Overcoming Difficulties in Learning Calculus Concepts: the Case of Grade 12 Students

by

Ashebir Sidelil Sebsibe

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Keywords

Calculus concepts; Concept test; Conceptual knowledge; Constructivism; Continuity; Derivative; Difficulties in calculus; Level of conceptual knowledge; Limit concept; Misconception; Overcoming difficulties; Procedural knowledge; Synthesized difficulties

ABSTRACT

Research has indicated the importance of calculus knowledge for undergraduate programs in science and technology fields. Unfortunately, one of the main challenges faced by students who join science and technology fields is their knowledge of calculus concepts. The main purpose of the study is to overcome students' difficulties in learning calculus concepts by developing a literature informed intervention model. A design-based research approach of three phases was conducted. Grade 12 natural science stream students in one administrative zone in Ethiopia were used as the study population.

Triangulated themes of students' difficulties and common conceptual issues that are causes of these synthesized difficulties in calculus were used as a foundation to propose an intervention model. Based on the proposed model, an intervention was prepared and administered. A pre post-test aimed to asses students' conceptual knowledge in calculus was used to examine the effect of the model. Quantitative analysis of the test revealed that the intervention has a positive effect. The experimental group score is better than the controlled group score with independent t-statistics, t = 4.195 with alpha =.05. In addition, qualitative analysis of the test revealed that students in the experimental group are able to overcome many of the difficulties. In particular, many students demonstrated process level conception, conceptual reasoning, qualitative justification, a consistency in reasoning, less algebraic error, and a proficiency in symbolic manipulation.

The study concludes with Implications for practice that includes the use of students' errors and misconceptions as an opportunity for progression. Besides, students should be assisted to make sense of concepts through real-life problems, including training teachers in problem-solving approaches and mathematical thinking practice.

Keywords: Calculus concepts; Concept test; Conceptual knowledge; Constructivism; Continuity; Derivative; Difficulties in calculus; Level of conceptual knowledge; Limit concept; Misconception; Overcoming difficulties; Procedural knowledge; Synthesized difficulties.

DECLARATION

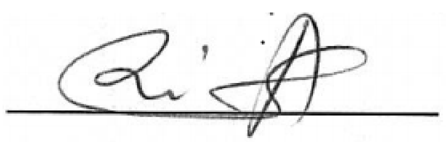
Name: Ashebir Sidelil Sebsibe

Student number: 57654972

Degree: PhD

Title: **Overcoming Difficulties in Learning Calculus Concepts: the Case of Grade 12 Students**

I declare that the above thesis is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.



October, 2019

SIGNATURE **DATE**

ACKNOWLEDGEMENT

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CHAPTER ONE: INTRODUCTION

This general introductory chapter consists of six parts. The first part (1.1) sets the background of the study. While the second part (1.2) presents the problem statement of the study, the next part (1.3) presents the purpose of the study, the significance it contributes, and the research questions that guide the study. The fourth part (1.4) gives an operational definition of terms and concepts used in the study. The fifth part (1.5) explains the design-based research approach, description of the organization of the study, and key findings of the study.

1.1. Background of the study

Calculus is a subdivision of mathematics, which emerged out of a need to be aware of continuously changing quantities. It deals with the infinitely small and the infinitely large quantities of a function (Muzangwa & Chifamba, 2012). Calculus concepts are a precondition for most science, technology, and engineering fields of undergraduate programmes. Students' conceptual knowledge of calculus concepts affects not only their performance and involvement in mathematics but also in these fields. It is a vital way to give rise to future scientists, technologists, mathematicians, and engineers (Bressoud, Carlson, Mesa & Rasmussen, 2013; Carlson & Oehrtman, 2005; Kinley, 2016; Roble, 2017; Sadler & Sonnert, 2016). Thus, it is critical that this topic has to be understood for helpful and proficient benefit of the good of it, for producing citizens who can engage in the production and service sectors with advance academic knowledge and vocational skills. As an instrument, calculus allows people to realize greater achievements than the mathematics courses that precede it (Kelley, 2006; Roble, 2017; Sadler & Sonnert, 2016).

Regardless of the comparative importance of calculus, it is very unsatisfactory that students' performance in calculus is destitute and there are many difficulties, which are previously examined and still take place in a good number of students' test scripts. Researchers, in different contexts of the world, have shown that students have problems in gaining a deep and accurate understanding of the limit concept in particular and calculus concepts in general (For instance, Çetin, 2009; Jordaan, 2005; Juter, 2006; Moru, 2006; Muzangwa & Chifamba, 2012). In the traditional approach, most mathematics teachers and students' centre of attention are rules and procedures. Because of this practice, most of the students perform rules and procedures without internalization and focusing on the embedded concepts (Berry & Nyman, 2003; Bezuidenhout, 2001; Kinley, 2016; Makgakga & Makwakwa, 2016).

From a constructivist view of knowledge construction, the approaches that students make sense in order to visualize concepts and mental images that they form has a major contribution to the existing difficulties (Aspinwell & Shaw, 2002). It is well recognized that the traditional approach to calculus is not effective in reducing those difficulties and misconceptions. Previous studies (For example, Herbert, 2013; Idris, 2009; Naidoo & Naidoo, 2007) test different approaches in which a good number of them are computer integrated. With all those efforts, the challenges of teaching calculus are still persistent and students' performance is below the expected level (Herbert, 2013; Naidoo & Naidoo, 2007; Reinholz, 2015). In the U.S.A., in which all students supported by appropriate technology and reform efforts, every fall semester, 27% of post-secondary students are not successful in calculus courses (Bressoud et al., 2013). In Malaysia, the failure rates of college students in consecutive calculus courses are above 30% for nearly every semester (Ahmad, Mahadi, Yusri, Yusop, Ali, & Heng, 2017). In Ethiopia, each year around 44% of pre-engineering students fails to get the pass grades in a refreshment calculus course and 14% drop out of the course before sitting for the final exam.

Besides these observations, students get good marks in teachers made tests and classroom evaluations do not mean they have the required conceptual knowledge in calculus. Researchers (for items designed to diagnose the existence of systematic errors) find evidence of students' difficulty and lack of knowledge in calculus. Thus, while students' performance on teachers made test and examination papers demonstrate some evidence of learning and understanding, researchers' findings confirm misconceptions, rote learning, and lack of conceptual knowledge (Idris, 2009). This gap is more visible to teachers of non-mathematics courses in which mathematics is a pre-requisite for the course that they teach (Bezuidenhout, 2001; Idris, 2009).

Thus, the extent teachers and researchers are aware, identify and react to students' difficulties is very important. Accordingly, the demand for an alternative approach to overcome the difficulties, especially in the areas where the practice of educational technology is not well developed, is compulsory. As Ethiopia is a part of the world, the case is not different. Research findings revealed similar results as found elsewhere (Areaya & Sidelil, 2012; Denbel, 2015; Walelign, 2014).

Currently, the country has acknowledged that its growth depends very much on the expansion of science and technology personnel, and thus on science and mathematics education (MoFED, 2010)[1]. One of the country's strategy states that a seventy percent of the university enrolment would be in natural science, engineering, and technology fields. This situation demands unique attention to science and mathematics at the secondary education level. On the contrary, science and mathematics at the secondary level encounter various challenges that seek urgent enhancements (Asfaw, Otore, Ayele, & Gebremariam, 2009, p.2).

An evaluation of students' mathematical ability conducted at Dire Dawa University revealed that a great number of students have been attending the university with an inadequate background of mathematical proficiency (Walelign, 2014) and their point of view towards the subject is not positive. In the same study, it has been mentioned that only 14.96% of the students accomplished 55% and above in the test prepared to assess their mathematical knowledge. The study concluded that at entry-level, a large number of students have a poor achievement in mathematics.

The trend of national learning assessment carried out every four years since 2000 showed that students' performance in science and mathematics was very low (Gebrekidan, 2010). It is also believed that failure in cognitive performance and psychological disappointment in science and mathematics contributes to repeating class years and eventually leads to terminating the academic track. According to the 2010 national learning assessment, 42.3% of grade 12 students' score in mathematics was found to be below the pass mark (Gebrekidan, 2010). Besides, the

[1] Ministry of Finance and Economic Development

World Bank document disclosed that national averages of the mathematics learning assessment scores at grade 12 declined from 2010 to 2014 (World Bank, 2017).

1.2. Statement of the problem

The study conducted by Areaya and Sidelil (2012), in calculus at upper secondary schools revealed that students have difficulties and misconceptions similar to those found in the literature. Moreover, teachers' opinion and practice, the focus of contents in textbooks, and locally prepared reference books is more procedural than conceptual as the duality of mathematics knowledge is concerned. Experience and observation also illustrates that besides the nature of the concepts that cause some inherent difficulties, the approaches used by the teachers to introduce these calculus concepts have an impact on the difficulties that students encounter. Besides, a large number of students blameworthy their engagement in the hard science fields of study due to the challenges that they face in calculus courses.

The evidence in the above paragraph together with the discussion in the background of the study reveals the gap between what is intended and the inadequate approaches employed for developing the required conceptual knowledge of calculus for benefiting the goods in it. Besides, acknowledging the nature of students' difficulties in calculus, it is apparent that such a profound cognitive difficulty will not be resolved unless the students get actual support from their guide that will provide them with practical tasks which are suitable for the perceptive formation of notions (Aspinwall & Shaw, 2002; Keri, Liston, Selden, Salomone, & Zorn, 2010; Tall, 1993). Moreover, understanding in general and concept formation, in particular, is context laden. It can be affected by the education system, teachers' training, school culture, and accessibility of technology.

The beginning of the calculus teaching improvement programme, which started in the U.S.A. and later extended to elsewhere in the world, initiated the introduction of calculus in high schools. Currently, in many countries, calculus is part of the high school curriculum. For example, the work of Brijlall and Ndlovu (2013), Çetin (2009), and Idris (2009) where evidence in South Africa, Turkey, and Malaysia respectively. One of the objectives of the reform was enabling students to grasp the basic underpinning concepts and to prepare them adequately for higher-level courses (Engelbrecht, Harding & Potgieter, 2005). Thus, students entering university are expected to join the university-level courses with the basic conceptual knowledge of calculus. Contrary to the objective, both theoretical and empirical analysis (For instance: Bezuidenhout, 2001; Brijlall & Ndlovu, 2013; Ferrini-Mundy & Gaudard, 1992; Idris, 2009; Juter, 2006; Kinley, 2016; Muzangwa & Chifamba, 2012) revealed that students learning is procedural skill dominated and lack conceptual knowledge. However, whether one views mathematical concepts as a foundation for applications (as tools for other disciplines) or as pure mathematics, procedural skill is necessary but not sufficient for the course (Lauritzen, 2012; Hiebert *et al*., 2000; Mahir, 2009).

In Ethiopia too, since the 1994 new education policy, calculus has been taught starting from secondary school (grade 12) in addition to university freshman course. The topics in calculus at grade twelve include the limit of number sequence, the limit of functions, continuity, derivatives, integrals, and their applications in the intuitive approach. At first-year in Universities, all science and engineering field students have been taking all of these topics as a refreshment course.

Experience and observation revealed that difficulties in calculus brought from grade 12 challenge students' progress at a university. The literature noted that those difficulties are due to teaching-learning practices that focus to a great level with the procedural part and neglected a solid ground in the underpinning concepts (Aspinwall & Miller, 2001). Thus, the question that remains to be answered is whether there are any other alternative strategies to approach calculus so that students gain better conceptual knowledge. The researcher thinks that observed difficulties could provide valuable learning opportunities for students provided appropriately utilized and this study is aimed to take advantage of this potential. Therefore, the claim of the researcher is that empirical students' learning is more procedure-oriented than conceptual. Therefore, to make a balance, the practice should give more attention to the deficient one. Of course, associated with this and other expectations of students, and what is intended in a curriculum, innovative activities are expected from teachers; shifting the perspective of knowledge from memorizing and replicating information and procedures into being able to dig and able to use it in any way required. Thus, the issue becomes problematic and needs research to design a strategy that will combine the procedural knowledge and the conceptual knowledge of those calculus concepts so that students gain knowledge that is adaptable to different contexts.

From the national issue and personal concern raised above, carrying out research by analysing the students' difficulties at the upper secondary school level in Ethiopia may shed light to minimize the problem. Thus, on successful completion, this study will be useful to improve the practice in teaching-learning calculus concepts. This, in turn, has a benefit to the successful progress of the national agenda, and as a result to influence positively the social and economic condition of the country.

1.3. Purpose of the study

The purpose of the study is twofold: (1) to explore and synthesize students' difficulties in learning calculus concepts (2) based on their difficulties, to develop an intervention model that enhances students' conceptual knowledge of calculus concepts. In particular, the study will address the following specific objectives:

I. To investigate and synthesise students' difficulties in calculus from current literature.
II. To investigate common conceptual issues that causes students' difficulties in calculus.
III. To identify components of an intervention model that enhances students' knowledge of calculus concepts.
IV. To determine the possible effect of the proposed intervention model on students' level of conceptual knowledge in calculus.

