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$\mathbb{Z}_2 \times \mathbb{Z}_2$-CORDIAL CYCLE-FREE HYPERGRAPHS

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Abstract

Hovey introduced A-cordial labelings as a generalization of cordial and harmonious labelings [7]. If A is an Abelian group, then a labeling $f\colon V(G) \to A$ of the vertices of some graph G induces an edge labeling on G; the edge uv receives the label $f(u)+f(v)$. A graph G is A-cordial if there is a vertex-labeling such that (1) the vertex label classes differ in size by at most one and (2) the induced edge label classes differ in size by at most one.

The problem of A-cordial labelings of graphs can be naturally extended for hypergraphs. It was shown that not every 2-uniform hypertree (i.e., tree) admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$-cordial labeling [8]. The situation changes if we consider p-uniform hypertrees for a bigger p. We prove that a p-uniform hypertree is $\mathbb{Z}_2 \times \mathbb{Z}_2$-cordial for any $p > 2$, and so is every path hypergraph in which all edges have size at least 3. The property is not valid universally in the class of hypergraphs of maximum degree 1, for which we provide a necessary and sufficient condition.

Keywords: V_4-cordial graph, hypergraph, labeling of hypergraph, hypertree.

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1. Introduction

A *hypergraph* H is a pair $H = (V, E)$ where V is a set of vertices and E is a set of non-empty subsets of V called hyperedges. The order (number of vertices) of a hypergraph H is denoted by $|H|$ and the size (number of edges) is denoted by $\|H\|$. If all edges have the same cardinality p, the hypergraph is said to be *p-uniform*. Hence a graph is a 2-uniform hypergraph. The degree of a vertex v, denoted by $d(v)$, is defined as $d(v) = |\{e \in E : v \in e\}|$; i.e., the degree of v is the number of edges to which it belongs. Two vertices in a hypergraph are adjacent if there is an edge containing both of them.

In order to avoid some trivialities, we assume in most of this paper that every edge of a hypergraph has at least two vertices. The only exception will be Section 3.2.

A *walk* in a hypergraph is a sequence $v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$, where $v_i \in V$, $e_i \in E$ and $v_{i-1}, v_i \in e_i$ for all i. We define a *path* in a hypergraph to be a walk with all v_i distinct and all e_i distinct. A *cycle* is a walk containing at least two edges, all e_i are distinct and all v_i are distinct except $v_0 = v_n$. A hypergraph is connected if for every pair of its vertices v, u, there is a path starting at v and ending at u. A *hypertree* is a connected hypergraph with no cycles.

A *star* is a hypertree in which one vertex — called the *center* of the star — is contained in all edges (and the edges are mutually disjoint outside this vertex). Observe that a p-uniform hypertree with $\|T\|$ edges always has exactly $1 + (p-1)\|T\|$ vertices. An even simpler structure is a *matching* — frequently called 'packing' in the literature — in which any two edges are vertex-disjoint. (Here we allow that isolated vertices may also occur.)

For a p-uniform hypergraph $H = (V, E)$, an Abelian group A and an A-labeling $c : V \to A$ let $v_c(a) = |c^{-1}(a)|$. The labeling c is said to be *A-friendly* if $|v_c(a) - v_c(b)| \leq 1$ for any $a, b \in A$. The labeling c induces an edge labeling $c^* : E \to A$ defined by $c^*(e) = \sum_{v \in e} c(v)$. Let $e_{c^*}(a) = |{c^*}^{-1}(a)|$. A hypergraph is said to be *A-cordial* if it admits an A-friendly labeling c such that $|e_{c^*}(a) - e_{c^*}(b)| \leq 1$ for any $a, b \in A$. Then we say that the edge labeling c^* is *A-cordial.*

Cordial labeling of graphs was introduced by Cahit [1] as a weakened version of graceful labeling and harmonious labeling. This notion was generalized by Hovey for any Abelian group of order k [7]. So far research on A-cordiality has mostly focused on the case where A is cyclic and so called *k-cordial.* Hovey [7] showed that all caterpillars are k-cordial for all k and all trees are k-cordial for $k = 3, 4, 5$. Moreover, he showed that cycles are k-cordial for any odd k. He raised the conjectures that if H is a tree graph, it is k-cordial for every k, and that all connected graphs are 3-cordial [7]. In the last twenty-five years there was little progress towards a solution to either of these conjectures. However, Driscoll, Krop and Nguyen proved recently that all trees are 6-cordial [4].

Note that this result does not extend even to the smallest noncyclic group, the Klein four-group (i.e., $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$); the paths P_4 and P_5 are not V_4-cordial what is shown in the following theorem.

Theorem 1 [8]**.** *The path P_n is V_4-cordial unless $n \in \{4, 5\}$.*

In [3] we investigated a problem analogous to Hovey's problem for hypertrees (connected hypergraphs without cycles) and presented various sufficient conditions on H to be k-cordial. From our theorems it follows that every uniform hyperpath is k-cordial for any k, and every k-uniform hypertree is k-cordial. We conjectured that all hypertrees are k-cordial for all k. Recently Tuczyński, Wenus and Węsek proved this conjecture for $k = 2, 3$ [9].

However, a 2-uniform hypertree is not V_4-cordial in general by Theorem 1.

In this paper we show that such counterexamples no longer exist in case of p-uniform hypertrees for $p \geq 3$. Namely, we prove that any p-uniform hypertree is V_4-cordial for all $p \geq 3$. Beyond that, for stars we can even drop the condition of uniformity. We also characterize V_4-cordial hypergraphs whose edges are mutually disjoint (i.e., matchings).

2. Extension Lemma and Uniform Hypertrees

We begin this section with some sufficient conditions under which a V_4-cordial labeling can be derived from that of a subhypergraph. This result will be applied later in several situations, leading to substantial shortening of various arguments. We use it first for uniform hypertrees, proving that all of them are V_4-cordial.

Before we present the results, we introduce a notation for convenience. Let the edge set of the hypergraph under consideration be $E = \{e_1, e_2, \dots, e_m\}$. For all $1 \leq i \leq m$, let us denote $X_i = \bigcup_{1 \leq j \leq i} e_j$. We will assume without loss of generality that the edges are indexed in such a way that e_i meets at most one connected component of the subhypergraph with vertex set X_{i-1} and edge set $\{e_1, \dots, e_{i-1}\}$. In particular, for hypertrees it means that each e_i has exactly one vertex in common with the set X_{i-1}; hence every $\{e_1, e_2, \dots, e_i\}$ forms a hypertree in which e_i is a pendant edge. For hypertrees it can also be assumed that e_m is the last edge in a longest path in T.

Theorem 2 (Extension Lemma)**.** *Let $H = (V, E)$ be a hypergraph with edge set $E = \{e_1, \dots, e_m\}$, and let $e_m^- := e_m \setminus (e_1 \cup \cdots \cup e_{m-1})$. Assume that $|e_m^-| \geq 2$, and that the following conditions hold:*

1. *If $|V| \equiv 0 \pmod 4$, then $m \equiv 1 \pmod 4$.*
2. *If $|V| \equiv 2 \pmod 4$, then $m \not\equiv 0 \pmod 4$.*
3. *If $|V| \equiv 3 \pmod 4$ and $|e_m^-| = 2$, then $m \not\equiv 0 \pmod 4$.*

If the hypergraph H^- obtained from H by omitting e_m from E and deleting the vertices of e_m^- from V is V_4-cordial, then H is V_4-cordial.

