Computation of Mean-Field Equilibria with Correlated Stochastic Processes

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Abstract. The numerical algorithm is presented for solving differential problem formulated as the Mean-Field Game (MFG) with the coupled system of two parabolic partial differential equations: the Fokker-Plank-Kolmogorov equation and the Hamilton-Jacobi-Bellman one. The case is considered with correlation of the considered stochastic processes. The description focuses on the discrete semi-Lagrangian approximation of these equations and on the application of the MFG theory directly at discrete level. The constructed algorithm is implemented to the problem of carbon dioxide pollution as an illustration.

Keywords: optimal control, Mean-Field Game, numerical approximation and algorithms, finite differences, carbon dioxide pollution

1 Introduction

The Mean-Field Game (MFG) approach is theoretically developed in papers by J.-M. Lasry and P.-L. Lions [1-2]. A “historical” information on the rapid development of this area of investigations is presented in the brief monograph [3] providing mathematical analysis. This approach has been adapted to many problems in physics, biology, engineering, and economics [1-9]. In the problem formulated here, the mean-field equilibrium is described by the coupled system of two parabolic partial differential equations: the Fokker-Plank-Kolmogorov (FPK) equation and the Hamilton-Jacobi-Bellman (HJB) one.

In this paper, we focus on the discrete approximation of these equations and on an application of the MFG theory directly at discrete level. Contrary to difference schemes applied by other authors, we propose the semi-Lagrangian approximation which improves some properties of a discrete problem of this type. Earlier this approximation was used for solving the same type of one-dimensional problem [6].

2 The mathematical model

We shall not derive the differential statement of the MFG problem and refer the reader to comprehensive book [3] for the general description. We begin with the Kolmogorov equation (which is called Fokker-Plank one in other content) for the density $m(t,x,y)$ of “atomized” agents on the rectangle $(0,H_1)\times(0,H_2)$ at time segment $t\in[0,T]$. The term “atomized” means that each of infinite agents has no influence on the situation (because of its zero measure support) but chooses the rational strategy that takes into account its own position and the agents’ distribution $m(t,x,y)$. This approach produces the following managerial problem [3].

Put $\Omega = (0, H_1) \times (0, H_2)$ with the boundary Γ and the closure $\bar{\Omega} = \Omega \bigcup \Gamma$. First, we introduce the forward FPK problem [8]:

$$\frac{\partial m}{\partial t} - \frac{\sigma_1^2}{2}\frac{\partial^2 m}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 m}{\partial y^2} - \gamma\sigma_1\sigma_2\frac{\partial^2 m}{\partial x \partial y} + \frac{\partial(\alpha m)}{\partial x} + \frac{\partial(\beta m)}{\partial y} = 0 \quad \text{in } (0,T)\times\Omega \tag{1}$$

$$m(0,x,y) = m_0(x,y) \quad \text{on } \bar{\Omega}, \qquad \partial m/\partial n = 0 \quad \text{on } [0,T]\times\Gamma \tag{2}$$

where $\partial m/\partial n$ means the normal derivative at boundary points of Γ. Here $\sigma_1 > 0,\ \sigma_2 > 0$ are fixed constants characterizing the probable noises produced by the Brownian motion [7] and $\gamma \in (-1, 1)$ is a coefficient of correlation for these stochastic processes. The control functions $\alpha(t,x,y), \beta(t,x,y)$ reflect the efforts directed towards the decreasing of m. Besides, (2) defines the initial density of agents on $\bar{\Omega}$ and provides the stay of agents on a "feasible" closed domain $\bar{\Omega}$ at each moment of time and interferes their exit out of limits.

In the model used, we want to minimize the cost functional

$$J(m,\alpha,\beta) = \int_0^T \exp(-rt)\int_\Omega \left(d_1\alpha^2 m/2 + d_2\beta^2 m/2 + g(t,x,y,m)\right)\mathrm{d}\Omega\,\mathrm{d}t. \tag{3}$$

Here d_1, d_2 are positive constants; nonnegative r is the risk-free discount rate. For function $g(t,x,y,m)$ we demand its concavity in argument m:

$$g(t,x,y,\tilde{m}) - g(t,x,y,m) \le \left(\tilde{m} - m\right) f(t,x,y,m) \ \text{ with } \ f(t,x,y,m) = \partial g/\partial m(t,x,y,m) \tag{4}$$

for all admissible values of other arguments.