Although studies of this type have been conducted by other researchers elsewhere, it has some differences with respect to the problem outlook, the research approach, content covered, the context of the study, methodology, population, and instruments used (both for data collection and intervention). In the first place, there is no research that integrates a synthesise of the literature on students' difficulties, plans an intervention, and tests the effect of the intervention in a sequential or developmental approach. On the other hand, most currently emerging interventions in calculus are educational technology and related infrastructure demanding. Nevertheless, contrary to what Tall and Mejia-Ramos (2004) described, still large parts of the world population have no such educational technology at secondary school level including this study population. For instance, Çetin (2009) and Naidoo and Naidoo (2007) in the study aimed to enhance the conceptual understanding of undergraduate students in calculus, computer-assisted interactive teaching was used. The study by Luneta and Makonye (2010), Pillay (2008), Przenioslo (2003), and Siyepu (2015) all were focused on the inquiry of the existing misconceptions on college students. On the other hand, some other researchers (Maharajh, Brijilall & Govender, 2008; Rabadi, 2015; Roh, 2005) found promising results without the use of such technology.

A study of this kind has not been conducted in the study area before. The population is limited to grade 12 natural science stream students and these students have no experience of using educational technology like graphic calculators, or computers in mathematics classrooms. Besides, the study integrated exploring of existing difficulties, designing of overcoming strategy, and testing of the possible effects of the proposed model.

This study has a potential benefit to practitioners, students, researchers, and as reference material in particular as well as to the policymakers in general. Accordingly, the information originated from this research study, besides addressing a national concern, will contribute to as the source of literature review for the filed.

1.3.1. Research questions

With the above objectives, the main question guiding the research is, based on students' difficulties in learning calculus concepts, what intervention model could be developed to overcome the identified difficulties and enhance their conceptual knowledge. The specific research questions are formulated as follows.

I. What does the current literature reveal about students' difficulties in learning calculus concepts?

II. What are the common conceptual issues that cause students' difficulties in calculus?

III. What are the components of an intervention model of learning calculus concepts that could be developed to enhance students' conceptual knowledge of calculus?

V. Is there a significant difference in the students' level of conceptual knowledge of calculus after learning with the proposed model? Explicitly, this question has the following null hypotheses:

   i. Ho: There is no significant difference between the mean scores of students in the experimental group and the control group during the pre-test.
   ii. Ho: There is no significant difference between the mean scores of students in the experimental group and the control group during the post-test.

1.4. Definition of key terms

Activity- a set of exercises and problems that are designed based on the constructs of conceptual knowledge and fairly different from exercises in textbooks and reference books (Breen and O'Shea, 2010).

Applied mathematics I- is a university refreshment course given to all incoming engineering, also called pre-engineering students. Sixty percent of this course content is calculus concepts, i.e. limit and continuity, derivatives and application of derivatives, integration, and application of integrations (HESC[2], 2013).

Conceptual items- assessment items that are designed based on the constructs of conceptual knowledge and aimed to asses' presence of conceptual knowledge and fairly different from the usual teachers made assessment items or exercises in textbooks and reference books.

Conceptual knowledge- is knowledge of how or why to apply a concept that is adaptable, modifiable and applicable to a variety of circumstances (based on Engelbrecht *et al.* (2005)).

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[2] Higher Education Strategy Centre

Intervention model- a set of purposeful constructs of conceptual knowledge that could be incorporated in the teaching-learning platform and accomplished, including the description of how the constructs are labelled and connected.

Learning difficulties in calculus- deficit in students' mathematical knowledge, includes the presence of misconceptions, interference of past knowledge or lack of a pre-requisite knowledge. For instance:

i. For the item, "compute $\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$". If a student answered $\frac{0}{0}$, then she/he has a misconception that limit is the value of the function at the limit point. Moreover, if she/he answered 0 i.e. if simplify $\frac{0}{0} = 0$ then this is a lack of the pre-requisite knowledge that number over zero is indeterminate form.

ii. For the item "is $f(x) = x^2$ continuous on [0, 4]? Justify your answer" if she/he answered yes the function is continuous because I can draw the graph without lifting my pen from $x = 0$ to $x = 4$. The answer is correct, but the reasoning has the difficulty that occurs due to past knowledge interference i.e. confusing continuity with connectedness.

Overcoming difficulties- a group of students is said to be have improved their conceptual knowledge (and hence overcome difficulties):

i. If the mean score of the experimental group students is greater than their counterparts in the control group.

ii. If the experimental group students' qualitative performance and justification for reasoning level items on the test are better than their counterparts in the control group including a correct answer for a correct reason.

Procedural knowledge- is the ability to compute the solution of a problem associated with exploring a set of rules and procedures in a coherent, consistent, and flexible mathematical practices.

Upper secondary (preparatory) school - a two-year programme (grade 11 and grade 12) that the students are expected to attend after they completed grade 10 and that prepares them for university (FDRGE[3], 1994).

1.5. Research approach and key findings

The approach of research emerges out of the purpose and nature of the research questions. To deal with the stated purpose and to answer the outlined research questions, the study demanded to synthesise literature on students' difficulties, explore common conceptual issues that cause those difficulties, propose an intervention model to overcome those difficulties, prepare an intervention based on the proposed model, and evaluate the possible effect. Thus, a design-based research approach (Plomp, 2007) was applied. For that reason, the study has been organized into three mutually reliant sub-studies (phases) that are in alignment with the research questions.

During the preliminary research phase, using systematic review, students' difficulties and strengths have been identified and synthesised. Informed by the literature and theory, a concept test was prepared and a diagnostic assessment was conducted to triangulate students' difficulties and to explore the causes of those difficulties.

At the prototyping phase, based on the difficulties, the causes of those difficulties, and the theoretical perspective components of an intervention strategy that could be implemented to overcome observed difficulties were identified. Those components were classified and structured. The structure is proposed as an intervention model that enhances students' conceptual knowledge in calculus.

The third is an assessment phase. An intervention based on the proposed intervention model was prepared and implemented on experimental group students. A pre-post test was administered to the students who avail themselves in two classes. The phase ended up with an analysis of the possible effects of the model. Finally, discussion, conclusion, and recommendation of the study were provided in a separate chapter. Figure 1 presents the procedure and layout of the study.

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[3] Federal Democratic Republic Government of Ethiopia

As discussed above, the study started by sharing synthesis of studies that elicit challenges faced and shown by students while learning calculus. The diagnostic assessment revealed that students of the study area have difficulties that are not far from those in the literature. Triangulated themes of difficulties revealed that students' learning involves a static view of a dynamic process. Additionally, a lack of describing definitions and relationships of terms was investigated as difficulties. Moreover, overgeneralization and inconsistent cognitive structure, over-dependence on procedural learning, and lack of making a logical connection between conceptual aspects were found as students' difficulties. Further, a lack of a coherent framework of reasoning and lack of computational proficiency were found as students' difficulties.

Besides, the diagnostic assessment revealed the way students' approach conceptual issue and causes of the difficulties. In particular, an arithmetic thinking than algebraic, linguistic ambiguity, compartmentalized learning, dependent on concept image than concept definition, obtains a correct answer for the wrong reasons, focuses only on an algebraic form of representations, and focuses on lower-level cognitive demanding exercises and in general surface learning approaches were identified as conceptual issues behind the difficulties. Thus, the researcher, guided by all these data, i.e. the literature, the empirical evidence, and his experience developed an intervention model. The model was intended to enhance conceptual knowledge through focusing on mathematical thinking practice conjecturing and convincing, reflection and communication via think-pair-share technique, and on the dual nature of concepts, reconstructive generalization vis-à-vis cognitive conflict strategies. In addition, incorporating reasoning level and real-life problems, widening students thinking through counterexamples, and error analysis have included.

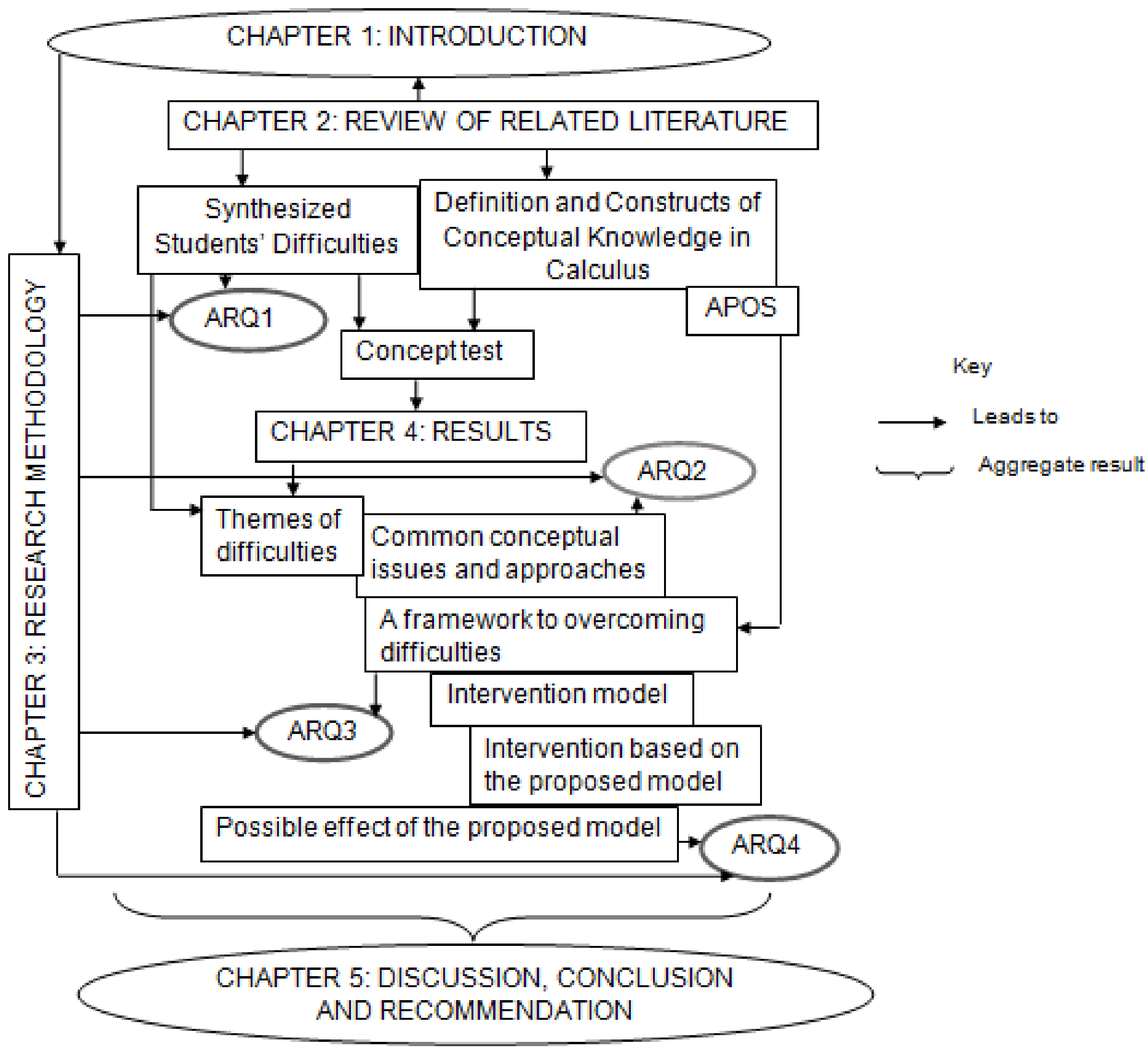

Figure 1: Approach and layout of the study[4]

After the implementation of the model, the post-test result showed that students in the experimental group scored (mean=28.10, SD=9.680) better than the controlled group (mean=20.26, SD=9.451). The independent t-statistics result indicates t = 4.195 with alpha = .05. This result suggests that students in the experimental group performed significantly better than the control group. The text analysis on students' test script showed that many students in the experimental group showed a process level conception, conceptual reasoning, qualitative justification, a consistency in

[4] ARQ i =Answer to research question i , where $1 \leq i \leq 4$

reasoning, and less algebraic and symbolic manipulation errors. The study concluded with recommendations for practitioners. In particular, it is recommended to include mathematical thinking practice and problem-solving skills in the curriculum and incorporate the Certainty of Response Index (CRI) in tests. Additionally, assessing teachers’ awareness and opinion about the emerging pedagogical and theoretical frameworks and incorporating real-life activities in the students’ tasks are points that seek further research. Moreover, replicating the study in a different context to assure generalization of the results, checking the retention of knowledge after using the model, and comparing the effectiveness of the intervention used in this study with an intervention based on computer programs are issues that need further research.

CHAPTER TWO: REVIEW OF RELATED LITERATURE

This chapter accounts for two components of the study. The first section is a literature review of students' difficulties and strengths in learning calculus of three conceptual areas. These conceptual areas are limits (both finite the limit at a point and limit involving infinity including the limit of a sequence), continuity, and derivatives. The review aims to present a synthesis of the difficulties that students demonstrate in the learning of calculus. The difficulties later used as a point of reference to prepare a concept test. The concept test, in turn, will be used to examine the conceptual knowledge of students in the study area. Besides, the test results will be used as an input to design an intervention model that must be implemented so that students overcome synthesised difficulties.

