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Ample Spectrum Contractions and Related Fixed Point Theorems

Antonio Francisco Roldán López de Hierro [1,*] **and Naseer Shahzad** [2]

[1] Department of Statistics and Operations Research, University of Granada, 18010 Granada, Spain
[2] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O.B. 80203, Jeddah 21589, Saudi Arabia; nshahzad@kau.edu.sa
* Correspondence: aroldan@ugr.es

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Abstract: Simulation functions were introduced by Khojasteh et al. as a method to extend several classes of fixed point theorems by a simple condition. After that, many researchers have amplified the knowledge of such kind of contractions in several ways. R-functions, $(R,\mathcal{S})$-contractions and $(\mathcal{A},\mathcal{S})$-contractions can be considered as approaches in this direction. A common characteristic of the previous kind of contractive maps is the fact that they are defined by a strict inequality. In this manuscript, we show the advantages of replacing such inequality with a weaker one, involving a family of more general auxiliary functions. As a consequence of our study, we show that not only the above-commented contractions are particular cases, but also another classes of contractive maps correspond to this new point of view.

Keywords: R-function; simulation function; manageable function; fixed point; contractivity condition; binary relation

1. Introduction

Fixed point theory is a branch of mathematics that has multiple applications in almost all scientific fields of study. Mainly, it is used to prove the existence (and, in many cases, also uniqueness) of solutions of great variety of equations arising in theoretical and practical disciplines: matrix equations, differential equations, integral equations, etc. One of its best advantage is the fact that it permits us to deal with linear and nonlinear problems, which makes this discipline into an essential part of nonlinear analysis.

Although it was not the first result in this line of research, Banach contractive mapping principle is widely considered the pioneering statement. Any new result in this area must generalize such principle. There are many directions in which it has been extended and improved: by using weaker contractivity conditions, more general families of auxiliary functions, by involving a partial order, by considering abstract metric spaces, etc.

In recent times, Khojasteh et al. [1] introduced a new class of auxiliary functions, called *simulation functions*, that let us consider a family of contractivity conditions that only involve two arguments: the distance between two points ($d(x,y)$) and the distance between their corresponding images ($d(Tx,Ty)$) under the considered operator. This work quickly attracted the attention of several researchers because of its potential applications (see, for instance, the work of Roldán López de Hierro et al. [2], who slightly modified the original definition, and those of Roldán López de Hierro and Shahzad [3,4], who presented *R-functions* as extensions of simulation functions).

The above-mentioned classes of contractions have been included in a new family of contractive mappings, called $(\mathcal{A},\mathcal{S})$-*contractions*, that extend and unify several results in fixed point theory (see [5]). Theoretical notions introduced in such manuscript were later developed by other researchers (see [6]) even with applications to fuzzy partial differential equations (see [7]) and optimal solutions and applications to nonlinear matrix and integral equations (see [8]). However, in the original definition of $(\mathcal{A},\mathcal{S})$-contractions, inspired by the previous contributions, the authors established a strict inequality that must be verified for some pairs of points related under a binary relation. In this manuscript, we improve such results in several ways: (1) the given family of auxiliary functions is more general; (2) coherently, the presented contractivity condition is weaker; and (3) the set of points that have to satisfy the contractivity condition is smaller. These improvements let us show that not only the above-commented contractions are particular cases of our study, but also new families of contractive maps correspond to this new approach (see [9–11]). The presented contractions are called *ample spectrum contractions* because they are an attempt to generalize all known contractions that are defined by contractivity conditions that involve only the terms $d(x,y)$ and $d(Tx,Ty)$.

2. Preliminaries

Basic notions and notations for a good understanding of this manuscript are given in [5]. Nevertheless, we recall here the essential facts. Throughout this manuscript, X always stands for a nonempty set. A *binary relation on* X is a nonempty subset $\mathcal{S}$ of the product space $X\times X$. If $(x,y)\in\mathcal{S}$, we denote it by $x\mathcal{S}y$. We write $x\mathcal{S}^*y$ when $x\mathcal{S}y$ and $x\neq y$. Notice that $\mathcal{S}^*$, if it is nonempty, is another binary relation on X. Two points x and y are *$\mathcal{S}$-comparable* if $x\mathcal{S}y$ or $y\mathcal{S}x$. A binary relation $\mathcal{S}$ is:

- *transitive*: If from $x\mathcal{S}y$ and $y\mathcal{S}z$ it follows $x\mathcal{S}z$,
- *reflexive*: If $x\mathcal{S}x$ for each $x\in\mathbb{R}$,
- *antisymmetric*: If from $x\mathcal{S}y$ and $y\mathcal{S}x$ it follows $x=y$.

Reflexive and transitive binary relations are called *preorders* (or *quasiorders*), and, if they are also antisymmetric, then they are *partial orders*. The trivial partial order $\mathcal{S}_X$ is defined by $x\mathcal{S}_Xy$ for each $x,y\in X$.

From now on, $\mathbb{N}=\{0,1,2,3,\ldots\}$ stands for the set of all nonnegative integers and $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$. Henceforth, let $T:X\to X$ be a map from X into itself, let (X,d) be a metric space and let $A\subseteq\mathbb{R}$ be a nonempty subset of the set of all real numbers. The range (or image) of d is $\operatorname{ran}(d)=\{d(x,y):x,y\in X\}\subseteq[0,\infty)$.

If $Tx=x$, then x is a *fixed point of* T. The maps $\{T^n:X\to X\}_{n\in\mathbb{N}}$ defined by $T^0=$identity, $T^1=T$ and $T^{n+1}=T\circ T^n$ for all $n\geq 2$ are known as the *iterates* of T. The *Picard sequence of* T *based on* $x_0\in X$ is the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$ (hence, $x_n=T^nx_0$ for each $n\in\mathbb{N}$). When any Picard sequence of T converges to a fixed point of T, we say that T is a *weakly Picard operator*, and if it has a unique fixed point, then T is known as *Picard operator*.

In [5], the authors used the following terminology. Let $\mathcal{S}$ be a binary relation on a metric space (X,d), let $Y\subseteq X$ be a nonempty subset, let $\{x_n\}$ be a sequence in X and let $T:X\to X$ be a self-mapping. We say that:

- A sequence $\{x_n\}\subseteq X$ is *asymptotically regular on* (X,d) if $\{d(x_n,x_{n+1})\}\to 0$.
- T is *$\mathcal{S}$-nondecreasing* if $Tx\mathcal{S}Ty$ for all $x,y\in X$ such that $x\mathcal{S}y$.
- $\{x_n\}$ is *$\mathcal{S}$-nondecreasing* if $x_n\mathcal{S}x_m$ for all $n,m\in\mathbb{N}$ such that $n<m$.
- $\{x_n\}$ is *$\mathcal{S}$-strictly-increasing* if $x_n\mathcal{S}^*x_m$ for all $n,m\in\mathbb{N}$ such that $n<m$.
- T is *$\mathcal{S}$-nondecreasing-continuous* if $\{Tx_n\}\to Tz$ for all $\mathcal{S}$-nondecreasing sequence $\{x_n\}\subseteq X$ such that $\{x_n\}\to z\in X$.
- T is *$\mathcal{S}$-strictly-increasing-continuous* if $\{Tx_n\}\to Tz$ for all $\mathcal{S}$-strictly-increasing sequence $\{x_n\}\subseteq X$ such that $\{x_n\}\to z\in X$.
- Y is *$(\mathcal{S},d)$-strictly-increasing-complete* if every $\mathcal{S}$-strictly-increasing and d-Cauchy sequence $\{y_n\}\subseteq Y$ is d-convergent to a point of Y.
- Y is *$(\mathcal{S},d)$-strictly-increasing-precomplete* if there exists a set Z such that $Y\subseteq Z\subseteq X$ and Z is $(\mathcal{S},d)$-strictly-increasing-complete;
- (X,d) is *$\mathcal{S}$-strictly-increasing-regular* if, for all $\mathcal{S}$-strictly-increasing sequence $\{x_n\}\subseteq X$ such that $\{x_n\}\to z\in X$, it follows that $x_n\mathcal{S}z$ for all $n\in\mathbb{N}$.

We follow the notation given in [12,13]. Next, we list a collection of properties that can be satisfied by a function $\phi : [0,\infty) \to [0,\infty)$.

$(\mathcal{P}_1)$ ϕ is non-decreasing, that is, if $0 \le t \le s$, then $\phi(t) \le \phi(s)$.
$(\mathcal{P}_{10})$ The series $\sum_{n\geq 1} \phi^n(t)$ converges for all $t > 0$.
$(\mathcal{P}_{11})$ $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.
$(\mathcal{P}_{12})$ $\phi(t) < t$ for all $t > 0$.
$(\mathcal{P}_{13})$ $\lim_{t\to 0^+} \phi(t) = 0$.
$(\mathcal{P}')$ $\phi(0) = 0$.

It is clear that $(\mathcal{P}_{10}) \Rightarrow (\mathcal{P}_{11})$ and, on the other hand, $(\mathcal{P}_{12}) \Rightarrow (\mathcal{P}_{13})$.

Proposition 1 ([12,13])**.** *If $(\mathcal{P}_1)$ holds, then $(\mathcal{P}_{10}) \Rightarrow (\mathcal{P}_{11}) \Rightarrow (\mathcal{P}_{12}) \Rightarrow (\mathcal{P}_{13}) \Rightarrow (\mathcal{P}')$.*

Given a function $\alpha : X \times X \to [0,\infty)$, it is possible to redefine the previous notions in terms of α (transitivity, α-admissibility, α-nondecreasing character, α-nondecreasing-continuity, α-strictly-increasing-regularity, (α,d)-strictly-increasing-completeness, (α,d)-strictly-increasing-precompleteness, etc.). For details, see [5]. Such properties can be translated to the previous setting by using the binary relation $\mathcal{S}_\alpha$ on X given, for $x,y \in X$, by

$$x\mathcal{S}_\alpha y \quad \text{if} \quad \alpha(x,y) \geq 1. \tag{1}$$

Lemma 1. *Let (X,d) be a metric space, let $T : X \to X$ be a self-mapping and let $\alpha : X \times X \to [0,\infty)$ be a function. Then, the following properties hold.*

1. *The binary relation $\mathcal{S}_\alpha$ is transitive if, and only if, α is transitive.*
2. *T is α-admissible if, and only if, T is $\mathcal{S}_\alpha$-nondecreasing.*
3. *Given $z_0 \in X$, the mapping T is $(d,\mathcal{S}_\alpha)$-nonincreasing-continuous at z_0 if, and only if, it is (d,α)-right-continuous at z_0.*
4. *T is $(d,\mathcal{S}_\alpha)$-nonincreasing-continuous if, and only if, T is (d,α)-right-continuous.*

In [5], Shahzad et al. introduced the following notions.

Definition 1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. We say that $\{(a_n,b_n)\}$ is a* $(T,\mathcal{S})$-sequence *if there exist two sequences $\{x_n\},\{y_n\} \subseteq X$ such that*

$$x_n\mathcal{S}y_n, \quad a_n = d(Tx_n,Ty_n) > 0 \quad \textit{and} \quad b_n = d(x_n,y_n) > 0 \quad \textit{for all } n \in \mathbb{N}.$$

If $\mathcal{S}$ is the trivial binary relation $\mathcal{S}_X$, then $\{(a_n,b_n)\}$ is called a T-sequence.