So, we get the optimization problem

$$\begin{cases} \inf\limits_{\alpha,\beta} J(m,\alpha,\beta) = \int_0^T \exp(-rt)\int_\Omega \left(d_1\alpha^2 m/2 + d_2\beta^2 m/2 + g(t,x,y,m)\right)\mathrm{d}\Omega\,\mathrm{d}t, \\ \frac{\partial m}{\partial t} - \frac{\sigma_1^2}{2}\frac{\partial^2 m}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 m}{\partial y^2} - \gamma\sigma_1\sigma_2\frac{\partial^2 m}{\partial x \partial y} + \frac{\partial(\alpha m)}{\partial x} + \frac{\partial(\beta m)}{\partial y} = 0 \quad \text{in } (0,T)\times\Omega \end{cases} \tag{5}$$

for the initial and the boundary conditions (2).

Here we briefly describe a formal way to get the optimality conditions for this differential problem. The rigorous derivation can be found in [3]. We will not use these differential justifications in our algorithms and give them only as the clear illustration for our considerations at discrete level.

Take an arbitrary function $v \in C^\infty([0,T]\times\bar{\Omega})$, multiply (1) by it, and integrate by parts with respect to t and x:

$$\begin{aligned} &-\int_0^T\int_\Omega \left(\frac{\partial v}{\partial t} + \frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \gamma\sigma_1\sigma_2\frac{\partial^2 m}{\partial x \partial y} + \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y}\right) m\,\mathrm{d}\Omega\,\mathrm{d}t \\ &+\int_\Omega \left(v(T,x,y)m(T,x,y) - v(0,x,y)m_0(x,y)\right)\mathrm{d}\Omega = 0 \end{aligned} \tag{6}$$

taking into account the boundary condition similar to (2)

$$\partial v/\partial n = 0 \quad \text{on } [0,T]\times\Gamma. \tag{7}$$

In addition to the cost functional, we also formulate the Lagrangian of problem (5)

$$\Im(m,\alpha,\beta,v) := J(m,\alpha,\beta) - \int_\Omega \big(v(T,x,y)m(T,x,y) - v(0,x,y)m_0(x,y)\big)\,\mathrm{d}\Omega$$
$$+\int_0^T\int_\Omega\left(\frac{\partial v}{\partial t}+\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\gamma\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}+\alpha\frac{\partial v}{\partial x}+\beta\frac{\partial v}{\partial y}\right)m\,\mathrm{d}\Omega\,\mathrm{d}t. \tag{8}$$

Thus, the minimization problem (5) may be rewritten [7] as the saddle point problem

$$\inf_{(m,\alpha,\beta)}\sup_v \Im(m,\alpha,\beta,v). \tag{9}$$

After "differentiation" with respect to some functions, we get the backward HJB equation with the initial and boundary conditions:

$$\frac{\partial v}{\partial t}+\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\gamma\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}+\alpha\frac{\partial v}{\partial x}+\beta\frac{\partial v}{\partial y}$$
$$=-\exp(-rt)\big(f+d_1\alpha^2/2+d_2\beta^2/2\big) \text{ on } [0,T]\times\Omega, \tag{10}$$

$$v(T,x,y)=0 \quad\text{on}\quad \bar\Omega, \qquad \partial v/\partial n = 0 \quad\text{on}\quad [0,T]\times\Gamma, \tag{11}$$

$$\alpha = -\frac{\exp(rt)}{d_1}\frac{\partial v}{\partial x},\ \beta = -\frac{\exp(rt)}{d_2}\frac{\partial v}{\partial y} \text{ on } [0,T]\times\bar\Omega, \tag{12}$$

which characterizes a saddle point [3] in addition to (1)-(2).

3 The numerical solution of the FPK equation

So, we have to solve problem (1)-(2) with the functions α and β satisfying the property $\alpha=\beta=0$ on $[0,T]\times\Gamma$ due to (11) and (12). Introduce discrete uniform grids in time and in space: $t_k = k\tau,\quad k=0,\dots,M,\ \tau=T/M;\ x_{i+1/2}=(i+1/2)h_1,\quad i=-1,\dots,N_1,\ h_1=H_1/N_1;$ $y_{j+1/2}=(j+1/2)h_2,\quad j=-1,\dots,N_2,\ h_x=H_2/N_2;$ for integers $M,N_1,N_2\ge 2.$ Denote $\mathbf{z}_{i+1/2,j+1/2}=(x_{i+1/2},y_{j+1/2})$ and put $\bar\Omega_h=\{\mathbf{z}_{i+1/2,j+1/2};\ i=-1,\dots,N_1,\ j=-1,\dots,N_2\}$ and $\Omega_h=\{\mathbf{z}_{i+1/2,j+1/2};\ i=0,\dots,N_1-1,\ j=0,\dots,N_2-1\}$. Introduce also the points $x_i = ih_1,\quad i=0,\dots,N_1;$ $y_j = jh_2,\quad j=0,\dots,N_2.$