There are various aspects of students' difficulties in understanding mathematical concepts such as cognitive, epistemological, didactical, and psychological (Moru, 2006). In this study, however, the term 'difficulty' is limited to a cognitive aspect of learning difficulties.

In the first section of the chapter, the first part (2.1.1) explains the scope of the review, the procedure followed in the searching of the literature and description of the literature used for the final analysis. The literature search was conducted iteratively. The second part (2.1.2) presents quotations, and initial codes obtained from the literature. The third part (2.1.3) presents the formation of descriptive themes and details of students' difficulties in each descriptive theme. The fourth part (2.1.4) presents analytical themes of difficulties on students' learning of the calculus concepts as analysed from the literature as identified.

The second section of the chapter presents the theoretical aspect of the study. The theoretical analysis presented in this chapter was used to describe the framework through which the students' activities are analysed, to construct definitions of key terms of the study and identify key constructs of conceptual knowledge from a different perspective. The theoretical framework of this study is the constructivism perspective of learning and its bridge theories. The section begins with (2.2.1) the discussion of constructivism learning theory followed by a discussion of its bridging theories as a model of concept formation. The views on conceptual and procedural knowledge in mathematics (2.2.2) are then discussed and evaluated with the purpose to identify the contextual definition of conceptual knowledge in the study. The section ends with (2.2.3) discussion of the basic constructs of conceptual knowledge in calculus.

2.1. Students' difficulties in understanding calculus concepts

2.1.1. Scope and procedures of the review

Scope of the review

This practical review focused on investigating literature on difficulties and strengths of learning calculus concepts among students taking the course at secondary school or at first-year university courses. Since the participants of this study are, grade 12 students, studies on advanced level calculus courses are not appropriate. In this study area, a new mathematics curriculum was implemented at all levels of the education system following the new education and training policy formulated in 1994. The final phase of the secondary school curriculum implementation occurred with mathematics in grade 12 in 2002. The new curriculum pulled the introduction of calculus from university freshman course to grade 12 (FDRGE, 1994). The review considered the starting of the new curriculum implementation year as a benchmark for inclusion of studies for the review. Thus, all local and international literature since September 2002 constituted the population of the review.

Thematic review, which is one type of systematic review, is a powerful tool to make informed decisions about challenging claims based on a qualitative explanation of the existing information about a problem (Thomas & Harden, 2008). The explanatory nature does not depend on the number of studies included rather on the depth and breadth of the studies selected for the review (ibid). Based on this background, relevant studies of the review were selected purposively.

Sampling of literature

With purposive sampling, before the individual studies were selected the following criteria for inclusion were set, i.e. a study was considered if it:

1. It is carried out in any country from around the world, but published/written in the English language.
2. Is non-intervention study on the limit (including the limit of a sequence), continuity, derivative or calculus (i.e. Involving more than one concept).
3. Previous systematic reviews on any one or more than one of the concepts; limit, continuity or derivative.
4. Has a year of publication (from 2002 to 2016).
5. Has the education level of participants (at upper secondary or first-year university).
6. It has a clear description and an explanation of the research purpose, number of participants, data collection instrument used, and source.
7. Is aimed to describe students' difficulty of learning calculus.
8. It is done in a context where classroom technology is not exhaustively used.

Procedures of the literature search

The review was guided by a coding and iterative process as proposed by Miles, Huberman, and Saldana (2014). Multiple literature searches were conducted in electronic databases. Keyword searches on the website Google, Google Scholar, UNISA's institutional repository, Education Resources Information Centre (ERIC) were used as a primary stage. Initial searching terms like the limit concept, derivative, difficulty in calculus, student difficulties in the limit, cognitive obstacles in calculus were implemented. Referring to reference lists of pre-accessed literature, by contacting the authors of some studies via research gate and academia web pages the searching was extended. Subsequent keyword searches were expanded by using combinations of alternative terms such as obstacles, misconceptions, alternative conceptions, errors in calculus, learning difficulties, calculus, limit, continuity, derivative, infinity, and the chain rule.

The Majority of the articles were identified through searches of electronic databases including: UNISA Library e-journals (Educational Studies in Mathematics, The Online Journal of Science and Technology, Canadian Journal of Science, Mathematics and Technology Education, International Journal of Science and Mathematics Education, Primus, African Journal of Research in SMT Education, Research in Collegiate Mathematics Education, Mathematical Association of America, The College Mathematics Journal, Journal of Mathematical Behaviour, Mathematical Thinking and Learning), Google, Google Scholar, ERIC: Clearinghouse for Science Mathematics and Environmental Education, ERIC: Educational Resources Information Centre, UNISA's institutional repository, EBSCO, Academic Search Premier, research gate, and ProQuest Dissertations and Theses.

To find local literature university web sites such as Addis Ababa University, Jimma University, Hawasa University, and electronic databases, Ethiopian Journal of Education, Ethiopian Journal of Education & Science, and manual searches by the researcher and contact to colleagues were implemented.

The broad search passed through title and abstract screening, resulted in the collection of over 207 studies, including journal articles, conference papers, book chapters, master's and doctoral dissertations, and unpublished papers. More than 71% of the materials talk about calculus at university and the remaining about calculus at secondary school students. The studies were then organized into groups dealing with the same concept (limit, continuity, derivative, or calculus) and then within each of the groups by date of publication. While collecting the literature, both intervention studies and duplicated works were excluded.

To screen the collected materials, the parameters that are listed on page 16 that are eight in number, were used and figure 2 presents the flow of the literature screening where the numbers 1 to 8 refer to the criteria set for inclusion.

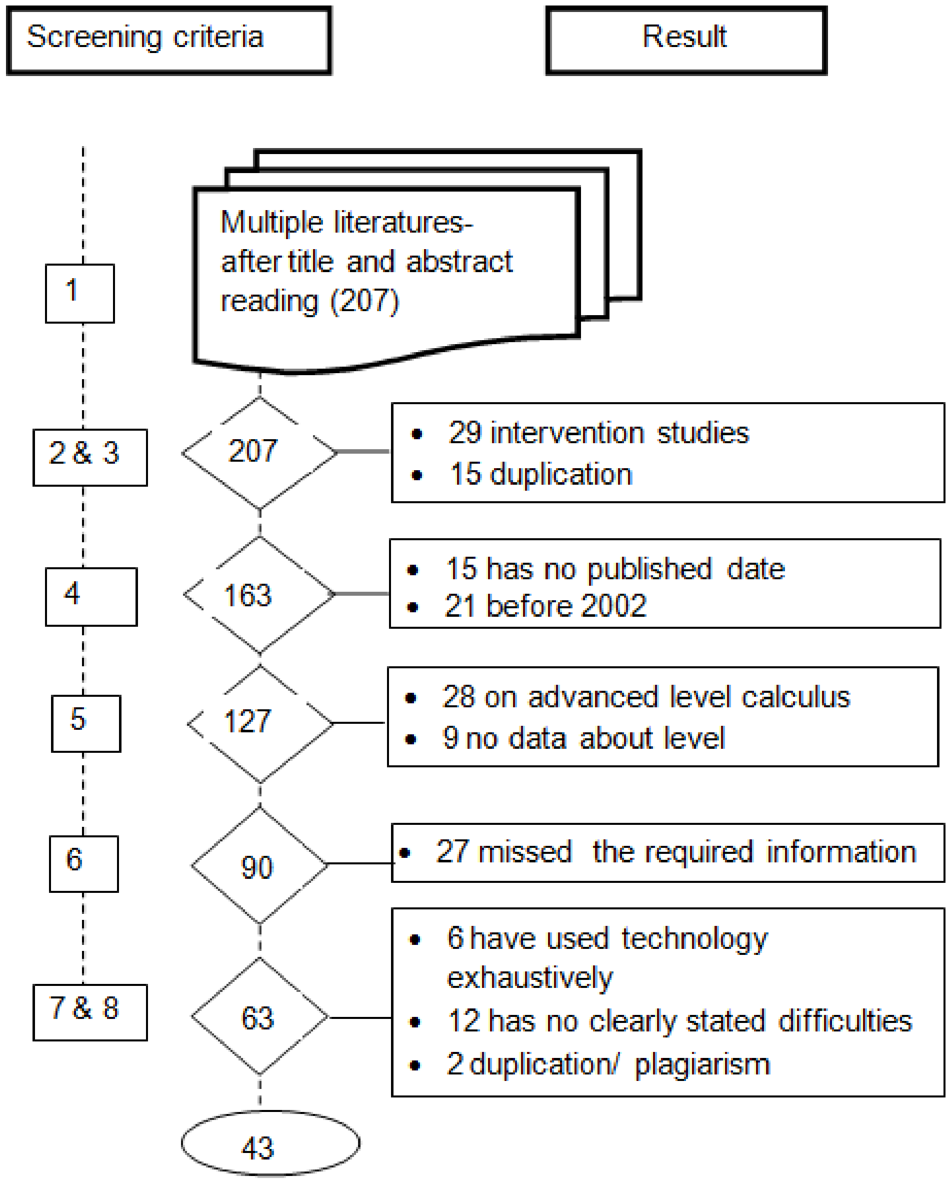

Figure 2: Flow of the literature search

After screening the materials through these inclusion criteria, 43 studies, which met the inclusion criteria for the final review, were selected. Figure 3 represents the percentage of the 43 studies used for the final analysis per each concept area.

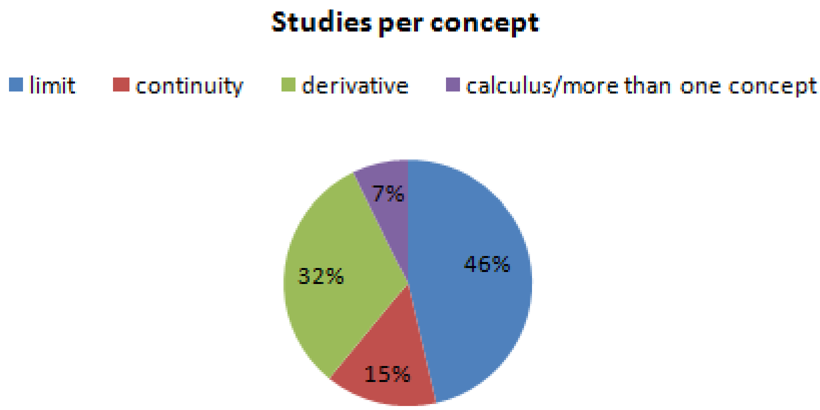

Figure 3: Percentage of literature used for each concept

2.1.2. Quotations and initial codes

To attain a broad narrative of students' difficulties in learning calculus concepts, the researcher treated each article as a case and explored what is inside in the following steps. These are to identify quotations of difficulties (mentioned errors, way of thinking or alternative conceptions/misconceptions) from each case and triangulate these quotations from each article to build an initial code followed by finding for similarity and difference among the initial codes to categorize them in a more general code called second-level codes or "descriptive themes" (Thomas & Harden, 2008).

Initial codes are labels used to describe a segment of text or an image (Miles *et al.*, 2014). The codes used in this study are aimed to address students' ways of thinking about a specific topic/concept, common errors demonstrated on the given tasks, alternative conceptions demonstrated and strategies mentioned in parts of students in solving given tasks. In the second stage of coding, the researcher concentrated on the similarity and difference of the initial codes so that new code capturing the meaning of a group of initial codes can be formed. This leads to less number of codes, but each code with span in interpretation.

At the initial review, the result sections (depending on the format, it can be the result, finding, summary, discussion of each study) have been read to capture a holistic picture of individual studies. Identification of quotations was started in the second round of reading and that was done using highlighting of texts identifies to be quotations. This initial stage of identifying quotes finalized resulting in 237 quotations all over the 43 articles included.

The coding was aimed to reduce the number of quotations to a more manageable size without losing meaning but looking similarity and difference of these codes within each of the three concepts. For instance, the following three quotations (i.e. students' were observed to "insert infinity in for x", "infinity as one big number" and "plugged in infinity as a number") taken from three different articles were coded as one "image of infinity" based on the ground that these conceptions are arising from confusing the image of infinity. The process ended reducing the 237 quotations to 36 initial codes (see appendix A for the details of the 43 literature, Appendix B for the 237 quotations, and Appendix C for the 36 initial codes generated from these 237 quotations).

2.1.3. Descriptive themes

After generating the 36 initial codes, the researcher tried to look for different second-level codes aimed to merge the above mentioned 36 initial codes into meaningful and careful units of difficulties or concept image. The researcher has done this categorization three different times but all were different. Then after discussing with two colleagues (both PhD students one at UNISA and the other at Addis Ababa University) and let one of them try to categorize, the researcher got a better picture to compress the codes, i.e. decided to follow the sequence of the course flow. Thus, the categorization was done in the order of pre-calculus concepts followed by limit, continuity, derivative images and a more general topic named "the collective image". Accordingly, the 36 initial codes were reduced to 10 descriptive themes that fall into five categories. Table 1 presents the 10 descriptive themes into five categories and their corresponding initial codes.

Before presenting the themes of difficulties that emerge from the literature, the next section will present the detail of the second level codes hereafter called descriptive themes one by one in the five categories.

