depending on the degree of involvement for these two workers. Full involvement by the three suppose a win of* 4 *thousand euros. The following characteristic function specifies the gain for all combinations*

$$v(S) = \begin{cases} 4 - |S_2| - 2|S_3| & \text{if } \ a \in S_1 \\ \max\{0, |S_1| - |S_3|\} & \text{if } \ a \in S_2 \\ 0 & \text{if } \ a \in S_3 \end{cases}$$

where $S = (S_1, S_2, S_3)$ *and* S_1 *contains the workers with full involvement,* S_2 *contains the workers with an intermediate involvement, and* S_3 *contains the rest of the workers with the lowest level of involvement.*

If we do not have any information about workers' attitude and the workers, how should the total gain be distributed among them? The value we propose in this paper assigns to them: $(2, 1, 1)$ *where the payment* 2 *is for the qualified worker* a*.*

Example 1.3 (A two-part test) *A test has two parts, a and b, which consist of ten questions each. Each question is binary scored by:* 1 *if it is correct and* 0 *otherwise. Thus the result for each part is the number of corrected answers which is a number from* 0 *to* 10*. The aggregated result for the test is a weighted mean of the well-answered questions. Part a is weighted as a* 60% *and part b is weighted as a* 40%*.*

Let $N = \{a, b\}$ *be the set of parts of the test. Let* $S = (S_1, S_2, \dots, S_{10}, S_{11})$ *be a* 11*-partition of* N *in which* S_i *contains the parts of the test with a score of* $11 - i$ *for* $i = 1, 2, \dots, 10, 11$*. If* $a \in S_h$ *and* $b \in S_i$*, then the aggregated score is given by* $V(S) = 6(11 - h) + 4(11 - i) = 110 - 6h - 4i$*, which scales the student's test mark between* 0 *and* 100*.*

If we do not have any information about possible differences, if they would exist, between both tests, which is the importance of each test for the exam? The value we propose in this paper assigns the intuitive answer: $(60, 40)$ *which preserves the relative importance between the two parts.*

Example 1.4 (The UNSC voting system) *As noted by [30], the United Nations Security Council (UNSC) can be modeled as a* 3*-simple game: a resolution is approved if there are at least nine members in favor and permanent members are not against it. This means that also if some of the permanent members abstain, without explicitly imposing the veto, a resolution can be carried on. The resulting game* v *has* 15 *players, with the subset* P *of the five permanent members, and a tripartition* $S = (S_1, S_2, S_3)$ *is winning (i.e.,* $v(S) = 1$*)* if and only if

$$|S_1| \geq 9 \quad and \quad S_3 \cap P = \emptyset.$$

where S_1 *contains the members in favor of the resolution,* S_3 *the members against it, and* S_2 *the abstainers. For further discussion on this significant system, see for example [22].*

Example 1.5 (A modified voting system for the UNSC) *The UNSC is critical to global peace and security, yet more than twenty years of negotiations over its reform have proved fruitless; see in [37] a survey on several proposed reforms that have not been implemented.*

A simple modified version of the UNSC voting game is proposed here that does not involve changes in the world countries forming it, would consist in just modifying the possibility of approval of a resolution if one permanent member is against it but all the other members are in favor of it. This means that for any permanent member $p \in P$*, the five losing tripartitions* $(N \setminus \{p\}, \emptyset, \{p\})$ *of the current system convert into winning tripartitions, and this is the only difference between the current and the proposed UNSC voting system. The inclusion of these five tripartitions in the set of winning tripartitions prevents the permanent members to have veto-right, but this situation only occurs when the other fourteen countries agree to vote in favor of the resolution at hand.*

The next section is devoted to formally introduce the class of games we deal with in this chapter.

1.3 Preliminaries: j-cooperative games

Let N be a finite set of *players.* A *j-partition* of N is a collection of j mutually disjoint subsets of N, $S_1, \ldots, S_j$ such that $\bigcup_{k=1}^{j} S_k = N$. Note that any S_i may be empty. Any subset S of N is called a *coalition* and we denote its cardinality by s.

A j-partition describes a division of players among j alternatives or j levels of voting approval or j possible actions or choices players can realize or choose. We assume that these j different alternatives are ordered and convey that level 1 corresponds to the highest level of performance, while the last, level j, corresponds to the lowest level. Thus, players in S_1 are those who work at the highest level, while those in S_j work at the lowest level of activity. In a voting context, voters in S_1 are those who vote for the highest level of approval, whereas those in S_j are those who vote for the lowest level of approval. Thus, the convention chosen is *ordinal* rather than numerical.

From now on we denote with J^N the set of all j-partitions on N endowed with an (strict) order from the first (highest) order of performance or activity to the last (lowest) one. Although we assume an order of the levels of activity, we do not do any assumption over the quantification of these levels. Thus, acting at the second level just means that such level of activity is lower than in level 1 but greater than in level 3.

A partial order $\subseteq^j$ on the set J^N is considered. If $S, T \in J^N$, then $S \subseteq^j T$ means $S_k \subseteq^j \bigcup_{i=1}^{k} T_i$ for any $k = 1, \ldots, j-1$. In words, S is contained in T if players in T are working or voting in the same or in a higher level than in S. We use $S \subset^j T$ if $S \subseteq^j T$ and $S \neq T$. The j-partitions $\mathcal{N} = (\emptyset, \ldots, \emptyset, N)$ and $\mathcal{M} = (N, \emptyset, \ldots, \emptyset)$ are respectively the minimum and maximum for the order $\subseteq^j$.

A binary voting situation in which voters (we use the term voters instead of the term players in the voting context) can vote among several ordered alternatives can be formalized by a $(j, 2)$-simple game, i.e., voters can vote in j different ordered ways to approve or reject a resolution and the aggregate output is binary. As previously said, we refer to $(j, 2)$-simple game as j-simple games throughout this article.

Definition 1.1 *[[33]] Let N be a finite set and J^N be the set of all totally ordered j-partitions on N. A* j-simple game *is a function $v : J^N \to \{0, 1\}$ such that:* (i) *it is monotonic: if $S \subset^j T$, then $v(S) \leq v(T)$;* (ii) $v(\mathcal{N}) = 0$ *and $v(\mathcal{M}) = 1$.*

We denote with $\mathcal{SJ}_N$ the space of all j-simple games on the finite set N. Note that $(2, 2)$-simple games are simple games since for any bipartition $S = (S_1, S_2)$ the first component S_1 is identified with the set of 'yes'-voters and $S_2 = N \setminus S_1$ with the set of 'no'-voters. Thus, any bipartition is in one-to-one correspondence with coalition S_1. Note also that $(3, 2)$-simple games can be interpreted as ternary voting games, as considered by [29], if the first level of approval corresponds to voting 'yes', the second level to abstain and the third level to voting 'no'.

In any j-simple game, the aggregated output set is binary and represented by $\{0, 1\}$, where these two numbers have the respective meaning that the submitted proposal is either defeated or passed.

