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Numerical solution of fractional differential equations

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Chapter 1

Introduction

In the present thesis we investigate theoretical and computational aspects of piecewise polynomial collocation methods for the numerical solution of fractional differential and integro-differential equations containing Caputo type fractional (non-integer) order derivatives of the unknown function.

The concept of a fractional derivative can be traced back [20, 77] to the end of the seventeenth century, the time when Newton and Leibniz developed the foundations of differential and integral calculus. In particular, Leibniz introduced the symbol

$$\frac{d^n}{dt^n} f(t)$$

to denote the n-th order derivative of a function $f = f(t)$. When he reported this in a letter to de L'Hospital (apparently with the implicit assumption that n is a non-negative integer), de L'Hospital replied: "What does $\frac{d^n}{dt^n} f(t)$ mean if $n = 1/2$?". To this question Leibniz had no satisfactory answer. In his reply, dated September 30, 1695, Leibniz wrote to de L'Hospital that "... this is an apparent paradox from which, one day, useful consequences will be drawn..." [45]. The letter from de L'Hospital is nowadays commonly accepted as the first occurrence of what we today call a fractional derivative, and the fact that de L'Hospital specifically asked for $n = 1/2$, which is a fraction (a rational number), gave rise to the name " fractional derivative". This name has remained in use, even if n is an arbitrary positive rational or irrational number, that is, $n \in \mathbb{R} := (-\infty, \infty)$, $n > 0$. As a matter of fact, even complex numbers may be allowed [45], but this is beyond the scope of this thesis.

The question raised by de L'Hospital motivated many scientists to search for a possibility to generalize the concept of integer order derivatives to fractional order derivatives. However, for a long time (nearly three centuries), considerations regarding fractional derivatives were purely theoretical treatments for which there were no serious practical applications. Therefore the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. In contrast to this, during the last decades the attitude has cardinally changed. It turns out that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. As a matter of fact, there has been a spectacular increase of studies regarding fractional derivatives and differential equations with such derivatives, mainly because of new applications of fractional derivatives in physics, chemistry, mechanics, electricity, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, etc.

In particular, some early examples are given in the works [61] (diffusion processes), [14, 15, 87] (modelling of the mechanical properties of materials), [57] (signal processing), [26, 27, 37] (modelling the behaviour of viscoelastic materials), [36, 55] (bioengineering), [40] (description of mechanical systems subject to damping), [41, 58] (kinetics of polymers). Some more recent results are described in the works [4, 80] (modelling the behaviour of humans), [31] (fractional processes in financial economics), [85] (atomic wall dynamics), [98] (viscoelastic laws for arterial wall mechanics), [48] (models of supercapacitor energy storage), [42] (transition of flow in fluid dynamics). A reader interested in additional applications and studies on fractional derivatives and fractional differential equations may consult the monographs [8, 19, 20, 45, 56, 59, 76, 79] and review papers [54, 78, 84].

Currently we know (see, for example, [20, 45, 91]) that there are many possible different generalizations of the concept of $\frac{d^n}{dt^n} f(t)$ to the case $n \notin \mathbb{N} := \{1, 2, \dots\}$. That is, there are many definitions for fractional derivatives, which are not always equivalent to each other. The two most frequently used fractional derivatives of order $n = \alpha > 0$ are defined by Riemann-Liouville and Caputo fractional differential operators D_{RL}^α and D_{Cap}^α, respectively (see Chapter 2 for their precise definitions). The former concept is historically the first (developed by Riemann and Liouville in the middle of 19th century) and for which the mathematical theory

has been by now established quite well (see [20, 45, 76, 79]). However, as pointed out in [20], Riemann-Liouville derivatives have certain disadvantages when trying to model some "real-world" phenomena. For example, when a real-world situation is modelled by an initial value problem involving Riemann-Liouville fractional derivatives, then one has to specify the values of certain Riemann-Liouville fractional derivatives at the initial point. In practical applications these values are frequently not available and their physical meaning might not be clear [21, 23, 26]. Moreover, rather non-natural is also the fact that the Riemann-Liouville derivative $D^\alpha_{RL}c$ ($\alpha \notin \mathbb{N}$) of a non-zero constant c does not vanish (see Section 2.5 below). The Caputo fractional derivative (introduced by Caputo in [14]), despite being closely related to the Riemann-Liouville derivative, does not exhibit the above-mentioned difficulties [20]. In the present thesis we will follow Caputo's approach.

When working with problems stemming from real-world applications, it is only rarely possible to find the solution of a given fractional differential equation in closed form, and even if such an analytic solution is available, it is typically too complicated to be used in practice [8]. Therefore, in general, numerical methods are required for solving fractional differential equations. As a consequence, the last decades have witnessed a steadily increasing development and analysis of numerical methods for fractional differential equations, of which a good deal are concerned with the numerical solution of initial and boundary value problems with one fractional derivative in the equation, see, for example, the works [10, 23, 24, 33, 38, 39, 52, 62, 66, 97] for initial value problems and [16, 32, 34, 35, 49, 67, 83] for boundary value problems. Considerations regarding the existence and uniqueness results for such problems can be found, for example, in [1, 2, 20, 21, 45]. A comprehensive survey of the most important methods for fractional initial value problems, along with a detailed introduction to the subject and a brief summary about the convergence behaviour of the methods is given in the monograph [8], see also [17, 20, 25]. Less attention has been paid to numerical methods for solving equations with multiple fractional derivatives (the so-called multi-term equations) [22, 28, 46, 47, 51, 53, 64, 65] and fractional differential equations with non-local boundary conditions [3, 5, 93, 94, 96], although the latter has been widely considered for integer order differential equations (see the survey paper [86]). A classical example of a multi-term problem is the Bagley-Torvik

equation [20, 76]

$$y''(t) + d_1 D^{\frac{3}{2}}_{Cap}(t) + d_0 y(t) = f(t),$$

where d_0 and d_1 are known constants and f is a given function. This equation arises, for example, in the modelling of the motion of a rigid plate immersed in a Newtonian fluid [87]. Finally, we note that very little has been written on solving fractional integro-differential equations with weakly singular kernels [70, 99].

One of the main objects of study in the present thesis are non-local boundary value problems for linear multi-term fractional differential equations and weakly singular integro-differential equations in the following form:

$$(D^{\alpha_p}_{Cap}y)(t) + \sum_{i=0}^{p-1} d_i(t)(D^{\alpha_i}_{Cap}y)(t) + \int_0^t (t-s)^{-\kappa}K(t,s)y(s)ds = f(t), \quad 0 \le t \le b,$$

$$(1.0.1)$$

$$\sum_{j=0}^{n_0} \beta_{ij0}\, y^{(j)}(0) + \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk}\, y^{(j)}(b_k) + \beta_i \int_0^{\bar{b}_i} y(s)ds = \gamma_i, \quad i = 0,\ldots,n-1,$$

$$(1.0.2)$$

where $\beta_{ij0}, \beta_{ijk}, \beta_i, \gamma_i \in \mathbb{R}$,

$$0 \le \kappa < 1, \quad 0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_p \le n, \quad n := \lceil \alpha_p \rceil, \quad p \in \mathbb{N},$$

$$n_0 < n,\ n_1 < n, \quad n_0, n_1 \in \mathbb{N}_0, \quad 0 < b_1 < \cdots < b_l \le b, \quad 0 < \bar{b}_i \le b, \quad l \in \mathbb{N},$$

$d_i : [0,b] \to \mathbb{R}\ (i = 0,\ldots,p-1)$, $f : [0,b] \to \mathbb{R}$, $K : \Delta \to \mathbb{R}$ are some given continuous functions, $\Delta := \{(s,t) : 0 \le s \le t \le b\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to a real number α and $D^{\alpha_i}_{Cap}y\ (i = 0,\ldots,p)$ are Caputo derivatives of order α_i of an unknown function y.

Note that for certain values of coefficients β_{ij0}, β_{ijk} and β_i the problem (1.0.1)–(1.0.2) takes the form of an initial value problem or a terminal value problem or a multi-point boundary value problem.

We also consider a non-linear fractional differential equation

$$(D^{\alpha}_{Cap}y)(t) = f(t,y), \quad 0 \le t \le b, \quad \alpha > 0, \tag{1.0.3}$$

subject to the conditions

$$\sum_{j=0}^{n_0} \beta_{ij0}\, y^{(j)}(0) + \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk}\, y^{(j)}(b_k) = \gamma_i, \quad i = 0,\ldots,n-1,\ n := \lceil \alpha \rceil, \tag{1.0.4}$$

where $\beta_{ij0}, \beta_{ijk}, \gamma_i \in \mathbb{R}$,

$$0 < b_1 < \cdots < b_l \leq b, \quad l \in \mathbb{N}, \quad n_0, n_1 \in \mathbb{N}_0, \quad n_0 < n, \ n_1 < n, \quad n-1 < \alpha < n,$$

$f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $D_{Cap}^\alpha y$ is the Caputo derivative of order α of an unknown function $y = y(t)$.

The main purpose of this thesis is to construct high order numerical methods for solving problems (1.0.1)–(1.0.2) and (1.0.3)–(1.0.4). To this end, first of all we need some information about the regularity of the exact solutions of (1.0.1)–(1.0.2) and (1.0.3)–(1.0.4). This becomes even more significant since we aim to study the optimal order of convergence of the proposed algorithms. However, fractional differential equations pose an extra challenge compared to integer order differential equations. For example, it is well known that, in the case of integer order differential equations, the smoothness properties of a solution are determined by certain assumptions on the given data (mainly on the given function on the right hand side of the equation). A typical result is the following (see, for example, [18]): if $k \in \mathbb{N}$, $b > 0$ and $f \in C^{k-1}(G)$, that is, the function $f = f(t, y)$ is $k-1$ times continuously differentiable on the region $G = \{(t, y) : t \in [0, b], |y - y_0| \leq \eta\}$, $y_0 \in \mathbb{R}$, $\eta > 0$, then the solution $y = y(t)$ of the initial value problem $\left\{ \frac{dy}{dt} = f(t, y), \quad y(0) = y_0 \right\}$ is k times continuously differentiable on an interval $[0, h]$ for some $h \in (0, b]$, that is, $y \in C^k[0, h]$.

A simple example shows that, in general, we can not expect this result to be true for fractional differential equations: if $y = y(t)$ is a solution of an initial value problem

$$(D_{Cap}^\alpha y)(t) = f(t, y), \quad y(0) = y_0, \quad 0 < \alpha < 1,$$

then it may happen that even for $f \in C^\infty(G)$ we have $y \notin C^1[0, h]$.

Indeed, it follows from Chapter 2 below (see (2.5.8) and (2.5.11)), that the non-differentiable at $t = 0$ function $y(t) = t^{0.5} + y_0$ is the unique solution of the initial value problem $\left\{ (D_{Cap}^{0.5} y)(t) = \frac{\sqrt{\pi}}{2}, y(0) = y_0 \right\}$, whose given function $f = \frac{\sqrt{\pi}}{2}$ (the right-hand side of the differential equation) is analytic.

In fact, below we see that the non-smooth behaviour of solutions to problems (1.0.1)–(1.0.2) and (1.0.3)–(1.0.4) is typical (see Theorems 4.2.1 and 5.2.2, respectively). Thus, when constructing high order numerical methods for fractional differential equations, one should take into account, in some way, the possible non-smooth behaviour of an exact solution. Numerical methods which assume

smooth solutions for fractional differential equations are valid only for a tiny subclass of problems, as is made clear in [81, 82].

In this thesis, using integral equation reformulations of the boundary value problems (1.0.1)–(1.0.2) and (1.0.3)–(1.0.4), we first study the regularity of their exact solutions. Based on the obtained regularity properties and piecewise polynomial collocation techniques, the numerical solution of the obtained weakly singular integral equations is discussed. In general, a collocation method is a projection method, in which we first choose a finite dimensional space of basis functions and a number of points in the domain (the so-called collocation points). The collocation solution to an equation is determined by the requirement that the equation must be satisfied at the collocation points. This leads to a system of algebraic equations for finding the collocation solution.

In collocation methods the possible non-smooth behaviour of the exact solution of the underlying problem near the boundary of the domain where the problem is posed can be taken into account by using special non-uniform grids reflecting the singular behaviour of the exact solution. In the numerical solution of integral and integro-differential equations with singularities this approach has been analyzed by many authors. We refer to the monographs [6, 11, 13, 90, 92], see also [7, 43, 44, 74, 95]. However, as pointed out in [13, 30], using strongly non-uniform grids may create significant round-off errors in the calculations and lead to implementation difficulties. Therefore, it is our aim in this thesis to construct and analyze high order numerical methods for solving (1.0.1)–(1.0.2) and (1.0.3)–(1.0.4) which do not need strongly non-uniform grids. Our approach is based on the idea of killing the singularities of the derivatives of the exact solution to the underlying problem by a suitable smoothing transformation. Note that in the case of integral and integro-differential equations similar ideas have been successfully used in [9, 29, 60, 63, 75, 88].

The thesis consists of seven chapters. Chapters 1 and 2 have an introductory character. In Chapter 2 we introduce notation, basic definitions and preliminary results. In particular, in Section 2.5 we give the definitions for Riemann-Liouville integral operators, Riemann-Liouville fractional differential operators, Caputo fractional differential operators and present some of their properties which we will use later.

Chapters 3, 4 and 6 are devoted to the numerical solution of linear problems, while Chapter 5 is concerned with non-linear problems. Our approach is based on

an idea often used in the numerical solution of integer-order differential equations, where the original equation is converted to an equivalent integral equation.

The purpose of Chapter 3 is to give an introductory overview of the approach used in the Chapters 4 and 5 of this thesis. To this end we consider a simplified form of problem (1.0.1)–(1.0.2). In particular, in Chapter 3 we restrict ourselves to the case where in equation (1.0.1) there are at most two fractional derivatives $D_{Cap}^{\alpha_2} y$ and $D_{Cap}^{\alpha_1} y$ of the unknown function y. Furthermore, we assume that $\alpha_1, \alpha_2 \in (0, 1)$, the case which is relevant to the majority of the classical applications [8].

In Chapter 4 we consider the full problem (1.0.1)–(1.0.2). Using an integral equation reformulation of this problem with respect to the Caputo derivative $z := D_{Cap}^{\alpha_p} y$ of y, the exact solution to (1.0.1)–(1.0.2), we first study the existence, uniqueness and regularity of y and its Caputo derivative z. We observe that (usual) derivatives of y and z may be unbounded near the left endpoint of the interval $[0, b]$, even if $d_0, \ldots, d_{p-1}, f \in C^\infty[0, b]$ and $K \in C^\infty(\Delta)$ (see Theorem 4.2.1). We then solve the reformulated problem with respect to z by a piecewise polynomial collocation method. Due to the lack of regularity of z, piecewise polynomial collocation methods on uniform grids for solving this type of integral equations will show poor convergence behaviour. A better convergence can be established by using special non-uniform grids with the grid points

$$t_j = b \left(\frac{j}{N} \right)^r, \quad j = 0, 1, \ldots, N, \tag{1.0.5}$$

where $N + 1$ is the number of grid points and $r \in [1, \infty)$ is the so called grading exponent. The parameter r describes the non-uniformity of the grid: if $r = 1$, then the grid points (1.0.5) are distributed uniformly on $[0, b]$; for $r > 1$ they are more densely located near the left endpoint of the interval $[0, b]$. High order methods use larger values of r (see Theorems 3.3.1 and 3.3.2), which may lead to unstable behaviour of numerical results. In order to avoid strongly graded grids, we modify our approach as follows: before applying a collocation method to the obtained integral equation with respect to $z = z(t)$, we introduce in the integral equation a change of variables

$$t = b^{1-\rho} \tau^\rho, \quad \tau \in [0, b],$$

depending on the parameter $\rho \in [1, \infty)$. This transformation of variables possesses a smoothing property for z (see Lemma 2.8.1). We then apply a piecewise

polynomial collocation method to the transformed integral equation on a uniform (or midly graded) grid and get an approximation $z_{\rho,N}$ to z_ρ, the exact solution of the transformed integral equation. After that we find an approximation y_N for y, the solution of (1.0.1)–(1.0.2) by the formula (4.3.14). The main results of Chapter 4 are given by Theorems 4.2.1, 4.4.1 and 4.4.2.

In Chapter 5 similar ideas have been used for the numerical solution of nonlinear fractional boundary value problems (1.0.3)–(1.0.4). The main results of this chapter are given by Theorems 5.2.2, 5.4.1 and 5.4.2.

In Chapter 6 an alternative approach for the numerical solution of linear fractional boundary value problems has been considered. We apply the Riemann-Liouville integral operator to the fractional differential equation and instead of z we derive an equivalent weakly singular integral equation for y, the exact solution of the underlying differential equation. After that, with the help of a suitable smoothing transformation and collocation techniques, we construct a numerical method for solving the boundary value problem under consideration. The attainable order the proposed algorithms is studied and the corresponding results are given by Theorem 6.2.1.

In Chapter 7 we introduce some test problems and compare the computational results of the numerical experiments with the theoretical ones obtained in Chapters 3, 4, 5 and 6. The numerical experiments completely support the theoretical analysis.

Most of the results given in Chapters 3 to 7 of this thesis are published in [68–73, 93], the thesis also contains new results that have not been published yet.

Chapter 2

Preliminary results

In this chapter we introduce basic notations and formulate some results which we will need later.

2.1 Notations

Throughout this work $c, c_0, c_1 \ldots$ denote positive constants that may have various values in different occurrences. By $\mathbb{N}$ we denote the set of all positive integers $\{1, 2, \ldots\}$, by $\mathbb{N}_0$ the set of all non-negative integers $\{0, 1, 2, \ldots\}$, by $\mathbb{Z}$ the set of all integers $\{\ldots, -1, 0, 1, \ldots\}$ and by $\mathbb{R}$ the set of all real numbers $(-\infty, \infty)$. By $\lceil \alpha \rceil$ we denote the smallest integer greater or equal to a real number α. By I we denote the identity mapping.

By $L^1(a, b)$ we denote the Banach space of measurable functions $u : [a, b] \to \mathbb{R}$ such that

$$\|u\|_{L^1(a,b)} = \int_a^b |u(t)| dt < \infty.$$

By $L^\infty(a, b)$ we denote the Banach space of measurable functions $u : [a, b] \to \mathbb{R}$ such that

$$\inf_{\Omega \subset [a,b]; \mu(\Omega)=0} \sup_{t \in [a,b] \backslash \Omega} |u(t)| < \infty,$$

where $\mu(\Omega)$ is the Lebesgue measure of set Ω. The norm of this space is defined as

$$\|u\|_{L^\infty(a,b)} = \|u\|_\infty = \inf_{\Omega \subset [a,b]; \mu(\Omega)=0} \sup_{t \in [a,b] \backslash \Omega} |u(t)|.$$

By $C[a, b]$ we denote the Banach space of continuous functions $u : [a, b] \to \mathbb{R}$ with the norm

$$\|u\|_{C[a,b]} = \|u\|_\infty = \max_{a \leq t \leq b} |u(t)|.$$

By $C^m[a, b]$ we denote the Banach space of m times ($m \in \mathbb{N}_0$, for $m = 0$ we set $C^0[a, b] = C[a, b]$) continuously differentiable functions $u : [a, b] \to \mathbb{R}$ with the norm

$$\|u\|_{C^m[a,b]} = \sum_{i=0}^m \|u^{(i)}\|_\infty.$$

2.2   Linear operators and operator equations

In this section we introduce some well-known results from the theory of linear operators (see, for example, [6, 50]).

Let E and F be normed vector spaces. A linear operator $A : E \to F$ is called bounded if there exists a constant $M \geq 0$ such that

$$\|Ax\|_F \leq M \|x\|_E, \quad \forall x \in E.$$

The smallest such M is called the operator norm $\|A\|$ of A. An operator $A : E \to F$ is said to be continuous if

$$\|x_n - x\|_E \to 0, \quad n \to \infty$$

implies

$$\|Ax_n - Ax\|_F \to 0, \quad n \to \infty.$$

A linear operator $A : E \to F$ is continuous if and only if it is bounded.

One says that a linear operator $A : E \to F$ has the inverse $A^{-1} : F \to E$ if $A^{-1}A = I_E$ and $AA^{-1} = I_F$, where I_E and I_F are the identity mappings in E and F, respectively.

Let E and F be Banach spaces. By $\mathcal{L}(E, F)$ we denote the Banach space of linear bounded operators $A : E \to F$ with the norm

$$\|A\|_{\mathcal{L}(E,F)} = \sup\{\|Ax\|_F : x \in E, \|x\|_E \leq 1\}.$$

Theorem 2.2.1. *Let E and F be Banach spaces. If the operators $A, B \in \mathcal{L}(E, F)$ are such that $A^{-1} \in \mathcal{L}(F, E)$ and $\|B\|_{\mathcal{L}(E,F)}\|A^{-1}\|_{\mathcal{L}(F,E)} < 1$, then $A + B$ is invertible and the estimate*

$$\|(A+B)^{-1}\|_{\mathcal{L}(F,E)} \leq \frac{\|A^{-1}\|_{\mathcal{L}(F,E)}}{1 - \|B\|_{\mathcal{L}(E,F)}\|A^{-1}\|_{\mathcal{L}(F,E)}}$$

holds.

Let E and F be normed spaces. A linear operator $A : E \to F$ is called compact if for every bounded sequence $(x_n)_{n \geq 1} \subset E$ the sequence $(Ax_n)_{n \geq 1} \subset F$ has a convergent subsequence in F. Note that every compact operator is bounded and thus continuous.

Theorem 2.2.2. *Let E, F and G be normed spaces and let $A : E \to F$ and $B : F \to G$ be bounded linear operators. Then the product $BA : E \to G$ is compact if one of the two operators A or B is compact.*

Theorem 2.2.3. *(Fredholm alternative theorem). Let E be a Banach sapce, and let $A \in \mathcal{L}(E, E)$ be a compact operator. Then the equation $x = Ax + f$, $f \in E$ has a unique solution $x \in E$ if and only if the homogeneous equation $z = Az$ has only the trivial solution $z = 0$. In this case the operator $I - A$ has a bounded inverse $(I - A)^{-1} \in \mathcal{L}(E, E)$.*

2.3 Non-linear operator equations

Let E be a Banach space with a norm $\|x\|$, $x \in E$. A sequence $\{A_n\}_{n \in \mathbb{N}}$ of operators $A_n \in \mathcal{L}(E, E)$ is called compactly converging to $A \in \mathcal{L}(E, E)$ (we write $A_n \to A$ compactly) if $A_n x \to Ax$ as $n \to \infty$ for every $x \in E$ and for any bounded sequence $\{x_n\}_{n \in \mathbb{N}}$, $x_n \in E$, it follows that the sequence $\{A_n x_n\}_{n \in \mathbb{N}}$ is relatively compact in E (i.e. every subsequence $\{A_n x_n\}_{n \in \mathbb{N}' \subset \mathbb{N}}$ contains a subsequence $\{A_n x_n\}_{n \in \mathbb{N}'' \subset \mathbb{N}'}$ converging in E).

Let us consider the nonlinear equations

$$x = Sx \tag{2.3.1}$$

and

$$x = S_n x, \quad n \in \mathbb{N}, \tag{2.3.2}$$

where $S : \mathcal{B} \to E$ and $S_n : \mathcal{B} \to E$ are nonlinear operators defined on an open set $\mathcal{B} \subset E$.

We recall that $S : \mathcal{B} \to E$ is called Fréchet differentiable at $x^0 \in \mathcal{B}$ if there exists a linear operator $S'(x^0) \in \mathcal{L}(E, E)$ such that

$$\|Sx - Sx^0 - S'(x^0)(x - x^0)\|/\|x - x^0\| \to 0 \quad \text{as} \quad \|x - x^0\| \to 0;$$

in this case $S'(x^0)$ is the (unique) Fréchet derivative of S at x^0.

We shall later need the following result adapted from the approximation theory by Vainikko (see Theorem 4.3 in [89] or Theorem 2 in [90]).

Theorem 2.3.1. *Let the following conditions be fulfilled:*
1^0 *equation (2.3.1) has a solution $x^* \in \mathcal{B}$, and the operator S is Fréchet differentiable at x^*;*
2^0 *there is a positive number δ such that the operators S_n $(n \in \mathbb{N})$ are Fréchet differentiable in the ball $\|x - x^*\| \leq \delta$, which is assumed to be contained in $\mathcal{B}$, and for any $\varepsilon > 0$ there is a $\delta_\varepsilon \in (0, \delta]$ such that for every $n \in \mathbb{N}$*

$$\|S_n'(x) - S_n'(x^*)\|_{\mathcal{L}(E,E)} \leq \varepsilon \quad \text{whenever} \quad \|x - x^*\| \leq \delta_\varepsilon;$$

3^0 $\|S_n x^* - S x^*\| \to 0$ *as* $n \to \infty$;
4^0 $S_n'(x^*) \to S'(x^*)$ *compactly, whereby $S_n'(x^*) \in \mathcal{L}(E, E)$ $(n \in \mathbb{N})$ are compact and the homogeneous equation $x = S'(x^*)x$ has in E only the trivial solution $x = 0$.*

Then there exist $n_0 \in \mathbb{N}$ and $\delta_0 \in (0, \delta]$ such that equation (2.3.2) has for $n \geq n_0$ a unique solution x_n in the ball $\|x - x^*\| \leq \delta_0$. Thereby $x_n \to x^*$ as $n \to \infty$ and the following error estimate holds:

$$\|x_n - x^*\| \leq c \|S_n x^* - S x^*\|, \quad n \geq n_0. \tag{2.3.3}$$

Here c is a positive constant not depending on n.

2.4 Gamma, beta and Mittag-Leffler functions

In this section we recall the definitions and some properties of the gamma, beta and Mittag-Leffler functions (more details can be found, for example, in [20, 76]).

The gamma function $\Gamma = \Gamma(x)$ is defined by the formula

$$\Gamma(x) := \int_0^\infty s^{x-1}e^{-s}ds, \quad x \in (0,\infty);$$

elementary considerations from the theory of improper integrals reveal that the integral $\int_0^\infty s^{x-1}e^{-s}ds$ exists for all $x > 0$. An important property of the gamma function is the recurrence relation

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0. \tag{2.4.1}$$

It is easy to see that $\Gamma(1) = \Gamma(2) = 1$ and

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}.$$

An interesting property of $\Gamma(x)$ is given by the equality

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \quad 0 < x < 1.$$

In particular, if $x = \frac{1}{2}$, then we have $(\Gamma(0.5))^2 = \pi$ and thus

$$\Gamma(0.5) = \sqrt{\pi}. \tag{2.4.2}$$

The beta function $B = B(x,y)$ is defined by the formula

$$B(x,y) := \int_0^1 s^{x-1}(1-s)^{y-1}ds, \quad x,y \in (0,\infty).$$

Functions Γ and B are related by the equality

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0. \tag{2.4.3}$$

The function $E_{\alpha,\beta}$ defined by

$$E_{\alpha,\beta}(x) := \sum_{j=0}^\infty \frac{x^j}{\Gamma(j\,\alpha + \beta)}, \quad x \in \mathbb{R}, \tag{2.4.4}$$

is called the two-parameter Mittag-Leffler function with parameters $\alpha > 0$ and $\beta > 0$. Note that power series defining $E_{\alpha,\beta}(x)$ in (2.4.4) is convergent for all $x \in \mathbb{R}$. It follows from (2.4.4) that $E_{1,1}(x) = e^x$, $x \in \mathbb{R}$.

2.5 Fractional differential operators

In this section we present the definitions and some properties of Riemann-Liouville integrals and Riemann-Liouville and Caputo fractional differential operators, see [20, 45]. Let $b \in \mathbb{R}$, $b > 0$.

For given $\delta \in (0, \infty)$ by J^δ we denote the Riemann–Liouville integral operator of order δ, defined as

$$(J^\delta y)(t) := \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \, y(s) \, ds, \quad t \in [0, b], \quad y \in L^1(0, b). \qquad (2.5.1)$$

For $\delta = 0$ we set $J^0 := I$. If $\delta > 0$, then the integral $(J^\delta y)(t)$ exists for almost all $t \in [0, b]$ and the function $J^\delta y$ is also an element of $L^1(0, b)$. Moreover, we have for any $y \in L^\infty(0, b)$ that

$$(J^\delta y)^{(k)} \in C[0, b], \quad (J^\delta y)^{(k)}(0) = 0, \quad \delta > 0, \quad k = 0, \dots, \lceil \delta \rceil - 1, \qquad (2.5.2)$$

$$J^\alpha J^\beta y = J^\beta J^\alpha y = J^{\alpha+\beta} y, \quad \alpha > 0, \quad \beta > 0. \qquad (2.5.3)$$

Note that for $J^\delta y \in C[0, b]$ we have $y \in C^{n-1}[0, b]$, where $n - 1 < \delta \leq n$, $n \in \mathbb{N}$. Note also that the operator J^δ ($\delta > 0$) is linear, bounded and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$ (see, e.g. [12]).

By D_{RL}^δ we denote the Riemann–Liouville fractional differentiation operator of order $\delta > 0$, defined as

$$(D_{RL}^\delta y)(t) := \frac{d^n}{dt^n} (J^{n-\delta} y)(t), \quad t \in [0, b], \quad n = \lceil \delta \rceil. \qquad (2.5.4)$$

Often it is assumed that $J^{n-\delta} y \in C^n[0, b]$. Note that for $\delta \in \mathbb{N}$ we have $(D_{RL}^\delta y)(t) = y^{(\delta)}(t)$, $t \in [0, b]$.

Let $m \in \mathbb{N}$. By $\mathcal{Q}_{m-1}[y]$ we denote the Taylor polynomial of degree $m - 1$ for the function $y \in C^{m-1}[0, b]$ at the point 0:

$$(\mathcal{Q}_{m-1}[y])(s) := \sum_{i=0}^{m-1} \frac{y^{(i)}(0)}{i!} \, s^i.$$

By D_{Cap}^δ we denote the Caputo fractional differential operator of order $\delta > 0$, defined by the formula

$$(D_{Cap}^\delta y)(t) := (D_{RL}^\delta (y - \mathcal{Q}_{n-1}[y]))(t), \quad t \in [0, b], \quad n := \lceil \delta \rceil. \qquad (2.5.5)$$

In the definition (2.5.5) we assume that $y \in C^{n-1}[0, b]$. If $\delta \in \mathbb{N}$ and $y \in C^\delta[0, b]$ then we have $(D^\delta_{Cap} y)(t) = y^{(\delta)}(t)$, $t \in [0, b]$.

A sufficient condition for the existence of $D^\delta_{Cap} y \in C[0, b]$ is $y \in C^{\lceil \delta \rceil}[0, b]$. However, this is not a necessary condition. In [91], Vainikko gives a comprehensive description of the range $J^\delta C[0, b]$ ($\delta > 0$) of J^δ as an operator from $C[0, b]$ into $C[0, b]$. In particular, he has derived necessary and sufficient conditions for the existence of $D^\delta_{Cap} y \in C[0, b]$ for a function $y \in C^{\lceil \delta \rceil - 1}[0, b]$, $\delta > 0$. As an example, if $0 < \delta < 1$ and $y \in C[0, b]$, then the following conditions (i) and (ii) are equivalent:

(i) the fractional derivative $D^\delta_{Cap} y \in C[0, b]$ exists;

(ii) a limit $\lim_{t \to 0} t^{-\delta}[y(t) - y(0)]$ exists, is finite and the Riemann improper integrals

$$\int_0^t (t - s)^{-\delta - 1}(y(t) - y(s))ds \quad (0 < t \leq b)$$

equiconverge in the sense that

$$\lim_{\substack{\theta \to 1 \\ 0 < \theta < 1}} \sup_{0 < t \leq b} \left| \int_0^t (t - s)^{-\delta - 1}(y(t) - y(s))ds - \int_0^{\theta t} (t - s)^{-\delta - 1}(y(t) - y(s))ds \right| = 0.$$

For any $y \in L^\infty(0, b)$ we have

$$D^\beta_{RL} J^\alpha y = D^\beta_{Cap} J^\alpha y = J^{\alpha - \beta} y, \quad 0 < \beta \leq \alpha. \tag{2.5.6}$$

Note that a function $y^{\lceil \delta \rceil - 1} \in C[0, b]$ such that $D^\delta_{Cap} y \in C[0, b]$ ($\delta > 0$) has the form (cf. [20])

$$y(t) = (J^\delta z)(t) + \sum_{\lambda = 0}^{n-1} c_\lambda t^\lambda, \quad t \in [0, b], \quad n = \lceil \delta \rceil \in \mathbb{N}, \tag{2.5.7}$$

where $z := D^\delta_{Cap} y$ and $c_\lambda \in \mathbb{R}$ ($\lambda = 0, \ldots, n - 1$) are some constants.