Remark 1. *Notice that $\{(a_n = d(Tx_n,Ty_n), b_n = d(x_n,y_n))\}$ is a $(T,\mathcal{S})$-sequence if, and only if,*

$$x_n\mathcal{S}^*y_n \quad \textit{and} \quad a_n > 0 \quad \textit{for all } n \in \mathbb{N}.$$

Definition 2. *We say that $T : X \to X$ is an* $(\mathcal{A},\mathcal{S})$-contraction *if there exists a function $\varrho : A \times A \to \mathbb{R}$ such that T and ϱ satisfy the following four conditions:*

$(\mathcal{A}_1)$ $\operatorname{ran}(d) \subseteq A$.
$(\mathcal{A}_2)$ *If $\{x_n\} \subseteq X$ is a Picard $\mathcal{S}$-nondecreasing sequence of T such that*

$$x_n \neq x_{n+1} \quad \textit{and} \quad \varrho\left(d\left(x_{n+1},x_{n+2}\right), d\left(x_n,x_{n+1}\right)\right) > 0 \quad \textit{for all } n \in \mathbb{N},$$

then $\{x_n\}$ is asymptotically regular on (X,d) (that is, $\{d(x_n,x_{n+1})\} \to 0$).

($\mathcal{A}_3$) *If $\{(a_n, b_n)\} \subseteq A \times A$ is a $(T, \mathcal{S})$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.*

($\mathcal{A}_4$) $\varrho\left(d(Tx, Ty), d(x, y)\right) > 0$ *for all $x, y \in X$ such that $x\mathcal{S}^*y$ and $Tx\mathcal{S}^*Ty$.*

*In such a case, we say that T is an $(\mathcal{A}, \mathcal{S})$-*contraction with respect to ϱ. *We denote the family of all $(\mathcal{A}, \mathcal{S})$-contractions from (X, d) into itself with respect to ϱ by $\mathcal{A}_{X,d,\mathcal{S},\varrho,A}$ or, for simplicity, by $\mathcal{A}_\varrho$ when no confusion is possible.*

If $\mathcal{S}$ is the trivial binary relation $\mathcal{S}_X$, then T is called an $\mathcal{A}$-contraction (with respect to ϱ).

Condition ($\mathcal{A}_1$) implies that A is a nonempty set. In some cases, we also consider the following properties.

($\mathcal{A}_2'$) If $x_1, x_2 \in X$ are two points such that

$$T^n x_1 \mathcal{S}^* T^n x_2 \quad \text{and} \quad \varrho(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d\left(T^n x_1, T^n x_2\right)) > 0 \quad \text{for all } n \in \mathbb{N},$$

then $\{d\left(T^n x_1, T^n x_2\right)\} \to 0$.

($\mathcal{A}_5$) If $\{(a_n, b_n)\}$ is a $(T, \mathcal{S})$-sequence such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.

3. Ample Spectrum Contractions

In this section, we slightly modify the axioms given in [5] in a subtle way in order to consider a wider class of contractions. In what follows, let (X, d) be a metric space, let $\mathcal{S}$ be a binary relation on X and let $T : X \to X$ be a self-mapping.

Definition 3. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. We say that $\{(a_n, b_n)\}$ is a* $(T, \mathcal{S}^*)$-sequence *if there exist two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that*

$$x_n \mathcal{S}^* y_n, \quad Tx_n \mathcal{S}^* Ty_n, \quad a_n = d(Tx_n, Ty_n) > 0 \quad \textit{and} \quad b_n = d(x_n, y_n) > 0 \quad \textit{for all } n \in \mathbb{N}.$$

Proposition 2. *Every $(T, \mathcal{S}^*)$-sequence is a $(T, \mathcal{S})$-sequence.*

Definition 4. *We say that $T : X \to X$ is a* ample spectrum contraction *if there exists a function $\varrho : A \times A \to \mathbb{R}$ such that T and ϱ satisfy the following four conditions:*

($\mathcal{B}_1$) *A is nonempty and $\{\, d(x, y) \in [0, \infty) : x, y \in X,\ x\mathcal{S}^*y \,\} \subseteq A$.*

($\mathcal{B}_2$) *If $\{x_n\} \subseteq X$ is a Picard $\mathcal{S}$-nondecreasing sequence of T such that*

$$x_n \neq x_{n+1} \quad \textit{and} \quad \varrho\left(d\left(x_{n+1}, x_{n+2}\right), d\left(x_n, x_{n+1}\right)\right) \geq 0 \quad \textit{for all } n \in \mathbb{N},$$

then $\{d\left(x_n, x_{n+1}\right)\} \to 0$.

($\mathcal{B}_3$) *If $\{(a_n, b_n)\} \subseteq A \times A$ is a $(T, \mathcal{S}^*)$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$, then $L = 0$.*

($\mathcal{B}_4$) $\varrho\left(d(Tx, Ty), d(x, y)\right) \geq 0$ *for all $x, y \in X$ such that $x\mathcal{S}^*y$ and $Tx\mathcal{S}^*Ty$.*

In such a case, we say that T is a ample spectrum contraction with respect to $\mathcal{S}$ and ϱ. *We denote the family of all ample spectrum contractions from (X, d) into itself with respect to $\mathcal{S}$ and ϱ by $\mathcal{B}_{X,d,\mathcal{S},\varrho,A}$.*

In some cases, we also consider the following properties:

($\mathcal{B}_2'$) If $x_1, x_2 \in X$ are two points such that

$$T^n x_1 \mathcal{S}^* T^n x_2 \quad \text{and} \quad \varrho(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d\left(T^n x_1, T^n x_2\right)) \geq 0 \quad \text{for all } n \in \mathbb{N},$$

then $\{d\left(T^n x_1, T^n x_2\right)\} \to 0$.

($\mathcal{B}_5$) If $\{(a_n, b_n)\}$ is a $(T, \mathcal{S}^*)$-sequence such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.

Remark 2. *The reader can observe the following facts about the previous assumptions:*

1. *Notice that conditions* $(\mathcal{B}_2)$, $(\mathcal{B}_3)$, $(\mathcal{B}_2')$ *and* $(\mathcal{B}_5)$ *establish that, if there exists a sequence (or one point, or two points) verifying some assumptions, then a thesis must hold. However, we point out that, if such kind of sequences (or points) does not exist, then conditions* $(\mathcal{B}_2)$, $(\mathcal{B}_3)$, $(\mathcal{B}_2')$ *and* $(\mathcal{B}_5)$ *hold.*
2. *Condition* $(\mathcal{B}_2)$ *follows from* $(\mathcal{B}_2')$ *using* $x_2 = Tx_1$.
3. *None of the previous conditions establishes a constraint about the values* $\{\,\varrho(0,s) : s \in A\,\}$ *because the first argument is always positive. In fact, it is possible that* $0 \notin A$.
4. *If* $x\mathcal{S}^*y$, *then* $d(x,y) > 0$. *Hence,* $0 \notin \{\, d\,(x,y) \in [0,\infty) : x,y \in X,\ x\mathcal{S}^*y\,\}$. *Nevertheless,* 0 *may belong to* A.
5. *If* $\mathcal{S}$ *is the binary relation such that* $x\mathcal{S}y$ *if, and only if,* $x = y$, *then* $\{\, d\,(x,y) \in [0,\infty) : x,y \in X,\ x\mathcal{S}^*y\,\}$ *is empty. This is the reason we must impose that* A *is nonempty.*
6. *Condition* $(\mathcal{B}_1)$ *guarantees that the function* ϱ *can be applied in the other assumptions. For instance, in* $(\mathcal{B}_2)$, *it is clear that* $x_n\mathcal{S}^*x_{n+1}$ *and* $x_{n+1}\mathcal{S}^*x_{n+2}$ *because* $\{x_n\}$ *is* $\mathcal{S}$*-nondecreasing and* $x_n \neq x_{n+1}$ *for all* $n \in \mathbb{N}$.
7. *As the reader can easily check in the proofs of the following results, we could also have supposed in Condition* $(\mathcal{B}_3)$ *that* $\{x_n\}$ *and* $\{y_n\}$ *are appropriate subsequences of the same Picard sequence* $\{z_n = T^n z_0\} \subseteq X$ *(in the sense that* $x_n = z_{p(n)}$ *and* $y_n = z_{q(n)}$ *being* $n \leq p(n) < q(n)$ *for all* $n \in \mathbb{N}$*). In order not to complicate the proofs, we do not include such assumption.*

Proposition 3. *If* $\varrho\,(t,s) \leq s - t$ *for all* $t,s \in A \cap (0,\infty)$, *then* $(\mathcal{B}_5)$ *holds.*

Proof. Assume that $\{a_n\}, \{b_n\} \subset (0,\infty) \cap A$ are two sequences such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. Since $a_n, b_n \in (0,\infty) \cap A$, then $0 < \varrho(a_n, b_n) \leq b_n - a_n$ for all $n \in \mathbb{N}$. As a consequence, $0 < a_n \leq b_n$ for all $n \in \mathbb{N}$, which means that $\{a_n\} \to 0$. □

The previous definition generalizes the notion of $(\mathcal{A}, \mathcal{S})$-contraction, as we prove in the following result:

Theorem 1. *Every* $(\mathcal{A}, \mathcal{S})$*-contraction is an ample spectrum contraction (with respect to the same function* ϱ*). Furthermore, if it satisfies* $(\mathcal{A}_2')$ *(respectively,* $(\mathcal{A}_5)$*), then it also verifies* $(\mathcal{B}_2')$ *(respectively,* $(\mathcal{B}_5)$*).*

In particular, we prove the following implications:

$$
\begin{aligned}
(\mathcal{A}_1) &\Rightarrow (\mathcal{B}_1)\,, \\
(\mathcal{A}_4) &\Rightarrow (\mathcal{B}_4)\,, \\
(\mathcal{A}_2) + (\mathcal{A}_4) &\Rightarrow (\mathcal{B}_2)\,, \\
(\mathcal{A}_3) + (\mathcal{A}_4) &\Rightarrow (\mathcal{B}_3)\,, \\
(\mathcal{A}_4) + (\mathcal{A}_5) &\Rightarrow (\mathcal{B}_5)\,, \\
\left(\mathcal{A}_2'\right) + (\mathcal{A}_4) &\Rightarrow \left(\mathcal{B}_2'\right).
\end{aligned}
$$

Proof. Let (X,d) be a metric space, let $T : X \to X$ be a mapping and let $\varrho : A \times A \to \mathbb{R}$ be a function. Clearly, $(\mathcal{A}_1) \Rightarrow (\mathcal{B}_1)$ and $(\mathcal{A}_4) \Rightarrow (\mathcal{B}_4)$. Next, we prove the rest of conditions.

