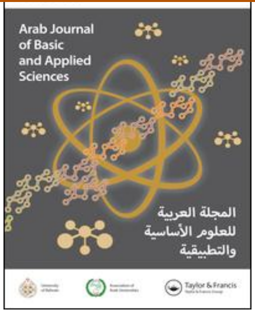

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ARTICLE

An unbiased estimator with prior information

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ABSTRACT

The ordinary least square (OLS) estimator suffers a breakdown in the presence of multicollinearity. The estimator is still unbiased but possesses a significant variance. In this study, we proposed an unbiased modified ridge-type estimator as an alternative to the OLS estimator and the biased estimators for handling multicollinearity in linear regression models. The properties of this new estimator were derived. The estimator is also unbiased with minimum variance. A real-life application to the higher heating value of poultry waste from proximate analysis and simulation study generally supported the findings.

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Unbiased; modified; efficient; ridge-type; simulation study

1. Introduction

Consider the linear regression model

$$y = X\beta + \varepsilon, \varepsilon \sim N(0, \sigma^2 I) \quad (1)$$

where y is a $n \times 1$ vector of the dependent variable, X is a known $n \times p$ full rank matrix of explanatory variables, β is a $p \times 1$ vector of regression coefficients and I is an $n \times n$ identity matrix. The ordinary least squares estimator (OLS) of β in model (1) is defined as:

$$\hat{\beta}_{OLS} = (S)^{-1} X'y \quad (2)$$

where $S = X'X$.

This estimator is the most widely used method to estimate the parameters in a linear regression model. It performs best when certain assumptions are satisfied. One of them is that the independent variables are not associated. However, in practice, there often exist strong or perfect linear relationships among the independent variables. This situation is called multicollinearity. The OLS estimator suffers a breakdown in the presence of multicollinearity. The estimator is still unbiased but possesses a significant variance (Ayinde, Lukman, Samuel, & Attah, 2018). Different approaches are available in the literature to handle this problem. These include Hoerl and Kennard (1970), Swindel (1976), Farebrother (1976), Liu (1993), Sakallioglu and Akdeniz (2003), Ozkale and Kacıranlar (2007), Yang and Chang (2010), Li and Yang (2012), Wu and Yang (2013), Wu (2014) and recently, Arumairajan and Wijekoon (2017), Ayinde et al. (2018), Lukman, Ayinde, Binuomote, and Onate (2019). The estimators by these authors are biased. Crouse, Jin, and Hanumara (1995) and Sakalloglu and Akdeniz (2003) proposed the unbiased version of the ridge estimator and Liu estimator, respectively, with the addition of prior information. These methods effectively handle the problem of multicollinearity and eliminate bias.

In this article, we proposed an unbiased modified ridge-type estimator (UMRT) with prior information and derived its properties. Furthermore, we discuss the performance of the proposed estimator over the OLS estimator, the Ridge estimator (RE) and the modified ridge-type estimator (MRT) using the mean square error matrix (MSEM) criteria.

The remaining part of this article is as follows. In Section 2, we proposed the unbiased modified ridge-type estimator and compared its performance with some existing estimators using the mean square error matrix (MSEM) criterion in Section 3. We estimate the biasing parameter k and d in Section 4. We conducted a simulation study and a real-life data application in Section 5. Finally, we provide some concluding remarks in Section 6.

2. Unbiased modified ridge-type estimator with prior information

Hoerl and Kennard (1970) defined the ridge estimator of β as:

$$\hat{\beta}_{RE}(k) = (S + kI)^{-1} X'y, \quad k > 0 \quad (3)$$

where k is the biasing parameter.

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Swindel (1976) defined the ridge estimator with prior information b

$$\hat{\beta}_{MRE}(k,b) = (S+kI)^{-1}(X'y+kb) \quad (4)$$

Crouse et al. (1995) introduced the unbiased ridge estimator based on the ridge estimator and prior information J. This is defined as

$$\hat{\beta}_{UMRE} = (S+kI)^{-1}(X'y+kJ) \quad (5)$$

where J and $\hat{\beta}_{OLS}$ are uncorrelated and J~N(β, V) such that $V = \left(\frac{\sigma^2}{k}\right)I_p$ and I_p is $p \times p$ identity matrix. J is estimated by $J = \frac{\sum_{i=1}^{p}\hat{\beta}_i}{p}$.

Lukman et al. (2019) proposed the modified ridge-type estimator which is defined as follows:

$$\hat{\beta}_{MRT}(k,d) == [S+k(1+d)]^{-1}S\hat{\beta}_{OLS} = F_{kd}\hat{\beta}_{OLS} \quad (6)$$

where $F_{kd} = [S+k(1+d)]^{-1}S$

Studying the following convex estimator

$$\hat{\beta}(C,J) = C\hat{\beta}_{OLS} + (I-C)J \quad (7)$$

where C is a $p\times p$ matrix and I is a $p\times p$ identity matrix. Consequently, the mean square error of $\hat{\beta}(C,J)$ is

$$MSE(\hat{\beta}(C,J)) = \sigma^2CS^{-1}C' + (I-C)V(I-C)' \quad (8)$$

Then,

$$\frac{\partial_{MSE}(\beta(C,J))}{\partial C} = 2C(\sigma^2S^{-1}+V) - 2V = 0 \quad (9)$$

From (9), C is obtained to be $C = V(\sigma^2S^{-1}+V)^{-1}$. Accordingly, $V = \sigma^2(I-C)^{-1}CS^{-1})$. The convex estimator $\beta(C,J)$ has minimum MSE for optimal value of C and it's an unbiased estimator of β. Therefore, the new estimator in this study is defined as

$$\begin{aligned}\hat{\beta}_{UMRT}(F_{kd},J) &= F_{kd}\hat{\beta}_{OLS} + (I-F_{kd})J \\ &= \hat{\beta}_{MRT}(k,d) + (I-F_{kd})J \end{aligned} \quad (10)$$

where $F_{kd} = [S+k(1+d)]^{-1}S$, then, the value of $V = \frac{\sigma^2}{k(1+d)}$. Consequently, $J \sim N(\beta, \frac{\sigma^2}{k(1+d)})$ for $k>0, 0<d<1$.