Finally, we give some examples of fractional derivatives. From the definition (2.5.4) it follows that for a constant $c \in \mathbb{R}$ we have ($\delta > 0$, $n = \lceil \delta \rceil$)

$$(D^\delta_{RL} c)(t) = \frac{c}{\Gamma(n - \delta)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n - \delta - 1} ds = \frac{c}{\Gamma(n - \delta)} \frac{d^n}{dt^n} \left(\frac{t^{n - \delta}}{n - \delta} \right)$$

$$= \frac{c}{\Gamma(1 + n - \delta)} (n - \delta)(n - \delta - 1) \cdots (1 - \delta) t^{-\delta}, \quad t > 0.$$

We see that the Riemann-Liouville fractional derivative of a constant function does not necessarily vanish (it only vanishes if $c = 0$ or $\delta \in \mathbb{N}$). In contrast

to this, it follows from (2.5.5) that the Caputo fractional derivative $D_{Cap}^{\delta} c$ of a constant $c \in \mathbb{R}$ vanishes:

$$(D_{Cap}^{\delta} c)(t) = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\delta-1}(c-c)ds = 0, \quad \delta > 0, \; n = \lceil \delta \rceil, \quad t > 0.$$
(2.5.8)

Further, let $\beta \in \mathbb{R}$ and denote

$$v_\beta(t) = t^\beta, \; t > 0.$$

If $\delta > 0$, $n = \lceil \delta \rceil$, $\beta > n-1$, $\beta \in \mathbb{R}$, then

$$(D_{Cap}^{\delta} v_\beta)(t) = \frac{\Gamma(1+\beta)}{\Gamma(\beta+1-\delta)} t^{\beta-\delta}, \quad t > 0;$$
(2.5.9)

if $\beta \in \mathbb{N}_0$, then

$$(D_{Cap}^{\delta} v_\beta)(t) = \begin{cases} 0 & \text{if} \quad \beta = 0, \ldots, n-1, \\ \frac{\beta!}{\Gamma(\beta+1-\delta)} t^{\beta-\delta} & \text{if} \quad \beta \geq n. \end{cases}$$
(2.5.10)

Indeed, let first $\beta > n-1$, $\beta \in \mathbb{R}$, where $n = \lceil \delta \rceil, \delta > 0$. Then $Q_{n-1}[v_\beta] = 0$ and (2.5.9) follows from the definition (2.5.5) and (2.4.1), (2.4.3):

$$\begin{aligned}
(D_{Cap}^{\delta} v_\beta)(t) &= \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\delta-1} s^\beta ds \\
&= \frac{1}{\Gamma(n-\delta)} \int_0^1 (1-\tau)^{n-\delta-1} \tau^\beta d\tau \frac{d^n}{dt^n} t^{\beta+n-\delta} \\
&= \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+n-\delta)} (\beta+n-\delta)(\beta+n-1-\delta) \cdots (\beta+1-\delta) t^{\beta-\delta} \\
&= \frac{\Gamma(1+\beta)}{\Gamma(\beta+1-\delta)} t^{\beta-\delta}, \quad t > 0.
\end{aligned}$$

For (2.5.10) it now suffices to consider only the case $\beta \leq n-1$, $\beta \in \mathbb{N}_0$, for which we have $(Q_{n-1} v_\beta)(s) = s^\beta$ $(s \geq 0)$ and hence

$$(D_{Cap}^{\delta} v_\beta)(t) = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\delta-1}(s^\beta - s^\beta)ds = 0.$$

In particular, we see that (2.5.9) together with (2.4.1) and (2.4.2) yields

$$D_{Cap}^{0.5} t^{0.5} = \Gamma(0.5+1) = 0.5\Gamma(0.5) = \frac{\sqrt{\pi}}{2}, \; t > 0.$$
(2.5.11)

2.6 Weighted spaces of functions

In order to characterize the behaviour of a solution of a fractional differential equation, we introduce a weighted space $C^{m,\nu}(0,b]$ of smooth functions on $(0,b]$ (cf. [12, 90]).

For given $b \in \mathbb{R}, b > 0$, $m \in \mathbb{N}$ and $\nu \in \mathbb{R}$, $\nu < 1$, by $C^{m,\nu}(0,b]$ we denote the set of continuous functions $u : [0,b] \to \mathbb{R}$ which are m times continuously differentiable in $(0,b]$ and such that for all $t \in (0,b]$ the following estimates hold:

$$\left| u^{(i)}(t) \right| \leq c \left\{ \begin{array}{ll} 1 & \text{if } i < 1-\nu \\ 1 + |\log t| & \text{if } i = 1-\nu \\ t^{1-\nu-i} & \text{if } i > 1-\nu \end{array} \right\}, \quad i = 1, \ldots, m.$$

In other words, $u \in C^{m,\nu}(0,b]$ if $u \in C[0,b] \cap C^m(0,b]$ and

$$|u|_{m,\nu} := \sum_{i=1}^{m} \sup_{0 < t \leq b} \omega_{i-1+\nu}(t) \left| u^{(i)}(t) \right| < \infty,$$

where, for $t > 0$,

$$\omega_\rho(t) := \left\{ \begin{array}{ll} 1 & \text{if } \rho < 0, \\ \frac{1}{1+|\log t|} & \text{if } \rho = 0, \\ t^\rho & \text{if } \rho > 0. \end{array} \right.$$

Equipped with the norm

$$\|u\|_{C^{m,\nu}(0,b]} := \|u\|_\infty + |u|_{m,\nu}, \quad u \in C^{m,\nu}(0,b],$$

the set $C^{m,\nu}(0,b]$ becomes a Banach space.

Note that $C^m[0,b]$ ($m \in \mathbb{N}$) belongs to $C^{m,\nu}(0,b]$ for arbitrary $\nu < 1$. Some other examples are given by $y_1(t) = t^{\frac{7}{2}}$, $y_2(t) = t^{\frac{3}{4}}$ and $y_3(t) = t \log t$ with $y_3(0) = 0$. Clearly, $y_1 \in C^{m,-\frac{5}{2}}(0,b]$, $y_2 \in C^{m,\frac{1}{4}}(0,b]$ and $y_3 \in C^{m,0}(0,b]$.

Moreover, a function of the form

$$y(t) = g_1(t) t^\delta + g_2(t) \ (\delta > 0)$$

belongs to $C^{m,\nu}(0,b]$ for all $\nu \in [1-\delta, 1)$ and $g_1, g_2 \in C^m[0,b]$, $m \in \mathbb{N}$. Note also

$$C^q[0,b] \subset C^{q,\nu}(0,b] \subset C^{m,\mu}(0,b] \subset C[0,b], \quad q \geq m \geq 1, \quad \nu \leq \mu < 1. \quad (2.6.1)$$

Observe that as ν increases so does the singular behaviour of the derivatives of the functions in $C^{q,\nu}(0,b]$.

Next we formulate two lemmas which we will need later. Their proofs can be found in [12].

Lemma 2.6.1. *If $y_1, y_2 \in C^{q,\nu}(0,b], q \in \mathbb{N}, \nu < 1$, then $y_1 y_2 \in C^{q,\nu}(0,b]$, and*

$$\|y_1 y_2\|_{C^{q,\nu}(0,b]} \leq c\|y_1\|_{C^{q,\nu}(0,b]}\|y_2\|_{C^{q,\nu}(0,b]},$$

with a constant c which is independent of y_1 and y_2.

Lemma 2.6.2. *Let $\eta \in \mathbb{R}, \eta < 1$ and let $K \in C(\Delta), \Delta := \{(s,t) : 0 \leq s \leq t \leq b\}$. Then operator S defined by*

$$(Sy)(t) := \int_0^t (t-s)^{-\eta} K(t,s) y(s) ds, \ t \in [0,b], \tag{2.6.2}$$

is compact as an operator from $L^\infty(0,b)$ into $C[0,b]$. If, in addition, $K \in C^q(\Delta), q \in \mathbb{N}$, then S is compact as an operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$, where $\eta \leq \nu < 1$.

2.7 Graded grids and interpolation operators

For $N \in \mathbb{N}$ and $1 \leq r < \infty$, let $\Pi_N := \{t_0, \ldots, t_N\}$ be a partition (a graded grid) of the interval $[0,b]$ with the grid points

$$t_j = b\left(\frac{j}{N}\right)^r, \quad j = 0, 1, \ldots, N, \tag{2.7.1}$$

where $r \in [1, \infty)$ is the so called grading parameter. If $r = 1$, then the grid points (2.7.1) are distributed uniformly; for $r > 1$ the grid points (2.7.1) are more densely clustered near the left endpoint of the interval $[0,b]$.

For a given integer $m \in \mathbb{N}$ by $S_{m-1}^{-1}(\Pi_N)$ we denote the standard space of piecewise polynomial functions:

$$S_{m-1}^{(-1)}(\Pi_N) := \left\{v : v\big|_{[t_{j-1},t_j]} \in \pi_{m-1}, j = 1, \ldots, N\right\}. \tag{2.7.2}$$

Here $v\big|_{[t_{j-1},t_j]}$ $(j = 1, \ldots, N)$ is the restriction of $v : [0,b] \to \mathbb{R}$ onto the subinterval $[t_{j-1}, t_j] \subset [0,b]$ and π_{m-1} denotes the set of polynomials of degree not exceeding $m-1$. Note that the elements of $S_{m-1}^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \ldots, t_{N-1}$ of the grid Π_N.

In every interval $[t_{j-1}, t_j]$ $(j = 1, \ldots, N)$, we define $m \in \mathbb{N}$ interpolation (collocation) points $t_{j1}, \ldots, t_{jm}$ by formula

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \ldots, m, \ j = 1, \ldots, N, \tag{2.7.3}$$

where $\eta_1 \ldots, \eta_m$ are some fixed (collocation) parameters which do not depend on j and N and satisfy

$$0 \leq \eta_1 < \eta_2 < \ldots < \eta_m \leq 1. \qquad (2.7.4)$$

For given $N, m \in \mathbb{N}$ let $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0,b] \to S_{m-1}^{(-1)}(\Pi_N)$ be an interpolation operator such that

$$\mathcal{P}_N v \in S_{m-1}^{(-1)}(\Pi_N), \ (\mathcal{P}_N v)(t_{jk}) = v(t_{jk}), \ k = 1, \ldots, m, \ j = 1, \ldots, N, \quad (2.7.5)$$

for any continuous function $v \in C[0,b]$. If $\eta_1 = 0$, then by $(\mathcal{P}_N v)(t_{j1})$ we denote the right limit $\lim_{t \to t_{j-1}, t > t_{j-1}} (\mathcal{P}_N v)(t)$. If $\eta_m = 1$, then by $(\mathcal{P}_N v)(t_{jm})$ we denote the left limit $\lim_{t \to t_j, t < t_j} (\mathcal{P}_N v)(t)$.

The proof of the following three lemmas can be found in [12, 90].

Lemma 2.7.1. *Let $\mathcal{P}_N : C[0,b] \to S_{m-1}^{(-1)}(\Pi_N)$ ($N \in \mathbb{N}$) be defined by (2.7.5). Then $\mathcal{P}_N \in \mathcal{L}(C[0,b], L^\infty(0,b))$ and the norms of $\mathcal{P}_N$ are uniformly bounded:*

$$\|\mathcal{P}_N\|_{\mathcal{L}(C[0,b],L^\infty(0,b))} \leq c, \quad N \in \mathbb{N},$$

with a positive constant c which is independent of N. Moreover, for every $z \in C[0,b]$ we have

$$\|z - \mathcal{P}_N z\|_{L^\infty(0,b)} \to 0 \quad as \quad N \to \infty.$$

Lemma 2.7.2. *Let $S : L^\infty(0,b) \to C[0,b]$ be a linear compact operator. Let $\mathcal{P}_N : C[0,b] \to S_{m-1}^{(-1)}(\Pi_N)$ ($N \in \mathbb{N}$) be defined by (2.7.5). Then*

$$\|S - \mathcal{P}_N S\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \to 0 \quad as \quad N \to \infty.$$

Lemma 2.7.3. *Let $z \in C^{m,\nu}(0,b]$, $m \in \mathbb{N}$, $\nu \in (-\infty, 1)$. Let $\mathcal{P}_N : C[0,b] \to S_{m-1}^{(-1)}(\Pi_N)$ ($N \in \mathbb{N}$) be defined by (2.7.5). Then*

$$\|z - \mathcal{P}_N z\|_\infty \leq c \begin{cases} N^{-m} & for \quad m < 1 - \nu, \ r \geq 1, \\ N^{-m}(1 + \log N) & for \quad m = 1 - \nu, \ r = 1, \\ N^{-m} & for \quad m = 1 - \nu, \ r > 1, \\ N^{-r(1-\nu)} & for \quad m > 1 - \nu, \ 1 \leq r < \frac{m}{1-\nu}, \\ N^{-m} & for \quad m > 1 - \nu, \ r \geq \frac{m}{1-\nu}. \end{cases}$$

where $r \in [1, \infty)$ is the grading exponent in (2.7.1) and c is a positive constant not depending on N.

2.8 Smoothing transformation

In this section we introduce a mapping

$$t = b^{1-\rho}\tau^\rho, \quad \tau \in [0, b], \tag{2.8.1}$$

depending on a parameter $\rho \in [1, \infty)$. From the definition (2.8.1) we get that $\tau = b^{(\rho-1)/\rho}t^{1/\rho} \in [0, b]$ for $t \in [0, b]$. In the case $\rho = 1$ it follows from (2.8.1) that $t = \tau$. We are interested in transformations (2.8.1) with $\rho > 1$, since this transformation then possesses a smoothing property for $z \in C^{q,\nu}(0, b]$. From [75] we obtain the following result.

Lemma 2.8.1. *Let* $z \in C^{q,\nu}(0, b]$ *(*$q \in \mathbb{N}$, $-\infty < \nu < 1$*) and let* $z_\rho(\tau) := z(b^{1-\rho}\tau^\rho)$, $\tau \in [0, b]$, *where* $\rho \in [1, \infty)$ *if* $\nu \in (0, 1)$ *and* $\rho \in \mathbb{N}$ *if* $\nu \le 0$. *Then* $z_\rho \in C^{q,\nu_\rho}(0, b]$, *where* $\nu_\rho := 1 - \rho(1 - \nu)$.

Remark 2.8.1. *Instead of (2.8.1) other transformations can also be used (see, e.g. [75]). For simplicity of presentation we restricts ourselves only to the transformation (2.8.1).*

Introducing in the definition of the Riemann-Liouville integral operator J^α ($\alpha > 0$) (see (2.5.1)) the change of variables

$$t = b^{1-\rho}\tau^\rho, \quad s = b^{1-\rho}\sigma^\rho, \quad \tau, \sigma \in [0, b], \quad \rho \in [1, \infty), \tag{2.8.2}$$

we obtain for $x \in L^\infty(0, b)$ that

$$(J^\alpha x)(t) = (J_\rho^\alpha x_\rho)(\tau), \quad t = b^{1-\rho}\tau^\rho, \quad \tau \in [0, b], \quad \alpha > 0, \tag{2.8.3}$$

where

$$x_\rho(\tau) := x(b^{1-\rho}\tau^\rho)$$

and

$$(J_\rho^\alpha x_\rho)(\tau) := \frac{\rho b^{(1-\rho)\alpha}}{\Gamma(\alpha)} \int_0^\tau (\tau^\rho - \sigma^\rho)^{\alpha-1} \sigma^{\rho-1} x_\rho(\sigma)\, d\sigma, \ \tau \in [0, b], \quad \alpha > 0. \tag{2.8.4}$$

The following lemma presents some properties of J_ρ^α which follow from the corresponding properties of the Riemann-Liouville integral operator J^α.

Lemma 2.8.2. *Let $\alpha > 0$ and $\rho \geq 1$ be some given real numbers. Then J_ρ^α defined by (2.8.4) is linear, bounded and compact as an operator from $L^\infty(0,b)$ into $C[0,b]$. Moreover, we have for any $z \in L^\infty(0,b)$ that*

$$J_\rho^\alpha J_\rho^\beta z = J_\rho^{\alpha+\beta} z, \quad \alpha > 0, \quad \beta > 0, \quad \rho \geq 1.$$

Finally, we have the following result (see [47]).

Lemma 2.8.3. *Let $z \in C^{m+1,\nu}(0,b]$, $m \in \mathbb{N}$, $\nu \in (-\infty, 1)$. Let $N \in \mathbb{N}$, $\alpha \in (0,1]$, $r \in [1,\infty)$, $\rho \in [1,\infty)$ if $\nu \in (0,1)$ and $\rho \in \mathbb{N}$ if $\nu \leq 0$, $z_\rho(\tau) = z(b^{1-\rho}\tau^\rho)$, $\tau \in [0,b]$. Let J_ρ^α $(\alpha > 0)$ and $\mathcal{P}_N$ $(N \in \mathbb{N})$ be defined by (2.8.4) and (2.7.5), respectively. Assume that the collocation points (2.7.3) with grid points (2.7.1) and parameters $\eta_1, \ldots, \eta_m$ satisfying (2.7.4) are used. Moreover, assume that $\eta_1, \ldots, \eta_m$ are such that a quadrature approximation*

$$\int_0^1 F(x)dx \approx \sum_{k=1}^m w_k F(\eta_k), \;\; 0 \leq \eta_1 < \eta_2 < \ldots < \eta_m \leq 1, \qquad (2.8.5)$$

with appropriate weights $\{w_k\}$ is exact for all polynomials F of degree m.

Then the following estimate holds:

$$\|J_\rho^\alpha(\mathcal{P}_N z_\rho - z_\rho)\|_\infty \leq c \left\{ \begin{array}{lll} E_N(m,\alpha,\nu,\rho,r) & if & 0 < \alpha < 1 \\ E_N^*(m,\nu,\rho,r) & if & \alpha = 1 \end{array} \right\}. \qquad (2.8.6)$$

Here c is a constant not depending on N,

$$E_N(m,\alpha,\nu,\rho,r) := \left\{ \begin{array}{lll} N^{-m-\alpha} & for & m < \rho(1+\alpha-\nu), \;\; r \geq 1, \\ N^{-m-\alpha}(1+\log N) & for & m = \rho(1+\alpha-\nu), \;\; r = 1, \\ N^{-m-\alpha} & for & m = \rho(1+\alpha-\nu), \;\; r > 1, \\ N^{-\rho r(1+\alpha-\nu)} & for & m > \rho(1+\alpha-\nu) \quad and \\ & & \quad 1 \leq r < \frac{m+\alpha}{\rho(1+\alpha-\nu)}, \\ N^{-m-\alpha} & for & m > \rho(1+\alpha-\nu) \quad and \\ & & \quad r \geq \frac{m+\alpha}{\rho(1+\alpha-\nu)} \end{array} \right.$$

$$(2.8.7)$$

and

$$
E_N^*(m, \nu, \rho, r) := \begin{cases}
N^{-m-1} & for \quad m < \rho(2 - \nu), \quad r \geq 1\,, \\
N^{-m-1}(1 + \log N)^2 & for \quad m = \rho(2 - \nu), \quad r = 1\,, \\
N^{-m-1}(1 + \log N) & for \quad m = \rho(2 - \nu), \quad r > 1\,, \\
N^{-\rho r(2-\nu)} & for \quad m > \rho(2 - \nu) \quad and \\
& \qquad 1 \leq r < \frac{m+1}{\rho(2-\nu)}, \\
N^{-m-1} & for \quad m > \rho(2 - \nu), \quad and \\
& \qquad r \geq \frac{m+1}{\rho(2-\nu)}.
\end{cases}
$$

$$(2.8.8)$$

Chapter 3

Linear fractional integro-differential equations with two fractional derivatives

In this chapter we discuss a possibility to construct high order numerical methods for solving initial and boundary value problems for linear weakly singular fractional integro-differential equations. In order to give an outline of the method that can be applied to a wide class of equations, we first restrict ourselves to equations involving only up to two fractional derivatives, both of order less than one. Later, in Chapter 4, we extend our study to general multi-term problems. More precisely, we will consider an equation in the form

$$(D_{Cap}^{\alpha_2}y)(t) + (D_{Cap}^{\alpha_1}y)(t)d_1(t) + y(t)d_0(t) + \int_0^t (t-s)^{-\kappa}K(t,s)y(s)ds = f(t),$$

$$(3.0.1)$$

subject to the condition

$$\beta_0 y(0) + \sum_{k=1}^{l}\beta_k y(b_k) + \beta\int_0^{\bar{b}} y(s)ds = \gamma,$$

$$(3.0.2)$$

where $0 \le t \le b$, $b > 0$, $l \in \mathbb{N}$, $0 < b_1 < \cdots < b_l \le b$, $\bar{b} \in (0,b]$, $\gamma, \beta, \beta_k \in \mathbb{R}$ ($k = 0, \ldots, l$) and $D_{Cap}^{\alpha_1}$ and $D_{Cap}^{\alpha_2}$ are Caputo differential operators of order α_1 and α_2, respectively. We assume that

$$0 < \alpha_1 < \alpha_2 < 1, \quad 0 \le \kappa < 1, \quad \sum_{k=0}^{l}\beta_k + \beta\,\bar{b} \ne 0$$

$$(3.0.3)$$

and $d_0, d_1, f \in C[0, b]$, $K \in C(\Delta)$, where

$$\Delta := \{(t, s) : 0 \le s \le t \le b\}. \tag{3.0.4}$$

Clearly, (3.0.1)–(3.0.2) is a special form of (1.0.1)–(1.0.2). In particular, the case where (3.0.1)–(3.0.2) is an initial value problem ($\beta_0 \ne 0$, $\beta_1 = ... = \beta_l = \beta = 0$) or a terminal value problem ($\beta_0 = \cdots = \beta_{l-1} = \beta = 0$, $\beta_l = b$, cf. [20, 32, 34]) is under consideration.

Using an integral equation reformulation of problem (3.0.1)–(3.0.2), we first present some results about the existence, uniqueness and regularity of its exact solution (Sections 3.1 and 3.2). On the basis of this information we then construct a class of numerical methods to solve such problems. After that we give an overview of the convergence and superconvergence results of the proposed algorithms (Sections 3.3 and 3.4). These results follow from the more general results proven in Chapter 4. Numerical experiments verifying the theoretical results are presented in Chapter 7.

3.1 Integral equation reformulation

First, let $y \in C[0, b]$ be an arbitrary function such that $D_{Cap}^{\alpha_2} y \in C[0, b]$ and let us denote $z := D_{Cap}^{\alpha_2} y$. Then

$$y(t) = (J^{\alpha_2} z)(t) + c, \tag{3.1.1}$$

where J^{α_2} is the Riemann-Liouville integral operator of order α_2 defined by (2.5.1) and c is a constant. By using properties (2.5.2) and (2.5.6) we see that a function in the form (3.1.1) satisfies the condition (3.0.2) if and only if

$$\beta_0 \, c + \sum_{k=1}^{l} \beta_k \, c + \beta \, \bar{b} \, c = \gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha_2} z)(b_k) - \beta \int_0^{\bar{b}} (J^{\alpha_2} z)(s) ds. \tag{3.1.2}$$

Due to the definition of J^{α_2} we can write

$$\int_0^{\bar{b}} (J^{\alpha_2} z)(s) ds = \frac{1}{\Gamma(\alpha_2)} \int_0^{\bar{b}} \int_0^s (s - \tau)^{\alpha_2 - 1} z(\tau) d\tau ds.$$

We simplify this double integral as follows. By changing the order of integration we find that

$$\frac{1}{\Gamma(\alpha_2)} \int_0^{\bar{b}} \int_0^s (s - \tau)^{\alpha_2 - 1} z(\tau) d\tau ds = \frac{1}{\Gamma(\alpha_2)} \int_0^{\bar{b}} z(\tau) \int_\tau^{\bar{b}} (s - \tau)^{\alpha_2 - 1} ds d\tau.$$

It is easy to see that

$$\int_{\tau}^{\bar{b}} (s - \tau)^{\alpha_2 - 1} ds = \frac{(\bar{b} - \tau)^{\alpha_2}}{\alpha_2}$$

and, by using the recurrence relation $\alpha_2 \Gamma(\alpha_2) = \Gamma(\alpha_2 + 1)$ (see (2.4.1)), we obtain

$$\int_0^{\bar{b}} (J^{\alpha_2} z)(s) ds = (J^{\alpha_2 + 1} z)(\bar{b}).$$

By denoting

$$\beta_* = \sum_{k=0}^{l} \beta_k, \tag{3.1.3}$$

we get from (3.1.2) that

$$c = \frac{\gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha_2} z)(b_k) - \beta\, (J^{\alpha_2 + 1} z)(\bar{b})}{\beta_* + \beta\, \bar{b}}.$$

We see that an arbitrary continuous function $y \in C[0, b]$ with $z = D_{Cap}^{\alpha_2} y \in C[0, b]$ satisfies the condition (3.0.2) if and only if it is in the form

$$y(t) = (J^{\alpha_2} z)(t) + (\beta_* + \beta\, \bar{b})^{-1} \left[\gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha_2} z)(b_k) - \beta\, (J^{\alpha_2 + 1} z)(\bar{b}) \right],$$
$$\tag{3.1.4}$$

where $0 \le t \le b$.

Let now $y \in C[0, b]$ be a solution to problem (3.0.1)-(3.0.2) so that $z = D_{Cap}^{\alpha_2} y \in C[0, b]$. Keeping in mind that $(D_{Cap}^{\alpha_2} y)(t) = z(t)$, by substituting (3.1.4) into (3.0.1) and using properties (2.5.6) and (2.5.8), we obtain

$$z(t) + (J^{\alpha_2 - \alpha_1} z)(t) d_1(t)$$

$$+ \left((J^{\alpha_2} z)(t) + (\beta_* + \beta\, \bar{b})^{-1} \left[\gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha_2} z)(b_k) - \beta\, (J^{\alpha_2 + 1} z)(\bar{b}) \right] \right) d_0(t)$$

$$+ \int_0^t (t - s)^{-\kappa} K(t, s) (J^{\alpha_2} z)(s) ds$$

$$+ (\beta_* + \beta\, \bar{b})^{-1} \left[\gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha_2} z)(b_k) - \beta\, (J^{\alpha_2 + 1} z)(\bar{b}) \right] \int_0^t (t - s)^{-\kappa} K(t, s) ds = f(t),$$

where $0 \le t \le b$. Note that

$$\int_0^t (t - s)^{-\kappa} K(t, s) (J^{\alpha_2} z)(s) ds = \frac{1}{\Gamma(\alpha_2)} \int_0^t (t - s)^{-\kappa} K(t, s) \int_0^s (s - \tau)^{\alpha_2 - 1} z(\tau) d\tau ds$$

$$= \frac{1}{\Gamma(\alpha_2)} \int_0^t z(s) \int_s^t (t - s)^{-\kappa} (\tau - s)^{\alpha_2 - 1} K(t, \tau) d\tau ds.$$

Next, by using a change of variables $\tau = (t - s)\sigma + s$ we get

$$\int_s^t (t-s)^{-\kappa}(\tau-s)^{\alpha_2-1}K(t,\tau)d\tau = (t-s)^{\alpha_2-\kappa}\int_0^1 \sigma^{\alpha_2-1}(1-\sigma)^{-\kappa}K(t,(t-s)\sigma+s)d\sigma.$$

Thus, we have

$$\int_0^t (t-s)^{-\kappa}K(t,s)(J^{\alpha_2}z)(s)ds = \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-\kappa}L(t,s)z(s)ds, \quad 0 \le t \le b, \tag{3.1.5}$$

where

$$L(t,s) := \int_0^1 \sigma^{\alpha_2-1}(1-\sigma)^{-\kappa}K(t,(t-s)\sigma+s)d\sigma \tag{3.1.6}$$

is a continuous function for $(t,s) \in \Delta$ since $K \in C(\Delta)$, $\alpha_2 \in (0,1]$ and $\kappa \in [0,1)$.

Remark 3.1.1. *Let $\alpha_2 \in (0,1]$, $\kappa \in [0,1)$, $K \in C^q(\Delta)$, $q \in \mathbb{N}$. Let L be defined by (3.1.6). Then $L \in C^q(\Delta)$.*

We see that we have obtained an integral equation with respect to z, which we write in the operator form

$$z = Tz + g, \tag{3.1.7}$$

where

$$(Tz)(t) = -(J^{\alpha_2-\alpha_1}z)(t)d_1(t) - (J^{\alpha_2}z)(t)d_0(t) - \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-\kappa}L(t,s)z(s)ds$$

$$+ (\beta_* + \beta\,\bar{b})^{-1}\left[\sum_{k=1}^l \beta_k(J^{\alpha_2}z)(b_k) + \beta\,(J^{\alpha_2+1}z)(\bar{b})\right]$$

$$\times \left(d_0(t) + \int_0^t (t-s)^{-\kappa}K(t,s)ds\right) \quad (0 \le t \le b) \tag{3.1.8}$$

and

$$g(t) = f(t) - \frac{\gamma}{\beta_* + \beta\bar{b}}\left(d_0(t) + \int_0^t (t-s)^{-\kappa}K(t,s)ds\right), \quad 0 \le t \le b. \tag{3.1.9}$$

In other words, we have shown that if a continuous function $y \in C[0,b]$ with $D_{Cap}^{\alpha_2}y \in C[0,b]$ is a solution to problem (3.0.1)-(3.0.2), then $z = D_{Cap}^{\alpha_2}y$ is a solution to integral equation (3.1.7).

It turns out that the converse is also true. Indeed, suppose that $z \in C[0,b]$ is a solution to integral equation (3.1.7). If we now define y by (3.1.4), we see that $y \in C[0,b]$, $z = D_{Cap}^{\alpha_2}y$ and that y satisfies the condition (3.0.2). We see that y determined by (3.1.4) is a solution to (3.0.1)-(3.0.2). Thus equation (3.1.7) is equivalent to the problem (3.0.1)-(3.0.2).

3.2 Existence, uniqueness and smoothness of the solution

The existence, uniqueness and regularity of the solution to (3.0.1)-(3.0.2) can be characterized by the following theorem.

Theorem 3.2.1. (i) *Assume that* $0 < \alpha_1 < \alpha_2 < 1$, $0 \leq \kappa < 1$, $d_0, d_1, f \in C[0, b]$, $K \in C(\Delta)$. *Moreover, let* $\sum_{k=0}^{l} \beta_k + \beta\, \bar{b} \neq 0$ *and assume that the problem* (3.0.1)-(3.0.2) *with* $f = 0$ *and* $\gamma = 0$ *has in* $C[0, b]$ *only the trivial solution* $y = 0$.

Then problem (3.0.1)-(3.0.2) *has a unique solution* $y \in C[0, b]$. *Moreover, we have* $D_{Cap}^{\alpha_2} y \in C[0, b]$.

(ii) *Assume that* (i) *holds and let* $K \in C^q(\Delta)$, $d_0, d_1, f \in C^{q,\mu}(0, b]$, $q \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$.

Then problem (3.0.1)-(3.0.2) *possesses a unique solution* y *such that* $y \in C^{q,\nu}(0, b]$ *and* $D_{Cap}^{\alpha_2} y \in C^{q,\nu}(0, b]$, *where*

$$\nu := \max\{1 - (\alpha_2 - \alpha_1), \mu, \kappa\}. \tag{3.2.1}$$

Proof. This theorem is a consequence of the more general Theorem 4.2.1 which we will prove in Chapter 4. □

Remark 3.2.1. *If* $K = 0$, *then we may in Theorem 3.2.1 set* $\nu = \max\{1 - (\alpha_2 - \alpha_1), \mu\}$. *If* $d_0, d_1, f \in C^q[0, b]$ ($q \in \mathbb{N}$), *then we may in Theorem 3.2.1 set* $\nu = \max\{1 - (\alpha_2 - \alpha_1), \kappa\}$, *and if also* $K = 0$, *then* $\nu = 1 - (\alpha_2 - \alpha_1)$.

Theorem 3.2.1 states that the regularity properties of y, the solution of problem (3.0.1)-(3.0.2), depend on the smoothness of functions d_0, d_1, f and K. However, as noted in Remark 3.2.1, even when we have $d_0, d_1, f \in C^q[0, b]$ and $K \in C^q(\Delta)$ (for some $q \in \mathbb{N}$), we cannot claim that $y \in C^q[0, b]$ – we may only say that $y \in C^{q,\max\{1-(\alpha_2-\alpha_1),\kappa\}}(0, b]$. That is, the solution of the problem (3.0.1)–(3.0.2) can, in general, exhibit singular behaviour even when the data of the problem is smooth. This complicates the construction of high order methods for solving such equations numerically.