[$(\mathcal{A}_2') + (\mathcal{A}_4) \Rightarrow (\mathcal{B}_2')$] Let $x_1, x_2 \in X$ be two points such that

$$T^n x_1 \mathcal{S}^* T^n x_2 \quad \text{and} \quad \varrho(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d\,(T^n x_1, T^n x_2)) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Let us denote

$$x_n^1 = T^n x_1 \quad \text{and} \quad x_n^2 = T^n x_2 \quad \text{for all } n \in \mathbb{N}.$$

Hence, by hypothesis, $x_n^1 = T^n x_1 \mathcal{S}^* T^n x_2 = x_n^2$ and $Tx_n^1 = T^{n+1} x_1 \mathcal{S}^* T^{n+1} x_2 = Tx_n^2$. Applying Condition $(\mathcal{A}_4)$, for all $n \in \mathbb{N}$,

$$\varrho(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d\left(T^n x_1, T^n x_2\right)) = \varrho(d\left(Tx_n^1, Tx_n^2\right), d\left(x_n^1, x_n^2\right)) > 0.$$

Therefore, Condition $(\mathcal{A}_2')$ implies that $\{d(T^n x_1, T^n x_2)\} \to 0$.

$[\,(\mathcal{A}_2) + (\mathcal{A}_4) \Rightarrow (\mathcal{B}_2)\,]$ It follows as in the previous implication by using $x_1 = x_0$ and $x_2 = Tx_0$.

$[\,(\mathcal{A}_3) + (\mathcal{A}_4) \Rightarrow (\mathcal{B}_3)\,]$ Let $\{(a_n, b_n)\} \subseteq A \times A$ be a $(T, \mathcal{S}^*)$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. By definition, there are two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that

$$x_n \mathcal{S}^* y_n, \quad Tx_n \mathcal{S}^* Ty_n, \quad a_n = d(Tx_n, Ty_n) > 0 \quad \text{and} \quad b_n = d(x_n, y_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

As $x_n \mathcal{S}^* y_n$ and $Tx_n \mathcal{S}^* Ty_n$, then it follows from $(\mathcal{A}_4)$ that

$$\varrho(a_n, b_n) = \varrho(d(Tx_n, Ty_n), d(x_n, y_n)) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, applying $(\mathcal{A}_3)$, we conclude that $L = 0$.

$[\,(\mathcal{A}_4) + (\mathcal{A}_5) \Rightarrow (\mathcal{B}_5)\,]$ Let $\{(a_n, b_n)\}$ be a $(T, \mathcal{S}^*)$-sequence such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. By definition, there exist two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that

$$x_n \mathcal{S}^* y_n, \quad Tx_n \mathcal{S}^* Ty_n, \quad a_n = d(Tx_n, Ty_n) > 0 \quad \text{and} \quad b_n = d(x_n, y_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

As $x_n \mathcal{S}^* y_n$ and $Tx_n \mathcal{S}^* Ty_n$, then it follows from $(\mathcal{A}_4)$ that

$$\varrho(a_n, b_n) = \varrho(d(Tx_n, Ty_n), d(x_n, y_n)) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, applying $(\mathcal{A}_5)$, we conclude that $\{a_n\} \to 0$. □

The previous theorem provides us a large list of ample spectrum contractions because every $(\mathcal{A}, \mathcal{S})$-contraction is an ample spectrum contraction. In particular, as the authors proved in [3,5], the following ones are examples of ample spectrum contractions:

- Banach contractions;
- Meir–Keeler contractions (see [14,15]);
- $\mathcal{Z}$-contractions involving *simulation functions* (see [1,2]);
- manageable contractions (see [16]);
- Geraghty contractions (see [17]); and
- R-contractions (see [3,5]).

The converse of Theorem 1 is false, as we show in the following example:

Example 1. *Let $X = \{0, 1, 3\}$ be endowed with the Euclidean metric $d_E(x, y) = |x - y|$ and the usual order $\leq$. Hence, (X, d_E) is a complete metric space. Let $A = \mathrm{ran}(d_E) = \{0, 1, 2, 3\}$ and let $T : X \to X$ and $\varrho : A \times A \to \mathbb{R}$ be defined by*

$$Tx = \begin{cases} 0, & \text{if } x \in \{0, 1\}, \\ 1, & \text{if } x = 3; \end{cases} \qquad \varrho(t, s) = 0 \text{ for all } t, s \in A.$$

Then, T is not an $(\mathcal{A}, \leq)$-contraction with respect to ϱ because, if $x = 1$ and $y = 3$, then $x < y$ and $Tx < Ty$, but $\varrho\left(d(Tx, Ty), d(x, y)\right) = 0$. Let us show that T is an ample spectrum contraction with respect to ϱ and $\leq$. Condition $(\mathcal{B}_4)$ is obvious. Properties $(\mathcal{B}_2)$ and $(\mathcal{B}_2')$ follows from the fact that any Picard sequence $\{x_n\}$ of T must verify $x_n = 0$ for all $n \geq 3$. Taking into account that any convergent sequence on A is almost constant (because it is discrete), Axioms $(\mathcal{B}_3)$ and $(\mathcal{B}_5)$ are satisfied because such kind of sequences do not exist. Hence, T is an ample spectrum contraction with respect to ϱ and $\leq$.

The notion of $(T,\mathcal{S}^*)$-sequence plays a key role in the definition of ample spectrum contraction. In fact, if we had not changed the notion of $(T,\mathcal{S})$-sequence by the concept of $(T,\mathcal{S}^*)$-sequence in Definition 4, then there would have not been any relationship between $(\mathcal{A},\mathcal{S})$-contractions and ample spectrum contractions. We illustrate this affirmation with the following example.

Example 2. *Let $X=[0,2]\cup C\cup D$, where $C=\{\,10m\in\mathbb{N}:m\in\mathbb{N}^*\,\}$ and $D=\{\,10m+4\in\mathbb{N}:m\in\mathbb{N}^*\,\}$. Assume that X is endowed with the Euclidean metric $d_E(x,y)=|\,x-y\,|$ and the usual order $\leq$. Hence, (X,d_E) is a complete metric space. The range of d_E can be expressed as*

$$\operatorname{ran}(d_E)=[0,2]\cup\{4\}\cup B\quad\textit{where } B\subset[6,\infty)\,.$$

Let $A=\operatorname{ran}(d_E)$ and let $T:X\to X$ and $\varrho:A\times A\to\mathbb{R}$ be defined by

$$Tx=\begin{cases}\dfrac{x}{4}, & \textit{if } x\in[0,2]\,,\\ 1+\dfrac{1}{m}, & \textit{if } x=10m\in C\quad(\textit{for some } m\in\mathbb{N}^*),\\ 0, & \textit{if } x=10m+4\in D\quad(\textit{for some } m\in\mathbb{N}^*);\end{cases}$$

$$\varrho(t,s)=\begin{cases}0, & \textit{if } t>1 \textit{ and } s\geq 1,\\ \dfrac{s}{2}-t, & \textit{otherwise.}\end{cases}$$

Notice that T satisfies the following properties.

(p_1) *$T(X)\subset[0,2]$. In particular, $|\,Tx-Ty\,|\leq 2$ for all $x,y\in X$.*
(p_2) *If $x,y\in X$ are two different points such that $x\in C\cup D$ or $y\in C\cup D$, then $|\,x-y\,|\geq 4$. In particular, if $|\,x-y\,|<4$, then $x,y\in[0,2]$.*
(p_3) *For all $x_0\in X$, the Picard sequence of T based on x_0 verifies $x_{n+1}=\frac{Tx_0}{4^n}$ for all $n\in\mathbb{N}$. Thus, every Picard sequence of T converges to zero.*

Let us show that T is an ample spectrum contraction with respect to ϱ and $\leq$.

$(\mathcal{B}_2)$ *Let $\{x_n\}\subseteq X$ be a Picard $\mathcal{S}$-nondecreasing sequence of T such that*

$$x_n\neq x_{n+1}\quad\textit{and}\quad\varrho\left(d_E\left(x_{n+1},x_{n+2}\right),d_E\left(x_n,x_{n+1}\right)\right)\geq 0\quad\textit{for all } n\in\mathbb{N}.$$

Since $\{x_n\}\to 0$, $\{d_E\left(x_n,x_{n+1}\right)\}\to 0$.

$(\mathcal{B}_3)$ *Let $\{(a_n,b_n)\}\subseteq A\times A$ be a $(T,<)$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L\geq 0$ and verifying that $L<a_n$ and $\varrho(a_n,b_n)\geq 0$ for all $n\in\mathbb{N}$. By definition, there are two sequences $\{x_n\},\{y_n\}\subseteq X$ such that*

$$x_n<y_n,\quad Tx_n<Ty_n,\quad a_n=d_E(Tx_n,Ty_n)>0\quad\textit{and}\quad b_n=d(x_n,y_n)>0\quad\textit{for all } n\in\mathbb{N}.$$

As $a_n=d_E(Tx_n,Ty_n)\in[0,2]$, then $L\leq 2$. Since $\{b_n=d_E(x_n,y_n)\}\to L\leq 2$, there exists $n_0\in\mathbb{N}$ such that $d_E(x_n,y_n)<4$ for all $n\geq n_0$. By (p_2), we have that $x_n,y_n\in[0,2]$ for all $n\geq n_0$. Therefore, for all $n\geq n_0$,

$$a_n=d_E(Tx_n,Ty_n)=\left|\,\frac{x_n}{4}-\frac{y_n}{4}\,\right|=\frac{|\,x_n-y_n\,|}{4}=\frac{b_n}{4}.$$

Letting $n\to\infty$, we deduce that $L=L/4$, so $L=0$.

$(\mathcal{B}_4)$ *Let $x,y\in X$ be two points such that $x<y$ and $Tx<Ty$. To prove that $\varrho\left(d(Tx,Ty),d(x,y)\right)\geq 0$, we observe three cases.*

► *If $\varrho\left(d(Tx,Ty),d(x,y)\right)=0$, then $(\mathcal{B}_4)$ holds. Hence, in what follows, we can assume that*

$$\varrho\left(d(Tx,Ty),d(x,y)\right)=\frac{|\,x-y\,|}{2}-|\,Tx-Ty\,|=\frac{y-x}{2}-\left(Ty-Tx\right),$$

which corresponds to the case in which $|\,Tx-Ty\,|\leq 1$ or $|\,x-y\,|<1$.

► *If* $|x-y| \geq 4$, *then, by* (p_1),

$$\varrho\left(d(Tx,Ty),d(x,y)\right)=\frac{|x-y|}{2}-|Tx-Ty|\geq\frac{4}{2}-2=0.$$

► *On the contrary case, if* $|x-y|<4$, *then x or y cannot belong to* $C\cup D$. *Then, necessarily,* $x,y\in[0,2]$, *thus*

$$\varrho\left(d(Tx,Ty),d(x,y)\right)=\frac{|x-y|}{2}-|Tx-Ty|=\frac{|x-y|}{2}-\left|\frac{x}{4}-\frac{y}{4}\right|=\frac{|x-y|}{4}>0,$$

which means that $(\mathcal{B}_4)$ *holds.*

In any case,$(\mathcal{B}_4)$ *holds.*

The following result is useful in order to study when an ample spectrum contraction can have multiple fixed points.

Proposition 4. *Let* (X,d) *be a metric space endowed with a binary relation* $\mathcal{S}$ *and let* $T:X\to X$ *and* $\varrho:A\times A\to\mathbb{R}$ *be two maps such that* $(\mathcal{B}_1)$, $(\mathcal{B}_2')$ *and* $(\mathcal{B}_4)$ *holds. If* $\omega,\omega'\in X$ *are two* $\mathcal{S}$*-comparable fixed points of* T*, then* $\omega=\omega'$.