Definition 1.2 *Let N be a finite set and J^N be the set of all totally ordered j-partitions on N. A j-*cooperative game *is a function $v : J^N \to \mathbb{R}$ such that $v(\mathcal{N}) = 0$.*

We denote by $\mathcal{J}_N$ the space of j-cooperative games on the finite set N. Note that a 2-cooperative game corresponds to a cooperative game in which the bipartition $S = (S_1, N \setminus S_1)$ is identified with the coalition S_1 formed by players who decide to cooperate.

The previous definition is almost equivalent to that of a multi-choice game as defined in [39, 40]. A distinction is that in the multi-choice setting an input level is distinguished from the others and reserved for lack of activity. In our context the last input level does not necessarily mean a total lack of activity and this becomes clear in the voting context, for j-simple games. For instance, for ternary voting games ($j = 3$ with three input choices: voting 'yes', 'abstain' or voting 'no') the last input level means voting against the submitted proposal, which would not be coherent with the multi-choice model and the same happens for other choices of j. Moreover, the restriction from j-cooperative games to j-simple games becomes natural.

There are many interesting subclasses of cooperative games that can easily be extended to j-cooperative games for $j > 2$. Here we just refer to monotonicity.

A j-cooperative game is *monotonic*, if for any pair of j-partitions S and T, such that $S \subset^j T$ then $v(S) \leq v(T)$.

Clearly, $\mathcal{J}_N$ is a vectorial space of dimension $j^n - 1$ and a basis formed by monotonic j-cooperative games is the one of *unanimity games* defined as:

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq^j T \\ 0, & \text{otherwise,} \end{cases}$$

for all j-partition $S \neq \mathcal{N}$.

1.4 A value for j-cooperative games

Let us introduce the following notation. From a given j-partition S, we define the j-partition $S_{a\uparrow_k}$ in which player a has moved from the lowest level j to the superior level k ($k < j$), and the j-partition $S_{a\downarrow_k}$ in which player a

has moved from the highest level of activity 1 to the inferior level k ($k > 1$). If $a \in S_j$:

$$S_{a\uparrow_k} = (S_1, \dots, S_k \cup \{a\}, \dots, S_j \setminus \{a\})$$

for any $k = 1, \cdots, j-1$; and if $a \in S_1$:

$$S_{a\downarrow_k} = (S_1 \setminus \{a\}, \dots, S_k \cup \{a\}, \dots, S_j)$$

for any $k = 2, \cdots, j$.

The idea we pursue with these two definitions is to consider two special types of *marginal contributions* for j-partitions in a given game v:

$$\begin{array}{ll} m^k(v, S, a) = v(S_{a\uparrow_k}) - v(S) & \text{if} \quad a \in S_j \\ m_k(v, S, a) = v(S) - v(S_{a\downarrow_k}) & \text{if} \quad a \in S_1 \end{array}$$

In the next definition, we propose a value for j-cooperative games inspired with the ideas of the Shapley value, [48], for cooperative games. The explicit formula for the proposed value depends on the marginal contributions $m^k(v, S, a)$ and $m_k(v, S, a)$. Before showing its explicit formulation, we give an intuitive idea that later will be justified.

In her turn, player a can achieve in choosing the input k an additional gain of $m^k(v, S, a)$ with respect to the gain obtained from her predecessors with the choice of the input each made. But, with the choice of input k, player a also prevents her predecessors from obtaining the extra gain of $m_k(v, S, a)$. Thus in some sense, player a has a double capacity: that of direct gain and that of blocking extra gain.

Definition 1.3 (A value for j-cooperative games) *For any $v \in \mathcal{J}_N$ and any player $a \in N$, the $\mathcal{F}$-*value *is defined as*

$$\mathcal{F}_a(v) = \frac{1}{j^n n!} \left[\sum_{\substack{S \in J^N: \\ a \in S_j}} \sum_{k=1}^{j-1} \gamma_j^n(s_j - 1) m^k(v, S, a) + \sum_{\substack{S \in J^N: \\ a \in S_1}} \sum_{k=2}^{j} \gamma_j^n(s_1 - 1) m_k(v, S, a) \right] \tag{1.1}$$

where

$$\gamma_j^n(t) = t! j^t \sum_{i=0}^{t} \frac{(n - t - 1 + i)!}{j^i i!}, \tag{1.2}$$

for $t = 0, 1, \dots, n-1$.

We show the coefficients in (1.2) in the next three tables for small values of n, $n \leq 6$ and for: $j = 2$ (Table 1.1), $j = 3$ (Table 1.2), and $j = 4$ (Table 1.3).

1.5 Probabilistic justification of the $\mathcal{F}$-value

In the following we mainly use the notation from [31] and also refer to [28, 29, 30] for precise definitions when the number of input alternatives is 3. We

$n \downarrow$ | $t \rightarrow$	0	1	2	3	4	5
1	1					
2	1	3				
3	2	4	14			
4	6	10	22	90		
5	24	36	64	156	744	
6	120	168	264	504	1368	7560

TABLE 1.1: Numerical coefficients $\gamma_2^n(t)$ for 2-cooperative games up to 6 players.

$n \downarrow$ | $t \rightarrow$	0	1	2	3	4	5
1	1					
2	1	4				
3	2	5	26			
4	6	12	36	240		
5	24	42	96	348	2904	
6	120	192	372	984	4296	43680

TABLE 1.2: Numerical coefficients $\gamma_3^n(t)$ for 3-cooperative games up to 6 players.

$n \downarrow$ | $t \rightarrow$	0	1	2	3	4	5
1	1					
2	1	5				
3	2	6	42			
4	6	14	54	510		
5	24	48	136	672	8184	
6	120	216	504	1752	10872	163800

TABLE 1.3: Numerical coefficients $\gamma_4^n(t)$ for 2-cooperative games up to 6 players.

consider a probabilistic model in which two relevant data for each player $a \in N$ are taken: the ordering in the queue for a and the input alternative chosen for a in her turn. A *roll-call* specifies these two data for each player, so that the number of roll-calls is $n!j^n$. Let $\mathcal{R}_j^N$ be the set of all roll-calls and $\mathcal{R} \in \mathcal{R}_j^n$.

When we are restricted to j-simple games the notion of pivotal voter is crucial and extendable to j-cooperative games.

Voter a is *pivotal* in the j-simple game if she is the only one who decides the (binary) outcome after her election of the input, no matter how the others following her in the queue will vote. The idea of a value that has all the ingredients of the Shapley-Shubik power index for j-simple games is based on the definition given in [31].

For any $v \in \mathcal{J}_N$ and any player $a \in N$, the f-power index

$$\mathfrak{f}_a(v) = \frac{|\{\mathcal{R} \in \mathcal{R}_j^n \ : \ a = piv(\mathcal{R}, v)\}|}{j^n n!}.$$

This formula measures the probability of being a pivotal voter in the space of all roll-calls with the uniform distribution. It has the disadvantage that does not depend on the characteristic function v.

Although there is a single pivotal player in a roll-call, we can distinguish between two types of being a pivotal player in a j-simple game. A player a is *positively* pivotal if after voting for the k-input the j-partition of those who voted before her with the rest of the players voting for the lowest level j is winning. Instead, a player a is *negatively* pivotal if after voting for the k-input the j-partition of those who voted before her with the rest of players voting for the first level of approval is losing, i.e., although all voters following a in the queue were to vote for the first level of approval, the result of the vote would still be 'losing'.

