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ON THE WEIGHTED FRACTIONAL POINCARÉ-TYPE INEQUALITIES

BY

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Abstract. Weighted fractional Poincaré-type inequalities are proved on John domains whenever the weights defined on the domain depend on the distance to the boundary and to an arbitrary compact set in the boundary of the domain.

1. Introduction. In this article we study a version of the classical fractional Poincaré-type inequality where the domain in the double integral in the Gagliardo seminorm is replaced by a smaller one:

$$(1.1)\qquad \Big(\int_{\Omega} |u(x)-u_{\Omega}|^p \,\mathrm{d}x\Big)^{1/p} \le C\Big(\int_{\Omega}\int_{B(x,\tau d(x))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}.$$

The parameter τ in the double integral belongs to $(0,1)$, and $d(x)$ denotes the distance from x to $\partial\Omega$. Inequality (1.1) was introduced in [HV1]. It is well-known that the classical fractional Poincaré inequality is valid for any bounded domain, while this new version (1.1) depends on the geometry of the domain. In [HV1] it was proved that (1.1) is valid on John domains, and hence in particular on Lipschitz domains. An example of a domain where (1.1) is not valid was also given. We refer the reader to [HV2] and [DIV] where fractional Sobolev–Poincaré versions of (1.1) are considered. For a weighted version of (1.1) where weights are power functions of the distance to the boundary we refer to [DD].

The main result of our paper is the following theorem, where the distance to an arbitrary set of the boundary has been added as a weight.

Theorem 1.1. *Let Ω in $\mathbb{R}^n$ be a bounded John domain and $1<p<\infty$. Given a compact set F in $\partial\Omega$, and parameters $\beta\ge 0$ and $s,\tau\in(0,1)$,*

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there exists a constant C *such that*

$$\Big(\int\limits_{\Omega} |u(x)-u_{\Omega,\omega^p}|^p d_F(x)^{p\beta}\,\mathrm{d}x\Big)^{1/p} \le C\bigg(\int\limits_{\Omega}\int\limits_{B(x,\tau d(x))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} d(x)^{ps} d_F(x)^{p\beta}\,\mathrm{d}y\,\mathrm{d}x\bigg)^{1/p} \tag{1.2}$$

for all functions $u \in L^p(\Omega, d(x)^{p\beta})$, *where* $d(x)$ *and* $d_F(x)$ *denote the distances from* x *to* $\partial\Omega$ *and* F *respectively, and* u_{Ω,ω^p} *is the weighted average* $d_F(\Omega)^{-p\beta}\int_\Omega u(z)d_F(z)^{p\beta}\,\mathrm{d}z$.

In addition, the constant C *in* (1.2) *can be written as*

$$C = C_{n,p,\beta}\tau^{s-n}K^{n+\beta},$$

where K *is the geometric constant introduced in* (5.1).

We would like to emphasize two points in this result: First, no extra conditions are required for the compact set F in $\partial\Omega$. The second point is that the estimate shows how the constant depends on the given τ and a certain geometric condition on the domain.

Some of the essential auxiliary parts for the proofs for weighted inequalities are taken from [L1] and [L2], where a useful decomposition technique was introduced by the second author. Our work was stimulated by the papers of Augusto C. Ponce [P1], [P2], [P3], where more general fractional Poincaré inequalities for functions defined on Lipschitz domains were investigated.

The paper is organized as follows: In Section 2, we introduce some definitions and preliminary results. In Section 3, we show how to use decompositions of functions to extend the validity of certain inequalities on "simple domains", such as cubes, to more complex ones. We are interested in extending the results from cubes to John domains. In Section 4, we apply the results obtained in the previous section to estimate the constant in the unweighted version of (1.2) on cubes. Especially we are interested in how the constant depends on τ. This result is auxiliary for our main theorem but it might be of independent interest. In Section 5, we show the validity of the weighted fractional Poincaré inequality studied in this paper with an estimate of the constant, and a generalization to the type of inequalities considered by Ponce.

2. Notation and preliminary results. Throughout the paper, Ω in $\mathbb{R}^n$ is a bounded domain with $n \ge 2$, $1 < p, q < \infty$ with $1/p + 1/q = 1$, unless otherwise stated. Moreover, given a *weight* (i.e., a positive measurable function) $\eta : \Omega \to \mathbb{R}$ and $1 \le r \le \infty$, we denote by $L^r(\Omega, \eta)$ the space of

Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ equipped with the norm

$$\|u\|_{L^r(\Omega,\eta)} := \Big(\int_\Omega |u(x)|^r \eta(x)\,\mathrm{d}x\Big)^{1/r}$$

if $1 \le r < \infty$, and

$$\|u\|_{L^\infty(\Omega,\eta)} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)\eta(x)|.$$

Finally, given a set A we denote by $\chi_A(x)$ its characteristic function.

Definition 2.1. Let $\mathcal{C}$ be the space of constant functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\{U_t\}_{t\in\Gamma}$ a collection of open subsets of Ω that covers Ω except for a set of Lebesgue measure zero; Γ is an index set. It also satisfies the additional requirement that for each $t \in \Gamma$ the set U_t intersects a finite number of U_s with $s \in \Gamma$. This collection $\{U_t\}_{t\in\Gamma}$ is called an *open covering of Ω*. Given $g \in L^1(\Omega)$ orthogonal to $\mathcal{C}$ (i.e., $\int g\varphi = 0$ for all $\varphi \in \mathcal{C}$), we say that a collection $\{g_t\}_{t\in\Gamma}$ of functions in $L^1(\Omega)$ is a *$\mathcal{C}$-orthogonal decomposition of g subordinate to* $\{U_t\}_{t\in\Gamma}$ if the following three properties are satisfied:

(1) $g = \sum_{t\in\Gamma} g_t$.
(2) $\operatorname{supp}(g_t) \subset U_t$ for all $t \in \Gamma$.
(3) $\int_{U_t} g_t = 0$ for all $t \in \Gamma$.