Proof. Assume that c' is a V_4-labeling of X_{m-1} that induces a V_4-cordial labeling c'^* of H^-. If $m-1 \equiv 0 \pmod 4$, then every V_4-friendly extension of c' to $X_m = V$ verifies that H is V_4-cordial. Otherwise, if $m \not\equiv 1 \pmod 4$, assumption 1 of the theorem implies $|X_m| \not\equiv 0 \pmod 4$. If $|X_{m-1}| \not\equiv 0 \pmod 4$, we first assign $a := 4 - (|X_{m-1}| \pmod 4)$ vertices of $e_m^- = e_m \setminus X_{m-1}$ to those elements of V_4 which occur on one fewer vertex of X_{m-1} than the other $4-a$ elements. Here $1 \le a \le 3$, and the step is feasible unless $|e_m^-| = 2$ and $a = 3$, because apart from this exception $|e_m^-| \ge a$ holds and there is enough room to have the current partial labeling completely balanced for the elements of V_4.

Suppose first that either $|e_m^-| \ge 3$ or $a \le 2$. Let $b = |e_m^-| - a$ denote the number of vertices unlabeled so far. We next distribute equally the elements of V_4 on $b - (b \pmod 4)$ vertices of e_m. There still remain some r unlabeled vertices in e_m, where $1 \le r \le 3$ since $|X_m| \not\equiv 0 \pmod 4$. We choose $q \in V_4$ such that the current partial sum on e_m plus q occurs fewer times than some other label(s) in c'^* on the edge set $e_1, \dots, e_{m-1}$. By assumption 2 that $|X_m| \equiv 2 \pmod 4$ implies $m \not\equiv 0 \pmod 4$, we can take $q \neq (0,0)$ if $r = 2$. Therefore we can easily select r distinct elements $l_1, \dots, l_r \in V_4$ such that $l_1 + \dots + l_r = q$. Assigning them to the remaining vertices, a V_4-cordial labeling of the entire T is obtained.

Consider now the case $|e_m^-| = 2$ with $a = 3$. Here $a = 3$ means that $|X_{m-1}| \equiv 1 \pmod 4$, and then $|e_m^-| = 2$ yields $|X_m| \equiv 3 \pmod 4$. Hence so far three elements of V_4 are used one fewer than the fourth element, and we have to use two of them on the unlabeled vertices of e_m. Now $m \not\equiv 0 \pmod 4$ by assumption 3, thus at least two sums are feasible on e_m. Consequently, by the pigeonhole principle, one of two feasible sums coincides with one of three sums which can be generated by the sum of labels on the vertices in $X_{m-1} \cap e_m$ together with the pairs of the three usable elements of V_4. ■

Theorem 3. *Let $p \ge 3$. Then every p-uniform hypertree is V_4-cordial.*

Proof. The theorem obviously holds for any hypertree with size one, this case is the anchor of induction. Let T be a p-uniform hypertree with size $m = \|T\| \ge 2$ and assume that the theorem holds for every p-uniform hypertree with size less than m. Let $T' = T - \{e_m\}$ be the p-uniform hypertree with vertex set $V' = X_{m-1}$. By induction there exists a V_4-friendly labeling c' for T' which induces a V_4-cordial labeling c'^*. Below we show that c' can be extended to a V_4-friendly labeling c of T in such a way that c induces a V_4-cordial labeling for T.

Recall that we have $|T| = (p-1)\|T\| + 1$, therefore the residue of $|X_m|$ modulo 4 is obtained according to Table 1. Column $m \equiv 0$ shows that the second and third conditions in Theorem 2 automatically hold, moreover only one of the two occurrences of 0 violates the first condition. Hence, to complete the proof, we may restrict our attention to $p \equiv 2 \pmod 4$ and $m \equiv 3 \pmod 4$, in which case we have $|X_m| \equiv 0 \pmod 4$. We will consider three subcases.

(mod 4)		$m \equiv$	0	1	2	3
$p \equiv$	0		1	0	3	2
	1		1	1	1	1
	2		1	2	3	0
	3		1	3	1	3

Table 1. The value of $|X_m| \pmod 4$.

Case 1. $e_{m-1} \cap e_m \neq \emptyset$. Note that in this situation $T'' = T - \{e_{m-1}, e_m\}$ is a p-uniform hypertree with the vertex set $V'' = X_{m-2}$. By the induction hypothesis there exists a V_4-friendly labeling c'' for T'' which induces a V_4-cordial labeling c''^*. We show that c'' can be extended to a V_4-friendly labeling c of T in such a way that c induces a V_4-cordial labeling for T. Note that in this case there are exactly two elements $x, y \in V_4$ that occur one time fewer in the labeling c'' of the vertices of T'' than the other two elements of V_4; and there is exactly one element $z \in V_4$ that occurs one time more in the labeling of the edges of T'' induced by c'' than the other three elements of V_4. Let $e_{m-1} = \{v, v_1^{m-1}, v_2^{m-1}, \dots, v_{p-1}^{m-1}\}$ and $e_m = \{v, v_1^m, v_2^m, \dots, v_{p-1}^m\}$.

Suppose first that $X_{m-2} \cap e_{m-1} = \{v\}$. If now $z \notin \{x + c''(v), y + c''(v)\}$ then we put the label x on v_1^{m-1} and y on v_1^m, and on the remaining vertices of the edges e_{m-1} and e_m each element of V_4 exactly $(p-2)/4$ times. Obviously we obtain a V_4-cordial labeling of T. If $z \in \{x + c''(v), y + c''(v)\}$, then there exists $\alpha \in V_4$ such that $z \notin \{x + c''(v) + \alpha, y + c''(v) + \alpha\}$. Label the vertices as follows: v_1^{m-1} by x, v_2^{m-1} by α, and v_3^{m-1}, v_4^{m-1}, v_5^{m-1} by the elements $(0,1), (1,0), (1,1)$, whereas v_1^m by y, v_2^m by $(0,0)$, and v_3^m, v_4^m, v_5^m by the elements of $V_4 - \{\alpha\}$; and on the remaining vertices put each element of V_4 exactly $(p-6)/4$ times in each of e_{m-1} and e_m.

Suppose now that $X_{m-2} \cap e_{m-1} \neq \{v\}$, say $X_{m-2} \cap e_{m-1} = \{v_1^{m-1}\}$. We can assume that $x + c''(v_1^{m-1}) \neq z$ because $y \neq x$. Label v_2^{m-1} by x and put on the remaining vertices of the edge e_{m-1} each element of V_4 exactly $(p-2)/4$ times in such a way that $y + c(v) \notin \{z, x + c''(v_1^{m-1})\}$. Now label v_1^m by y and put on the remaining vertices of the edge e_m each element of V_4 exactly $(p-2)/4$ times.

Case 2. $e_{m-1} \cap e_m = \emptyset$. One can easily see (and it also follows from the inductive step described below) that if $m = 3$, then the hypertree (path) T is V_4-cordial. Therefore we can assume that $m \geq 7$. Observe that this time $T'' =$

$T-\{e_{m-2},e_{m-1},e_m\}$ is a p-uniform hypertree with the vertex set $V''=X_{m-3}$. By induction there exists a V_4-friendly labeling c'' for T'' which induces a V_4-cordial labeling c''^*. Note that in this case there are exactly three elements $x,y,z\in V_4$ that occur one time fewer in the labeling c'' of vertices T'' than the other element of V_4, and all the elements of V_4 occur the same times in the labeling of edges of T'' induced by c''. We show that the labeling c'' can be extended to a V_4-friendly labeling c of T in such a way that c induces a V_4-cordial labeling for T.

Assume first that $e_{m-2}\cap e_m\neq\emptyset$ and $e_{m-2}\cap e_{m-1}\neq\emptyset$. Let $v\in X_{m-3}\cap e_{m-2}$, $u\in e_{m-2}\cap e_{m-1}$ and $w\in e_{m-2}\cap e_m$. For the moment we assume that $v\notin\{u,w\}$. Put the label x on the vertex u, and on the remaining vertices of the edge e_{m-2} each element of V_4 exactly $(p-2)/4$ times in such a way that $c(w)=c(u)$. For the edges e_{m-1} and e_m proceed the same way now as in Case 1.