Proof. Reasoning by contradiction, assume that ω and ω' are two distinct fixed points of T. As ω and ω' are $\mathcal{S}$-comparable, we can suppose, without loss of generality, that $\omega\mathcal{S}\omega'$. Hence, $\omega\mathcal{S}^*\omega'$ and also $T\omega\mathcal{S}^*T\omega'$. Let $a_n=d(\omega,\omega')>0$ for all $n\in\mathbb{N}$. By using $(\mathcal{B}_4)$, for all $n\in\mathbb{N}$,

$$\varrho\left(a_{n+1},a_n\right)=\varrho\left(d\left(\omega,\omega'\right),d\left(\omega,\omega'\right)\right)=\varrho\left(d\left(T\omega,T\omega'\right),d\left(\omega,\omega'\right)\right)\geq 0.$$

Therefore, it follows from $(\mathcal{B}_2')$ that $\{a_n=d(\omega,\omega')\}\to 0$, which is a contradiction. Thus, $\omega=\omega'$. □

4. Fixed Point Theorems Involving Ample Spectrum Contractions

Once we have changed the notions of $(T,\mathcal{S})$-sequence and $(\mathcal{A},\mathcal{S})$-contraction by the concepts of $(T,\mathcal{S}^*)$-sequence and ample spectrum contraction, we are ready to introduce the main results of the manuscript, which is the aim of the current section. Concretely, as we show below, the following one is the most general theorem of this manuscript.

Theorem 2. *Let* (X,d) *be a metric space endowed with a transitive binary relation* $\mathcal{S}$ *and let* $T:X\to X$ *be an* $\mathcal{S}$*-nondecreasing ample spectrum contraction with respect to* $\varrho:A\times A\to\mathbb{R}$. *Suppose that* $T(X)$ *is* $(\mathcal{S},d)$*-strictly-increasing-precomplete and there exists a point* $x_0\in X$ *such that* $x_0\mathcal{S}Tx_0$. *Assume that at least one of the following conditions is fulfilled:*

(a) T *is* $\mathcal{S}$*-strictly-increasing-continuous.*
(b) (X,d) *is* $\mathcal{S}$*-strictly-increasing-regular and Condition* $(\mathcal{B}_5)$ *holds.*
(c) (X,d) *is* $\mathcal{S}$*-strictly-increasing-regular and* $\varrho(t,s)\leq s-t$ *for all* $t,s\in A\cap(0,\infty)$.

Then, the Picard sequence of T *based on* x_0 *converges to a fixed point of* T*. In particular,* T *has at least a fixed point.*

Notice that the metric space (X,d) needs not to be complete.

Proof. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and let $\{x_{n+1}=Tx_n\}_{n\geq 0}$ be the Picard sequence of T based on x_0. If there exists some $n_0\in\mathbb{N}$ such that $x_{n_0+1}=x_{n_0}$, then x_{n_0} is a fixed point of T, and $\{x_n\}$ converges to such point. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. As T is $\mathcal{S}$-nondecreasing and $x_0 \mathcal{S} T x_0 = x_1$, then $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$, and, as $\mathcal{S}$ is transitive,

$$x_n \mathcal{S} x_m \quad \text{for all } n, m \in \mathbb{N} \text{ such that } n < m. \tag{2}$$

In fact, as $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then

$$x_n \mathcal{S}^* x_{n+1} \quad \text{and} \quad T x_n \mathcal{S}^* T x_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Let consider the sequence $\{d(x_n, x_{n+1})\} \subseteq A$. Taking into account Equation (3) and the fact that T is an ample spectrum contraction, Condition $(\mathcal{B}_4)$ implies that, for all $n \in \mathbb{N}$,

$$\varrho(d\left(T^{n+1}x_0, T^{n+2}x_0\right), d\left(T^n x_0, T^{n+1}x_0\right)) = \varrho\left(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})\right) \geq 0.$$

Applying $(\mathcal{B}_2)$,we deduce that $\{x_n = T^n x_0\}$ is an asymptotically regular sequence on (X, d), that is, $\{d(x_n, x_{n+1})\} \to 0$.

Let us show that $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing sequence. Indeed, in view of Equation (2), assume that there exists $n_0, m_0 \in \mathbb{N}$ such that $n_0 < m_0$ and $x_{n_0} = x_{m_0}$. If $p_0 = m_0 - n_0 \in \mathbb{N} \setminus \{0\}$, then $x_{n_0} = x_{n_0 + k\,p_0}$ for all $k \in \mathbb{N}$. In particular, the sequence $\{d(x_n, x_{n+1})\}$ contains the constant subsequence

$$\left\{d(x_{n_0+k\,p_0}, x_{n_0+k\,p_0+1}) = d(x_{n_0}, x_{n_0+1}) > 0\right\}_{k \in \mathbb{N}},$$

which contradicts the fact that $\{d(x_n, x_{n+1})\} \to 0$. This contradiction guarantees that $x_n \neq x_m$ for all $n \neq m$, thus $x_n \mathcal{S}^* x_m$ for all $n, m \in \mathbb{N}$ such that $n < m$, that is, $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing sequence.

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that

$$\begin{aligned} &k \leq n(k) < m(k), \quad d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) \quad \text{for all } k \in \mathbb{N}, \\ &\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon_0. \end{aligned}$$

Let $L = \varepsilon_0 > 0$, $\{a_k = d(x_{n(k)}, x_{m(k)})\} \to L$ and $\{b_k = d(x_{n(k)-1}, x_{m(k)-1})\} \to L$. As $n(k) < m(k)$ (and $n(k) - 1 < m(k) - 1$), then $x_{n(k)} \mathcal{S}^* x_{m(k)}$ and $x_{n(k)-1} \mathcal{S}^* x_{m(k)-1}$. Thus, $\{(a_k, b_k)\}$ is a $(T, \mathcal{S}^*)$-sequence. Since $L = \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) = a_k$ and

$$\begin{aligned} \varrho\left(a_k, b_k\right) &= \varrho\left(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)-1})\right) \\ &= \varrho\left(d(Tx_{n(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, x_{m(k)-1})\right) \geq 0 \end{aligned}$$

for all $k \in \mathbb{N}$, Condition $(\mathcal{B}_3)$ guarantees that $\varepsilon_0 = L = 0$, which is a contradiction. As a consequence, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}_{n \geq 1} \subseteq T(X)$ and $T(X)$ is $(\mathcal{S}, d)$-strictly-increasing-precomplete, there is a subset $Z \subseteq X$ such that $T(X) \subseteq Z \subseteq X$ and Z is $(\mathcal{S}, d)$-strictly-increasing-complete. In particular, as $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing and Cauchy sequence, there exists $z \in Z \subseteq X$ such that $\{x_n\} \to z$. Let us show that z is a fixed point of T considering three cases.

Case 1. Assume that T is $\mathcal{S}$-strictly-increasing-continuous. In this case, $\{x_{n+1} = Tx_n\} \to Tz$, so $Tz = z$.

Case 2. Assume that (X, d) is $\mathcal{S}$-strictly-increasing-regular and condition $(\mathcal{B}_5)$ holds. In this case, as $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing sequence such that $\{x_n\} \to z$, it follows that

$$x_n \mathcal{S} z \text{ for all } n \in \mathbb{N}. \tag{4}$$

Since T is $\mathcal{S}$-nondecreasing,

$$Tx_n\mathcal{S}Tz \text{ for all } n \in \mathbb{N}. \tag{5}$$

Let $a_n = d(x_{n+1}, Tz) = d(Tx_n, Tz)$ and $b_n = d(x_n, z)$ for all $n \in \mathbb{N}$. Clearly, $\{b_n\} \to 0$. Notice that

$$b_n = 0 \ \Rightarrow \ a_n = 0 \tag{6}$$

because

$$b_n = 0 \quad \Leftrightarrow \quad x_n = z \quad \Rightarrow \quad x_{n+1} = Tx_n = Tz \quad \Leftrightarrow \quad a_n = 0.$$

Let consider the set

$$\Omega = \{n \in \mathbb{N} : a_n = 0\} = \{n \in \mathbb{N} : d(x_{n+1}, Tz) = 0\}.$$

Subcase 2.1. Assume that Ω is finite. In this case, there exists $n_0 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) = a_n > 0$ for all $n \geq n_0$. By (6), $d(x_n, z) = b_n > 0$ for all $n \geq n_0$. In this case, $\{(a_n, b_n)\}_{n \geq n_0}$ is a $(T, \mathcal{S})$-sequence (because $a_n = d(Tx_n, Tz) > 0$ and $b_n = d(x_n, z) > 0$ for all $n \geq n_0$). In particular, $x_n \neq z$ and $Tx_n \neq Tz$ for all $n \geq n_0$. By Equations (4) and (5), we deduce that $x_n\mathcal{S}^*z$ and $Tx_n\mathcal{S}^*Tz$ for all $n \geq n_0$. It follows from $(\mathcal{B}_4)$ that

$$\varrho(a_n, b_n) = \varrho\left(d(Tx_n, Tz), d(x_n, z)\right) \geq 0 \quad \text{for all } n \geq n_0.$$

As a consequence, as $(\mathcal{B}_5)$ holds, we conclude that $\{a_n = d(x_{n+1}, Tz)\} \to 0$, that is, $\{x_{n+1}\} \to Tz$, which guarantees that $Tz = z$.

Subcase 2.2. Assume that Ω is not finite. In this case, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$d(x_{n(k)+1}, Tz) = 0 \quad \text{for all } k \in \mathbb{N}.$$

Hence, $x_{n(k)+1} = Tz$ for all $k \in \mathbb{N}$. Since $\{x_n\} \to z$ and $\left\{x_{n(k)+1}\right\} \to Tz$, $Tz = z$.

Case 3. Assume that (X, d) is $\mathcal{S}$-strictly-increasing-regular and $\varrho(t, s) \leq s - t$ for all $t, s \in A \cap (0, \infty)$. Proposition 3 guarantees that Item (b) is applicable.

In any case, we conclude that z is a fixed point of T. □

In the following result, we describe sufficient conditions in order to guarantee uniqueness of the fixed point.

Theorem 3. *Under the hypotheses of Theorem 2, assume that the following properties are fulfilled:*

- *Condition $(\mathcal{B}_2')$ holds; and*
- *for all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that z is, at the same time, $\mathcal{S}$-comparable to x and $\mathcal{S}$-comparable to y.*

Then, T has a unique fixed point.

Proof. Let $x, y \in \mathrm{Fix}(T)$ be two fixed points of T. By hypothesis, there exists $z_0 \in X$ such that z_0 is, at the same time, $\mathcal{S}$-comparable to x and $\mathcal{S}$-comparable to y. Let $\{z_n\}$ be the Picard sequence of T based on z_0, that is, $z_{n+1} = Tz_n$ for all $n \in \mathbb{N}$. We prove that $x = y$ by showing that $\{z_n\} \to x$ and $\{z_n\} \to y$. We first use x, but the same argument is valid for y.