This idea of pivotal player and its two versions for a roll-call is easily extendible to j-cooperative games. Apart of doing such extension, we also wish to express the proposed value for a j-cooperative game in terms of the marginal contributions $m^k(v, S, a)$ and $m_k(v, S, a)$ that involve j-partitions rather than roll-calls. Thus, the idea is to associate a set of roll-calls with each j-partition with the idea described above when adapting from a positively pivotal player (for j-simple games) to the marginal contribution $m^k(v, S, a)$ (for j-cooperative games). Similarly, a set of roll-calls is associated with each j-partition when adapting from a negatively pivotal player (for j-simple games) to the marginal contribution $m_k(v, S, a)$ (for j-cooperative games). This is collected by the coefficient $\gamma_j^n(t)$ given in Equation (1.2).

Given a subset T of N with cardinality t and a player $a \notin T$, the coefficient $\gamma_j^n(t)$ counts the number of roll-calls such that:

- all players in $N \backslash (T \cup \{a\})$ precede a in the queue and thus, have already chosen the input level;
- players in T either precede or follow a in the queue:

- if they precede a in the queue, they have already chosen the input level, while,
- if they follow a in the queue, they have not yet chosen the input level and thus all j input alternatives are counted.

Let us call this set the T-*free* set of roll-calls for a, since no matter if players in T precede or not a in the queue. Players preceding a are the only ones who have already chosen their input alternative.

Lemma 1.1 *The cardinality of the T-*free *set of roll-calls for a given player $a \notin T$ is $\gamma_j^n(t)$.*

Proof. Consider the T-*free* set of roll-calls for a given player $a \notin T$. Let i be the number of players in T preceding a, thus i can be any number between 0 and t.

The number of players preceding a in the queue are $n - t - 1 + i$ since $a \in N \setminus T$. As all orderings for these players are allowed, we have for them $(n - t - 1 + i)!$ possible orderings. Any subset of i players in T may precede a, thus $\binom{t}{i}$ is the number of elections for them.

The number of players following a in the queue are then $t - i$, again as all orderings for these players are allowed we have for them $(t - i)!$ possible orderings. Moreover, these players can choose any input alternative, so that we have for them j^{t-i} choices.

By applying the multiplication principle, it follows that the T-*free* set of roll-calls for a given player not belonging to T is:

$$\sum_{i=0}^{t} (n - t - 1 + i)! \binom{t}{i} (t - i)! j^{t-i}$$

and after taking out common factors

$$\gamma_j^n(t) = t! j^t \sum_{i=0}^{t} \frac{(n - t - 1 + i)!}{j^i i!}$$

as stated. ∎

Theorem 1.1 *The value based on marginal contributions under the uniform probability scheme for roll-calls is the $\mathcal{F}$-value.*

Proof. The marginal contribution $m^k(v, S, a)$ for player $a \in S_j$ is the gain that player a can assure to j-partition S when the player in her turn chooses the k-level of activity instead of the lowest level j, i.e., it is the gain capacity for a in her turn. Such gain capacity after choosing the k-level is quantified as $v(S_{a\uparrow k}) - v(S)$ with $a \in S_j$.

The multiplication factor of $m^k(v, S, a)$ only depends on the number of players in $S_j \setminus \{a\}$, $s_j - 1$, and counts all $(S_j - \{a\})$-free roll-calls that can be formed according to Lemma 1.1, i.e., the number of roll-calls for which a adds the value $m^k(v, S, a)$. As k is any number in between 1 and $j-1$ and $a \in S_j$ is the only requirement for S, we consider the two former addends in the first part of Equation (1.1). After dividing by the total number of roll-calls $j^n n!$ we obtain the total gain capacity for a.

Similarly, the marginal contribution $m_k(v, S, a)$ for player $a \in S_1$ is the lost gain that player a causes to j-partition S when the player in her turn chooses the k-level of activity instead of the highest level of activity 1, i.e., it is the blocking capacity for a in her turn. Such blocking capacity after choosing the k-level is quantified as $v(S) - v(S_{a\downarrow k})$ with $a \in S_1$.

The multiplication factor of $m_k(v, S, a)$ only depends on the number of players in $S_1 \setminus \{a\}$, $s_1 - 1$, and counts all $(S_1 - \{a\})$-free roll-calls that can be formed according to Lemma 1.1, i.e., the number of roll-calls for which player a causes a loss of $m_k(v, S, a)$. As k is any number in between 2 and j and $a \in S_1$ is the only requirement for S, we consider the two last addends in the second part of Equation (1.1). After dividing by the total number of roll-calls $j^n n!$ we obtain the total blocking capacity for a. ■

1.6 The $\mathcal{F}$-value restricted to cooperative games is the Shapley value

The purpose of this section is to prove that the $\mathcal{F}$-value for 2-cooperative games is the Shapley value for cooperative games. Cooperative games are 2-cooperative games in our context and therefore the value of coalition $S \subseteq N$ in a cooperative game is the value of the bipartition $(S, N \setminus S)$. Thus, we can indistinctly write $v(S)$ or $v(S, N \setminus S)$.

Thus, to prove our claim, we need to demonstrate the coincidence of the value in (1.1) with the Shapley value.

The well-known formula of the Shapley value in terms of the marginal contributions of the characteristic function is given by:

$$\phi_a(v) = \sum_{S \subseteq N \setminus \{a\}} \rho^n(s)[v(S \cup \{a\}) - v(S)], \tag{1.3}$$

where $s = |S|$ and

$$\rho^n(s) = \frac{s!(n-s-1)!}{n!}.$$

Less known is the equivalent expression for the Shapley value [7]. For any $a \in N$:

$$\phi_a(v) = \sum_{S \subseteq N \setminus \{a\}} \Gamma^n(s)[v(S \cup \{a\}) - v(S)], \tag{1.4}$$

where $s = |S|$ and for any $s = 0, \ldots, n-1$:

$$\Gamma^n(s) = \frac{1}{2^n n!}\left[s!\sum_{k=0}^{s}\frac{(n-k-1)!}{(s-k)!}2^k + (n-s-1)!\sum_{k=0}^{n-s-1}\frac{(n-k-1)!}{(n-s-1-k)!}2^k\right] \quad (1.5)$$

By using the coefficients: $\lambda^n(s) = s!\sum_{k=0}^{s}\frac{(n-k-1)!}{(s-k)!}2^k$ for $s = 0, 1, \ldots, n-1$ then equation (1.5) can be expressed as:

$$\Gamma^n(s) = \frac{1}{2^n n!}\left[\lambda^n(s) + \lambda^n(n-s-1)\right]$$

Theorem 1.2 *The $\mathcal{F}$-value for 2-cooperative games coincides with the Shapley value.*

Proof. We need to prove the equivalence of formulas (1.1) and (1.3). Formula (1.1) for $j = 2$ becomes

$$\Phi_a(v) = \frac{1}{2^n n!}\left[\sum_{\substack{S\in 2^N:\\ a\in S_2}}\gamma_2^n(s_2-1)m^1(v,S,a) + \sum_{\substack{S\in 2^N:\\ a\in S_1}}\gamma_2^n(s_1-1)m_2(v,S,a)\right]$$

which is equivalent to

$$\Phi_a(v) = \frac{1}{2^n n!}\left[\sum_{S_1\subseteq N\setminus\{a\}}(\gamma_2^n(n-s_1-1)+\gamma_2^n(s_1-1))(v(S_1\cup\{a\})-v(S_1))\right]$$

where in the last expression the characteristic function v is applied to coalition S_1 instead of the 2-partition $(S_1, N\setminus S_1)$.