In the other situation, if $X_{m-3}\cap e_{m-2}$ coincides with $e_{m-2}\cap e_{m-1}$, we apply essentially the same strategy, imposing the condition that the vertex w gets the label $c''(v)$.

Next, let $e_{m-2}\cap e_m=\emptyset$ and $e_{m-2}\cap e_{m-1}\neq\emptyset$. This situation can be reduced to Case 1 by a modification of the indexing of the edges, viewing e_{m-1} as the new e_m, also e_{m-2} as the new e_{m-1}, and the old e_m (which is disjoint from both other edges) as the new e_{m-2}. Using the new indices we have $e_{m-1}\cap e_m\neq\emptyset$, which has already been settled. A similar re-indexing works if $e_{m-2}\cap e_{m-1}=\emptyset$ and $e_{m-2}\cap e_m\neq\emptyset$.

Finally, assume that $e_{m-2}\cap e_m=\emptyset$ and $e_{m-2}\cap e_{m-1}=\emptyset$. Then let $e_{m-2}=\big\{v_1^{m-2},v_2^{m-2},\dots,v_p^{m-2}\big\}$, $e_{m-1}=\big\{v_1^{m-1},v_2^{m-1},\dots,v_p^{m-1}\big\}$ and $e_m=\big\{v_1^m,v_2^m,\dots,v_p^m\big\}$ such that $X_{m-3}\cap e_{m-2}=\big\{v_1^{m-2}\big\}$, $X_{m-3}\cap e_{m-1}=\big\{v_1^{m-1}\big\}$ and $X_{m-3}\cap e_m=\big\{v_1^m\big\}$. Suppose first that

$$\big|\big\{c''\big(v_1^{m-2}\big),c''\big(v_1^{m-1}\big),c''\big(v_1^m\big)\big\}\big|<3,$$

then without loss of generality we can assume that $c''\big(v_1^{m-1}\big)=c''(v_1^m)$. Put the label x on the vertex v_p^{m-2} and on the remaining vertices of the edge e_{m-2} each element of V_4 exactly $(p-2)/4$ times. For the edges e_{m-1} and e_m proceed the same way now as in Case 1.

Otherwise, if $\big\{c''\big(v_1^{m-2}\big),c''\big(v_1^{m-1}\big),c''\big(v_1^m\big)\big\}=\{a,b,c\}$ is a set of three distinct labels, let us denote by β the element of $V_4-\{a,b,c\}$. On $p-2$ vertices in each of e_{m-2},e_{m-1},e_m we distribute the elements of V_4 equally, using $(p-2)/4$ times each. The current partial sums on these edges are a,b,c, and we need to assign x,y,z (one of them in each edge) in a way that the sums remain mutually distinct. If $\beta\notin\{x,y,z\}$, then in fact $\{a,b,c\}=\{x,y,z\}$, and we can obviously create the sums $x+y$, $y+z$, and $z+x$, which satisfy the conditions. Else, if say $\beta=x$, we have $\{a,b,c\}=\{a,y,z\}$ where $a\neq x$. We then create two nonzero sums $a+y$ and $y+x$, and the zero sum $z+z$. The corresponding labeling satisfies the conditions and completes the proof of the theorem. ■

3. Stars, Matchings, Paths

In this section we consider hypergraphs also with smaller edges than in the previous sections, because even such extensions allow characterizations for the existence of V_4-cordial labelings in some subclasses. In particular, stars need no restriction, whereas V_4-cordial hypergraphs of maximum degree 1 admit a simple characterization. The case of paths seems to be more complicated to handle, here we only exhibit an infinite family which is not V_4-cordial.

3.1. Stars

Recall that the edge set of a star is a collection of sets of size at least 2 each, which are mutually disjoint apart from a single vertex which is contained in all of them. Hence each edge e_i contains precisely $|e_i| - 1$ private vertices, and with the notation of the Extension Lemma (Theorem 2) we have $|e_m^-| = |e_m| - 1$, no matter which indexing order $e_1, \ldots, e_m$ of the edges we take.

Theorem 4. *Every star is V_4-cordial.*

Proof. Let H be a star with m edges $e_1, \ldots, e_m$. We can associate the m-tuple $(f_1, \ldots, f_m)$ of integers with H, where $f_i = |e_i| - 1$ for all $1 \le i \le m$. It is clear that every m-tuple of positive integers uniquely determines the corresponding star up to isomorphism, moreover $|H| = 1 + \sum_{i=1}^m f_i$. This representation can further be simplified to one which still determines H, namely we can denote by m_k the number of indices i such that $f_i = k$.

The proof will be an induction on $|H|$, anchored by approximately 30 small cases. We are going to introduce several reductions, along which it will turn out which of the small cases are relevant to be checked separately. Below we describe the situations and explain why they are reducible.

(1) If there is a $k \ge 5$ with $m_k > 0$, then it reduces to $m_k := m_k - 1$ and $m_{k-4} := m_{k-4} + 1$.

The reason is that inside an edge with 5 or more non-center vertices we can assign four to the elements of V_4, hence creating a partial sum equal to zero and decreasing $|H|$ by four, still having a star with m edges. Hence it suffices to consider stars represented by 4-tuples (m_1, m_2, m_3, m_4).

(2) If there is a $k \le 4$ with $m_k \ge 4$, then it reduces to $m_k := m_k - 4$.

Assume that $|e_1| = |e_2| = |e_3| = |e_4| = k + 1$. Table 2 shows how the non-center vertices of e_1, e_2, e_3, e_4 can be labeled to induce four distinct edge labels, and hence eliminate those four edges. In this way all remaining stars to be considered are represented by 4-tuples $(m_1, m_2, m_3, m_4) \in \{0, 1, 2, 3\}^4$, that is already a finite collection of basic configurations.

	$k=1$	$k=2$	$k=3$	$k=4$
$e_1=(0,0)$	$(0,0)$	$(0,0)$, $(0,0)$	$(0,0)$, $(0,0)$, $(0,0)$	$(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$
$e_2=(0,1)$	$(0,1)$	$(1,0)$, $(1,1)$	$(0,1)$, $(0,1)$, $(0,1)$	$(0,0)$, $(0,1)$, $(0,1)$, $(0,1)$
$e_3=(1,0)$	$(1,0)$	$(0,1)$, $(1,1)$	$(1,0)$, $(1,0)$, $(1,0)$	$(0,0)$, $(1,0)$, $(1,0)$, $(1,0)$
$e_4=(1,1)$	$(1,1)$	$(0,1)$, $(1,0)$	$(1,1)$, $(1,1)$, $(1,1)$	$(0,0)$, $(1,1)$, $(1,1)$, $(1,1)$

Table 2. Eliminating four edges of equal size. The label of center vertex, when different from $(0,0)$, permutes the edge sums indicated in the first column.

(3) If $f_1+f_2+f_3+f_4 \equiv 0 \pmod 4$, then e_1,e_2,e_3,e_4 can be eliminated. More explicitly, if in each position the 4-tuple (m_1,m_2,m_3,m_4) is at least as large as one or more of

$$(0,1,2,1),\quad (0,2,0,2),\quad (1,0,1,2),\quad (1,2,1,0),\quad (2,0,2,0),\quad (2,1,0,1)$$

then the configuration is reducible.

