

Backreaction in the Lemaître-Tolman-Bondi model

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Abstract.

We study backreaction analytically using the parabolic Lemaître-Tolman-Bondi universe as a toy model. We calculate the average expansion rate and energy density on two different hypersurfaces and compare the results. We also consider the Hubble law and find that backreaction slows down the expansion if measured with proper time, but speeds it up if measured with energy density.

PACS numbers: 04.40.Nr, 98.80.-k, 98.80.Jk

1. Introduction

The homogeneous and isotropic Friedmann-Robertson-Walker models are usually thought to give a good description of the average behaviour of the universe, since the universe appears to be homogeneous and isotropic on sufficiently large scales (though see [1]). The reasoning behind the FRW equations is that one takes the average of the real (inhomogeneous and anisotropic) metric and energy-momentum tensor, and plugs these into the Einstein equation. However, the physically correct thing would be to first plug the real inhomogeneous and anisotropic quantities into the Einstein equation, and only then take the average. Because the Einstein equation is non-linear, these two procedures are not equivalent. In other words, the averages of the real quantities do not satisfy the Einstein equation. The feature that the average behaviour of an inhomogeneous spacetime is not the same as the behaviour of the corresponding smooth spacetime (that is, one with the same average initial conditions) is called backreaction. In the context of cosmology, the issue was highlighted in [2] (it had been discussed earlier in [3]) as the *fitting problem*: how do we find the homogeneous and isotropic model which best fits the real inhomogeneous and anisotropic universe?

The relation between the average sources (energy density, pressure, etc.) and average geometric quantities (expansion rate, shear, etc.) in a general spacetime filled with a perfect irrotational fluid is known [4, 5]. However, the system of equations is not closed, which means that different inhomogeneous spacetimes with the same initial averages evolve differently even as far as average quantities are concerned. Since there is no procedure for finding the average behaviour of a given spacetime (short of solving it exactly), it is difficult to quantitatively evaluate the importance of backreaction in

cosmology. One notable calculation is [6], where the behaviour of an universe-sized box filled with inhomogeneous expanding dust is followed numerically using the Newtonian limit of the exact backreaction formalism of [4, 7]. The results show unambiguously that backreaction is a real phenomenon which can have a large impact: for example, it can make regions with no initial overdensity turn around and collapse. However, the Newtonian limit is not expected to fully capture the relativistic backreaction. (In particular, the global backreaction vanishes identically for periodic boundary conditions, unlike in the relativistic case.)

Fully relativistic quantitative studies of backreaction have usually been done in the context of perturbative solutions around a FRW background [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] (see [13] for a more extensive list of references). Of particular interest has been the possibility that backreaction of long wavelength perturbations could lead to a dynamical relaxation of vacuum energy, at the same stroke providing an elegant inflationary mechanism and explaining why the vacuum energy is so much smaller than theoretically expected [14, 17, 18, 19, 20]. It has also been suggested that this could solve the coincidence problem of the effective vacuum energy density being of the order of the matter energy density today [19, 20]. Another possible explanation for the coincidence problem is backreaction from structure formation, that is, from small wavelength modes [9, 13, 16, 21]. Since the acceleration of the universe seems to have started around the era when structure formation is important, it seems a natural possibility that this deviation from the simple prediction of deceleration in the homogeneous and isotropic models with normal matter could be related to the growth of inhomogeneities in the universe.

Perturbative treatment of backreaction has two drawbacks. First, naive perturbative results break down when perturbations have an effect on the background, in other words when backreaction is important. In addition to the usual issues of cosmological perturbation theory such as gauge-invariance [8, 11, 22], one has to worry about new problems such as convergence and consistency of the perturbative expansion. These make consistent backreaction calculations in the perturbative framework an involved task. Second, present-day universe is not perturbatively close to homogeneity, but contains non-linear structures. If the coincidence problem is to be solved by backreaction from structure formation, one would have to go beyond perturbation theory (though if the effect involves mainly perturbations breaking away from the linear regime, a quasi-perturbative treatment might still be possible).