We also refer to this collection of functions as a *$\mathcal{C}$-decomposition*. We say that $\{g_t\}_{t\in\Gamma}$ is a *finite $\mathcal{C}$-decomposition* if $g_t \not\equiv 0$ only for a finite number of $t \in \Gamma$. Notice that condition (3) is equivalent to orthogonality to the space $\mathcal{C}$ of constant functions. Indeed, this condition can be replaced by $\int_{U_t} g_t(x)\varphi(x)\,\mathrm{d}x = 0$ for all $\varphi \in \mathcal{C}$ and $t \in \Gamma$. Inequality (1.2) is based on the operator of fractional derivatives whose zeros are the constant functions. In other inequalities that involve other operators, such as the Korn inequality with the symmetric part of a differential operator, the decompositions of functions must be orthogonal to other spaces, for instance the space of infinitesimal rigid displacements in the case of the Korn inequality. We refer to [L2], [L3] for examples of decompositions orthogonal to some other spaces.

Inequality (1.2), and similar Poincaré-type inequalities, can be written in terms of a distance to the space $\mathcal{C}$ of constant functions by replacing its left hand side by

$$\inf_{\alpha\in\mathcal{C}} \Big(\int_\Omega |u(x)-\alpha|^p d_F(x)^{p\beta}\,\mathrm{d}x\Big)^{1/p}.$$

The technique used in this paper may also be considered when the distances to other vector spaces $\mathcal{V}$ are involved, in which case a $\mathcal{V}$-orthogonal decomposition of functions is required. We direct the reader to [L3] where a generalized version of the Korn inequality is studied by using decomposition of functions.

Let us denote by $G = (V, E)$ a graph with vertices V and edges E. Graphs in this paper have neither multiple edges nor loops and the number of vertices in V is at most countable.

A *rooted tree* (or simply a tree) is a connected graph G in which any two vertices are connected by exactly one simple path, and a *root* is simply a distinguished vertex $a \in V$. Moreover, if $G = (V, E)$ is a rooted tree with a root a, it is possible to define a *partial order* "$\preceq$" in V as follows: $s \preceq t$ if and only if the unique path connecting t to the root a passes through s. The *height* or *level* of any $t \in V$ is the number of vertices in $\{s \in V : s \preceq t \text{ with } s \neq t\}$. The *parent* of a vertex $t \in V$ is the vertex s such that $s \preceq t$ and its height is one unit smaller than the height of t. We denote the parent of t by t_p. It can be seen that each $t \in V$ different from the root has a unique parent, but several elements in V could have the same parent. Note that two vertices are connected by an edge (*adjacent vertices*) if one is the parent of the other.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that an open covering $\{U_t\}_{t\in\Gamma}$ of Ω is a *tree covering* if it also satisfies the properties:

(1) $\chi_\Omega(x) \leq \sum_{t\in\Gamma} \chi_{U_t}(x) \leq N\chi_\Omega(x)$ for almost every $x \in \Omega$, where $N \geq 1$.
(2) Γ is the set of vertices of a rooted tree (Γ, E) with a root a.
(3) There is a collection $\{B_t\}_{t\neq a}$ of pairwise disjoint open cubes with $B_t \subseteq U_t \cap U_{t_p}$.

Definition 2.3. Given a tree covering $\{U_t\}_{t\in\Gamma}$ of Ω we define the following *Hardy-type operator* T on L^1-functions:

$$Tg(x) := \sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} |g|, \tag{2.1}$$

where

$$W_t := \bigcup_{s\succeq t} U_s, \tag{2.2}$$

and χ_t is the characteristic function of B_t for all $t \neq a$.

We may refer to W_t as the *shadow* of U_t.

Note that the definition of T is based on the a priori choice of a tree covering $\{U_t\}_{t\in\Gamma}$ of Ω. Thus, whenever T is mentioned in this paper, there is a tree covering $\{U_t\}_{t\in\Gamma}$ of Ω explicitly or implicitly associated to it.

The following fundamental result was proved in [L2, Theorem 4.4]; it shows the existence of a $\mathcal{C}$-decomposition of functions subordinate to a tree covering of the domain.

Theorem 2.4. *Let Ω in $\mathbb{R}^n$ be a bounded domain with a tree covering $\{U_t\}_{t\in\Gamma}$. Given $g \in L^1(\Omega)$ such that $\int_\Omega g\varphi = 0$, for all $\varphi \in \mathcal{C}$, and*

$\operatorname{supp}(g) \cap U_s \neq \emptyset$ *for a finite number of* $s \in \Gamma$*, there exists a* $\mathcal{C}$*-decomposition* $\{g_t\}_{t\in\Gamma}$ *of* g *subordinate to* $\{U_t\}_{t\in\Gamma}$ *(refer to Definition* 2.1*).*

Moreover, let $t \in \Gamma$*. If* $x \in B_s$ *where* $s = t$ *or* $s_p = t$*, then*

$$|g_t(x)| \le |g(x)| + \frac{|W_s|}{|B_s|} Tg(x), \tag{2.3}$$

where W_t *denotes the shadow of* U_t *defined in* (2.2)*. Otherwise*

$$|g_t(x)| \le |g(x)|. \tag{2.4}$$

Remark 2.5. The $\mathcal{C}$-decomposition stated in Theorem 2.4 is finite. This fact is not in the statement of [L2, Theorem 4.4] but it is easily deduced from its proof.

In the next lemma, the continuity of the operator T is shown. We refer the reader to [L1, Lemma 3.1] for a proof.

Lemma 2.6. *The operator* $T : L^q(\Omega) \to L^q(\Omega)$ *defined in* (2.1) *is continuous for any* $1 < q \le \infty$*. Moreover, its norm is bounded by*

$$\|T\|_{L^q \to L^q} \le 2\left(\frac{qN}{q-1}\right)^{1/q}.$$

Here N *is the overlapping constant from Definition* 2.2*.*

If $q = \infty$, the above inequality means $\|T\|_{L^\infty \to L^\infty} \le 2$. Actually, for T being an averaging operator, it can be easily observed that $\|T\|_{L^\infty \to L^\infty} = 1$, but this does not affect our work. Notice that $L^q(\Omega, \omega^{-q}) \subset L^1(\Omega)$ if the weight $\omega : \Omega \to \mathbb{R}_{>0}$ has $\omega^p \in L^1(\Omega)$. Then the operator T introduced in Definition 2.3 for functions in $L^1(\Omega)$ is well-defined in $L^q(\Omega, \omega^{-q})$.