It is easy to show that $\hat{\beta}_{UMRT}(F_{kd},J)$ is an unbiased estimator of β. The expectation vector, bias vector, dispersion matrix and mean square error matrix of the proposed estimator are:

$$\begin{aligned}E(\hat{\beta}_{UMRT}(F_{kd},J)) &= E(\hat{\beta}_{MRT} + (I-F_{kd})J) \\ &= [S+k(1+d)]^{-1}S\beta + [S+k(1+d)]^{-1}k(1+d)\beta \\ &= [S+k(1+d)]^{-1}[S\beta + k(1+d)\beta] \\ &= [S+k(1+d)]^{-1}\beta[S+k(1+d)] \\ &= \beta\end{aligned} \quad (11)$$

$$\text{Bias}(\hat{\beta}_{\text{UMRT}}(F_{\text{kd}},J)) = E(\hat{\beta}_{\text{UMRT}}(F_{\text{kd}},J)) - \beta = \beta - \beta = 0 \quad (12)$$

$$\begin{aligned}D(\beta_{UMRT}(F_{kd},J)) &= D(\beta_{MRT} + (I-F_{kd})J) \\ &= \sigma^2[S+k(1+d)]^{-1}\end{aligned} \quad (13)$$

Since Bias = 0, then,

$$\text{MSEM}\ ((\beta_{\text{UMRT}}(F_{\text{kd}},J)) = D(\hat{\beta}_{\text{UMRT}}(F_{kd},J)) \quad (14)$$

Consequently, the estimator $\hat{\beta}_{UMRT}(F_{kd},J)$ is an unbiased estimator of β.

Suppose there exist an orthogonal matrix Q such that $Q'X'XQ = \Lambda = diag(\lambda_1,\lambda_2,...,\lambda_p)$ where λ_i is the ith eigenvalue of $X'X$. Λ and Q are the matrices of eigenvalues and eigenvectors of $X'X$, respectively. Model (1) can be written in canonical form as:

$$y = Z\alpha + \varepsilon \quad (15)$$

where $Z = XQ$, $\alpha = Q'\beta$ and $Z'Z = \Lambda$. For model (15), we get the following representations:

$$\hat{\alpha}_{OLS} = \Lambda^{-1}Z'y \quad (16)$$

$$\hat{\alpha}_{RE}(k) = (\Lambda+k)^{-1}Z'y \quad (17)$$

$$\hat{\alpha}_{MRT}(k,d) = [\Lambda+k(1+d)]^{-1}\Lambda\hat{\alpha}_{OLS} \quad (18)$$

$$\hat{\alpha}_{UMRT}(F_{kd},J) = \hat{\alpha}_{MRT}(k,d) + (I-F_{kd})J \quad (19)$$

Lemma 2.1. *Let M be an $n\times n$ positive definite matrix, that is $M>0$, and α be some vector, then $M-\alpha\alpha' \geq 0$ if and only if $\alpha'M^{-1}\alpha \leq 1$ (*Farebrother, 1976*).*

Lemma 2.2. *Let $\hat{\beta}_i = A_iyi = 1,2$ be two linear estimators of β. Suppose that $D = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2) > 0$, where $Cov(\hat{\beta}_i), i = 1,2$ denotes the covariance matrix of $\hat{\beta}_i$ and $b_i = Bias(\hat{\beta}_i) = (A_iX-I)\beta, i = 1,2$. Consequently,*

$$\begin{aligned}\Delta(\hat{\beta}_1 - \hat{\beta}_2) &= MSEM(\hat{\beta}_1) - MSEM(\hat{\beta}_2) \\ &= \sigma^2D + b_1b'_1 - b_2b'_2 > 0\end{aligned} \quad (20)$$

if and only if $b'_2[\sigma^2D + b_1b'_1]^{-1}b_2 < 1$ where $MSEM(\hat{\beta}_i) = Cov(\hat{\beta}_i) + b_ib'_i$ (Trenkler & Toutenburg, 1990).

3. Theoretical Comparisons

3.1. Comparison of the OLS estimator and the unbiased modified ridge-type estimator

Theorem 3.1. *The unbiased modified ridge-type estimator $\hat{\beta}_{UMRT}(F_{kd},J)$ is superior to the OLS estimator in the mean square error sense for $k>0$ and $0<d<1$*

Proof. By Definition,

$$MSEM(\hat{\beta}_{OLS}) = \sigma^2\Lambda^{-1} \quad (21)$$

The MSEM difference between Eqs. (14) and (21)

$$\begin{aligned}MSEM(\hat{\beta}_{OLS}) - MSEM(\hat{\beta}_{UMRT}(F_{kd},J)) &= \sigma^2\Lambda^{-1} - \sigma^2[\Lambda+k(1+d)]^{-1} \\ &= \sigma^2\left(\Lambda^{-1} - (\Lambda+k(1+d))^{-1}\right) \\ &= \sigma^2 diag\left[\frac{1}{\lambda_i} - \frac{1}{(\lambda_i+k(1+d))}\right]_{i=1}^{p}\end{aligned} \quad (22)$$

It was observed that $\Lambda^{-1} - [\Lambda+k(1+d)]^{-1}$ will be positive definite if and only if $\lambda_i + k(1+d) - \lambda_i > 0$. However, for $k>0$ and $0<d<1$, $\lambda_i +$

$k(1+d)-\lambda_i$ will be positive definite. By Lemma 2.2, the proof is completed.

3.2. Comparison of ridge estimator and the unbiased modified ridge-type estimator

From the representation, $\hat{\beta}_{RE}(k)=(\Lambda+kI)^{-1}Z'y$, the mean square error matrix is

$$D(\hat{\beta}_{RE}(k)=\sigma^2 B_k \Lambda B'_k \tag{23}$$

$$MSEM(\hat{\beta}_{RE}(k))=\sigma^2 B_k\Lambda B'_k+k^2B_k\alpha\alpha'B'_k \tag{24}$$

where $B_k=(\Lambda+kI)^{-1}$.