Indeed, the condition $f_1+f_2+f_3+f_4 \equiv 0 \pmod 4$ actually means that $f_1+f_2+f_3+f_4$ equals 8 or 12, because 4 and 16 would only occur as 4×1 and 4×4, respectively, and these cases have just been settled by (2). Simple enumeration yields that there are six possible 4-tuples (f_1,f_2,f_3,f_4) apart from permutations. Table 3 exhibits an ad hoc labeling from the many possibilities for each of them, showing that all these subconfigurations can be eliminated. There is a direct one-to-one correspondence between the 4-tuples (m_1,m_2,m_3,m_4) and (f_1,f_2,f_3,f_4), for example $(m_1,m_2,m_3,m_4)=(1,0,2,1)$ — the third case listed above — means $f_1=1,\ f_2=3,\ f_3=3,\ f_4=4$.

(f_1,f_2,f_3,f_4)	$e_1=(0,0)$	$e_2=(0,1)$	$e_3=(1,0)$	$e_4=(1,1)$
$(1,1,2,4)$	$(0,0)$	$(0,1)$	$(0,0)$, $(1,0)$	$(0,1)$, $(1,0)$, $(1,1)$, $(1,1)$
$(1,1,3,3)$	$(0,0)$	$(0,1)$	$(0,0)$, $(0,1)$, $(1,1)$	$(1,0)$, $(1,0)$, $(1,1)$
$(1,2,2,3)$	$(0,0)$	$(0,0)$, $(0,1)$	$(0,1)$, $(1,1)$	$(1,0)$, $(1,0)$, $(1,1)$
$(1,3,4,4)$	$(0,0)$	$(0,1)$, $(0,1)$, $(0,1)$	$(0,0)$, $(1,0)$, $(1,0)$, $(1,0)$	$(0,0)$, $(1,1)$, $(1,1)$, $(1,1)$
$(2,2,4,4)$	$(0,0)$, $(0,0)$	$(0,0)$, $(0,1)$	$(0,1)$, $(1,0)$, $(1,0)$, $(1,1)$	$(0,1)$, $(1,0)$, $(1,1)$, $(1,1)$
$(2,3,3,4)$	$(0,0)$, $(0,0)$	$(0,1)$, $(0,1)$, $(0,1)$	$(1,0)$, $(1,0)$, $(1,0)$	$(0,0)$, $(1,1)$, $(1,1)$, $(1,1)$

Table 3. Eliminating four edges whose total number of non-center vertices is 8 or 12.

Since the theorem claims V_4-cordiality of stars without any exceptions, all the situations described above provide an inductive step when they occur as subconfigurations. It follows that, for an anchor of the induction, a V_4-cordial labeling has to be presented for only those stars which are not reducible by any of (1)–(3). There are 79 such cases, as listed in Table 4. Below we show how they can be handled.

O — Obvious cases are the stars with just one edge ($m_1 + m_2 + m_3 + m_4 = 1$, the V_4-cordial labelings are precisely the V_4-friendly ones) and the star graphs ($m_2 = m_3 = m_4 = 0$, a labeling is V_4-cordial if and only if it is V_4-friendly on the set of leaves and also on the entire vertex set). There are 6 such cases.

T — Trivial reduction applies for stars with 5 edges ($m \equiv 1 \pmod 4$, hence the last edge admits any V_4-friendly extension from a V_4-cordial labeling for the first $m-1$ edges); and also for stars of order 5 or 9 or 13 ($n \equiv 1 \pmod 4$, hence the last vertex can get an arbitrary label needed for a V_4-cordial extension from $m-1$ edges to m edges). This reduction settles 24 cases.

F — Four vertices can be eliminated if $m_4 \geq 1$ and $m_2 + m_3 + m_4 \geq 2$ (here extension goes from $n-4$ to n, while m remains unchanged). Indeed, inside a 5-element edge we can label three non-center vertices with $(0,1), (1,0), (1,1)$ while assigning the label $(0,0)$ to a vertex in another edge of size at least 3. This reduction settles further 24 cases.

R — Reduction applies by Theorem 2 for stars with $n \equiv 2 \pmod 4$ unless $m \equiv 0 \pmod 4$; and also with $n \equiv 3 \pmod 4$ except when $m \equiv 0 \pmod 4$ and the star contains no edges of 4 or 5 vertices (i.e., $m_3 = m_4 = 0$). This reduction settles further 13 cases.

***** — There are 12 cases not covered by the previous considerations; Table 5 exhibits a V_4-cordial labeling for each of them. Although there are several cases, all are very easy to construct.

Together with this last set of labelings *****, all cases are exhausted and the theorem is proved. ■

3.2. Matchings

Recall that a matching (also called packing) in a hypergraph is a collection of mutually disjoint edges. We now consider hypergraphs whose entire edge set is a matching. Contrary to the previous parts of the paper, in this particular section we allow singleton edges (edges consisting of just one vertex), and either exclude or allow isolated vertices. Let us denote by $\mathcal{M}$ the class of hypergraphs with maximum degree 1, i.e., hypergraphs whose edge set is a matching, possibly together with one or more vertices of degree 0. More restrictively let $\mathcal{M}_0 \subset \mathcal{M}$ denote the subclass consisting of the 1-regular hypergraphs, the subscript

$(0,0,0,1)$	$1,5$	**O**	$(0,2,1,0)$	$3,8$	*	$(1,1,1,1)$	$4,11$	**F**
$(0,0,0,2)$	$2,9$	**T**	$(0,2,1,1)$	$4,12$	**F**	$(1,1,2,0)$	$4,10$	*
$(0,0,0,3)$	$3,13$	**T**	$(0,2,2,0)$	$4,11$	**R**	$(1,1,3,0)$	$5,13$	**T**
$(0,0,1,0)$	$1,4$	**O**	$(0,2,3,0)$	$5,14$	**T**	$(1,2,0,0)$	$3,6$	**R**
$(0,0,1,1)$	$2,8$	**F**	$(0,3,0,0)$	$3,7$	**R**	$(1,2,0,1)$	$4,10$	**F**
$(0,0,1,2)$	$3,12$	**F**	$(0,3,0,1)$	$4,11$	**F**	$(1,3,0,0)$	$4,8$	*
$(0,0,1,3)$	$4,16$	**F**	$(0,3,1,0)$	$4,10$	*	$(1,3,0,1)$	$5,12$	**T**
$(0,0,2,0)$	$2,7$	**R**	$(0,3,1,1)$	$5,14$	**T**	$(2,0,0,1)$	$3,7$	**R**
$(0,0,2,1)$	$3,11$	**F**	$(0,3,2,0)$	$5,13$	**T**	$(2,0,0,2)$	$4,11$	**F**
$(0,0,2,2)$	$4,15$	**F**	$(0,3,3,0)$	$6,16$	*	$(2,0,0,3)$	$5,15$	**T**
$(0,0,2,3)$	$5,19$	**T**	$(1,0,0,0)$	$1,2$	**O**	$(2,0,1,0)$	$3,6$	**R**
$(0,0,3,0)$	$3,10$	**R**	$(1,0,0,1)$	$2,6$	**R**	$(2,0,1,1)$	$4,10$	**F**
$(0,0,3,1)$	$4,14$	**F**	$(1,0,0,2)$	$3,10$	**F**	$(2,1,0,0)$	$3,5$	**T**
$(0,0,3,2)$	$5,18$	**T**	$(1,0,0,3)$	$4,14$	**F**	$(2,1,1,0)$	$4,8$	*
$(0,0,3,3)$	$6,22$	**F**	$(1,0,1,0)$	$2,5$	**T**	$(2,2,0,0)$	$4,7$	*
$(0,1,0,0)$	$1,3$	**O**	$(1,0,1,1)$	$3,9$	**T**	$(2,3,0,0)$	$5,9$	**T**
$(0,1,0,1)$	$2,7$	**F**	$(1,0,2,0)$	$3,8$	*	$(3,0,0,0)$	$3,4$	**O**
$(0,1,0,2)$	$3,11$	**F**	$(1,0,2,1)$	$4,12$	**F**	$(3,0,0,1)$	$4,8$	*
$(0,1,0,3)$	$4,15$	**F**	$(1,0,3,0)$	$4,11$	**R**	$(3,0,0,2)$	$5,12$	**T**
$(0,1,1,0)$	$2,6$	**R**	$(2,0,0,0)$	$2,3$	**O**	$(3,0,0,3)$	$6,16$	**F**
$(0,1,1,1)$	$3,10$	**F**	$(1,0,3,1)$	$5,15$	**T**	$(3,0,1,0)$	$4,7$	**R**
$(0,1,1,2)$	$4,14$	**F**	$(1,1,0,0)$	$2,4$	*	$(3,0,1,1)$	$5,11$	**T**
$(0,1,1,3)$	$5,18$	**T**	$(1,1,0,1)$	$3,8$	**F**	$(3,1,0,0)$	$4,6$	*
$(0,1,2,0)$	$3,9$	**T**	$(1,1,0,2)$	$4,12$	**F**	$(3,1,1,0)$	$5,9$	**T**
$(0,1,3,0)$	$4,12$	*	$(1,1,0,3)$	$5,16$	**T**	$(3,2,0,0)$	$5,8$	**T**
$(0,2,0,0)$	$2,5$	**T**	$(1,1,1,0)$	$3,7$	**R**	$(3,3,0,0)$	$6,10$	**R**
$(0,2,0,1)$	$3,9$	**T**						