Lemma 2.7. *Let* Ω *in* $\mathbb{R}^n$ *be a bounded domain,* $\{U_t\}_{t\in\Gamma}$ *a tree covering of* Ω *and* $\omega : \Omega \to \mathbb{R}$ *a weight which satisfies* $\omega^p \in L^1(\Omega)$*. If*

$$\operatorname*{ess\,sup}_{y\in W_t} \omega(y) \le C_2 \operatorname*{ess\,inf}_{x\in B_t} \omega(x) \tag{2.5}$$

for all $a \neq t \in \Gamma$*, then the Hardy-type operator* T *defined in* (2.1) *and subordinate to* $\{U_t\}_{t\in\Gamma}$ *is continuous from* $L^q(\Omega, \omega^{-q})$ *to itself. Moreover, its norm for* $1 < q < \infty$ *is bounded by*

$$\|T\|_{L \to L} \le 2\left(\frac{qN}{q-1}\right)^{1/q} C_2,$$

where L *denotes* $L^q(\Omega, \omega^{-q})$*, and* N *is the overlapping constant from Definition* 2.2*.*

Proof. Given $g \in L^q(\Omega, \omega^{-q})$ we have

$$\begin{aligned}\int_\Omega |Tg(x)|^q \omega(x)^{-q}\,\mathrm{d}x &= \int_\Omega \omega(x)^{-q} \bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|}\int_{W_t} |g(y)|\,\mathrm{d}y\bigg|^q \mathrm{d}x\\ &= \int_\Omega \bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|}\omega(x)^{-1}\int_{W_t} |g(y)|\omega(y)^{-1}\omega(y)\,\mathrm{d}y\bigg|^q \mathrm{d}x.\end{aligned}$$

Now, condition (2.5) implies that $\omega(y) \le C_2\omega(x)$ for almost every $x \in B_t$ and $y \in W_t$. Thus,

$$\begin{aligned}&\int_\Omega |Tg(x)|^q \omega(x)^{-q}\,\mathrm{d}x\\ &\qquad\le \int_\Omega \bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|}\omega(x)^{-1}C_2\omega(x)\int_{W_t} |g(y)|\omega(y)^{-1}\,\mathrm{d}y\bigg|^q \mathrm{d}x\\ &\qquad= C_2^q\int_\Omega \bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|}\int_{W_t} |g(y)|\omega(y)^{-1}\,\mathrm{d}y\bigg|^q \mathrm{d}x = C_2^q\int_\Omega |T(g\omega^{-1})|^q\,\mathrm{d}x.\end{aligned}$$

Finally, $g\omega^{-1}$ belongs to $L^q(\Omega)$ and T is continuous from $L^q(\Omega)$ to itself; we refer to Lemma 2.6 to conclude that

$$\int_\Omega |Tg(x)|^q\omega(x)^{-q}\,\mathrm{d}x \le 2^q\frac{qN}{q-1}C_2^q\|g\|^q_{L^q(\Omega,\omega^{-q})}. \ \blacksquare$$

3. A decomposition and fractional Poincaré inequalities. Let Ω in $\mathbb{R}^n$ be an arbitrary bounded domain and $\{U_t\}_{t\in\Gamma}$ an open covering of Ω. The weight $\omega : \Omega \to \mathbb{R}_{>0}$ satisfies $\omega^p \in L^1(\Omega)$. In addition, u_Ω denotes the average $|\Omega|^{-1}\int_\Omega u(z)\,\mathrm{d}z$. For weighted spaces of functions, $u_{\Omega,\omega}$ represents the weighted average $(\omega(\Omega))^{-1}\int_\Omega u(z)\omega(z)\,\mathrm{d}z$, where $\omega(\Omega) := \int_\Omega \omega(z)\,\mathrm{d}z$.

Now, given a bounded domain U in $\mathbb{R}^n$ and a nonnegative measurable function $\mu : U \times U \to \mathbb{R}$ we introduce the Poincaré-type inequality

$$\inf_{c\in\mathbb{R}} \|u-c\|_{L^p(U,\omega^p)} \le C\Big(\int_U\int_U |u(x)-u(y)|^p\mu(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}, \tag{3.1}$$

where $u \in L^p(U,\omega^p)$. Notice that the right hand side might be infinite. The validity of (3.1) depends on U, p, μ and ω. The function $\mu(x,y)$ might be zero, but $\omega(x)$ is strictly positive almost everywhere in Ω.

Let us mention three examples.

EXAMPLES 3.1. (1) The unweighted fractional Poincaré inequality with $\mu(x,y) = 1/|x-y|^{n+sp}$, where $s \in (0,1)$, is the classical fractional Poincaré inequality, which is clearly valid for any bounded domain.

(2) If $\mu(x,y) = \chi_{B_x}(y)/|x-y|^{n+sp}$, where B_x is the ball centered at x with radius $\tau d(x)$ for $s,\tau \in (0,1)$, then the inequality represents a more

recently studied fractional Poincaré inequality whose validity depends on the geometry of the domain (refer to [HV1] for details).

(3) Finally, $\mu(x,y) = \rho(|x-y|)/|x-y|^p$, where ρ is a certain nonnegative radial function, yields another inequality which has also been studied recently (refer to [P1] for details).

Inequality (3.1) deals with an estimation of the distance to $\mathcal{C}$ of an arbitrary function u in $L^p(\Omega,\omega^p)$. The local-to-global argument used in this paper to study Poincaré-type inequalities is based on the fact that $L^p(\Omega,\omega^p)$ is the dual space of $L^q(\Omega,\omega^{-q})$ and on the existence of decompositions of functions in $L^q(\Omega,\omega^{-q})$ orthogonal to $\mathcal{C}$. Let us properly define this set and a subspace:

$$\mathcal{W} := \left\{ g \in L^q(\Omega,\omega^{-q}) : \int g\varphi = 0 \text{ for all } \varphi \in \mathcal{C} \right\}, \tag{3.2}$$

$$\mathcal{W}_0 := \{ g \in \mathcal{W} : \operatorname{supp}(g) \text{ intersects a finite number of } U_t \}. \tag{3.3}$$

The integrability of ω^p implies that $L^q(\Omega,\omega^{-q}) \subset L^1(\Omega)$; then $\mathcal{W}$ and $\mathcal{W}_0$ are well-defined. For a similiar condition we refer to [KO]. Following Remark 2.5, the $\mathcal{C}$-decomposition of functions in $\mathcal{W}_0$ stated in Theorem 2.4 is finite, which is not valid in general for functions in $\mathcal{W}$. This property satisfied by functions in $\mathcal{W}_0$ simplifies the proof of Lemma 3.3, which motivates the definition of this space.