The difference between $\hat{\beta}_{RE}(k)$ and $\hat{\beta}_{UMRT}(F_{kd},J)$ in term of the MSEM is

$$\begin{aligned}&MSEM(\hat{\beta}_{RE}(k))-MSEM(\hat{\beta}_{UMRT}(F_{kd},J))\\&=\sigma^2B_k\Lambda B'_k+k^2B_k\alpha\alpha'B'_k-\sigma^2[\Lambda+k(1+d)]^{-1}\\&==\sigma^2(B_k\Lambda B'_k-(\Lambda+k(1+d))^{-1})+k^2B_k\alpha\alpha'B'_k\end{aligned} \tag{25}$$

Let $k>0$, $0<d<1$, thus, we have the following theorem.

Theorem 3.2. *Let us consider two estimators $\hat{\beta}_{RE}(k)$ and $\hat{\beta}_{UMRT}(F_{kd},J)$. If $k>0$ and $0<d<1$, the estimator $\hat{\beta}_{UMRT}(F_{kd},J)$ is superior to the estimator $MSEM(\hat{\beta}_{RE}(k))$ in the MSEM if and only if $B_k\Lambda B'_k-(\Lambda+k(1+d))^{-1}\geq 0$.*

Proof: The difference between Eqs. (14) and (23)

$$\begin{aligned}&=D(\hat{\beta}_{RE}(k))-D(\hat{\beta}_{UMRT}(F_{kd},J))\\&=\sigma^2(\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1}-\sigma^2(\Lambda+k(1+d))^{-1}\\&=\sigma^2\ diag\left[\frac{\lambda_i}{(\lambda_i+k)^2}-\frac{1}{\lambda_i+k(1+d)}\right]_{i=1}^{p}\end{aligned} \tag{26}$$

We observed that $(\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1}-(\Lambda+k(1+d))^{-1}$ will be positive definite if and only if $\lambda_i(\lambda_i+k(1+d))-(\lambda_i+k)^2>0$ or $\lambda_i(d-1)>k$. where $k>0$ and $0<d<1$.

3.3. Comparison of modified ridge-type estimator and unbiased modified ridge-type estimator

From the representation, $\hat{\alpha}_{MRT}(k,d)=[\Lambda+k(1+d)]^{-1}Z'y$, the dispersion and MSEM is defined as follows:

$$D(\hat{\beta}_{MRT}(k,d))=\sigma^2\overleftrightarrow{R}_k\Lambda^{-1}\overleftrightarrow{R}_k \tag{27}$$

where $\overleftrightarrow{R}=\Lambda(\Lambda+k(1+d)I)^{-1}$

$$\begin{aligned}&MSEM(\hat{\beta}_{MRT}(k,d)\\&=\sigma^2\overleftrightarrow{R}_k\Lambda^{-1}R_k+(\overleftrightarrow{R}_k-I)\alpha\alpha'(\overleftrightarrow{R}_k-Z)'\end{aligned} \tag{28}$$

Theorem 3.3. *The unbiased modified ridge type estimator always dominates the modified ridge type estimator in the MSEM sense for $k>0$ and $0<d<1$.*

Proof : The difference between (14) and (28)

$$\begin{aligned}&MSEM(\hat{\beta}_{MRT}(k,d))-MSEM(\hat{\beta}_{UMRT}(F_{kd},J)\\&=\sigma^2k(1+d)[\Lambda+k(1+d)I]^{-1}[I+k(1+d)\alpha\alpha']\\&[\Lambda+k(1+d)I]^{-1}\end{aligned} \tag{29}$$

Therefore, $MSEM(\hat{\beta}_{MRT}(k,d))-MSEM(\hat{\beta}_{UMRT}(F_{kd},J))$ is a non-negative matrix for $k>0$ and $0<d<1$. The proof of Theorem 3.3 is completed.

4. Estimation of the biasing parameters *k* and *d*

In this section, we discuss the estimation of the biasing parameter k and d.

4.1. The estimation of parameter *d*

In the definition of the new estimator, J and $\hat{\alpha}_{OLS}$ are uncorrelated. Therefore, $(\hat{\alpha}_{OLS}-J)\sim N(0,\frac{\sigma^2}{k(1+d)}[\Lambda^{-1}k(1+d)+1])$ and

$$E\left[(\hat{\alpha}_{OLS}-J)(\hat{\alpha}_{OLS}-J)'\right]=\frac{\sigma^2}{k(1+d)}\left[p+k(1+d)tr(\Lambda^{-1})\right] \tag{30}$$

From (30), if σ^2 is known for a fixed k, we can get an unbiased estimator of d as follows:

$$\hat{d}=\frac{p\sigma^2}{k\left[(\hat{\beta}_{OLS}-J)(\hat{\beta}_{OLS}-J)'-\sigma^2tr(\Lambda^{-1})\right]}-1 \tag{31}$$

When σ^2 is unknown, s^2 is used as an estimate of σ^2.

$$s^2=\frac{(Y-X\hat{\beta}_{OLS})'(Y-X\hat{\beta}_{OLS})}{n-p} \tag{32}$$

Consequently,

$$\hat{d}=\frac{ps^2}{k\left[(\hat{\beta}_{OLS}-J)(\hat{\beta}_{OLS}-J)'-s^2tr(\Lambda^{-1})\right]}-1 \tag{33}$$

where $tr(\Lambda^{-1})=\sum_{i=1}^{p}\frac{1}{\lambda_i}$ and λ_i is the eigen-value of $X'X$. It was observed that the estimator of d in (33) can return a negative value. To eliminate the negative value, Wu (2014) suggests replacing $\hat{d}$ with one (1) when its estimate is negative. Here, in this study, when d in Eq. (33) is negative, we adopt the estimator of $\hat{d}$ suggested by Ozkale and Kaciranlar (2007) as follows:

$$\hat{d}^*=\min\left[\frac{\alpha_i^2}{\frac{\sigma^2}{\lambda_i}+\alpha_i^2}\right] \tag{34}$$

4.2. Estimating the biasing parameter *k*

From Eq. (30), if σ^2 is known and d is assumed to be fixed, an unbiased estimate of k is defined as follows:

Table 1. Correlation matrix of the variables.