Table 4. The 79 cases of (m_1, m_2, m_3, m_4) which are not excluded by (1)–(3), the corresponding pairs m, n (number of edges $m = m_1 + m_2 + m_3 + m_4$, number of vertices $n = f_1 + f_2 + f_3 + f_4 + 1$), and a way how they can be settled. The 12 cases marked with * need labelings to be constructed separately.

indicating that the number of 0-degree vertices is zero.

Despite the fact that the removal of the center from a star does not change the relative value of edge sums — equal edge sums remain equal, distinct ones remain distinct — this operation is not invariant with respect to V_4-cordiality. This fact, supported by an infinite family of examples, is expressed in the following proposition as opposed to Theorem 4.

(m_1, m_2, m_3, m_4)	m, n	$f_i = 1$	$f_i = 2$	$f_i = 3$	$f_i = 4$
$(0,1,3,0)$	$4,12$		$(0,0)$, $(0,0)$	$(0,1)$, $(0,1)$, $(0,1)$ $(1,0)$, $(1,0)$, $(1,0)$ $(1,1)$, $(1,1)$, $(1,1)$	
$(0,2,1,0)$	$3,8$		$(0,0)$, $(0,1)$ $(0,0)$, $(1,0)$	$(0,1)$, $(1,0)$, $(1,1)$	
$(0,3,1,0)$	$4,10$		$(0,1)$, $(1,0)$ $(0,1)$, $(1,1)$ $(1,0)$, $(1,1)$	$(0,0)$, $(0,0)$, $(0,0)$	
$(0,3,3,0)$	$6,16$		$(0,0)$, $(0,0)$ $(0,0)$, $(0,0)$ $(0,1)$, $(1,0)$	$(0,1)$, $(0,1)$, $(0,1)$ $(1,0)$, $(1,0)$, $(1,0)$ $(1,1)$, $(1,1)$, $(1,1)$	
$(1,0,2,0)$	$3,8$	$(0,1)$		$(0,0)$, $(0,1)$, $(1,0)$ $(1,0)$, $(1,1)$, $(1,1)$	
$(1,1,0,0)$	$2,4$	$(0,0)$	$(0,1)$, $(1,0)$		
$(1,1,2,0)$	$4,10$	$(0,0)$	$(0,1)$, $(1,0)$	$(0,0)$, $(0,0)$, $(0,1)$ $(1,0)$, $(1,1)$, $(1,1)$	
$(1,3,0,0)$	$4,8$	$(0,0)$	$(0,1)$, $(1,0)$ $(0,1)$, $(1,1)$ $(1,0)$, $(1,1)$		
$(2,1,1,0)$	$4,8$	$(0,1)$ $(1,0)$	$(0,0)$, $(1,1)$	$(0,1)$, $(1,0)$, $(1,1)$	
$(2,2,0,0)$	$4,7$	$(0,1)$ $(1,0)$	$(0,1)$, $(1,0)$ $(1,1)$, $(1,1)$		
$(3,0,0,1)$	$4,8$	$(0,1)$ $(1,0)$ $(1,1)$			$(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$
$(3,1,0,0)$	$4,6$	$(0,0)$ $(0,1)$ $(1,0)$	$(0,0)$, $(1,1)$		

Table 5. Labeling for the 12 small cases which remain after the reductions **O**, **T**, **F**, and **R**. If $n \equiv 0 \pmod 4$, then the center gets the unique label occurring fewer in the list than the other elements of V_4, and if $n \equiv 2 \pmod 4$, then it has three options for its label. In $(2,2,0,0)$ the center vertex gets the label $(0,0)$; this is an exceptional case where only three labels can be used on the non-centers and the fourth element of V_4 can occur only on the center (cf. Proposition 5).

Proposition 5. If $H \in \mathcal{M}_0$ is a hypergraph consisting of mutually disjoint edges, such that both $|H|$ and $\|H\|$ are even, moreover $|H| \not\equiv \|H\| \pmod 4$, then H is not V_4-cordial.

Proof. Let $E(H) = \{e_1, \ldots, e_m\}$ and $V(H) = \{v_1, \ldots, v_n\} = e_1 \cup \cdots \cup e_m$, where the edges $e_1, \ldots, e_m$ are mutually disjoint. Consider any vertex labeling $c : V(H) \to V_4$ and its induced edge labeling $c^* : E(H) \to V_4$.

Assume that the labeling is V_4-cordial, i.e., c is V_4-friendly on $V(H)$ and the induced edge labeling $c^* \colon E(H) \to V_4$ fulfils $|e_{c^*}(a) - e_{c^*}(b)| \le 1$ for any $a, b \in V_4$. Since each vertex belongs to precisely one edge, the sum S of all labels satisfies

$$S = \sum_{i=1}^{n} c(v_i) = \sum_{j=1}^{m} c^*(e_j).$$

Now the conditions on $|H|$ and $\|H\|$ imply that precisely one of the order and size is a multiple of 4, the other is congruent to $2 \pmod 4$. For the multiple of 4, every element of V_4 occurs the same number of times as a vertex label or as an edge label, thus

$$S = (0, 0).$$

On the other hand, in the "$2 \pmod 4$" set precisely two elements of V_4 occur one fewer times than the other two elements. Since the overall sum of labels should also be $S = (0, 0)$, it follows that the sum of two distinct $a, b \in V_4$ should be zero, which is impossible. ■

It turns out that this proposition characterizes the exceptions, apart from which all matchings are V_4-cordial.

Theorem 6. *Let H be a matching, where* 1*-element edges are also allowed.*

(i) *If $H \in \mathcal{M}_0$, then H is V_4-cordial if and only if H does not satisfy the conditions of Proposition* 5; *i.e., if either at least one of $|H|$ and $\|H\|$ is odd, or both are even and $|H| \equiv \|H\| \pmod 4$.*

(ii) *If $H \in \mathcal{M} \setminus \mathcal{M}_0$, then H is V_4-cordial.*

Proof. Let $H = (V, E)$, with n vertices and m edges, say $E = \{e_1, \ldots, e_m\}$. The argument mostly applies the ideas of the proof of Theorem 4, keeping in mind that now $e_m^- = e_m$ holds in any indexing order of the edges. If $H \in \mathcal{M}_0$, then H can be extended to a star H^+ by inserting a center vertex, say x ($x \notin V$), and enlarging each edge e_i to $e_i^+ := e_i \cup \{x\}$. We already know that H^+ has a V_4-cordial labeling c^+. If H itself is not V_4-cordial, then it must be the case that the label of the center occurs one fewer than the most frequent vertex label; otherwise we would simply forget about the center and its label. We are going to prove that this situation can be avoided, unless the conditions of Proposition 5 hold.