Now, we introduce the spaces

$$\begin{aligned} \mathcal{W} \oplus \omega^p\mathcal{C} &= \{ g + \alpha\omega^p : g \in \mathcal{W} \text{ and } \alpha \in \mathcal{C} \}, \\ \mathcal{S} := \mathcal{W}_0 \oplus \omega^p\mathcal{C} &= \{ g + \alpha\omega^p : g \in \mathcal{W}_0 \text{ and } \alpha \in \mathcal{C} \}. \end{aligned} \tag{3.4}$$

It is not difficult to observe that $L^q(\Omega,\omega^{-q}) = \mathcal{W} \oplus \omega^p\mathcal{C}$ and $\mathcal{S}$ is a subspace of $L^q(\Omega,\omega^{-q})$. The following lemma, proved in [L2, Lemma 3.1], states that $\mathcal{S}$ is also dense in $L^q(\Omega,\omega^{-q})$, and uses in its proof the requirement that for each $t \in \Gamma$ the set U_t intersects a finite number of U_s with $s \in \Gamma$.

LEMMA 3.2. *Suppose that $\omega^p \in L^1(\Omega)$. The space $\mathcal{S}$ is dense in $L^q(\Omega,\omega^{-q})$. Moreover, if $g + \alpha\omega^p$ is an element in $\mathcal{S}$, then*

$$\|g\|_{L^q(\Omega,\omega^{-q})} \le 2\|g + \alpha\omega^p\|_{L^q(\Omega,\omega^{-q})}.$$

LEMMA 3.3. *Suppose that $\omega^p \in L^1(\Omega)$. If there exists an open covering $\{U_t\}_{t\in\Gamma}$ of Ω such that (3.1) is valid on U_t for all $t \in \Gamma$, with a uniform constant C_1, and there exists a finite $\mathcal{C}$-orthogonal decomposition of any function g in $\mathcal{W}_0$ subordinate to $\{U_t\}_{t\in\Gamma}$, with the estimate*

$$\sum_{t\in\Gamma} \|g_t\|^q_{L^q(U_t,\omega^{-q})} \le C_0^q \|g\|^q_{L^q(\Omega,\omega^{-q})},$$

then there exists a constant C such that

$$\|u-u_{\Omega,\omega^p}\|_{L^p(\Omega,\omega^p)} \le C\Big(\sum_{t\in\Gamma}\int_{U_t}\int_{U_t}|u(x)-u(y)|^p\mu(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \tag{3.5}$$

for any $u\in L^p(\Omega,\omega^p)$. Moreover, $C=2C_0C_1$ works in (3.5).

Proof. Without loss of generality we can assume that $u_{\Omega,\omega^p}=0$. We use Lemma 3.2 to estimate the norm on the left hand side of (3.5) by duality. Thus, let $g+\omega^p\psi$ be an arbitrary function in $\mathcal{S}$. Then, by using the finite $\mathcal{C}$-orthogonal decomposition of g, we conclude that

$$\begin{aligned}\int_\Omega u(g+\alpha\omega^p) &= \int_\Omega ug = \int_\Omega u\sum_{t\in\Gamma} g_t\\ &= \sum_{t\in\Gamma}\int_{U_t} ug_t = \sum_{t\in\Gamma}\int_{U_t}(u-c_t)g_t.\end{aligned}\tag{3.6}$$

Notice that the identity in the second line is valid for any $t\in\Gamma$ and $c_t\in\mathbb{R}$.

Next, by using the Hölder inequality for (3.6), the fact that (3.1) is valid on U_t with a uniform constant C_1, and finally the Hölder inequality over the sum, we obtain

$$\begin{aligned}\int_\Omega u(g+\alpha\omega^p) &\le \sum_{t\in\Gamma}\inf_{c\in\mathbb{R}}\|u-c\|_{L^p(U_t,\omega^p)}\|g_t\|_{L^q(U_t,\omega^{-q})}\\
&\le C_1\sum_{t\in\Gamma}\Big(\int_{U_t}\int_{U_t}|u(x)-u(y)|^p\mu(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}\|g_t\|_{L^q(U_t,\omega^{-q})}\\
&\le C_1\Big(\sum_{t\in\Gamma}\int_{U_t}\int_{U_t}|u(x)-u(y)|^p\mu(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}\Big(\sum_{t\in\Gamma}\|g_t\|^q_{L^q(U_t,\omega^{-q})}\Big)^{1/q}\\
&\le C_0C_1\Big(\sum_{t\in\Gamma}\int_{U_t}\int_{U_t}|u(x)-u(y)|^p\mu(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}\|g\|_{L^q(U,\omega^{-q})}\\
&\le 2C_0C_1\Big(\sum_{t\in\Gamma}\int_{U_t}\int_{U_t}|u(x)-u(y)|^p\mu(x,y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}\|g+\alpha\omega^p\|_{L^q(U,\omega^{-q})}.\end{aligned}$$

Finally, as $\mathcal{S}$ is dense in $L^q(\Omega,\omega^{-q})$, by taking the supremum over all the functions $g+\alpha\omega^p$ in $\mathcal{S}$ with $\|g+\alpha\omega^p\|_{L^q(\Omega,\omega^{-q})}\le 1$ we get the result. ∎

4. On fractional Poincaré inequalities on cubes. In this section, we use the results stated in the previous two sections to show a certain fractional Poincaré inequality on an arbitrary cube Q. Thus, in order to show the existence of the $\mathcal{C}$-decomposition, which is used later to apply Lemma 3.3, we define a tree covering $\{U_t\}_{t\in\Gamma}$ of Q. This covering is only used in this section and for cubes. In the following section, we work with a different bounded domain, an arbitrary bounded John domain, which requires a different covering. However, let us warn the reader that we will keep the notation $\{U_t\}_{t\in\Gamma}$ used in Section 3.

The local inequality stated in the following proposition is well-known. We refer the reader to [DD] for its proof.

Proposition 4.1. *The fractional Poincaré inequality*

$$\inf_{c\in\mathbb{R}} \|u(x)-c\|_{L^p(U)} \le \left(\frac{\operatorname{diam}(U)^{n+sp}}{|U|}\int_U\int_U \frac{|u(y)-u(x)|^p}{|y-x|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p}$$

holds for any bounded domain U in $\mathbb{R}^n$ and $1\le p<\infty$.