3.3 Numerical methods based on graded grids

In order to take into account the potential non-smoothness of the exact solution $y = y(t)$ of (3.0.1)-(3.0.2) at the origin $t = 0$, we introduce on the interval $[0, b]$ a

graded grid Π_N ($N \in \mathbb{N}$) with the grid points (2.7.1) and look for an approximate solution y_N to (3.0.1)-(3.0.2) in the form (cf. (3.1.4))

$$y_N(t) = (J^{\alpha_2} z_N)(t) + (\beta_* + \beta\,\bar{b})^{-1} \left[\gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha_2} z_N)(b_k) - \beta\,(J^{\alpha_2+1} z_N)(\bar{b}) \right],$$
(3.3.1)

where $0 \leq t \leq b$ and $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ ($m \in \mathbb{N}$) is determined by the following collocation conditions:

$$z_N(t_{jk}) = (T z_N)(t_{jk}) + g(t_{jk}), \quad k = 1, \ldots, m, \; j = 1, \ldots, N. \qquad (3.3.2)$$

Here T, g and $\{t_{jk}\}$ are defined by (3.1.8), (3.1.9) and (2.7.3), respectively. If $\eta_1 = 0$, then by $z_N(t_{j1})$ we denote the right limit $\lim_{t \to t_{j-1}, t > t_{j-1}} z_N(t)$. If $\eta_m = 1$, then $z_N(t_{jm})$ denotes the left limit $\lim_{t \to t_j, t < t_j} z_N(t)$. Conditions (3.3.2) have an operator equation representation

$$z_N = \mathcal{P}_N T z_N + \mathcal{P}_N g \qquad (3.3.3)$$

with an operator $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0,T] \to S_{m-1}^{(-1)}(\Pi_N)$ defined by (2.7.5).

The collocation conditions (3.3.2) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$. In particular, if $\eta_1 > 0$ or $\eta_m < 1$, then we can use the Lagrange fundamental polynomial representation:

$$z_N(t) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu} \varphi_{\lambda\mu}(t), \quad t \in [0, b], \qquad (3.3.4)$$

where, for $\mu = 1, \ldots, m$, $\lambda = 1, \ldots, N$,

$$\varphi_{\lambda\mu}(t) := \begin{cases} 0 & \text{for} \quad t \notin [t_{\lambda-1}, t_\lambda], \\ \prod_{i=1, i \neq \mu}^{m} \frac{t - t_{\lambda i}}{t_{\lambda\mu} - t_{\lambda i}} & \text{for} \quad t \in [t_{\lambda-1}, t_\lambda]. \end{cases} \qquad (3.3.5)$$

Then $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and $z_N(t_{jk}) = c_{jk}$, $k = 1, \ldots, m$, $j = 1, \ldots, N$. Searching for the solution of (3.3.2) in the form (3.3.4), we obtain a system of linear algebraic equations with respect to the coefficients $c_{jk} = z_N(t_{jk})$:

$$c_{jk} = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} (T \varphi_{\lambda\mu})(t_{jk}) c_{\lambda\mu} + g(t_{jk}), \quad k = 1, \ldots, m, \; j = 1, \ldots, N. \qquad (3.3.6)$$

Note that for the computation of $(T \varphi_{\lambda\mu})(t_{jk})$ we need the weakly singular integrals $\left(J^\delta \varphi_{\lambda\mu} \right)(t_{jk})$ ($\delta > 0$) (see (3.1.8)), which can be found exactly.

After solving the linear system (3.3.6) for the unknown coefficients $\{c_{jk}\}$, by using (3.3.1) and (3.3.4) we get the following expression for the approximate solution y_N of y, the exact solution of (3.0.1)-(3.0.2):

$$
y_N(t) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu}(J^{\alpha_2}\varphi_{\lambda\mu})(t) + (\beta_* + \beta\bar{b})^{-1}
$$
$$
\times \left[\gamma - \sum_{k=1}^{l} \beta_k \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu}(J^{\alpha_2}\varphi_{\lambda\mu})(b_k) - \beta \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu}(J^{\alpha_2+1}\varphi_{\lambda\mu})(\bar{b}) \right].
$$
$$(3.3.7)$$

Note that this algorithm can also be used in the case if in (2.7.4) $\eta_1 = 0$ and $\eta_m = 1$. In this case we have $t_{jm} = t_{j+1,1} = t_j$, $c_{jm} = c_{j+1,1} = z_N(t_j)$ ($j = 1, \ldots, N-1$), and hence in the system (3.3.6) there are $(m-1)N+1$ equations and unknowns.

For method (3.3.1)–(3.3.2) we can formulate Theorems 3.3.1 and 3.3.2 below. Their proofs are given in Chapter 4. More precisely, Theorems 3.3.1 and 3.3.2 follow from (for $\rho = 1$) the more general Theorems 4.4.1 and 4.4.2, respectively.

Theorem 3.3.1. (i) *Let $N, m \in \mathbb{N}$ and assume that the grid points (2.7.1) with collocation points (2.7.3) and arbitrary parameters $\eta_1, \ldots, \eta_m$ satisfying (2.7.4) are used. Assume that conditions (3.0.3) are satisfied, $d_0, d_1 \in C[0, b]$, $f \in C[0, b]$ and $K \in C(\Delta)$. Moreover, assume that the problem (3.0.1)-(3.0.2) with $f = 0$ and $\gamma = 0$ has in $C[0, b]$ only the trivial solution $y = 0$.*

Then (3.0.1)-(3.0.2) has a unique solution $y \in C[0, b]$ such that $D_{Cap}^{\alpha_2} y \in C[0, b]$. Moreover, there exists an integer N_0 such that for all $N \geq N_0$ equation (3.3.3) possesses a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and

$$
\|y - y_N\|_\infty \to 0 \quad \text{as} \quad N \to \infty \tag{3.3.8}
$$

where y_N is defined by (3.3.1).

(ii) If, in addition, $d_0, d_1, f \in C^{q,\mu}(0, b]$, $K \in C^q(\Delta)$, where $q := m$ and with $\mu \in \mathbb{R}, \mu < 1$, then for all $N \geq N_0$ and $r \geq 1$ (given by (2.7.1)) the following error estimate holds:

$$
\|y - y_N\|_\infty \leq c \begin{cases} N^{-r(1-\nu)} & for \quad 1 \leq r < \frac{m}{1-\nu}, \\ N^{-m} & for \quad r \geq \frac{m}{1-\nu}. \end{cases} \tag{3.3.9}
$$

Here ν is given by formula (3.2.1) and c is a constant which does not depend on N.

Theorem 3.3.2. *Let $N, m \in \mathbb{N}$ and let the following conditions be fulfilled:*
(i) the assumptions (i)-(ii) of Theorem 3.3.1 hold with $q := m + 1$;
(ii) the quadrature approximation

$$\int_0^1 F(x) \, dx \approx \sum_{k=1}^m w_k \, F(\eta_k),$$

with the knots $\{\eta_k\}$ satisfying (2.7.4) and appropriate weights $\{w_k\}$ is exact for all polynomials F of degree m.

Then (3.0.1)-(3.0.2) has a unique solution $y \in C[0, b]$ such that $D_{Cap}^{\alpha_2} y \in C^{q,\nu}(0, b]$. There exists an integer N_0 such that, for all $N \geq N_0$, equation (3.3.3) possesses a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$, determining by (3.3.1) a unique approximation y_N to y, the solution of (3.0.1)-(3.0.2), and the following error estimate holds:

$$\|y - y_N\|_\infty \leq c \begin{cases} N^{-r(1+\alpha_2-\alpha_1-\nu)} & for \quad 1 \leq \; r < \frac{m+\alpha_2-\alpha_1}{1+\alpha_2-\alpha_1-\nu}, \\ N^{-m-(\alpha_2-\alpha_1)} & for \qquad r \geq \frac{m+\alpha_2-\alpha_1}{1+\alpha_2-\alpha_1-\nu}. \end{cases} \qquad (3.3.10)$$

Here $r \in [1, \infty)$ is the grading exponent of the grid (see (2.7.1)), ν is given by formula (3.2.1) and c is a positive constant not depending on N.

Remark 3.3.1. *To satisfy condition (ii) in Theorem 3.3.2 for all polynomials of degree m it is sufficient to have the collocation parameters $\eta_1, \ldots, \eta_m$ be the points of an m-point Gaussian quadrature rule applied on the interval $[0, 1]$, which is exact for all polynomials of degree $2m - 1$. As an example, for $m = 2$ and $m = 3$ we can use collocation parameters*

$$\eta_1 = \frac{3 - \sqrt{3}}{6}, \quad \eta_2 = 1 - \eta_1$$

and

$$\eta_1 = \frac{5 - \sqrt{15}}{10}, \quad \eta_2 = \frac{1}{2}, \quad \eta_3 = 1 - \eta_1,$$

respectively, see [13].

3.4 Numerical methods based on smoothing transformations and graded grids

The theorems in the previous section show that with sufficiently large values of the grading exponent r in (2.7.1), it is possible to obtain an optimal convergence rate for given m. However, as previously noted, using large values of r can lead to computational difficulties. In order to avoid strongly graded grids, we modify our approach as follows: before applying a collocation method to the obtained integral equation with respect to $z = z(t)$, we introduce in the integral equation a change of variables

$$t = b^{1-\rho}\tau^\rho, \quad \tau \in [0, b],$$

depending on the parameter $\rho \in [1, \infty)$. We then apply a piecewise polynomial collocation method to the transformed integral equation on a uniform (or midly graded) grid.

More precisely, we choose a smoothing parameter $\rho \in [1, \infty)$ and consider for equation (3.1.7) a change of variables introduced in Section 2.8 by (2.8.2):

$$t = b^{1-\rho}\tau^\rho, \quad s = b^{1-\rho}\sigma^\rho, \quad \tau, \sigma \in [0, b].$$

Using in (3.1.7) this change of variables we get for $z_\rho(\tau) = z(b^{1-\rho}\tau^\rho)$ an integral equation in the form

$$z_\rho = T_\rho z_\rho + g_\rho, \tag{3.4.1}$$

where, for $0 \le \tau \le b$, we have

$$
\begin{aligned}
(T_\rho z_\rho)(\tau) = {} & -(J_\rho^{\alpha_2-\alpha_1} z_\rho)(\tau) d_{1,\rho}(\tau) - (J_\rho^{\alpha_2} z_\rho)(\tau) d_{0,\rho}(\tau) \\
& - \frac{\rho b^{(1-\rho)(1+\alpha-\kappa)}}{\Gamma(\alpha_2)} \int_0^\tau (\tau^\rho - \sigma^\rho)^{\alpha_2-\kappa} \sigma^{\rho-1} L_\rho(\tau, \sigma) z_\rho(\sigma) d\sigma \\
& + (\beta_* + \beta\,\bar{b})^{-1} \left[\sum_{k=1}^l \beta_k (J_\rho^{\alpha_2} z_\rho)(b_{k,\rho}) + \beta\,(J_\rho^{\alpha_2+1} z_\rho)(\bar{b}_\rho) \right] \\
& \times \left(d_{0,\rho}(\tau) + b^{(1-\rho)(1-\kappa)}\rho \int_0^\tau (\tau^\rho - \sigma^\rho)^{-\kappa} \sigma^{\rho-1} K_\rho(\tau, \sigma) d\sigma \right) \tag{3.4.2}
\end{aligned}
$$

and

$$
g_\rho(\tau) = f_\rho(\tau) - \frac{\gamma}{\beta_* + \beta\,\bar{b}} \left(d_{0,\rho}(\tau) + b^{(1-\rho)(1-\kappa)}\rho \int_0^\tau (\tau^\rho - \sigma^\rho)^{-\kappa} \sigma^{\rho-1} K_\rho(\tau, \sigma) d\sigma \right). \tag{3.4.3}
$$

Here

$$f_\rho(\tau) := f(b^{1-\rho}\tau^\rho), \quad d_{0,\rho}(\tau) := d_0(b^{1-\rho}\tau^\rho), \quad d_{1,\rho}(\tau) := d_1(b^{1-\rho}\tau^\rho),$$

$$K_\rho(\tau,\sigma) := K(b^{1-\rho}\tau^\rho, b^{1-\rho}\sigma^\rho), \quad L_\rho(\tau,\sigma) := L(b^{1-\rho}\tau^\rho, b^{1-\rho}\sigma^\rho),$$

$$b_{i,\rho} := b^{(\rho-1)/\rho}b_i^{1/\rho} \in (0,b], \; i = 1,\ldots,l, \quad \bar{b}_\rho := b^{(\rho-1)/\rho}\bar{b}^{1/\rho} \in (0,b]$$

and J_ρ^δ ($\delta > 0$) is defined by the formula (2.8.4).

Let $y(t)$ be the solution of problem (3.0.1)–(3.0.2). Using in (3.1.4) the change of variables (2.8.2) we see that $y_\rho(\tau) := y(b^{1-\rho}\tau^\rho)$ can be expressed in the form

$$y_\rho(\tau) = (J_\rho^{\alpha_2}z_\rho)(\tau) + (\beta_* + \beta\,\bar{b})^{-1}\left[\gamma - \sum_{k=1}^l \beta_k(J_\rho^{\alpha_2}z_\rho)(b_{k,\rho}) - \beta\,(J_\rho^{\alpha_2+1}z_\rho)(\bar{b}_\rho)\right],$$

(3.4.4)

where $0 \le \tau \le b$.

Approximations $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ $(m, N \in \mathbb{N})$ to the exact solution z_ρ of equation (3.4.1) we find by collocation conditions

$$z_{\rho,N}(t_{jk}) = (T_\rho z_{\rho,N})(t_{jk}) + g_\rho(t_{jk}), \quad k = 1,\ldots,m, \; j = 1,\ldots,N. \qquad (3.4.5)$$

Here T_ρ, g_ρ and t_{jk} are defined by (3.4.2), (3.4.3) and (2.7.3), respectively. Conditions (3.4.5) have an operator equation representation

$$z_{\rho,N} = \mathcal{P}_N T_\rho z_{\rho,N} + \mathcal{P}_N g_\rho, \qquad (3.4.6)$$

with an interpolation operator $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0,T] \to S_{m-1}^{(-1)}(\Pi_N)$ defined by (2.7.5).

As in Section 3.3, the collocation conditions (3.4.5) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$. If $\eta_1 > 0$ or $\eta_m < 1$ then we can again use the Lagrange fundamental polynomial representation:

$$z_{\rho,N}(\tau) = \sum_{\lambda=1}^N \sum_{\mu=1}^m c_{\lambda\mu}\varphi_{\lambda\mu}(\tau), \quad \tau \in [0,b], \qquad (3.4.7)$$

where $\varphi_{\lambda\mu}$ ($\lambda = 1,\ldots,N$, $\mu = 1,\ldots,m$) are defined by (3.3.5). In this case $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ and $z_{\rho,N}(t_{jk}) = c_{jk}$, $k = 1,\ldots,m$, $j = 1,\ldots,N$. Searching for the solution of (3.4.5) in the form (3.4.7), we obtain a system of linear algebraic equations with respect to the coefficients $\{c_{jk}\}$:

$$c_{jk} = \sum_{\lambda=1}^N \sum_{\mu=1}^m (T_\rho\varphi_{\lambda\mu})(t_{jk})c_{\lambda\mu} + g_\rho(t_{jk}), \quad k = 1,\ldots,m, \; j = 1,\ldots,N. \quad (3.4.8)$$

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Approximation $y_{\rho,N}$ to y_ρ we find by the formula

$$y_{\rho,N}(\tau) = (J_\rho^{\alpha_2} z_{\rho,N})(\tau) + (\beta_* + \beta\,\bar{b})^{-1}$$
$$\times \left[\gamma - \sum_{k=1}^{l} \beta_k (J_\rho^{\alpha_2} z_{\rho,N})(b_{k,\rho}) - \beta\,(J_\rho^{\alpha_2+1} z_{\rho,N})(\bar{b}_\rho)\right], \qquad (3.4.9)$$

where $0 \le \tau \le b$ and $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ is determined by (3.4.5). After solving the linear system (3.4.8) for the unknown coefficients $\{c_{jk}\}$, by using (3.4.7) and (3.4.9) we get the following formula for the approximate solution $y_{\rho,N}$:

$$y_{\rho,N}(\tau) = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}(J_\rho^{\alpha_2}\varphi_{\lambda\mu})(\tau) + (\beta_* + \beta\,\bar{b})^{-1}$$
$$\times \left[\gamma - \sum_{k=1}^{l} \beta_k \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}(J_\rho^{\alpha_2}\varphi_{\lambda\mu})(b_{k,\rho}) - \beta \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}(J_\rho^{\alpha_2+1}\varphi_{\lambda\mu})(\bar{b}_\rho)\right].$$
$$(3.4.10)$$

Approximations $y_N(t)$ to the solution $y(t)$ of problem (3.0.1)–(3.0.2) we find by setting

$$y_N(t) := y_{\rho,N}(b^{(\rho-1)/\rho}t^{1/\rho}), \quad t \in [0,b]. \qquad (3.4.11)$$

For method (3.4.5),(3.4.9) we can formulate the following Theorems 3.4.1 and 3.4.2 below. These theorems follow from the more general Theorems 4.4.1 and 4.4.2, respectively.

Theorem 3.4.1. (i) *Let $N, m \in \mathbb{N}$, $\rho \ge 1$, $r \ge 1$ and assume that the grid points (2.7.1) with collocation points (2.7.3) and arbitrary parameters $\eta_1, \ldots, \eta_m$ satisfying (2.7.4) are used. Assume that conditions (3.0.3) are satisfied, $d_0, d_1 \in C[0,b]$, $f \in C[0,b]$ and $K \in C(\Delta)$. Moreover, assume that the problem (3.0.1)-(3.0.2) with $f = 0$ and $\gamma = 0$ has in $C[0,b]$ only the trivial solution $y = 0$.*

Then problem (3.0.1)-(3.0.2) has a unique solution $y \in C[0,b]$ such that $D_{Cap}^{\alpha_2} y \in C[0,b]$. There exists an integer N_0 such that for all $N \ge N_0$ equation (3.4.6) possesses a unique solution $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$, determining by (3.4.9) and (3.4.11) a unique approximation y_N to y, the solution of (3.0.1)-(3.0.2), and

$$\|y - y_N\|_\infty \to 0 \quad \text{as} \quad N \to \infty. \qquad (3.4.12)$$

(ii) *If, in addition, $h, f \in C^{q,\mu}(0, b]$, $K \in C^q(\Delta)$, where $q := m$ and with $\mu \in \mathbb{R}$, $\mu < 1$, then for all $N \geq N_0$ the following error estimate holds:*

$$\|y - y_N\|_\infty \leq c \begin{cases} N^{-m} & for \quad m < \rho(1 - \nu), \quad r \geq 1, \\ N^{-m}(1 + \log N) & for \quad m = \rho(1 - \nu), \quad r = 1, \\ N^{-m} & for \quad m = \rho(1 - \nu), \quad r > 1, \\ N^{-\rho r(1-\nu)} & for \quad m > \rho(1 - \nu), \quad 1 \leq r < \frac{m}{\rho(1-\nu)}, \\ N^{-m} & for \quad m > \rho(1 - \nu), \quad r \geq \frac{m}{\rho(1-\nu)}. \end{cases}$$
(3.4.13)

Here ν is determined by (3.2.1) (see Theorem 3.2.1), r is the grid parameter in (2.7.1), ρ is the smoothing parameter in (2.8.2) and c is a positive constant which does not depend on N.

Theorem 3.4.2. *Let $m \in \mathbb{N}$, $\rho \geq 1$ and let the following conditions be fulfilled:*
(i) *the assumptions (i)-(ii) of Theorem 3.4.1 hold with $q := m + 1$;*
(ii) *the quadrature approximation*

$$\int_0^1 F(x)\, dx \approx \sum_{k=1}^m w_k\, F(\eta_k),$$

with the knots $\{\eta_k\}$ satisfying (2.7.4) and appropriate weights $\{w_k\}$ is exact for all polynomials F of degree m.

Then problem (3.0.1)-(3.0.2) has a unique solution $y \in C[0, b]$ such that $D_{Cap}^{\alpha_2} y \in C^{q,\nu}(0, b]$. There exists an integer N_0 such that, for $N \geq N_0$, equation (3.4.6) possesses a unique solution $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$, determining by (3.4.9) and (3.4.11) a unique approximation y_N to y, the solution of (3.0.1)-(3.0.2), and the following error estimate holds:

$$\|y - y_N\|_\infty \leq c \begin{cases} N^{-m-(\alpha_2-\alpha_1)} & for \quad m < \rho(1 + \alpha_2 - \alpha_1 - \nu) \\ & and \quad r \geq 1, \\ N^{-m-(\alpha_2-\alpha_1)}(1 + \log N) & for \quad m = \rho(1 + \alpha_2 - \alpha_1 - \nu) \\ & and \quad r = 1, \\ N^{-m-(\alpha_2-\alpha_1)} & for \quad m = \rho(1 + \alpha_2 - \alpha_1 - \nu) \\ & and \quad r > 1, \\ N^{-\rho r(1+\alpha_2-\alpha_1-\nu)} & for \quad m > \rho(1 + \alpha_2 - \alpha_1 - \nu) \\ & and \quad 1 \leq r < \frac{m+\alpha_2-\alpha_1}{\rho(1+\alpha_2-\alpha_1-\nu)}, \\ N^{-m-(\alpha_2-\alpha_1)} & for \quad m > \rho(1 + \alpha_2 - \alpha_1 - \nu) \\ & and \quad r \geq \frac{m+\alpha_2-\alpha_1}{\rho(1+\alpha_2-\alpha_1-\nu)}. \end{cases}$$
(3.4.14)

Here ν is determined by (3.2.1) (see Theorem 3.2.1), r is the grid parameter in (2.7.1), ρ is the smoothing parameter in (2.8.2) and c is a positive constant which does not depend on N.

Chapter 4

General multi-term fractional linear integro-differential equations

In the previous chapter we introduced a way how to construct effective numerical methods for solving fractional linear differential and integro-differential equations with one or two fractional differential operators. Our goal now is to extend this approach to solve a much wider class of problems. More precisely, in the present chapter we consider fractional linear multi-term weakly singular integro-differential equations of the form

$$(D_{Cap}^{\alpha_p} y)(t) + \sum_{i=0}^{p-1} d_i(t)(D_{Cap}^{\alpha_i} y)(t) + \int_0^t (t-s)^{-\kappa} K(t,s)y(s)ds = f(t), \quad 0 \le t \le b,$$

$$(4.0.1)$$

with non-local boundary conditions

$$\sum_{j=0}^{n_0} \beta_{ij0}\, y^{(j)}(0) + \sum_{k=1}^l \sum_{j=0}^{n_1} \beta_{ijk}\, y^{(j)}(b_k) + \beta_i \int_0^{\bar{b}_i} y(s)ds = \gamma_i, \quad i = 0, \ldots, n-1.$$

$$(4.0.2)$$

Here $\beta_{ij0}, \beta_{ijk}, \beta_i, \gamma_i \in \mathbb{R}$, $p \in \mathbb{N}$, $n_0, n_1 \in \mathbb{N}_0$,

$$0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_p \le n, \quad n := \lceil \alpha_p \rceil, \quad 0 \le \kappa < 1,$$

$$0 < b_1 < \cdots < b_l \le b, \quad 0 < \bar{b}_i \le b, \quad i = 0, \ldots, n-1, \quad n_0 < n, \ n_1 < n, \ (4.0.3)$$

$d_i : [0, b] \to \mathbb{R}$ $(i = 0, \ldots, p - 1)$, $f : [0, b] \to \mathbb{R}$, $K : \Delta \to \mathbb{R}$ are some given continuous functions, $\Delta := \{(s, t) : 0 \leq s \leq t \leq b\}$, and $D_{Cap}^{\alpha_i} y$ $(i = 0, \ldots, p)$ are Caputo derivatives of an unknown function y.

Following the ideas of [68–70, 73], we construct in this chapter a class of high-order methods for the numerical solution of (4.0.1)–(4.0.2). Similarly to Chapter 3, we first introduce an integral equation reformulation of the underlying problem and prove some results about the existence, uniqueness and regularity of the exact solution of (4.0.1)–(4.0.2). Using this information we regularize the solution by a suitable smoothing transformation. After that we solve the transformed equation by a piecewise polynomial collocation method on a mildly graded or uniform grid. Optimal global convergence estimates are derived and a superconvergence result for a special choice of collocation parameters is established. Numerical illustrations confirming the convergence estimates are given in Chapter 7.

4.1 Integral equation reformulation

Let $n = \lceil \alpha_p \rceil \in \mathbb{N}$ and let $y \in C^{n-1}[0, b]$ be an arbitrary function such that $D_{Cap}^{\alpha_p} y \in C[0, b]$. We denote $z := D_{Cap}^{\alpha_p} y$. Then (see Section 2.5)

$$y(t) = (J^{\alpha_p} z)(t) + \sum_{\lambda=0}^{n-1} c_\lambda \, t^\lambda, \quad t \in [0, b], \tag{4.1.1}$$

where $c_\lambda \in \mathbb{R}$ $(\lambda = 0, \ldots, n - 1)$ are some constants. From properties (2.5.2) and (2.5.6) we see that for y in the form (4.1.1) we can write

$$y^{(j)}(0) = j! c_j, \quad y^{(j)}(c) = (J^{\alpha_p - j} z)(c) + \sum_{\lambda=j}^{n-1} \frac{\lambda! \, c_\lambda}{(\lambda - j)!} \, c^{\lambda - j}, \quad c \in [0, b],$$

where $j = 0, \ldots, n - 1$, and, using (2.5.3), we have that

$$\int_0^a y(s) ds = (J^{\alpha_p + 1} z)(a) + \sum_{\lambda=0}^{n-1} \frac{c_\lambda}{\lambda + 1} \, a^{\lambda+1}, \quad a \in [0, b].$$

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Thus, a function y in the form (4.1.1) satisfies the conditions (4.0.2) if and only if

$$\sum_{j=0}^{n_0} \beta_{ij0}\, j!\, c_j + \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk}\left[(J^{\alpha_p-j}z)(b_k) + \sum_{\lambda=j}^{n-1} \frac{\lambda!}{(\lambda-j)!}\, b_k^{\lambda-j}\, c_\lambda\right]$$

$$+ \beta_i \left[(J^{\alpha_p+1}z)(\bar{b}_i) + \sum_{\lambda=0}^{n-1} \frac{c_\lambda}{\lambda+1}\, \bar{b}_i^{\lambda+1}\right] = \gamma_i, \qquad (4.1.2)$$

where $i = 0, \ldots, n-1$. By setting $\beta_{ij0} = 0$ for $j = n_0 + 1, \ldots, n-1$ and $\beta_{ijk} = 0$ for $j = n_1 + 1, \ldots, n-1$ $(k = 1, \ldots, l)$, we can write

$$\sum_{j=0}^{n_0} \beta_{ij0}\, j!\, c_j = \sum_{j=0}^{n-1} \beta_{ij0}\, j!\, c_j \quad (i = 0, \ldots, n-1)$$

and

$$\sum_{j=0}^{n_1} \beta_{ijk}\left[(J^{\alpha_p-j}z)(b_k) + \sum_{\lambda=j}^{n-1} \frac{\lambda!}{(\lambda-j)!}\, b_k^{\lambda-j}\, c_\lambda\right]$$

$$= \sum_{j=0}^{n_1} \beta_{ijk}(J^{\alpha_p-j}z)(b_k) + \sum_{j=0}^{n-1}\sum_{\lambda=0}^{j} \beta_{i\lambda k} \frac{j!}{(j-\lambda)!}\, b_k^{j-\lambda}\, c_j,$$

for $k = 1, \ldots, l$, $i = 0 \ldots, n-1$. The conditions (4.1.2) can thus be rewritten in the form

$$\sum_{j=0}^{n-1}\left[j!\, \beta_{ij0} + \sum_{k=1}^{l}\sum_{\lambda=0}^{j} \beta_{i\lambda k} \frac{j!}{(j-\lambda)!}\, b_k^{j-\lambda} + \frac{\beta_i}{j+1}\, \bar{b}_i^{j+1}\right] c_j$$

$$= \gamma_i - \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk}(J^{\alpha_p-j}z)(b_k) - \beta_i(J^{\alpha_p+1}z)(\bar{b}_i), \quad i = 0, \ldots, n-1, \quad (4.1.3)$$

giving us an algebraic linear system of n equations with respect to $c_0, \ldots, c_{n-1}$. Let

$$M := \left(j!\, \beta_{ij0} + \sum_{k=1}^{l}\sum_{\lambda=0}^{j} \beta_{i\lambda k} \frac{j!}{(j-\lambda)!}\, b_k^{j-\lambda} + \frac{\beta_i}{j+1}\, \bar{b}_i^{j+1}\right)_{i,j=0}^{n-1}$$

be the matrix of the system (4.1.3).

In the sequel we assume that the matrix M is regular. Observe that the matrix M is regular if and only if from all polynomials y of degree $n-1$ only

$y = 0$ satisfies the homogeneous conditions

$$\sum_{j=0}^{n_0} \beta_{ij0} y^{(j)}(0) + \sum_{k=1}^{l} \sum_{j=0}^{n_1} \beta_{ijk} y^{(j)}(b_k) + \beta_i \int_0^{\bar{b}_i} y(s) ds = 0 , \quad i = 0, \ldots, n-1 ,$$

$$(4.1.4)$$

corresponding to the conditions (4.0.2) by $\gamma_i = 0$, $i = 0, \ldots, n-1$.

Indeed, substituting (4.1.1) with $z = 0$ into (4.1.4) we obtain a homogeneous system of algebraic equations with respect to $c_0, c_1, \ldots, c_{n-1}$. This system coincides with (4.1.3) by $\gamma_i = 0$ ($i = 0, \ldots, n-1$) and $z = 0$. Therefore, the homogeneous system corresponding to (4.1.3) has only the trivial solution $c_0 = c_1 = \cdots = c_{n-1} = 0$ (and thus M is regular) if and only if from all polynomials y of degree $n-1$ only $y = 0$ satisfies (4.1.4).

Let $M^{-1} = (p_{ij})_{i,j=0}^{n-1}$ be the inverse of M. Using M^{-1}, the solution of the system (4.1.3) can be written in the form

$$c_\lambda = \delta_\lambda - \sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk} (J^{\alpha_p - j} z)(b_k) - \omega_\lambda (J^{\alpha_p + 1} z)(\bar{b}_\lambda) , \quad \lambda = 0, \ldots, n-1,$$

where

$$\delta_\lambda := \sum_{\mu=0}^{n-1} p_{\lambda\mu} \gamma_\mu , \quad \kappa_{\lambda jk} := \sum_{\mu=0}^{n-1} p_{\lambda\mu} \beta_{\mu jk}, \quad \omega_\lambda := \sum_{\mu=0}^{n-1} p_{\lambda\mu} \beta_\lambda. \quad (4.1.5)$$

Therefore a function y in the form (4.1.1) satisfies the conditions (4.0.2) if and only if it can be expressed by the formula

$$y = Gz + Q, \quad (4.1.6)$$

where (for $t \in [0, b]$)

$$(Gz)(t) := (J^{\alpha_p} z)(t) - \sum_{\lambda=0}^{n-1} t^\lambda \left[\sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk} (J^{\alpha_p - j} z)(b_k) + \omega_\lambda (J^{\alpha_p + 1} z)(\bar{b}_\lambda) \right] ,$$

$$(4.1.7)$$

$$Q(t) := \sum_{\lambda=0}^{n-1} \delta_\lambda t^\lambda. \quad (4.1.8)$$

Suppose now that $y \in C^{n-1}[0, b]$ is a solution of the boundary value problem (4.0.1)–(4.0.2) such that $D_{Cap}^{\alpha_p} y \in C[0, b]$. Then it follows from the observations

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above that y has the form (4.1.6) where $z = D_{Cap}^{\alpha_p} y \in C[0, b]$ and G and Q are defined by the formulas (4.1.7) and (4.1.8), respectively. Inserting (4.1.6) into (4.0.1) we see that

$$z(t) + \sum_{i=0}^{p-1} d_i(t)(D_{Cap}^{\alpha_i}[Gz + Q])(t) + \int_0^t (t-s)^{-\kappa} K(t, s)[(Gz)(s) + Q(s)]ds = f(t).$$

Therefore $z = D_{Cap}^{\alpha_p} y$ satisfies the equation

$$z = Tz + g \tag{4.1.9}$$

with

$$(Tz)(t) := -\sum_{i=0}^{p-1} d_i(t)(D_{Cap}^{\alpha_i} Gz)(t) - \int_0^t (t-s)^{-\kappa} K(t, s)(Gz)(s)ds, \tag{4.1.10}$$

$$g(t) := f(t) - \sum_{i=0}^{p-1} d_i(t)(D_{Cap}^{\alpha_i} Q)(t) - \int_0^t (t-s)^{-\kappa} K(t, s)Q(s)ds, \tag{4.1.11}$$

where $0 \le t \le b$. Conversely, it turns out that if $z \in C[0, b]$ is a solution of equation (4.1.9) then y defined by (4.1.6) belongs to $C^{n-1}[0, b]$ and is a solution to (4.0.1)–(4.0.2). In this sense equation (4.1.9) is equivalent to the boundary value problem (4.0.1)–(4.0.2).