In the same way as in the proof of Theorem 4, one can verify that the following reductions are feasible inside the class $\mathcal{M}$. For easier comparison we keep the sequence of properties in the same order.

1. If $|e_i| \geq 5$ for some $1 \leq i \leq m$, then we can reduce n to $n-4$ by assigning each element of V_4 to one vertex of e_i, while the status of the conditions with respect to $|H|$ and $\|H\|$ remain unchanged. This eliminates all edges larger than 4.
2. If $|e_1| = |e_2| = |e_3| = |e_4|$, then we can apply the labeling scheme given in Table 2 inside these four edges. Then n decreases by a multiple of 4, and m decreases by exactly 4. Hence again the conditions with respect to $|H|$ and $\|H\|$ remain unchanged.
3. If $|e_1| + |e_2| + |e_3| + |e_4|$ equals 8 or 12, then we can apply the labeling scheme given in Table 3 inside these four edges. More explicitly, this step is applicable whenever the edges can be indexed in such a way that the sequence $(|e_1|, |e_2|, |e_3|, |e_4|)$ is one of $(1,1,2,4)$, $(1,1,3,3)$, $(1,2,2,3)$, $(1,3,4,4)$, $(2,2,4,4)$, $(2,3,3,4)$. Then again n decreases by a multiple of 4, and m decreases by exactly 4. Hence the conditions with respect to $|H|$ and $\|H\|$ remain unchanged.
4. If all edges are singletons, or if H has only one edge, an obvious labeling verifies that H is V_4-cordial. Note that in these cases the conditions of Proposition 5 do not hold because here we have either $|H| = \|H\|$ or $|H| = 1$.
5. If $|e_1| = 4$ and $|e_2| > 1$, then $(0,0)$ can be assigned to a vertex of e_2, and the other three elements of V_4 to vertices of e_1; in this way zero partial sums are inserted in both edges and n is reduced to $n-4$, while m is kept unchanged. Since n and m do not change modulo 4, the status of the conditions on $|H|$ and $\|H\|$ remains the same.

Steps 1–3 of this list are analogous to (1)–(3) in the proof of Theorem 4, while the parts 4 and 5 correspond to the reductions **O** and **F**, respectively.

Hence only some of those 49 cases remain to be considered which are marked with **T** or **R** or * in Table 4. For the case of matchings they are summarized in Table 6. Among them there are 14 further ones which are reducible by step 5; we indicate them with $\mathbf{F'}$. This leaves 35 cases, among which there are 6 satisfying the congruence conditions of Proposition 5 and hence we know that they are not V_4-cordial. These are marked with $\times$.

Note that in the current situation we have $n = f_1 + f_2 + f_3 + f_4$, without the $+1$ term; this is the reason why the pairs m, n differ by 1 when compared in Tables 4 and 6. Now a natural analogue of **T** is the following reduction, which necessarily is slightly more restrictive.

$(0,0,0,2)$	$2,8$	$\mathbf{F'}$	$(0,3,1,0)$	$4,9$	$\mathbf{T'}$	$(1,1,2,0)$	$4,9$	$\mathbf{T'}$
$(0,0,0,3)$	$3,12$	$\mathbf{F'}$	$(0,3,1,1)$	$5,13$	$\mathbf{F'}$	$(1,1,3,0)$	$5,12$	$\mathbf{T'}$
$(0,0,2,0)$	$2,6$	$\mathbf{R'}$	$(0,3,2,0)$	$5,12$	$\mathbf{T'}$	$(1,2,0,0)$	$3,5$	$\mathbf{T'}$
$(0,0,2,3)$	$5,18$	$\mathbf{F'}$	$(1,0,0,1)$	$2,5$	$\mathbf{T'}$	$(1,3,0,0)$	$4,7$	**
$(0,0,3,0)$	$3,9$	$\mathbf{T'}$	$(1,0,1,0)$	$2,4$	$\times$	$(1,3,0,1)$	$5,11$	$\mathbf{F'}$
$(0,0,3,2)$	$5,17$	$\mathbf{F'}$	$(1,0,1,1)$	$3,8$	$\mathbf{F'}$	$(2,0,0,3)$	$5,14$	$\mathbf{F'}$
$(0,1,1,0)$	$2,5$	$\mathbf{T'}$	$(1,0,2,0)$	$3,7$	**	$(2,0,1,0)$	$3,5$	$\mathbf{T'}$
$(0,1,1,3)$	$5,17$	$\mathbf{F'}$	$(2,1,1,0)$	$4,7$	$\mathbf{R'}$	$(2,2,0,0)$	$4,6$	$\times$
$(0,1,2,0)$	$3,8$	**	$(1,0,3,0)$	$4,10$	$\times$	$(2,3,0,0)$	$5,8$	$\mathbf{T'}$
$(0,1,3,0)$	$4,11$	$\mathbf{R'}$	$(2,1,0,0)$	$3,4$	**	$(3,0,0,1)$	$4,7$	$\mathbf{R'}$
$(0,2,0,0)$	$2,4$	$\times$	$(3,2,0,0)$	$5,7$	$\mathbf{T'}$	$(3,0,0,2)$	$5,11$	$\mathbf{F'}$
$(0,2,0,1)$	$3,8$	$\mathbf{F'}$	$(2,0,0,1)$	$3,6$	$\mathbf{R'}$	$(3,0,1,0)$	$4,6$	$\times$
$(0,2,1,0)$	$3,7$	$\mathbf{R'}$	$(1,0,3,1)$	$5,14$	$\mathbf{F'}$	$(3,0,1,1)$	$5,10$	$\mathbf{F'}$
$(0,2,2,0)$	$4,10$	$\times$	$(1,1,0,0)$	$2,3$	$\mathbf{R'}$	$(3,1,0,0)$	$4,5$	$\mathbf{T'}$
$(0,2,3,0)$	$5,13$	$\mathbf{T'}$	$(1,1,0,3)$	$5,15$	$\mathbf{F'}$	$(3,1,1,0)$	$5,8$	$\mathbf{T'}$
$(0,3,0,0)$	$3,6$	**	$(1,1,1,0)$	$3,6$	$\mathbf{R'}$	$(3,3,0,0)$	$6,9$	$\mathbf{T'}$
$(0,3,3,0)$	$6,15$	$\mathbf{R'}$						

Table 6. The 4-tuples (m_1, m_2, m_3, m_4) not eliminated by steps 1–6, the pairs m, n, and a way how they can be settled.

$\mathbf{T'}$ — Trivial reduction applies if we have $n \equiv 1 \pmod 4$ or $m \equiv 1 \pmod 4$ or both, and H contains an edge whose deletion (also deleting its vertices) does not lead to a case marked with $\times$.

The reason is that the last vertex can get any label when we have a completely balanced labeling on $n-1$ vertices, hence the needed label on the last edge can surely be generated; or, the last edge can get any label, hence any V_4-friendly extension of a V_4-cordial labeling of the hypergraph with $m-1$ edges will do the job. This operation settles 15 further cases.

As a further simplification, Theorem 2 leads to the following reduction.

$\mathbf{R'}$ — If there is a non-singleton edge e_i such that $H - e_i$ is a matching not marked with $\times$, then the following conditions are sufficient for reduction: $n \equiv 2 \pmod 4$ unless $m \equiv 0 \pmod 4$, or $n \equiv 3 \pmod 4$ unless $|e_i| = 2$ and $m \equiv 0 \pmod 4$.

This eliminates 9 further cases.

** — There are 5 cases not covered by the previous considerations; Table 7 exhibits a V_4-cordial labeling for each of them.