By rearranging properly the subscripts, we obtain the two next equalities:

$$\lambda^n(n-s_1-1) = \gamma_2^n(n-s_1-1) \quad \text{and} \quad \lambda^n(s_1) = \gamma_2^n(s_1)$$

This shows the equivalence of the $\mathcal{F}$-value with the value in (1.4). The proof of Corollary 3 in [7] shows the equality of the coefficients $\rho^n(s)$ and $\Gamma^n(s)$ for every $0 \le s \le n-1$ and therefore the equivalence of the $\mathcal{F}$-value for 2-cooperative games with the Shapley value for cooperative games.

1.7 Another formulation for the $\mathcal{F}$-value

The value $\mathcal{F}$ for j-cooperative games proposed in the previous section is given in terms of some marginal contributions as shown in (1.1), but it also can be expressed as a linear combination of the different values of the characteristic function on each j-partition.

Indeed, the next proposition is directly obtained from (1.1) by conveniently grouping the coefficients of $v(S)$ for each j-partition $S \neq \mathcal{N}$. Therefore, we omit its simple proof.

Proposition 1.1 *For any $v \in \mathcal{J}_N$ and any player $a \in N$, the $\mathcal{F}$-value admits the expression*

$$\mathcal{F}_a(v) = \sum_{S \in J^N} b_j^n(s_1, s_j) v(S) \tag{1.6}$$

where

$$b(s_1, s_j) = \begin{cases} \dfrac{\gamma_j^n(s_j) + (j-1)\gamma_j^n(s_1 - 1)}{j^n n!}, & \text{if } a \in S_1 \\[2ex] \dfrac{\gamma_j^n(s_j) - \gamma_j^n(s_1)}{j^n n!}, & \text{if } a \in S_i,\ 1 < i < j \\[2ex] -\dfrac{\gamma_j^n(s_1) + (j-1)\gamma_j^n(s_j - 1)}{j^n n!}, & \text{if } a \in S_j \end{cases} \tag{1.7}$$

and $s_1 \geq 0$, $n > s_j \geq 0$ and $s_1 + s_j \leq n$.

Note that $b(s_1, s_j) = 0$ for every S with $s_1 = s_j$ (with $a \in S_i$ for some $1 < i < j$).

The next equation shows Formula (1.6) for $j = 3$ and player set $N = \{a, b\}$.

$$\begin{aligned} \mathcal{F}_a(v) \quad = \quad & \frac{1}{2}v(\{a,b\},\emptyset,\emptyset) + \frac{1}{6}v(\{a\},\{b\},\emptyset) + \frac{1}{3}v(\{a\},\emptyset,\{b\}) - \frac{1}{6}v(\{b\},\{a\},\emptyset) \\ & + \frac{1}{6}v(\emptyset,\{a\},\{b\}) - \frac{1}{3}v(\{b\},\emptyset,\{a\}) - \frac{1}{6}v(\emptyset,\{b\},\{a\}). \end{aligned}$$

As a simple illustration on the different types of computing the value proposed, we revisit the first example described in Section 14.2.

The voting system in Example 1.1 is a 3-simple game and it can be described by the set of minimal winning tripartitions (i.e., minimal winning tripartitions with respect to the inclusion $\subseteq^3$) trivially defined from the characteristic function v:

$$W^m(v) \ = \ \{(\{a\},\{b\},\{c\})\ ,\ (\{a\},\{c\},\{b\})\}$$

by monotonicity it is easy to generate the six remaining winning tripartitions.

We start with this example by showing three ways to compute the $\mathfrak{f}$-power index. We will see that these three successive methods are becoming simpler since the first involves all roll-calls, the second all tripartitions, whereas the third only winning tripartitions. Thus, the gain in each step is significant.

The first procedure described in [30] involves all roll-calls and is based on the definition of pivotal player

$$\mathfrak{f}_a(v) = \frac{|\{\mathcal{R} \in \mathcal{R}_j^n \ : \ a = piv(\mathcal{R}, v)\}|}{j^n n!}, \tag{1.8}$$

for $j = 3$. Following [30] to compute (1.8), it follows that:

1. a votes first and does not vote 'yes'. This probability is $\frac{2}{9}$.
2. a votes second, the first voter voted 'no' and a does not vote 'yes'. This has probability $\frac{2}{27}$.
3. a votes second and the first voter did not vote 'no'. The probability is $\frac{6}{27}$.
4. a votes last, and the other two did not vote 'no'. This has probability $\frac{8}{27}$.

Thus, $\mathfrak{f}_a(v) = \frac{22}{27}$; and by anonymity and efficiency $\mathfrak{f}_b(v) = \mathfrak{f}_c(v) = \frac{5}{54}$.

The second procedure uses (1.1) directly for $n = j = 3$ so that the coefficients are: $\gamma_3^3(0) = 2$, $\gamma_3^3(1) = 5$, $\gamma_3^3(2) = 26$ (see the third row in Table 1.2) which need to be accounted only for the marginal contributions being equal to 1 and for tripartitions S with either $a \in S_1$ or $a \in S_3$:

1. 2 of these marginal contributions for a have coefficient 26,
2. 12 of these marginal contributions for a have coefficient 5, and
3. 12 of these marginal contributions for a have coefficient 2.

Thus we obtain, as expected, the same result. As shown in this example, in general it becomes simpler to deal with j-partitions, with a total number of j^n elements, than roll-calls that count $n!j^n$ elements.

The third procedure involves only winning tripartitions since we apply Equation (1.6) and its coefficients in (1.7). As the number of winning tripartitions in this example is 8, the expression in (1.6) is just the sum of the coefficients in (1.7) corresponding to winning tripartitions. All these coefficients have as a denominator the number of roll-calls: $3!3^3 = 162$. Thus we just need to compute the numerators in (1.7) for the winning tripartitions (we ignore the superscript and subscript since for this game $n = j = 3$). These numerators, $b'(s_1, s_j) = 162 \cdot b(s_1, s_j)$, are shown in Table 1.4. The sum of these coefficients is, as expected, 132 so that $\mathfrak{f}_a(v) = \frac{22}{27}$.

It is important to note that some existing values that are called 'Shapley value' for some extensions of cooperative games do not coincide with the $\mathcal{F}$-value. For instance, we have checked for this simple example the values given by Hsiao and Raghavan for multi-choice games [40], by Bolger [13, 14, 16] for games with r-alternatives or by Bilbao et al. [11] for bicooperative games. We have obtained different results. Note that the $\mathcal{F}$-value coincides with the power index $\mathfrak{f}$ when we are restricted to the ternary case ($j = 3$ with abstention as intermediate input).

winning tripartitions	$b'(s_1, s_j)$
$(\{a,b,c\},\emptyset,\emptyset)$	$b'(3,0) = 54$
$(\{a,b\},\{c\},\emptyset)$	$b'(2,0) = 12$
$(\{a,c\},\{b\},\emptyset)$	$b'(2,0) = 12$
$(\{a,b\},\emptyset,\{c\})$	$b'(2,1) = 15$
$(\{a,c\},\emptyset,\{b\})$	$b'(2,1) = 15$
$(\{a\},\{b,c\},\emptyset)$	$b'(1,0) = 6$
$(\{a\},\{b\},\{c\})$	$b'(1,1) = 9$
$(\{a\},\{c\},\{b\})$	$b'(1,1) = 9$

TABLE 1.4: Numerators of the coefficients $b(s_1, s_j)$ in (1.7) for this game.