Table 1: Interrelated descriptive themes

Category	Descriptive themes	Initial codes
Pre-calculus knowledge	Variable and function image	Co-variational reasoning
		Function image
		Computational ability
	Image of infinity	Infinity image (actual versus potential)
		Infinity, the undefined and indeterminate interplay
Limit image	Concept definition	Concept definition
		Linguistic ambiguity
	The dynamic-static interplay of limit	The limit value is not attainable
		The limit value is a boundary
		The limit value is an approximation
		Conflicting concept image
		A static view of the limit process
	The discrete-continuous interplay of limit	The discrete thinking of continuous idea
		Continuous view of discrete idea
	Over-generalization	Alternative conception
		Monotonic- convergence interplay
		Domain-limit interplay
		Limit value means the same as a function value
		Non-existence case of limit
		Point wise thinking of limit
Continuity concept	Continuity concept image	Domain- continuity interplay
		Limit-continuity interplay
		Confusing continuity with connectedness
		Continuity concept image
		Continuity-asymptote interplay
Derivative concept	Derivative concept image	Definition of terms
		Difficulties in rules and procedures of derivatives
		Symbolic interpretation
		Infinity small
		Continuity- differentiability interplay
The collective image	Procedural knowledge and routine exercises	Procedural learning
		Unsynchronized knowledge structure
		Lack of conceptual knowledge
	Representation	Algebraic representation
		Visualization
		Problem-solving

2.1.3.1. Pre-calculus knowledge

Variable and function image

One of the basic pre-calculus underpinning for beginning calculus students is conceptual knowledge and reasoning ability of function concept. Carlson, Oehrtman, and Engelke (2010) describe the function concept as the main pillar of the mathematics curriculum from elementary to advanced concepts like calculus. A strong understanding of variables as generalized figures and as sequentially co-varying objects (Gray, Loud & Sokolowski, 2009), a process view of functions, the ability to justify as co-varying and computational abilities (Carlson *et al.,* 2010) are identified as essential knowledge that facilitates conceptual learning in calculus.

Gray *et al.* (2009) found that the majority of calculus students included in their study have faced difficulty in using variables as generalized and changeable quantities. In addition, they found that students focus on or influence by arithmetic approach for items demanding an algebraic approach, practice "point-by-point or static way" of evaluating an independent variable of a function with the real domain. The ability to use variables as varying quantities showed a positive correlation with students' performance in calculus.

In calculus, it is common to see students evaluate a function "f" at the first few points (usually, integers) close to "a" to compute $\lim_{x \to a} f(x)$. This sequence based thinking of variables (as integers) than the real number domain of functions corresponds to "action view of function" (Carlson et al., 2010). But, calculus learning demands beyond action level conception. According to APOS theory, computing value of the function "f" at a finitely many successive discrete points should be followed by an "interiorization" of these actions to establish a domain process in which the input values approaches "a" and the subsequent output values approaches the limit value "L" (Moru, 2006).

Students are said to have attained process view of function provided they begin to imagine quantities that are potentially changing simultaneously or according to Jones (2015) when they use "co-variational reasoning". The literature (Jones, 2015; Oehrtman, 2002; Roh, 2005; Wangle, 2013) has documented that students have difficulty with the limit that originates from lack of the co-variational reasoning or lack of having a process view of functions.

The literature (e.g., Jayakody, 2012; Luneta & Makonye, 2010; Makonye, 2012; Takaci, Pesic & Tatar, 2006), has also documented that students Inadequate concept image of function challenge their performance in calculus. Most students in those studies have demonstrated narrow example space (usually, they do well only on polynomials) believe that a function must be in one piece and think that a function as "chunky, not smooth". Especially, simplification of rational functions, the issue of continuity and discontinuity of rational functions, issue of the derivative when come to compound functions and piecewise or split defined functions were identified frequently troublesome.

Wangle (2013), found that only some students who are considered as strong have qualities such as providing real-life examples while learning, have good reasoning skills of function, and able to move flexibly among representations. Due to the belief, a function must be in one piece, studies (Maharajh *et al.,* 2008; Takaci *et al.,* 2006; Wangle, 2013), have found that students face difficulty to compute the limit or to demonstrate continuity and discontinuity of split-functions irrespective of forms of representation.

Maharaj (2013) has found that most calculus students face a challenge to learn calculus concepts due to a lack of function understanding that is not developed to a process level while calculus-learning demand beyond the process level conception. In an item that asks to express $y = \frac{1}{x^2-7}$ as a composition of two functions f and g such that $y = f[g(x)]$, Maharaj found that 17.4% of students lack the appropriate mental structure of function i.e. the conception of function developed to process level.

The limit of the number sequence is a base for the discussion and application of infinite series in analysis courses. Even though a sequence is a function, the discrete nature of number sequence (Jones, 2015) distinguishes the limit of sequences (which usually denote by $\lim_{n\to\infty} a_n$) from the limit of real-valued functions at infinity (i.e. $\lim_{x\to\infty} f(x)$). Since the two topics are treated differently, some students even did not have an understanding of a sequence as a function (Moru, 2006). Thus, some of the difficulties are overlapping and some others are unique. In this review, it is found that (e.g., Moru, 2006) many students consider:

- A sequence is well defined provided it has a single algebraic representation and hence an alternating sequence is two distinct sequences.
- The same sequence given in different modes of representation stands for different sequences. Thus, the function image takes a good share of students' difficulties.

The literature (Jordaan, 2005; Juter, 2006; Maharaj, 2010; Pillay, 2008; Siyepu, 2015), showed that students' computational abilities or algebraic manipulation skill gap from pre-calculus algebra bound their performance in calculus. Siyepu (2015), found that some students manipulate $f(x+h)$ as $f(x)+f(h)$, treat $y=4x^{\frac{3}{2}}$ and $y=2x+3lnx$ as compound functions, convert $6x^{\frac{1}{2}}$ to $\sqrt{6x}$. Siyepu concluded that students' attention of prior learning, i.e. prior learning "subjected to rote learning of familiar exercises" (p.15) are the source of errors and difficulties observed during learning calculus.

Pillay (2008) found that many students demonstrate incorrect algebraic manipulation, provide incomplete solution, and have problems with the "symbolism associated with calculus". Accordingly, some of the observed difficulties were:

- two subjects incorrectly factorized $y=\frac{x^2+6x-4}{x}$ as $y=\frac{x(x+6-4)}{x}$.
- two subjects incorrectly simplified $-7x(x-2)$ as $-7x^2-14$.
- two other subjects incorrectly manipulated $\frac{f(x+h)-f(x)}{h}$ for $f(x)=-x^2+1$, as $\frac{f(-x+h)^2+1-f(-x^2+1)}{h}$.

Pillay in her conclusion mentioned that such a "lack of procedural fluency" was an obstacle for students in coming to understand calculus concepts. Luneta and Makonye (2010) documented that most difficulties of calculus students were due to knowledge gaps in basic algebra and unsynchronized conceptual and procedural knowledge. Some students demonstrated procedural errors (wrote $f(x)+f(h)$ instead of $f(x+h)$ to determine the derivative of $f(x)=x^2$, or incorrectly simplified

$x^5 - x^2 = x^7$ and $y = \frac{\sqrt{x}-4}{\sqrt{x}} = x^{\frac{-1}{4}} - 4x^{\frac{-1}{4}}$). They concluded that the observed lack of conceptual understanding and skill gaps in computational ability undermined students' performance in calculus.

Concept image of infinity

In calculus, Infinity may appear with real-valued function as a limit point or as a limit value, which is called limit at infinity and infinite limit denoted by $\lim_{x \to \pm\infty} f(x)$ and $\lim_{x \to a} f(x) = \pm\infty$ respectively. Such a limit involving infinity has many significant applications in mathematics and science (Jones, 2015). However, students face more difficulties with the limit involving infinity as compared to the limit without infinity (Elia, Gagatsis, Panaoura, Zachariades & Zoulinak, 2009; Jaffar & Dindyal, 2011; Nair, 2010).

The literature on infinity describes the dual nature of the notion of infinity- potential infinity versus actual infinity (Jones, 2015). Potential infinity refers to an on-going process without an end. We do not actually come across in our daily lives; it is entirely a mental construct. In contrast, actual infinity refers to the idea of a finite entity to this infinite process (Jones, 2015). Jones, states that the mental structure in "potential infinity" has a resemblance to a process whereas, the "actual infinity" has much in common with an object-level concept formation. This nature of the notion of infinity corresponds to the dual nature of the limit, i.e. limit is both a dynamic process and a static object (Gray & Tall, 1994). The limit at infinity requires thinking of the infinity as a potential process and the infinite limit requires thinking of the infinity as an object.

The literature (Areaya & Sidelil, 2012; Jones, 2015; Moru, 2006; Oehrtman, 2002; Parameswaran, 2007; Roh, 2005) has revealed that for limit at infinity, students recognize infinity as a number i.e. object conception of infinity. They plugged in infinity as a number to calculate the required limit value. According to Jones (2015, p.112) students usually approach infinity as an actual value that can be manipulated. He further states that "each student applies this approach at least once, whereas many students apply the approach so many times during the interview".

Another difficulty related to the notion of infinity is confusing “infinite” with “the limit does not exist” and “indeterminate form” in computing limits. The literature (Bergsten, 2006; Elia et al., 2009; Jaffar & Dindyal, 2011; Juter, 2006; Moru, 2006; Nair, 2010) has found that most students are not aware enough when to use these terms. In the limit, the term “infinity” is used to express being unbounded and “does not exist” is used to mean that the one-sided limits are different. However, the literature revealed that students didn’t differentiate accordingly. Such confusion may emerge from the discussion of real numbers (Jaffar & Dindyal, 2011). In real numbers, sometimes $\frac{\boldsymbol{a}}{\mathbf{0}}$ $(\boldsymbol{a} \neq \mathbf{0})$ may be written as ∞ or undefined. Further, they mentioned that pre-calculus conception of indeterminate forms and individual learning models as additional factors for the formation of these confused cognitive structures.

2.1.3.2. Limit image

Concept definition

While calculus is a gateway to advanced science and mathematics (Roble, 2017; Sadler & Sonnert, 2016), the limit is a gateway to calculus (Zollman, 2014). Although derivatives and integrals make up the majority of calculus, a sound understanding of the limit is necessary to learn these major concepts in calculus (Maharaj, 2010; Muzangwa & Chifamba, 2012; Rabadi, 2015). One distinction between complex mathematics and elementary mathematics is the role of definitions in advanced mathematics (Tall, 2002). When introducing a new concept, an ordinary starting point is through a definition. This demands relating terms in a mathematical language and terms in the medium of instruction.

The terms ‘approach to’, ‘tends to’, ‘reach’, and ‘converge’ are frequently used to define or describe the limit. These are not only terms with a technical and formal definition in mathematics, but also have everyday uses not connected to their mathematical meanings (Fernandez-Plaza, Rico & Ruiz-Hidalgo, 2013). Several researchers confirmed that due to the conflicts between formal and colloquial uses of these terms, students face the challenge to express accurately the mathematical meaning of the concept of the limit (Jaffar & Dindyal, 2011; Moru, 2006; Oehrtman

2002). Thus, cognitive structures of the limit of a function formed by the students contained a lot of inconsistency and are often stumped (Jordaan, 2005; Moru, 2006).

The literature (Cetin, 2009; Elia *et al.,* 2009; Jordaan, 2005) has found that students lack to state definitions of the limit in their own words. This implies that students lack the mental structure that can be translated into word expressions. On an item asking students to define the limit of a function in their own words, Jordaan (2005) found that students showed a low response rate. In most cases, even better-performing students missed items asking definitions and theorems. This indicates that students' concept image about the limit is incompatible with a concept definition. This gap may lead to developing an alternative conception.

The literature (Denbel, 2015; Jayakody, 2012; Maharajh *et al.,* 2008) mentioned that students fail to pay attention to the contextual meaning of terms during problem-solving in calculus. Areaya and Sidelil (2012) found that most students do not believe that a constant sequence is monotonic due to linguistic ambiguity. Fernandez-Plaza *et al.* (2013, p.699) conducted a study aimed to investigate students' interpretation of terms approach to', 'tend to', 'reach to', and 'to exceed' in learning limit at a point. The study identified the following difficulties that students encountered due to confusion of these terms with their common language use- the limit value cannot be reached, the limit value is an upper bound, and the limit is an approximation. Of course, these are the most frequently occurring difficulties in the literature of the limit.

The dynamic-static interplay of the limit

The intuitive introduction of limit $lim_{x \to a} f(x) = L$ is an interpretation of the behaviour of the function f as $x \to a$. In the literature, this is described as the dynamic notion of limit (e.g. Jones, 2015). This dynamic nature of thinking demands focusing on the behaviour of function values about the point rather than on the function value exactly at the limit point. Students with dynamic thinking of limit may then recognize that the function being defined does not guarantee the existence of a limit. A good conception is then when students distinguish among the dynamic limit process and the resulting static limit value.