This completes the proof of the theorem. ■

(m_1, m_2, m_3, m_4)	m, n	$f_i = 1$	$f_i = 2$	$f_i = 3$
$(0,1,2,0)$	$3,8$		$(0,0)$, $(1,1)$	$(0,0)$, $(0,1)$, $(1,1)$ $(0,1)$, $(1,0)$, $(1,0)$
$(0,3,0,0)$	$3,6$		$(0,0)$, $(0,0)$ $(0,1)$, $(1,1)$ $(1,0)$, $(1,1)$	
$(1,0,2,0)$	$3,7$	$(0,1)$		$(0,0)$, $(0,1)$, $(1,0)$ $(1,0)$, $(1,1)$, $(1,1)$
$(1,3,0,0)$	$4,7$	$(0,0)$	$(0,1)$, $(1,0)$ $(0,1)$, $(1,1)$ $(1,0)$, $(1,1)$	
$(2,1,0,0)$	$3,4$	$(0,1)$ $(1,0)$	$(0,0)$, $(1,1)$	

Table 7. Labeling for the 5 final cases of matchings. (Edges of size 4 do not occur.)

3.3. Paths

Inside the class of path hypergraphs we define a *hyperpath* as a path in which all edges have size at least 3. The main result of this section is that every hyperpath is V_4-cordial. Before proving this, we exhibit an infinite family of paths which are not V_4-cordial, hence showing that edges of size 2 create more problems than the sporadic examples P_4 and P_5 themselves. The complete characterization of V_4-cordial paths remains open.

Proposition 7. If H is a path with three edges e_1, e_2, e_3, such that e_2 is the middle edge having size $|e_2| = 2$, moreover $|H| \equiv 0 \pmod 4$, then H is not V_4-cordial.

Proof. Let $V(H) = \{v_1, \ldots, v_n\}$, and consider any V_4-friendly vertex labeling $c : V(H) \to V_4$ with the corresponding induced edge labeling $c^* : E(H) \to V_4$. Since $e_1 \cup e_3 = V(H)$ and $|H|$ is a multiple of 4, we now have

$$c^*(e_1) + c^*(e_3) = \sum_{i=1}^{n} c(v_i) = (0,0).$$

This implies $c^*(e_1) = c^*(e_3)$, hence the labeling cannot be V_4-cordial. ■

Theorem 8. *Every hyperpath is V_4-cordial.*

Proof. Consider a hyperpath $H = (V, E)$, with $E = \{e_1, \ldots, e_m\}$. We apply induction on the number m of edges, from $m - 4$ to m. The base of induction will be $m = 1, 2, 3$; and a special interpretation will be given to the case $m = 0$ to make it possible that the inductive step works for $m = 4$, hence avoiding the need to verify the assertion separately for the many different paths with four edges.

Inside this proof, we simplify the notation to denote the three elements of $V_4 \setminus \{(0,0)\}$ by a, b, c and write 0 for $(0,0)$.

Case $m = 1$. Every V_4-friendly labeling is V_4-cordial.

Case $m = 2$. Sequentially creating a V_4-friendly labeling, for the last vertex we still have at least two choices — which ensure that the sums on e_1 and e_2 can be made different — unless $n \equiv 0 \pmod 4$. In this exceptional case, however, the sum over the vertex set is equal to $0 \in V_4$. Then we assign a nonzero element b to the vertex $e_1 \cap e_2$; this guarantees that the two sums differ, because the sum over e_1 plus the sum over e_2 is equal to b.

Case $m = 3$. Let us start with the periodic labeling $0, a, b, c, 0, a, b, c, \ldots$ along the vertices of the path, and see whether the sums s_1, s_2, s_3 on e_1, e_2, e_3 are distinct or not. If some equalities occur, we eliminate them in two steps as follows.

First, to eliminate $s_1 = s_3$ if it occurs, we switch the label between vertex $e_1 \cap e_2$ and its successor (which is only in e_2, not in $e_1 \cup e_3$, because $|e_2| \geq 3$). This keeps s_2 (and also s_3) unchanged, but modifies the sum over e_1 to a new updated value of s_1, which is then different from s_3.

Second, to maintain $s_1 \neq s_3$ and eliminate $s_1 = s_2$ or $s_2 = s_3$ if it holds after the first step, we switch the label between vertex $e_2 \cap e_3$ and one of its next two successors. (Recall that $|e_3| \geq 3$ holds, hence $|e_3^-| \geq 2$.) These are two possibilities, each keeping s_3 (and also s_1) unchanged, but offering two new values for an updated s_2. At least one of the two will be different from both s_1 and s_3, hence satisfying the requirement. (After any of the two switches the original equality $s_1 = s_2$ or $s_2 = s_3$ automatically disappears, we only have to ensure that a new equality with the other end will not arise.)

Inductive step from $m - 4$ to m. Instead of dealing with the last four edges, we omit the first two and last two edges from the hyperpath $e_1, \ldots, e_m$. Hence let H' be the hyperpath with vertex set $X' = \bigcup_{j=3}^{m-2} e_j$ and edge set $E' = \{e_j \mid 3 \leq j \leq m-2\}$, with $|X'| = n'$ and $|E'| = m' = m - 4$. By the induction hypothesis there exists a V_4-cordial labeling (c', c'^*) on (X', E'). Our goal is to assign $n - n'$ labels to the vertices of $V \setminus X'$ and generate four distinct sums on e_1, e_2, e_{m-1}, e_m. The $n - n'$ labels have to be selected from a multiset S' of $4 \cdot \lceil n/4 \rceil - n'$ elements over V_4; namely, starting with $\lceil n/4 \rceil$ copies of V_4 we delete the elements which have been assigned to X', and from the remaining multiset we need to select $n - n'$ labels properly. Note that the multiplicities of any two elements in S' differ by at most 1, because c' is V_4-friendly by assumption, hence what remains after omitting its labels from $\lceil n/4 \rceil$ times V_4 is also balanced in the sense of tolerating difference at most 1. We can assume without loss of generality that $n \equiv 0 \pmod 4$, because any other case would give us some flexibility in selecting the set of labels, whereas in this case the multiset of labels to be used is determined.

Assume that the vertices in $e_2 \cap X'$ and in $e_{m-1} \cap X'$ are labeled with x and y, respectively, and that the sum of all labels over X' is z. (Some or all of x, y, z may coincide.) Then the label x' of the vertex in $e_1 \cap e_2$ and y' of the vertex in $e_{m-1} \cap e_m$ should satisfy

$$x' + y' = x + y + z. \tag{1}$$

Indeed, since $n \equiv 0 \pmod 4$, the sum z of labels inside X' is equal to the sum outside X', moreover — as said above — the intention is to achieve that the four sums on the edges e_i $(i = 1, 2, m-1, m)$ are all distinct. If this holds, then those four sums on e_1, e_2, e_{m-1}, e_m sum up to $0 \in V_4$, what implies $z + x + y + x' + y' = \sum_{v \in V(H) \setminus X'} c'(v) + x + y + x' + y' = 0$.

We proceed in three steps, after which a V_4-cordial labeling will be obtained.

Step 1. Determine x', y'.

We choose x' and y' in such a way that one of them is an element which is most frequent in S', moreover the remaining multiset $S' \setminus \{x', y'\}$ still contains at least one occurrence of 0. We argue that this can always be done. Indeed, the condition on edge sizes implies $|S'| \geq 8$. Assume first that equality $|S'| = 8$ holds; then each element occurs precisely twice in S'. If equation (1) requires $x' = y'$ (that is, if $x + y + z = 0$), then we can use any of the three labels different from 0 for x'. On the other hand if $x' \neq y'$, the required sum $x' + y'$ can be formed in two ways, each of them leaving two elements of V_4 with multiplicity 2 and two with 1 in $S' \setminus \{x', y'\}$, hence either choice is feasible. Finally if $|S'| > 8$, the most frequent element occurs at least three times. We choose it for x', and assign $x + y + z - x'$ for y'. This is feasible because all elements have multiplicity at least 2 in $S' \setminus \{x'\}$.