We will study backreaction analytically and free of perturbative ambiguities with an exact toy model, the Lemaître-Tolman-Bondi model [23, 24, 25] (see also [26, 27]). The LTB model is the spherically symmetric dust solution of the Einstein equation (or rather the family of such solutions). Like the perturbed FRW spacetime, the LTB spacetime is a generalisation of the FRW universe. The LTB model can be viewed as an Einstein-de Sitter universe with a single spherically symmetric perturbation which can be arbitrarily large, as opposed to the linearly perturbed FRW universe which has an ensemble of small perturbations. The LTB model can describe the collapse of an overdensity or the formation of a void in an expanding universe [28], and it has also

been used to describe the local inhomogeneous universe [29]. As a model for the entire universe, it is a toy model which is useful because the backreaction problem can be studied quantitatively and without any approximations, as has been previously done in [30] (averaging in the LTB model has also been discussed in [31]; see also [32]).

In a perturbative framework one aspect of backreaction is that inhomogeneities change the behaviour of a smooth background. This feature is obviously not present in an exact inhomogeneous solution, but the difference between the average behaviour and the behaviour of the corresponding smooth spacetime can be studied unambiguously. With an exact solution one can also study the choice of the hypersurface of averaging in isolation of the different issue of the choice of gauge [8, 10, 11, 12, 13, 22].

The present study is a simple exact non-perturbative counterpart of [13], where we calculated perturbatively the average expansion rate of a linearly perturbed Einstein-de Sitter universe. In section 2 we calculate the average expansion rate in the simplest LTB solution for two different choices of hypersurface, and compare the results with each other and the FRW universe. In section 3 we look at the impact of backreaction on the Hubble law and discuss our results.

2. The backreaction calculation

2.1. The Lemaître-Tolman-Bondi model

The metric. We will proceed as in [13]: we will write down the metric, find the proper time and the expansion rate and take the average over the hypersurface of proper time. We will also take the average over the hypersurface of constant coordinate time, and compare the results.

The LTB model is the most general spherically symmetric dust solution. The Einstein equation is (we take the cosmological constant to be zero)

$$G_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu} = \frac{1}{M_{Pl}^2} \rho u_\mu u_\nu, \quad (1)$$

where $M_{Pl} = 1/\sqrt{8\pi G_N}$ is the (reduced) Planck mass, ρ is the energy density and u^μ is the velocity of the matter fluid, with $u_\mu u^\mu = -1$. The spherical symmetry allows the Einstein equation (1) to be solved exactly. The metric turns out to be

$$ds^2 = -dt^2 + \frac{R'(t, r)^2}{1 + E(r)} dr^2 + R(t, r)^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2)$$

where the functions $R(t, r)$ and $E(r)$ are related to each other and to the energy density $\rho(t, r)$ as follows

$$\begin{aligned} \dot{R}(t, r)^2 &= \frac{1}{M_{Pl}^2} \frac{m(r)}{R(t, r)} + E(r) \\ \rho(t, r) &= \frac{m'(r)}{R(t, r)^2 R'(t, r)}, \end{aligned} \quad (3)$$

where dots and primes denote derivatives with respect to t and r , respectively, and $m(r)$ is a function which describes how much energy there is within the radius r . (The

freedom to choose the r -coordinate means that $m(r)$ can be redefined at will.) The velocity of the matter fluid is simply

$$u^\mu = (1, 0, 0, 0) . \quad (4)$$

There are three different classes of solutions to (3), for $E > 0$, $E < 0$ and $E = 0$. We will consider the solution with $E = 0$, known as the parabolic solution, which is the simplest and the most analogous to the Einstein-de Sitter universe. We assume that $m' > 0$ and choose the radial coordinate such that $m(r) = 4M_{Pl}^2 r^3 / (9t_1^2)$, where t_1 is a positive constant with the dimension of time. We then have

$$R(t, r) = r \left(\frac{t - t_0(r)}{t_1} \right)^{\frac{2}{3}} , \quad (5)$$

where $t_0(r)$ is an arbitrary function. It is transparent that the solution reduces to the FRW case if $t_0(r) = \text{constant}$. There is a singularity at $t = t_0(r)$, and we have chosen to look at the case $t > t_0(r)$, which describes matter expanding away from the big bang rather than collapsing towards a future singularity. The function $t_0(r)$ is the big bang time for a comoving observer at constant r : different observers find their part of the universe to have emerged from the singularity at different times in the past. In order to avoid a singularity caused by shells of matter crossing, we must have $t'_0(r) \leq 0$ [33].