Once students are introduced to the notion of limit, they form their own “concept image”. That concept image is then shaped probably by the choice of examples that teachers use in the class, examples in textbooks or reference books. At the introduction of limit, the selection of simple and continuous functions like $f(x) = x^2$ or $f(x) = sinx$ creates the impression that at the limit point both limit value and function value are the same or the limit exist provided the function is defined at the limit point. This led to the incorrect generalization that the limit process and the computation of function value are exactly the same things (Jordaan, 2005). The computation process involves only finite specific actions. When students are restricted to treating ordinary computations of a function, they are said to have a static view of the limit process (Çetin, 2009; Maharaj, 2010; Moru, 2006). Accordingly, a student having a static conception of the limit of a function consider $\lim_{x \to a} f(x) = L$ as either $f(a)$ or evaluate f for a finite number of points close to “a”. Students with this conception may conclude that the limit is the same as the function value. According to Roh (2008), “misconceptions” happen when students fail to internalize these infinite processes instead demonstrate the static view to compute the limit value.

Furthermore, the computation of limit value is not limited to a finite sequential and discrete step that provides a specific answer. Rather, it involves the imagination to get a pattern from continuous and infinite coordination. This is precisely where the one at the process level performs better than the one at the action. However, process level conception by itself is not an end. Frequently cited students’ difficulties are that they think the limit of a function at a point is not attainable.

Jones (2015) found that some students focus on what happens at infinity than as x approaches to infinity to find the limit at infinity, which is an indication of a static view of the limit process. Several researchers (Çetin, 2009; Duru, 2011; Elia et al., 2009; Jayakody, 2012; Jordaan, 2005; Moru, 2006; Nair, 2010) finding have revealed that most students conceive the limit process as static which falls into action level conception of the limit. Most students’ computation of a limit or their expression revealed that they understood the limit of a function at a point “a” as $f(a)$. Belongingness of “a” to the domain of f is an essential and enough state for the existence of a limit at "a" (Przenioslo, 2003) and is defined at the point "a" is essential state to compute limit at $x = a$ (Duru, 2011; Elia et al., 2009; Nair, 2010) were also ways used to express a static view of the limit process.

The literature also revealed students a dynamic view of limit value which frequently expressed by the phrases like limit value is "unreachable", "an approximation" or "a boundary". These difficulties are also mentioned as linguistic ambiguity by several researchers. While the work of Fernandez-Plaza *et al.* (2013), Jordaan (2005), Moru (2006), and Roh (2005) documented that most students have the conception of limit value as a dynamic object, the work of Elia *et al.* (2009), Jaffar and Dindyal (2011), Oehrtman (2002), and Parameswan (2007) documented that most students expressed the limit value as the value being approximated. Others work (Fernandez-Plaza *et al.,* 2013; Jordaan, 2005; Moru, 2006) documented that students described the limit value as an upper bound, as a border, or a boundary that is not surpassed. Some studies also showed students have a confused image of the limit, which depends on context (Juter, 2005b). Thus, although some students demonstrated a clear distinction of limit as a dynamic process and static value, most students have trouble with understanding this dual nature of the limit.

The discrete-continuous interplay of the limit

According to Ferrini-Mundy and Gaudard (1992), one cause of students' difficulties in calculus is that they attained a calculus course with a discrete orientation of continuous ideas. The review has also revealed that not only discrete thinking of continuous idea, but also continuous thinking of discrete ideas affect students' performance in calculus. Moru (2006) and Roh (2005) documented that students think discrete idea as continuous. Particularly, Moru (2006, p. 126) continues saying many students join points on the graph of a sequence by a line. On the other hand, the literature (Gray *et al.,* 2009; Jones, 2015; Wangle, 2013) documented that students have a point-by-point or discreet thinking of continuous ideas.

Overgeneralization

At the introduction of a new concept, students learning almost certainly influenced by information provided by teachers, textbooks, worksheets, assessment trends, and so on. If the activities on these resources often involve maximal intellectual engagement, then it helps students to develop conceptual knowledge that can be further manipulated (Konicek-Moran & Keeley, 2015). Learning an advanced concept, like limit, engage a construction process. This means that students modify and reconstruct their existing cognitive structure based on their current exposure. The resulting cognitive structure may vary from the formal concept definition. It is also possible for an individual to have more than one cognitive structures of a concept that conflict with each other. This leads to the over-generalization of existing knowledge or the formation of an alternative conception.

Several researchers have documented that students develop overgeneralization in the learning of calculus concepts in general and limit and continuity in particular. The following are the basic overgeneralizations identified in the review:

- Convergence imlies monotonic (Areaya & Sidelil, 2012; Fernandez-Plaza *et al*., 2013).
- Being defined at "$\boldsymbol{a}$" is an essential condition to compute limit at the point "$\boldsymbol{a}$" (Duru, 2011; Elia *et al.,* 2009; Nair, 2010; Przenioslo, 2003).
- Limit and function values are the same (Bergsten, 2006; Elia *et al.,* 2009; Jayakody, 2012; Juter, 2005b; Maharajh *et al.,* 2008; Moru, 2006; Nair, 2010).

Some of these overgeneralizations occurred due to the introduction of limit using simple and continuous functions in which the limit and the function value is the same at any real number. Other overgeneralizations comprise, the limit of a function f does not exist at $x = a$ only when the two side limits are different (Elia et al., 2009), divergent means tend to infinity (Moru, 2006), and oscillating behaviour always leads to divergence (Roh, 2005).

The development of alternative conception may lead students to have conflicting concept images. In calculus, it is common to see the correct answer for the wrong reasons. For instance, students may compute the limit of a continuous function using an overgeneralization that the limit is the same as the function value. Some researchers used qualitative analysis of students' reasoning to examine the true nature of students' cognitive structure. The literature documented that students' performance indicates the correct answers for the wrong reasons and wrong answer with high confidence (Çetin, 2009; Juter, 2006; Luneta & Makonye, 2010).

2.1.3.3. Continuity concept image

Continuity is the next major concept that plays an important role in calculus. Students' conception of continuity may be influenced by their knowledge of continuity definition in lower secondary schools, knowledge of the graph, algebraic manipulation, the concept of asymptote, and one-sided limit (Rabadi, 2015).

The literature has documented that students have difficulty with domain continuity interplay, limit-continuity interplay, and continuity-connectedness confusion. Students think that if a function is defined at a given point, then it is necessarily continuous at that point (Takaci *et al.,* 2006; Vela, 2011; Wangle, 2013) continuity is an issue only for functions defined for all real numbers (Nair, 2010; Wangle, 2013). On the other hand, students did not associate continuity with limits; rather associate continuity with "connectedness" which is the most frequently mentioned difficulty (Maharajh *et al.,* 2008; Takaci *et al.,* 2006; Vela, 2011; Wangle, 2013). Due to this thinking and lack of linking continuity with limit, most students conclude that a piecewise-defined function is discontinuous and they frequently associate continuity with smoothness or differentiability (Nair, 2010; Maharajh *et al.,* 2008; Vela, 2011; Wangle, 2013). Students also lack the awareness to demonstrate proofs and counterexamples of continuity and discontinuity (Ko & Knuth, 2009).

In addition to a lack of explaining continuity in terms of limit, some students confuse the role of limit and continuity, i.e. confuse limit-continuity interplay. Other difficulties related to continuity includes the limited conception that if f is discontinuous at a, then f is not defined at a (Ko & Knuth, 2009), reversing the limit-continuity interplay (Duru, 2011; Jordaan, 2005), and existence of the limit is sufficient for continuity at a point (Maharajh et al., 2008; Nair, 2010; Vela, 2011; Wangle, 2013). Moreover, in Przenioslo's (2003) study it is found that a good number of students think that the continuity at a point is necessary for the existence of a limit.

Another area of difficulty is continuity-asymptote interplay. Wrong understandings such as if a function is unqualified to have limit at a point then it should have a vertical asymptote (Areaya & Sidelil, 2012), low response rate to compute limit at point of discontinuity, and the understanding that every point of discontinuity is a vertical asymptote (Nair, 2010), was documented. Besides, point of discontinuity means asymptote (Takaci *et al.*, 2006), difficulty to identify vertical asymptote of a rational function, non-existence of vertical asymptote is a sufficient condition for continuity (Vela, 2011), and more confused with jump discontinuity (Parameswaran, 2007) were documented challenges in students' progress.

2.1.3.4. Derivative concept image

The subject of derivation being an important part of the analysis is a mathematically hidden topic in calculus (Herbert, 2013; Orhun, 2012). A derivative has different representations. It can be introduced geometrically as the slope of a tangent to a curve, symbolically as the limit of the different quotient of a given function or numerically using physical problems like distance or velocity data.

The process of introducing the derivative concept demands using new and familiar concepts and notation (algebraic and graphic representation of function, rate, limit, continuity, infinitesimal quantities, a scant line, tangent line, and variables), and notations ($\frac{\Delta f}{\Delta x}$, f' ,$\frac{dy}{dx}$) all are incorporated. Thus, students' backgrounds on these concepts and notations accompany the learning of the derivative concept. According to Naidoo and Naidoo (2007), the derivative is one of the concepts at a higher level of conceptual hierarchy in calculus. For instance, in the first principles of differentiation, $\mathrm{f}'(\mathrm{x}) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, which is later denoted by $\frac{df}{dx}$ or $\frac{dy}{dx}$, demand prerequisite conception of limit, rate, algebraic manipulation, variables, and infinitesimal quantities. It can be interpreted as a function $f'(x)$, a number $f'(a)$ if evaluated at "a", slope of the tangent line as a limited position of secant line (Pillay, 2008; Siyepu, 2013). Thus, the layers and the parts the derivative concept demands not only are making 'connections between representations' but also 'connections within representations' (Hähkiöniemi, 2006).

Students' difficulties in derivative start from definitions and notations, confusing notation or symbol and meaning. In an item asking what is the meaning of the expression" $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ "and what this expression is usually used for, Jordaan (2005) concluded that many students can use the formula to compute the derivative function but they cannot explain the embedded conceptual issues behind the procedures. On a similar item, Areaya and Sidelil (2012) found that on average only 56.8% of participants successfully identified the symbols used to denote the quantity, the name of the quantities, and the meaning of the quantity obtained after computing the calculation. The literature (Hashemi, Abu, Kashefi & Rahimi, 2014; Makgakga & Makwakwa, 2016) argues that students of the derivative come back to learning focused on procedural and symbolic aspects more than the embedded conceptual issues.

In a study that aimed at analysing grade 12 students' difficulties in calculus, Luneta and Makonye (2010) administered a test to 45 participants. They classified errors that occur into two as (i) on task (OT) errors that occur when dealing with the embedded calculus concept and (ii) not on task errors (NOT) errors that are not directly related to the concept. The study indicated that NOT on task errors (which account 40% of the errors) mostly occurred due to lack of algebraic manipulation and function notation. The following are two examples from NOT errors:

- $x^{\frac{1}{2}} - x^{\frac{-1}{2}} = x^{\frac{-1}{4}}$ (Misapply exponents).
- wrote $f(x)$+$f(h)$ instead of $f(x+h)$.

On task errors (which account for the remaining 60% of the errors) occurred due to one or more than one of the following reasons- "stuck thinking on a concept, failure to recognize differentiation rules, lack of conceptual bases of differentiation, unbalance conceptual and procedural knowledge, and parallel conflicting but calculus conceptual knowledge" (p.44). The following are two examples from OT errors:

- $\left(\frac{f}{g}\right)' = \frac{f\prime}{g\prime}$ thus, $\left(\frac{\sqrt{y}-4}{\sqrt{y}}\right)' = \frac{\frac{1}{2}y^{-\frac{1}{2}}}{\frac{1}{2}y^{-\frac{1}{2}}} = 1.$
- On the first item which asks to apply first principles to show that if $f(x) = -x^2$, then $f'(x) = -2x$ the following was part of a learners solution:

  $\ldots\ 2(x+h)^2$ (instead of $f(x+h)$ i.e. $-(x+h)^2$)

$-2x^2 - 2xh - 2h^2 + x^2$ (instead of $f(x+h) - f(x)$ $i.e. -(x+h)^2 + x^2$)

$4xh - 2h^2 = h(4x - 2h)$ (Instead of $-2xh - h^2 = -h(2x+h)$)

$f'(x) = 4x$ (Instead of $f'(x) = -2x$)

Based on these observations they comment that, “students do not ask themselves why their answers are different from the one given. They only believe that their answers are correct, and the one given is wrong'' (p.39). Such a wrong answer with high confidence implies the existence of an alternative conception. Thus, some of the students also have an alternative conception of derivatives. Their recommendation includes attention to equip students with solid algebraic skills at pre-calculus courses, to shift the practice of teaching toward a balance between routine and embedded ideas, to give attention to the geometric/graphical basis of the derivative.

With the rules and procedures of derivatives, the literature identified the following difficulties:

- Misinterpret derivative rules and procedures specially confusing composition and combination rules (Horvath, 2008; Luneta & Makonye, 2010; Makonye, 2012).
- Carry out an incorrect algebraic simplification. In particular, unable to manipulate trigonometric identities (Pillay, 2008; Usman, 2012).
- Ignore rule restrictions in algebraic expressions (Luneta & Makonye, 2010).
- Interference, i.e. misinterpret an object due to an already existing overgeneralization (Siyepu, 2013).