Since z_0 is $\mathcal{S}$-comparable to x, assume that $z_0\mathcal{S}x$ (the case $x\mathcal{S}z_0$ is similar). As T is $\mathcal{S}$-nondecreasing, $z_n\mathcal{S}x$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $z_{n_0} = x$, then $z_n = x$ for all $n \geq n_0$. In particular, $\{z_n\} \to x$ and the proof is finished. On the contrary case, assume that $z_n \neq x$ for all $n \in \mathbb{N}$. Therefore $z_n\mathcal{S}^*x$ and $Tz_n\mathcal{S}^*Tx$ for all $n \in \mathbb{N}$. Using the contractivity Condition $(\mathcal{B}_4)$, for all $n \in \mathbb{N}$,

$$0 \leq \varrho(d(Tz_n, Tx), d(z_n, x)) = \varrho(d(T^{n+1}z_0, T^{n+1}x), d(T^n z_0, T^n x)).$$

It follows from $(\mathcal{B}_2')$ that $\{d(T^n z_0, T^n x)\} \to 0$, that is, $\{z_n\} \to x$. □

5. Consequences

In this section, we illustrate how many well known theorems in fixed point theory (that involve only $d(x,y)$ and $d(Tx,Ty)$ in their contractivity conditions) can be deduced from our main results.

5.1. Meir–Keeler Contractions

Meir and Keeler generalized the Banach theorem in a way that have attracted much attention in the last 40 years.

Definition 5 (Meir and Keeler [15]). *A* Meir–Keeler contraction *is a mapping $T : X \to X$ from a metric space (X,d) into itself such that for all $\varepsilon > 0$, there exists $\delta > 0$ verifying that if $x, y \in X$ and $\varepsilon \leq d(x,y) < \varepsilon + \delta$, then $d(Tx,Ty) < \varepsilon$.*

Lim characterized this kind of mappings in terms of a contractivity condition using the following class of auxiliary functions.

Definition 6 (Lim [14]). *A function $\phi : [0,\infty) \to [0,\infty)$ is called an* L-function *if*

(a) $\phi(0) = 0$;
(b) $\phi(t) > 0$ *for all* $t > 0$; *and*
(c) *for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $phi(t) \leq \varepsilon$ *for all* $t \in [\varepsilon, \varepsilon + \delta]$.

Each L-function must satisfy:

$$\phi(t) \leq t \quad \text{for all } t \in [0,\infty)\,. \tag{7}$$

Theorem 4 (Lim [14], Theorem 1). *Let (X,d) be a metric space and let $T : X \to X$ be a self-mapping. Then, T is a Meir–Keeler mapping if, and only if, there exists an (non-decreasing, right-continuous) L-map ϕ such that*

$$d(Tx,Ty) < \phi(d(x,y)) \quad \textit{for all } x,y \in X \textit{ verifying } d(x,y) > 0. \tag{8}$$

Meir and Keeler [15] demonstrated the following fixed point theorem by using a result of Chu and Diaz [18].

Theorem 5 (Meir and Keeler [15]). *Every Meir–Keeler contraction from a complete metric space into itself has a unique fixed point.*

We prove that this result can be immediately deduced from our main statements.

Theorem 6. *Every Meir–Keeler contraction is an ample spectrum contraction that also verifies $(\mathcal{B}_2')$ and $(\mathcal{B}_5)$.*

Proof. Let (X,d) be a metric space and let $T : X \to X$ be a Meir–Keeler contraction. By Theorem 4, there exists an L-map $\phi : [0,\infty) \to [0,\infty)$ verifying Equation (8). Let $A = \operatorname{ran}(d)$ and let define $\varrho_\phi : A \times A \to \mathbb{R}$ by $\varrho_\phi(t,s) = \phi(s) - t$ for all $t,s \in A$. Let us show that T is an ample spectrum contraction with respect to ϱ_ϕ.

$(\mathcal{B}_2')$ Let $x_1, x_2 \in X$ be two points such that

$$T^n x_1 \neq T^n x_2 \quad \text{and} \quad \varrho_\phi(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d\left(T^n x_1, T^n x_2\right)) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

As $d\left(T^n x_1, T^n x_2\right) > 0$, it follows from Equations (7) and (8) that, for all $n \in \mathbb{N}$,

$$d\left(T^{n+1}x_1, T^{n+1}x_2\right) = d\left(TT^n x_1, TT^n x_2\right) < \phi(d\left(T^n x_1, T^n x_2\right)) \leq d(T^n x_1, T^n x_2).$$

As $\{d(T^n x_1, T^n x_2)\}$ is a bounded-below decreasing sequence of real numbers, it is convergent. Let $L \geq 0$ be its limit. To prove that $L = 0$, we reason by contradiction. Assume that $L > 0$. Hence,

$$0 < L \leq d(T^{n+1}x_1, T^{n+1}x_2) < \phi(d(T^n x_1, T^n x_2)) \leq d(T^n x_1, T^n x_2) \quad \text{for all } n \in \mathbb{N}.$$

Letting $\varepsilon = L > 0$ in Condition (c) of Definition 6, there exists $\delta > 0$ such that $\phi(t) \leq \varepsilon = L$ for all $t \in [\varepsilon, \varepsilon + \delta]$. As $\{d(T^n x_1, T^n x_2)\} \searrow L^+$, there exists $n_0 \in \mathbb{N}$ such that $L < d(T^{n_0}x_1, T^{n_0}x_2) < L + \delta$ for all $n \geq n_0$. Therefore,

$$\phi(d(T^{n_0}x_1, T^{n_0}x_2)) \leq \varepsilon = L < \phi(d(T^{n_0}x_1, T^{n_0}x_2)),$$

which is a contradiction. Thus, $L = 0$ and $\{d\,(T^n x_1, T^n x_2)\} \to 0$.

$(\mathcal{B}_2)$ It follows from $(\mathcal{B}_2')$.

$(\mathcal{B}_3)$ Let $\{(a_n, b_n)\} \subseteq A \times A$ be a T-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho_\phi(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. By definition, there exist two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that

$$a_n = d(Tx_n, Ty_n) > 0 \quad \text{and} \quad b_n = d(x_n, y_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Notice that, from Equation (8), for all $n \in \mathbb{N}$,

$$L < a_n = d(Tx_n, Ty_n) < \phi(d(x_n, y_n)) = \phi\,(b_n) \leq b_n.$$

To prove that $L = 0$, assume that $L > 0$. Letting $\varepsilon = L > 0$ in Condition (c) of Definition 6, there exists $\delta > 0$ such that

$$\phi(t) \leq \varepsilon = L \quad \text{for all } t \in [\varepsilon, \varepsilon + \delta]\,.$$

As $\{d\,(x_n, y_n)\} \searrow L^+$, there exists $n_0 \in \mathbb{N}$ such that $L < d\,(x_n, y_n) < L + \delta$ for all $n \geq n_0$. Therefore,

$$\phi(d\,(x_{n_0}, y_{n_0})) \leq \varepsilon = L < \phi(d\,(x_{n_0}, y_{n_0})),$$

which is a contradiction. Thus, $L = 0$.

$(\mathcal{B}_4)$ It is clear that, for all $x, y \in X$ such that $d(x, y) > 0$ and $d(Tx, Ty) > 0$, Theorem 4 guarantees that

$$\varrho_\phi\,(d(Tx, Ty), d(x, y)) = \phi\,(d(x, y)) - d(Tx, Ty) > 0.$$

$(\mathcal{B}_5)$ Let $\{(a_n, b_n)\}$ be a T-sequence such that $\{b_n\} \to 0$ and $\varrho_\phi(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$0 \leq \varrho_\phi(a_n, b_n) = \phi\,(b_n) - a_n,$$

which means that $0 \leq a_n \leq \phi\,(b_n) \leq b_n$. Therefore, $\{b_n\} \to 0$ implies $\{a_n\} \to 0$. □

Theorem 7. *Theorem 5 follows from Theorems 2 and 3.*

Proof. From Theorem 6, every Meir–Keeler contraction is an ample spectrum contraction that also verifies $(\mathcal{B}_2')$ and $(\mathcal{B}_5)$, thus Theorems 2 and 3 are applicable in order to conclude that every Meir–Keeler contraction has a unique fixed point. □

5.2. Samet et al.'s Contractions

In [9], Samet et al. introduced the following kind of contractions and proved the following results. Let us denote by Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\Sigma_{n \in \mathbb{N}} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Definition 7. *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,y)\, d(Tx,Ty) \leq \psi(d(x,y)) \quad \textit{for all } x,y \in X. \tag{9}$$

The main results in [9] can be summarized as follows.

Theorem 8 (Samet, Vetro and Vetro [9], Theorems 2.1, 2.2 and 2.3)**.** *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) T is α-admissible (that is, if $\alpha(x,y) \geq 1$, then $\alpha(Tx,Ty) \geq 1$);
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; and
(iii) at least, one of the following conditions holds:

$(iii.1)$ T is continuous; or
$(iii.2)$ if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.
Furthermore, adding the condition:

(H) for all $x,y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$,

we obtain uniqueness of the fixed point of T.

To show that the previous theorem can be seen as a consequence of our main results, we present the following statement in which we use a more general class of auxiliary functions.

Theorem 9. *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. Assume that there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ such that ψ is nondecreasing, $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, and also*

$$\alpha(x,y)\, d(Tx,Ty) \leq \psi(d(x,y)) \quad \textit{for all } x,y \in X. \tag{10}$$

Then, T is an ample spectrum contraction with respect to $\mathcal{S}_\alpha$ that also verifies $(\mathcal{B}_2')$ and $(\mathcal{B}_5)$.

Proof. Let $\mathcal{S}_\alpha$ be the binary relation on X given in (1). Let $A = \mathrm{ran}(d)$ and let define $\gamma : A \to \mathbb{R}$ and $\varrho_\gamma : A \times A \to \mathbb{R}$ by, for all $t,s \in A$,

$$\gamma(s) = \inf(\{\, \alpha(x,y) : d(x,y) = s \,\}),$$
$$\varrho_\gamma(t,s) = \psi(s) - t\,\gamma(s).$$

Notice that γ is well defined because if $s \in A = \mathrm{ran}(d)$, then there exist $x_s, y_s \in X$ such that $d(x_s,y_s) = s$, and we can take infimum in a nonempty, subset of non-negative real numbers. Furthermore, as $\gamma(d(x,y)) \leq \alpha(x,y)$ for all $x,y \in X$, then, by (10),

$$\begin{aligned} \varrho_\gamma(d(Tx,Ty), d(x,y)) &= \psi(d(x,y)) - d(Tx,Ty)\,\gamma(d(x,y)) \\ &\geq \psi(d(x,y)) - d(Tx,Ty)\,\alpha(x,y) \geq 0. \end{aligned}$$

Hence, $(\mathcal{B}_4)$ holds. Let us prove the rest of properties.
$(\mathcal{B}_2')$ Let $x_1, x_2 \in X$ be two points such that

$$T^n x_1 \mathcal{S}_\alpha^* T^n x_2 \quad \text{and} \quad \varrho_\phi(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d(T^n x_1, T^n x_2)) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Since $T^n x_1 \mathcal{S}_\alpha^* T^n x_1$, then $\alpha(T^n x_1, T^n x_2) \geq 1$ and $T^n x_1 \neq T^n x_2$ for all $n \in \mathbb{N}$. By using Equation (10) and Proposition 1, for all $n \in \mathbb{N}$,

$$\begin{aligned} d(T^{n+1}x_1, T^{n+1}x_2) &\leq \alpha(T^n x_1, T^n x_2)\, d(TT^n x_1, TT^n x_2) \\ &\leq \psi\left(d(T^n x_1, T^n x_2)\right) \leq d(T^n x_1, T^n x_2). \end{aligned}$$

As $\{d(T^n x_1, T^n x_2)\}$ is a bounded-below non-increasing sequence of real numbers, it is convergent. Let $L \geq 0$ be its limit. Hence,

$$0 \leq L \leq d(T^{n+1}x_1, T^{n+1}x_2) \leq \psi(d(T^n x_1, T^n x_2)) \leq d(T^n x_1, T^n x_2) \quad \text{for all } n \in \mathbb{N}.$$

As ψ is nondecreasing, for all $n \in \mathbb{N}$,

$$d(T^n x_1, T^n x_2) \leq \psi(d(T^{n-1}x_1, T^{n-1}x_2)) \leq \psi^2(d(T^{n-2}x_1, T^{n-2}x_2)) \leq \ldots \leq \psi^n(d\left(x_1, x_2\right)).$$

Taking into account that $d\left(x_1, x_2\right) > 0$, then $\lim_{n\to\infty} \psi^n\left(d\left(x_1, x_2\right)\right) = 0$, and letting $n \to \infty$ in

$$0 \leq L \leq d(T^n x_1, T^n x_2) \leq \psi^n(d\left(x_1, x_2\right)),$$

we conclude that $L = \lim_{n\to\infty} d(T^n x_1, T^n x_2) = 0$.