We shall find a solution to this problem as a grid function $m^h(t,x,y)$ at each time level t_k on $\bar\Omega_h$. Split the approximation of equation (1) into two parts. First, consider the operator along axis *Ox*: $1/2\,\partial m/\partial t+\partial(\alpha m)/\partial x$. To approximate it at time level t_k for each node $\mathbf{z}_{i+1/2,j+1/2}\in\Omega_h$, fix $y_{j+1/2}$ and use the approximation by analogy to paper [10] at plane $y=y_{j+1/2}$:

$$m^h_{k,i+1/2,j+1/2}/\tau-\beta^{i+1/2,j+1/2}_{k-1,i-1/2}m^h_{k-1,i-1/2,j+1/2}-\beta^{i+1/2,j+1/2}_{k-1,i+1/2}m^h_{k-1,i+1/2,j+1/2}-\beta^{i+1/2,j+1/2}_{k-1,i+3/2}m^h_{k-1,i+3/2,j+1/2} \tag{14}$$

where

$$\beta^{i+1/2,j+1/2}_{k-1,i-1/2,j+1/2}=\big(x_i-x^-_{k-1,i,j+1/2}\big)/2\tau h_1,\quad \beta^{i+1/2,j+1/2}_{k-1,i+1/2,j+1/2}=\big(x^-_{k-1,i+1,j+1/2}-x^+_{k-1,i,j+1/2}\big)/2\tau h_1,$$
$$\beta^{i+1/2,j+1/2}_{k-1,i+3/2,j+1/2}=\big(x^+_{k-1,i+1,j+1/2}-x_{i+1}\big)/2\tau h_1, \tag{15}$$
$$x^\pm_{k-1,i,j+1/2}=x_i\pm\max\{0,2\tau a_{k-1,i,j+1/2}\}\ \text{ and }\ x^\pm_{k-1,i+1,j+1/2}=x_i\pm\max\{0,2\tau a_{k-1,i+1,j+1/2}\}.$$

The point in a subscript means that any *appropriate* value may be taken in this position. Now consider the operator along axis *Oy:* $1/2\,\partial m/\partial t+\partial(\beta m)/\partial y$. To approximate it at time level t_k for each node $\mathbf{z}_{i+1/2,\,j+1/2}\in\Omega_h$, fix $x_{i+1/2}$ and use the approximation analogous to (14) at plane $x=x_{i+1/2}$:

$$m^h_{k,i+1/2,\,j+1/2}\big/\tau-\beta^{i+1/2,\,j+1/2}_{k-1,\,j-1/2}m^h_{k-1,i+1/2,\,j-1/2}-\beta^{i+1/2,\,j+1/2}_{k-1,\,j+1/2}m^h_{k-1,i+1/2,\,j+1/2}-\beta^{i+1/2,\,j+1/2}_{k-1,\,j+3/2}m^h_{k-1,i+1/2,\,j+3/2} \tag{16}$$

where

$$\begin{aligned}&\beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j-1/2}=\left(y_j-y^-_{k-1,i+1/2,\,j}\right)\big/2\tau h_2,\quad \beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2}=\left(y^-_{k-1,i+1/2,\,j+1}-y^+_{k-1,i+1/2,\,j}\right)\big/2\tau h_2,\\&\beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+3/2}=\left(y^+_{k-1,i+1/2,\,j+1}-y_{j+1}\right)\big/2\tau h_2,\end{aligned} \tag{17}$$

$$y^\pm_{k-1,i+1/2,\,j}=y_j\pm\max\left\{0,2\tau b_{k-1,i+1/2,\,j}\right\}\quad\text{and}\quad y^\pm_{k-1,i+1/2,\,j+1}=y_{j+1}\pm\max\left\{0,2\tau b_{k-1,i+1/2,\,j+1}\right\}.$$

Now consider the approximation of diffusion part

$$-\frac{\sigma_1^2}{2}\frac{\partial^2 m}{\partial x^2}-\frac{\sigma_2^2}{2}\frac{\partial^2 m}{\partial y^2}-\gamma\sigma_1\sigma_2\frac{\partial^2 m}{\partial x\partial y} \tag{18}$$