1.8 Axiomatization

The first idea that comes to mind is whether Shapley's classic axioms or their adaptation to j-cooperative games serves to characterize the considered value.

It is considerably simple to verify that these axioms are met for the considered value (see the list in next subsection) and it is also quite simple to verify that these are not enough to uniquely characterize it. In cooperative games the axioms of efficiency, anonymity and that of null player determine the Shapley value of the unanimity games, which by induction and the axiom of additivity (or transfer for simple games) uniquely extend the value to the rest of the games.

Thus, if we search for an axiomatic set including these axioms, it seems reasonable to add a conclusive property for determining the value on unanimity games.

1.8.1 Classical axioms for j-cooperative games

In the following, $\psi : \mathcal{J}_N \to \mathbb{R}^n$ is a value for j-cooperative games.

Anonymity (briefly denoted by An) The value ψ satisfies *anonymity* if for all game $v \in \mathcal{J}_N$, any permutation π of N and any $a \in N$

$$\psi_a(v) = \psi_{\pi(a)}(\pi v)$$

where $(\pi v)(S) = v(\pi(S))$.

Null Player (N) The value ψ satisfies the *null player* axiom if given a null player[1] a in the game v, then

$$\psi_a(v) = 0.$$

Efficiency (E) The index ψ satisfies *efficiency* if for any $v \in \mathcal{J}_N$

$$\sum_{a \in N} \psi_a(v) = v(N).$$

Additivity (Ad) The value ψ satisfies *additivity* if for any $v, w \in \mathcal{J}_N$

$$\psi(v + w) = \psi(v) + \psi(w).$$

Transfer (T) The index ψ satisfies *transfer* if for any $v, w \in \mathcal{S}J_N$

$$\psi(v) + \psi(w) = \psi(v \wedge w) + \psi(v \vee w),$$

where $(v \wedge w)(S) = \min\{v(S), w(S)\}$ and $(v \vee w)(S) = \max\{v(S), w(S)\}$ for all $S \in J^N$.

We remark that in the characterization we provide in Theorem 1.3, the weaker condition that can replace anonymity is symmetry. Two players a and b are *equivalent* if for every S such that $\{a, b\} \subseteq S_j$ it holds $m^k(v, S, a) = m^k(v, S, b)$ for all $k = 1, \ldots, j-1$. The value ψ satisfies *symmetry* if for any $a, b \in N$ and game $v \in \mathcal{J}_N$ it holds: $\psi_a(v) = \psi_b(v)$ if a and b are equivalent.

A particular case, for 3-simple games, has been proven in detail in [8] and its extension to arbitrary j-cooperative games does not represent any difficulty so that the tedious but simple proof is omitted. The following trivial result is left for the reader.

Lemma 1.2 *(i) The $\mathfrak{f}$-power index for j-simple games satisfies the axioms of: anonymity, transfer, efficiency and null player.*

(ii) The $\mathcal{F}$-value for j-cooperative games satisfies the axioms of: anonymity, additivity, efficiency and null player.

The basic idea of the classical proof for the Shapley value for cooperative games or the Shapley-Shubik power index for simple games is that the axioms of anonymity, null player and efficiency uniquely characterize the value or

[1] Player a is *null* in the j-cooperative game $v \in \mathcal{J}_N$ if $m^1(v, S, a) = 0$ for all $a \in S_j$.

index on unanimity games, and as these games form a basis or a lattice of the set of games the value or index uniquely extends to the rest of the games by additivity for cooperative games and transfer for both types of games.

If we intend to follow the same thread as in the original respective proofs by Shapley [48] and Dubey [23], we must ascertain how the $\mathcal{F}$-value works on unanimity games. The following lemma establishes the case for which anonymity, null-player and efficiency axioms are sufficient to determine a value on unanimity games.

Lemma 1.3 *Let u_S be the unanimity j-simple game. A value on u_S is uniquely determined by the axioms of anonymity, efficiency and null player* if and only if *there is a unique $i < j$ such that $S_i \neq \emptyset$.*

Proof. ($\Leftarrow$) It is clear that all players in S_j are nulls in u_S, while all players in S_i are anonymous in u_S and as $S_i \cup S_j = N$ by efficiency follows that all players in S_i receive $1/s_i$, while the players in S_j receive 0 for the value.

($\Rightarrow$) We proceed by the way of contradiction. Assume that for at least two indices $i < i' < j$ we have $S_i \neq \emptyset$ and $S_{i'} \neq \emptyset$ in the unanimity game u_S. Consider the value ψ which assigns $1/s_i$ to all players in S_i and zero to the others. Consider the value ψ' which assigns $1/s_{i'}$ to all players in $S_{i'}$ and zero to the others. These two different values satisfy anonymity, efficiency and null-player axioms, a contradiction with the uniqueness assumption. ■

The need of a new axiom to uniquely characterize the value on unanimity games is now clear. Indeed, according to Lemma 1.3 only if $j = 2$ (i.e., for cooperative games) the three axioms uniquely determine the value on unanimity games.

We propose a new axiom on unanimity games that together with the other four uniquely characterize the $\mathcal{F}$-value and the $\mathfrak{f}$-power index for j-cooperative games and j-simple games, respectively.

1.8.2 An axiom on unanimity games

Assume now $j \geq 3$. Let S be any j-partition with $a \in S_1$. When player a shifts her vote to the lower input level i ($i = 2, \ldots, j-1$), we have the following expression:

$$\mathcal{F}_a(u_S) - g_a(u_S) = \frac{j-1}{j-i}(\mathcal{F}_a(u_S) - g_a(u_S)) \tag{1.9}$$

where $g_a(u_s)$ is the value derived by $\mathcal{F}$ in u_S when $a \in S_1$ is the last non-null player in the queue of the roll-calls. Thus, it just lacks to find the value of $g_a(u_S)$ which is the proportion of roll-calls in which a is pivotal for u_S and occupies the last position among the non-null players in u_S. Thus $g_a(u_S)$ is the product of the following three numbers:

1. The proportion of roll-calls for which a is the last non-null player in the queue. This number is

$$\frac{(s_1 + s_2 + \cdots + s_{j-1} - 1)!}{(s_1 + s_2 + \cdots + s_{j-1})!} = \frac{1}{(s_1 + s_2 + \cdots + s_{j-1})} = \frac{1}{n - s_j}.$$

2. The proportion of roll-calls in which a is pivotal in the last non-null player position in the queue. To be pivotal in the last position, it is necessary that the rest of non-null players, who all precede her in the queue, have chosen the same or a better input level than in S. Thus, in her turn, player a can decide either to make a partition T winning by choosing level 1 or losing by choosing levels $2, \ldots, j$. Thus, in order for a to be pivotal, any j-partition T in which the non-null players in u_S different from a have already chosen the input level, with $S \subseteq^j T$ must be pivotal. In fact, T is winning in u_S but it could be losing if a changes her mind to vote for an inferior input level. Consider

$$\frac{|W(u_S)|}{j^n} = \delta(u_S).$$

   in which $\delta(v)$ is the structural decisiveness index of the game v which gives the proportion of winning j-partitions in the game, this extension to j-simple games leads to the *structural decisiveness* index. The structural decisiveness index for simple games was introduced by Coleman [21] and studied in depth in Carreras [19, 20].