Step 2. Distribute all but 6 labels from $S' \setminus \{x', y'\}$.

If $|S'| = 8$, there is nothing to do in this step, the remaining multiset is

$$0, 0, a, a, b, b \quad \text{or} \quad 0, a, a, b, b, c \quad \text{or} \quad 0, 0, a, a, b, c$$

whose sum is

$$0 \quad \text{or} \quad c \quad \text{or} \quad a,$$

respectively. If $|S'| > 8$, we distribute $|S'| - 8$ elements from $S' \setminus \{x', y'\}$ almost arbitrarily, but in such a way that the following conditions are met:

- either $0, a, a, b, b, c$ or $0, 0, a, a, b, c$ remains;
- e_2 and e_{m-1} have just one unlabeled vertex;
- each of e_1 and e_m has two unlabeled vertices.

After this, let us denote the current sums of labels in e_1, e_2, e_{m-1}, e_m by s_1, s_2, s_{m-1}, s_m, respectively. From these four partial sums we shall have to create four distinct final sums by properly distributing the remaining six labels. From this point of view (s_1, s_2, s_{m-1}, s_m) and $(a+s_1, a+s_2, a+s_{m-1}, a+s_m)$ are equivalent. For this reason we may assume without loss of generality that 0 is most frequent among s_1, s_2, s_{m-1}, s_m. Hence, apart from the permutation of subscripts, only the following five types of 4-tuples are relevant for (s_1, s_2, s_{m-1}, s_m).

1. $(0, 0, 0, 0)$, sum $= 0$;
2. $(0, 0, 0, a')$, sum $= a' \neq 0$;
3. $(0, 0, a', a')$, sum $= 0$;
4. $(0, 0, a', b')$, sum $= a' + b' \neq 0$;
5. $(0, a', b', c')$, sum $= a' + b' + c' = 0$.

Here we use prime notation to mean that a', b' may be other than a, b in the remaining 6-element multiset; but different primed letters mean different elements. Observe, however, that $s_1 + s_2 + s_{m-1} + s_m$ must be equal to the sum of the six elements in the multiset, because the total sum over the four edges will eventually be zero; this is implied by the choice of x' and y'. This fact yields, in particular, that not all 4-tuples fit together with all 6-tuples. Namely, $0, 0, a, a, b, b$ is compatible with the cases $1, 3, 5$, while $0, a, a, b, b, c$ and $0, 0, a, a, b, c$ admit the cases $2, 4$.

Step 3. Complete the labeling on e_1, e_2, e_{m-1}, e_m.

This step is a little time consuming, but easy. The selection rules described above already imply that if three edges have mutually distinct final sums, then the fourth edge has the missing value for its sum. To achieve this, we systematically enumerate the 4-tuples listed in 1–5 above with their compatible 6-tuples of labels, and — up to symmetry — the possible positions of $0, a', b'$ and the elements that can play the role of 0, a', and b'. Tables 8 and 9 exhibit a suitable way of extending c' to a V_4-cordial labeling of the entire path.

Case $m = 4$. Let us artificially introduce the 0-path as a single vertex with no edges. It is of course V_4-cordial, any label x can be assigned to the vertex. Now, for $m = 4$ we identify the vertex with $e_2 \cap e_3$, and apply the inductive step above as described for the case $x = y$. This completes the proof of the theorem. ■

(s_1, s_2, s_{m-1}, s_m)	6-tuple	$a' =$	e_1	e_2	e_{m-1}	e_m
$(0,0,0,0)$	$0,0,a,a,b,b$	$-$	$0,a \to a$	$0 \to 0$	$b \to b$	$a,b \to c$
$(0,0,0,a')$	$0,a,a,b,b,c$	c	$a,b \to c$	$0 \to 0$	$b \to b$	$a,c \to a$
	$0,0,a,a,b,c$	a	$0,a \to a$	$0 \to 0$	$b \to b$	$a,c \to c$
$(0,0,a',0)$	$0,a,a,b,b,c$	c	$a,b \to c$	$0 \to 0$	$b \to a$	$a,c \to b$
	$0,0,a,a,b,c$	a	$0,a \to a$	$0 \to 0$	$b \to c$	$a,c \to b$
$(0,0,a',a')$	$0,0,a,a,b,b$	a	$0,a \to a$	$0 \to 0$	$b \to c$	$a,b \to b$
		c	$0,b \to b$	$0 \to 0$	$b \to a$	$a,a \to c$
$(0,a',0,a')$		a	$a,b \to c$	$0 \to a$	$b \to b$	$0,a \to 0$
		c	$0,a \to a$	$0 \to c$	$b \to b$	$a,b \to 0$
$(0,a',a',0)$		a	$0,b \to b$	$0 \to a$	$b \to c$	$a,a \to 0$
		c	$0,b \to b$	$0 \to c$	$b \to a$	$a,a \to 0$
$(a',0,0,a')$		a	$0,b \to c$	$0 \to 0$	$b \to b$	$a,a \to a$
		c	$0,b \to a$	$0 \to 0$	$b \to b$	$a,a \to c$

Table 8. Labels inserted into e_1, e_2, e_{m-1}, e_m starting from at most two distinct sums, and the final sum of labels inside e_i.

(s_1, s_2, s_{m-1}, s_m)	6-tuple	$a', b' =$	e_1	e_2	e_{m-1}	e_m
$(0,0,a',b')$	$0,a,a,b,b,c$	a,b	$a,a \to 0$	$b \to b$	$b \to c$	$0,c \to a$
	$0,0,a,a,b,c$	b,c	$a,a \to 0$	$b \to b$	$c \to a$	$0,0 \to c$
$(0,a',0,b')$	$0,a,a,b,b,c$	a,b	$a,a \to 0$	$b \to c$	$b \to b$	$0,c \to a$
	$0,0,a,a,b,c$	b,c	$a,a \to 0$	$0 \to b$	$c \to c$	$0,b \to a$
$(0,a',b',0)$	$0,a,a,b,b,c$	a,b	$a,a \to 0$	$b \to c$	$c \to a$	$0,b \to b$
	$0,0,a,a,b,c$	b,c	$a,a \to 0$	$0 \to b$	$0 \to c$	$b,c \to a$
$(a',0,0,b')$	$0,a,a,b,b,c$	a,b	$b,c \to 0$	$a \to a$	$b \to b$	$0,a \to c$
	$0,0,a,a,b,c$	b,c	$0,b \to 0$	$a \to a$	$c \to c$	$0,a \to b$
$(0,a',b',c')$	$0,0,a,a,b,b$	a,b	$a,a \to 0$	$0 \to a$	$0 \to b$	$b,b \to c$
		b,c	$a,a \to 0$	$0 \to b$	$0 \to c$	$b,b \to a$
$(a',0,b',c')$		a,b	$0,a \to 0$	$a \to a$	$0 \to b$	$b,b \to c$
		b,c	$0,b \to 0$	$b \to b$	$0 \to c$	$a,a \to a$

Table 9. Labels inserted into e_1, e_2, e_{m-1}, e_m starting from 3 or 4 distinct sums, and the final sum of labels inside e_i.

4. CONCLUSIONS

We finish the paper with some simple open problems.

Conjecture 9. *Let $T = (V, E)$ be a hypertree. If $|e| \geq 3$ for every $e \in E(T)$, then T is V_4-cordial.*

Problem 10. Characterize the class of hypergraphs which are cycle-free and V_4-cordial.

Problem 11. Give necessary and sufficient conditions for V_4-cordial path hypergraphs.

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