The most part of discrete approximations is not appropriate for our demands since the approximation must produce self-adjoint operator with M-property. Therefore, we take the stencil of different scheme dependent on sign of γ. Introduce the expressions

$$A_{i+1/2,\,j+1/2}=\begin{bmatrix}a_{i-1/2,\,j+3/2}&a_{i+1/2,\,j+3/2}&a_{i+3/2,\,j+3/2}\\a_{i-1/2,\,j+1/2}&a_{i+1/2,\,j+1/2}&a_{i+3/2,\,j+1/2}\\a_{i-1/2,\,j-1/2}&a_{i+1/2,\,j-1/2}&a_{i+3/2,\,j-1/2}\end{bmatrix},\;M_{k,i+1/2,\,j+1/2}=\begin{bmatrix}m_{k,i-1/2,\,j+3/2}&m_{k,i+1/2,\,j+3/2}&m_{k,i+3/2,\,j+3/2}\\m_{k,i-1/2,\,j+1/2}&m_{k,i+1/2,\,j+1/2}&m_{k,i+3/2,\,j+1/2}\\m_{k,i-1/2,\,j-1/2}&m_{k,i+1/2,\,j-1/2}&m_{k,i+3/2,\,j-1/2}\end{bmatrix}$$

with the 9-point scalar product between corresponding entries and first, treat the case $\gamma\ge 0$. Put

$$\begin{aligned}&a_{i-1/2,\,j+3/2}=a_{i+3/2,\,j-1/2}=0;\quad a_{i+1/2,\,j+3/2}=a_{i+1/2,\,j-1/2}=-\sigma_1^2\big/2h_1^2+\gamma\sigma_1\sigma_2\big/2h_1h_2;\\&a_{i-1/2,\,j+1/2}=a_{i+3/2,\,j+1/2}=-\sigma_2^2\big/2h_2^2+\gamma\sigma_1\sigma_2\big/2h_1h_2;\\&a_{i+3/2,\,j+3/2}=a_{i-1/2,\,j-1/2}=-\gamma\sigma_1\sigma_2\big/2h_1h_2;\quad a_{i+1/2,\,j+1/2}=1/\tau+\sigma_1^2\big/h_1^2+\sigma_2^2\big/h_2^2-\gamma\sigma_1\sigma_2\big/h_1h_2.\end{aligned} \tag{19}$$

Off-diagonal coefficients will be non-positive if

$$|\gamma|\sigma_2\big/\sigma_1\le h_2\big/h_1\le\sigma_2\big/|\gamma|\sigma_1. \tag{20}$$

In principle, these inequalities are satisfied when $h_2\simeq h_1\sigma_2/\sigma_1$. And the less $\gamma\in[0,1)$ the wider boundary of the ratio h_2/h_1. In the case $\gamma<0$ we take other coefficients:

$$\begin{aligned}&a_{i+3/2,\,j+3/2}=a_{i-1/2,\,j-1/2}=0;\quad a_{i+1/2,\,j+3/2}=a_{i+1/2,\,j-1/2}=-\sigma_1^2\big/2h_1^2-\gamma\sigma_1\sigma_2\big/2h_1h_2;\\&a_{i-1/2,\,j+1/2}=a_{i+3/2,\,j+1/2}=-\sigma_2^2\big/2h_2^2-\gamma\sigma_1\sigma_2\big/2h_1h_2;\\&a_{i-1/2,\,j+3/2}=a_{i+3/2,\,j-1/2}=\gamma\sigma_1\sigma_2\big/2h_1h_2;\quad a_{i+1/2,\,j+1/2}=1/\tau+\sigma_1^2\big/h_1^2+\sigma_2^2\big/h_2^2+\gamma\sigma_1\sigma_2\big/h_1h_2.\end{aligned} \tag{21}$$

Inequalities (20) ensure non-positive off-diagonal coefficients. In both cases we have the approximation of order $O(h_1^2+h_2^2)$.