From (2.5.6), (2.5.10) and (4.1.7) it follows that for $i = 0, \dots, n-1$

$$(D_{Cap}^{\alpha_i} Gz)(t) = (J^{\alpha_p - \alpha_i} z)(t)$$

$$- \sum_{\substack{\lambda = \lceil \alpha_i \rceil \\ \alpha_i \le n-1}}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} t^{\lambda - \alpha_i} \left(\sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk}(J^{\alpha_p - j} z)(b_k) + \omega_\lambda (J^{\alpha_p + 1} z)(\bar{b}_\lambda) \right);$$

$$\int_0^t (t-s)^{-\kappa} K(t, s)(Gz)(s)ds = \int_0^t (t-s)^{-\kappa} K(t, s)(J^{\alpha_p} z)(t)ds$$

$$- \sum_{\lambda=0}^{n-1} \left(\sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk}(J^{\alpha_p - j} z)(b_k) + \omega_\lambda (J^{\alpha_p + 1} z)(\bar{b}_\lambda) \right) \int_0^t (t-s)^{-\kappa} K(t, s)s^\lambda ds$$

$$= \frac{1}{\Gamma(\alpha_p)} \int_0^t (t-s)^{\alpha_p - \kappa} L(t, s)z(s)ds$$

$$- \sum_{\lambda=0}^{n-1} t^{1+\lambda-\kappa} \left(\sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk}(J^{\alpha_p - j} z)(b_k) + \omega_\lambda (J^{\alpha_p + 1} z)(\bar{b}_\lambda) \right) K_\lambda(t);$$

from (2.5.6), (2.5.10) and (4.1.8) it follows that for $i = 0, \ldots, n-1$

$$(D_{Cap}^{\alpha_i} Q)(t) = \sum_{i=0}^{p-1} d_i(t) \sum_{\substack{\lambda = \lceil \alpha_i \rceil \\ \alpha_i \leq n-1}}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} \, \delta_\lambda \, t^{\lambda - \alpha_i};$$

$$\int_0^t (t-s)^{-\kappa} K(t,s) Q(s) ds = \sum_{\lambda=0}^{n-1} \delta_\lambda \, K_\lambda(t) \, t^{1 + \lambda - \kappa}.$$

Here $t \in [0,b]$, $\kappa_{\lambda j k}$, δ_λ, ω_λ are given by (4.1.5) and

$$L(t,s) := \int_0^1 \tau^{\alpha_p - 1} (1-\tau)^{-\kappa} K(t, (t-s)\tau + s) d\tau, \quad 0 \leq s \leq t \leq b, \qquad (4.1.12)$$

$$K_\lambda(t) := \int_0^1 (1-s)^{-\kappa} s^\lambda K(t, ts) ds, \quad 0 \leq t \leq b, \quad \lambda \geq 0. \qquad (4.1.13)$$

Thus, from (4.1.10) and (4.1.11) we have for $t \in [0,b]$ that

$$(Tz)(t) = -\sum_{i=0}^{p-1} d_i(t) \Bigg[(J^{\alpha_p - \alpha_i} z)(t)$$

$$- \sum_{\substack{\lambda = \lceil \alpha_i \rceil \\ \alpha_i \leq n-1}}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} \, t^{\lambda - \alpha_i} \left(\sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda j k} (J^{\alpha_p - j} z)(b_k) + \omega_\lambda (J^{\alpha_p + 1} z)(\bar{b}_\lambda) \right) \Bigg]$$

$$- \frac{1}{\Gamma(\alpha_p)} \int_0^t (t-s)^{\alpha_p - \kappa} L(t,s) z(s) ds$$

$$+ \sum_{\lambda=0}^{n-1} t^{1 + \lambda - \kappa} \left(\sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda j k} (J^{\alpha_p - j} z)(b_k) + \omega_\lambda (J^{\alpha_p + 1} z)(\bar{b}_\lambda) \right) K_\lambda(t) \qquad (4.1.14)$$

and

$$g(t) = f(t) - \sum_{i=0}^{p-1} d_i(t) \sum_{\substack{\lambda = \lceil \alpha_i \rceil \\ \alpha_i \leq n-1}}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} \, \delta_\lambda \, t^{\lambda - \alpha_i} - \sum_{\lambda=0}^{n-1} \delta_\lambda \, K_\lambda(t) \, t^{1 + \lambda - \kappa}.$$

$$(4.1.15)$$

Remark 4.1.1. *Note that if $K \in C^q(\Delta)$, $q \in \mathbb{N}_0$, then $L \in C^q(\Delta)$ and $K_\lambda \in C^q[0,b]$, $\lambda \geq 0$.*

Remark 4.1.2. *A special case of problem (4.0.1)–(4.0.2) is the initial value problem:*

$$(D_{Cap}^{\alpha_p} y)(t) + \sum_{i=0}^{p-1} d_i(t) (D_{Cap}^{\alpha_i} y)(t) + \int_0^t (t-s)^{-\kappa} K(t,s) y(s) ds = f(t), \quad 0 \leq t \leq b,$$

$$y^{(i)}(0) = \gamma_i, \quad i = 0, \ldots, n-1, \quad n = \lceil \alpha_p \rceil, \quad 0 \leq \kappa < 1, \qquad (4.1.16)$$

where d_i $(i = 1, \ldots, p-1)$, K and f are given continuous functions on $[0, b]$. The solution of (4.1.16) can be expressed by the formula (see (4.1.6))

$$y = Gz + Q,$$

where

$$z = D_{Cap}^{\alpha_p} y, \quad G = J^{\alpha_p}, \quad Q(t) = \sum_{i=0}^{n-1} \frac{\gamma_i}{i!} t^i.$$

The integral equation $z = Tz + g$ (see (4.1.9)) corresponding to the initial value problem (4.1.16) is given by

$$(Tz)(t) = -\sum_{i=0}^{p-1} d_i(t)(J^{\alpha_p - \alpha_i} z)(t) - \frac{1}{\Gamma(\alpha_p)} \int_0^t (t-s)^{\alpha_p - \kappa} L(t, s) z(s) ds$$

and

$$g(t) = f(t) - \sum_{i=0}^{p-1} d_i(t) \sum_{\substack{\lambda = \lceil \alpha_i \rceil \\ \alpha_i \leq n-1}}^{n-1} \frac{\gamma_\lambda}{\Gamma(\lambda + 1 - \alpha_i)} t^{\lambda - \alpha_i} - \sum_{\lambda=0}^{n-1} \frac{\gamma_\lambda}{\lambda!} K_\lambda(t) t^{1 + \lambda - \kappa},$$

where L and K_λ $(\lambda = 0, \ldots, n-1)$ are defined by (4.1.12) and (4.1.13), respectively.

4.2 Existence, uniqueness and smoothness of the solution

The following theorem characterizes the existence, uniqueness and regularity properties of the solution of (4.0.1)-(4.0.2).

Theorem 4.2.1. (i) *Assume that* $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_p \leq n$, $0 \leq \kappa < 1$, $d_i \in C[0, b]$ $(i = 0, \ldots, p-1)$, $f \in C[0, b]$, $K \in C(\Delta)$ *and* $n_0 < n$, $n_1 < n$, *where* $n := \lceil \alpha_p \rceil$. *Moreover, assume that problem* (4.0.1)–(4.0.2) *with* $f = 0$ *and* $\gamma_i = 0$ $(i = 0, \ldots, n-1)$ *has in* $C[0, b]$ *only the trivial solution* $y = 0$, *and from all polynomials* y *of degree* $n-1$ *only* $y = 0$ *satisfies the conditions* (4.1.4).

Then problem (4.0.1)-(4.0.2) *has a unique solution* $y \in C^{n-1}[0, b]$. *Moreover, we have* $D_{Cap}^{\alpha_p} y \in C[0, b]$.

(ii) *Assume that* (i) *holds and let* $d_i \in C^{q,\mu}(0, b]$ $(i = 0, \ldots, p-1)$, $f \in C^{q,\mu}(0, b]$, $K \in C^q(\Delta)$, *where* $q \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$.

Then problem (4.0.1)–(4.0.2) possesses a unique solution $y \in C^{n-1}[0, b]$ such that $y \in C^{q,\nu}(0, b]$ and $D_{Cap}^{\alpha_p}y \in C^{q,\nu}(0, b]$, where

$$\nu := \max\{\mu, \nu_1, \nu_2, \kappa\} \tag{4.2.1}$$

with

$$\nu_1 \quad := \quad \max\{1 - (\alpha_p - \alpha_i) : \ \alpha_p - \alpha_i \notin \mathbb{N}, \ i = 0, \dots, p-1\},$$
$$\nu_2 \quad := \quad \max\{1 - (\lceil \alpha_i \rceil - \alpha_i) : \ \alpha_i < n-1, \ \alpha_i \notin \mathbb{N}_0, \ i = 0, \dots, p-1\}.$$

If for all indices $i = 0, \dots, p-1$ we have $\alpha_p - \alpha_i \in \mathbb{N}$, then we may set $\nu = \{\mu, \nu_2, \kappa\}$. Analogously, if we have $\alpha_i \in \mathbb{N}_0$ for all indices $i = 0, \dots, p-1$ such that $\alpha_i < n-1$, then we may set $\nu = \{\mu, \nu_1, \kappa\}$.

Proof. (i) First, we observe that the forcing function g of equation $z = Tz + g$ (see (4.1.9) and (4.1.15)) belongs to $C[0, b]$. This follows from $f \in C[0, b]$, $d_i \in C[0, b]$, $\lambda = \lceil \alpha_i \rceil \geq \alpha_i$ ($i = 0, \dots, p-1$) and from Remark 4.1.1 with $q = 0$.

Next, due to (4.1.14), operator T can be rewritten in the form

$$T = -\sum_{i=0}^{p-1} D_i\left[J^{\alpha_p - \alpha_i} - \sum_{\substack{\lambda = \lceil \alpha_i \rceil \\ \alpha_i \leq n-1}}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} A_{\lambda - \alpha_i} M_\lambda\right] - B_L + \sum_{\lambda=0}^{n-1} A_{1+\lambda-\kappa} M_\lambda \overline{K}_\lambda,$$

$$\tag{4.2.2}$$

with $D_i, A_\sigma, M_\lambda, B_L$ and $\overline{K}_\lambda$ defined by

$$(D_i x)(t) := d_i(t)x(t), \quad i = 0, \dots, p-1, \quad (A_\sigma x)(t) := t^\sigma x(t), \quad \sigma \in \mathbb{R}, \sigma \geq 0,$$

$$(M_\lambda x)(t) := \sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk}(J^{\alpha_p - j}x)(b_k) + \omega_\lambda(J^{\alpha_p + 1}x)(\bar{b}_\lambda), \quad \lambda = 0, \dots, n-1,$$

$$(B_L x)(t) := \frac{1}{\Gamma(\alpha_p)} \int_0^t (t-s)^{\alpha_p - \kappa} L(t, s)x(s)ds,$$

$$(\overline{K}_\lambda x)(t) := K_\lambda(t)x(t), \quad \lambda = 0, \dots, n-1,$$

where $t \in [0, b]$ and $x \in C[0, b]$. Here (see Remark 4.1.1) $L \in C(\Delta)$ and $K_\lambda \in C[0, b]$ are given by (4.1.12) and (4.1.13), respectively. Using Lemma 2.6.2 we obtain that J^δ ($\delta > 0$), M_λ ($\lambda = 0, \dots, n-1$) and B_L are compact as operators from $C[0, b]$ into $C[0, b]$. Clearly D_i ($i = 0, \dots, p-1$), $\overline{K}_\lambda$ ($\lambda = 0, \dots, n-1$) and A_σ ($\sigma \in \mathbb{R}, \sigma \geq 0$) are bounded as operators from $C[0, b]$ into $C[0, b]$. This

together with Theorem 2.2.2 yields that T, given by (4.2.2), is compact as an operator from $C[0, b]$ into $C[0, b]$.

Note that if $f = 0$ and $\gamma_i = 0$ $(i = 0, \ldots, n-1)$, then $\delta_\lambda = 0$ $(\lambda = 0, \ldots, n-1)$ (see (4.1.5)) and thus $g = 0$ (see (4.1.15)). From this we obtain that if the homogeneous equation corresponding to problem (4.0.1)–(4.0.2) has only the trivial solution $y = 0$, then $z = Tz$ has in $C[0, b]$ only the trivial solution $z = 0$. Since $g \in C[0, b]$, we obtain by Theorem 2.2.3 that equation $z = Tz + g$ possesses a unique solution $z \in C[0, b]$. This together with (4.1.6) yields that problem (4.0.1)-(4.0.2) has a unique solution $y \in C^{n-1}[0, b]$ such that $D_{Cap}^{\alpha_p} y = z \in C[0, b]$ (see Section 2.5).

(ii) Let us prove that $z = D_{Cap}^{\alpha_p} y$ belongs to $C^{q,\nu}(0, b]$ (with $q \in \mathbb{N}$ and ν given by (4.2.1)) for $K \in C^q(\Delta)$, d_i $(i = 0, \ldots, p-1) \in C^{q,\mu}(0, b]$, $f \in C^{q,\mu}(0, b]$, $\mu \in \mathbb{R}$, $\mu < 1$. To this end we first establish that g, the forcing function of equation $z = Tz + g$, belongs to $C^{q,\nu}(0, b]$. Indeed, it follows from (4.1.15) that $g = g_1 + g_2 + g_3$,

$$g_1(t) := f(t), \quad g_2(t) := -\sum_{i=0}^{p-1} d_i(t) \sum_{\substack{\lambda = \lceil \alpha_i \rceil \\ \alpha_i \le n-1}}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} \delta_\lambda \, t^{\lambda - \alpha_i},$$

$$g_3(t) := -\sum_{\lambda=0}^{n-1} \delta_\lambda \, K_\lambda(t) \, t^{1+\lambda-\kappa},$$

where $t \in [0, b]$. Clearly $g_1 = f \in C^{q,\mu}(0, b] \subset C^{q,\nu}(0, b]$. Note that, if $\delta \in \mathbb{N}_0$, then for all $\lambda \ge \lceil \delta \rceil, \lambda \in \mathbb{N}_0$ we have $t^{\lambda - \delta} \in C^q[0, b] \subset C^{q,\nu}(0, b]$ for arbitrary $q \in \mathbb{N}$ and $\nu < 1$. If $\delta \notin \mathbb{N}_0$, then for all $\lambda \ge \lceil \delta \rceil, \lambda \in \mathbb{N}_0$ we have $t^{\lambda - \delta} \in C^{q, 1 - \lceil \delta \rceil + \delta}(0, b]$ (see Section 2.6). Thus, since $d_i \in C^{q,\mu}(0, b]$ $(i = 0, \ldots, p-1)$, by using Lemma 2.6.1 we can write $g_2 \in C^{q,\nu}(0, b]$ with ν defined by (4.2.1). Finally, since for all $\lambda \in \mathbb{N}_0$ it holds that $t^{1+\lambda-\kappa} \in C^{q,\kappa}(0, b] \subset C^{q,\nu}(0, b]$ and $K_\lambda \in C^q[0, b]$, we have $g_3 \in C^{q,\nu}(0, b]$ and hence $g = g_1 + g_2 + g_3 \in C^{q,\nu}(0, b]$.

If there exists $i \in \{0, \ldots, p-1\}$ such that $\alpha_i \notin \mathbb{N}_0$, then it follows from the definition of ν that $1 - (\alpha_p - \alpha_i) \le \nu$ and therefore from Lemma 2.6.2 we have that $J^{\alpha_p - \alpha_i}$ is compact as an operator from $C^{q,\nu}(0, b]$ into $C^{q,\nu}(0, b]$. If $\alpha_p \in \mathbb{N}, \alpha_i \in \mathbb{N}_0$, then $J^{\alpha_p - \alpha_i}$ is compact as an operator from $C^{q,\nu}(0, b]$ into $C^{q,\nu}(0, b]$. Clearly D_i $(i = 0, \ldots, p-1)$ and $A_{\lambda-\alpha_i}$ $(\lambda = \lceil \alpha_i \rceil, \ldots, n-1, \alpha_i \le n-1, i = 0, \ldots, p-1)$ are linear and bounded as operators from $C^{q,\nu}(0, b]$ into $C^{q,\nu}(0, b]$. Linear operators (functionals) $M_\lambda : C^{q,\nu}(0, b] \to \mathbb{R}$ $(\lambda = 0, \ldots, n-1)$ are bounded

and consequently compact in $C^{q,\nu}(0,b]$. Thus, as the composition of a compact and bounded operator is compact, we see that $D_i J^{\alpha_p - \alpha_i}$ ($i = 0, \dots, p-1$) and $D_i A_{\lambda - \alpha_i} M_\lambda$ ($\lambda = \lceil \alpha_i \rceil, \dots, n-1, \alpha_i \leq n-1, i = 0, \dots, p-1$) are linear and compact as operators from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$.

Similarly, we see that operators $\overline{K}_\lambda$, $A_{1+\lambda-\kappa}$ ($\lambda = 0, \dots, n-1$) are linear and bounded as operators from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$, thus the operators $A_{1+\lambda-\kappa} M_\lambda \overline{K}_\lambda$ ($\lambda = 0, \dots, n-1$) are linear and compact as operators from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Finally, since $\kappa - \alpha_p < \kappa \leq \nu$, it follows from Lemma 2.6.2 that B_L is compact as an operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Thus, T defined by (4.2.2) is linear and compact as an operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Since the homogeneous equation $z = Tz$ has in $C^{q,\nu}(0,b] \subset C[0,b]$ only the trivial solution $z = 0$, it follows from Theorem 2.2.3 that equation $z = Tz + g$ has a unique solution $z \in C^{q,\nu}(0,b]$.

Note that the inclusion $z \in C^{q,\nu}(0,b]$ together with (4.1.6) and Lemma 2.6.2 yields that problem (4.0.1)-(4.0.2) possesses a unique solution $y \in C^{q,\nu}(0,b]$ such that $D_{Cap}^{\alpha_p} y = z \in C^{q,\nu}(0,b]$. $\qquad\square$

4.3 Smoothing transformation and approximate solutions

Similarly to Section 3.4 (see also Section 2.8), let us consider for equation (4.1.9) a change of variables (see (2.8.2))

$$ t = b^{1-\rho} \tau^\rho, \quad s = b^{1-\rho} \sigma^\rho, \quad \tau, \sigma \in [0, b], $$

where $\rho \in [1, \infty)$. Using in (4.1.9) this change of variables, we get for

$$ z_\rho(\tau) := z(b^{1-\rho} \tau^\rho) $$

an integral equation in the form

$$ z_\rho = T_\rho z_\rho + g_\rho. \tag{4.3.1} $$

Here T_ρ is an integral operator defined by the formula (cf. (4.1.14))

$$(T_\rho z_\rho)(\tau) := -\sum_{i=0}^{p-1} d_{i,\rho}(\tau)\left[(J_\rho^{\alpha_p - \alpha_i} z_\rho)(\tau) - \sum_{\substack{\lambda=\lceil \alpha_i\rceil \\ \alpha_i \le n-1}}^{n-1} \frac{\lambda! b^{(1-\rho)(\lambda-\alpha_i)}}{\Gamma(\lambda+1-\alpha_i)}\, \tau^{\rho(\lambda-\alpha_i)}\right.$$

$$\left.\times \left(\sum_{k=1}^{l}\sum_{j=0}^{n_1} \kappa_{\lambda jk}(J_\rho^{\alpha_p - j} z_\rho)(b_{k,\rho}) + \omega_\lambda(J_\rho^{\alpha_p+1} z_\rho)(\bar{b}_{\lambda,\rho})\right)\right]$$

$$-\frac{\rho\, b^{(1-\rho)(1+\alpha_p-\kappa)}}{\Gamma(\alpha_p)}\int_0^\tau (\tau^\rho - \sigma^\rho)^{\alpha_p - \kappa}\sigma^{\rho-1} L_\rho(\tau,\sigma) z_\rho(\sigma) d\sigma$$

$$+\sum_{\lambda=0}^{n-1} b^{(1-\rho)(1+\lambda-\kappa)}\tau^{\rho(1+\lambda-\kappa)} K_{\lambda,\rho}(\tau)$$

$$\times \left(\sum_{k=1}^{l}\sum_{j=0}^{n_1} \kappa_{\lambda jk}(J_\rho^{\alpha_p - j} z_\rho)(b_{k,\rho}) + \omega_\lambda(J_\rho^{\alpha_p+1} z_\rho)(\bar{b}_{\lambda,\rho})\right), \tag{4.3.2}$$

$g_\rho(\tau) := g(b^{1-\rho}\tau^\rho)$ is given by (cf. (4.1.15))

$$g_\rho(\tau) = f_\rho(\tau) - \sum_{i=0}^{p-1} d_{i,\rho}(\tau) \sum_{\substack{\lambda=\lceil \alpha_i\rceil \\ \alpha_i \le n-1}}^{n-1} \frac{\lambda!\, b^{(1-\rho)(\lambda-\alpha_i)}}{\Gamma(\lambda+1-\alpha_i)}\, \delta_\lambda\, \tau^{\rho(\lambda-\alpha_i)}$$

$$-\sum_{\lambda=0}^{n-1} \delta_\lambda\, b^{(1-\rho)(1+\lambda-\kappa)}\, K_{\lambda,\rho}(\tau)\, \tau^{\rho(1+\lambda-\kappa)}, \tag{4.3.3}$$

with $\tau \in [0,b]$,

$$K_{\lambda,\rho}(\tau) := K_\lambda(b^{1-\rho}\tau^\rho),\ \lambda = 0,\dots,n-1, \quad L_\rho(\tau,\sigma) := L(b^{1-\rho}\tau^\rho, b^{1-\rho}\sigma^\rho),$$

$$f_\rho(\tau) := f(b^{1-\rho}\tau^\rho), \quad d_{i,\rho}(\tau) := d_i(b^{1-\rho}\tau^\rho), \quad i = 0,\dots,p-1,$$

$$b_{k,\rho} := b^{(\rho-1)/\rho} b_k^{1/\rho} \in (0,b], \quad k = 1,\dots,l,$$

$$\bar{b}_{\lambda,\rho} := b^{(\rho-1)/\rho}\bar{b}_\lambda^{1/\rho} \in (0,b], \quad \lambda = 0,\dots,n-1$$

and J_ρ^α $(\alpha > 0)$ is given by (2.8.4). Here L and K_λ $(\lambda = 0,\dots,n-1)$ are defined by (4.1.12) and (4.1.13), respectively.

Remark 4.3.1. *On the basis of Remark 4.1.2 we see that for initial value problem (4.1.16) the operator T_ρ and the forcing function g_ρ can be written in the forms*

$$(T_\rho z_\rho)(\tau) = -\sum_{i=0}^{p-1} d_{i,\rho}(\tau)(J_\rho^{\alpha_p - \alpha_i} z_\rho)(\tau)$$

$$-\frac{\rho\, b^{(1-\rho)(1+\alpha_p-\kappa)}}{\Gamma(\alpha_p)}\int_0^\tau (\tau^\rho - \sigma^\rho)^{\alpha_p - \kappa}\sigma^{\rho-1} L_\rho(\tau,\sigma) z_\rho(\sigma) d\sigma$$

and

$$g_\rho(\tau) = f_\rho(\tau) - \sum_{i=0}^{p-1} d_{i,\rho}(\tau) \sum_{\substack{\lambda=\lceil\alpha_i\rceil \\ \alpha_i \leq n-1}}^{n-1} \frac{\lambda! b^{(1-\rho)(\lambda-\alpha_i)}}{\Gamma(\lambda+1-\alpha_i)} \gamma_\lambda \, \tau^{\rho(\lambda-\alpha_i)}$$

$$- \sum_{\lambda=0}^{n-1} \frac{b^{(1-\rho)(\lambda-\alpha_i)}}{\lambda!} \gamma_\lambda \, K_{\lambda,\rho}(\tau) \, \tau^{\rho(1+\lambda-\kappa)},$$

respectively.

Let $y(t)$ be the solution of problem (4.0.1)–(4.0.2). Using in (4.1.6) the change of variables (2.8.2), we see that

$$y_\rho(\tau) := y(b^{1-\rho}\tau^\rho), \quad \tau \in [0, b] \tag{4.3.4}$$

can be expressed in the form

$$y_\rho = G_\rho z_\rho + Q_\rho, \tag{4.3.5}$$

where (cf. (4.1.7))

$$(G_\rho z_\rho)(\tau) := (J_\rho^{\alpha_p} z_\rho)(\tau)$$

$$- \sum_{\lambda=0}^{n-1} b^{(1-\rho)\lambda} \tau^{\rho\lambda} \left[\sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk} (J_\rho^{\alpha_p-j} z_\rho)(b_{k,\rho}) + \omega_\lambda (J_\rho^{\alpha_p+1} z_\rho)(\bar{b}_{\lambda,\rho}) \right],$$

$$\tag{4.3.6}$$

and (cf. (4.1.8))

$$Q_\rho(\tau) := \sum_{\lambda=0}^{n-1} b^{\lambda(1-\rho)} \delta_\lambda \, \tau^{\rho\lambda}, \tag{4.3.7}$$

with $\tau \in [0, b]$.

Approximations $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ $(m, N \in \mathbb{N})$ to the solution z_ρ of (4.3.1) we find by collocation conditions

$$z_{\rho,N}(t_{jk}) = (T_\rho z_{\rho,N})(t_{jk}) + g_\rho(t_{jk}), \quad k = 1, \ldots, m, \ j = 1, \ldots, N, \tag{4.3.8}$$

where T_ρ, g_ρ and t_{jk} are defined by (4.3.2), (4.3.3) and (2.7.3), respectively. Note that conditions (4.3.8) for finding $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ have an operator equation representation

$$z_{\rho,N} = \mathcal{P}_N T_\rho z_{\rho,N} + \mathcal{P}_N g_\rho, \tag{4.3.9}$$

where $\mathcal{P}_N$ is defined by (2.7.5).

The collocation conditions (4.3.8) form a system of equations whose exact form is determined by the choice of a basis in the space $S_{m-1}^{(-1)}(\Pi_N)$. If $\eta_1 > 0$ or $\eta_m < 1$ then we can use the Lagrange fundamental polynomial representation:

$$z_{\rho,N}(\tau) = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}\varphi_{\lambda\mu}(\tau), \quad \tau \in [0,b], \qquad (4.3.10)$$

where $\varphi_{\lambda\mu}$ ($\lambda = 1,\ldots,N, \mu = 1,\ldots,m$) are defined by (3.3.5). Then $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ and $z_{\rho,N}(t_{jk}) = c_{jk}$, $k = 1,\ldots,m$, $j = 1,\ldots,N$. Substituting $z_{\rho,N}(\tau)$ in the form (4.3.10) to (4.3.8), we obtain a system of linear algebraic equations with respect to the coefficients $\{c_{jk}\}$:

$$c_{jk} = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m}(T_\rho\varphi_{\lambda\mu})(t_{jk})c_{\lambda\mu} + g_\rho(t_{jk}), \quad k = 1,\ldots,m, \ j = 1,\ldots,N. \quad (4.3.11)$$

Solving this system of equations we obtain the coefficients $\{c_{jk}\}$ and thus have found the approximation $z_{\rho,N}$ in the form (4.3.10).

Approximation $y_{\rho,N}$ to y_ρ we find by the formula

$$y_{\rho,N} = G_\rho z_{\rho,N} + Q_\rho \qquad (4.3.12)$$

and by substituting $z_{\rho,N}$ in the form (4.3.10) into (4.3.12), we get that

$$y_{\rho,N} = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}G_\rho\varphi_{\lambda\mu} + Q_\rho. \qquad (4.3.13)$$

Finally, approximation y_N to the solution y of problem (4.0.1)–(4.0.2) we find by setting

$$y_N(t) = y_{\rho,N}(b^{(\rho-1)/\rho}t^{1/\rho}), \quad t \in [0,b]. \qquad (4.3.14)$$

4.4 Convergence analysis

The following two theorems characterize the convergence rate of the proposed method.

Theorem 4.4.1. (i) *Let* $m, N \in \mathbb{N}$, $r \geq 1$ *and assume that the grid points* (2.7.1) *with collocation points* (2.7.3) *and arbitrary parameters* $\eta_1,\ldots,\eta_m$ *satisfying* (2.7.4) *are used. Assume that conditions* (4.0.3) *are satisfied,* $d_i \in C[0,b]$ ($i =$

$0 \ldots, p-1$), $f \in C[0, b]$ and $K \in C(\Delta)$. Moreover, assume that problem (4.0.1)–(4.0.2) with $f = 0$ and $\gamma_i = 0$ ($i = 0, \ldots, n-1$) has only the trivial solution $y = 0$ and from all polynomials y of degree $n - 1$ only $y = 0$ satisfies the conditions (4.1.4).

Then problem (4.0.1)–(4.0.2) has a unique solution $y \in C^{n-1}[0, b]$ such that $D_{Cap}^{\alpha_p} y \in C[0, b]$. There exists an integer N_0 such that for all $N \geq N_0$ equation (4.3.9) possesses a unique solution $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$, determining by (4.3.12) and (4.3.14) a unique approximation y_N to y, the solution of (4.0.1)–(4.0.2), and

$$\|y_N - y\|_\infty \to 0 \quad \text{as} \quad N \to \infty. \tag{4.4.1}$$

(ii) If, in addition, $d_i \in C^{q,\mu}(0, b]$ ($i = 0 \ldots, p-1$), $f \in C^{q,\mu}(0, b]$, $K \in C^q(\Delta)$, where $q := m$ and with $\mu \in \mathbb{R}$, $\mu < 1$, then for all $N \geq N_0$ the following error estimate holds:

$$\|y - y_N\|_\infty \leq c \begin{cases} N^{-m} & for \quad m < \rho(1-\nu), \quad r \geq 1, \\ N^{-m}(1 + \log N) & for \quad m = \rho(1-\nu), \quad r = 1, \\ N^{-m} & for \quad m = \rho(1-\nu), \quad r > 1, \\ N^{-\rho r(1-\nu)} & for \quad m > \rho(1-\nu), \quad 1 \leq r < \frac{m}{\rho(1-\nu)}, \\ N^{-m} & for \quad m > \rho(1-\nu), \quad r \geq \frac{m}{\rho(1-\nu)}, \end{cases}$$
$$\tag{4.4.2}$$

where ν is determined by the formula (4.2.1), $r \geq 1$ is the grading parameter in (2.7.1), $\rho \in [1, \infty)$ if $\nu \in (0, 1)$ and $\rho \in \mathbb{N}$ if $\nu \leq 0$, and c is a constant not depending on N.

Proof. (i) First we prove the convergence (4.4.1). To this end, we need to show that equation $z_\rho = T_\rho z_\rho + g_\rho$ (see (4.3.1)), with T_ρ and g_ρ given by (4.3.2) and (4.3.3), is uniquely solvable in $L^\infty(0, b)$. We observe that T_ρ is compact as an operator from $L^\infty(0, b)$ to $C[0, b]$, thus also from $L^\infty(0, b)$ to $L^\infty(0, b)$. Further, $g_\rho \in C[0, b] \subset L^\infty(0, b)$ and the homogeneous equation $z_\rho = T_\rho z_\rho$ has in $C[0, b]$ only the trivial solution $z_\rho = 0$. This together with $T_\rho \in \mathcal{L}(L^\infty(0, b), C[0, b])$ yields that $z_\rho = T_\rho z_\rho$ possesses also in $L^\infty(0, b)$ only the trivial solution $z_\rho = 0$. Consequently, by Theorem 2.2.3, equation $z_\rho = T_\rho z_\rho + g_\rho$ with $g_\rho \in L^\infty(0, b)$ possesses a unique solution $z_\rho \in L^\infty(0, b)$. In other words, operator $I - T_\rho$ is invertible in $L^\infty(0, b)$ and its inverse $(I - T_\rho)^{-1}$ is bounded: $(I - T_\rho)^{-1} \in \mathcal{L}(L^\infty(0, b), L^\infty(0, b))$. From Lemma 2.7.2 and from the boundedness of $(I - T_\rho)^{-1}$

in $L^\infty(0,b)$ we obtain that $I - P_N T_\rho$ is invertible in $L^\infty(0,b)$ for all sufficiently large N, say $N \geq N_0$, and

$$\|(I - \mathcal{P}_N T_\rho)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c, \quad N \geq N_0, \qquad (4.4.3)$$

where c is a constant not depending on N. Thus, for $N \geq N_0$, equation (4.3.9) provides a unique solution $z_{\rho,N} \in S_{m-1}^{-1}(\Pi_N)$. Note that, for z_ρ, the solution of equation $z_\rho = T_\rho z_\rho + g_\rho$, it holds $P_N z_\rho = P_N T_\rho z_\rho + P_N g_\rho$. We have for it and $z_{\rho,N}$ that

$$(I - \mathcal{P}_N T_\rho)(z_\rho - z_{\rho,N}) = z_\rho - z_{\rho,N} - P_N T_\rho z_\rho + P_N T_\rho z_{\rho,N} = z_\rho - \mathcal{P}_N z_\rho, \quad N \geq N_0.$$

Therefore, by (4.4.3),

$$\|z_\rho - z_{\rho,N}\|_\infty \leq c\|z_\rho - \mathcal{P}_N z_\rho\|_\infty, \quad N \geq N_0, \qquad (4.4.4)$$

where c is a positive constant not depending on N. Using (4.3.5) and (4.3.13) we obtain that

$$y(t) - y_N(t) = y_\rho(\tau) - y_{\rho,N}(\tau) = (G_\rho(z_\rho - z_{\rho,N}))(\tau), \quad t = b^{1-\rho}\tau^\rho, \ N \geq N_0. \tag{4.4.5}$$

From this, (4.4.4) and Lemma 2.8.2 it follows that

$$\|y - y_N\|_\infty \leq c\|z_\rho - z_{\rho,N}\|_\infty \leq c_1 \|z_\rho - \mathcal{P}_N z_\rho\|_\infty, \ N \geq N_0, \qquad (4.4.6)$$

where c and c_1 are some positive constants not depending on N. This together with $z_\rho \in C[0,b]$ and Lemma 2.7.1 yields the convergence (4.4.1).