$(\mathcal{B}_2)$ It follows from $(\mathcal{B}_2')$.

$(\mathcal{B}_3)$ Let $\{(a_n, b_n)\} \subseteq A \times A$ be a $(T, \mathcal{S}_\alpha)$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho_\gamma(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. By definition, there are two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that

$$x_n \mathcal{S}_\alpha y_n, \quad a_n = d(Tx_n, Ty_n) > 0 \quad \text{and} \quad b_n = d(x_n, y_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\alpha\left(x_n, y_n\right) \geq 1$ for all $n \in \mathbb{N}$. To prove that $L = 0$, we reason by contradiction. Assume that $L > 0$. By Property $(\mathcal{P}_{12})$ of Proposition 1, $\psi(L) < L$. It follows from Equation (10) that

$$\psi(L) < L < a_n = d(Tx_n, Ty_n) \leq \alpha\left(x_n, y_n\right)\, d(Tx_n, Ty_n) \leq \psi\left(d\left(x_n, y_n\right)\right) \leq d\left(x_n, y_n\right) = b_n. \tag{11}$$

Since $\{b_n\} \to L$, then $\lim_{n\to\infty} \psi\left(d\left(x_n, y_n\right)\right) = L$. As ψ is nondecreasing, the following limit exists and takes the value

$$\lim_{s\to L^+} \psi\left(s\right) = \lim_{n\to\infty} \psi\left(d\left(x_n, y_n\right)\right) = L.$$

As ψ is nondecreasing, $\psi(L) \leq \psi(s) \leq \psi(t)$ for all $L \leq s \leq t$, so

$$\psi(L) < L = \lim_{s\to L^+} \psi\left(s\right) \leq \psi\left(t\right) \quad \text{for all } t \in (L, \infty)\,.$$

Taking in mind that $L \leq \psi\left(t\right)$ for all $t \in (L, \infty)$, next, we distinguish two cases.

(Case 1) Assume that $\psi(t) > L$ for all $t \in (L, \infty)$. In this case, let $t_0 \in (L, \infty)$ be arbitrary. Then, $\psi(t_0) > L$. Therefore, $\psi^2(t_0) = \psi(\psi(t_0)) > L$. Repeating this argument, $\psi^3(t_0) = \psi(\psi^2(t_0)) > L$. Similarly, by induction, $\psi^n(t_0) > L$ for all $n \in \mathbb{N}$, which contradicts the fact that $\lim_{n\to\infty} \psi^n\left(t_0\right) = 0$.

(Case 2) Assume that there exists $L' > L$ such that $\psi(L') = L$. In this case, as ψ is nondecreasing, for all $t \in (L, L']$, we have that $L \leq \psi\left(t\right) \leq \psi(L') = L$, so $\psi(t) = L$ for all $t \in (L, L']$. Since $\{b_n = d\left(x_n, y_n\right)\} \searrow L^+$, there exists $n_0 \in \mathbb{N}$ such that $d\left(x_{n_0}, y_{n_0}\right) \in (L, L']$. Hence, $\psi\left(d\left(x_{n_0}, y_{n_0}\right)\right) = L$, which contradicts the strict inequality in Equation (11) because

$$L < a_{n_0} \leq \psi\left(d\left(x_{n_0}, y_{n_0}\right)\right).$$

In any case, we get a contradiction, so $L = 0$.

($\mathcal{B}_5$) Let $\{(a_n, b_n)\}$ be a $(T, \mathcal{S}_\alpha)$-sequence such that $\{b_n\} \to 0$ and $\varrho_\gamma(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. By definition, there exist two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that

$$x_n \mathcal{S}_\alpha y_n, \quad a_n = d(Tx_n, Ty_n) > 0 \quad \text{and} \quad b_n = d(x_n, y_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

In particular, $\alpha(x_n, y_n) \geq 1$ for all $n \in \mathbb{N}$. It follows from Equation (10) that

$$0 < a_n = d(Tx_n, Ty_n) \leq \alpha(x_n, y_n)\, d(Tx_n, Ty_n) \leq \psi(d(x_n, y_n)) \leq d(x_n, y_n) = b_n.$$

Since $\{b_n\} \to 0$, then $\{a_n\} \to 0$. □

Corollary 1. *Every Samet et al.'s α-ψ-contraction (in the sense of Definition 7) is an ample spectrum contraction with respect to $\mathcal{S}_\alpha$ that also verifies $(\mathcal{B}_2')$ and $(\mathcal{B}_5)$.*

Proof. It follows from the fact that, if $\psi \in \Psi$, then Theorem 9 is applicable because ψ is nondecreasing and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$ (recall Proposition 1). □

Theorem 10. *Theorem 8 immediately follows from Theorems 2 and 3.*

Proof. By Corollary 1, every Samet et al.'s α-ψ-contraction is an ample spectrum contraction with respect to $\mathcal{S}_\alpha$ that also verifies $(\mathcal{B}_2')$ and $(\mathcal{B}_5)$, thus Theorems 2 and 3 are applicable. □

5.3. Some Meditations about a Nonsymmetric Condition

In [1], Khojasteh et al. introduced the notion of simulation function as a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$; and
(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Shortly after, Roldán López de Hierro et al. [2] pointed out that Condition (ζ_3) is symmetric in both arguments of ζ, which is not necessary. Hence, these authors introduced the following variation in Axiom (ζ_3):

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

In this way, they removed the symmetry of a key function involved in the contractivity condition. After that, Roldán López de Hierro and Shahzad [3] presented the concept of R-contraction, which is intimately associated to an R-function $\varrho : A \times A \to \mathbb{R}$. Such kind of functions must satisfy the following conditions (see [3], Definition 12):

(ϱ_1) If $\{a_n\} \subset (0, \infty) \cap A$ is a sequence such that $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.
(ϱ_2) If $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$ are two sequences converging to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

Questions immediately arise: Why did the authors impose

$$L < a_n \quad \text{for all } n \in \mathbb{N} \tag{12}$$

in Assumption (ϱ_2)? Why did they not consider

$$L < b_n \quad \text{for all } n \in \mathbb{N} \tag{13}$$

rather than Equation (12)? A first response we can give is that both assumptions are interesting in order to remove the symmetry in the variables of ϱ in Assumption (ϱ_2) because the role of the sequence $\{a_n\}$ is different from the role of $\{b_n\}$. However, are Equations (12) and (13) equivalent? The response is no: we do believe that the condition in Equation (12) is better than the one in Equation (13). We justify it by the following fact: using the hypothesis in Equation (12), it is easy to check that every Meir–Keeler condition is an R-condition (see Theorem 25 in [3]). However, if we have only assumed that Equation (13) holds, then some Meir–Keeler contractions would not have been R-contractions. To illustrate it, we modify Example 2 in the following way.

Example 3. *Let $X = [0,1] \cup C \cup D$, where $C = \{\, 10m \in \mathbb{N} : m \in \mathbb{N}^* \,\}$ and $D = \{\, 10m + 1 + \frac{1}{m} \in \mathbb{N} : m \in \mathbb{N}^* \,\}$. If X is furnished with the Euclidean metric $d_E(x,y) = |\, x - y \,|$ for all $x, y \in X$, then (X, d_E) is a complete metric space. Let $T : X \to X$ be the self-mapping defined by*

$$Tx = \begin{cases} \dfrac{x}{4}, & \text{if } x \in [0,1]\,, \\ 0, & \text{if } x = 10m \in C \quad (\text{for some } m \in \mathbb{N}^*), \\ 1 - \dfrac{1}{2m}, & \text{if } x = 10m + 1 + \frac{1}{m} \in D \quad (\text{for some } m \in \mathbb{N}^*); \end{cases}$$

Notice that $Tx \in [0,1)$ for all $x \in X$. Therefore,

$$d_E(Tx, Ty) < 1 \quad \text{for all } x, y \in X. \tag{14}$$

Let us show that T is a Meir–Keeler contraction in (X, d_E). Indeed, let $\phi : [0,\infty) \to [0,\infty)$ be the function given by

$$\phi\,(t) = \begin{cases} \dfrac{t}{2}, & \text{if } t \in [0,1]\,, \\ 1, & \text{if } t > 1. \end{cases}$$

Clearly, ϕ is an L-function, and we claim that Equation (8) holds. Let $x, y \in X$ be such that $d\,(x,y) > 0$. Suppose, without loss of generality, that $x < y$.

- *If $x, y \in [0,1]$, then $d_E(x,y) \leq 1$ and*

$$d_E(Tx,Ty) = d_E\left(\frac{x}{4}, \frac{y}{4}\right) = \left|\, \frac{x}{4} - \frac{y}{4} \,\right| = \frac{|\, x - y \,|}{4} < \frac{|\, x - y \,|}{2} = \phi(d_E(x,y)).$$

- *If $x \in [0,1]$ and $y \in C \cup D$, then $d_E(x,y) > 1$, and it follows from Equation (14) that*

$$d_E(Tx,Ty) < 1 = \phi(d_E(x,y)).$$

- *If $x, y \in C \cup D$, then $d_E(x,y) > 1$ and, similarly, $d_E(Tx,Ty) < 1 = \phi(d_E(x,y))$.*

In any case, Equation (8) holds and Theorem 4 ensures us that T is a Meir–Keeler contraction in (X, d_E). In fact, Theorem 21 in [3] guarantees that the function $\varrho_\phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ given by

$$\varrho_\phi(t,s) = \phi(s) - t \quad \text{for all } t, s \in [0,\infty)\,,$$

is an R-function on $[0,\infty)$ verifying (ϱ_3). In particular, it satisfies Axiom (ϱ_2). Let us show that ϱ_ϕ would not satisfy (ϱ_2) if we replace Equation (12) with Equation (13). Indeed, let $\{x_n\}_{n \in \mathbb{N}^*}$ and $\{y_n\}_{n \in \mathbb{N}^*}$ be the sequences in X given by

$$x_n = 10n \quad \text{and} \quad y_n = 10n + 1 + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$,

$$a_n = d_E(Tx_n, Ty_n) = d_E\left(0, 1 - \frac{1}{2n}\right) = 1 - \frac{1}{2n} > 0 \quad \text{and}$$
$$b_n = d_E(x_n, y_n) = d_E\left(10n,\, 10n + 1 + \frac{1}{n}\right) = 1 + \frac{1}{n} > 1.$$

Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned}\varrho_\phi(a_n, b_n) &= \varrho_\phi\left(1 - \frac{1}{2n}, 1 + \frac{1}{n}\right) = \phi\left(1 + \frac{1}{n}\right) - \left(1 - \frac{1}{2n}\right) \\ &= 1 - \left(1 - \frac{1}{2n}\right) = \frac{1}{2n} > 0\end{aligned}$$

However, $L = 1$ is not zero. Therefore, ϱ_ϕ does not satisfy (ϱ_2) if we replace Equation (12) with Equation (13). Thus, in this case, there would be Meir–Keeler contractions that are not R-contractions.