3. The number of input levels for which player a is pivotal when she is the last non-null player in the queue is

$$j.$$

The product of these three numbers defines the unknown $g_a(u_S)$ which is

$$g_a(u_S) = j\delta(u_S)\frac{1}{n - s_j} = \frac{j\delta(u_S)}{n - s_j} \tag{1.10}$$

Thus, we can formulate the last axiom for an arbitrary value from (1.9) and the last expression. Note that from (1.10) the expression $g_a(u_S)$ can be interpreted as the decisiveness per capita with respect to non-null players of game u_S multiplied by the number of available inputs for each player.

Axiom of level change effect on unanimity games for $j \geq 3$ (U) Let u_S be a unanimity game and $a \in S_1$. Then

$$\psi_a(u_{S_{a\downarrow i}}) = \frac{1}{j-1}\left[(j-i)\psi_a(u_S) + (i-1)g_a(u_S)\right] \tag{1.11}$$

The next result gives sense what we intend to.

Lemma 1.4 *A value ψ for j-cooperative games that satisfies anonymity, null-player, efficiency and level change effect on unanimity games is uniquely determined on the set of unanimity games.*

Proof. By the Axioms of (An) and (N), the value of ψ in any u_S, where S is a j-partition, depends only on the numbers s_i for all $i = 1, \ldots, j-1$ since $\psi_a(u_S) = 0$ if $a \in S_j$ and $\psi_a(u_S) = \psi_b(u_S)$ if $a, b \in S_i$ for some i. Thus, form now on the vector $\overline{s} := (s_1, s_2, \ldots, s_{j-1}, s_j)$ represents all j-partitions S with these respective cardinalities. In particular, the vector $(n, 0, \ldots, 0)$ represents the j-partition $\mathcal{M}$ which assigns a value of $1/n$ to each player according to Lemma 1.3 which only assumes (An), (N) and (E). Now we consider all vectors lexicographically ordered so that $(n, 0, \ldots, 0)$ is the first in the ranking. The value ψ is then uniquely determined by (An), (N), (E) and (U) on the unanimity games corresponding to the subsequent vectors in the ordering: $(n-1, 1, 0, \ldots, 0)$,...,$(n-1, 0, \ldots, 0, 1)$. From the value of ψ on all these unanimity games we can obtain the value of ψ for all the unanimity games whose vectors verify that $s_1 = n-2$ by applying by (An), (N), (E) and (U). If m is the number of non-null components of $\overline{s}$ in between 2 and $j-1$, both included, then the Axiom (U) is applied m times so that m unanimity games with known ψ with $n-1$ as a vectorial first component intervene. By the finiteness of the number of vectors, the process stops with the determination of ψ for all the unanimity games. ■

To clarify the preceding proof note that the value of ψ on u_S with $\overline{s} := (s_1, s_2, \ldots, s_{j-1}, s_j)$ $(s_1 < n)$ is determined from the values of ψ on unanimity games preceding $\overline{s}$ in lexicographic ordering and with a vectorial first component of $s_1 + 1$. Assume for example that $j = 6$, $n = 8$ and $\overline{s} := (3, 0, 1, 2, 1, 0)$. By Axiom (U), which is given by the recurrence relation in (1.11), ψ is determined in u_S for the player in the third level from the value of ψ in u_T of a player in the first level of $\overline{t} := (4, 0, 0, 2, 1, 0)$. Analogously, by Axiom (U) ψ is determined in u_S for a player in the fourth level from the value of ψ in u_R of a player in the first level of $\overline{r} := (4, 0, 1, 1, 1, 0)$; and by Axiom (U) ψ is determined in u_S for the player in the fifth level from the value of ψ in u_X of a player in the first level of $\overline{x} := (4, 0, 1, 2, 0, 0)$. Finally, the value of ψ for the three players in the first level of $\overline{s}$ are determined by (E) and (An). Thus, ψ is determined on u_S.

1.8.3 An axiomatization for the $\mathcal{F}$-value

The last step for uniquely characterizing the value $\mathcal{F}$-value and the $\mathfrak{f}$-power index on j-cooperative and j-simple games respectively is the extension to all games, but this follows the same guidelines as in the seminal papers by Shapley [48] and Dubey [23], respectively. In our framework the unanimity games also form a basis of the set of j-cooperative games and by additivity (and transfer for j-simple games) the value ψ uniquely extends to the rest of games. We also refer to [8] for the proof for 3-simple games and whose extension to the broader case of multiple input alternatives becomes tedious but simple. The following just states the result.

Theorem 1.3 *(i) A value ψ on j-cooperative games satisfies anonymity, null player, efficiency, level change effect on unanimity games and additivity* if and only if $\psi = \mathcal{F}$.

(ii) A value ψ on j-simple games satisfies anonymity, null player, efficiency, level change effect on unanimity games and transfer if and only if $\psi = \mathfrak{f}$.

We conclude by pointing out that these five axioms are independent as shown in [8] for 3-simple games. The examples used there easily extend to greater values for j.

1.9 The $\mathcal{F}$-value on constant-sum j-cooperative games

Given a j-cooperative game (N, v), we consider

$$a(k) := v(\emptyset, \ldots, \emptyset, \underbrace{\{a\}}_{k}, \emptyset, \ldots, \emptyset, N \setminus \{a\})$$

which is the value that player a can obtain by choosing input level k and without any degree of collaboration by the others. As v is requested to be monotonic, it holds $a(1) \geq a(2) \geq \cdots \geq a(j-1) \geq a(j) = 0$. Thus, the maximum achievement player a can obtain by herself without the collaboration of the others is $a(1)$.

A j-cooperative game (N, v) is of *constant-sum* if

$$v(S) := \sum_{i=1}^{j} \sum_{a \in S_i} a(i)$$

for all $S \in J^N$.

The players do not take advantage of cooperation in this type of games, cooperation does not provide any surplus to them. The following result is quite intuitive and any reasonable value for j-cooperative games should give the same assignment.