(ii) If $K \in C^m(\Delta)$, $h, f \in C^{m,\mu}(0,b]$, $m \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$, then it follows from the part (ii) of Theorem 4.2.1 for $q = m$ that $z \in C^{m,\nu}(0,b]$, with ν given by (4.2.1). By Lemma 2.8.1 we get that $z_\rho \in C^{m,\nu_\rho}(0,b]$, where $\nu_\rho = 1 - \rho(1-\nu)$. This together with (4.4.6) and Lemma 2.7.3 yields the estimate (4.4.2). $\qquad \square$

It follows from Theorem 4.4.1 that in the case of sufficiently smooth f, d_i ($i = 0, \ldots, p-1$) and K, using sufficiently large values of ρ and r, by every choice of collocation parameters $0 \leq \eta_1 < \cdots < \eta_m \leq 1$ a convergence of order $O(N^{-m})$ can be expected. From Theorem 4.4.2 below we see that by a careful choice of parameters $\eta_1, \ldots, \eta_m$ and by a slightly more restrictive smoothness requirement on f, d_i ($i = 0, \ldots, p-1$) and K it is possible to establish a faster convergence.

Theorem 4.4.2. *Let $m \in \mathbb{N}$ and let the following conditions be fulfilled:*

(i) *the assumptions* (i)-(ii) *of Theorem 4.4.1 hold with $q := m + 1$;*

(ii) *the quadrature approximation*

$$\int_0^1 F(x)\,dx \approx \sum_{k=1}^m w_k\, F(\eta_k),$$

with the knots $\{\eta_k\}$ satisfying (2.7.4) *and appropriate weights $\{w_k\}$ is exact for all polynomials F of degree m.*

Then problem (4.0.1)-(4.0.2) *has a unique solution $y \in C^{n-1}[0,b]$ such that $y \in C^{q,\nu}(0,b]$ and $D_{Cap}^{\alpha_p} y \in C^{q,\nu}(0,b]$. There exists an integer N_0 such that, for all integers $N \geq N_0$, equation* (4.3.9) *possesses a unique solution $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$, determining by* (4.3.12) *and* (4.3.14) *a unique approximation y_N to y, the solution of* (4.0.1)-(4.0.2), *and the following error estimates hold:*

$$\|y - y_N\|_\infty \leq c \begin{cases} N^{-m-\alpha^*} & for \quad m < \rho(1 + \alpha^* - \nu), \quad r \geq 1, \\ N^{-m-\alpha^*}(1 + \log N) & for \quad m = \rho(1 + \alpha^* - \nu), \quad r = 1, \\ N^{-m-\alpha^*} & for \quad m = \rho(1 + \alpha^* - \nu), \quad r > 1, \\ N^{-\rho r(1+\alpha^*-\nu)} & for \quad m > \rho(1 + \alpha^* - \nu) \quad and \\ & \qquad 1 \leq r < \frac{m+\alpha^*}{\rho(1+\alpha^*-\nu)}, \\ N^{-m-\alpha^*} & for \quad m > \rho(1 + \alpha^* - \nu) \quad and \\ & \qquad r \geq \frac{m+\alpha^*}{\rho(1+\alpha^*-\nu)}, \end{cases}$$

$$(4.4.7)$$

for $\alpha^* := \min\{\alpha_p - \alpha_{p-1}, \alpha_p - n_1\} < 1$ and

$$\|y - y_N\|_\infty \leq c_1 \begin{cases} N^{-m-1} & for \quad m < \rho(2 - \nu), \quad r \geq 1, \\ N^{-m-1}(1 + \log N)^2 & for \quad m = \rho(2 - \nu), \quad r = 1, \\ N^{-m-1}(1 + \log N) & for \quad m = \rho(2 - \nu), \quad r > 1, \\ N^{-\rho r(2-\nu)} & for \quad m > \rho(2 - \nu) \quad and \\ & \qquad 1 \leq r < \frac{m+1}{\rho(2-\nu)}, \\ N^{-m-1} & for \quad m > \rho(2 - \nu) \quad and \\ & \qquad r \geq \frac{m+1}{\rho(2-\nu)}, \end{cases}$$

$$(4.4.8)$$

for $\alpha^* \geq 1$. Here ν is determined by (4.2.1) *(see Theorem 4.2.1), $\rho \in [1, \infty)$ if $\nu \in (0,1)$ and $\rho \in \mathbb{N}$ if $\nu \leq 0$, r is the grid parameter in* (2.7.1), *and c, c_1 are positive constants which do not depend on N.*

Proof. From Theorem 4.4.1 we know that problem (4.0.1)–(4.0.2) has a unique solution $y \in C^{n-1}[0, b]$ such that $z = D_{Cap}^{\alpha_p} y \in C[0, b]$ and that for any $\rho \in [1, \infty)$ there exists an integer N_0 such that for all $N \geq N_0$ equation (4.3.9) has a unique solution $z_{\rho, N}$ for which (4.4.4) with $z_\rho(\tau) = z(b^{1-\rho}\tau^\rho)$ ($\tau \in [0, b]$) is valid. Denote

$$\hat{z}_{\rho, N} := T_\rho z_{\rho, N} + g_\rho, \quad N \geq N_0, \tag{4.4.9}$$

where T_ρ and g_ρ are defined by (4.3.2) and (4.3.3), respectively. From (4.3.9) we see that $z_{\rho, N} = \mathcal{P}_N \hat{z}_{\rho, N}$. Substituting this expression of $z_{\rho, N}$ into (4.4.9) we get that

$$\hat{z}_{\rho, N} = T_\rho \mathcal{P}_N \hat{z}_{\rho, N} + g_\rho, \quad N \geq N_0. \tag{4.4.10}$$

From (4.3.1) and (4.4.10) follows the identity

$$(I - T_\rho \mathcal{P}_N)(\hat{z}_{\rho, N} - z_\rho) = T_\rho(\mathcal{P}_N z_\rho - z_\rho), \quad N \geq N_0.$$

Since

$$(I - T_\rho \mathcal{P}_N)^{-1} = I + T_\rho(I - \mathcal{P}_N T_\rho)^{-1}\mathcal{P}_N, \quad N \geq N_0,$$

we get with the help of (4.4.3) that

$$\|\hat{z}_{\rho, N} - z_\rho\|_\infty \leq c\, \|T_\rho(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad N \geq N_0.$$

This together with (4.3.2) yields, for $N \geq N_0$,

$$\|\hat{z}_{\rho, N} - z_\rho\|_\infty \leq c \sum_{i=0}^{p-1} \|J_\rho^{\alpha_p - \alpha_i}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty + c_1\, \|T_{\rho, L}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty$$

$$+ c_2 \sum_{k=1}^{l} \sum_{j=0}^{n_1} |J_\rho^{\alpha_p - j}(\mathcal{P}_N z_\rho - z_\rho)(b_{k\rho})| + c_3 \sum_{i=0}^{p-1} |J_\rho^{\alpha_p + 1}(\mathcal{P}_N z_\rho - z_\rho)(\bar{b}_{i,\rho})|, \tag{4.4.11}$$

where $b_{k,\rho} = b^{(\rho-1)/\rho} b_k^{1/\rho} \in (0, b]$ $(k = 1, \dots, l)$, $\bar{b}_{i,\rho} = b^{(\rho-1)/\rho} \bar{b}^{1/\rho} \in (0, b]$ $(i = 0, \dots, n-1)$ and

$$(T_{\rho, L}x)(\tau) := \int_0^\tau (\tau^\rho - \sigma^\rho)^{\alpha_p - \kappa} \sigma^{\rho-1} L_\rho(\tau, \sigma) x(b^{1-\rho}\sigma^\rho) d\sigma, \quad x \in L^\infty(0, b),$$

with $L_\rho(\tau, \sigma) := L(b^{1-\rho}\tau^\rho, b^{1-\rho}\sigma^\rho)$ and L defined by (4.1.12). Here and below c, c_1, c_2 and c_3 are generic positive constants which are independent of N. With

the help of Lemma 2.8.2 we obtain for $N \geq N_0$ the following estimates:

$$\|J_\rho^{\alpha_p-\alpha_i}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty \leq c\|J_\rho^{\alpha_p-\alpha_{p-1}}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad i = 0\ldots,p-1,$$
(4.4.12)

$$\|T_{\rho,L}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty \leq c_1\|J_\rho^{\alpha_p+1-\kappa}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty$$
$$\leq c_2\|J_\rho^{\alpha_p-\alpha_{p-1}}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty,$$
(4.4.13)

$$|J_\rho^{\alpha_p-j}(\mathcal{P}_N z_\rho - z_\rho)(b_{k\rho})| \leq \|J_\rho^{\alpha_p-j}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty$$
$$\leq c\|J_\rho^{\alpha_p-n_1}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad j = 0,\ldots,n_1, \quad (4.4.14)$$

$$|J_\rho^{\alpha_p+1}(\mathcal{P}_N z_\rho - z_\rho)(\bar{b}_{i,\rho})| \leq \|J_\rho^{\alpha_p+1}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty$$
$$\leq c\|J_\rho^{\alpha_p-n_1}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad i = 0,\ldots,n-1.$$
(4.4.15)

It follows from (4.4.11)–(4.4.15) and Lemma 2.8.2 that

$$\|\hat{z}_{\rho,N} - z_\rho\|_\infty \leq c\|J_\rho^{\alpha^*}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad N \geq N_0,$$
(4.4.16)

where $\alpha^* = \min\{\alpha_p - \alpha_{p-1}, \alpha_p - n_1\}$. Since $z_{\rho,N} = \mathcal{P}_N \hat{z}_{\rho,N}$, we get with the help of (4.3.5) that

$$\|y_N - y\|_\infty = \|y_{\rho,N} - y_\rho\|_\infty = \|G_\rho(z_{\rho,N} - z_\rho)\|_\infty$$
$$\leq \|G_\rho \mathcal{P}_N(\hat{z}_{\rho,N} - z_\rho)\|_\infty + \|G_\rho(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \ N \geq N_0. \quad (4.4.17)$$

Using (4.3.6) and (4.4.14)–(4.4.15) we obtain

$$\|G_\rho(\mathcal{P}_N z_\rho - z_\rho)\|_\infty \leq \|J_\rho^{\alpha_p}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty + c\|J_\rho^{\alpha_p-n_1}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty$$
$$\leq c_1\|J_\rho^{\alpha^*}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad N \geq N_0.$$

This together with (4.4.16) and (4.4.17) yields

$$\|y_N - y\|_\infty \leq c\|J_\rho^{\alpha^*}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad N \geq N_0.$$
(4.4.18)

Because of Theorem 4.2.1 we have $z \in C^{m+1,\nu}(0,b]$ and due to Lemma 2.8.2

$$\|J_\rho^{\alpha^*}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty \leq c\|J_\rho^1(\mathcal{P}_N z_\rho - z_\rho)\|_\infty, \quad N \geq N_0,$$

for $\alpha^* \geq 1$. Therefore it follows from (4.4.18) and Lemma 2.8.3 that the estimates (4.4.7) and (4.4.8) are true. $\qquad\square$

Chapter 5

Nonlinear fractional differential equations

Let us consider the boundary value problem for non-linear fractional differential equations of the form

$$(D_{Cap}^{\alpha} y)(t) = f(t, y(t)), \quad 0 \le t \le b, \tag{5.0.1}$$

with

$$\sum_{j=0}^{n_0} \beta_{ij0}\, y^{(j)}(0) + \sum_{k=1}^{l} \sum_{j=0}^{n_1} \beta_{ijk}\, y^{(j)}(b_k) = \gamma_i\,, \quad i = 0, \dots, n-1\,, \tag{5.0.2}$$

where

$$
\begin{aligned}
& n := \lceil \alpha \rceil \in \mathbb{N}, \quad n-1 < \alpha < n\,, \quad 0 < b_1 < \cdots < b_l \le b, \quad l \in \mathbb{N}, \\
& \beta_{ij0}, \beta_{ijk}, \gamma_i \in \mathbb{R}, \quad n_0, n_1 \in \mathbb{N}_0, \quad n_0 < n,\ n_1 < n,
\end{aligned} \tag{5.0.3}
$$

$f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $D_{Cap}^{\alpha} y$ is the Caputo derivative of order $\alpha > 0$ of an unknown function y.

Following the ideas of [71, 72], we construct a class of high-order methods for the numerical solution of (5.0.1)–(5.0.2). Using an integral equation reformulation (see Chapter 4) of the boundary value problem, we first regularize the solution by a suitable smoothing transformation. After that we find a numerical solution to the problem (5.0.1)–(5.0.2) by a piecewise polynomial collocation method on a mildly graded or uniform grid. Finally, we give global convergence estimates and a global superconvergence result for suitably chosen collocation parameters. Numerical examples supporting the theoretical results are given in Chapter 7.

5.1 Integral equation reformulation

We first find an integral equation reformulation of the problem (5.0.1)–(5.0.2), by using similar ideas as those presented in Sections 3.1 and 4.1. Let $n = \lceil \alpha \rceil \in \mathbb{N}$ and let $y \in C^{n-1}[0, b]$ be such that $D^{\alpha}_{Cap} y \in C[0, b]$. Introduce a new unknown function $z := D^{\alpha}_{Cap} y$. Then (see (2.5.7))

$$y(t) = (J^{\alpha} z)(t) + \sum_{\lambda=0}^{n-1} c_{\lambda} t^{\lambda}, \quad t \in [0, b], \tag{5.1.1}$$

where $c_{\lambda} \in \mathbb{R}$ ($\lambda = 0, \ldots, n-1$) are some constants. In analogy to Section 4.1, we see that a function y in the form (5.1.1) satisfies the boundary conditions (5.0.2) if and only if

$$\sum_{j=0}^{n-1} \left[j! \, \beta_{ij0} + \sum_{k=1}^{l} \sum_{\lambda=0}^{j} \beta_{i\lambda k} \frac{j!}{(j-\lambda)!} b_k^{j-\lambda} \right] c_j = \gamma_i - \sum_{k=1}^{l} \sum_{j=0}^{n_1} \beta_{ijk} (J^{\alpha-j} z)(b_k),$$

$$i = 0, \ldots, n-1, \quad (5.1.2)$$

by setting $\beta_{ij0} = 0$ for $j > n_0$ and $\beta_{ijk} = 0$ for $j > n_1$ $k = 1, \ldots, l$. Clearly, (5.1.2) is a linear system of n equations with respect to $c_0, \ldots, c_{n-1}$. Let

$$M := \left(j! \, \beta_{ij0} + \sum_{k=1}^{l} \sum_{\lambda=0}^{j} \beta_{i\lambda k} \frac{j!}{(j-\lambda)!} b_k^{j-\lambda} \right)_{i,j=0}^{n-1}$$

be the matrix of the system (5.1.2). In the sequel we assume that the matrix M is regular. Observe that M is regular if and only if from all polynomials y of degree $n-1$ only $y = 0$ satisfies the homogeneous boundary conditions

$$\sum_{j=0}^{n_0} \beta_{ij0} y^{(j)}(0) + \sum_{k=1}^{l} \sum_{j=0}^{n_1} \beta_{ijk} y^{(j)}(b_k) = 0, \quad i = 0, \ldots, n-1, \tag{5.1.3}$$

corresponding to the conditions (5.0.2) with $\gamma_i = 0$, $i = 0, \ldots, n-1$.

Let $M^{-1} = (p_{\lambda\mu})_{\lambda,\mu=0}^{n-1}$ be the inverse of M. Using M^{-1}, the solution of the system (5.1.2) can be written in the form

$$c_{\lambda} = \delta_{\lambda} - \sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk} (J^{\alpha-j} z)(b_k), \quad \lambda = 0, \ldots, n-1,$$

where

$$\delta_\lambda := \sum_{\mu=0}^{n-1} p_{\lambda\mu}\, \gamma_\mu \,, \quad \kappa_{\lambda jk} := \sum_{\mu=0}^{n-1} p_{\lambda\mu}\, \beta_{\mu jk}\,. \tag{5.1.4}$$

Therefore a function y in the form (5.1.1) satisfies the conditions (5.0.2) if and only if it can be expressed by the formula

$$y = Gz + Q, \tag{5.1.5}$$

where

$$(Gz)(t) := (J^\alpha z)(t) \ - \ \sum_{\lambda=0}^{n-1} t^\lambda \sum_{k=1}^{l} \sum_{j=0}^{n_1} \kappa_{\lambda jk}(J^{\alpha-j}z)(b_k), \quad t \in [0,b]\,, \tag{5.1.6}$$

$$Q(t) := \sum_{\lambda=0}^{n-1} \delta_\lambda\, t^\lambda, \quad t \in [0,b]\,. \tag{5.1.7}$$

Remark 5.1.1. *For the case of an initial value problem*

$$(D_{Cap}^\alpha y)(t) = f(t, y(t)), \quad 0 \le t \le b,$$

$$y^{(i)}(0) = \gamma_i\,, \quad i = 0, \ldots, n-1, \quad n = \lceil \alpha \rceil,$$

we have

$$G = J^\alpha, \quad Q(t) = \sum_{\lambda=0}^{n-1} \frac{\gamma_\lambda}{\lambda!}\, t^\lambda.$$

Suppose now that $y^* \in C[0,b]$ is a solution of the boundary value problem (5.0.1)–(5.0.2) such that $z^* := D_{Cap}^\alpha y^* \in C[0,b]$. Then it follows from the observations above that y^* has the form (see (5.1.5)) $y^* = Gz^* + Q$, where G and Q are defined by the formulas (5.1.6) and (5.1.7), respectively. Inserting (5.1.5) into (5.0.1), we see that $z^* = D_{Cap}^\alpha y^*$ satisfies the equation

$$z = Tz, \tag{5.1.8}$$

where

$$(Tz)(t) := f(t, (Gz)(t) + Q(t)), \quad t \in [0,b]. \tag{5.1.9}$$

Conversely, if $z^* \in C[0,b]$ is a solution of equation (5.1.8), then $y^* = Gz^* + Q$ is a solution to (5.0.1)–(5.0.2). In this sense equation (5.1.8) is equivalent to the boundary value problem (5.0.1)–(5.0.2). Observe that (5.1.8) is a nonlinear integral equation with respect to z.

5.2 Smoothness of the solution

The regularity of a solution to (5.0.1)–(5.0.2) can be characterized by Theorem 5.2.2 below, for the proof of which we first formulate the following lemma from [66].

Lemma 5.2.1. *Assume that the following conditions for equation (5.0.1) are fulfilled:*
(i) $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $f : \bar{\Omega} \to \mathbb{R}$ is a continuous function which is q times ($q \in \mathbb{N}$) continuously differentiable in Ω where

$$\Omega := \{(t, y) : t \in (0, b], y \in \mathbb{R}\}, \quad \bar{\Omega} := \{(t, y) : t \in [0, b], y \in \mathbb{R}\}; \qquad (5.2.1)$$

(ii) there exist a monotonic increasing function $\psi : [0, \infty) \to \mathbb{R}$ and a real number $\nu \in [1 - \alpha, 1)$ such that for all nonnegative integers i and j with $i + j \leq q$ and for all $(t, y) \in \Omega$

$$\left| \frac{\partial^{i+j}}{\partial t^i \partial y^j} f(t, y) \right| \leq \psi(|y|) \left\{ \begin{array}{lll} 1 & \text{if} & i < 1 - \nu \\ 1 + |\log t| & \text{if} & i = 1 - \nu \\ t^{1-\nu-i} & \text{if} & i > 1 - \nu \end{array} \right\}; \qquad (5.2.2)$$

if $\alpha \in (0, 1)$, then we assume in addition to (5.2.2) that for all nonnegative integers i and j with $i + j \leq q$ and for all $(t, y_1), (t, y_2) \in \Omega$ it holds

$$\left| \frac{\partial^{i+j}}{\partial t^i \partial y^j} [f(t, y_1) - f(t, y_2)] \right| \leq \psi(\max\{|y_1|, |y_2|\})|y_1 - y_2| \left\{ \begin{array}{lll} 1 & \text{if} & i = 0 \\ t^{1-\nu-i} & \text{if} & i > 0 \end{array} \right\}; \qquad (5.2.3)$$

(iii) for arbitrary given constants $\theta_i \in \mathbb{R}$ ($i = 0, \ldots, n - 1$) equation (5.0.1) possesses a solution $y^{**} \in C^{n-1}[0, b]$ such that $D_{Cap}^\alpha y^{**} \in C[0, b]$ and

$$(y^{**})^{(i)}(0) = \theta_i, \quad i = 0, \ldots, n - 1, \quad n = \lceil \alpha \rceil.$$

Then $y^{**} \in C^{q,\nu}(0, b]$ and $D_{Cap}^\alpha y^{**} \in C^{q,\nu}(0, b]$.

Theorem 5.2.2. *Let the conditions* (i) *and* (ii) *of Lemma 5.2.1 be fulfilled. Moreover, assume that (5.0.3) is true and from all polynomials y of degree $\lceil \alpha \rceil - 1$ only $y = 0$ satisfies the conditions (5.1.3). Finally, suppose that the boundary value problem (5.0.1)–(5.0.2) possesses a solution $y^* \in C^{\lceil \alpha \rceil - 1}[0, b]$ such that $D_{Cap}^\alpha y^* \in C[0, b]$.*
Then $y^* \in C^{q,\nu}(0, b]$ and $D_{Cap}^\alpha y^* \in C^{q,\nu}(0, b]$.

Proof. Let $y^* \in C^{n-1}[0, b]$ be a solution to (5.0.1)–(5.0.2) such that $D_{Cap}^\alpha y^* \in C[0, b]$. Denote $z^* := D_{Cap}^\alpha y^*$ and

$$c_\lambda^* := \delta_\lambda - \sum_{k=1}^l \sum_{j=0}^{n-1} \kappa_{\lambda jk} (J^{\alpha-j} z^*)(b_k), \quad \lambda = 0, \ldots, n-1, \quad n = \lceil \alpha \rceil,$$

where δ_λ and $\kappa_{\lambda jk}$ are defined by (5.1.4). Then (see (5.1.1)),

$$y^*(t) = (J^\alpha z^*)(t) + \sum_{\lambda=0}^{n-1} c_\lambda^* t^\lambda, \quad t \in [0, b].$$

Consequently, y^* is a solution to equation (5.0.1) which satisfies the following initial conditions (see (2.5.2) and (2.5.6)):

$$(y^*)^{(i)}(0) = i! \, c_i^*, \quad i = 0, \ldots, n-1.$$

This together with Lemma 5.2.1 yields the assertions of Theorem 5.2.2. □

5.3   Smoothing transformation and numerical solutions

For the numerical solution of the boundary value problem (5.0.1)–(5.0.2) we use the following method. Let us consider for equation (5.1.8) a change of variables (see (2.8.2))

$$t = b^{1-\rho} \tau^\rho, \quad s = b^{1-\rho} \sigma^\rho, \quad \tau, \sigma \in [0, b],$$

where $\rho \in [1, \infty)$.

Using in (5.1.8) this change of variables, we get for

$$z_\rho(\tau) := z(b^{1-\rho} \tau^\rho)$$

an integral equation of the form

$$z_\rho = T_\rho z_\rho, \tag{5.3.1}$$

where

$$(T_\rho z_\rho)(\tau) \quad := \quad f\left(b^{1-\rho} \tau^\rho, (G_\rho z_\rho)(\tau) + Q_\rho(\tau)\right), \tag{5.3.2}$$

$$(G_\rho z_\rho)(\tau) \quad := \quad (J_\rho^\alpha z_\rho)(\tau) - \sum_{\lambda=0}^{n-1} b^{(1-\rho)\lambda} \tau^{\rho\lambda} \sum_{k=1}^l \sum_{j=0}^{n_1} \kappa_{\lambda jk}(J_\rho^{\alpha-j} z_\rho)(b_{k\rho}) \tag{5.3.3}$$

$$Q_\rho(\tau) \quad := \quad Q(b^{1-\rho} \tau^\rho), \quad b_{k\rho} := b^{(\rho-1)/\rho} b_k^{1/\rho} \in (0, b], \quad \tau \in [0, b],$$

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with $\kappa_{\lambda jk}$, Q and J_ρ^α given by (5.1.4), (5.1.7) and (2.8.4), respectively.

Let $y^*(t)$ ($t \in [0, b]$) be a solution of problem (5.0.1)–(5.0.2). Using in (5.1.5) the change of variables (2.8.2) we see that

$$y_\rho^*(\tau) := y^*(b^{1-\rho}\tau^\rho)$$

can be expressed in the form

$$y_\rho^* = G_\rho z_\rho^* + Q_\rho, \tag{5.3.4}$$

where G_ρ is defined by (5.3.3) and $Q_\rho(\tau) = Q(b^{1-\rho}\tau^\rho)$, $\tau \in [0, b]$.

Approximations $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ ($m, N \in \mathbb{N}$) to the solution z_ρ^* of equation (5.3.1) we find by collocation method from the conditions

$$z_{\rho,N}(t_{jk}) = (T_\rho z_{\rho,N})(t_{jk}), \quad k = 1, \ldots, m, \; j = 1, \ldots, N, \tag{5.3.5}$$

with $\{t_{jk}\}$, defined by (2.7.3). If $\eta_1 = 0$, then by $z_{\rho,N}(t_{j1})$ we denote the right limit $\lim_{t \to t_{j-1}, t > t_{j-1}} z_{\rho,N}(t)$. If $\eta_m = 1$, then $z_{\rho,N}(t_{jm})$ denotes the left limit $\lim_{t \to t_j, t < t_j} z_{\rho,N}(t)$.

Note that the conditions (5.3.5) for finding $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ have an operator equation representation

$$z_{\rho,N} = \mathcal{P}_N T_\rho z_{\rho,N}, \tag{5.3.6}$$

where $\mathcal{P}_N$ and T_ρ are defined by (2.7.5) and (5.3.2), respectively.

The collocation conditions (5.3.5) form a system of equations whose exact form is determined by the choice of a basis in the space $S_{m-1}^{(-1)}(\Pi_N)$. If $\eta_1 > 0$ or $\eta_m < 1$, then we can use the Lagrange fundamental polynomial representation:

$$z_{\rho,N}(\tau) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu} \varphi_{\lambda\mu}(\tau), \quad \tau \in [0, b], \tag{5.3.7}$$

where $\varphi_{\lambda\mu}$ ($\lambda = 1, \ldots, N$, $\mu = 1, \ldots, m$) are defined by (3.3.5). Then $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ and $z_{\rho,N}(t_{jk}) = c_{jk}$, $k = 1, \ldots, m$, $j = 1, \ldots, N$. Searching the solution to (5.3.5) in the form (5.3.7), we obtain a system of nonlinear algebraic equations with respect to the coefficients $\{c_{jk}\}$:

$$c_{jk} = f\left(b^{1-\rho}t_{jk}^\rho, \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} (G_\rho \varphi_{\lambda\mu})(t_{jk}) c_{\lambda\mu} + Q_\rho(t_{jk})\right), \quad k = 1, \ldots, m, \; j = 1, \ldots, N. \tag{5.3.8}$$

Remark 5.3.1. *In the case of an initial value problem we have $G_\rho = J_\rho^\alpha$ (see Remark 5.1.1). Since $(J_\rho^\alpha \varphi_{\lambda\mu})(t_{jk}) = 0$ if $\lambda > j$, the coefficients $c_{j1}, \ldots, c_{jm}$ can be found for every $j = 1, \ldots, N$ from the systems*

$$c_{jk} = f\left(b^{1-\rho}t_{jk}^\rho, \sum_{\lambda=1}^{j}\sum_{\mu=1}^{m}(J_\rho^\alpha \varphi_{\lambda\mu})(t_{jk})c_{\lambda\mu} + Q_\rho(t_{jk})\right), \quad k = 1, \ldots, m, \quad (5.3.9)$$

with m unknowns c_{jk}, $k = 1, \ldots, m$. That is, we can find the coefficients $\{c_{jk}\}$ ($k = 1, \ldots, m$, $j = 1, \ldots, N$) step-by-step by solving N systems of non-linear algebraic equations with m unknowns.

Approximations $y_{\rho,N}$ to y_ρ^* we find by the formula

$$y_{\rho,N} = G_\rho z_{\rho,N} + Q_\rho, \quad (5.3.10)$$

where $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ is determined by (5.3.5). Having $\{c_{jk}\}$ in hand, with the help of (5.3.7) we can rewrite (5.3.10) as

$$y_{\rho,N}(\tau) = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m}c_{\lambda\mu}(G_\rho \varphi_{\lambda\mu})(\tau) + Q_\rho(\tau), \quad \tau \in [0, b]. \quad (5.3.11)$$

Finally, approximations $y_N(t)$ to the solution $y^*(t)$ of problem (5.0.1)–(5.0.2) we find by setting

$$y_N(t) := y_{\rho,N}(b^{(\rho-1)/\rho}t^{1/\rho}), \quad t \in [0, b]. \quad (5.3.12)$$

5.4 Convergence analysis

The convergence behaviour of the proposed algorithms can be expressed by Theorems 5.4.1 and 5.4.2.

Theorem 5.4.1. *Suppose (5.0.3) and let problem (5.0.1)–(5.0.2) have a solution $y^* \in C[0, b]$ such that $z^* := D_{Cap}^\alpha y^* \in C[0, b]$. Let $f : \bar\Omega \to \mathbb{R}$ be a continuous function such that $\frac{\partial}{\partial y}f(t, y)$ is continuous for $(t, y) \in \bar\Omega$, $\frac{\partial^2}{\partial y^2}f(t, y)$ is continuous for $(t, y) \in \Omega$ and*

$$\left|\frac{\partial^j}{\partial y^j}f(t, y)\right| \le \psi(|y|), \quad (t, y) \in \Omega, \quad j = 0, 1, 2.$$

Here Ω and $\bar\Omega$ are defined by (5.2.1) and $\psi : [0, \infty) \to \mathbb{R}$ is a monotonically increasing function. Moreover, assume that from all polynomials y of degree $n -$

1 ($n = \lceil \alpha \rceil$) *only* $y = 0$ *satisfies the homogeneous boundary conditions* (5.1.3) *and from all solutions* $y \in C[0,b]$ *of the linear homogeneous (fractional) differential equation*

$$(D^\alpha_{Cap} y)(t) = \frac{\partial}{\partial y} f(t, y^*(t)) y(t), \quad t \in [0, b], \tag{5.4.1}$$

only $y = 0$ *satisfies the conditions* (5.1.3). *Finally, let* $m \in \mathbb{N}$ *and assume that the collocation points* (2.7.3) *with grid points* (2.7.1) *and arbitrary parameters* $\eta_1, \ldots, \eta_m$ *satisfying* (2.7.4) *are used.*

Then there exist $N_0 \in \mathbb{N}$ *and* $\delta_0 > 0$ *such that, for all* $N \geq N_0$, *equation* (5.3.6) *possesses a unique solution* $z_{\rho,N} \in S^{(-1)}_{m-1}(\Pi_N)$ *in the ball* $\|x - z^*_\rho\|_\infty \leq \delta_0$, *where* $z^*_\rho(\tau) = z^*(b^{1-\rho}\tau)$, $\tau \in [0, b]$, $z^* = D^\alpha_{Cap} y^*$ *and* $\rho \geq 1$. *Moreover,*

$$\|y_N - y^*\|_\infty \to 0 \quad \text{as} \quad N \to \infty, \tag{5.4.2}$$

where $y_N(t) = y_{\rho,N}(b^{(\rho-1)/\rho} t^{1/\rho})$, $t \in [0, b]$, *and* $y_{\rho,N}$ *is defined with the help of* $z_{\rho,N}$ *by* (5.3.10).

If, in addition, the assumptions of Theorem 5.2.2 are fulfilled with $q = m$ *and some* $\nu \in [1 - \alpha, 1)$, *then for all* $N \geq N_0$ *and* $r \geq 1$ *the following error estimate holds:*

$$\|y_N - y^*\|_\infty \leq c \begin{cases} N^{-m} & \text{for} \quad m < 1 - \nu, \quad r \geq 1, \\ N^{-m}(1 + \log N) & \text{for} \quad m = 1 - \nu, \quad r = 1, \\ N^{-m} & \text{for} \quad m = 1 - \nu, \quad r > 1, \\ N^{-r(1-\nu)} & \text{for} \quad m > 1 - \nu, \quad 1 \leq r < \frac{m}{1-\nu}, \\ N^{-m} & \text{for} \quad m > 1 - \nu, \quad r \geq \frac{m}{1-\nu}. \end{cases} \tag{5.4.3}$$

Here c *is a constant not depending on* N, $\rho \in [1, \infty)$ *if* $\nu \in (0, 1)$ *and* $\rho \in \mathbb{N}$ *if* $\nu \leq 0$.