The following proposition is a special case of [HV1, Lemma 2.2]. In the present paper, we give a different proof which lets us estimate the dependence of the constant with respect to τ.

Proposition 4.2. *Let Q in $\mathbb{R}^n$ be a cube with side length $l(Q)=L$, let $1<p<\infty$ and $\tau\in(0,1)$. Then*

$$\inf_{c\in\mathbb{R}} \|u(x)-c\|_{L^p(Q)} \le C_{n,p}\,\tau^{s-n}L^s\left(\int_Q\int_{Q\cap B(x,\tau L)} \frac{|u(y)-u(x)|^p}{|y-x|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p},$$

where $C_{n,p}$ depends only on n and p.

Proof. This result follows from Lemma 3.3 applied to Q, where $\mu(x,y)=1/|x-y|^{n+sp}$ and $\omega\equiv 1$. So, let us start by defining an appropriate tree covering of Q to obtain, via Theorem 2.4 and Remark 2.5, a finite $\mathcal{C}$-decomposition of any function in $\mathcal{W}_0$. Let $m\in\mathbb{N}$ be such that $\sqrt{n+3}/\tau<m\le 1+\sqrt{n+3}/\tau$ and $\{A_t\}_{t\in\Gamma}$ the regular partition of Q with m^n open cubes. The side length of each cube is $l(A_t)=L/m$. In the example shown in Figure 1, $n=2$, $m=4$, and the index set Γ has 16 elements.

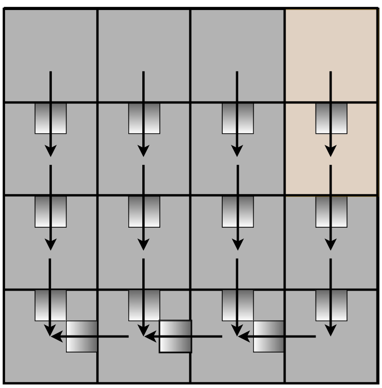

Fig. 1. A tree covering of Q

The tree covering of Q that we are looking for will be defined by enlarging the sets in the covering $\{A_t\}_{t\in\Gamma}$ in an appropriate way but keeping the tree structure of Γ, which is introduced in the following lines. Indeed, we pick a cube A_a, whose index will be the root, and inductively define a tree structure in Γ such that the unique chain connecting t to a is associated to a chain of cubes connecting Q_t to Q_a, with minimal number of cubes, such that two consecutive cubes share an $n-1$-dimensional face. In Figure 1, the cube A_a is in the lower left corner and the tree structure is represented by black arrows that "descend" to the root. Now that Γ has a tree structure, we define the tree covering $\{U_t\}_{t\in\Gamma}$ of Q with the rectangles $U_t := (\overline{A}_t \cup \overline{A}_{t_p})^\circ$ if $t \neq a$ and $U_a := A_a$. (Here B° means the interior of B.) In order to have a better understanding of the construction, notice that $U_t \cap U_{t_p} = A_{t_p}$ for all $t \neq a$. Moreover, the index set Γ in the example with its tree structure has seven levels, from level 0 to level 6 (refer to page 4 for definitions), with only one index of level 6, whose rectangle U_t appears in Figure 1 in a different color.

Now, let us define the collection $\{B_t\}_{t\neq a}$ of pairwise disjoint open cubes $B_t \subseteq U_t \cap U_{t_p}$ or equivalently $B_t \subseteq A_{t_p}$. Given $t \neq a$, we split A_{t_p} into 3^n cubes with the same size. The open set B_t is the cube in the regular partition of A_{t_p} whose closure intersects the $n-1$-dimensional face A_{t_p} in $\overline{A}_t \cap \overline{A}_{t_p}$. There are 3^{n-1} cubes with that property but we pick B_t to be the one which does not share any part with any other $n-1$-dimensional face of $\overline{A}_{t_p}$.

The cubes in $\{B_t\}_{t\neq a}$ have side length equal to $L/(3m)$ and are represented in Figure 1 by the 15 grey gradient small cubes. By construction, it is easy to check that $\{B_t\}_{t\neq a}$ is a collection of pairwise disjoint open cubes $B_t \subseteq U_t \cap U_{t_p}$, hence $\{U_t\}_{t\in\Gamma}$ is a tree covering of Q with $N = 2n$ (it could also be less).

By Theorem 2.4, there is a finite $\mathcal{C}$-decomposition $\{g_t\}_{t\in\Gamma}$ of $g = \sum_{t\in\Gamma} g_t$ subordinate to $\{U_t\}_{t\in\Gamma}$ which satisfies (2.3) and (2.4). Moreover, it can be seen that

$$\frac{|W_s|}{|B_s|} \leq \frac{|Q|}{|B_s|} = (3m)^n$$

for all $s \in \Gamma$, thus

$$|g_t(x)| \leq |g(x)| + (3m)^n Tg(x)$$

for all $t \in \Gamma$ and $x \in U_t$. Next, using the continuity of T stated in Lemma 2.6 and some straightforward calculations we conclude

$$\begin{aligned}\sum_{t\in\Gamma} \|g_t\|^q_{L^q(U_t)} &\leq 2^{q-1} N \left(1 + (3m)^{nq} 2^q \frac{qN}{q-1}\right) \|g\|^q_{L^q(Q)} \\ &\leq \frac{2^{2q+2} n^2 q}{q-1}\,(3m)^{nq} \|g\|^q_{L^q(Q)} \\ &\leq \frac{2^{2q+2} 3^{nq} n^2 q}{q-1} (1+\sqrt{n+3})^{nq}\,\tau^{-nq} \|g\|^q_{L^q(Q)}.\end{aligned}$$

Hence, we have a finite $\mathcal{C}$-decomposition of any function in $\mathcal{W}_0$ subordinate to $\{U_t\}_{t\in\Gamma}$ with the constant in the estimate equal to

$$C_0 = \left(\frac{2^{2q+2}3^{nq}n^2 q}{q-1}\right)^{1/q}(1+\sqrt{n+3})^n\tau^{-n}.$$

Now, from Proposition 4.1 and using the fact that $m > \sqrt{n+3}/\tau$ and $\operatorname{diam}(U_t) \le \sqrt{n+3}\,L/m \le \tau L$, we can conclude that inequality (3.1) is valid on each U_t with a uniform constant

$$C_1 = (n+3)^{n/(2p)}(\tau L)^s.$$

Thus, using Lemma 3.3 we can see that

$$\|u-u_Q\|_{L^p(Q)} \le 2C_0C_1\left(\sum_{t\in\Gamma}\int_{U_t}\int_{U_t}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p}.$$