Thus, the combination of (15)-(18) and (19), (21) gives the difference scheme

$$\begin{aligned}A_{i+1/2,\,j+1/2}\cdot M_{k,i+1/2,\,j+1/2}&=\beta^{i+1/2,\,j+1/2}_{k-1,i-1/2,\,j+1/2}m^h_{k-1,i-1/2,\,j+1/2}+\beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2}m^h_{k-1,i+1/2,\,j+1/2}\\&+\beta^{i+1/2,\,j+1/2}_{k-1,i+3/2,\,j+1/2}m^h_{k-1,i+3/2,\,j+1/2}+\beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j-1/2}m^h_{k-1,i+1/2,\,j-1/2}+\beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2}m^h_{k-1,i+1/2,\,j+1/2}\\&+\beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+3/2}m^h_{k-1,i+1/2,\,j+3/2}\quad\forall\;i=0,\ldots,N_1-1,\;j=0,\ldots,N_2-1,\end{aligned} \tag{22}$$

with initial condition

$$m^h_{k,i+1/2,\,j+1/2}=m_0(x_{i+1/2},y_{j+1/2})\;\forall\;i=0,\ldots,N_1-1,\;j=0,\ldots,N_2-1. \tag{23}$$

For any node $\boldsymbol{z}_{i+1/2,\,j+1/2} \in \bar{\Omega}^h \setminus \Omega^h$ outside of the domain Ω, we put

$$m^h_{k,i+1/2,\,j+1/2} = m^h_{k,i'+1/2,\,j'+1/2} \tag{24}$$

with the nearest node $\boldsymbol{z}_{i'+1/2,\,j'+1/2} \in \Omega^h$ to ensure Neumann discrete boundary condition. We will exclude values with arguments out of a rectangle by means of (24) and designate the remained system of the linear algebraic equations as

$$\mathfrak{A} m^h_{\cdot,\cdot,\cdot} = \mathfrak{F} m^h_{0,\cdot,\cdot} \tag{25}$$

Remark. Let inequalities (21) hold and

$$\tau\left|\alpha^h_{k,i,\,j+1/2}\right| \le h_1/8, \quad \tau\left|\beta^h_{k,i+1/2,\,j}\right| \le h_2/8 \quad \forall\ k = 0,\dots,M-1,\ \ i = 1,\dots,N_1-1,\ \ j = 1,\dots,N_2-1. \tag{26}$$

Then, all coefficients in the right-hand side of (23) are nonnegative. Let all values $m^h_{k-1,i+1/2,\,j+1/2}$ at time level $t = t_{k-1}$ be nonnegative too. Then, due to the property of an M-matrix, all values $m^h_{k,i+1/2,\,j+1/2}$ at time level $t = t_k$ are also nonnegative.

4 The discrete optimal conditions

Instead of cost functional (3), introduce the discrete one

$$\begin{aligned} J^h(m^h,\alpha^h,\beta^h) = \sum_{k=0}^{M-1}\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} \exp(-rt_k)\Big(d_1 r^h_{k,i+1/2,\,j+1/2}\, m^h_{k,i+1/2,\,j+1/2}\big/2 \\ + d_2 s^h_{k,i+1/2,\,j+1/2} m^h_{k,i+1/2,\,j+1/2}\big/2 + g^h_{k,i+1/2,\,j+1/2}\Big) h_1 h_2 \tau \end{aligned} \tag{27}$$

for

$$\begin{aligned} &r^h_{k,i+1/2,\cdot} = (\alpha^h_{k,i,\cdot})^2\big/2 + (\alpha^h_{k,i+1,\cdot})^2\big/2,\ s^h_{k,\cdot,\,j+1/2} = (\beta^h_{k,\cdot,\,j})^2\big/2 + (\beta^h_{k,\cdot,\,j+1})^2\big/2, \\ &g^h_{k,i+1/2,\,j+1/2} = g(t_k, x_{i+1/2}, y_{j+1/2}, m^h_{k,i+1/2,\,j+1/2}). \end{aligned} \tag{28}$$

Note that we have the approximations of the second order in x, y and the first order in t.

Thus, we have the discrete problem for the minimization of (27) with condition (25):

$$\begin{cases} \inf\limits_{\alpha^h,\beta^h} J^h(m^h,\alpha^h,\beta^h), \\ \mathfrak{A} m^h_{\cdot,\cdot,\cdot} = \mathfrak{F} m^h_{0,\cdot,\cdot}. \end{cases} \tag{29}$$

To formulate the discrete optimal control problem, introduce a discrete function $v^h_{\cdot,\cdot,\cdot} = \left\{v^h_{k,i+1/2,\,j+1/2};\ k = 0,\dots,M,\ i = -1,\dots,N_1,\ \ j = -1,\dots,N_2\right\}$ with the property like (24).