HHV	FC	VM	ASH	
1	0.7844	−0.0307	−0.5995	HHV
	1	−0.5538	−0.1485	FC
		1	−0.7409	VM
			1	ASH

Table 2. Regression coefficients and MSE.

	$\hat{k}_{HM} = 0.007676$, $\hat{d}$=0.0006			
Coeff.	$\hat{\alpha}_{OLS}$	$\hat{\alpha}_{RE}$	$\hat{\alpha}_{MRT}$	$\hat{\alpha}_{UMRT}$
α_0	167.7189	102.0991	8.0166	167.7189
α_1	−1.2704	−0.6143	−0.6141	−1.2704
α_2	−1.5311	−0.8763	−0.8760	−1.5311
α_3	−1.6840	−1.0267	−1.0265	−1.6840
Bias	0	0.6589	0.6591	0
MSE	4521.101	1675.788	1674.967	1674.533
$AMSE_{CV}$	349.9209	5.6562	5.6528	3.9485

$$\hat{k} = \frac{p\sigma^2}{(1+d)\left[(\hat{\beta}_{OLS} - J)(\hat{\beta}_{OLS} - J)' - \delta^2 tr(\Lambda^{-1})\right]} \tag{35}$$

When $\hat{k}$ is negative, estimate $\hat{k}$ as follows:

$$\hat{k} = \frac{p\sigma^2}{\sum_{i=1}^{p} \alpha_i^2} \tag{36}$$

5. Numerical example and Monte–Carlo simulation

5.1. Application to poultry waste data

The theoretical results are illustrated with real-life data which was analyzed in the study of Qian, Lee, Soto, and Chen (2018). A total of 48 samples of poultry waste were collected from different published open literature reviews to form a database for derivation, evaluation and validation of proximate-based higher heating value (HHV) models. Six samples (#43, 44, 45, 46, 47 and 48) were deleted due to incomplete information. The linear regression model is:

$$HHV = \beta_0 + \beta_1 FC + \beta_2 VM + \beta_3 A + \varepsilon \tag{37}$$

where HHV denotes Higher Heating Value, FC denotes Fixed Carbon, VM denotes Volatile Matter, A denotes ASH and ε is the random error term that is expected to be normally distributed. The relationship between the variables were obtained by the correlation matrix as follows.

From Table 1, there is a strong positive relationship between higher heating value and Fixed Carbon while a negative relationship exists between HHV and VM; HHV and Ash. To identify the distribution of the error term, we used the Jarque-Bera (JB) test. The test statistic and the corresponding p value are JB $=$ 0.6409 and p value $=$.7258, respectively. Since this p value is larger than any reasonable alpha value used in the literature, we conclude that the error term follows the normal distribution. We diagnosed the model for a possible presence of multicollinearity. The variance inflation factor (VIF) values are $VIF_{FC} = 997.819$, $VIF_{VM} = 2163.504$, $VIF_{ASH} = 1533.782$. Literature shows that a model suffers from multicollinearity when $VIF_i > 10$. Since the values of the VIF in the above model is higher than 10, we conclude that the model suffers from severe multicollinearity. Alternatively, we can use the condition number (CN) to examine if the explanatory variables are related where $CN = \frac{maximum(eigenvalue)}{minimum(eigenvalue)}$. If CN is between 100 and 1000 there is moderate to strong multicollinearity and if it exceeds 1000 there is severe multicollinearity (Arumairajan & Wijekoon, 2017; Gujarati, 1995). The condition number is 581291.39 which indicates the presence of severe multicollinearity. Therefore, it will be appropriate to predict higher heating value with an alternative unbiased estimator possessing minimum variance. We adopt K fold crossvalidation to validate the performances of the estimators. The data is partitioned into K equal size folds ($K=10$ in this study). In these K folds, onefold will be treated as the test set and use the remaining K – 1 (9) folds as the training set. The MSE is computed on the observations in the held-out fold. The process is repeated ten times, taking out a different part each time. The validation test error is obtained by computing the average K estimates of the test error, and we get an estimated validation (test) error rate for new observations. The estimator with the lowest validation MSE is the best. The average MSE of the validation error in this study is defined as:

$$AMSE_{CV} = \frac{\sum_{k=1}^{10} \frac{1}{n_k} \sum_{i=1}^{n} (y_i - \tilde{y}_i)^2}{10} \tag{38}$$

where n_k is the number of subsample in each fold, $\tilde{y}_i$ is the fitted value for observation i, obtained from the data with fold k removed. The result is presented in Table 2.

The result in Table 2 shows that the unbiased modified ridge-type estimator (UMRT) produced the same estimates with the OLS estimator. Also, the technique was able to circumvent the problem of large variance which is peculiar to the OLS estimator. The proposed estimator has the smallest mean square error and prediction error, respectively.

5.2. Monte–Carlo simulation

We carried out a Monte–Carlo simulation to investigate the performances of these estimators. The explanatory variables were generated in line with the study of McDonald and Galarneau (1975), Liu

Table 3. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 30$, Sig = 1 and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.85	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.7149	0.7149	0.7149	0.7149	0.7149	0.7149	0.7149	0.7149	0.7149
RIDGE	0.4992	0.4280	0.4015	0.4992	0.4280	0.4015	0.4992	0.4280	0.4015
MRT	0.4853	0.3970	0.3664	0.4009	0.3565	0.3379	0.4394	0.3669	0.3392
UMRT	0.4486	0.3767	0.3520	0.4186	0.3489	0.3264	0.4055	0.3376	0.3161
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ =0.95	$k = 0.5$	$k = 0.8$	$k = 0.95$	k = 0.5	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	2.0294	2.0294	2.0294	2.0294	2.0294	2.0294	2.0294	2.0294	2.0294
RIDGE	0.8523	0.6412	0.5758	0.8523	0.6412	0.5758	0.8523	0.6412	0.5758
MRT	0.7056	0.6173	0.5132	0.6309	0.6129	0.5124	0.6258	0.6123	0.5132
UMRT	0.6968	0.5200	0.4689	0.6173	0.4628	0.4200	0.5855	0.4409	0.4014
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.99	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	9.5571	9.5571	9.5571	9.5571	9.5571	9.5571	9.5571	9.5571	9.5571
RIDGE	1.0061	0.6503	0.5616	1.0061	0.6503	0.5616	1.0061	0.6503	0.5616
MRT	0.7978	0.6407	0.5298	0.7685	0.6287	0.5219	0.7585	0.6250	0.5196
UMRT	0.7333	0.4932	0.4365	0.6168	0.4301	0.3875	0.5741	0.4078	0.3703