Since $\operatorname{diam}(U_t) \le \tau L$, we have $U_t \subset B(x,\tau L)$ for any $x \in U_t$; thus, using the control on the overlapping of the tree covering given by $N = 2n$, we find that

$$\|u-u_Q\|_{L^p(Q)} \le C_{n,p}\,\tau^{-n}(\tau L)^s\left(\int_Q\int_{Q\cap B(x,\tau L)}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p},$$

where

$$C_{n,p} = 2\left(\frac{2^{2q+2}3^{nq}n^2 q}{q-1}\right)^{1/q}(1+\sqrt{n+3})^n(n+3)^{n/(2p)}(2n)^{1/p}. \tag{4.1}$$ ∎

5. On fractional Poincaré inequalities on John domains. In this section, we apply the results obtained in the previous sections to an arbitrary bounded John domain Ω. Its definition is recalled below. The weight $\omega(x)$ is defined as $d_F(x)^\beta$, where $d_F(x)$ denotes the distance from x to an arbitrary compact set F in $\partial\Omega$ and $\beta \ge 0$. In the particular case where $F = \partial\Omega$, $d_{\partial\Omega}(x)$ is simply denoted as $d(x)$. Notice that ω^p belongs to $L^1(\Omega)$ for Ω being bounded and β nonnegative.

A *Whitney decomposition* of Ω is a collection $\{Q_t\}_{t\in\Gamma}$ of closed pairwise disjoint dyadic cubes which satisfies:

1. $\Omega = \bigcup_{t\in\Gamma} Q_t$.
2. $\operatorname{diam}(Q_t) \le \operatorname{dist}(Q_t,\partial\Omega) \le 4\operatorname{diam}(Q_t)$.
3. $\frac{1}{4}\operatorname{diam}(Q_s) \le \operatorname{diam}(Q_t) \le 4\operatorname{diam}(Q_s)$ if $Q_s \cap Q_t \ne \emptyset$.

Here, $\operatorname{dist}(Q_t,\partial\Omega)$ is the Euclidean distance between Q_t and the boundary of Ω, denoted by $\partial\Omega$. The diameter of the cube Q_t is denoted by $\operatorname{diam}(Q_t)$ and the side length is written as $\ell(Q_t)$.

Two different cubes Q_s and Q_t with $Q_s \cap Q_t \ne \emptyset$ are called *neighbors*. This kind of covering exists for any proper open set in $\mathbb{R}^n$ (refer to [S, VI.1] for details). Moreover, each cube Q_t has $\leq 12^n$ neighbors. And if we fix $0 < \epsilon < 1/4$ and define $(1+\epsilon)Q_t$ as the cube with the same center as Q_t and side length $(1+\epsilon)\ell(Q_t)$, then $(1+\epsilon)Q_t$ touches $(1+\epsilon)Q_s$ if and only if Q_t and Q_s are neighbors.

Given a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ of Ω, an *expanded Whitney decomposition* of Ω is the collection $\{Q_t^*\}_{t\in\Gamma}$ of open cubes defined by

$$Q_t^* := \tfrac{9}{8}Q_t^\circ.$$

Observe that this collection of cubes satisfies

$$\chi_\Omega(x) \leq 12^n \sum_{t\in\Gamma} \chi_{Q_t^*}(x) \leq (12^n)^2 \chi_\Omega(x)$$

for all $x \in \mathbb{R}^n$.

We recall the definition of a *bounded John domain.* A bounded domain Ω in $\mathbb{R}^n$ is a John domain with constants a and b, $0 < a \leq b < \infty$, if there is a point x_0 in Ω such that for each point x in Ω there exists a rectifiable curve γ_x in Ω, parametrized by its arc length written as $\mathrm{length}(\gamma_x)$, such that

$$\mathrm{dist}(\gamma_x(t), \partial\Omega) \geq \frac{a}{\mathrm{length}(\gamma_x)} t \qquad \text{for all } t \in [0, \mathrm{length}(\gamma_x)]$$

and

$$\mathrm{length}(\gamma_x) \leq b.$$

Examples of John domains are convex domains, uniform domains, and also domains with slits, for example $B^2(0,1) \setminus [0,1)$. The John property fails in domains with zero angle outward spikes. John domains were introduced by Fritz John [J]; they were later named John domains by O. Martio and J. Sarvas.

There are other equivalent definitions of John domains. In these notes, we are interested in a definition in the style of Boman chain condition (see [BKL]) in terms of Whitney decompositions and trees. This equivalent definition is introduced in [L2].

Definition 5.1. A bounded domain Ω in $\mathbb{R}^n$ is a *John domain* if for any Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ there exists a constant $K > 1$ and a tree structure of Γ, with a root a, that satisfies

$$Q_s \subseteq KQ_t \tag{5.1}$$

for any $s, t \in \Gamma$ with $s \succeq t$. In other words, the shadow of Q_t written as W_t is contained in KQ_t (see (2.2)). Moreover, the intersection of cubes associated to adjacent indices, Q_t and Q_{t_p}, is an $n-1$-dimensional face of one of these cubes.

Now, given a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ of a bounded John domain Ω in $\mathbb{R}^n$, with constant K in the sense of (5.1), we define the tree covering

$\{U_t\}_{t\in\Gamma}$ of expanded Whitney cubes by

$$U_t := Q_t^*. \tag{5.2}$$

The overlapping is bounded by $N = 12^n$. Now, each open cube B_t in the collection $\{B_t\}_{t\neq a}$ shares the center with the $n-1$-dimensional face $Q_t \cap Q_{t_p}$ and has side length $l_t/64$, where l_t is the side length of Q_t. It follows from property (3) of the Whitney decomposition, and some calculations, that this collection is pairwise disjoint and

$$B_t \subset Q_t^* \cap Q_{t_p}^* = U_t \cap U_{t_p}.$$

Moreover, it can be seen that

$$\frac{|W_t|}{|B_t|} \leq \frac{\left(K\frac{9}{8}l_t\right)^n}{(l_t/64)^n} = 72^n K^n \tag{5.3}$$

for all $t \in \Gamma$ with $t \neq a$.