As it can be easily checked, Property (ϱ_2) that R-functions must satisfy leads to Condition $(\mathcal{A}_3)$ for $(\mathcal{A}, \mathcal{S})$-contractions and Condition $(\mathcal{B}_3)$ for ample spectrum contractions.

$(\mathcal{B}_3)$ If $\{(a_n, b_n)\} \subseteq A \times A$ is a $(T, \mathcal{S}^*)$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$, then $L = 0$.

If we have assumed the condition in Equation (13) rather than the condition in Equation (12) in $(\mathcal{B}_3)$, then the same arguments given in Example 3 prove that there would be Meir–Keeler contractions that are not ample spectrum contractions. As a consequence, we conclude that the assumption in Equation (12) is more appropriate than the one in Equation (13) in the context of fixed point theory.

Nevertheless, in the next subsection, we are going to show that, under some very recent contractivity conditions, they would be equivalent.

5.4. Shahzad et al.'s Contractions

In [10], Shahzad et al. presented some coincidence point results for a new class of contractive mappings that they called (α, ψ, ϕ)-*contractions*. They used the following kind of auxiliary functions.

Definition 8 (Roldán López de Hierro [10], Definition 3.5)**.** *Let $\mathcal{F}_A$ be the family of all pairs (ψ, ϕ) where $\psi, \phi : [0, \infty) \to [0, \infty)$ are two functions verifying the following two conditions:*

$(\mathcal{F}_A^1)$ *If $\{a_n\} \subset (0, \infty)$ is a sequence such that $\psi(a_{n+1}) \leq \phi(a_n)$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.*
$(\mathcal{F}_A^2)$ *If $\{a_n\}, \{b_n\} \subset [0, \infty)$ are two sequences converging to the same limit L and such that $L < a_n$ and $\psi(b_n) \leq \phi(a_n)$ for all $n \in \mathbb{N}$, then $L = 0$.*

As a consequence of their main coincidence results, they presented the following statement (see the necessary preliminaries in [10]).

Theorem 11 (Shahzad, Karapınar and Roldán López de Hierro [10], Theorem 6.1)**.** *Let (X, d) be a metric space, let $\alpha : X \times X \to [0, \infty)$ be a function and let $T : X \to X$ be a mapping such that the following conditions are fulfilled:*

1. *there exists a subset $A \subseteq X$ such that $T(X) \subseteq A$ and (A, d) is complete;*
2. *α is transitive and T is α-admissible;*
3. *there exists $(\psi, \phi) \in \mathcal{F}_A$ such that*

$$\alpha(x, y)\, \psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X; \tag{15}$$

and

4. *at least one of the following conditions holds:*
   (a) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* T *is* (d, α)*-right-continuous; or*
   (b) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0, x_0) \geq 1$ *and* T *is* (d, α)*-left-continuous.*

Then, T *has, at least, a fixed point.*
Additionally, assume that $\phi(0) = 0$*,* $\psi^{-1}(\{0\}) = \{0\}$*, and the following property holds:*

(U) *for all fixed points* x *and* y *of* T*, there exists* $z \in X$ *such that* z *is, at the same time,* α*-comparable to* x *and to* y*.*

Then, T *has a unique fixed point.*

In the following definition, we modify the second condition.

Definition 9. *Let* $\mathcal{G}_{\mathcal{A}}$ *be the family of all pairs* (ψ, ϕ) *where* $\psi, \phi : [0, \infty) \to [0, \infty)$ *are two functions verifying the following two conditions:*

$(\mathcal{F}_{\mathcal{A}}^1)$ *If* $\{a_n\} \subset (0, \infty)$ *is a sequence such that* $\psi(a_{n+1}) \leq \phi(a_n)$ *for all* $n \in \mathbb{N}$*, then* $\{a_n\} \to 0$*.*
$(\mathcal{G}_{\mathcal{A}}^2)$ *If* $\{a_n\}, \{b_n\} \subset [0, \infty)$ *are two sequences converging to the same limit* L *and such that* $L < b_n$ *and* $\psi(b_n) \leq \phi(a_n)$ *for all* $n \in \mathbb{N}$*, then* $L = 0$*.*

The same theorem can be proved in this case.

Theorem 12. *Let* (X, d) *be a metric space, let* $\alpha : X \times X \to [0, \infty)$ *be a function and let* $T : X \to X$ *be a mapping such that the following conditions are fulfilled:*

1. *There exists a subset* $A \subseteq X$ *such that* $T(X) \subseteq A$ *and* (A, d) *is complete.*
2. α *is transitive and* T *is* α*-admissible.*
3. *There exists* $(\psi, \phi) \in \mathcal{G}_{\mathcal{A}}$ *such that*

$$\alpha(x, y)\, \psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X. \tag{16}$$

4. *At least one of the following conditions holds:*
   (a) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* T *is* (d, α)*-right-continuous; or*
   (b) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0, x_0) \geq 1$ *and* T *is* (d, α)*-left-continuous.*

Then, T *has, at least, a fixed point.*
Additionally, assume that $\phi(0) = 0$*,* $\psi^{-1}(\{0\}) = \{0\}$*, and the following property holds:*

(U) *For all fixed points* x *and* y *of* T*, there exists* $z \in X$ *such that* z *is, at the same time,* α*-comparable to* x *and to* y*.*

Then, T *has a unique fixed point.*

Let us show how this last result can be deduced from Theorems 2 and 3. The key is the following result.

Lemma 2. *Let* (X, d) *be a metric space, let* $\alpha : X \times X \to [0, \infty)$ *be a function and let* $T : X \to X$ *be a mapping such that the following conditions are fulfilled:*

1. *There exists* $(\psi, \phi) \in \mathcal{G}_{\mathcal{A}}$ *such that*

$$\alpha(x, y)\, \psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X. \tag{17}$$

2. *There exist two distinct points $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq 1$.*

Then, T is an ample spectrum contraction with respect to a function ϱ and $\mathcal{S}_\alpha$ that also verifies $(\mathcal{B}_2')$.

Proof. Let us consider

$$\begin{aligned} A &= \{ d(x,y) \in [0,\infty) : x,y \in X,\ x\mathcal{S}_\alpha^* y \} \\ &= \{ d(x,y) \in [0,\infty) : x,y \in X,\ x \neq y,\ \alpha(x,y) \geq 1 \}. \end{aligned}$$

As $d(x_0, x_1) \in A$, then A is nonempty. Let us define the function $\gamma : A \to \mathbb{R}$, for all $t \in A$, by

$$\gamma(t) = \inf(\{ \alpha(x,y) : x,y \in X,\ x\mathcal{S}_\alpha^* y \text{ and } d(x,y) = t \}).$$

To prove that γ is well defined, let $t \in A$ be arbitrary and let

$$\Omega_t = \{ \alpha(x,y) : x,y \in X,\ x\mathcal{S}_\alpha^* y \text{ and } d(x,y) = t \}.$$

By definition, as $t \in A$, there exist $x_t, y_t \in X$ such that $x_t\mathcal{S}_\alpha^* y_t$ and $t = d(x_t, y_t)$. Therefore, $\alpha(x_t, y_t) \in \Omega_t$, so this set is nonempty. Moreover, let $x, y \in X$ be arbitrary points such that $x\mathcal{S}_\alpha^* y$ and $d(x,y) = t$. Hence, $\alpha(x,y) \geq 1$. This proves that $\alpha(x,y) \geq 1$ for all number $\alpha(x,y) \in \Omega_t$. Taking into account that Ω_t is nonempty and bounded below by 1, we can take infimum, which means that $\gamma(t)$ is well defined. In particular, we have proved the following facts:

$$\gamma(t) = \inf \Omega_t \geq 1 \quad \text{for all } t \in A; \tag{18}$$

$$\gamma(d(x,y)) \leq \alpha(x,y) \quad \text{for all } x,y \in X \text{ such that } x\mathcal{S}_\alpha^* y. \tag{19}$$

Considering the pair $(\psi, \phi) \in \mathcal{G}_{\mathcal{A}}$, let $\varrho : A \times A \to \mathbb{R}$ be defined, for all $t, s \in A$, by

$$\varrho(t,s) = \phi(s) - \gamma(s)\ \psi(t) \quad \text{for all } t,s \in A.$$

We claim that T is an ample spectrum contraction with respect to ϱ and $\mathcal{S}_\alpha$ that also verifies $(\mathcal{B}_2')$. We demonstrate each condition. $(\mathcal{B}_1)$ is obvious.

$(\mathcal{B}_4)$ Let $x, y \in X$ be arbitrary points such that $x\mathcal{S}_\alpha^* y$ and $Tx\mathcal{S}_\alpha^* Ty$, that is, $\alpha(x,y) \geq 1$, $\alpha(Tx, Ty) \geq 1$, $x \neq y$ and $Tx \neq Ty$. Therefore, applying Equation (17),

$$\alpha(x,y)\ \psi(d(Tx,Ty)) \leq \phi(d(x,y)). \tag{20}$$

In particular, it follows from Equations (19) and (20) that

$$\begin{aligned} \varrho(d(Tx,Ty), d(x,y)) &= \phi(d(x,y)) - \gamma(d(x,y))\ \psi(d(Tx,Ty)) \\ &\geq \phi(d(x,y)) - \alpha(x,y)\ \psi(d(Tx,Ty)) \geq 0, \end{aligned}$$

so $(\mathcal{B}_4)$ holds.

$(\mathcal{B}_2')$ Let $x_1, x_2 \in X$ be two points such that

$$T^n x_1 \mathcal{S}_\alpha^* T^n x_2 \quad \text{and} \quad \varrho(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d(T^n x_1, T^n x_2)) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Notice that $T^n x_1 \mathcal{S}_\alpha^* T^n x_2$ and $T^{n+1} x_1 \mathcal{S}_\alpha^* T^{n+1} x_2$ imply that $d(T^n x_1, T^n x_2)$ and $d\left(T^{n+1}x_1, T^{n+1}x_2\right)$ belong to A. Let

$$a_n = d(T^n x_1, T^n x_2) > 0 \quad \text{for all } n \in \mathbb{N}.$$

In particular, as $\gamma \geq 1$, then

$$0 \leq \varrho(d\left(T^{n+1}x_1, T^{n+1}x_2\right), d\left(T^n x_1, T^n x_2\right)) = \varrho\left(a_{n+1}, a_n\right) \\ = \phi\left(a_n\right) - \gamma\left(a_n\right)\psi\left(a_{n+1}\right) \leq \phi\left(a_n\right) - \psi\left(a_{n+1}\right),$$

that is, $\psi\left(a_{n+1}\right) \leq \phi\left(a_n\right)$, for all $n \in \mathbb{N}$. Since $(\phi, \psi) \in \mathcal{G}_{\mathcal{A}}$, Condition $\left(\mathcal{F}^1_{\mathcal{A}}\right)$ implies that $\{a_n\} \to 0$, that is, $\{d\left(T^n x_1, T^n x_2\right)\} \to 0$, which means that $(\mathcal{B}'_2)$ holds.