Table 4. Estimated MSE for OLS, RE, D and MRT when $n = 30$, sig = 5 and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.85	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	17.872	17.890	17.890	17.890	17.890	17.890	17.890	17.890	17.890
RIDGE	10.123	10.119	9.303	12.193	10.119	9.303	12.193	10.119	9.303
MRT	9.972	9.057	8.825	11.436	9.795	9.573	11.319	9.685	8.468
UMRT	8.517	8.510	7.684	9.833	7.578	6.771	9.431	7.178	6.385
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.95	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	50.749	50.749	50.749	50.749	50.749	50.749	50.749	50.749	50.749
RIDGE	20.139	14.149	12.214	20.139	14.149	12.214	20.139	14.149	12.214
MRT	19.573	13.922	11.414	18.847	13.352	12.005	19.549	13.126	10.704
UMRT	15.758	10.520	8.925	13.448	8.732	7.341	12.504	8.025	6.722
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.99	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	238.93	238.93	238.93	238.93	238.93	238.93	238.93	238.93	238.93
RIDGE	21.40	11.79	9.35	21.40	11.79	9.35	21.40	11.79	9.35
MRT	16.38	11.50	8.74	13.04	10.65	8.03	18.52	10.33	6.07
UMRT	14.05	7.44	5.84	10.87	5.66	4.43	9.70	5.02	3.93

Table 5. Estimated MSE for OLS, RE, D and MRT when $n = 50$, sig = 1 and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.85	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.3124	0.3124	0.3124	0.3124	0.3124	0.3124	0.3124	0.3124	0.3124
RIDGE	0.2617	0.2395	0.2301	0.2617	0.2395	0.2301	0.2617	0.2395	0.2301
MRT	0.2519	0.2243	0.2299	0.2451	0.2373	0.2240	0.2367	0.2140	0.1978
UMRT	0.2463	0.2206	0.2104	0.2362	0.2091	0.1988	0.2316	0.2040	0.1938
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.95	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.822	0.822	0.822	0.822	0.822	0.822	0.822	0.822	0.822
RIDGE	0.533	0.443	0.411	0.533	0.443	0.411	0.533	0.443	0.411
MRT	0.519	0.409	0.389	0.513	0.369	0.362	0.491	0.380	0.375
UMRT	0.469	0.381	0.352	0.432	0.348	0.322	0.416	0.335	0.310
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.99	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	3.775	3.775	3.775	3.775	3.775	3.775	3.775	3.775	3.775
RIDGE	0.973	0.676	0.593	0.973	0.676	0.593	0.973	0.676	0.593
MRT	0.792	0.582	0.487	0.623	0.528	0.438	0.721	0.533	0.445
UMRT	0.750	0.524	0.465	0.645	0.458	0.410	0.605	0.433	0.390

Table 6. Estimated MSE for OLS, RE, D and MRT when $n = 50$, sig $= 5$ and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	7.217	7.217	7.217	7.217	7.217	7.217	7.217	7.217	7.217
RIDGE	6.138	5.619	5.388	6.138	5.619	5.388	6.138	5.619	5.388
MRT	6.112	5.610	5.268	5.745	4.959	4.883	5.715	5.122	4.445
UMRT	5.783	5.146	4.873	5.540	4.836	4.542	5.426	4.693	4.391
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	19.912	19.912	19.912	19.912	19.912	19.912	19.912	19.912	19.912
RIDGE	12.910	10.505	9.580	12.910	10.505	9.580	12.910	10.505	9.580
MRT	12.327	9.613	8.359	11.032	8.326	8.084	11.903	9.206	7.740
UMRT	11.213	8.693	7.781	10.180	7.664	6.786	9.724	7.227	6.370
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	93.640	93.640	93.640	93.640	93.640	93.640	93.640	93.640	93.640
RIDGE	22.567	14.290	11.879	22.567	14.290	11.879	22.567	14.290	11.879
MRT	18.875	11.675	9.043	12.880	9.968	7.433	14.485	9.694	6.199
UMRT	16.390	9.869	8.065	13.402	7.851	6.360	12.232	7.085	5.720

Table 7. Estimated MSE for OLS, RE, D and MRT when $n = 100$, sig $= 1$ and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.1285	0.1285	0.1285	0.1290	0.1290	0.129	0.129	0.129	0.129
RIDGE	0.1195	0.1151	0.1132	0.1190	0.1150	0.113	0.119	0.115	0.113
MRT	0.1253	0.1245	0.1243	0.1250	0.1240	0.124	0.125	0.124	0.124
UMRT	0.1165	0.1112	0.1090	0.1140	0.1090	0.106	0.114	0.108	0.105
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.370	0.36964	0.370	0.370	0.370	0.370	0.36964	0.36964	0.36964
RIDGE	0.300	0.27206	0.261	0.300	0.272	0.261	0.30007	0.27206	0.26073
MRT	0.350	0.34732	0.348	0.348	0.348	0.349	0.34778	0.34842	0.3505
UMRT	0.281	0.24961	0.238	0.268	0.236	0.225	0.26251	0.23089	0.21993
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	1.7988	1.7988	1.7988	1.7988	1.7988	1.7988	1.7988	1.7988	1.7988
RIDGE	0.8014	0.6084	0.5472	0.8014	0.6084	0.5472	0.8014	0.6084	0.5472
MRT	1.5095	1.4892	1.4873	1.4985	1.4872	1.4886	1.4949	1.4875	1.4903
UMRT	0.6598	0.4944	0.4457	0.5861	0.4399	0.3986	0.5563	0.4188	0.3807