Lemma 5.2. *Let Ω in $\mathbb{R}^n$ be a John domain with constant K in the sense of* (5.1)*, F in $\partial\Omega$ a compact set and $d_F(x)$ the distance from x to F. Then*

$$\sup_{y\in W_t} d_F(y) \leq 3K\sqrt{n} \inf_{x\in B_t} d_F(x) \quad \textit{for all } t \in \Gamma.$$

A similar inequality is also valid if we consider the weight $d_F(x)^\beta$ with a nonnegative power of the distance to F. Thus, this lemma implies, via Lemma 2.7, the continuity of the operator T from $L^q(\Omega, d_F^{-q\beta})$ to itself with an estimation of its constant. Then, there exists a $\mathcal{C}$-decomposition with a weighted estimate for a certain weight.

Proof of Lemma 5.2. Given $t \in \Gamma$ with $t \neq a$, $x \in B_t$ and $y \in W_t := \bigcup_{s\succeq t} U_s$, we have to prove that $d_F(y) \leq 3K d_F(x)$. Notice that $d(x) \leq d_F(x)$ for all $x \in \Omega$. Moreover, $Q_s \subseteq KQ_t$ for all $s \succeq t$, then $W_t \subseteq KU_t$. In addition,

$$\begin{aligned} d_F(y) &\leq |y-x| + d_F(x) \leq \operatorname{diam}(W_t) + d_F(x) \\ &\leq K \operatorname{diam}(U_t) + d_F(x) = K\frac{9}{8} \operatorname{diam}(Q_t) + d_F(x). \end{aligned}$$

Finally, using property (2) of the Whitney decomposition we deduce that $3Q_t \subset \Omega$. Then, as

$$\operatorname{dist}(Q_t^*, \partial\Omega) \geq \operatorname{dist}(Q_t^*, (3Q_t)^c) \geq \tfrac{15}{16} l_t,$$

some calculations yield

$$\operatorname{diam}(Q_t) \leq \tfrac{16}{15}\sqrt{n} \operatorname{dist}(Q_t^*, \partial\Omega) \leq \tfrac{16}{9}\sqrt{n} \operatorname{dist}(Q_t^*, \partial\Omega).$$

Thus,

$$\begin{aligned} d_F(y) &\le 2K\sqrt{n}\,\mathrm{dist}(Q_t^*, \partial\Omega) + d_F(x) \\ &\le 2K\sqrt{n}\,d(x) + d_F(x) \le 2K\sqrt{n}\,d_F(x) + d_F(x). \ \blacksquare \end{aligned}$$

Now we are able to prove Theorem 1.1 and also to give the dependence of the constant C on the given value of τ and on the constant K from (5.1).

Proof of Theorem 1.1. This result follows from Lemma 3.3 with the tree covering $\{U_t\}_{t\in\Gamma}$ of Ω defined in (5.2), $\omega(x) := d_F(x)^\beta$ and

$$\mu(x,y) := \frac{d(x)^{ps} d_F(x)^{p\beta} \chi_{B(x,\tau d(x))}(y)}{|x-y|^{n+sp}}. \tag{5.4}$$

Notice that ω^p belongs to $L^1(\Omega)$, as assumed at the beginning of Section 3. The validity of (3.1) on a cube U_t, with a uniform constant C_1, follows from Proposition 4.2. Indeed, by using the fact that U_t is an expanded Whitney cube by a factor $9/8$ and $F \subseteq \partial\Omega$, we deduce that

$$\sup_{x\in U_t} d_F(x)^\beta \le 2^\beta \inf_{x\in U_t} d_F(x)^\beta.$$

Thus,

$$\begin{aligned} &\inf_{c\in\mathbb{R}} \|u(x)-c\|_{L^p(U_t, d_F^{p\beta})} \\ &\quad\le C_{n,p}\tau^{s-n} L_t^s\, 2^\beta \bigg(\int\limits_{U_t}\int\limits_{U_t} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} d_F(x)^{p\beta} \chi_{B(x,\tau L_t)}(y)\,\mathrm{d}y\,\mathrm{d}x \bigg)^{1/p}, \end{aligned}$$

where L_t is the side length of U_t and $C_{n,p}$ is the constant in (4.1).

Observe that $L_t \le d(x)$ for all $x \in U_t$. Indeed, if $x \in Q_t$, then

$$L_t = \tfrac{9}{8} l_t < \sqrt{n}\, l_t = \mathrm{diam}(Q_t) \le \mathrm{dist}(Q_t, \partial\Omega) \le d(x),$$

where l_t is the side length of Q_t. Now, if $x \in U_t \setminus Q_t$, then

$$\sqrt{n}\, l_t \le \mathrm{dist}(Q_t, \partial\Omega) \le \mathrm{dist}(U_t, \partial\Omega) + \tfrac{1}{16}\sqrt{n}\, l_t,$$

hence $\frac{15}{16}\sqrt{n}\, l_t \le \mathrm{dist}(U_t, \partial\Omega)$ and

$$L_t = \tfrac{9}{8} l_t < \tfrac{15}{16}\sqrt{n}\, l_t \le \mathrm{dist}(U_t, \partial\Omega) \le d(x).$$

Then the validity of $L_t \le d(x)$ for all $x \in U_t$ implies (3.1) for all U_t, where $\mu(x,y)$ is the function defined in (5.4), and with the uniform constant

$$C_1 = C_{n,p}\tau^{s-n} 2^\beta, \tag{5.5}$$

where $C_{n,p}$ is as in (4.1).

Next, by Theorem 2.4, there is a finite $\mathcal{C}$-decomposition $\{g_t\}_{t\in\Gamma}$ subordinate to $\{U_t\}_{t\in\Gamma}$ for any function g in $\mathcal{W}_0$ which satisfies (2.3) and (2.4).

Moreover, using (5.3), it can be seen that

$$|g_t(x)| \le |g(x)| + (72K)^n Tg(x)$$

for all $t \in \Gamma$ and $x \in U_t$.