We omit intermediate considerations at discrete level that are analogous to the differential ones (5)-(12) and derive the system of discrete equations characterizing optimal condition:

$$\begin{aligned} &A_{i+1/2,\,j+1/2}\cdot V_{k-1,i+1/2,\,j+1/2} = \beta^{i+1/2,\,j+1/2}_{k-1,i-1/2,\,j+1/2} v^h_{k,i-1/2,\,j+1/2} + \beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2} v^h_{k,i+1/2,\,j+1/2} \\ &+\beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2} v^h_{k,i+3/2,\,j+1/2} + \beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2} v^h_{k,i+1/2,\,j-1/2} + \beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2} v^h_{k,i+1/2,\,j+1/2} + \beta^{i+1/2,\,j+1/2}_{k-1,i+1/2,\,j+1/2} v^h_{k,i+1/2,\,j+3/2} \\ &+ b^h_{k,i+1/2,\,j+1/2} \quad \forall\ k = 0,\dots,M-1,\ \ i = 0,\dots,N_1-1,\ \ j = 0,\dots,N_2-1, \end{aligned} \tag{30}$$

for the initial data

$$v^h_{M,\cdot,\cdot} = \mathbf{0} \tag{31}$$

where

$$b^h_{k,i+1/2,\,j+1/2} = -\exp(-rt_k)\Big(f(t_k, x_{i+1/2}, y_{i+1/2},\ m^h_{k,i+1/2,\,j+1/2}) + d_1\, r^h_{k,i+1/2,\,j+1/2}\big/2 + d_2 s^h_{k,i+1/2,\,j+1/2}\big/2\Big). \tag{32}$$

Again, for any node $\boldsymbol{z}_{i+1/2,\,j+1/2} \in \bar{\Omega}^h \setminus \Omega^h$ (outside of the domain Ω), we put

$$v^h_{k,i+1/2,j+1/2} = v^h_{k,i'+1/2,j'+1/2} \tag{33}$$

with the nearest node $\boldsymbol{z}_{i'+1/2,j'+1/2} \in \Omega^h$ in accordance with zeroth Neumann boundary condition.

Differentiating with respect to $\alpha^h_{\cdot,\cdot,\cdot}$ and $\beta^h_{\cdot,\cdot,\cdot}$ gives equalities

$$\begin{aligned} \alpha^h_{k,i,\cdot} &= -\exp(rt_k)\left(v^h_{k,i+1/2,\cdot} - v^h_{k,i-1/2,\cdot}\right)\Big/ d_1 h_1 \quad \forall\, i = 1,\dots,N_1-1, \\ \beta^h_{k,\cdot,j} &= \exp(rt_k)\left(v^h_{k,\cdot,j+1/2} - v^h_{k,\cdot,j-1/2}\right)\Big/ d_2 h_2 \quad \forall\, j = 1,\dots,N_2-1. \end{aligned} \tag{34}$$

Note that the calculation in time is performed in the reverse order from t_{M-1} to t_0.

5 The numerical solution of the complete model

Suppose that some initial approximation $\alpha^h_{\cdot,\cdot,\cdot}, \beta^h_{\cdot,\cdot,\cdot}$, and $m^h_{\cdot,\cdot,\cdot}$ are given. For example, we can firstly take $\alpha^h_{\cdot,\cdot,\cdot} = \mathbf{0}$, $\beta^h_{\cdot,\cdot,\cdot} = \mathbf{0}$. Then, $m^h_{k,\cdot,\cdot} = m^h_{0,\cdot,\cdot}$ for any $k = 1,\dots,M-1$. The better approximations of the control grid functions $\alpha^h_{\cdot,\cdot,\cdot}$, $\beta^h_{\cdot,\cdot,\cdot}$ may be computed by the following iterative way.

Iterative algorithm.

1. Solve (30), (31), (33) $\forall\, k = M-1, M-2,\dots,0$ to get $v^h_{\cdot,\cdot}$.
2. Compute $\tilde{\alpha}^h_{k,\cdot,\cdot}$ and $\tilde{\beta}^h_{k,\cdot,\cdot}$ by (34) for $k = 0,\dots,M-1$.
3. Compute $\tilde{m}^h_{\cdot,\cdot,\cdot}$ by (25).
4. Compute $J^h(\tilde{m}^h, \tilde{\alpha}^h, \tilde{\beta}^h)$ by (27).
5. If $\left|J^h(m^h,\alpha^h,\beta^h) - J^h(\tilde{m}^h,\tilde{\alpha}^h,\tilde{\beta}^h)\right| > Tol$ then $\{\, \alpha^h_{\cdot,\cdot,\cdot} := \tilde{\alpha}^h_{\cdot,\cdot,\cdot};\ \ m^h_{\cdot,\cdot,\cdot} := \tilde{m}^h_{\cdot,\cdot,\cdot};$ go to 1$\}$.
6. Take $\tilde{\alpha}^h_{\cdot,\cdot,\cdot}, \tilde{\beta}^h_{\cdot,\cdot,\cdot}$, and $\tilde{m}^h_{\cdot,\cdot,\cdot}$ as an approximate solution of (29).