Table 8. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 100$, Sig $= 5$ and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	3.2125	3.2125	3.2125	3.2125	3.2125	3.2125	3.2125	3.2125	3.2125
RIDGE	2.9762	2.8490	2.7891	2.9762	2.8490	2.7891	2.9762	2.8490	2.7891
MRT	3.1210	3.0748	3.0537	3.1038	3.0508	3.0270	3.0955	3.0394	3.0145
UMRT	2.8903	2.7239	2.6471	2.8288	2.6365	2.5491	2.7989	2.5946	2.5025
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	9.2410	9.2410	9.2410	9.2410	9.2410	9.2410	9.2410	9.2410	9.2410
RIDGE	7.4076	6.5890	6.2371	7.4076	6.5890	6.2371	7.4076	6.5890	6.2371
MRT	8.5726	8.3126	8.2076	8.4709	8.1937	8.0852	8.4240	8.1406	8.0313
UMRT	6.8435	5.8766	5.4793	6.4679	5.4264	5.0111	6.2934	5.2237	4.8030
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	44.971	44.971	44.971	44.971	44.971	44.971	44.971	44.971	44.971
RIDGE	18.971	13.474	11.659	18.971	13.474	11.659	18.971	13.474	11.659
MRT	36.254	34.771	34.290	35.615	34.230	33.789	35.348	34.008	33.586
UMRT	14.969	10.053	8.526	12.819	8.340	6.996	11.932	7.658	6.395

Table 9. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 200$, Sig = 1 and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.0676	0.0676	0.0676	0.0676	0.0676	0.0676	0.0676	0.0676	0.0676
RIDGE	0.0654	0.0643	0.0638	0.0654	0.0643	0.0638	0.0654	0.0643	0.0638
MRT	0.0670	0.0668	0.0667	0.0669	0.0667	0.0667	0.0668	0.0667	0.0668
UMRT	0.0647	0.0633	0.0626	0.0642	0.0626	0.0619	0.0639	0.0622	0.0615
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.1998	0.1998	0.1998	0.1998	0.1998	0.1998	0.1998	0.1998	0.1998
RIDGE	0.1808	0.1718	0.1678	0.1808	0.1718	0.1678	0.1808	0.1718	0.1678
MRT	0.1942	0.1929	0.1927	0.1936	0.1927	0.1928	0.1933	0.1927	0.1930
UMRT	0.1747	0.1637	0.1592	0.1705	0.1586	0.1538	0.1685	0.1562	0.1514
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.9936	0.9936	0.9936	0.9936	0.9936	0.9936	0.9936	0.9936	0.9936
RIDGE	0.6317	0.5211	0.4808	0.6317	0.5211	0.4808	0.6317	0.5211	0.4808
MRT	0.8903	0.8783	0.8773	0.8840	0.8774	0.8791	0.8817	0.8779	0.8809
UMRT	0.5528	0.4435	0.4066	0.5068	0.4021	0.3685	0.4870	0.3851	0.3532

Table 10. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 200$, Sig = 5 and $p = 3$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	1.6888	1.6888	1.6888	1.6888	1.6888	1.6888	1.6888	1.6888	1.6888
RIDGE	1.6310	1.5980	1.5820	1.6310	1.5980	1.5820	1.6310	1.5980	1.5820
MRT	1.6680	1.6565	1.6510	1.6638	1.6502	1.6438	1.6617	1.6472	1.6403
UMRT	1.6089	1.5642	1.5426	1.5927	1.5396	1.5143	1.5847	1.5276	1.5005
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	4.9937	4.9937	4.9937	4.9937	4.9937	4.9937	4.9937	4.9937	4.9937
RIDGE	4.4848	4.2205	4.0982	4.4848	4.2205	4.0982	4.4848	4.2205	4.0982
MRT	4.8031	4.7109	4.6698	4.7685	4.6642	4.6186	4.7519	4.6423	4.5948
UMRT	4.3056	3.9668	3.8143	4.1791	3.7933	3.6228	4.1182	3.7114	3.5333
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	24.840	24.840	24.840	24.840	24.840	24.840	24.840	24.840	24.840
RIDGE	15.310	12.165	10.973	15.310	12.165	10.973	15.310	12.165	10.973
MRT	21.499	20.576	20.244	21.120	20.201	19.881	20.953	20.042	19.729
UMRT	13.084	9.839	8.684	11.745	8.538	7.439	11.158	7.989	6.922

Table 11. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 30$, Sig = 1 and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	1.478	1.478	1.478	1.478	1.478	1.478	1.478	1.478	1.478
RIDGE	1.031	0.884	0.828	1.031	0.884	0.828	1.031	0.884	0.828
MRT	0.983	0.811	0.770	0.977	0.870	0.671	0.975	0.770	0.672
UMRT	0.927	0.775	0.721	0.864	0.715	0.663	0.837	0.689	0.639
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	4.647	4.647	4.647	4.647	4.647	4.647	4.647	4.647	4.647
RIDGE	1.910	1.433	1.279	1.910	1.433	1.279	1.910	1.433	1.279
MRT	1.636	1.174	1.099	1.419	1.099	0.998	1.413	1.008	0.969
UMRT	1.560	1.146	1.021	1.377	1.006	0.898	1.302	0.951	0.851
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	23.779	23.779	23.779	23.779	23.779	23.779	23.779	23.779	23.779
RIDGE	2.391	1.493	1.269	2.391	1.493	1.269	2.391	1.493	1.269
MRT	1.954	1.255	0.976	1.943	0.979	0.909	1.943	0.993	0.826
UMRT	1.702	1.097	0.957	1.408	0.941	0.837	1.300	0.886	0.795

Table 12. Estimated MSE for OLS, RE, D and MRT when $n = 30$, sig $= 5$ and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	36.944	36.944	36.944	36.944	36.944	36.944	36.944	36.944	36.944
RIDGE	25.322	21.258	19.666	25.322	21.258	19.666	25.322	21.258	19.666
MRT	23.855	19.001	17.695	23.503	18.657	12.364	20.349	20.111	15.226
UMRT	22.464	18.117	16.498	20.701	16.289	14.692	19.915	15.499	13.922
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	116.17	116.17	116.17	116.17	116.17	116.17	116.17	116.17	116.17
RIDGE	46.06	33.15	28.89	46.06	33.15	28.89	46.06	33.15	28.89
MRT	38.88	27.83	22.20	33.97	24.13	21.59	27.60	23.86	19.36
UMRT	36.65	25.09	21.46	31.61	21.01	17.77	29.53	19.37	16.30
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	594.48	594.48	594.48	594.48	594.48	594.48	594.48	594.48	594.48
RIDGE	53.56	29.41	23.20	53.56	29.41	23.20	53.56	29.41	23.20
MRT	41.75	20.55	15.70	40.94	15.74	13.15	26.73	14.92	12.42
UMRT	35.14	18.35	14.27	27.08	13.80	10.67	24.09	12.16	9.38