Now, $\omega(x) := d_F(x)^\beta$ fulfills the hypothesis of Lemma 2.7 where the constant in (2.5) is $C_2 = (3K\sqrt{n})^\beta$ (this assertion uses Lemma 5.2). Thus, the operator T is continuous from $L := L^q(\Omega, d_F^{-q\beta})$ to itself with the norm

$$\|T\|_{L\to L} \le 2\left(\frac{qN}{q-1}\right)^{1/q}(3K\sqrt{n})^\beta.$$

Hence,

$$\begin{aligned}
\sum_{t\in\Gamma}\|g_t\|^q_{L^q(U_t,d_F^{-q\beta})} &\le 2^{q-1}\Big\{\Big(\sum_{t\in\Gamma}\int_{U_t}|g(x)|^q\, d_F(x)^{-q\beta}\,\mathrm{d}x\Big) \\
&\qquad + (72K)^{qn}\Big(\sum_{t\in\Gamma}\int_{U_t}|Tg(x)|^q\, d_F(x)^{-q\beta}\,\mathrm{d}x\Big)\Big\} \\
&\le 2^{q-1}N\Big\{\int_\Omega |g(x)|^q\, d_F(x)^{-q\beta}\,\mathrm{d}x + (72K)^{qn}\int_\Omega |Tg(x)|^q\, d_F(x)^{-q\beta}\,\mathrm{d}x\Big\} \\
&\le 2^{q-1}N\Big\{1 + (72K)^{qn}2^q\frac{qN}{q-1}(3K\sqrt{n})^{q\beta}\Big\}\|g\|^q_{L^q(\Omega,d_F^{-q\beta})} \\
&\le 4^q N^2 (72K)^{qn}\frac{q}{q-1}(3K\sqrt{n})^{q\beta}\|g\|^q_{L^q(\Omega,d_F^{-q\beta})} \\
&= 4^q\, 12^{2n}\, 72^{qn}\,(3\sqrt{n})^{q\beta}\frac{q}{q-1}K^{q(n+\beta)}\|g\|^q_{L^q(\Omega,d_F^{-q\beta})}.
\end{aligned}$$

Therefore, we have a $\mathcal{C}$-decomposition subordinate to $\{U_t\}_{t\in\Gamma}$ with constant

$$C_0 = 4(12)^{2n/q}(72)^n(3\sqrt{n})^\beta\left(\frac{q}{q-1}\right)^{1/q}K^{n+\beta}. \tag{5.6}$$

Finally, inequality (3.5) and control on the overlapping of the tree covering by $N = 12^n$ imply (1.2). ■

Remark 5.3. Notice that the proof of Theorem 1.1 provides an explicit constant $C = 2C_0C_1$ for inequality (1.2), where C_0 and C_1 are given respectively in (5.6) and (5.5).

The next result, similar to Proposition 4.1, follows from the Hölder inequality (equivalently, from Minkowski's integral inequality).

Proposition 5.4. *Let $\rho : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be a positive radial Lebesgue measurable function which is increasing with respect to the radius. Then the* *fractional Poincaré-type inequality*

$$\|u(x)-u_U\|_{L^p(U)} \leq \frac{\operatorname{diam}(U)^{n/p}\rho(\operatorname{diam}(U))}{|U|^{1/p}}\left(\int_U\int_U \frac{|u(y)-u(x)|^p}{|y-x|^n\rho(|y-x|)^p}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p} \tag{5.7}$$

holds for any bounded domain U in $\mathbb{R}^n$ and $1<p<\infty$, where $u_U := |U|^{-1}\int_U u(y)\,\mathrm{d}y$.

Proof. We compute

$$\begin{aligned}\int_U |u(x)-u_U|^p\,\mathrm{d}x &= \int_U\left|\frac{1}{|U|}\int_U u(x)-u(y)\,\mathrm{d}y\right|^p\mathrm{d}x \leq \frac{1}{|U|}\int_U\int_U |u(x)-u(y)|^p\,\mathrm{d}y\,\mathrm{d}x\\ &\leq \frac{\operatorname{diam}(U)^n\rho(\operatorname{diam}(U))^p}{|U|}\int_U\int_U \frac{|u(x)-u(y)|^p}{|x-y|^n\rho(|x-y|)^p}\,\mathrm{d}y\,\mathrm{d}x. \qquad \blacksquare\end{aligned}$$

REMARK 5.5. If $\rho(x)=|x|^s$ with $s\in(0,1)$, we recover the classical fractional Poincaré inequality.

We generalize the fractional Poincaré inequality stated in Theorem 1.1 by replacing the fractional derivatives given by the power functions $|x|^s$ with $0<s<1$ by general increasing and positive radial functions $\rho|x|$.

THEOREM 5.6. *Let Ω in $\mathbb{R}^n$ be a bounded John domain and $1<p<\infty$. Given an arbitrary compact set F in $\partial\Omega$, a parameter $\beta\geq 0$ and a positive radial Lebesgue measurable function $\rho:\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{R}$ increasing with respect to the radius, there exists a constant C such that*

$$\left(\int_\Omega |u(x)-u_{\Omega,\omega^p}|^p d_F(x)^{p\beta}\,\mathrm{d}x\right)^{1/p} \leq C\left(\int_\Omega\int_{\Omega\cap B(x,d(x))}\frac{|u(x)-u(y)|^p}{|x-y|^n(\rho|x-y|)^p}\rho(2d(x))^p d_F(x)^{p\beta}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p} \tag{5.8}$$

for all $u\in L^p(\Omega,d(x)^{p\beta})$. Here $d(x)$ and $d_F(x)$ are the distances from x to $\partial\Omega$ and F respectively, and u_{Ω,ω^p} is the weighted average

$$d_F(\Omega)^{-p\beta}\int_\Omega u(z)d_F(z)^{p\beta}\,\mathrm{d}z.$$

In addition, the constant C in (5.8) *can be written as*

$$C = C_{n,p,\beta}\,K^{n+\beta},$$

where K is the geometric constant introduced in (5.1).

Proof. This proof mimics the one of Theorem 1.1 with Proposition 5.4 instead of Proposition 4.2. Indeed, we will use again the tree covering $\{U_t\}_{t\in\Gamma}$ of Ω defined in (5.2) and the weight $\omega(x) = d_F(x)^\beta$, but, in this case $\mu(x,y)$ is defined as

$$\mu(x,y) := \frac{\rho(2d(x))^p d_F(x)^{p\beta}}{|x-y|^n \rho(|x-y|)^p}.$$

We only have to show that (3.1) is satisfied on U_t for all t, with a uniform constant. This follows from (5.7) by using the inequality $\mathrm{diam}(U_t) \le 2d(x)$ for all $x \in U_t$. ∎

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