Proof. To apply Theorem 2.3.1 we consider (5.3.1) and (5.3.6) as operator equations (2.3.1) and (2.3.2) in the space $E := L^\infty(0, b)$, where $S := T_\rho$ and $S_N := \mathcal{P}_N T_\rho$ with T_ρ and $\mathcal{P}_N$, defined by (5.3.2) and (2.7.5), respectively.

First we find the Fréchet derivative $T'_\rho(x^0)$ for T_ρ at $x^0 \in L^\infty(0, b)$. If x and x^0 belong to $L^\infty(0, b)$, then, due to Taylor formula,

$$(T_\rho x)(\tau) - (T_\rho x^0)(\tau) = \frac{\partial}{\partial y} f\big(t, (G_\rho x^0)(\tau) + Q_\rho(\tau)\big) \big(G_\rho(x - x^0)\big)(\tau)$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial y^2} f\big(t, (G_\rho x^0)(\tau) + \xi(\tau)(G_\rho(x - x^0))(\tau) + Q_\rho(\tau)\big) \big[(G_\rho(x - x^0))(\tau)\big]^2,$$

$$\tag{5.4.4}$$

where $\xi(\tau) \in [0, 1]$ and $t = b^{1-\rho}\tau \in (0, b]$. From this it follows that

$$\left(T'_\rho(x^0)x\right)(\tau) = \left[\frac{\partial}{\partial y}f(b^{1-\rho}\tau, y)\bigg|_{y=(G_\rho x^0)(\tau)+Q_\rho(\tau)}\right](G_\rho x)(\tau), \qquad (5.4.5)$$

where $\tau \in [0, b]$ and $x^0, x \in L^\infty(0, b)$. Observe that $T'_\rho(x^0) \in \mathcal{L}(L^\infty(0, b), C[0, b])$ if $x^0 \in L^\infty(0, b)$. From (5.4.4) we see that

$$S'_N(x^0)x = (\mathcal{P}_N T_\rho)'(x^0)z = \mathcal{P}_N\left(T'_\rho(x^0)x\right), \quad x^0, x \in L^\infty(0, b),$$

with $T'_\rho(x^0)x$ defined by (5.4.5).

It follows from Lemma 2.8.2 that G_ρ (defined by (5.3.3)) is linear, bounded and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$. Since from all solutions y of equation (5.4.1) only $y = 0$ satisfies (5.1.3), equation $x = T'_\rho(z^*_\rho)x$ has in $L^\infty(0, b)$ only the trivial solution $x = 0$. Using these observations and Lemma 2.8.2 we can check that the operators $S = T_\rho$ and $S_N = \mathcal{P}_N T_\rho$ satisfy the conditions $1^0 - 4^0$ of Lemma 2.3.1. Thus there exist $N_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that, for all $N \geq N_0$, equation (5.3.6) possesses a unique solution $z_{\rho,N}$ in the ball $\|x - z^*_\rho\|_\infty \leq \delta_0$, and

$$\|z_{\rho,N} - z^*_\rho\|_\infty \leq c\,\|\mathcal{P}_N z^*_\rho - z^*_\rho\|_\infty,$$

with a positive constant c which is independent of N. Since $y^*_\rho = G_\rho z^*_\rho + Q_\rho$ and $y_{\rho,N} = G_\rho z_{\rho,N} + Q_\rho$ (see (5.3.4) and (5.3.10), respectively), we obtain that

$$\|y_N - y^*\|_\infty \leq c\,\|z_{\rho,N} - z^*_\rho\|_\infty \leq c_1\,\|\mathcal{P}_N z^*_\rho - z^*_\rho\|_\infty, \qquad (5.4.6)$$

with some positive constants c and c_1 which are independent of N. This together with $z^*_\rho \in C[0, b]$ and Lemma 2.7.1 yields the convergence (5.4.2).

If the assumptions of Theorem 5.2.2 are fulfilled with $q = m$ and $\nu \in [1-\alpha, 1)$, then $z^* \in C^{m,\nu}(0, b]$, and we get by Lemma 2.8.1 that $z^*_\rho \in C^{m,\nu_\rho}(0, b]$, $\nu_\rho = 1 - \rho(1 - \nu)$. This together with (5.4.6) and Lemma 2.7.3 yields the estimate (5.4.3). $\qquad \square$

Theorem 5.4.2. *Assume that the following conditions are fulfilled:*
(i) problem (5.0.1)–(5.0.2) has a solution $y^* \in C[0, b]$ and (5.0.3) holds;
(ii) $m \in \mathbb{N}$ and problem (5.0.1)–(5.0.2) satisfies the conditions of Theorem 5.2.2 with $q = m + 1$ and $\nu \in [1 - \alpha, 1)$;
(iii) from all solutions $y \in C[0, b]$ to equation (5.4.1) only $y = 0$ satisfies the

boundary conditions (5.1.3);

(iv) ρ is a smoothing parameter (see (2.8.2)) such that $\rho \in [1,\infty)$ if $\nu \in (0, 1)$ and $\rho \in \mathbb{N}$ if $\nu \leq 0$;

(v) $\mathcal{P}_N$ is defined by (2.7.5), where the collocation points (2.7.3) are generated by grid points (2.7.1) and by parameters $0 \leq \eta_1 < \eta_2 < \cdots < \eta_m \leq 1$, so that a quadrature approximation (2.8.5) is exact for all polynomials of degree m.

Then there exist $N_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that for all $N \geq N_0$ equation (5.3.6) possesses a unique solution $z_{\rho,N}$ in the ball $\|x - z_\rho^*\| \leq \delta_0$, where $z_\rho^*(\tau) = z^*(b^{1-\rho}\tau)$, $\tau \in [0,b]$, $z^* = D_{Cap}^\alpha y^*$. Moreover, for $N \geq N_0$ the following error estimates hold:

$$
\|y_N - y^*\|_\infty \leq c \begin{cases} N^{-m-\alpha^*} & for & m < \rho(1+\alpha^* - \nu), \quad r \geq 1, \\ N^{-m-\alpha^*}(1 + \log N) & for & m = \rho(1+\alpha^* - \nu), \quad r = 1, \\ N^{-m-\alpha^*} & for & m = \rho(1+\alpha^* - \nu), \quad r > 1, \\ N^{-\rho r(1+\alpha^* - \nu)} & for & m > \rho(1+\alpha^* - \nu) \quad and \\ & & 1 \leq r < \frac{m+\alpha^*}{\rho(1+\alpha^* - \nu)}, \\ N^{-m-\alpha^*} & for & m > \rho(1+\alpha^* - \nu) \quad and \\ & & r \geq \frac{m+\alpha^*}{\rho(1+\alpha^* - \nu)}, \end{cases}
$$

$$(5.4.7)$$

for $\alpha^* := \alpha - n_1 < 1$ and

$$
\|y_N - y^*\|_\infty \leq c \begin{cases} N^{-m-1} & for & m < \rho(2 - \nu), \quad r \geq 1, \\ N^{-m-1}(1 + \log N)^2 & for & m = \rho(2 - \nu), \quad r = 1, \\ N^{-m-1}(1 + \log N) & for & m = \rho(2 - \nu), \quad r > 1, \\ N^{-\rho r(2-\nu)} & for & m > \rho(2 - \nu) \quad and \\ & & 1 \leq r < \frac{m+1}{\rho(2-\nu)}, \\ N^{-m-1} & for & m > \rho(2 - \nu) \quad and \\ & & r \geq \frac{m+1}{\rho(2-\nu)}, \end{cases}
$$

$$(5.4.8)$$

for $\alpha^* \geq 1$. Here y_N is given by equation (5.3.12), $r \in [1,\infty)$ is given by (2.7.1) and c is a positive constant not depending on N.

Proof. Since $q = m+1 \geq 2$, it follows from Theorem 5.4.1 that there exist $\hat{N}_0 \in \mathbb{N}$ and $\hat{\delta}_0 > 0$ such that for $N \geq \hat{N}_0$ equation (5.3.6) has a unique solution $z_{\rho,N}$ in the ball $\|x - z_\rho^*\|_\infty \leq \hat{\delta}_0$. Denote

$$\hat{z}_{\rho,N} := T_\rho z_{\rho,N}, \quad N \geq \hat{N}_0, \tag{5.4.9}$$

with T_ρ defined by the formula (5.3.2). Taking into account (5.3.6), we obtain that $\mathcal{P}_N \hat{z}_{\rho,N} = z_{\rho,N}$. Substituting $z_{\rho,N} = \mathcal{P}_{\rho,N} \hat{z}_{\rho,N}$ into (5.4.9) we see that $\hat{z}_{\rho,N}$ is a solution of the equation

$$\hat{z}_{\rho,N} = T_\rho \mathcal{P}_N \hat{z}_{\rho,N}, \quad N \geq \hat{N}_0. \tag{5.4.10}$$

Consider the equations (5.3.1) and (5.4.10) as operator equations (2.3.1) and (2.3.2) in the space $E := C[0,b]$, where $S := T_\rho$ and $S_N := T_\rho \mathcal{P}_N$. In a similar way as we obtained the formula (5.4.5), we get for the Fréchet derivative of $S_N = T_\rho \mathcal{P}_N$ at $x^0 \in C[0,b]$ the following formula:

$$\big(S_N'(x^0)x\big)(\tau) = \left[\frac{\partial}{\partial y} f(b^{1-\rho}\tau, y)\,\bigg|_{y=(G_\rho \mathcal{P}_N x^0)(\tau)+Q_\rho(\tau)}\right] (G_\rho \mathcal{P}_N x)(\tau), \ \tau \in [0,b],$$

where $x \in C[0,b]$. Moreover, we can check that the operators $S = T_\rho$ and $S_N = T_\rho \mathcal{P}_N$ satisfy the conditions $1^0 - 4^0$ of Lemma 2.3.1. From this Lemma it follows that there exist $N_0 \geq \hat{N}_0$ and $\delta_0 > 0$ such that, for $N \geq N_0$, equation (5.4.10) possesses a unique solution $\hat{z}_{\rho,N}$ in the ball $\|x - z_\rho^*\|_\infty \leq \delta_0$, whereby

$$\|\hat{z}_{\rho,N} - z_\rho^*\|_\infty \leq c\, \|T_\rho \mathcal{P}_N z_\rho^* - T_\rho z_\rho^*\|_\infty, \quad N \geq N_0. \tag{5.4.11}$$

Here and below by c and also c_1 we denote positive constants which are independent of N. Since

$$(T_\rho \mathcal{P}_N z_\rho^*)(\tau) \ - \ (T_\rho z_\rho^*)(\tau) = f\big(b^{1-\rho}\tau, (G_\rho \mathcal{P}_N z_\rho^*)(\tau) + Q_\rho(\tau)\big)$$
$$- \ f\big(b^{1-\rho}\tau, (G_\rho z_\rho^*)(\tau) + Q_\rho(\tau)\big), \quad \tau \in [0,b],$$

then with the help of Lagrange formula we obtain that

$$\|T_\rho \mathcal{P}_N z_\rho^* - T_\rho z_\rho^*\|_\infty \leq c\, \|G_\rho(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty. \tag{5.4.12}$$

Using (5.3.3) and Lemma 2.8.2 we get

$$\|G_\rho(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty \leq \|J_\rho^\alpha(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty$$

$$+c\sum_{k=1}^{l}\sum_{j=0}^{n_1} \big|\big(J_\rho^{\alpha-j}(\mathcal{P}_N z_\rho^* - z_\rho^*)\big)(b_{k\rho})\big| \leq c_1\, \|J_\rho^{\alpha-n_1}(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty, \tag{5.4.13}$$

where $b_{k\rho} = b^{(\rho-1)/\rho} b_k^{1/\rho} \in [0,b]$. From (5.4.11)–(5.4.13) it follows that

$$\|\hat{z}_{\rho,N} - z_\rho^*\|_\infty \leq c\, \|J_\rho^{\alpha-n_1}(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty, \quad N \geq N_0. \tag{5.4.14}$$

Further, as $y_{\rho,N} = G_\rho z_{\rho,N} + Q_\rho$, $y_\rho^* = G_\rho z_\rho^* + Q_\rho$ and $z_{\rho,N} = \mathcal{P}_N \hat{z}_{\rho,N}$, then

$$\|y_N - y^*\|_\infty = \|y_{\rho,N} - y_\rho^*\|_\infty = \|G_\rho(z_{\rho,N} - z_\rho^*)\|_\infty$$
$$\leq \|G_\rho \mathcal{P}_N(\hat{z}_{\rho,N} - z_\rho^*)\|_\infty + \|G_\rho(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty, \quad N \geq N_0.$$

This together with (5.4.13) and (5.4.14) yields

$$\|y_N - y^*\|_\infty \leq c\|J_\rho^{\alpha - n_1}(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty, \quad N \geq N_0. \tag{5.4.15}$$

Because of Theorem 5.2.2 we have $z^* \in C^{m+1,\nu}(0, b]$ and, due to Lemma 2.8.2,

$$\|J_\rho^{\alpha - n_1}(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty \leq c\|J_\rho^1(\mathcal{P}_N z_\rho^* - z_\rho^*)\|_\infty, \quad \alpha - n_1 \geq 1, \quad N \geq N_0.$$

This together with (5.4.15) and Lemma 2.8.3 yields the estimates (5.4.7) and (5.4.8). $\qquad\square$

Chapter 6

An alternative method to linear fractional differential equations

In Chapters 3, 4 and 5 we introduced numerical methods which are based on a re-formulation of the underlying problem with respect to the highest order fractional derivative z of the exact solution y. An alternative idea is to apply Riemann-Liouville integral operator to the fractional initial or boundary value problem and construct an equivalent weakly singular integral equation with respect to y. Based on [93] we use this approach to find numerical solutions to linear single-term differential equations of order $0 < \alpha < 1$. More precisely, for the sake of simplicity, we restrict ourselves to the following boundary value problem:

$$(D^{\alpha}_{Cap}y)(t) + d_0(t)y(t) = f(t), \ 0 \leq t \leq b, \tag{6.0.1}$$

$$\beta_0 y(0) + \sum_{k=1}^{l} \beta_k y(b_k) + \beta \int_0^{\bar{b}} y(s)ds = \gamma, \tag{6.0.2}$$

where $b > 0$, $0 < b_1 < \cdots < b_l \leq b$, $l \in \mathbb{N}$, $\bar{b} \in (0, b]$, $\gamma, \beta, \beta_k \ (k = 0, \ldots, l) \in \mathbb{R}$ and $D^{\alpha}_{Cap}y$ is the Caputo fractional derivative of order α of the unknown function y. We assume that $d_0, f \in C[0, b]$ and

$$0 < \alpha < 1, \quad \sum_{k=0}^{l} \beta_k + \beta \bar{b} \neq 0. \tag{6.0.3}$$

The rest of the chapter is organized as follows. We first introduce the alternative integral equation reformulation for the exact solution y. We then formulate Theorem 6.1.1, which characterizes the existence, uniqueness and smoothness properties of the exact solution y, based on the more general results obtained in Theorem 3.2.1. After that, by suitably transforming the integral equation of the exact solution y, we construct numerical methods for finding an approximate solution to y with the help of graded grids and piecewise polynomial collocation techniques. The attainable order of convergence of the proposed algorithms is given by Theorem 6.2.1. We note that there is no superconvergence result comparable to the methods introduced in Chapters 3, 4 and 5 and the numerical experiments conducted in Chapter 7 confirm that the method is, in general, not superconvergent on the interval $[0, b]$.

6.1   Integral equation reformulation

First, let $y \in C[0,b]$ be an arbitrary function such that $D_{Cap}^{\alpha} y \in C[0,b]$ and let us denote $z := D_{Cap}^{\alpha} y$. Then we can write

$$y(t) = (J^{\alpha} z)(t) + c, \tag{6.1.1}$$

where c is some constant. It then follows (see Section 3.1) that a function in the form (6.1.1) satisfies the boundary condition (6.0.2) if and only if

$$y(t) = (J^{\alpha} z)(t) + (\beta_* + \beta \bar{b})^{-1} \left[\gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha} z)(b_k) - \beta (J^{\alpha+1} z)(\bar{b}) \right], \quad 0 \le t \le b, \tag{6.1.2}$$

where $\beta_* := \sum_{k=0}^{l} \beta_k$.

Let now $y \in C[0,b]$ be a solution to problem (6.0.1)-(6.0.2) such that $z = D_{Cap}^{\alpha} y \in C[0,b]$. From (6.0.1) it follows that

$$z(t) = f(t) - d_0(t) y(t) \tag{6.1.3}$$

and by substituting (6.1.3) into (6.1.2) we obtain

$$\begin{aligned}
y(t) = {}& (J^{\alpha}[f - d_0 y])(t) \\
& + \frac{1}{\beta_* + \beta \bar{b}} \left[\gamma - \sum_{k=1}^{l} \beta_k (J^{\alpha}[f - d_0 y])(b_k) - \beta(J^{\alpha+1}[f - d_0 y])(\bar{b}) \right],
\end{aligned}$$

where $t \in [0, b]$. Thus y is also a solution of an integral equation in the form

$$y = Ty + g, \tag{6.1.4}$$

where

$$(Ty)(t) = -(J^\alpha(d_0 y))(t) + \frac{1}{\beta_* + \beta \bar{b}} \left[\sum_{k=1}^{l} \beta_k (J^\alpha(d_0 y))(b_k) + \beta(J^{\alpha+1}(d_0 y))(\bar{b}) \right] \tag{6.1.5}$$

and

$$g(t) = (J_\rho^\alpha f)(t) + \frac{1}{\beta_* + \beta \bar{b}} \left[\gamma - \sum_{k=1}^{l} \beta_k (J^\alpha f)(b_k) - \beta(J^{\alpha+1} f)(\bar{b}) \right], \tag{6.1.6}$$

with $t \in [0, b]$.

We have shown that if a continuous function $y \in C[0, b]$ satisfying $D_{Cap}^\alpha y \in C[0, b]$ is a solution to problem (6.0.1)-(6.0.2), then it is also a solution to integral equation (6.1.4). Let us now show that the converse is also true. Indeed, suppose that $y \in C[0, b]$ satisfying $D_{Cap}^\alpha y \in C[0, b]$ is a solution to integral equation (6.1.4). Then we can write

$$z(t) := (D_{Cap}^\alpha y)(t) = (D_{Cap}^\alpha [Ty + g])(t).$$

With the help of (2.5.8) we see from (6.1.5) and (6.1.6) that

$$(D_{Cap}^\alpha (Ty))(t) = -(D_{Cap}^\alpha J^\alpha (d_0 y))(t)$$

and

$$(D_{Cap}^\alpha g)(t) = (D_{Cap}^\alpha J^\alpha f)(t).$$

From property (2.5.6) we see that $D_{Cap}^\alpha J^\alpha = I$, and thus

$$(D_{Cap}^\alpha y)(t) = -d_0(t)y(t) + f(t),$$

which is the fractional differential equation (6.0.1). This shows that y satisfies the boundary condition (6.0.2), as (6.1.4) is a reordering of (6.1.2) with z substituted by $f - d_0 y$. Thus y determined by the integral equation (6.1.4) is also a solution to (6.0.1)-(6.0.2). In this sense the integral equation (6.1.4) is equivalent to the problem (6.0.1)-(6.0.2).

The existence, uniqueness and regularity of the solution to (6.0.1)-(6.0.2) can be characterized by the following theorem, which follows from Theorem 3.2.1.

Theorem 6.1.1. (i) *Assume that $0 < \alpha < 1$, $d_0, f \in C[0, b]$. Moreover, let $\sum_{k=0}^{l} \beta_k + \beta\,\bar{b} \neq 0$ and assume that the problem (6.0.1)-(6.0.2) with $f = 0$ and $\gamma = 0$ has in $C[0, b]$ only the trivial solution $y = 0$.*

Then problem (6.0.1)-(6.0.2) has a unique solution $y \in C[0, b]$. Moreover, we have $D_{Cap}^{\alpha} y \in C[0, b]$.

(ii) *Assume that* (i) *holds and let $d_0, f \in C^{q,\mu}(0, b]$, $q \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$.*

Then problem (6.0.1)-(6.0.2) possesses a unique solution y such that $y \in C^{q,\nu}(0, b]$ and $D_{Cap}^{\alpha} y \in C^{q,\nu}(0, b]$, where

$$\nu := \max\{1 - \alpha, \mu\}. \tag{6.1.7}$$

Remark 6.1.1. *If $d_0, f \in C^q[0, b]$ ($q \in \mathbb{N}$), then we may in Theorem 6.1.1 set $\nu = 1 - \alpha$.*

6.2   Numerical method

For the numerical solution of problem (6.0.1)-(6.0.2) we first transform the equivalent integral equation (6.1.4) by a suitable change of variables and then use a collocation method for the numerical solution of the transformed integral equation. More precisely, we choose a smoothing parameter $\rho \in [1, \infty)$ and consider for equation (6.1.4) a change of variables introduced in Section 2.8 by (2.8.2):

$$t = b^{1-\rho} \tau^{\rho}, \quad s = b^{1-\rho} \sigma^{\rho}, \quad \tau, \sigma \in [0, b].$$

Using in (6.1.4) this change of variables we get for $y_\rho(\tau) = y(b^{1-\rho} \tau^\rho)$ an integral equation in the form

$$y_\rho = T_\rho y_\rho + g_\rho, \tag{6.2.1}$$

where

$$
\begin{aligned}
(T_\rho y_\rho)(\tau) = &-(J_\rho^\alpha (d_{0,\rho} y_\rho))(\tau) + (\beta_* + \beta\,\bar{b})^{-1} \\
&\times \left[\sum_{k=1}^{l} \beta_k (J_\rho^\alpha (d_{0,\rho} y_\rho))(b_{k,\rho}) + \beta (J_\rho^{\alpha+1}(d_{0,\rho} y_\rho))(\bar{b}_\rho) \right]
\end{aligned} \tag{6.2.2}
$$

and

$$
g_\rho(\tau) = (J^\alpha f_\rho)(\tau) + (\beta_* + \beta\,\bar{b})^{-1} \left[\gamma - \sum_{k=1}^{l} \beta_k (J_\rho^\alpha f_\rho)(b_{k,\rho}) - \beta (J_\rho^{\alpha+1} f_\rho)(\bar{b}_\rho) \right],
$$
$$\tag{6.2.3}$$

with $\tau \in [0, b]$. Here

$$d_{0,\rho}(\tau) := d_0(b^{1-\rho}\tau^\rho), \quad f_\rho(\tau) := f(b^{1-\rho}\tau^\rho), \quad \tau \in [0, b],$$
$$b_{i,\rho} := b^{(\rho-1)/\rho}b_i^{1/\rho} \in (0, b], \ i = 1, \dots, l, \quad \bar{b}_\rho := b^{(\rho-1)/\rho}\bar{b}^{1/\rho} \in (0, b].$$

We look for an approximate solution $y_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ to the exact solution y_ρ of integral equation (6.2.1), where $y_{\rho,N}$ is determined by the following collocation conditions:

$$y_{\rho,N}(t_{jk}) = (T_\rho y_{\rho,N})(t_{jk}) + g_\rho(t_{jk}), \quad k = 1, \dots, m, \ j = 1, \dots, N. \quad (6.2.4)$$

Here T_ρ, g_ρ and t_{jk} are defined by (6.2.2), (6.2.3) and (2.7.3), respectively. Conditions (6.2.4) have an operator equation representation

$$y_{\rho,N} = \mathcal{P}_N T_\rho y_{\rho,N} + \mathcal{P}_N g_\rho, \quad (6.2.5)$$

with an interpolation operator $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0, T] \to S_{m-1}^{(-1)}(\Pi_N)$ defined by (2.7.5).

The collocation conditions (6.2.4) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$. If $\eta_1 > 0$ or $\eta_m < 1$ then we can use the Lagrange fundamental polynomial representation:

$$y_{\rho,N}(\tau) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu}\varphi_{\lambda\mu}(\tau), \quad \tau \in [0, b], \quad (6.2.6)$$

where $\varphi_{\lambda\mu}$ ($\lambda = 1, \dots, N, \ \mu = 1, \dots, m$) are defined by (3.3.5). Then $y_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ and $y_{\rho,N}(t_{jk}) = c_{jk}, \ k = 1, \dots, m, \ j = 1, \dots, N$. Searching for the solution of (6.2.4) in the form (6.2.6), we obtain a system of linear algebraic equations with respect to the coefficients $c_{jk} = y_{\rho,N}(t_{jk})$:

$$c_{jk} = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} (T_\rho\varphi_{\lambda\mu})(t_{jk})c_{\lambda\mu} + g_\rho(t_{jk}), \quad k = 1, \dots, m, \ j = 1, \dots, N. \quad (6.2.7)$$

After solving the system (6.2.7), we find the approximation $y_{\rho,N}$ with the help of the coefficients c_{jk} and (6.2.6). Finally, the approximation $y_N(t)$ to the solution $y(t)$ of problem (6.0.1)-(6.0.2) we find by setting

$$y_N(t) = y_{\rho,N}(b^{(\rho-1)/\rho}t^{1/\rho}), \quad t \in [0, b]. \quad (6.2.8)$$

The convergence of the proposed method is characterized by the following theorem.

Theorem 6.2.1. (i) *Let* $N, m \in \mathbb{N}$, $r \geq 1$ *and assume that the grid points* (2.7.1) *with collocation points* (2.7.3) *and arbitrary parameters* $\eta_1, \ldots, \eta_m$ *satisfying* (2.7.4) *are used. Assume that conditions* (6.0.3) *are satisfied and that* $d_0 \in C[0, b]$, $f \in C[0, b]$. *Moreover, assume that the problem* (6.0.1)-(6.0.2) *with* $f = 0$ *and* $\gamma = 0$ *has in* $C[0, b]$ *only the trivial solution* $y = 0$.

Then problem (6.0.1)-(6.0.2) *has a unique solution* $y \in C[0, b]$ *such that* $D_{Cap}^\alpha y \in C[0, b]$. *There exists an integer* N_0 *such that for all* $N \geq N_0$ *equation* (6.2.5) *possesses a unique solution* $y_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$, *determining by* (6.2.6) *and* (6.2.8) *a unique approximation* y_N *to* y, *the solution of* (6.0.1)-(6.0.2), *and*

$$\|y - y_N\|_\infty \to 0 \quad \text{as} \quad N \to \infty. \tag{6.2.9}$$

(ii) *If, in addition,* $d_0, f \in C^{q,\mu}(0, b]$, *where* $q := m$ *and with* $\mu \in \mathbb{R}$, $\mu < 1$, *then for all* $N \geq N_0$ *the following error estimate holds:*

$$\|y - y_N\|_\infty \leq c \begin{cases} N^{-m} & \text{for} \quad m < \rho(1-\nu), \quad r \geq 1, \\ N^{-m}(1 + \log N) & \text{for} \quad m = \rho(1-\nu), \quad r = 1, \\ N^{-m} & \text{for} \quad m = \rho(1-\nu), \quad r > 1, \\ N^{-\rho r(1-\nu)} & \text{for} \quad m > \rho(1-\nu), \quad 1 \leq r < \frac{m}{\rho(1-\nu)}, \\ N^{-m} & \text{for} \quad m > \rho(1-\nu), \quad r \geq \frac{m}{\rho(1-\nu)}. \end{cases} \tag{6.2.10}$$

Here ν *is determined by* (6.1.7) *(see Theorem 6.1.1),* r *is the grid parameter in* (2.7.1), $\rho \in [1, \infty)$ *if* $\nu \in (0, 1)$ *and* $\rho \in \mathbb{N}$ *if* $\nu \leq 0$, *and* c *is a positive constant which does not depend on* N.

Proof. (i) Let us first prove the convergence (6.2.9). Consider equation $y_\rho = T_\rho y_\rho + g_\rho$ (see (6.2.1)), with T_ρ and g_ρ given by (6.2.2) and (6.2.3), respectively. Observe that T_ρ is compact as an operator from $L^\infty(0, b)$ to $C[0, b]$, thus also from $L^\infty(0, b)$ to $L^\infty(0, b)$. The problem (6.0.1)-(6.0.2) with $f = 0$ and $\gamma = 0$ has in $C[0, b]$ only the trivial solution $y = 0$, implying that the homogeneous equation $y_\rho = T_\rho y_\rho$ has in $L^\infty(0, b)$ only the solution $y_\rho = 0$. Consequently, by Theorem 2.2.3, equation $y_\rho = T_\rho y_\rho + g_\rho$ with $g_\rho \in L^\infty(0, b)$ possesses a unique solution $y_\rho \in L^\infty(0, b)$. In other words, operator $I - T_\rho$ is invertible in $L^\infty(0, b)$ and its inverse $(I - T_\rho)^{-1}$ is bounded: $(I - T_\rho)^{-1} \in \mathcal{L}(L^\infty(0, b), L^\infty(0, b))$. This together with Lemma 2.7.2 and Theorem 2.2.1 yields that $I - \mathcal{P}_N T_\rho$ is invertible in $L^\infty(0, b)$ for all sufficiently large N, say $N \geq N_0$, and

$$\|(I - \mathcal{P}_N T_\rho)^{-1}\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \leq c, \quad N \geq N_0, \tag{6.2.11}$$

where c is a constant not depending on N. Thus, for $N \geq N_0$, equation (6.2.5) provides a unique solution $y_{\rho,N} \in S_{m-1}^{-1}(\Pi_N)$. We have for it and y_ρ, the solution of equation $y_\rho = T_\rho y_\rho + g_\rho$, that

$$(I - \mathcal{P}_N T_\rho)(y_\rho - y_{\rho,N}) = y_\rho - \mathcal{P}_N y_\rho, \quad N \geq N_0.$$

Therefore, by (6.2.11),

$$\|y_\rho - y_{\rho,N}\|_\infty \leq c \, \|y_\rho - \mathcal{P}_N y_\rho\|_\infty, \quad N \geq N_0, \tag{6.2.12}$$

where c is a positive constant not depending on N. Finally, from $y_\rho \in C[0,b]$, Lemma 2.7.3, $y_\rho(\tau) = y(b^{1-\rho}\tau^\rho)$ ($\tau \in [0,b]$) and (6.2.8) we get the convergence (6.2.9).

(ii) If $d_0, f \in C^{m,\mu}(0,b]$, $m \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$, then it follows from part (ii) of Theorem 6.1.1 that $y \in C^{m,\nu}(0,b]$, with ν given by (6.1.7). Thus from Lemma 2.8.1 we see that $y_\rho \in C^{m,\nu_\rho}(0,b]$, where $\nu_\rho = 1 - \rho(1 - \nu)$. This together with (6.2.12) and Lemma 2.7.3 yields the estimate (6.2.10). □

Chapter 7

Numerical experiments

When introducing a numerical method it is always prudent, whenever possible, to test its validity by numerical experiments. Thus, in this chapter we give a detailed overview of our numerical experiments based on the methods introduced in Chapters 3, 4, 5 and 6. We are particularly interested in verifying the error estimates derived in Chapters 3–6 by Theorems 3.4.1, 3.4.2, 4.4.1, 4.4.2, 5.4.1, 5.4.2 and 6.2.1, respectively. To this end we first give a general overview of the testing methodology used.

7.1 Introduction

All the numerical results in this chapter are calculated using Fortran programming language in 16-digit double precision. For the solution of linear system of equations we use the package LAPACK, which is a standard library for numerical linear algebra. For finding a solution to a non-linear system of equations we rely on Newton's iterative method. Where the numerical calculation of integrals is required we use the package QUADPACK, which introduces general-purpose adaptive and non-adaptive integration routines, including the handling of weak singularities. More precisely, we rely on QAGS, a globally adaptive quadrature based on 21-point Gauss–Kronrod quadrature within each subinterval.

For some theorems, namely Theorem 3.3.2, 3.4.2, 4.4.2 and 5.4.2, it is necessary to choose collocation parameters $0 \leq \eta_1 < \cdots < \eta_m \leq 1$ ($m \in \mathbb{N}$) such that the quadrature approximation (2.8.5) with appropriate weights $\{w_k\}$ is exact for all polynomials of degree m. For this it suffices that the collocation parameters

$\eta_1, \ldots, \eta_m$ be the node points of the m-point Gauss-Legendre quadrature rule applied on the interval $[0,1]$ (see Remark 3.3.1). We are mostly interested in the case $m = 2$ and $m = 3$, for which we use collocation parameters

$$\eta_1 = \frac{3 - \sqrt{3}}{6}, \quad \eta_2 = 1 - \eta_1 \quad (m = 2) \tag{7.1.1}$$

and

$$\eta_1 = \frac{5 - \sqrt{15}}{10}, \quad \eta_2 = \frac{1}{2}, \quad \eta_3 = 1 - \eta_1 \quad (m = 3), \tag{7.1.2}$$

respectively. If in some special cases we use different collocation parameters, we will explicitly state this.