The proper time. As emphasised in [10, 13, 16], it is important to cast things in terms of the proper time of the observer. Given (4), the derivative in the direction orthogonal to the hypersurface defined by the velocity of comoving observers is $\partial_\tau = u^\mu \nabla_\mu = \nabla_t$. From the condition $\partial_\tau \tau = 1$ we get the proper time $\tau = t + f(r)$, where $f(r)$ is an arbitrary function.

In the perturbed FRW case [13] there was a difference between the background coordinate time t and the proper time τ because the velocity u^μ of the perturbed comoving observers was different from the background velocity, leading to a perturbed τ . Here the situation is different. The velocity of comoving observers is the same as in the smooth case, but different observers start their clocks at different times. When discussing the expansion rate measured by cosmological observers, we want to compare observers whose clocks show the same time. We therefore choose $f(r) = -t_0(r)$, so the proper time is

$$\tau(t, r) = t - t_0(r) , \quad (6)$$

which is the ‘‘local age of the universe’’, the time since the big bang measured by a comoving observer at constant r . Note that in order for τ to be a time coordinate, the normal $\nabla_\mu \tau$ has to be timelike, which translates into the constraint $R'(t, r) > -t'_0(r)$.

The expansion rate. We are interested in the expansion rate measured by a comoving observer, given by

$$\theta(t, r) = \nabla_\mu u^\mu , \quad (7)$$

which in the FRW case reduces to $\theta = 3H$, where H is the Hubble parameter. Plugging in (2), (4) and (5) we have

$$\theta(t, r) = \frac{2}{\tau} \left(1 - \frac{r\tau'}{3\tau + 2r\tau'} \right). \quad (8)$$

The expansion rate of an Einstein-de Sitter universe is $2/\tau$, so the second term is the contribution of the inhomogeneity. Since $\tau' = -t'_0 > 0$, this term is negative, and the expansion in terms of the proper time is always slower than in the FRW case, and slows down with increasing r . In the converse situation with $\tau' < 0$, the expansion rate would be larger than in the FRW case, but the inner shells of matter would overtake the outer shells, resulting in a singularity. (In a more realistic model, shell-crossing would mean that the description of matter as a pressureless ideal fluid breaks down, not necessarily that there is a singularity.)

The energy density. From (3) and (5) we have

$$\rho(t, r) = \frac{4M_{Pl}^2}{3\tau^2} \frac{3\tau}{3\tau + 2r\tau'}. \quad (9)$$

In the solution with $E = 0$, the density is uniquely determined by the local big bang time t_0 , and one can see from ρ concretely what kind of a physical situation a given t_0 corresponds to. In particular, when t_0 is constant, (9) reduces to the FRW case, $\rho = 4M_{Pl}^2/(3\tau^2)$. It is possible to specify a LTB solution by giving the energy density on initial and final hypersurfaces [28], instead of giving E and t_0 as done here.

2.2. Taking the average

The average expansion rate. To obtain the average expansion rate, we should integrate the local expansion rate (8) over the appropriate hypersurface. To find the integration measure on the hypersurface of constant τ , let us write the metric (2) in terms of τ :

$$\begin{aligned} ds^2 &= -d\tau^2 - 2t'_0 d\tau dr + (R'^2 - t_0'^2) dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\frac{R'^2}{R'^2 - t_0'^2} d\tau^2 + (R'^2 - t_0'^2) \left(dr - \frac{t'_0}{R'^2 - t_0'^2} d\tau \right)^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \quad (10)$$

where we again see that $R' > -t'_0$ is required for τ to be a time coordinate. We can now read off the integration measure on the hypersurface of constant τ [34]:

$$|\det^{(\tau)} g_{ij}|^{1/2} = (R'^2 - t_0'^2)^{1/2} R^2 \sin\theta. \quad (11)$$