Theorem 1.4 *Let (N, v) be a constant-sum j-cooperative game. Then, $\mathcal{F}_a(v) = a(1)$ for all $a \in N$.*

Proof. Observe that in a constant-sum game $m^k(v, S) = a(k)$, while

$m_k(v,S) = a(1) - a(k)$. Then Equation (1.1) becomes

$$\mathcal{F}_a(v) = \frac{1}{j^n n!}\left[\sum_{\substack{S\in J^N:\\ a\in S_j}}\sum_{k=1}^{j-1}\gamma_j^n(s_j-1)a(k) + \sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=2}^{j}\gamma_j^n(s_1-1)(a(1)-a(k))\right]$$

As in the first addend there is one term with $a(1)$ and $a(j) = 0$ it follows:

$$\begin{aligned}\mathcal{F}_a(v) \quad = \quad & \frac{1}{j^n n!}\left[\sum_{\substack{S\in J^N:\\ a\in S_j}}\gamma_j^n(s_j-1)a(1) + \sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=2}^{j}\gamma_j^n(s_1-1)a(1)\right.\\ & \left. + \left(\sum_{\substack{S\in J^N:\\ a\in S_j}}\sum_{k=2}^{j-1}\gamma_j^n(s_j-1) - \sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=2}^{j-1}\gamma_j^n(s_1-1)\right)a(k)\right]\end{aligned} \tag{1.12}$$

As there is a bijection between the j-partitions in which $a \in S_1$ and those in which $a \in S_j$ we can group the terms in the first row of (1.12) and also deduce that the addends in the second row of (1.12) cancel. Thus, the previous expression is simplified to

$$\begin{aligned}\mathcal{F}_a(v) = \frac{a(1)}{j^n n!}\left[\sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=1}^{j}\gamma_j^n(s_1-1)\right] &= \frac{a(1)}{j^n n!}\left[\sum_{k=1}^{j}\sum_{\substack{S\in J^N:\\ a\in S_1}}\gamma_j^n(s_1-1)\right]\\ = \frac{a(1)}{j^n n!}\left[j\left(\sum_{\substack{S\in J^N:\\ a\in S_1}}\gamma_j^n(s_1-1)\right)\right]\end{aligned} \tag{1.13}$$

As the last addend in (1.13) counts the total number of roll-calls such that $a \in S_1$ which is $j^{n-1}n!$, we have: $\mathcal{F}_a(v) = a(1)$. ■

1.10 Generating functions for computing the $\mathcal{F}$-value for weighted j-simple games

In this section we show the method of generating functions to compute the value proposed in this paper. Although everything we do is extendible to j-simple games for any $j \geq 2$, we just consider, for avoiding more notation complications, the case $j = 3$ which includes ternary voting systems. We focus on this case because we are interested in computing the value for the UNSC voting system and a natural variation of it.

Formula (1.1) for ternary cooperative game reduces to

$$\begin{aligned}\mathcal{F}_a(v) &= \tfrac{1}{3^n n!}\left[\sum_{S:a\in S_1}(\gamma_3^n(s_3)+\gamma_3^n(s_1-1))\,[v(S)-v(S_{a\downarrow 3})]\right]\\ &+ \tfrac{1}{3^n n!}\left[\sum_{S:a\in S_1}\gamma_3^n(s_1-1)\,[v(S)-v(S_{a\downarrow 2})]\right] \qquad (1.14)\\ &+ \tfrac{1}{3^n n!}\left[\sum_{S:a\in S_2}\gamma_3^n(s_3)\,[v(S)-v(S_{a\downarrow 3})]\right].\end{aligned}$$

As in j-simple games, all marginal contributions are either 1 or 0, it is convenient to use the two sets:

$$\mathcal{C}_a^{YA}(v)=\{S\in 3^N:\ a\in S_1, S\in W, S_{a\downarrow 2}\notin W\}$$
$$\mathcal{C}_a^{AN}(v)=\{S\in 3^N:\ a\in S_1, S_{a\downarrow 2}\in W, S_{a\downarrow 3}\notin W\ \}$$

and then compute the power index as

$$\begin{aligned}\mathcal{F}_a(v) &= \tfrac{1}{3^n n!}\left[\sum_{S\in\mathcal{C}_a^{YA}(v)}(\gamma_3^n(s_3)+2\gamma_3^n(s_1-1))\right.\\ &\left. + \tfrac{1}{3^n n!}\sum_{S\in\mathcal{C}_a^{AN}(v)}(2\gamma_3^n(s_3)+\gamma_3^n(s_1-1))\right]. \qquad (1.15)\end{aligned}$$

The delay in the development of a convincing theory for simple games with ordered alternatives is possibly due to the lack of a consistent notion of weighted game in this context. This important issue was solved with the concept of weighted j-simple game provided in [33]. A characterization for it in terms of trade robustness was provided there, since then several alternative works deal with the notion of weighted j-simple game, among others [34, 35, 36].

Such definition for binary voting systems reduces to the existence of j ordered weights, that respect monotonicity, for each voter and a quota such that a j-partition S is winning if the sum of the weights of voters at the level of approval they choose is greater or equal than the quota. As observed in [33], one of these j weights can be normalized at zero. In the context of ternary voting games where the options for voters are: voting 'yes', 'abstaining' or voting 'no', it seems natural normalizing at the level of abstention and thus, every voter has a non-negative weight for voting yes and a non-positive weight for voting no. Thus we can associate to each voter $a\in N$ the triple $(w_a^{yes}, w_a^{abs}, w_a^{no})$ with $w_a^{yes}\geq w_a^{abs}\geq w_a^{no}$, and after normalization at the intermediate level, we have: $w_a^{yes}\geq 0$, $w_a^{abs}=0$ and $w_a^{no}\leq 0$.

Let us consider that a representation for the weighted game is:

$$v\equiv[q;(w_1^{yes},w_1^{no}),\cdots,(w_n^{yes},w_n^{no})]$$

where q is the *quota*. Thus,

$$v(S) = 1 \text{ if and only if } w(S) := \sum_{i \in S_1} w_i^{yes} + \sum_{i \in S_3} w_i^{no} \geq q$$

Since we have the explicit formula (1.15), in case of a weighted game we can compute the power index by using generating functions. Generating functions for computing power indices have been used in many works among others [1, 2, 3, 10, 18]. Generating functions for 3-simple games have been used in [32] for computing the Banzhaf power index and some other power indices. We now introduce generating functions for computing the power index $\mathfrak{f}$ for the UNSC voting system with abstention.

Definition 1.4 *Let $v \equiv [q; (w_1^{yes}, w_1^{no}), \cdots, (w_n^{yes}, w_n^{no})]$ be a representation of a weighted game with abstention. For any $a \in N$, the generating function is defined as*

$$F_a(x) = \prod_{p \in N, p \neq a} \left(y x^{w_p^{yes}} + 1 + t x^{w_p^{no}} \right) \tag{1.16}$$

Observe that the role of the variables y and t are the counting of the number of 'yes'-voters and 'no'-voters, respectively. Then, there is no need to count the number of abstainers since it can be deduced since the number of voters is known. Note also that the power of the variable x is the weight, which in the case of an abstainer is zero, which explains the 1 in the middle position.

The function $F_a(x)$ can also be written as

$$F_a(x) = \sum_{k=\underline{w}}^{\overline{w}} \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} b_{k,i,j} y^i t^j x^k$$

where $\underline{w} = \sum_{i \in N} w_i^{no}$ and $\overline{w} = \sum_{i \in N} w_i^{yes}$.