Table 13. Estimated MSE for OLS, RE, D and MRT when $n = 50$, sig $= 1$ and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.816	0.816	0.816	0.816	0.816	0.816	0.816	0.816	0.816
RIDGE	0.631	0.559	0.530	0.631	0.559	0.530	0.631	0.559	0.530
MRT	0.624	0.533	0.499	0.569	0.478	0.475	0.627	0.534	0.453
UMRT	0.581	0.501	0.470	0.549	0.466	0.435	0.534	0.451	0.421
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	2.377	2.377	2.377	2.377	2.377	2.377	2.377	2.377	2.377
RIDGE	1.259	1.002	0.914	1.259	1.002	0.914	1.259	1.002	0.914
MRT	1.145	0.908	0.798	1.129	0.997	0.789	1.122	0.992	0.785
UMRT	1.073	0.835	0.759	0.971	0.749	0.681	0.928	0.715	0.649
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	11.493	11.493	11.493	11.493	11.493	11.493	11.493	11.493	11.493
RIDGE	2.223	1.528	1.329	2.223	1.528	1.329	2.223	1.528	1.329
MRT	1.823	1.335	1.217	1.780	1.276	0.993	1.764	1.371	0.970
UMRT	1.702	1.165	1.021	1.454	1.004	0.888	1.358	0.944	0.839

Table 14. Estimated MSE for OLS, RE, D and MRT when $n = 50$, sig $= 5$ and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	20.393	20.393	20.393	20.393	20.393	20.393	20.393	20.393	20.393
RIDGE	15.748	13.875	13.102	15.748	13.875	13.102	15.748	13.875	13.102
MRT	15.284	13.496	12.743	14.130	12.723	11.567	14.060	12.647	11.490
UMRT	14.446	12.328	11.493	13.607	11.384	10.532	13.225	10.966	10.111
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	59.417	59.417	59.417	59.417	59.417	59.417	59.417	59.417	59.417
RIDGE	31.210	24.485	22.124	31.210	24.485	22.124	31.210	24.485	22.124
MRT	28.23	23.94	21.50	29.69	22.444	17.024	25.456	19.234	18.826
UMRT	26.368	19.948	17.782	23.643	17.509	15.481	22.485	16.495	14.532
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	287.31	287.31	287.31	287.31	287.31	287.31	287.31	287.31	287.31
RIDGE	53.30	34.43	28.82	53.30	34.43	28.82	53.30	34.43	28.82
MRT	43.80	29.38	23.47	36.23	28.37	27.67	40.61	30.03	20.39
UMRT	39.26	24.08	19.77	32.37	19.25	15.64	29.64	17.40	14.07

Table 15. Estimated MSE for OLS, RE, D and MRT when $n = 100$, sig = 1 and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ=0.85	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.27131	0.27131	0.27131	0.27131	0.27131	0.27131	0.27131	0.27131	0.27131
RIDGE	0.25222	0.24259	0.23822	0.25222	0.24259	0.23822	0.25222	0.24259	0.23822
MRT	0.26867	0.26824	0.26828	0.26843	0.26829	0.26855	0.26835	0.26839	0.26877
UMRT	0.2457	0.2336	0.2283	0.2411	0.2275	0.2217	0.2389	0.2247	0.2188
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.95	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.80889	0.80889	0.80889	0.80889	0.80889	0.80889	0.80889	0.80889	0.80889
RIDGE	0.6439	0.57816	0.55129	0.6439	0.57816	0.55129	0.6439	0.57816	0.55129
MRT	0.77317	0.76397	0.76117	0.76922	0.76085	0.75869	0.76754	0.7597	0.7579
UMRT	0.5981	0.5246	0.4962	0.5688	0.4925	0.464	0.5555	0.4784	0.4501
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.99	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	4.04968	4.04968	4.04968	4.04968	4.04968	4.04968	4.04968	4.04968	4.04968
RIDGE	1.74771	1.33466	1.19865	1.74771	1.33466	1.19865	1.74771	1.33466	1.19865
MRT	3.62855	3.58645	3.57717	3.60842	3.57623	3.57105	3.60081	3.57322	3.56986
UMRT	1.4465	1.0776	0.9619	1.2856	0.9477	0.8449	1.2191	0.8956	0.7987

Table 16. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 100$, Sig = 5 and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ=0.85	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	6.7828	6.7828	6.7828	6.7828	6.7828	6.7828	6.7828	6.7828	6.7828
RIDGE	6.2695	5.9981	5.8712	6.2695	5.9981	5.8712	6.2695	5.9981	5.8712
MRT	6.6616	6.6019	6.5750	6.6393	6.5713	6.5412	6.6286	6.5568	6.5254
UMRT	6.0858	5.7340	5.5733	5.9552	5.5511	5.3695	5.8920	5.4641	5.2732
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.95	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	20.222	20.222	20.222	20.222	20.222	20.222	20.222	20.222	20.222
RIDGE	15.908	14.121	13.375	15.908	14.121	13.375	15.908	14.121	13.375
MRT	19.279	18.936	18.799	19.143	18.781	18.641	19.081	18.713	18.572
UMRT	14.669	12.621	11.803	13.863	11.695	10.852	13.494	11.282	10.433
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.99	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	101.24	101.24	101.24	101.24	101.24	101.24	101.24	101.24	101.24
RIDGE	42.35	31.37	27.68	42.35	31.37	27.68	42.35	31.37	27.68
MRT	89.75	87.86	87.23	88.95	87.15	86.57	88.61	86.86	86.30
UMRT	34.38	24.35	21.10	30.05	20.69	17.73	28.24	19.20	16.37