For the numerical experiments we approximate the norm $\|u - v\|_\infty$ ($u, v \in C[0, b]$) with the quantities ε_N as follows:

$$\varepsilon_N := \max_{j=1,\ldots,N} \max_{k=0,\ldots,10} |u(s_{jk}) - v(s_{jk})|. \tag{7.1.3}$$

Here $s_{jk} := t_{j-1} + k(t_j - t_{j-1})/10$ ($k = 0, \ldots, 10$, $j = 1, \ldots, N$), $t_j = b\left(\frac{j}{N}\right)^r$ ($j = 1, \ldots, N$) are the grid points introduced in (2.7.1) with grading parameter $r \geq 1$, and $N \in \mathbb{N}$ is the parameter specifying the size of the partition $\Pi_N := \{t_0, \ldots, t_N\}$ of the interval $[0, b]$. We also compute the ratios

$$\Theta_N := \frac{\varepsilon_{N/2}}{\varepsilon_N}, \tag{7.1.4}$$

which we use to characterize the observed numerical convergence rate.

7.2 Numerical results for Chapter 3

In this section we look at some fractional differential equations of the type (3.0.1)–(3.0.2). That is, we look at linear fractional differential equations with at most two Caputo fractional derivatives $D_{Cap}^{\alpha_1} y$ and $D_{Cap}^{\alpha_2} y$, where $0 < \alpha_1 < \alpha_2 \leq 1$. We thus have only one boundary condition, which can be local, non-local or both. To solve such problems by method (3.4.5),(3.4.9) we set $z := D_{Cap}^{\alpha_2} y$. Using the change of variables introduced in (2.8.3), we have for $z_\rho(\tau) := z(\tau^\rho)$ the integral equation (3.4.1) with T_ρ and g_ρ given by (3.4.2) and (3.4.3), respectively. Approximations $z_{\rho,N} \in S_{m-1}^{(-1)}(\Pi_N)$ ($N, m \in \mathbb{N}$) to the solution z_ρ of equation (3.4.1) on the interval $[0, b]$ are found by (3.4.5) using (2.7.3) with (7.1.1) or (7.1.2), the knots of the Gaussian quadrature formula (2.8.5) for $m = 2$ or $m = 3$.

We determine c_{jk} $(k = 1, \ldots, m,\ j = 1, \ldots, N)$ and $z_{\rho,N}(t)$ $(t \in [0, b])$ by (3.4.8) and (3.4.7), respectively. After that we find the approximate solution y_N by using formula (3.4.10) and the relation (3.4.11).

Example 1. Consider the boundary value problem

$$(D_{Cap}^{\frac{1}{2}}y)(t) + d_0(t)y(t) = f(t),\ 0 \le t \le 1, \tag{7.2.1}$$

$$y(0) + y(1) + \int_0^{\frac{1}{2}} y(s)ds = 2 + \frac{2^{\frac{1}{4}}}{7} + \frac{2^{-\frac{1}{4}}}{9}, \tag{7.2.2}$$

with

$$d_0(t) = t^{\frac{1}{2}}$$

and

$$f(t) = \frac{3\,\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\,t^{\frac{1}{4}} + \frac{5\,\Gamma(\frac{1}{4})}{12\,\Gamma(\frac{3}{4})}\,t^{\frac{3}{4}} + t^{\frac{5}{4}} + t^{\frac{7}{4}},$$

where $0 \le t \le 1$. We see that (7.2.1)–(7.2.2) is a special problem of (3.0.1)–(3.0.2) with $d_1 = K = 0$,

$$\alpha_2 = \frac{1}{2}, \quad b = 1, \quad b_1 = 1, \bar{b} = \frac{1}{2}, \quad \beta_0 = \beta_1 = \beta = 1, \quad \gamma = 2 + \frac{2^{\frac{1}{4}}}{7} + \frac{2^{-\frac{1}{4}}}{9},$$

and

$$y(t) = t^{\frac{3}{4}} + t^{\frac{5}{4}} \quad (t \in [0, 1])$$

is its exact solution. We have that $d_0, d_1, f \in C^{q,\mu}(0, 1]$, $K \in C^q(\Delta)$ with $\mu = \frac{3}{4}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (3.2.1),

$$\nu = \max\{1 - \alpha_2, \mu\} = \frac{3}{4}.$$

In Table 7.1 $(m = 2)$ and Table 7.2 $(m = 3)$ some results of numerical experiments for different values of the parameters N, ρ and r are presented, using the collocation parameters (7.1.1) and (7.1.2), respectively. In the case $m = 2$ it follows from (3.4.14) in Theorem 3.4.2 with $\alpha_2 = \frac{1}{2}$ and $\nu = \frac{3}{4}$ that, for sufficiently large N,

$$\varepsilon_N \le c_0 \begin{cases} N^{-0.75\,\rho r} & \text{if}\quad 1 \le \rho r < \frac{10}{3} = 3.33(3), \\ N^{-2.5} & \text{if}\quad \rho r \ge \frac{10}{3}, \end{cases} \tag{7.2.3}$$

Table 7.1: Numerical results for problem (7.2.1)–(7.2.2) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

N	$r = 1,\ \rho = 1$		$r = 2,\ \rho = 1$		$r = 1,\ \rho = 2$		$r = 2,\ \rho = 2$	
	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$1.76 \cdot 10^{-2}$		$5.04 \cdot 10^{-3}$		$1.34 \cdot 10^{-3}$		$3.02 \cdot 10^{-3}$	
8	$1.02 \cdot 10^{-2}$	1.72	$1.85 \cdot 10^{-3}$	2.72	$5.08 \cdot 10^{-4}$	2.64	$6.21 \cdot 10^{-4}$	4.86
16	$5.94 \cdot 10^{-3}$	1.73	$6.68 \cdot 10^{-4}$	2.77	$2.21 \cdot 10^{-4}$	2.30	$1.18 \cdot 10^{-4}$	5.26
32	$3.45 \cdot 10^{-3}$	1.72	$2.39 \cdot 10^{-4}$	2.80	$8.56 \cdot 10^{-5}$	2.58	$2.12 \cdot 10^{-5}$	5.56
64	$2.02 \cdot 10^{-3}$	1.71	$8.47 \cdot 10^{-5}$	2.82	$3.16 \cdot 10^{-5}$	2.71	$3.77 \cdot 10^{-6}$	5.63
128	$1.18 \cdot 10^{-3}$	1.70	$3.00 \cdot 10^{-5}$	2.82	$1.14 \cdot 10^{-5}$	2.77	$6.65 \cdot 10^{-7}$	5.67
256	$6.97 \cdot 10^{-4}$	1.70	$1.06 \cdot 10^{-5}$	2.83	$4.06 \cdot 10^{-6}$	2.80	$1.17 \cdot 10^{-7}$	5.68
		1.68		2.83		2.83		5.66

Table 7.2: Numerical results for problem (7.2.1)–(7.2.2) with $m = 3$, $\eta_1 = \frac{5-\sqrt{15}}{10}$, $\eta_2 = \frac{1}{2}$, $\eta_3 = 1 - \eta_1$.

N	$r = 1,\ \rho = 1$		$r = 2,\ \rho = 1$		$r = 2,\ \rho = 2$		$r = 3,\ \rho = 2$	
	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$6.52 \cdot 10^{-3}$		$2.30 \cdot 10^{-3}$		$7.93 \cdot 10^{-5}$		$1.11 \cdot 10^{-4}$	
8	$3.84 \cdot 10^{-3}$	1.70	$7.86 \cdot 10^{-4}$	2.93	$1.20 \cdot 10^{-5}$	6.59	$1.08 \cdot 10^{-5}$	10.28
16	$2.25 \cdot 10^{-3}$	1.70	$2.73 \cdot 10^{-4}$	2.87	$1.56 \cdot 10^{-6}$	7.71	$1.07 \cdot 10^{-6}$	10.01
32	$1.33 \cdot 10^{-3}$	1.70	$9.60 \cdot 10^{-5}$	2.85	$1.95 \cdot 10^{-7}$	7.99	$9.94 \cdot 10^{-8}$	10.81
64	$7.82 \cdot 10^{-4}$	1.70	$3.38 \cdot 10^{-5}$	2.84	$2.43 \cdot 10^{-8}$	8.04	$8.96 \cdot 10^{-9}$	11.09
128	$4.62 \cdot 10^{-4}$	1.69	$1.19 \cdot 10^{-5}$	2.83	$3.02 \cdot 10^{-9}$	8.03	$7.98 \cdot 10^{-10}$	11.23
256	$2.73 \cdot 10^{-4}$	1.69	$4.21 \cdot 10^{-6}$	2.83	$3.75 \cdot 10^{-10}$	8.06	$7.11 \cdot 10^{-11}$	11.23
		1.68		2.83		8.00		11.31

where c_0 is a positive constant not depending on N. Due to (7.2.3), the ratios Θ_N for $\{\rho = 1,\ r = 1\}$, $\{\rho = 1,\ r = 2\}$, $\{\rho = 2,\ r = 1\}$ and $\{\rho = 2,\ r = 2\}$ ought to be approximatively $2^{0.75} \approx 1.68$, $2^{1.5} \approx 2.83$, $2^{1.5} \approx 2.83$ and $2^{2.5} \approx 5.66$, respectively. These values are given in the last row of Table 7.1.

In the case $m = 3$ it follows from (3.4.14) in Theorem 3.4.2 with $\alpha = \frac{1}{2}$ and

$\nu = \frac{3}{4}$ that, for sufficiently large N,

$$\varepsilon_N \le c_1 \begin{cases} N^{-0.75\,\rho r} & \text{if} \quad 1 \le \rho r < \frac{14}{3} = 4.66(6), \\ N^{-3.5} & \text{if} \quad \rho r \ge \frac{14}{3}, \end{cases} \tag{7.2.4}$$

where c_1 is a positive constant not depending on N. Due to (7.2.4), the ratios Θ_N for $\{\rho = 1,\, r = 1\}$, $\{\rho = 1,\, r = 2\}$, $\{\rho = 2,\, r = 2\}$ and $\{\rho = 2,\, r = 3\}$ ought to be approximatively $2^{0.75} \approx 1.68$, $2^{1.5} \approx 2.83$, $2^3 = 8.00$ and $2^{3.5} \approx 11.31$, respectively. These values are given in the last row of Table 7.2.

As we can see from Tables 7.1 and 7.2 the numerical results are in good agreement with theoretical estimates.

Example 2. Consider the following boundary value problem:

$$(D^{\frac{1}{2}}_{Cap} y)(t) + d_0(t)y(t) = f(t), \quad 0 \le t \le 1, \tag{7.2.5}$$

$$y(0) + y(1) + \int_0^{\frac{1}{2}} y(s)ds = 1 + e + e^{\frac{1}{2}}, \tag{7.2.6}$$

where

$$d_0(t) = 1, \quad f(t) = t^{\frac{1}{2}} E_{1,\frac{3}{2}}(t) + e^t, \quad 0 \le t \le 1.$$

Here $E_{1,\frac{3}{2}}(t)$ is the Mittag-Leffler function defined in (2.4.4). We see that (7.2.5)-(7.2.6) is a special problem of (3.0.1)-(3.0.2) with $d_1 = 0$, $K = 0$,

$$\alpha_2 = \frac{1}{2}, \quad b = 1, \quad b_1 = 1, \bar{b} = \frac{1}{2}, \quad \beta_0 = \beta_1 = \beta = 1, \quad \gamma = e + e^{\frac{1}{2}},$$

and

$$y(t) = e^t \quad (t \in [0,1])$$

is its exact solution. Clearly $K \in C^q(\Delta)$, $d_0, d_1, f \in C^{q,\mu}(0,1]$ for $\mu = \frac{1}{2}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (3.2.1) and Remark 3.2.1,

$$\nu = \max\{1 - \alpha_2, \mu\} = \frac{1}{2}.$$

In Tables 7.3–7.4 some results of numerical experiments with $m = 2$ and with different values of the parameters N, ρ and r are presented. The numerical results in Table 7.3 have been obtained with collocation points (7.1.1). Thus, in the case

$1 \leq \rho < 2$ it follows from (3.4.14) with $\alpha_2 = \frac{1}{2}$ and $\nu = \frac{1}{2}$ that, for sufficiently large N,

$$\varepsilon_N \leq c_0 \begin{cases} N^{-\rho r} & \text{if} \quad 1 \leq \rho r < 2.5, \\ N^{-2.5} & \text{if} \quad \rho r \geq 2.5, \end{cases} \tag{7.2.7}$$

where c_0 is a positive constant not depending on N. Due to (7.2.7), the ratios Θ_N for $\{\rho = 1,\ r = 1\}$, $\{\rho = 1,\ r = 1.5\}$, $\{\rho = 1.5,\ r = 1\}$ and $\{\rho = 1.5,\ r = 2\}$ ought to be approximatively 2, $2^{1.5} \approx 2.83$, $2^{1.5} \approx 2.83$ and $2^{2.5} \approx 5.66$, respectively. These values are given in the last row of Table 7.3. As we can see from Table 7.3 the numerical results are in good accord with theoretical estimates.

Table 7.3: Numerical results for problem (7.2.5)-(7.2.6) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1,\ \rho = 1$		$r = 1.5,\ \rho = 1$		$r = 1,\ \rho = 1.5$		$r = 2,\ \rho = 1.5$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$8.55 \cdot 10^{-3}$		$6.36 \cdot 10^{-3}$		$5.81 \cdot 10^{-3}$		$2.01 \cdot 10^{-2}$	
8	$4.52 \cdot 10^{-3}$	1.89	$2.09 \cdot 10^{-3}$	3.04	$1.14 \cdot 10^{-3}$	5.11	$5.05 \cdot 10^{-3}$	3.98
16	$2.36 \cdot 10^{-3}$	1.92	$6.98 \cdot 10^{-4}$	3.00	$2.71 \cdot 10^{-4}$	4.20	$1.06 \cdot 10^{-3}$	4.77
32	$1.22 \cdot 10^{-3}$	1.94	$2.38 \cdot 10^{-4}$	2.93	$7.72 \cdot 10^{-5}$	3.50	$2.03 \cdot 10^{-4}$	5.22
64	$6.22 \cdot 10^{-4}$	1.96	$8.28 \cdot 10^{-5}$	2.88	$2.40 \cdot 10^{-5}$	3.22	$3.71 \cdot 10^{-5}$	5.47
128	$3.16 \cdot 10^{-4}$	1.97	$2.91 \cdot 10^{-5}$	2.85	$7.91 \cdot 10^{-6}$	3.04	$6.64 \cdot 10^{-6}$	5.58
256	$1.60 \cdot 10^{-4}$	1.98	$1.02 \cdot 10^{-5}$	2.84	$2.70 \cdot 10^{-6}$	2.93	$1.18 \cdot 10^{-6}$	5.62
		2.00		2.83		2.83		5.66

For the data in in Table 7.4 we have used collocation parameters

$$\eta_1 = 0.1,\ \eta_2 = 0.9,$$

which do not satisfy the conditions imposed on collocation parameters by Theorem 3.4.2. While the numerical convergence rate for $\{\rho = 1,\ r = 1\}$, $\{\rho = 1,\ r = 1.5\}$ and $\{\rho = 1.5,\ r = 1\}$ is still in good accord with the theoretical estimates of Theorem 3.4.2, the proposed method does not obtain the global superconvergence rate $2^{2.5} \approx 5.66$ with $\{\rho = 1.5,\ r = 2\}$. Instead, the highest attained convergence rate is slightly above $2^2 = 4.00$, which is the maximal convergence rate predicted by Theorem 3.4.1. This shows that, in general, the conditions imposed by Theorem 3.4.2 are necessary for the superconvergence of the proposed method.

Table 7.4: Numerical results for problem (7.2.5)-(7.2.6) with $m = 2$, $\eta_1 = 0.1$, $\eta_2 = 0.9$.

N	$r = 1,\ \rho = 1$		$r = 1.5,\ \rho = 1$		$r = 1,\ \rho = 1.5$		$r = 2,\ \rho = 1.5$	
	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$9.52 \cdot 10^{-3}$		$7.12 \cdot 10^{-3}$		$1.09 \cdot 10^{-2}$		$3.28 \cdot 10^{-2}$	
8	$5.61 \cdot 10^{-3}$	1.70	$2.43 \cdot 10^{-3}$	2.93	$2.56 \cdot 10^{-3}$	4.26	$9.36 \cdot 10^{-3}$	3.51
16	$3.01 \cdot 10^{-3}$	1.86	$8.83 \cdot 10^{-4}$	2.75	$6.54 \cdot 10^{-4}$	3.91	$2.31 \cdot 10^{-3}$	4.06
32	$1.56 \cdot 10^{-3}$	1.92	$3.13 \cdot 10^{-4}$	2.82	$2.28 \cdot 10^{-4}$	2.87	$5.36 \cdot 10^{-4}$	4.30
64	$8.01 \cdot 10^{-4}$	1.95	$1.10 \cdot 10^{-4}$	2.85	$7.77 \cdot 10^{-5}$	2.93	$1.23 \cdot 10^{-4}$	4.37
128	$4.07 \cdot 10^{-4}$	1.97	$3.85 \cdot 10^{-5}$	2.85	$2.64 \cdot 10^{-5}$	2.94	$2.83 \cdot 10^{-5}$	4.34
256	$2.06 \cdot 10^{-4}$	1.98	$1.35 \cdot 10^{-5}$	2.85	$9.01 \cdot 10^{-6}$	2.93	$6.59 \cdot 10^{-6}$	4.29
		2.00		2.83		2.83		5.66

Example 3. Consider the boundary value problem

$$(D_{Cap}^{\frac{2}{5}}y)(t) + (D_{Cap}^{\frac{1}{5}}y)(t) + \int_0^t (t - s)^{-\frac{3}{5}}y(s)ds = f(t),\ 0 \le t \le 1, \qquad (7.2.8)$$

$$\int_0^1 y(s)ds = \frac{13}{8}, \qquad (7.2.9)$$

with

$$f(t) = \frac{3\,\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{5})}\,t^{\frac{1}{5}} + \frac{3\,\Gamma(\frac{3}{5})}{2\,\Gamma(\frac{2}{5})}\,t^{\frac{2}{5}} + \frac{3\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)}{5}t + \frac{5}{2}t^{\frac{2}{5}},$$

where $0 \le t \le 1$. We see that (7.2.8)–(7.2.9) is a special problem of (3.0.1)–(3.0.2) with $d_0 = 0$, $d_1 = 1$, $K = 1$,

$$\alpha_2 = \frac{2}{5}, \alpha_1 = \frac{1}{5}, \quad b = 1, \quad \kappa = \frac{3}{5}, \quad \bar{b} = 1, \quad \beta_0 = \beta_1 = 0, \beta = 1, \quad \gamma = \frac{13}{8},$$

and

$$y(t) = t^{\frac{3}{5}} + 1 \quad (t \in [0, 1])$$

is its exact solution. We have that $d_0, d_1, f \in C^{q,\mu}(0, 1]$, $K \in C^q(\Delta)$ with $\mu = \frac{4}{5}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (3.2.1),

$$\nu = \max\{1 - (\alpha_2 - \alpha_1), \mu, \kappa\} = \frac{4}{5}.$$

Table 7.5: Numerical results for problem (7.2.8)–(7.2.9) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1,\ \rho = 1$		$r = 2,\ \rho = 1$		$r = 3,\ \rho = 1$		$r = 4,\ \rho = 1$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$1.51 \cdot 10^{-2}$		$7.41 \cdot 10^{-3}$		$3.83 \cdot 10^{-3}$		$4.85 \cdot 10^{-3}$	
8	$1.06 \cdot 10^{-2}$	1.42	$3.40 \cdot 10^{-3}$	2.18	$1.19 \cdot 10^{-3}$	3.22	$1.02 \cdot 10^{-3}$	4.77
16	$7.36 \cdot 10^{-3}$	1.45	$1.52 \cdot 10^{-3}$	2.23	$3.53 \cdot 10^{-4}$	3.37	$1.92 \cdot 10^{-4}$	5.28
32	$5.02 \cdot 10^{-3}$	1.47	$6.74 \cdot 10^{-4}$	2.26	$1.03 \cdot 10^{-4}$	3.42	$3.55 \cdot 10^{-5}$	5.42
64	$3.39 \cdot 10^{-3}$	1.48	$2.97 \cdot 10^{-4}$	2.27	$3.00 \cdot 10^{-5}$	3.44	$6.54 \cdot 10^{-6}$	5.43
128	$2.28 \cdot 10^{-3}$	1.49	$1.30 \cdot 10^{-4}$	2.28	$8.69 \cdot 10^{-6}$	3.45	$1.22 \cdot 10^{-6}$	5.36
256	$1.52 \cdot 10^{-3}$	1.50	$5.72 \cdot 10^{-5}$	2.28	$2.51 \cdot 10^{-6}$	3.46	$2.33 \cdot 10^{-7}$	5.24
		1.32		1.74		2.30		3.03

Table 7.6: Numerical results for problem (7.2.8)–(7.2.9) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1,\ \rho = 2$		$r = 1,\ \rho = 3$		$r = 1,\ \rho = 4$		$r = 1,\ \rho = \frac{11}{2}$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$4.57 \cdot 10^{-3}$		$1.94 \cdot 10^{-3}$		$5.53 \cdot 10^{-4}$		$3.54 \cdot 10^{-4}$	
8	$1.81 \cdot 10^{-3}$	2.53	$5.31 \cdot 10^{-4}$	3.65	$1.04 \cdot 10^{-4}$	5.33	$6.15 \cdot 10^{-5}$	5.74
16	$7.44 \cdot 10^{-4}$	2.43	$1.48 \cdot 10^{-4}$	3.59	$1.93 \cdot 10^{-5}$	5.36	$1.06 \cdot 10^{-5}$	5.79
32	$3.12 \cdot 10^{-4}$	2.38	$4.16 \cdot 10^{-5}$	3.55	$3.62 \cdot 10^{-6}$	5.34	$1.86 \cdot 10^{-6}$	5.70
64	$1.33 \cdot 10^{-4}$	2.36	$1.18 \cdot 10^{-5}$	3.53	$6.81 \cdot 10^{-7}$	5.32	$3.33 \cdot 10^{-7}$	5.60
128	$5.66 \cdot 10^{-5}$	2.34	$3.35 \cdot 10^{-6}$	3.52	$1.29 \cdot 10^{-7}$	5.30	$6.05 \cdot 10^{-8}$	5.51
256	$2.46 \cdot 10^{-5}$	2.31	$9.56 \cdot 10^{-7}$	3.51	$2.45 \cdot 10^{-8}$	5.24	$1.10 \cdot 10^{-8}$	5.47
		1.74		2.30		3.03		4.60

In Tables 7.5 and 7.6 some results of numerical experiments for different values of the parameters N, ρ and r are presented, using $m = 2$ and the collocation parameters (7.1.1). It follows from (3.4.14) in Theorem 3.4.2 with $\alpha_2 = \frac{2}{5}$, $\alpha_1 = \frac{1}{5}$ and $\nu = \frac{4}{5}$ that, for sufficiently large N,

$$\varepsilon_N \leq c_0 \begin{cases} N^{-0.4\,\rho r} & \text{if} \quad 1 \leq \rho r < \frac{11}{2} = 5.5, \\ N^{-2.2} & \text{if} \quad \rho r \geq \frac{11}{2}, \end{cases} \tag{7.2.10}$$

where c_0 is a positive constant not depending on N. Due to (7.2.10), the ratios Θ_N for $\{\rho = 1,\, r = 1\}$, $\{\{\rho = 1,\, r = 2\},\, \{\rho = 2,\, r = 1\}\}$, $\{\{\rho = 1,\, r = 3\},\, \{\rho = 3,\, r = 1\}\}$, $\{\{\rho = 1,\, r = 4\},\, \{\rho = 4,\, r = 1\}\}$ and $\{\rho = 5.5,\, r = 1\}$ ought to be approximatively $2^{0.4} \approx 1.32$, $2^{0.8} \approx 1.74$, $2^{1.2} \approx 2.30$, $2^{1.6} \approx 3.03$ and $2^{2.2} \approx 4.60$, respectively. These values are given in the last row of Tables 7.5 and 7.6. We can see that the numerical method converges somewhat faster than the estimate (3.4.14) predicts.

7.3   Numerical results for Chapter 4

As the boundary value problem (3.0.1)–(3.0.2) studied in Chapter 3 is a special case of the more general problem (4.0.1)–(4.0.2) analyzed in Chapter 4, all the examples in the previous section remain valid for Theorems 4.4.1 and 4.4.2. Thus in this section we concentrate on a linear fractional differential equations of type (4.0.1)–(4.0.2) where $\alpha_p \geq 1$. More specifically, we look at the following fractional linear boundary value problem:

$$y''(t) + (D^{\frac{3}{2}}_{Cap}y)(t) + y(t) = f(t), \tag{7.3.1}$$

$$\begin{aligned} y(0) &= 2, \\ y'(0) + y'(1) &= \frac{9}{2}, \end{aligned} \tag{7.3.2}$$

with

$$f(t) = \frac{15}{4}t^{\frac{1}{2}} + \frac{15\,\Gamma(\frac{1}{2})}{8}t + t^{\frac{5}{2}} + 2.$$

Equation (7.3.1) is also known as the Bagley-Torvik equation [87]. Actually, this is a special problem of (4.0.1)–(4.0.2) with $d_0 = d_1 = 1$, $K = 0$, $n_0 = 1$, $n_1 = 1$, $p = 2$ and

$$\begin{aligned} \alpha_2 = 2,\ \alpha_1 &= \frac{3}{2}, \quad b = 1, \quad \bar{b}_0 = \bar{b}_1 = 0, \quad \gamma_0 = 2,\ \gamma_1 = \frac{9}{2}, \\ \beta_{000} = \beta_{100} &= \beta_{111} = 1,\ \beta_0 = \beta_1 = 0. \end{aligned}$$

The exact solution to problem (7.3.1)–(7.3.2) is

$$y(t) = t^{\frac{5}{2}} + 2, \quad t \in [0, 1].$$

Clearly $f \in C^{q,\mu}(0, b]$ for arbitrary $q \in \mathbb{N}$ and $\mu = 0.5$. Therefore, by (4.2.1), we have that $\nu = 0.5$.

Denote $z := y''$. Then the solution y of (7.3.1)–(7.3.2) can be presented in the form $y = Gz + Q$, where

$$(Gz)(t) := \left(J^2 z\right)(t) - 0.5t \left(J^{0.5} z\right)(1), \quad Q(t) := 2 + 2.25\, t, \quad t \in [0, 1],$$

and z is a solution of the integral equation

$$z(t) = Tz + g, \quad t \in [0, 1],$$

where

$$(Tz)(t) = -\left(J^2 z\right)(t) - \left(J^{0.5} z\right)(t) + 0.5t \left(J^{0.5} z\right)(1),\ t \in [0, 1]$$

and $g(t) = f(t) - 2.25\, t$, $t \in [0, 1]$.

Using the change of variables $t = \tau^\rho$, $\tau \in [0, 1]$, $\rho \geq 1$, and notations

$$y_\rho(\tau) := y(\tau^\rho), \ \ z_\rho(\tau) := z(\tau^\rho), \quad f_\rho(\tau) = f(\tau^\rho), \quad \tau \in [0, 1], \ \rho \geq 1,$$

we get that $y_\rho = (G_\rho z_\rho) + Q_\rho$, where

$$(G_\rho z_\rho)(\tau) := \left(J_\rho^2 z_\rho\right)(\tau) - 0.5\tau^\rho \left(J_\rho^{0.5} z_\rho\right)(1), \quad Q_\rho(\tau) := 2 + 2.25\, \tau^\rho, \quad \tau \in [0, 1],$$

and $z_\rho(\tau) = z(\tau^\rho)$ is the solution of the integral equation

$$z_\rho(\tau) = T_\rho z_\rho + g_\rho, \tag{7.3.3}$$

with

$$(T_\rho z_\rho)(\tau) = -\left(J_\rho^2 z_\rho\right)(\tau) - \left(J_\rho^{0.5} z_\rho\right)(\tau) + 0.5\tau^\rho \left(J_\rho^{0.5} z_\rho\right)(1),\ \tau \in [0, 1],$$
$$(g_\rho)(\tau) = f_\rho(\tau) - 2.25\tau^\rho,\ \tau \in [0, 1].$$

Since $\tau = t^{\frac{1}{\rho}}$, we have $y(t) = y_\rho(t^{\frac{1}{\rho}})$.

For the numerical solution of (7.3.1)–(7.3.2) we determine approximations $z_{\rho,N}$ to the solution z_ρ of (7.3.3) in the form (4.3.10), finding the coefficients $\{c_{jk}\}$ from collocation conditions (4.3.11):

$$c_{jk} = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} (T_\rho \varphi_{\lambda\mu})(t_{jk}) c_{\lambda\mu} + g_\rho(t_{jk}), \quad k = 1, \ldots, m, \ j = 1, \ldots, N. \tag{7.3.4}$$

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with grid points (2.7.1) and collocation points (2.7.3). After finding the coefficients $\{c_{jk}\}$, we find the approximations y_N to the solution y of problem (7.3.1)–(7.3.2) with the help of formula (cf. (4.3.13))

$$y_{\rho,N}(\tau) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu}(J_\rho^2 \varphi_{\lambda\mu})(\tau) - 0.5\tau^\rho(J_\rho^{0.5}\varphi_{\lambda\mu})(1) + 2 + 2.25\,\tau^\rho, \quad \tau \in [0,\,1]$$

and formula (4.3.14).

In Table 7.7 some results of numerical experiments for different values of the parameters N, ρ and r are presented, using $m = 2$ and the collocation parameters (7.1.1). It follows from (4.4.7) in Theorem 4.4.2 with $\alpha_2 = 2$, $\alpha_1 = 1.5$, $\alpha_0 = 0$, $\nu = 0.5$, $n_1 = 1$ and $\alpha^* = 0.5 < 1$ that, for sufficiently large N,

$$\varepsilon_N \le c \begin{cases} N^{-\rho r} & \text{if} \quad 1 \le \rho r < 2.5, \\ N^{-2.5} & \text{if} \quad \rho r \ge 2.5, \end{cases} \tag{7.3.5}$$

where c is a positive constant not depending on N.

Table 7.7: Numerical results for problem (7.2.8)–(7.2.9) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1,\ \rho = 1$		$r = 2,\ \rho = 1$		$r = 2.5,\ \rho = 1$		$r = 1,\ \rho = 2.5$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$1.12 \cdot 10^{-3}$		$2.26 \cdot 10^{-4}$		$2.57 \cdot 10^{-4}$		$1.21 \cdot 10^{-4}$	
8	$4.04 \cdot 10^{-4}$	2.78	$2.37 \cdot 10^{-5}$	9.54	$1.48 \cdot 10^{-5}$	17.34	$9.55 \cdot 10^{-6}$	12.66
16	$1.44 \cdot 10^{-4}$	2.80	$2.77 \cdot 10^{-6}$	8.56	$1.29 \cdot 10^{-6}$	11.48	$7.97 \cdot 10^{-7}$	11.98
32	$5.16 \cdot 10^{-5}$	2.80	$3.44 \cdot 10^{-7}$	8.03	$1.67 \cdot 10^{-7}$	7.73	$7.12 \cdot 10^{-8}$	11.19
64	$1.84 \cdot 10^{-5}$	2.80	$4.38 \cdot 10^{-8}$	7.86	$2.17 \cdot 10^{-8}$	7.70	$6.98 \cdot 10^{-9}$	10.20
128	$6.56 \cdot 10^{-6}$	2.81	$5.64 \cdot 10^{-9}$	7.77	$2.94 \cdot 10^{-9}$	7.38	$7.70 \cdot 10^{-10}$	9.07
256	$2.33 \cdot 10^{-6}$	2.81	$7.37 \cdot 10^{-10}$	7.65	$4.34 \cdot 10^{-10}$	6.78	$9.69 \cdot 10^{-11}$	7.94
		2.00		4.00		5.66		5.66

Due to (7.4.6), the ratios Θ_N for $\{\rho = 1,\ r = 1\}$, $\{\{\rho = 1,\ r = 2\}$, $\{\rho = 2,\ r = 1\}\}$ and $\{\{\rho = 1,\ r = 2.5\}, \{\rho = 2.5,\ r = 1\}\}$ ought to be approximatively $2^1 = 2.00$, $2^2 = 4$ and $2^{2.5} \approx 5.66$, respectively. These values are given in the last row of Table 7.7. As we can see, the the actual convergence rate is somewhat faster than the theoretical results predict.

7.4 Numerical results for Chapter 5

In this section we give two examples of a non-linear fractional differential equations of type (5.0.1)–(5.0.2), where the highest order α of the Caputo fractional derivatives satisfies $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively.