$(\mathcal{B}_2)$ It immediately follows from $(\mathcal{B}_2)$.

$(\mathcal{B}_3)$ Let $\{(a'_n, b'_n)\} \subseteq A \times A$ be a $(T, \mathcal{S}^*_\alpha)$-sequence such that $\{a'_n\}$ and $\{b'_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a'_n$ and $\varrho(a'_n, b'_n) \geq 0$ for all $n \in \mathbb{N}$. By definition, there exist two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that

$$x_n \mathcal{S}^* y_n, \quad Tx_n \mathcal{S}^* Ty_n, \quad a'_n = d(Tx_n, Ty_n) > 0 \quad \text{and} \quad b'_n = d(x_n, y_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

As $\gamma \geq 1$, then

$$0 \leq \varrho(a'_n, b'_n) = \phi\left(b'_n\right) - \gamma\left(b'_n\right)\psi\left(a'_n\right) \leq \phi\left(b'_n\right) - \psi\left(a'_n\right),$$

that is, $\psi\left(a'_n\right) \leq \phi\left(b'_n\right)$, for all $n \in \mathbb{N}$. Since $(\phi, \psi) \in \mathcal{G}_{\mathcal{A}}$, Condition $\left(\mathcal{G}^2_{\mathcal{A}}\right)$ (applied to $\{a_n\} = \{b'_n\}$ and $\{b_n\} = \{a'_n\}$) implies that $L = 0$, which means that $(\mathcal{B}_3)$ holds.

As a consequence, we conclude that T is an ample spectrum contraction with respect to ϱ and $\mathcal{S}_\alpha$ that also verifies $(\mathcal{B}'_2)$. □

Lemma 2 permits us to show that Theorem 12 is a particular case of the above-presented main statements.

Theorem 13. *Theorem 12 follows from Theorems 2 and 3.*

Proof. Assume that all the hypotheses of Theorem 12 hold. For instance, assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and T is (d, α)-right-continuous (notice that Condition $(4.b)$ requires a version of Theorems 2 and 3 in which T is non-increasing). Let $\{x_{n+1} = Tx_n\}_{n\geq 0}$ be the Picard sequence of T based on x_0. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T, and $\{x_n\}$ converges to such point. In this case, the part about existence of a fixed point of T is finished. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Let $\mathcal{S}_\alpha$ be the binary relation on X given, for $x, y \in X$, by

$$x \mathcal{S}_\alpha y \quad \text{if} \quad \alpha(x, y) \geq 1. \tag{21}$$

By Lemma 1:

- As α is transitive, $\mathcal{S}_\alpha$ is transitive.
- As T is α-admissible, T is $\mathcal{S}_\alpha$-nondecreasing.
- As T is (d, α)-right-continuous, T is $\mathcal{S}_\alpha$-nonincreasing-continuous, thus T is $\mathcal{S}_\alpha$-strictly-increasing-continuous (T satisfies Item (a) of Theorem 2).

By Hypothesis 1 of Theorem 12, there exists a subset $A \subseteq X$ such that $T(X) \subseteq A$ and (A, d) is complete. In particular, $T(X)$ is $(\mathcal{S}_\alpha, d)$-strictly-increasing-precomplete. Finally, Lemma 2 guarantees that T is a an ample spectrum contraction with respect to ϱ and $\mathcal{S}_\alpha$ that also verifies $(\mathcal{B}'_2)$. As all hypotheses of Theorem 2 are satisfied, T has at least a fixed point.

Following the statement of Theorem 12, additionally, assume that $\phi(0) = 0$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:

(U) For all fixed points x and y of T, there exists $z \in X$ such that z is, at the same time, α-comparable to x and to y.

Then, Theorem 3 is applicable, thus T has a unique fixed point. $\square$

Remark 3. *Notice that, in fact, we have proved that every Shahzad et al.'s contraction in the sense of Theorem 11 is an ample spectrum contraction with respect to an appropriate function* ϱ*.*

5.5. Wardowski's F-Contractions

Definition 10 (Wardowski [11], Definition 2.1)**.** *Given a function* $F:(0,\infty)\to\mathbb{R}$ *, let consider the following properties:*

(F_1) *F is strictly increasing, that is,* $F(t)<F(s)$ *for all* $t,s\in(0,\infty)$ *such that* $t<s$*.*
(F_2) *For each sequence* $\{t_n\}_{n\in\mathbb{N}}$ *of positive real numbers we have that* $\{t_n\}\to 0$ *if, and only if,* $\{F(t_n)\}\to-\infty$*.*
(F_3) *There exists* $\lambda\in(0,1)$ *such that* $\lim_{t\to 0^+} t^{\lambda}F(t)=0$*.*

If (X,d) *is a metric space, a mapping* $T:X\to X$ *is an* F-contraction *if there exist a positive number* $\tau>0$ *and a function* $F:(0,\infty)\to\mathbb{R}$ *satisfying properties* (F_1)*-*(F_3) *such that*

$$\tau+F\left(d\left(Tx,Ty\right)\right)\le F\left(d\left(x,y\right)\right)\qquad \text{for all } x,y\in X \text{ such that } d\left(Tx,Ty\right)>0.$$

Theorem 14 (Wardowski [11], Theorem 2.1)**.** *Let* (X,d) *be a complete metric space and let* $T:X\to X$ *be an F-contraction. Then, T has a unique fixed point* $x^*\in X$*, and for every* $x_0\in X$ *a sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *is convergent to* x^**.*

Lemma 3. *Every F-contraction is an ample spectrum contraction.*

Notice that in the following proof we do not use Property (F_3).

Proof. Let (X,d) be a metric space and let $T:X\to X$ be an F-contraction with respect to a constant $\tau>0$ and a function $F:(0,\infty)\to\mathbb{R}$. Let $\lambda=\mathrm{e}^{-\tau}\in(0,1)$, let $A=[0,\infty)$ and let $\phi:(0,\infty)\to(0,\infty)$ and $\varrho:A\times A\to\mathbb{R}$ be the functions:

$$\phi\left(t\right)=\begin{cases}\mathrm{e}^{F(t)}, & \text{if } t>0,\\ 0, & \text{if } t=0;\end{cases}$$
$$\varrho\left(t,s\right)=\lambda\,\phi\left(s\right)-\phi\left(t\right)\quad \text{for all } t,s\in[0,\infty)$$

Property (F_1) implies that ϕ is strictly increasing on $(0,\infty)$ and Property (F_2) guarantees that for each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers we have that

$$\{t_n\}\to 0 \text{ if, and only if, } \{\phi(t_n)\}\to 0. \tag{22}$$

We claim that T is an ample spectrum contraction with respect to ϱ and the trivial preorder $\mathcal{S}_X$. Property $(\mathcal{B}_1)$ is obvious.

$(\mathcal{B}_2)$ Let $\{x_n\}\subseteq X$ be a Picard sequence of T such that

$$x_n\neq x_{n+1}\quad\text{and}\quad \varrho\left(d\left(x_{n+1},x_{n+2}\right),d\left(x_n,x_{n+1}\right)\right)\ge 0\quad\text{for all } n\in\mathbb{N}.$$

Therefore, for all $n\in\mathbb{N}$, $d\left(x_n,x_{n+1}\right)>0$ and

$$0\le\varrho\left(d\left(x_{n+1},x_{n+2}\right),d\left(x_n,x_{n+1}\right)\right)=\lambda\,\phi\left(d\left(x_n,x_{n+1}\right)\right)-\phi\left(d\left(x_{n+1},x_{n+2}\right)\right),$$

so

$$0\le\phi\left(d\left(x_{n+1},x_{n+2}\right)\right)\le\lambda\,\phi\left(d\left(x_n,x_{n+1}\right)\right).$$

In particular, $\{\phi(d(x_n, x_{n+1}))\} \to 0$, and the property in Equation (22) guarantees that $\{d(x_n, x_{n+1})\} \to 0$.

$(\mathcal{B}_3)$ Let $\{(a_n, b_n)\} \subseteq A \times A$ be a $(T, \mathcal{S}_X^*)$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) \geq 0$ for all $n \in \mathbb{N}$. By Definition 3, $a_n > 0$ and $b_n > 0$ for all $n \in N$. To prove that $L = 0$, assume, by contradiction, that $L > 0$. Notice that for all $n \in \mathbb{N}$,

$$0 \leq \varrho(a_n, b_n) = \lambda\, \phi(b_n) - \phi(a_n).$$

As ϕ is strictly increasing,

$$0 < \phi(L) < \phi(a_n) \leq \lambda\, \phi(b_n) < \phi(b_n).$$

This means that $L < a_n < b_n$. Since ϕ is strictly increasing, the following limit exists:

$$L' = \lim_{s \to L^+} \phi(s).$$

Furthermore, $0 < \phi(L) \leq L'$. As $\{a_n\} \to L$, $\{b_n\} \to L$ and $L < a_n < b_n$ for all $n \in \mathbb{N}$, then

$$L' = \lim_{s \to L^+} \phi(s) = \lim_{n \to \infty} \phi(a_n) = \lim_{n \to \infty} \phi(b_n).$$

Taking limit as $n \to \infty$ in $\phi(a_n) \leq \lambda\, \phi(b_n)$, we deduce that $L' \leq \lambda\, L'$, which contradicts the fact that $L' > 0$. Therefore, $L = 0$.

$(\mathcal{B}_4)$ Let $x, y \in X$ be two points such that $Tx \neq Ty$. In particular, $d(Tx, Ty) > 0$. Hence,

$$\begin{aligned} \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \quad &\Leftrightarrow \quad e^{\tau + F(d(Tx,Ty))} \leq e^{F(d(x,y))} \\ \Leftrightarrow \quad e^{F(d(Tx,Ty))} \leq e^{-\tau}\, e^{F(d(x,y))} \quad &\Leftrightarrow \quad \phi(d(Tx, Ty)) \leq \lambda\, \phi(d(x, y)) \\ \Leftrightarrow \quad \lambda\, \phi(d(x, y)) - \phi(d(Tx, Ty)) \geq 0 \quad &\Leftrightarrow \quad \varrho(d(Tx, Ty), d(x, y)) \geq 0. \end{aligned}$$

Therefore, T is an ample spectrum contraction with respect to ϱ and $\mathcal{S}_X$. □

As a consequence, Theorem 14 is a simple consequence of Theorems 2 and 3.

Finally, we point out that the present techniques can be easily generalized to guarantee existence and uniqueness of multidimensional coincidence/fixed points following the techniques described in [19–25].

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