The derivative concept becomes more problematic when applied to the combination and composition of functions. Derivation of composition functions is not only conceptually but also procedural difficulties for many students (Maharaj, 2013; Siyepu, 2013). A common tool to treat the derivative of a composition function is the chain rule. Maharaj (2013) in a study aimed to explore natural science university students’ knowledge of derivatives, implement an APOS level of concept formation, and his own genetic decomposition as a framework. The study found that only 42.24% of students demonstrated an adequate schema for the composition function item. In parallel to the literature, what Maharaj wrote at the end of the analysis is that “the chain rule is amongst the most difficult concepts to transmit to the students in calculus” (p.12).

In a study on the graph of functions, Orhun (2012) found that many students lack to relate conceptual aspects of a function and its derivative graphically. Hashemi *et al.* (2014) documented that many students unable to identify the interplay between conceptual aspects and prefer specific and explicit instruction than to dealing with generalized conceptual issues. It is certain that students performed was better in familiar type exercises, which means they were at an action level of cognitive structure. From an interview, Hashemi *et al.* (2014) found that students might perform high in tests, but not have conceptual knowledge. Usually, students confuse the interplay between continuity and differentiability of a given function. In particular, students use smoothness of the graph as criteria for continuity (Maharajh *et al.,* 2008; Nair, 2010).

2.1.3.5. The collective image

Procedural learning and routine exercises

Several researchers confirmed that calculus teaching-learning lacks conceptual knowledge. The consequence of this practice is worthwhile when it is at secondary school because it may influence students to focus more on the routine aspect of the subsequent courses too (Ferrini-Mundy & Gaudard 1992; Naidoo & Naidoo, 2007). Calculus difficulties are patterns of error, approach to the concepts, and focuses of the learning materials. Several educators argue that most students’ difficulties in calculus emerge from teaching-learning which focuses on procedures and symbolic manipulations than the embedded concepts.

The literature (Abbey, 2008; Bergsten, 2006; Brijlall & Ndlovu, 2013; Makgakga & Makwakwa, 2016) documented that calculus teaching-learning focuses on applying memorized rules without attention to the context provided by tasks. In particular, the articles by Cetin (2009) and Elia *et al.* (2009) revealed that students fail to apply the limit concept to solve unfamiliar problems. Instead, they recognize the limit value of a function only as a number rather than a means of computing fairly accurate values of the function. Several researchers mention that the reason for such lack of conceptual knowledge is lack of mental structure developed to the required level (process and object level) of function, limit, and derivative (Cetin, 2009; Brijlall & Ndlovu, 2013; Maharaj, 2013; Siyepu, 2015). In some cases, even the existing conceptual knowledge and procedural knowledge lack synchronization (Luneta & Makonye, 2010).

The literature has also revealed that most students do not react at all or demonstrate low success for the unfamiliar task items and for items demanding higher levels of cognitive thinking (Horvath, 2008; Makonye, 2012; Roh, 2005; Usman, 2012). Besides, there are signs that students' thinking lacks' meta-cognition (Makonye, 2012; Usman, 2012). Several researchers mention basic factors that influence students' performance on unfamiliar task. Usually, students fail to grasp the concept of the problem, lack understanding the language of the problem, lack the knowhow of identifying the required, and lack skill to use the given information or modelling tasks, and fail to choose appropriate procedures to be used (Abbey, 2008; Brijlall & Ndlovu, 2013; Maharajh *et al.,* 2008; Siyepu, 2015; Usman, 2012). Thus, the points mentioned above are the reasons that many students have difficulty with problem-solving.

Representation

Among others that determine students' success in calculus is their conceptual ability in visualization and flexibility in the form of representations. Teaching this concept using different representations could prevent the formation of "misconceptions" (Maharaj, 2010). Research findings of Jaffar and Dindyal (2011) and Moru (2006) revealed that some students reacted differently to the same idea given in different representations. In addition, the literature (Elia *et al.,* 2009; Wangle, 2013) has revealed that most students have difficulty to translate between representations and they are very dependent only on the algebraic form of representation. Blaisdell (2012) on a study aimed to investigate the Influence of question format/representation found that students stimulate different concept images of the same idea given in different forms of representations used.

Besides addressing learning style preference, multiple representations are tools to visualize a given problem from different perspectives or to be able to express one's idea about a concept in different forms. In students, it is not common to use a blended approach to explain their idea through the problem at hand may best explain in such a way (Hashemi *et al*., 2014). Moreover, the inadequate schema of interpreting the graph of the derivative function or challenge to characterize a function based on information from a graph (Hashemi *et al.*, 2014; Maharaj, 2013; Orhun, 2012), lack to use appropriate mathematical language to describe information given in non-algebraic form (Abbey, 2008; Orhun, 2012) was also documented difficulties.

2.1.4. Analytical themes

The researcher reviewed the literature on three concepts (limit, continuity, and derivative) and noticed that across these concepts, some of the difficulties are overlapping and some others are unique to a concept. From what has been discovered about students' difficulties in learning calculus concepts, analytical themes are reported as follows:

Function image lacks process view

The literature has documented that the ability of co-variational reasoning as a result of a process view of functions and computational ability i.e. algebraic thinking than arithmetic (Carlson *et al.,* 2010; Maharaj, 2013) are identified as essential knowledge that facilitates conceptual learning in calculus. Though some calculus students demonstrate this pre-calculus knowledge, most students lack it. Jones (2015), Oehrtman (2002), Roh (2005), and Wangle (2013) have found that students' performance on the limit is largely affected by their action view of function. Students' computational abilities or algebraic manipulation skill of the functions in limit, continuity, or derivative takes the lion's share of students' difficulties in calculus learning (Juter, 2006; Maharaj, 2010; Pillay, 2008; Siyepu, 2015).

Image of infinity lacks process view

Other pre-calculus concepts that influence calculus learning are students' image of infinity. One difficulty with infinity is object conception of infinity while process conception is required; plugged in infinity as a number to calculate the limit at infinity (Jones, 2015; Moru, 2006; Oehrtman, 2002; Parameswaran, 2007; Roh, 2005). In addition, at one or another time, most students confuse infinity with undefined or indeterminate form during computation of limits, in particular, limits of rational functions and the limit of different-quotient (Bergsten, 2006; Vandebrouck & Leidwanger, 2016). While limits at infinity demand a process view of infinity and most students do not understand this view, infinite limit demands an object view of infinity and most students satisfy this view (Jones, 2015). Thus, the pre-calculus knowledge gap, i.e. function image, infinity image, and computational ability seems to be common areas of difficulty for beginning calculus students.

Depending on concept image than concept definition

The review noticed that most frequent difficulties in calculus originate from the role of definitions in advanced mathematics. Terms like: "a function does not attain its limit", "limit values are unreachable" or "limit is an approximation", and confusing continuity with connectedness are difficulties related with the linguistic ambiguity of terms in definition of concepts (Çetin, 2009; Jordaan, 2005; Moru, 2006; Vela, 2011; Wangle, 2013). A set of articles (Denbel, 2015; Jaffar & Dindyal, 2011; Jayakody, 2012; Maharajh *et al.,* 2008) have documented that students ignore the contextual meaning of terms in solving problems. Thus, lack of understanding definitions and the role of the contextual meaning of terms in problem-solving seems difficulty in calculus learning.

Lack of a consistent mental image of the limit

The literature has documented that students have trouble with making consistent cognitive structure of the limit. While some students conceive the dynamic limit process as static (Çetin, 2009; Duru, 2011; Jones, 2015; Moru, 2006; Nair, 2010), some others consider the static limit value as dynamic (Oehrtman. 2002; Parameswaran, 2007). While some students consider real-valued functions as discrete and hence point-by-point thinking of the limit process (Gray *et al.*, 2009;

Wangle, 2013), some others consider number sequences as continuous (Moru, 2006; Roh, 2005). Thus, lacking a consistent mental image of the limit is a difficulty in calculus.

Overgeneralized and immature conception

The literature has also documented that learning calculus involves a construction process (Çetin, 2009; Wangle, 2013). From a constructivist learning point of view, in coming to understand a concept, or when students fail to understand a concept, they may develop an alternative conception of overgeneralization (Konicek-Moran & Keeley, 2015). Whether it is the limit, continuity or derivative the literature documented that most students demonstrated overgeneralizations or immature conceptions (Duru, 2011; Jordaan, 2005; Maharajh *et a*l., 2008; Nair, 2010; Vela, 2011; Wangle, 2013). Due to those overgeneralizations, students sometimes demonstrate correct answers for wrong reasons and wrong answers with high confidence (Çetin, 2009; Juter, 2006; Luneta & Makonye, 2010). Thus, overgeneralized or immature knowledge but not noticed by students accordingly and hence conflicting concept images (Juter, 2005a) seems troublesome in learning calculus concepts.

Rote knowledge versus conceptual knowledge

A feature of advanced mathematics like calculus is the need for conceptual knowledge, as its ultimate goal (for non-mathematics major students) is the wide application in science, business, engineering, and technology subjects (Paramenswaran 2007; Siyepu, 2013). However, empirical research shows that students end up with rote and manipulative learning of one or the other concepts in calculus without an understanding of the core ideas (Cetin, 2009; Elia *et al.*, 2009; Hashemi *et al.,* 2014; Luneta & Makonye, 2010). The literature also revealed that most students didn’t react at all or demonstrate low success for unfamiliar task items or items demanding higher levels of cognitive thinking (Horvath, 2008; Juter, 2006; Makonye, 2012; Roh, 2005; Usman, 2012).

There are signs that students’ thinking lacks meta-cognition (Makonye, 2012; Usman, 2012). Some students write or speak contradicting answers without being aware that they are contradicting. Though students' learning focuses on the procedural aspects, they also demonstrate procedural difficulties. Most difficulties in derivative correspond to a lack of manipulation of rules and procedures (Horvath, 2008; Luneta & Makonye, 2010; Makonye, 2012). The literature also revealed that students could not make a link among two or more concepts or lack doing the logical link among different attributes of the same concept, and they demonstrate unsynchronized approach than explore generalized nature of concepts (Hashemi *et al.,* 2014). Thus, although strong students are concerned with the embedded idea in their learning and observed divergent thinking with their ability to answer problems, most students over depend on procedural learning and lack conceptual knowledge.

Focusing only on the algebraic form of representation

The literature has also documented the importance of multiple representations i.e. the same concepts represented in different ways that provide students an opportunity to build abstractions about the concepts and varied viewpoints. The ability to move among representations (numerical, algebraic, graphical and description or application problems) has been used as a sign of strong conceptual knowledge (Aspinwall & Miller, 2001; Lauritzen, 2012; Zollman, 2014). Though some students demonstrate the ability to use multiple representations in their answer to problems or demonstrate consistent understanding to the same idea in different representations (Wangle, 2013), most students, however, keep on with only one representation (usually, symbolic) and hard to see that these are different illustrations of identical mathematical concepts (Blaisdell, 2012; Moru, 2006; Wangle, 2013).

Specially, Blaisdell (2012) on a study aimed to investigate the Influence of question format/representation found that students stimulate different concept images of the same idea based on the type of representation. While the teaching of the limit is more of algebraic (Hashemi *et al.,* 2014), the study by Blaisdell (2012) and Duru (2011) found that higher scores in graphical representation than algebraic representation whereas Hashemi *et al.* (2014), Maharaj (2013), and Orhun (2012) found that students have difficulty to characterize a function from its graph. Thus, while multiple representations are an indication of the depth of knowledge and demonstrated by only a few students, a lack of it seems troublesome for most students.

Lacking problem-solving framework

One way to disclose depth and breadth of getting conceptual knowledge in learning calculus concepts is via the extent of using that knowledge in problem-solving (Hashemi, Abu, Kashefi & Mokhtar, 2015). Problem-solving by itself might be an instrument to overcome conceptual difficulties in calculus (Rabadi, 2015). The literature revealed that many students had difficulty to model the concepts in a problem (Brijlall & Ndlovu, 2013; Siyepu, 2015). Others documented that students lack the ability to integrate information to gain conditions which will satisfy given and required in a problem (Brijlall & Ndlovu, 2013; Maharajh *et al.,* 2008), lack making network of concepts toward solving a problem (Usman, 2012), and fails to choose appropriate procedures to be applied for a given problem (Siyepu, 2013). The literature also documented that all the teaching, learning, and textbooks approach contribute a share to these difficulties as their focus is largely on manipulation of symbolic aspects on routine exercises (Rabadi, 2015). Thus, lack of exposure to non-routine problems and problem-solving framework is the other dimension of difficulty.

Overall, the literature has documented the essential knowledge aspects in the learning of calculus concepts. Although only some students demonstrate this essential knowledge, most students lack this knowledge. The following are the identified themes of difficulties.

- A static view of a dynamic process.
- Lack of definitions and relationship of terms.
- Overgeneralizations or immature conceptions.
- Over-dependence on procedural learning.
- Lack of multiple representations.
- Lack of problem-solving framework.
- Lack of procedural proficiency.