Example 1. Consider the following non-linear initial value problem:

$$(D^{\frac{1}{3}}_{Cap}y)(t) = f(t, y(t)), \quad 0 \le t \le 1, \tag{7.4.1}$$

$$y(0) = 1, \tag{7.4.2}$$

where

$$f(t, y) = \frac{ty^3}{10} - \frac{t(t^{\frac{2}{3}} + 1)^3}{10} + \frac{\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})}t^{\frac{1}{3}}.$$

This is a special problem of (5.0.1)–(5.0.2) with $n = 1$, $n_0 = n_1 = 0$, $l = 1$,

$$\alpha = \frac{1}{3}, \quad b = 1, \quad \beta_{000} = 1, \ \beta_{001} = 0, \quad \gamma_0 = 1.$$

The function $f(t, y)$ satisfies the conditions of Theorem 5.2.2 with $\nu = \frac{1}{3}$ and arbitrary $q \in \mathbb{N}$. An exact solution to (7.4.1)–(7.4.2) is

$$y(t) = t^{\frac{2}{3}} + 1, \quad 0 \le t \le 1.$$

To solve (7.4.1)–(7.4.2) with method (5.3.6),(5.3.10) we set $z := D^{\frac{1}{3}}_{Cap}y$. Then a solution of (7.4.1)–(7.4.2) can be presented in the form $y^*(t) = (J^{\frac{1}{3}}z^*)(t) + 1$, where z^* is a solution of the integral equation

$$z(t) = f\left(t, (J^{\frac{1}{3}}z)(t) + 1\right), \quad t \in [0, 1].$$

Using a change of variables $t = \tau^\rho$, $\tau \in [0, 1]$, $\rho \ge 1$, and notations

$$y_\rho(\tau) := y(\tau^\rho), \ z_\rho(\tau) := z(\tau^\rho), \quad \tau \in [0, 1], \ \rho \ge 1, \tag{7.4.3}$$

we get that

$$y^*_\rho(\tau) = (J^{\frac{1}{3}}_\rho z^*_\rho)(\tau) + 1, \quad \tau \in [0, 1],$$

where $z^*_\rho = z^*(\tau^\rho)$ is a solution of the integral equation

$$z_\rho(\tau) = f\left(\tau^\rho, (J^{\frac{1}{3}}_\rho z_\rho)(\tau) + 1\right), \quad \tau \in [0, 1]. \tag{7.4.4}$$

Since $\tau = t^{\frac{1}{\rho}}$, we have $y^*(t) = y^*_\rho(t^{\frac{1}{\rho}})$, $\quad t \in [0,1]$.

For the numerical solution of problem (7.4.1)–(7.4.2) we first determine approximations $z_{\rho,N}$ to the solution z^*_ρ of (7.4.4) in the form (5.3.7), finding the coefficients $\{c_{jk}\}$ from collocation conditions (5.3.8):

$$c_{jk} = f\left(t^\rho_{jk}, \sum_{\beta=1}^{j}\sum_{\gamma=1}^{m}(J_\rho^{1/3}\varphi_{\beta\gamma})(t_{jk})c_{\beta\gamma} + 1\right), \quad k = 1,\dots,m, \ j = 1,\dots,N.$$

(7.4.5)

with grid points (2.7.1) and collocation points (2.7.3). Note that we can find the coefficients $\{c_{jk}\}$ $(k = 1,\dots,m, \ j = 1,\dots,N)$ by solving N systems of non-linear algebraic equations with m unknowns. For this we have used Newton's iterative method. After finding the coefficients $\{c_{jk}\}$, we find the approximations y_N to a solution y^* of problem (7.4.1)–(7.4.2) with the help of formula (cf. (5.3.11))

$$y_{\rho,N}(\tau) = \sum_{\beta=1}^{N}\sum_{\gamma=1}^{m}c_{\beta\gamma}(J_\rho^{\frac{1}{3}}\varphi_{\beta\gamma})(\tau) + 1, \quad \tau \in [0,1]$$

and formula (5.3.12).

Table 7.8: Numerical results for problem (7.4.1)–(7.4.2) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1, \ \rho = 1$		$r = 1, \ \rho = 2$		$r = 1, \ \rho = 3.5$		$r = 1, \ \rho = 4$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$2.27 \cdot 10^{-2}$		$2.80 \cdot 10^{-3}$		$6.10 \cdot 10^{-4}$		$9.17 \cdot 10^{-4}$	
8	$1.42 \cdot 10^{-2}$	1.59	$1.12 \cdot 10^{-3}$	2.51	$1.21 \cdot 10^{-4}$	5.04	$1.93 \cdot 10^{-4}$	4.75
16	$8.95 \cdot 10^{-3}$	1.59	$4.43 \cdot 10^{-4}$	2.52	$2.40 \cdot 10^{-5}$	5.04	$3.98 \cdot 10^{-5}$	4.84
32	$5.64 \cdot 10^{-3}$	1.59	$1.76 \cdot 10^{-4}$	2.52	$4.77 \cdot 10^{-6}$	5.04	$7.95 \cdot 10^{-6}$	5.01
64	$3.55 \cdot 10^{-3}$	1.59	$6.98 \cdot 10^{-5}$	2.52	$9.46 \cdot 10^{-7}$	5.04	$1.57 \cdot 10^{-6}$	5.07
128	$2.24 \cdot 10^{-3}$	1.59	$2.77 \cdot 10^{-5}$	2.52	$1.88 \cdot 10^{-7}$	5.04	$3.08 \cdot 10^{-7}$	5.09
256	$1.41 \cdot 10^{-3}$	1.59	$1.10 \cdot 10^{-5}$	2.52	$3.73 \cdot 10^{-8}$	5.04	$6.03 \cdot 10^{-8}$	5.10
		1.59		2.52		5.04		5.04

In Tables 7.8 and 7.9 some results of numerical experiments for different values of the parameters N, ρ and r are presented, using $m = 2$ and the collocation parameters (7.1.1). It follows from (5.4.7) in Theorem 5.4.2 with $\alpha = \frac{1}{3}$, $\nu = \frac{2}{3}$

Table 7.9: Numerical results for problem (7.4.1)–(7.4.2) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

N	$r = 2,\ \rho = 1$ ε_N	Θ_N	$r = 3,\ \rho = 1$ ε_N	Θ_N	$r = 3.5,\ \rho = 1$ ε_N	Θ_N	$r = 2,\ \rho = 2$ ε_N	Θ_N
4	$8.95 \cdot 10^{-3}$		$5.41 \cdot 10^{-3}$		$6.40 \cdot 10^{-3}$		$1.76 \cdot 10^{-3}$	
8	$3.55 \cdot 10^{-3}$	2.52	$1.35 \cdot 10^{-3}$	4.01	$1.28 \cdot 10^{-3}$	5.01	$3.91 \cdot 10^{-4}$	4.50
16	$1.41 \cdot 10^{-3}$	2.52	$3.37 \cdot 10^{-4}$	4.00	$2.49 \cdot 10^{-4}$	5.13	$8.05 \cdot 10^{-5}$	4.86
32	$5.59 \cdot 10^{-4}$	2.52	$8.43 \cdot 10^{-5}$	4.00	$4.81 \cdot 10^{-5}$	5.17	$1.60 \cdot 10^{-5}$	5.03
64	$2.22 \cdot 10^{-4}$	2.52	$2.11 \cdot 10^{-5}$	4.00	$9.33 \cdot 10^{-6}$	5.16	$3.15 \cdot 10^{-6}$	5.09
128	$8.81 \cdot 10^{-5}$	2.52	$5.27 \cdot 10^{-6}$	4.00	$1.82 \cdot 10^{-6}$	5.14	$6.16 \cdot 10^{-7}$	5.10
256	$3.50 \cdot 10^{-5}$	2.52	$1.32 \cdot 10^{-6}$	4.00	$3.55 \cdot 10^{-7}$	5.12	$1.21 \cdot 10^{-7}$	5.10
		2.52		4.00		5.04		5.04

and $\alpha^* = \alpha < 1$ that, for sufficiently large N,

$$\varepsilon_N \leq c \begin{cases} N^{-\frac{2}{3}\rho r} & \text{if } 1 \leq \rho r < 3.5, \\ N^{-\frac{7}{3}} & \text{if } \rho r \geq 3.5, \end{cases} \tag{7.4.6}$$

where c is a positive constant not depending on N.

Due to (7.4.6), the ratios Θ_N for $\{\rho = 1,\ r = 1\}$, $\{\{\rho = 1,\ r = 2\}$, $\{\rho = 2,\ r = 1\}\}$, $\{\rho = 1,\ r = 3\}$ and $\{\{\rho = 1,\ r = 3.5\}$, $\{\rho = 3.5,\ r = 1\}$, $\{\rho = 4,\ r = 1\}$, $\{\rho = 2,\ r = 2\}\}$ ought to be approximatively $2^{\frac{2}{3}} \approx 1.59$, $2^{\frac{4}{3}} \approx 2.52$, $2^2 = 4$ and $2^{\frac{7}{3}} \approx 5.04$, respectively. These values are given in the last row of Tables 7.8 and 7.9. As we can see, the numerical data is in very good agreement with theoretical results.

Example 2. Consider the following non-linear boundary value problem:

$$(D_{Cap}^{1.5}y)(t) = f(t, y(t)),\ 0 \leq t \leq 1, \tag{7.4.7}$$

$$y(0) = -1, \quad y'(1) = 1.9, \tag{7.4.8}$$

where

$$f(t, y) = \frac{1}{2}y^2 - \frac{1}{2}(t^{1.9} - 1)^2 + \frac{\Gamma(2.9)}{\Gamma(1.4)}t^{0.4}.$$

This is a special problem of (5.0.1)–(5.0.2) with $n = 2$, $n_0 = 0$, $n_1 = 1$, $l = 1$,

$$\alpha = 1.5, \quad b = 1, \ b_1 = 1, \quad \gamma_0 = -1, \ \gamma_1 = 1.9,$$

$$\beta_{000} = 1, \ \beta_{100} = 0, \ \beta_{001} = 0, \ \beta_{011} = 0, \ \beta_{101} = 0, \ \beta_{111} = 1,$$

and

$$y(t) = t^{1.9} - 1, \quad t \in [0, 1],$$

is its exact solution. The function $f(t, y)$ satisfies the conditions of Theorem 5.2.2 with $\nu = \frac{3}{5}$ and arbitrary $q \in \mathbb{N}$.

Denote $z := \left(D_{Cap}^{1.5} y \right)$. Then a solution y^* of (7.4.7)–(7.4.8) can be presented in the form $y^* = Gz^* + Q$, where

$$(Gz)(t) := \left(J^{1.5} z \right)(t) - t \left(J^{0.5} z \right)(1), \quad Q(t) := -1 + 1.9\, t, \quad t \in [0, 1],$$

and z^* is a solution of the integral equation

$$z(t) = f\left(t, (Gz)(t) + Q(t) \right), \quad t \in [0, 1].$$

Using the change of variables $t = \tau^\rho$, $\tau \in [0, 1]$, $\rho \geq 1$, and notations

$$y_\rho(\tau) := y(\tau^\rho), \ z_\rho(\tau) := z(\tau^\rho), \quad \tau \in [0, 1], \ \rho \geq 1,$$

we get that $y_\rho^* = \left(G_\rho z_\rho^* \right) + Q_\rho$, where

$$(G_\rho z_\rho)(\tau) := \left(J_\rho^{1.5} z_\rho \right)(\tau) - \tau^\rho \left(J_\rho^{0.5} z_\rho \right)(1), \quad Q_\rho(\tau) := -1 + 1.9\, \tau^\rho, \quad \tau \in [0, 1],$$

and $z_\rho^*(\tau) = z^*(\tau^\rho)$ is a solution of the integral equation

$$z_\rho(\tau) = f\left(\tau^\rho, \left(G_\rho z_\rho^* \right)(\tau) + Q_\rho(\tau) \right), \ \tau \in [0, 1]. \tag{7.4.9}$$

Approximations $z_{\rho,N}$ to the solution z_ρ^* of (7.4.9) we find in the form (5.3.7), where the coefficients $\{c_{jk}\}$ are found from the non-linear system (5.3.6) by Newton method, using grid points (2.7.1) and collocation points (2.7.3). After that we find the approximations y_N to the solution y^* of problem (7.4.7)–(7.4.8) by the formulas (5.3.11) and (5.3.12).

In Table 7.10 some results of numerical experiments for different values of the parameters N, ρ and r are shown, using $m = 2$ and the collocation parameters (7.1.1).

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Table 7.10: Numerical results for problem (7.4.7)–(7.4.8) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1,\ \rho = 1$		$r = 1,\ \rho = 2$		$r = 1,\ \rho = 3$		$r = 1,\ \rho = 4$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$1.44 \cdot 10^{-3}$		$3.84 \cdot 10^{-4}$		$6.47 \cdot 10^{-4}$		$3.05 \cdot 10^{-3}$	
8	$5.39 \cdot 10^{-4}$	2.68	$6.07 \cdot 10^{-5}$	6.33	$1.03 \cdot 10^{-4}$	6.28	$4.79 \cdot 10^{-4}$	6.38
16	$2.10 \cdot 10^{-4}$	2.56	$1.01 \cdot 10^{-5}$	5.99	$1.74 \cdot 10^{-5}$	5.91	$8.06 \cdot 10^{-5}$	5.94
32	$8.07 \cdot 10^{-5}$	2.60	$1.73 \cdot 10^{-6}$	5.84	$3.02 \cdot 10^{-6}$	5.77	$1.39 \cdot 10^{-5}$	5.78
64	$3.08 \cdot 10^{-5}$	2.62	$3.00 \cdot 10^{-7}$	5.77	$5.28 \cdot 10^{-7}$	5.72	$2.44 \cdot 10^{-6}$	5.72
128	$1.17 \cdot 10^{-5}$	2.63	$5.23 \cdot 10^{-8}$	5.74	$9.27 \cdot 10^{-8}$	5.69	$4.28 \cdot 10^{-7}$	5.70
256	$4.43 \cdot 10^{-6}$	2.64	$9.15 \cdot 10^{-9}$	5.72	$1.63 \cdot 10^{-8}$	5.69	$7.52 \cdot 10^{-8}$	5.69
		1.87		3.48		5.66		5.66

It follows from (5.4.7) in Theorem 5.4.2 with $\alpha = \frac{3}{2}$, $\nu = \frac{3}{5}$ and $\alpha^* = \alpha - n_1 = \frac{1}{2} < 1$ that, for sufficiently large N,

$$\varepsilon_N \leq c \begin{cases} N^{-\frac{9}{10}\rho r} & \text{if} \quad 1 \leq \rho r < \frac{25}{9} \approx 2.78, \\ N^{-\frac{5}{2}} & \text{if} \quad \rho r \geq \frac{25}{9}, \end{cases} \tag{7.4.10}$$

where c is a positive constant not depending on N.

Due to (7.4.10), the ratios Θ_N for $\{\rho = 1,\, r = 1\}$, $\{\rho = 2,\, r = 1\}$ and $\{\{\rho = 3,\, r = 1\},\ \{\rho = 4,\, r = 1\}\}$ ought to be approximatively $2^{\frac{9}{10}} \approx 1.87$, $2^{\frac{9}{5}} \approx 3.48$ and $2^{\frac{25}{9}} \approx 5.66$, respectively. These values are given in the last row of Table 7.10. As we can see, for smaller values of r and ρ the convergence is faster than predicted by the estimate (5.4.7).

7.5 Numerical results for Chapter 6

In this section we show that the numerical method introduced in Chapter 6 for a boundary value problem of type (6.0.1)–(6.0.2) does not attain superconvergence. For this we look at the boundary value problem (7.2.5)–(7.2.6) examined in Example 1 of Section 7.2. That is, we consider the following problem:

$$(D_{Cap}^{\frac{1}{2}} y)(t) + d_0(t) y(t) = f(t), \quad 0 \leq t \leq 1, \tag{7.5.1}$$

$$y(0) + y(1) + \int_0^{\frac{1}{2}} y(s)ds = 2 + \frac{2^{\frac{1}{4}}}{7} + \frac{2^{-\frac{1}{4}}}{9}, \qquad (7.5.2)$$

where

$$d_0(t) = t^{\frac{1}{2}}, \quad f(t) = \frac{3\,\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} t^{\frac{1}{4}} + \frac{5\,\Gamma(\frac{1}{4})}{12\,\Gamma(\frac{3}{4})} t^{\frac{3}{4}} + t^{\frac{5}{4}} + t^{\frac{7}{4}}, \quad 0 \le t \le 1.$$

We see that (7.2.1)–(7.2.2) is a special problem of (6.0.1)–(6.0.2) with

$$\alpha = \frac{1}{2}, \quad b = 1, \quad b_1 = 1, \bar{b} = \frac{1}{2}, \quad \beta_0 = \beta_1 = \beta = 1, \quad \gamma = 2 + \frac{2^{\frac{1}{4}}}{7} + \frac{2^{-\frac{1}{4}}}{9},$$

and

$$y(t) = t^{\frac{3}{4}} + t^{\frac{5}{4}} \quad (t \in [0,1])$$

is its exact solution. Clearly $d_0, f \in C^{q,\mu}(0,1]$ for $\mu = \frac{1}{2}$ and arbitrary $q \in \mathbb{N}$. Therefore, by Theorem 6.1.1,

$$\nu = \max\{1 - \alpha, \mu\} = \frac{1}{2}.$$

Table 7.11: Numerical results for problem (7.5.1)–(7.5.2) with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

N	$r = 1, \rho = 1$		$r = 2, \rho = 1$		$r = 1, \rho = 2$		$r = 2, \rho = 2$	
	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$2.87 \cdot 10^{-2}$		$1.25 \cdot 10^{-2}$		$2.21 \cdot 10^{-2}$		$6.45 \cdot 10^{-2}$	
8	$1.94 \cdot 10^{-2}$	1.48	$4.86 \cdot 10^{-3}$	2.58	$6.43 \cdot 10^{-3}$	3.44	$1.95 \cdot 10^{-2}$	3.31
16	$1.25 \cdot 10^{-2}$	1.55	$1.79 \cdot 10^{-3}$	2.71	$2.10 \cdot 10^{-3}$	3.06	$5.35 \cdot 10^{-3}$	3.65
32	$7.87 \cdot 10^{-3}$	1.59	$6.47 \cdot 10^{-4}$	2.77	$7.13 \cdot 10^{-4}$	2.95	$1.40 \cdot 10^{-3}$	3.82
64	$4.86 \cdot 10^{-3}$	1.62	$2.31 \cdot 10^{-4}$	2.80	$2.47 \cdot 10^{-4}$	2.89	$3.58 \cdot 10^{-4}$	3.91
128	$2.96 \cdot 10^{-3}$	1.64	$8.21 \cdot 10^{-5}$	2.81	$8.63 \cdot 10^{-5}$	2.86	$9.05 \cdot 10^{-5}$	3.95
256	$1.79 \cdot 10^{-3}$	1.65	$2.91 \cdot 10^{-5}$	2.82	$3.04 \cdot 10^{-5}$	2.84	$2.28 \cdot 10^{-5}$	3.98
		1.41		2.00		2.00		4.00

In Tables 7.11 and 7.12 some results of numerical experiments for different values of the parameters N, ρ and r are presented, using for $m = 2$ and $m = 3$

Table 7.12: Numerical results for problem (7.5.1)–(7.5.2) with $m = 3$, $\eta_1 = \frac{5-\sqrt{15}}{10}$, $\eta_2 = \frac{1}{2}$, $\eta_3 = 1 - \eta_1$.

N	$r = 1,\ \rho = 1$		$r = 2,\ \rho = 1$		$r = 3,\ \rho = 2$		$r = 3,\ \rho = 3$	
	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	$1.84 \cdot 10^{-2}$		$7.19 \cdot 10^{-3}$		$1.05 \cdot 10^{-2}$		$2.23 \cdot 10^{-2}$	
8	$1.16 \cdot 10^{-2}$	1.58	$2.66 \cdot 10^{-3}$	2.70	$1.80 \cdot 10^{-3}$	5.84	$4.80 \cdot 10^{-3}$	4.65
16	$7.19 \cdot 10^{-3}$	1.62	$9.63 \cdot 10^{-4}$	2.77	$2.63 \cdot 10^{-4}$	6.87	$7.86 \cdot 10^{-4}$	6.10
32	$4.39 \cdot 10^{-3}$	1.64	$3.44 \cdot 10^{-4}$	2.80	$3.54 \cdot 10^{-5}$	7.42	$1.13 \cdot 10^{-4}$	6.99
64	$2.66 \cdot 10^{-3}$	1.65	$1.22 \cdot 10^{-4}$	2.81	$4.59 \cdot 10^{-6}$	7.71	$1.50 \cdot 10^{-5}$	7.48
128	$1.60 \cdot 10^{-3}$	1.66	$4.34 \cdot 10^{-5}$	2.82	$5.85 \cdot 10^{-7}$	7.85	$1.95 \cdot 10^{-6}$	7.74
256	$9.63 \cdot 10^{-4}$	1.67	$1.54 \cdot 10^{-5}$	2.82	$7.38 \cdot 10^{-8}$	7.93	$2.47 \cdot 10^{-7}$	7.87
		1.41		2.00		8.00		8.00

the collocation parameters (7.1.1) and (7.1.2), respectively. For $m = 2$ it follows from (6.2.10) in Theorem 6.2.1 with $\nu = \frac{1}{2}$ that, for sufficiently large N,

$$\varepsilon_N \leq c_0 \begin{cases} N^{-0.5\rho r} & \text{if} \quad 1 \leq \rho r < 4, \\ N^{-2} & \text{if} \quad \rho r \geq 4, \end{cases} \tag{7.5.3}$$

where c_0 is a positive constant not depending on N. Due to (7.5.3), the ratios Θ_N for $\{\rho = 1,\ r = 1\}$, $\{\{\rho = 1,\ r = 2\}, \{\rho = 2,\ r = 1\}\}$ and $\{\rho = 2,\ r = 2\}$ ought to be approximatively $2^{0.5} \approx 1.41$, $2^1 = 2$ and $2^2 = 4$, respectively. These values are given in the last row of Table 7.11.

In the case $m = 3$ it follows from (6.2.10) in Theorem 6.2.1 with $\nu = \frac{1}{2}$ that, for sufficiently large N,

$$\varepsilon_N \leq c_1 \begin{cases} N^{-0.5\rho r} & \text{if} \quad 1 \leq \rho r < 6, \\ N^{-3} & \text{if} \quad \rho r \geq 6, \end{cases} \tag{7.5.4}$$

Here c_1 is a constant which is independent of N. Due to (7.5.4), the ratios Θ_N for $\{\rho = 1,\ r = 1\}$, $\{\rho = 1,\ r = 2\}$ and $\{\{\rho = 2,\ r = 3\}, \{\rho = 3,\ r = 3\}\}$ ought to be approximatively $2^{0.5} \approx 1.41$, $2^1 = 2$ and $2^3 = 8$, respectively.

As we can see from Tables 7.11 and 7.12, for small values of r and ρ the actual convergence rate is faster than predicted by Theorem 6.2.1. However, the maximal numerical convergence rate agrees with the estimate (6.2.10) and thus the method is not superconvergent on the interval $[0, 1]$.

7.6 Concluding remarks

In this thesis we have introduced and analyzed high order numerical methods for solving boundary value problems for linear and nonlinear fractional differential equations with Caputo fractional derivatives. For this we have studied the regularity of their exact solutions and have shown that, despite the lack of regularity, it is possible to recover the optimal convergence order of the proposed algorithms, by using special graded grids and by introducing a smoothing variable transformation which allows a new equation with a smoother solution. Moreover, by a judicious choice of smoothing, grid and collocation parameters it is possible to obtain global superconvergence results for methods based on the integral equation reformulation with respect to the highest order fractional derivative. The obtained theoretical results have been verified by extensive numerical examples.

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Sisukokkuvõte

Murruliste tuletistega diferentsiaalvõrrandite ligikaudne lahendamine

Murrulised tuletised (s.t. tuletised, mille järk ei ole täisarv) on pakkunud huvi juba alates ajast, millal I. Newton ja G. W. Leibniz rajasid matemaatilise analüüsi aluseks oleva diferentsiaal- ja integraalarvutuse. Kaua aega käsitleti murruliste tuletistega seotud küsimusi vaid teoreetilisest vaatepunktist, sest ei olnud näha, millised võiksid olla murruliste tuletiste rakendusvõimalused. Viimastel aastakümnetel on aga leitud, et murrulisi tuletisi sisaldavad diferentsiaalvõrrandid kirjeldavad mitmesuguste materjalide ja protsesside käitumist paremini kui täisarvulist järku tuletistega diferentsiaalvõrrandid. See on kaasa toonud suure huvi murruliste tuletiste ja nende rakendamise võimalikkuse kohta.

Murruliste tuletistega diferentsiaalvõrrandite täpse lahendi leidmine ei ole enamasti võimalik ja seega peame nende lahendeid leidma ligikaudselt. See nõuab aga spetsiaalsete meetodite väljatöötamist, kuna murruliste tuletistega diferentsiaalvõrrandite korral ei ole reeglina rakendatavad täisarvuliste tuletistega diferentsiaalvõrrandite vallast tuntud tulemused.

Käesoleva töö põhieesmärk on välja töötada efektiivsed lahendusalgoritmid murruliste tuletistega diferentsiaalvõrrandite ligikaudseks lahendamiseks võimalikult laia ülesannete klassi korral. Selle saavutamiseks formuleeritakse esialgne ülesanne ümber teatava integraalvõrrandina ja uuritakse kõigepealt selle (seega ka lähteülesande) lahendi siledust ning võimalikku singulaarset käitumist. Lahendi sileduse informatsiooni põhjal rakendatakse saadud integraalvõrrandi ligikaudseks lahendamiseks tükiti polünomiaalset kollokatsioonimeetodit. Lähislahendite kiire koonduvuse saavutamiseks kasutatakse spetsiaalselt gradueeritud ebaühtlaseid võrke, milles võrgu sõlmed paiknevad tihedamalt integreerimislõigu alguspunkti

läheduses, kus ülesande lahendi tavatuletised võivad tõkestamatult kasvada. Tugevalt ebaühtlaste võrkude kasutamine võib soodustada ümardamisvigade kuhjumist ning praktiliste arvutuste läbiviimisel põhjustada teatavat numbrilist ebastabiilsust, kui võrgupunktide arv on küllalt suur. Seetõttu on töös sisse toodud sobiv siluv muutujavahetus, mille abil teisendatakse aluseks olevat integraalvõrrandit nii, et uue integraalvõrrandi lahendi tavatuletiste singulaarsused on nõrgemad või koguni puuduvad. Seejärel leitakse uue integraalvõrrandi ligikaudne lahend ühtlasel või nõrgalt gradueeritud võrgul tükiti polünomiaalse kollokatsioonimeetodi abil.

Töös vaadeldakse kahte varianti lähteülesande ümberformuleerimiseks temaga samaväärse integraalvõrrandina. Peamiselt käsitletakse situatsiooni, kus saadava integraalvõrrandi otsitava suuruse osas ei ole mitte lähteülesande täpne lahend, vaid lähte-ülesandes esinev otsitava funktsiooni kõrgeimat järku murruline tuletis. Töös näidatakse, et sellise lähenemise korral on võimalik teatavate siluvate teisenduste ja sobivate võrkude abil saavutada lähislahendite ülikiire koondumine. Vaadeldakse ka alternatiivset meetodit, kus lähteülesande lahendamine on taandatud sellise integraalvõrrandi lahendamisele, milles otsitavaks funktsiooniks on lähteülesande täpne lahend. Sel korral on esitatud analüüs, mis ei näita lähislahendite ülikiiret koonduvust. Toodud numbrilised eksperimendid kinnitavad, et antud meetodi korral lähislahendite ülikiiret koonduvust tõepoolest ei saavutata.

Käesolev doktoritöö koosneb seitsmest peatükist. Esimeses peatükis antakse lühike kirjeldus murruliste tuletiste ajaloost ning murruliste tuletistega diferentsiaalvõrrandite numbrilise lahendamisega seotud probleemidest. Teises peatükis esitatakse rida töös vajalikke tähistusi, mõisteid ja tulemusi. Järgnevad kolm peatükki on pühendatud meetodile, mis kasutab esialgse ülesande taandamist integraalvõrrandile, kus otsitavaks suuruseks on esialgses ülesandes esinev kõrgeimat järku murruline tuletis. Kolmanda peatüki eesmärk on anda sissejuhatav ülevaade neljandas ja viiendas peatükis kasutatavast lähenemisest.

Neljandas peatükis vaadeldakse võimalikult laia murruliste tuletistega lineaarsete ülesannete klassi kujul (4.0.1)–(4.0.2):

$$(D_{Cap}^{\alpha_p} y)(t) + \sum_{i=0}^{p-1} d_i(t)(D_{Cap}^{\alpha_i} y)(t) + \int_0^t (t-s)^{-\kappa} K(t,s)y(s)ds = f(t) \,, \quad 0 \leq t \leq b,$$

$$\sum_{j=0}^{n_0} \beta_{ij0}\, y^{(j)}(0) + \sum_{k=1}^{l} \sum_{j=0}^{n_1} \beta_{ijk}\, y^{(j)}(b_k) + \beta_i \int_0^{\bar{b}_i} y(s)ds = \gamma_i\,, \quad i = 0,\ldots,n-1,$$

kus $\beta_{ij0}, \beta_{ijk}, \beta_i, \gamma_i \in \mathbb{R}$, $p \in \mathbb{N}$, $n_0, n_1 \in \mathbb{N}_0$,

$$0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_p \le n, \quad n := \lceil \alpha_p \rceil, \quad 0 \le \kappa < 1,$$

$$0 < b_1 < \cdots < b_l \le b, \quad 0 < \bar{b}_i \le b, \quad i = 0,\ldots,n-1, \quad n_0 < n,\ n_1 < n,$$

$d_i : [0,b] \to \mathbb{R}$ $(i = 0,\ldots,p-1)$, $f : [0,b] \to \mathbb{R}$, $K : \{(s,t) : 0 \le s \le t \le b\} \to \mathbb{R}$ on etteantud pidevad funktsioonid ja $D_{Cap}^{\alpha_i} y$ $(i = 0,\ldots,p)$ on vastavate järkudega Caputo murrulised tuletised otsitavast funktsioonist y. Peatüki põhitulemused on antud teoreemidega 4.2.1, 4.4.1 ja 4.4.2. Teoreemiga 4.2.1 on antud vaadeldava ülesande täpse lahendi y olemasolu, ühesus ja regulaarsuse omadused. Teoreemides 4.4.1 ja 4.4.2 on esitatud tulemused välja töötatud numbriliste meetodite abil leitud lähislahendite koondumis ja veahinnangute kohta. Diferentsiaalvõrrandis olevate kordajate $d_0(t), \ldots, d_{p-1}(t)$ ja vabaliikme $f(t)$ sileduse kohta tehakse teatud eeldused, mis on rahuldatud kõigi lõigus $[0,b]$ pidevate ja m korda $(m \ge 1)$ pidevalt diferentseeruvate funktsioonide korral ning võimaldavad käsitleda ka selliseid funktsioone, mille tuletised alates mingist järgust võivad olla tõkestamata integreerimislõigu alguspunkti ümbruses.

Viiendas peatükis uuritakse mittelineaarse murrulise tuletisega rajaülesande (5.0.1)–(5.0.2) ligikaudset lahendamist peatükkides kolm ja neli vaadeldud meetodite abil. Vaadeldava ülesande lahendi siledus on antud teoreemiga 5.2.2 ja lähislahendite koonduvus ja koonduvuskiirust iseloomustavad teoreemid 5.4.1 ja 5.4.2.

Kuuendas peatükis on esitatud alternatiivne meetod murrulise tuletisega diferentsiaalvõrrandi rajaülesande ligikaudseks lahendamiseks. Saadud lähislahendite koonduvuskiiruse hinnangud on antud teoreemiga 6.2.1. Me näeme, et erinevalt varem vaadeldud meetodist tulemus lähislahendite ülikiire koonduvuse kohta puudub, mida kinnitavad ka järgmises peatükis läbi viidud numbrilised eksperimendid.

Viimases peatükis on töös saadud teoreetilisi tulemusi testitud numbriliste eksperimentide abil. Testülesannete lahendamisel saadud arvulistest tulemustest järeldub, et töös saadud veahinnangud on järgu poolest mitteparandatavad.

Enamus käesoleva töö tulemustest sisalduvad autori poolt avaldatud publikatsioonides [68–73, 93]. Seejuures osa avaldatud tulemusi on käesolevas töös

laiendatud üldisemale juhule. Saadud tulemusi on tutvustatud üheksal rahvusvahelisel teaduskonverentsil.

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List of original publications

This thesis is based on the following publications:

- A. Pedas, E. Tamme and M. Vikerpuur. Piecewise polynomial collocation for a class of fractional integro-differential equations. *Integral Methods in Science and Engineering*, pages 471–482. Birkhäuser, 2015.

- A. Pedas, E. Tamme and M. Vikerpuur. Spline collocation for fractional integro-differential equations. *Lecture Notes in Computer Science*, 9045:315–322, 2015.

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