The average of a scalar observable $F(t, r)$ over the hypersurface of constant τ is then

$$\begin{aligned} \langle F \rangle_{\tau_c} &= \frac{\int d^4x |\det^{(\tau)} g_{ij}|^{1/2} \delta(t - t_0(r) - \tau_c) F(t, r)}{\int d^4x |\det^{(\tau)} g_{ij}|^{1/2} \delta(t - t_0(r) - \tau_c)} \\ &= \frac{\int_0^\infty dr r^2 \left[\left(\frac{\tau_c}{t_1} \right)^{\frac{4}{3}} \left(\frac{3\tau_c - 2rt'_0}{3\tau_c} \right)^2 - t_0'^2 \right]^{1/2} F(t_0(r) + \tau_c, r)}{\int_0^\infty dr r^2 \left[\left(\frac{\tau_c}{t_1} \right)^{\frac{4}{3}} \left(\frac{3\tau_c - 2rt'_0}{3\tau_c} \right)^2 - t_0'^2 \right]^{1/2}}, \end{aligned} \quad (12)$$

where $\delta(t - t_0(r) - \tau_c)$ is the delta function, τ_c is the constant value of proper time which labels the hypersurface, and on the second line we have plugged in (5) and (11). To keep the analogy with the FRW case as close as possible, we will consider only spacetimes where the coordinate r ranges from 0 to ∞ .

In comparison, the integration measure on the hypersurface of constant t is

$$|\det {}^{(t)}g_{ij}|^{1/2} = R'R^2 \sin \theta, \quad (13)$$

and the average over the hypersurface of constant t reads

$$\begin{aligned} \langle F \rangle_{t_c} &= \frac{\int d^4x |\det {}^{(t)}g_{ij}|^{1/2} \delta(t - t_c) F(t, r)}{\int d^4x |\det {}^{(t)}g_{ij}|^{1/2} \delta(t - t_c)} \\ &= \frac{\int_0^\infty dr r^2 (t_c - t_0) [3(t_c - t_0) - 2rt'_0] F(t_c, r)}{\int_0^\infty dr r^2 (t_c - t_0) [3(t_c - t_0) - 2rt'_0]}, \end{aligned} \quad (14)$$

where t_c is again the constant labelling the hypersurface. There are two differences between the averages (12) and (14). The integration measure is different, and the quantity being held constant in the integral is different. In general, both are expected to be important, but in the present case the integration measure will in fact turn out not to matter. We will now evaluate the average of the expansion rate θ over both hypersurfaces for some simple choices of t_0 .

Comparing the averages. Our LTB solution is specified once we give the function t_0 , which has to satisfy $R' > -t'_0 \geq 0$. We consider three examples of t_0 . They are given in Table 1, along with the average expansion rates and average densities (r_0 is a positive constant). The exponential factors in the first two cases are needed for satisfying the condition $R' > -t'_0$; without them, the averages would be the same but the hypersurfaces would not be spacelike for all τ_c . We also give the effective scale factor a_τ defined by $3\partial_\tau a_\tau / a_\tau = \langle \theta \rangle_\tau$, in analogy with [4, 5, 6, 7]‡ (t_2 is a positive constant).

Table 1. Average expansion rates and densities.

	$t_0(r)$	$\langle \theta \rangle_\tau$	$\langle \theta \rangle_t$	$\langle \rho \rangle_\tau$	$\langle \rho \rangle_t$	a_τ
Case 1	$-re^{-\frac{2t_1}{r}}$	$\frac{2}{\tau} \frac{1}{2}$	0	0	0	$(\tau/t_2)^{1/3}$
Case 2	$-r_0 e^{-\frac{2t_1}{r}} \ln \frac{r+r_0}{r_0}$	$\frac{2}{\tau} \frac{3\tau+r_0}{3\tau+2r_0}$	0	$\frac{4M_{Pl}^2}{3\tau^2} \frac{3\tau}{3\tau+2r_0}$	0	$[\tau(\tau + 2r_0/3)/t_2^2]^{1/3}$
Case 3	$\frac{r_0}{1+r^3/(2t_1^2 r_0)}$	$\frac{2}{\tau}$	$\frac{2}{t}$	$\frac{4M_{Pl}^2}{3\tau^2}$	$\frac{4M_{Pl}^2}{3t^2}$	$(\tau/t_2)^{2/3}$

In cases 1 and 2 the big bang occurred at $t = 0$ in the center at $r = 0$, while the far away shells of constant r asymptotically approach being infinitely old. In contrast, in case 3 there is only a finite difference (of r_0) in the big bang time between the center

‡ Note that this definition is different from the one used in [13, 16].

and the region at asymptotic infinity. The three cases cover the possible asymptotic behaviours for the quantity $r\tau'$: divergent, finite and zero.