2.2. Theoretical Framework

The purpose of this section is to present the theoretical aspect of the study. The theoretical analysis presented will be used to: describe the framework through which the students' activities are analysed, construct definitions of key terms of the study, and establish key constructs of conceptual knowledge from different perspectives.

The theoretical framework of the study is constructivism perspective of learning and its bridge theories. The section begins with the discussion of constructivism learning theory followed by a discussion of its bridging theories as a model of concept formation. The views on the duality of knowledge (conceptual and procedural) in mathematics then discussed and evaluated with the purpose to identify contextual definition of conceptual knowledge to the study. The section ends with a discussion on basic constructs of conceptual knowledge in calculus.

2.2.1. Constructivism

The Constructivist theory of learning is a perspective that focuses on how students actively create knowledge based on their existing cognitive framework (Seifert & Sutton, 2009). Opposing the argument that students are a tabula rasa, constructivism gives great attention to prior knowledge already present in the students and to the role of students and relevant information during the knowledge construction process. This theory states, "Reality is an individual matter and hence learning is a factor of experiences and previous knowledge" (Pritchard & Woollard, 2011, p.4). Pritchard and Woollard used this statement as justifications of why two students attend the same lesson demonstrate different learning outcomes. Particularly, the prior knowledge about the subject, how tasks and instructional activities were interpreted (the thinking), and how activities during the lesson were carried out (including psychological factors) are factors that determine the output of learning. Thus, the individual experience, the thinking, and the environment are central to the learning process.

Constructivism has two different but complementary forms: radical and social (Ernest, 1994; Liu & Matthews, 2005; Pritchard & Woollard, 2011). While both support the active role of the individual in constructing knowledge out of the experience, there is a profound distinction on the role of socio-cultural context and hence on how learning takes place (Nair, 2010).

2.2.1.1. Radical constructivism

Radical constructivists view knowledge as an entirely individual construct, learning as an individual-oriented mental process, and students as independent investigators (Von Glasersfeld, 1995). For the radical constructivist Von Glasersfeld, establishing knowledge is an independent issue. Thus, knowledge is a reality that an individual creates based on her/his experience and it is located in the mind. Students are considered as independent investigators of knowledge based on their experience with no concern about the knowledge exterior to their coverage (Von Glasersfeld, 1995). It is also characterized by its emphasis on students and "discovery-oriented" knowledge construction. The interactions with the surrounding community serve only as motivation for the cognitive argument (Liu & Matthews, 2005).

Radical constructivism has got recognition due to its contribution to shifting the view of learning from teachers' centre to student focused and recognizing students' learning style preferences (Ernest, 1994). As a result, it changed students' role from being passive receivers to being construct meaning for their own. In this context, students are also responsible for construction errors and encountered difficulties (ibid). Nevertheless, its idiosyncratic nature exposed it to criticisms. Particularly, its ignorance of the cultural components of the world and the social interactions are taken as limitations (Ernest, 1994; Thomas, 1994). Thomas in his critics entitled, "Abandonment of Knowledge" and "Social Constructivism," describes that while the former refers to ignorance of the knowledge out of the individual and in surrounding the later refers to the ignorance of the social interaction and its contribution to the sustainability of the constructed knowledge (including parents, friends, and teacher's role). It is also described as confused due to the attempt to incorporate a social view of knowledge into it while it is said to be idiosyncratic (Ernest, 1994).

2.2.1.2. Social constructivism

For social constructivist, social and cultural interactions are means for knowledge creation. Thus, learning is a social process which is largely context and situation laden (Liu & Matthews, 2005). An individual student is a member of a community of students' and should have to collaborate among fellow students and appreciate different perspectives. Social constructivism views learning as "changes in thinking" that takes place because of guidance and interaction with others and the student as assisted performer (ibid). Accordingly, learning occurs through appropriate guidance and resources from those having the knowledge and experience to do so, teachers in the case of formal classroom learning.

The recognition for the foundation and development of social constructivism, also called socio-cultural theory goes to the work of Vygotsky, Piaget, Bruner, and Bandura (Pritchard & Woollard, 2011). Specially, Vygotsky's idea of "the zone of proximal development" describes the gap between an individual's potential to learn independently and the scale-up of that potential to a higher level when the learning is supported by a capable adult or collaborates with peer groups (Seifert & Sutton, 2009, p.36). Such support to scaffold students' potential to a higher level is said to be "instructional scaffolding" (ibid).

Although social constructivism has got popularity since it recognizes both individual and private meanings of knowledge and widely implemented in formal and non-formal classrooms, it is also not free of criticism (Ernest, 1994). From a theoretical and practical point of view, its socio-cultural perspective can limit diversity in the classroom. In particular, if the assistance provider is not competent, she/he either limits the potential to progress or misguide the students. As a result, students become dependent on the social environment for performance assessment rather than an independent investigator and self-controller (Confrey, 1995).

From a philosophical point of view, both radical constructivists and social constructivists claim that an individual constructs her/his own world-view and can do that reconstruct based on pre-existing structure and newly acquired experience. However, the construction process for the former it is individual, and for the later it is both individual and shared, and hence, culture and context have roles (Pritchard & Woollard, 2011). Moreover, social constructivists emphasize that reality cannot manifest without the societal argument. Thus, knowledge is a product of social interaction and learning is a socially mediated process for advancing mental processes (Ernest, 1994). Regardless of the differences mentioned above, there is a significant comparison among most constructivists in both camps with regard to the role of students' position, individual experience, learning tasks, and social interaction for knowledge construction (Liu & Matthews, 2005).

2.2.2. The constructivist perspective of a classroom environment

In a constructivist learning context, the students have to attempt to make sense of classroom activities, interact with others, reflect based on her/his perception and appreciate different perspectives. The teachers' role is beyond presenting new information. The teacher has to view each student as unique individual with unique need and backgrounds, diagnose and acknowledge their prior conceptual knowledge, design the teaching-learning environment in a way that facilitates social interaction, provide timely support and feedback, and see for contradictions if there is any for further actions (Bransford, Brown & Cocking, 2000).

Constructivism emphasizes the role of pre-existing cognitive structure in the students. The prior mental representation is a foundation in which the new information is to be built-in. Piaget (as in Pritchard & Woollard, 2011) called each mental representation a schema. Thus, “schemas are assimilated net of ideas which are accumulated in long-term memory and potential source to be reminded whenever necessary (p.11)”. Any further new concept is recognized depending on its extent of fitness to the schema.

Hence, learning can take place only by relating the unknown to what is already known. According to Piaget's genetic epistemology (as in Pritchard & Woollard, 2011), the process of constructing knowledge has to undertake three mental activities: assimilation, accommodation, and equilibration. Assimilation is an awareness of the latest experiences with regard to existing conceptual structure (Glasersfeld, 1995). The new information is measured by the degree, which it relates to an existing schema, and either it fit well or even maybe contradicting the existing one. Despite the apparent contradiction, contradicting information also may be assimilated if it seems reasonable from the students' perspective. When the contradiction is much more than compromise with the existing structure, accommodation will be happen, i.e. accommodation is the modification or alteration of pre-existing conceptual structure so that new or contradictory pieces of knowledge to be established (Seifert & Sutton, 2009).

The two stages of knowledge construction are not always smooth. Cognitive equilibration is a process of resolving contradictions in students' mental structure (Glasersfeld, 1995). There are different conditions to be focused on to attain cognitive equilibrium. An individual may be satisfied about the link between the existing one and the new knowledge and hence being in a state of equilibrium, aware of the contradiction in the existing thinking and being in a state of experience cognitive conflict. This crossroad differentiates students as successful or unsuccessful in learning a given concept. The one that capable to eliminate the contradiction will re-establish a state of equilibrium and would be successful. The way to regain equilibrium even leads the student to a more sophisticated mode of thought (ibid).

Glasersfeld (1995, p. 68) in his summarized learning theory contribution of Piaget's work connected the triple stages of concept formation as "cognitive modification and learning in an explicit direction occur once a scheme, rather than built-up the expected result, results in conflict and cognitive conflict, in turn, link accommodation that re-establishes equilibrium". As a result, cognitive equilibrium is the process of making stability between existing mental structures and new knowledge. While cognitive conflict is a means for learning, the resulting cognitive equilibrium is an end of learning a specific concept. Here, teachers' role will be to design activities that motivate cognitive conflict (but not societal), follow-up students' interpretation and provided guidance, design assessment activities that help to make check and balance between conflict and equilibrium and to administer accordingly (Bransford *et at.*, 2000).

2.2.3. Models of concept formation in mathematics

Constructivism outlook on learning has been central to several of the recent empirical and theoretical works in mathematics education (Ernest, 1994). Within its inquiry approach to learning, constructivism motivates students to be active during learning and acquire knowledge that can be transferred beyond classroom context (ibid). Due to this, several educators in mathematics education prefer it.

Many scholars (For example Bezuidenhout, 2001; Dubinsky, 2002; Ernest, 1994, Tall & Mejia-Ramos, 2004) argued that due to the constructive nature of mathematics cognition, there is a strong tie among students' prior knowledge, concept formation process, and mathematical difficulties. Students at formal schools are not free of social influences. On the other hand, students within the same context and culture demonstrate different knowledge and performance. Thus, learning occurs individually and socially. As a result, students make difficulties during the knowledge acquisition process by their internal construction and sense-making of their natural thoughts and experiences (Ernest, 1994).

Based on the constructivist perspective of learning, researchers in mathematics education have derived frameworks to deal with concept formation in mathematics. The most widely used constructivism frameworks are APOS (Cotterill *et al.,* 1996), the three worlds of mathematical thinking (Tall & Mejia-Ramos, 2004) and concept image and concept definition (Tall & Vinner, 1981). This study uses the first framework, and the details will be discussed next.

2.2.3.1. APOS (Action, Process, Object, and Schema) theory

APOS is a constructivist framework of learning developed by Dubinsky and his colleagues based on Piaget's reflective abstraction. The notion of reflective abstraction focuses on the actions or operations done by students on physical or mental objects. That is, reflective abstraction is a set of mental operations that are directly invisible but only be inferred from prolonged observation or qualitative actions of students (Dubinsky, 2002; Glasersfeld, 1995).

Reflective abstraction has three components: (i) expansion of the existing mental structure (ii) reconstruction of existing knowledge structures and (iii) a process of resolving contradictions in an individual's mental structure (Pritchard & Woollard, 2011). Therefore, reflective abstraction is a progression through construction, and Dubinsky (2002) identified five types of construction in reflective abstraction. These are interiorization, coordination, encapsulation, generalization, and reversal.

Dubinsky in collaboration with other researchers in Research in Undergraduate Mathematics Education Community (RMEC) used these five constructs to describe how process and object conception are constructed and formulate APOS theory.

According to Asiala *et al*. (1997, p. 9) the formation of a mathematical knowledge, "initiates through the exploitation of existing mental objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects". The whole cognitive configuration is said to be a schema". The descriptions of action, process, object and schema, and the constructs involved in the formation of such knowledge are discussed below.

Action- is explained as "a repeatable mental or physical manipulation of objects" (Moru, 2006, p.49). In this stage, the conversion of an object is thought of as exterior, and the student is only conscious about the execution of routine procedures (Dubinsky & McDonald, 2001). It is like assembling equipment using a manual or according to Moru (2006) the ability to pick a number for a variable and compute the value of an algebraic expression. For instance, in the learning of the limit of functions for a student at action level, $lim_{x \to a} f(x) = f(a)$ (Cottrill et al., 1996). Although action level conception is restricted, it can serve as a foundation for the concept formation process. For instance, as in the above example to introduce limit dynamically, one can use sequence of such actions (evaluating f at a sufficient number of points both from the right and from left close to a) so that students' can predict the result.

Process- when the student is aware of the actions she/he is performing, the actions, then is interiorized to a process (Cottrill et al., 1996). Thus, the process stage is relatively internal and involves visualising a conversion of mental or physical objects without actually computing but by deduction. At this stage, students can carry out the same action without external stimuli (without a manual, a guide, or a teacher). In this stage, students can also have a mental representation of a process, turn around the process, as well as use it with other processes. Coordination is the creation of a process by bringing together two or more processes (Cotrill et al., 1996). The computation of the limit involves the coordination of the input process, and the corresponding output through the given mapping (ibid). Thus, a student at the process level can evaluate (say $\lim_{x\to 0^+}\frac{1}{x}=\infty$) without consideration of specific values at a time or by computing the first few elements and contemplating the remaining. The essential difference between an action and a process is that in action, it is external and students need systematic direction to carry out the transformation, whereas, in a process, the transformation carried out is internal and conceived with regard to relationships among cognitive structures of an individual student (Carlson & Oehrtman, 2005). For instance, for the items

- $lim_{x\to 0}(x^3+2x)=$ _______ and
- If $f(x)=\begin{cases} x+2\ if\ x\le 3 \\ 6-x\ if\ x>3 \end{cases}$ then $lim_{x\to 3}f(x)=$ _______

A student at action level of computing the limit of a function at a point can answer the first, but not the second. She/he possibly answers the second as either five or three. Nevertheless, one at the process level of computing the limit most possibly will answer both correctly.