Now we give the hint to demonstrate that (34) ensures the steepest descent of the above estimate for the difference between values of discrete cost functional. Consider two controls: fix α^h, β^h and vary $\tilde{\alpha}^h, \tilde{\beta}^h$. Take $r^h, \tilde{r}^h, s^h, \tilde{s}^h$ from (28) and the solutions m^h, $\tilde{m}^h$ of discrete problem (25). Find the difference between two values of the discrete functional $J^h(\tilde{m}^h,\tilde{\alpha}^h,\tilde{\beta}^h) - J^h(m^h,\alpha^h,\beta^h)$. Imply the concavity of the function g with respect to m and disintegrate each addend in the right-hand side into independent parts with respect to every $\tilde{\alpha}^h_{k,i,j+1/2}$ and $\tilde{\beta}^h_{k,i+1/2,j}$ like in [6]. First, the minimization of this difference independently decays into the minimization of the quadratic polynomials. Second, the coefficients of the principal term in $\tilde{\alpha}^h_{k,i,j+1/2}, \tilde{\beta}^h_{k,i+1/2,j}$ are strictly positive. Thus, the minimum for each of them is reached at points

$$\begin{aligned} \bar{\alpha}^h_{k,i,\cdot} &= -q^1_{k,i,\cdot}\big/ 2p^1_{k,i,\cdot} = -\exp(rt_k)\left(v^h_{k,i+1/2,\cdot} - v^h_{k,i-1/2,\cdot}\right)\Big/ d_1 h_1 \quad \forall\, i = 1,\dots,N_1-1, \\ \bar{\beta}^h_{k,\cdot,j} &= -q^2_{k,\cdot,j}\big/ 2p^2_{k,\cdot,j} = \exp(rt_k)\left(v^h_{k,\cdot,j+1/2} - v^h_{k,\cdot,j-1/2}\right)\Big/ d_2 h_2 \quad \forall\, j = 1,\dots,N_2-1. \end{aligned}$$

One can see that these minimization points indeed coincide with the computed in the iterative algorithm.

At first glance, it seems that the coincidence of equalities (34) and the condition of minimization is a natural consequence of approximation of the differential problem. However, this coincidence is not automatic for many other difference schemes (see, for example, [5, 6]) for which additional inner iterations or corrections are necessary.

Taylor expansion at the point $\mathbf{z}_{k,i+1/2,j+1/2}$ demonstrates the approximation order of $O(\tau+h)$ for scheme (22)-(24). Due to the M-matrix property, the scheme is stable and provides the accuracy of $O(\tau+h)$ in $L_1(\Omega)-$norm. The same conclusion is valid for the difference scheme (30)-(33).

6 The model

The agents of our model are the producers. Each of them is associated to a production at time t, denoted by $q(t)$ which results in amount of emission $e(t)$. Generally speaking, an increase in production can result in more emissions and vice versa. So, it is reasonable to use the emission as a state variable instead of production (see, e.g., [9]). The dynamics of agent's emission $e(t)$ corresponds to the following controlled process:

$$de(t) = -l(t)\,dt + \sigma_1 dW_1(t) + dN_t\left(e(t)\right), \quad e(0) = e_0, \tag{35}$$

where $l(t)$ is the level of emission reduction and $\sigma_1 dW_1(t)$ is its stochastic disturbance. It results from technological innovations, market fluctuation, and some other uncertain factors. Here $W_1(t)$ is a standard Brownian motion and σ_1 is a noise parameter. $N_t\left(e(t)\right)$ is the reflection part which guarantees for the process to stay in $[e_{\min}, e_{\max}]$ determined by production capacity of an agent [6, 7, 9]. Another dynamic state of the system is the amount of the permitted emission $x(t)$. Let the agents try their best to negotiate between them to pursue the highest level of the permitted emission. The effort level is represented by $\mu(t)$. The permitted emission $x(t)$ follows the stochastic process

$$dx(t) = \mu(t)\,dt + \sigma_2 dW_2(t) + dN_t\left(x(t)\right), \quad x(0) = x_0. \tag{36}$$

Here $\sigma_2 dW_2(t)$ is the stochastic disturbance in which $W_2(t)$ is dependent on $W_1(t)$ with correlation $\gamma \in [0,1)$. $N_t\left(x(t)\right)$ is the reflection part which guarantees for the process to stay in $[x_{\min}, x_{\max}]$. It is determined by the police makers of the cap-and-trade police.