Scalar quantities (such as the expansion rate) are invariant under coordinate transformations, but spatial averaging is not a covariant procedure, and the choice of hypersurface makes a physical difference. Cases 1 and 2 illustrate the drastic difference between the hypersurfaces of constant τ and constant t . In case 1 the average expansion rate on the hypersurface of constant τ is 1/2 of the FRW rate, and in case 2 it starts at 1/2 of the FRW rate and asymptotically approaches it as τ grows, and the scale factor correspondingly interpolates between $(\tau/t_1)^{1/3}$ (case 1) and $(\tau/t_1)^{2/3}$ (the FRW case). However, in both cases 1 and 2 the average expansion rate on the hypersurface of constant t is zero. In case 3 the average over both hypersurfaces yields the unperturbed expansion rate in terms of the appropriate time.

It may seem strange that the average of the expansion rate is zero, given that the expansion rate is positive definite. The resolution is that the average is dominated by the region at asymptotic infinity, since it contains infinitely more observers than the region within any given finite radius. Since the age of the region at large r is asymptotically infinite, its expansion rate is asymptotically zero, so the average expansion rate goes to zero. It is for the same reason that the averages turn out to be the unperturbed values in case 3: the big bang time t_0 goes asymptotically to zero, so there is asymptotically no difference between τ and t , and therefore no backreaction. In fact, the averages over the two hypersurfaces are simply the asymptotic limits of the quantities (8), (9) with the appropriate time coordinate held fixed. Since the three cases we have studied cover all possible limits of the quantity $r\tau'$, we have exhausted the possibilities for the averages: any permissible function t_0 leads to one of the average quantities in cases 1 to 3 (apart from a trivial global change in the normalisation of t). The averages over all space do not depend on the integration measure, since in the asymptotic limit the measure in the nominator and the denominator cancels. This feature is peculiar to the spherical symmetry of the LTB model. In a realistic model of the universe where all points are statistically equivalent, the integration measure is expected to make a difference. It would also be more physical to average over the finite region that has been in causal contact with a given observer, in which case the backreaction would not depend only on the asymptotic limit, and would be finite in all cases.

The above examples show that even in the LTB model, where (unlike in the perturbed FRW universe) the coordinate time t is a physically measurable quantity, taking averages over the hypersurface of constant t can give misleading results. We are interested in the average expansion rate measured by local observers as a function of the proper time they measure, so we should compare observers whose clocks show the same time. Assuming that cosmological observers normalise their clocks to the locally inferred big bang time t_0 instead of synchronising them to some global constant, observers further out in r will reach the same value of τ earlier in terms of t : the averaging surface tilts to the past with increasing r .

3. Discussion

The Hubble law. To get another viewpoint on the backreaction, let us look at the relation between the average expansion rate and average density in case 2. From the expressions given in Table 1 we can solve for τ in terms of $\langle\rho\rangle_\tau$, and insert this into the expression for $\langle\theta\rangle_\tau$. The resulting Hubble law is

$$\langle\theta/3\rangle_\tau^2 = \frac{4}{9\tau^2} \left(\frac{3\tau + r_0}{3\tau + 2r_0} \right)^2 \quad (15)$$

$$= \frac{\langle\rho\rangle_\tau}{3M_{Pl}^2} \left(1 + \frac{r_0^2}{12M_{Pl}^2} \langle\rho\rangle_\tau \right) \quad (16)$$

$$= \frac{4}{9t_1^2} \frac{1}{a_\tau^3} \left(1 + \frac{r_0^2}{9t_1^2} \frac{1}{a_\tau^3} \right), \quad (17)$$

where we have written the square of the expansion rate in three different forms: as a function of the proper time, the average energy density and the effective scale factor. The FRW relations are recovered as r_0/τ goes to zero.