In the previous formula, the coefficient $b_{k,i,j}$ counts the number of tripartitions S of total weight k such that there are i players in S_1 and h players in S_3. Using these coefficients, Equation (1.15) becomes

$$\mathcal{F}_a(v) = \frac{1}{3^n n!} \left[\sum_{k=q-w_a^{yes}}^{q-1} b_{k,i,h} (2\gamma_j^n(i) + \gamma_j^n(h)) + \sum_{k=q}^{q-w_a^{no}-1} b_{k,i,h} (\gamma_j^n(i) + 2\gamma_j^n(h)) \right] \tag{1.17}$$

for any player a such that $\mathcal{C}_a^{YA}(v) \neq \emptyset$ and $\mathcal{C}_a^{AN}(v) \neq \emptyset$. If voter a is null, then the $\mathcal{F}$-value is zero. If voter a is null in the YA-level (which implies $\mathcal{C}_a^{YA}(v) = \emptyset$) but not in the AN-level (which implies $\mathcal{C}_a^{AN}(v) \neq \emptyset$), then the first addend in (1.17) must be replaced by 0; and conversely, if voter a is not null in the YA-level (which implies $\mathcal{C}_a^{YA}(v) \neq \emptyset$) but it is in the AN-level (which implies $\mathcal{C}_a^{AN}(v) = \emptyset$), then the second addend in (1.17) must be replaced by 0.

1.11 Examples revisited

Example 1.6 (Example 1.2 revisited) *As $n = j = 3$, the coefficients in (1.2) are: $\gamma_3^3(0) = 2$, $\gamma_3^3(1) = 5$ and $\gamma_3^3(2) = 26$; we then obtain $\mathcal{F}(v) = (2, 1, 1)$ after the substitution in (1.1) where the payment 2 is for the qualified worker a and 1 is the payment for each of the other two.*

Example 1.7 (Example 1.3 revisited) *Each test plays the role of a player. As we did in the previous example, we could use (1.1) with its coefficients $\gamma_{11}^2(0) = 1$ and $\gamma_{11}^2(1) = 12$ to obtain $\mathcal{F}(v) = (60, 40)$. However, the result directly follows from Theorem 1.4 since v is a constant-sum game. Thus, the importance of each test for the exam is given by the intuitive assignment $(60, 40)$ that preserves the relative importance between the two parts.*

Example 1.8 (Example 1.4 revisited) *Recall that for the UNSC voting system, the winning tripartitions S satisfy*

$$|S_1| \geq 9 \ \text{ and } S_3 \cap P = \emptyset.$$

We compute the value by using the method of generating functions. A weighted representation for this voting system, see [33], is given by a threshold of 9 a weight of $(1, 0, -6)$ for each permanent member and a weight of $(1, 0, 0)$ for a non-permanent member.

We now compute the power index by using its expression in Equation (1.15). It is then clear that for a permanent member p it holds:

$$\mathcal{C}_p^{YA}(v) = \{S : p \in S_1, |S_1| = 9, \text{ and } |S_3 \cap P| = \emptyset\}$$

and

$$\mathcal{C}_p^{AN}(v) = \{S : p \in S_1, |S_1| > 9 \text{ and } |S_3 \cap P| = \emptyset\}.$$

So,

$$\mathfrak{f}_p(v) = \sum_{s_3=0}^{6} \left\{ [2\gamma_3^{15}(8) + \gamma_3^{15}(s_3)] \sum_{j=\max\{0, s_3-2\}}^{4} \binom{4}{j} \binom{10}{8-j} \binom{j+2}{s_3} \right\} + \sum_{s_1=10}^{15} \sum_{s_3=0}^{15-s_1} \left\{ [\gamma_3^{15}(s_1 - 1) + 2\gamma_3^{15}(s_3)] \sum_{j=\max\{0, s_1+s_3-11\}}^{4} \binom{4}{j} \binom{10}{s_1 - 1 - j} \binom{11 - s_1 + j}{s_3} \right\}.$$

On the other hand, for a non-permanent r we have

$$\mathcal{C}_r^{YA}(v) = \{S : r \in S_1, |S_1| = 9, \text{ and } |S_3 \cap P| = \emptyset\}$$

and $\mathcal{C}_r^{AN}(v) = \emptyset$. Thus,

$$\mathfrak{f}_r(v) = \sum_{s_3=0}^{6} \left\{ [2\gamma_3^{15}(8) + \gamma_3^{15}(s_3)] \sum_{j=\max\{0, s_3-1\}}^{5} \binom{5}{j} \binom{9}{8-j} \binom{j+1}{s_3} \right\}.$$

Using these formulas we obtain

$$\mathfrak{f}_p(v) = 0.16338987329859317, \qquad \mathfrak{f}_r(v) = 0.01830506335070341.$$

$$\mathfrak{f}_p(v) \approx 0.16339, \qquad \mathfrak{f}_r(v) \approx 0.018305.$$

It is a close result to the one computed in [30] by using (1.8)*, although it differs a bit from it. Likely the difference lies in a rounding problem. Observe that the relative importance according to this index for the two types of voters is given by*

$$\frac{\mathfrak{f}_p(v)}{\mathfrak{f}_r(v)} \approx 8.93,$$

which is still too big in favor of the permanent nations.

Example 1.9 (Example 1.5 revisited) *Recall that the modification of the UNSC we have proposed converts the five losing tripartitions* $(N \setminus \{p\}, \emptyset, \{p\})$ *for all* $p \in P$ *into winning. The remaining tripartitions do not change its status.*

This new 3*-simple game can still be represented as a weighted game with quota* $q = 9$ *and vector of weights for the permanent members* $(1, 0, -5)$ *and* $(1, 0, 0)$ *for non-permanent members.*

Using again the generating function method, the values we obtain for a permanent member p *and for a non-permanent member* r *are:*

$$\mathfrak{f}_p(v) = 0.013958034451108942, \qquad \mathfrak{f}_q(v) = 0.030209827744455294.$$

$$\mathfrak{f}_p(v) \approx 0.013958, \qquad \mathfrak{f}_q(v) \approx 0.03021.$$

Observe that the relative importance according to this index for the two types of voters is in this slightly modified example:

$$\frac{\mathfrak{f}_p(v)}{\mathfrak{f}_r(v)} \approx 4.62.$$

i.e., the relative importance has been reduced to almost half with respect to the standard model.

The United Nations Security Council is critical to global peace and security, yet more than twenty years of negotiations over its reform have proved fruitless. The change proposal we do for the UNSC voting system only alters five tripartitions over more than 14.3 million. As shown, this has two effects. On the one hand, it reduces the relative power to the half between the two types of voters and, on the other hand, it avoids veto power by permanent members in an acceptable way:

'if everyone thinks differently, it is that I must be wrong'.

1.12 Conclusion

The value proposed in this paper for j-cooperative games or multi-choice games has ingredients to be a generalization of the Shapley value and it can make stake out which is the most reasonable extension for the well-known value to the broader context considered. Among the arguments supporting the value proposed here, we can find the following: it is totally consistent in its particularization from j-cooperative games to j-simple games; it admits an explicit formula in terms of the characteristic function; it is supported by a probabilistic model; it is supported by an axiomatic characterization; it assigns to each player a single numerical value that does not depend on input alternatives.

The capacity of theoretical studies and applications of the value on the contexts described are enormous and future research is encouraged.

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