Assume that all agents have the same capacity in production and negotiation but make different decisions. The states e and x of the producers continuum at time t are distributed in $\bar{\Omega} = [e_{\min}, e_{\max}] \times [x_{\min}, x_{\max}]$ with the probability density function $m(t,e,x)$. The initial density $m_0(e,x)$ is given. We mention that a given agent cannot influence the distribution of all players' state, but it can produce a piece of information which has an effect on decision made by others.

The agents' net revenues include five parts: the production revenue, the cost of emission reduction, the cost of pursuing permitted emission, the cost of the carbon tax, and the carbon trading. The production revenue of agent is $Y(t,e,m) = \left(e_{\max} e - e^2/2\right)\big/\left(c_1 + c_2 m\right)$, $e \in [e_{\min}, e_{\max}]$, where c_1 and c_2 are positive constants. Agent's marginal revenue is positive and decreasing with respect to the density m. This property is related to an economic concept "negative externality". The cost of emission reduction and negotiation for the permitted emission are $C_e(l) = d_1 l^2(t)$ and $C_n(l) = d_2 \mu^2(t)$, respectively. Here d_1 and d_2 are positive constants and the quadratic form guarantees increasing marginal costs.

The cost of carbon tax is $C_a(t,e,x) = p_a(t)\min\{e(t), x(t)\}$ where $p_a(t)$ denotes the tax rate. If the emission amount $e(t)$ is less than permission level $x(t)$, the agent should pay the carbon tax of his emission. If the emission amount $e(t)$ is more than $x(t)$, the agent should pay the carbon tax of the basic part. However, the permission of the exceed part should be brought by the carbon trading mechanism expressed by the fourth part of revenue $C_b(t,e,x) = p_b(t)\left(e(t)-x(t)\right)^+$ where the emission permits price $p_b(t) > p_a(t)$.

After summation, the net revenue of agent at time t is

$$f(t,e,x,l,\mu,m) = Y(t,e,m(t,e,x)) - d_1 l^2 - d_2\mu^2 - p_a(t)\min\{e,x\} - p_b(t)\left(e-x\right)^+.$$

The agents can adjust their strategies $l(t)$ and $\mu(t)$ to maximize their expected discounted revenues over the time segment $[0,T]$. When the probability density function is not related with the control variables, we have the optimal control problem [7]

$$\max_{l(t),\mu(t)} \mathbb{E}\left\{\int_0^T \exp(-rt) f\left(t,e(t),x(t),l(t),\mu(t),m\left(t,e(t),x(t)\right)\right)dt\right\} \tag{37}$$

subject to (35), (36) where $r \geq 0$ is the risk-free discount rate.

Consider the above problem from the microscopic point of view, then the optimal control for the continuum of agents is expressed as

$$\sup_{l(t),\mu(t)} \left\{\int_0^T \exp(-rt)\int_\Omega f\left(t,e(t),x(t),l(t),\mu(t),m\left(t,e(t),x(t)\right)\right) m\,\mathrm{d}\Omega\,\mathrm{d}t\right\}. \tag{38}$$

In order to rewrite this problem in the form of (5), we produce linear substitution $(e-e_{\min}, x-x_{\min})$ by variables (x,y) and take the minus before f to change supremum in (38) for infimum in (5). Then put $l=-\alpha$ and $\mu=\beta$; etc. After these substitutions, this task coincides with the statement of previous sections.

So, the formulated problem (35)–(38) can be written in the form (5) with the application of numerical algorithms of section 5.

7 Conclusion

Thus, we suggest the discrete approximation of the MFG problem which completely inherited the basic properties of the differential problem simultaneously with the approximation of each differential equation. The approximations of direct FPK and inverse HJB problems have adjoint operators which are monotone in corresponding adjoint vector spaces. And the discrete approximation (34) of differential connection (12) for the minimization of the cost functional gives the condition for steepest descent of the discrete cost functional.

Finally, this approach gives a direct and simple rule for the minimization of the cost functional, which accelerates the convergence of algorithm in comparison with other difference approximations [5, 6].

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