From the first expression, (15), it seems that the expansion is slower than in the homogeneous case. However, from the second and third expressions, (16) and (17), it seems as if the universe would instead expand faster as there is an additional positive $\langle\rho\rangle_\tau^2$ -term driving the expansion. We have the paradoxical-seeming situation that the expansion is slower than in the FRW case if measured in terms of the proper time, but faster if measured in terms of the energy density (or the scale factor, since conservation of mass implies $\langle\rho\rangle_\tau \propto a_\tau^{-3}$). The reason is that the backreaction reduces (in terms of τ) $\langle\rho\rangle_\tau$ more than it reduces $\langle\theta\rangle_\tau$, as we see from Table 1. Therefore, the dependence of $\langle\theta\rangle_\tau$ on $\langle\rho\rangle_\tau$ has to be stronger than in the FRW case to get the dependence on τ right. This underlines the point, mentioned in [10, 13], that one should be careful to identify which quantity is being measured in terms of what parameters when considering backreaction.

A measure of backreaction. In [13] it was suggested that since the Weyl tensor gives a measure of the inhomogeneity and anisotropy of a spacetime it could provide some indication of whether backreaction is important. For the LTB solution with $E = 0$, the ratio of the square of the Weyl tensor to the square of the scalar curvature is, from (2) and (5),

$$\frac{C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}}{\mathcal{R}^2} = \frac{16}{27} \left(\frac{r\tau'}{\tau} \right)^2, \quad (18)$$

which does measure the importance of backreaction. For example, in case 2 it is, when averaged over the hypersurface of constant τ , essentially the ratio $(r_0/\tau)^2$. Note that the information entropy measure introduced in [35] to quantify the inhomogeneity of a spacetime measures the difference between a spacetime and its average, not the difference between the average and the corresponding smooth spacetime. (At any rate, the measure was defined for compact domains, and so is not applicable to the present case.)

One can calculate the backreaction for geometric quantities other than the expansion rate and the Weyl tensor. One interesting candidate is the shear, which

is zero in the FRW case. Like the square of the Weyl tensor, the shear turns out to be a measure of the backreaction both in the LTB case and in the perturbative case of [13]. In the perturbative case, the shear becomes of the order of the scalar curvature, just like the Weyl tensor. This is in conflict with observations, but the perturbative analysis of [13] is at any rate expected to break down when backreaction is indicated to be important.

Conclusion. We have calculated the average expansion rate and energy density in the simplest, parabolic, LTB solution analytically for some example cases (which turn out to cover all possibilities). Comparison of the averages with each other demonstrates the importance of the choice of hypersurface, and comparison with the FRW case shows that backreaction slows down the expansion if measured in terms of the proper time, but speeds it up if measured in terms of the energy density or the scale factor. The calculation is an example of an exact, quantitative study in backreaction.

Unlike the perturbed FRW metric, the LTB metric is not meant to be a realistic model of the universe, but it is a useful toy model for studying backreaction because one can obtain analytical results. One way towards a more realistic backreaction calculation from the first order perturbed FRW model used in [13] is to do a consistent second order perturbative analysis. Another possibility could be a quasi-perturbative approach where one keeps to first order perturbation theory for scales which are in the linear regime, but applies a non-perturbative solution for scales which have gone non-linear. Since the LTB model can describe the formation of clusters and voids smoothly starting from small perturbations [28], it is ideal for this purpose. (A simple version of embedding LTB solutions into a FRW universe was done in [36], and the idea goes back to [23].) This would involve the LTB solutions with $E \neq 0$, and it would be interesting to first study how the average expansion rate behaves in those solutions themselves.

Acknowledgments

I thank Marco Bruni, Jai-chan Hwang and Dominik Schwarz for discussions, a number of people including Carsten van de Bruck and Joe Silk for suggesting looking at the LTB model, and the Helsinki Institute of Physics for hospitality.

The research has been supported by PPARC grant PPA/G/O/2002/00479, by a grant from the Magnus Ehrnrooth Foundation and by the European Union network HPRN-CT-2000-00152, ‘‘Supersymmetry and the Early Universe’’.

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