Table 17. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 200$, Sig = 1 and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ=0.85	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.1583	0.1583	0.1583	0.1583	0.1583	0.1583	0.1583	0.1583	0.1583
RIDGE	0.1500	0.1457	0.1437	0.1500	0.1457	0.1437	0.1500	0.1457	0.1437
MRT	0.1564	0.1554	0.1550	0.1560	0.1550	0.1545	0.1558	0.1548	0.1543
UMRT	0.1471	0.1415	0.1390	0.1450	0.1387	0.1359	0.1440	0.1373	0.1344
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.95	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	0.4750	0.4750	0.4750	0.4750	0.4750	0.4750	0.4750	0.4750	0.4750
RIDGE	0.4017	0.3697	0.3561	0.4017	0.3697	0.3561	0.4017	0.3697	0.3561
MRT	0.4601	0.4545	0.4524	0.4579	0.4522	0.4501	0.4569	0.4511	0.4491
UMRT	0.3796	0.3423	0.3274	0.3650	0.3254	0.3100	0.3583	0.3178	0.3024
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
γ = 0.99	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	2.3623	2.3623	2.3623	2.3623	2.3623	2.3623	2.3623	2.3623	2.3623
RIDGE	1.2441	1.0011	0.9184	1.2441	1.0011	0.9184	1.2441	1.0011	0.9184
MRT	2.1774	2.1518	2.1453	2.1656	2.1447	2.1406	2.1610	2.1424	2.1395
UMRT	1.0681	0.8434	0.7702	0.9714	0.7612	0.6943	0.9309	0.7276	0.6637

Table 18. Estimated MSE for OLS, Ridge, MRT and UMRT when $n = 200$, Sig = 5 and $p = 6$.

	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.85$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	3.9583	3.9583	3.9583	3.9583	3.9583	3.9583	3.9583	3.9583	3.9583
RIDGE	3.7453	3.6289	3.5736	3.7453	3.6289	3.5736	3.7453	3.6289	3.5736
MRT	3.9083	3.8822	3.8701	3.8987	3.8684	3.8545	3.8940	3.8618	3.8471
UMRT	3.6668	3.5130	3.4412	3.6102	3.4312	3.3487	3.5827	3.3918	3.3045
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	11.876	11.876	11.876	11.876	11.876	11.876	11.876	11.876	11.876
RIDGE	10.000	9.148	8.778	10.000	9.148	8.778	10.000	9.148	8.778
MRT	11.494	11.339	11.275	11.434	11.266	11.199	11.406	11.233	11.165
UMRT	9.414	8.396	7.972	9.021	7.915	7.466	8.837	7.697	7.239
	$d = 0.4$			$d = 0.7$			$d = 0.85$		
$\gamma = 0.99$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$	$k = 0.5$	$k = 0.8$	$k = 0.95$
OLS	59.056	59.056	59.056	59.056	59.056	59.056	59.056	59.056	59.056
RIDGE	30.589	24.123	21.865	30.589	24.123	21.865	30.589	24.123	21.865
MRT	54.236	53.316	52.992	53.851	52.951	52.635	53.685	52.794	52.484
UMRT	25.926	19.782	17.704	23.317	17.442	15.484	22.210	16.464	14.561

(1993) and Lukman and Ayinde (2017). This is defined as:

$$X_{ij} = (1 - \gamma^2)^{1/2} z_{ij} + \gamma z_{ip} i = 1, 2, ..., n, j = 1, 2, ..., p \tag{39}$$

where z_{ij} is independent standard normal distribution with mean zero and unit variance, γ^2 is the correlation between any two explanatory variables and p is the number of explanatory variables. The values of γ were taken as 0.85, 0.95 and 0.99, respectively. In this study, the number of explanatory variable (p) was taken to be three and six.

The response variable is defined as:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon_i \tag{40}$$

where $\varepsilon_i \sim (0, \sigma^2)$. The values of β were chosen such that $\beta'\beta = 1$ (Newhouse & Oman, 1971). The sample size used are 30 and 50. Two different values of σ: 1 and 5. The experiment is repeated 1000 times. The estimated MSE is calculated as

$$MSE(\hat{\beta}) = \frac{1}{1000} \sum_{j=1}^{1000} (\hat{\beta}_{ij} - \beta_i)'(\hat{\beta}_{ij} - \beta_i) \tag{41}$$

where $\hat{\beta}_{ij}$ denotes the estimate of the ith parameter in jth replication and β_i is the true parameter values. The estimated MSEs of the estimators for different values of n, k, d, σ and γ are shown in Tables 3–18. The following observations were made:

1. The unbiased estimator is superior to OLS in all the cases. OLS estimator has the least performance when there is multicollinearity.
2. Also, the unbiased estimator consistently outperforms the ridge and modified ridge estimators. Even though, ridge and modified ridge estimators dominate OLS in all cases.
3. When the sample size increase, the MSE decreases even when the correlation between the explanatory variables increases.
4. As sample sizes remain constant, increasing the value of σ increases the mean square errors of each of the estimators.
5. As the number of explanatory variables increases, the mean squared error of all the estimators' increases for a given level of multicollinearity and σ.

Generally, we confirm the superiority of the unbiased estimator over other estimators at the different level of multicollinearity and error variance. The performance of the modified-ridge estimator dominates the ridge estimator and OLS.

6. Conclusion

The OLS estimator suffers a breakdown in the presence of multicollinearity. The estimator is unbiased but possesses a significant variance. An alternative estimator called unbiased modified ridge-type estimator with prior information was proposed in this study. This estimator was proved to be unbiased and possess minimum variance theoretically. Also, a simulation study and real-life application were conducted to establish the superiority of this estimator over the existing estimators in terms of the MSEM criterion and crossvalidation prediction error. The performance of this new estimator is better than the OLS estimator and ridge estimator for all degree of multicollinearity. This estimator was able to circumvent the problem of inflated variance that faces the OLS estimator. Finally, this estimator should be adopted as a replacement to the OLS estimator and the biased estimators when there is multicollinearity in a linear model.

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