Object- object level concept formation is a level where the student perceives the concept as something to which actions and processes may be performed. A student in this stage conceives the totality of the process as unit and understands that conversions can be performed on it (Cottrill et al., 1996). The construction of a cognitive object through awareness of totality of a process, either by manipulation or imagination of it as a whole without performing subsequent actions is said to be Encapsulation (ibid). A student who encapsulated a process in to an object level of the limit, for instance as in the above example, have object view of the limit value so can act on it. Thus, given $\lim_{x\to a}f(x)$ and $\lim_{x\to a}g(x)$ then she/he can easily compute $lim_{x\to a}(f+g)_{(x)}$.

Schema- is described as the complete conceptual structure that is a result of consistent compilation of actions, processes, and objects (Cottrill *et al.,* 1996). As it is compilation of the preceding levels, a student at schema level is competent enough to move flexibly back and forth among all the levels. *Generalization* is the ability to extend the acquired schema on a higher level of the phenomenon (Dubinsky, 2002). *Reversal*, on the other hand, is the ability to visualise an existing mental structure in reverse to extend it or make a new mental process. For instance, in calculus, pair of processes that are reversal are differentiation and integration. According to Stewart (2008, p. 26) the description of the schema in APOS is analogous to Tall and Vinner's (1981) idea of concept image.

These basic constructs, the piece of knowledge that could involve in learning a concept and the interplay among them is presented in Figure 4 taken from Dubinsky (2002, p. 107).

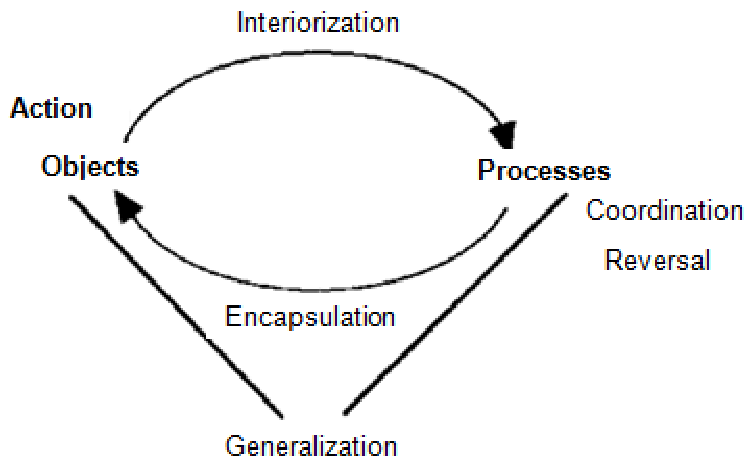

Figure 4: Constructs of mathematical knowledge and their interplay

Source: Dubinsky (2002, p.107)

Asiala et al. (1997, p. 8) outlined that the "genetic decomposition" of a concept is a planned mental model that probably will explain cognitive structures of the concept in a student's mind. Therefore, a genetic decomposition consisting of specific actions, processes, or objects that might involve in the cognitive schema to deal with a given concept.

APOS theory has recognised not only as a research framework but also for designing mathematical curricula (Dubinsky & McDonald, 2001; Stewart, 2008). Several researchers used APOS framework to describe the level of students' difficulties and use a "genetic decomposition" of a specific topic to prepare an intervention and reported positive results (Maharaj, 2010; Stewart, 2008). However, this does not mean it is free of limitation (Maharaj, 2010; Pinto & Tall, 2001; Tall, 1999).

To begin with, explanations offered by an APOS analysis may not explain what actually occurs in an individual's mind. On the other hand, an individual may have a certain mental structure in the mind but may not apply it in a given learning or problem-solving (Maharaj, 2010). According to Pinto and Tall (2001), there is also an issue of learning style preferences. There are two forms of learning style preferences- formal and natural (ibid). Those labelled as formal attempts to base their learning in deductive approach from concept definition. They form their concept image by focusing on rules and procedures, and then they deductively build their formal theory. For those labelled as natural learners', concept formation is based on an existing concept image gained from perception (ibid). Pinto and Tall further contend that formal thinkers are well-matched with APOS theory, but it does not make clear the method of natural thinkers' learning.

Within these limitations, APOS has many applications in algebra and calculus as a tool of analysis for researchers. Particularly, APOS has recognition to explain students' difficulties in calculus and to suggest pedagogical strategies that promote conceptual learning. For instance, the work of Çetin (2009), Cottrill *et al.* (1996), Maharaj (2010), and Moru (2006) in limit; Wangle (2013) in continuity; Jojo *(*2011) and Maharaj (2013) in the derivative, and Stewart (2008) in linear algebra were evidence.

2.2.4. Conceptual knowledge in mathematics

A substantial number of studies regarding students learning of mathematics in general and above all calculus concepts involve two dimensions of knowledge-conceptual and procedural (See for instance, Engelbrecht *et al.,* 2005; Hiebert & Lefevre, 1986; Lauritzen, 2012; Schneider & Stern, 2005; Star, 2005; Star & Stylianides, 2013). There are also scholars who use different terms to name the duality for instance, relational and instrumental (Skemp, 1976). In the more recent literature, the conceptual and procedural terms to name the duality are dominantly used (Star & Stylianides, 2013).

2.2.4.1. Conceptual knowledge

Conceptual knowledge is defined as the ability to demonstrate, interpret, and relate the verity of mathematical concepts correctly to a variety of problem-solving situations (Engelbrecht *et al.,* 2005). Rittle-Johnson, Siegler, and Alibali (2001) define conceptual knowledge as a set of pieces of knowledge about a concept and skill of interconnecting these pieces into a whole or network. The essentials of these networks can be rules or procedures, and even problems given in various representations. One with conceptual knowledge in mathematics demonstrates the ability to decompose a given mathematical expression into pieces or express the network in verbal statements. Built-in to such knowledge is associated network of knowledge so that the whole is as important as the individual elements that connected to give the whole (Engelbrecht *et al*., 2005).

An influential theme that is common among several definitions of conceptual knowledge is, “making connection or relation.” The term “relational” has also used by Skemp (1976) to name one type of mathematical understanding as will be discussed later. This theme originated from the definition of conceptual knowledge given by Hiebert and Lefevre, which by itself is seen as a foundation for the subsequent definitions of conceptual knowledge (as in Star & Stylianides, 2013). Hiebert and Lefevre (1986, p. 3) define conceptual knowledge as “a type of knowledge that is loaded in associations”. It can be considered as an associated network of knowledge so that the whole is as important as the individual elements that connected to give the whole. Its connected nature promotes awareness and the ability to move from particular to general and flexibility during task performance.

According to Tall (2002), mathematical thinking is a cognitive composition that is friendly to the “biological structure of the human brain” (p. 16). It is massive store of knowledge and inner associations, which systematically deals with various cognitive tasks. This definition of mathematical thinking is more like the definition of conceptual knowledge by Hiebert and Lefevre (1986). In both definitions, the focus is not only the amount of knowledge available, but also the connection and integration among those pieces of knowledge. In Konicek-Moran and Keeley (2015) view, a student is said to have conceptual knowledge if she/he is able to- think with it, extend it to similar situations, verbalize it, and get a similar or different way of expressing it.

Students use their conceptual knowledge to identify what and when to use definitions, rules, and procedures, and to distinction associated concepts and evaluate results (Schneider & Stern, 2005). It is accumulated in some forms of relational representations or hierarchies and is not attached to specific problem types, rather can be adapted to the different context of problems. It is rich in relationships or webs of correlated ideas and allows individuals to distinguish between these correlations (Lauritzen, 2012; Mahir, 2009). In addition, it can be easily verbalized, flexibly transformed in the course of deduction and reflection (Schneider & Stern, 2005).

2.2.4.2. Procedural knowledge

Procedural knowledge is commonly associated with knowledge of procedures, and the setting where the procedures can be executed (Star & Stylianides, 2013). Engelbrecht et al. (2005) define it as the ability to explain the solution to a problem via the exploitation of a set of rules and procedures that associated with algorithms and symbols. According to Rittle-Johnson *et al.* (2001), procedural knowledge is the ability to perform algorithms quickly and efficiently as a part of problem-solving. This knowledge type is attached to a specific problem type and therefore is not easy to generalize it to different arrangement of problems in the same domain.

Hiebert and Lefevre (1986) describe procedural knowledge in mathematics into two components. The first component involves being familiar with the language of mathematics which is the symbolic representation. The other component is the knowledge of rules and procedures of those symbols to solve problems. The main quality of the procedural knowledge is to be “executed in a predetermined linear sequence” (p. 6). Thus, procedural knowledge as compared to conceptual knowledge engages minimal cognitive awareness and a little cognitive resources. It is easy to learn, and it allows students to execute possible actions that could be properly performed to solve a given problem. Nevertheless, it is less connected and shallow in representation and hence hard to reflect and communicate (Schneider & Stern, 2005). That is why Hiebert and Lefevre (1986, p. 6) emphasized that procedural knowledge is the “narrative of managing mathematical signs and syntaxes”. It requires only a consciousness of rules and not interpretations or analysis. However, this does not mean that procedural knowledge has no relevance. Rather, students must learn to master fundamental concepts and computation of procedures (Mahir, 2009; Schoenfeld, 1992). Each one is quite limited unless it is connected to the other (Lauritzen, 2012; Rittle-Johnson *et al.*, 2001).

2.2.4.3. Relational understanding

Skemp’s relational understanding refers to both the ability to perform procedures and to justify why those procedures and rules are used, whereas, instrumental understanding represents knowing the rules, and procedures of mathematics. He argues that in the short run the later may be more pleasing because learning how to do something is usually easier to memorize than learning something with deep meaning attached and then relating that to how it works. Moreover, even for teachers, instrumental understanding is easier to make assessment than relational understanding. In the long run, however, relational understanding is more helpful.

With regard to retention period of knowledge as mentioned above, Crowley (2000) as in Tall (2002, p. 16) comments that even average ability student works in a “cognitive kit-bag” that lack connection and perform explicit procedures. Resulting in the spot success and satisfaction and possibly, “long-term cognitive load and failure”. Instrumental oriented students can be identified from their performance in classroom tasks. Those students can perform simple routine exercises very well, but stack for items that are different in nature from the usual classroom and textbook items (Gray & Tall, 1994). Thus, Skemp strongly argues that teaching should promote relational understanding.

Going back to the conceptual and procedural duality of knowledge, in the more recent research literature, Skemp’s instrumental and relational understanding referred to procedural and conceptual knowledge respectively (Wangle, 2013). Thus, in this study, too conceptual understanding and relational understanding is considered synonymous.

2.2.4.4. Contextual definition of conceptual Knowledge

Star (2005) points out that the term conceptual knowledge includes both what is known, and the way that it can be built-in. Likewise, the term procedural knowledge specifies knowledge of procedures and the way that procedures can be known. In conceptual knowledge, the construction can be deep and rich in the association of the networks, whereas, in the procedural knowledge it is shallow and less in connection. Thus, Star argues that the description of knowledge dually like this encompasses both knowledge types and knowledge quality. These two aspects of knowledge and the interplay between them is presented in Table 2 taken from Star (2005, p. 408).

Table 2: Types and qualities of procedural and conceptual knowledge

Knowledge type	Knowledge quality	
	Superficial	Deep
Procedural	Common usage of procedural knowledge	? Genuine conceptual Knowledge
Conceptual	?	Common usage of conceptual knowledge

Source: Star 2005, p. 408.

Star further argues that the present practice on the duality of knowledge makes it hard to think and denote the knowledge that is deep in quality and procedural in type. Duffin and Simpson (2000) describe, “Depth of understanding” as the ability to explain and justify each step of a problem-solving in mathematical terms. While the surface level procedural knowledge is automated skills on ordinary rules of algorithms, the deep level serves the purpose of creating and modifying the superficial level. Thus, deep procedural knowledge is as important as conceptual knowledge.

The intention of the researcher in this study is not to claim that conceptual knowledge is more essential than procedural knowledge. He strongly believes both are important aspects of students' knowledge. Therefore, both deserve careful attention. With regard to the significance of both types of mathematical expertise, Hiebert and Lefevre's (1986) comment that: what makes a mathematical knowledge complete is not only the existence of both types of knowledge but also the strength of the integration among them. When both exist but lack integration, students may show interest and initiation to participate in problem-solving but remain unsuccessful.

In this study, conceptual knowledge refers to knowledge of both concepts and procedures, which is integrated and deep in quality. In particular, conceptual knowledge about a mathematical concept consists of the knowledge to compute procedures and to justify the reasoning employed within relevant representation forms together with the ability to communicate in written, in a coherent, consistent, and flexible mathematical practice. According to Star, "types and qualities" description of the duality, this definition refers to the area where the deep in procedural, and deep in conceptual overlaps. Defined in this way, conceptual knowledge deserves the description that it is an adequate competence to solve all types of problems and tasks. The next section presents constructs that are the manifestations of this conceptual knowledge in calculus.

2.2.5. Basic constructs of conceptual knowledge in calculus

There are common themes among the different definitions that are given to the term conceptual knowledge by different educators and researchers. However, there is no objective rule that answers the question 'what does it means to have conceptual knowledge of a specific topic like the limit or derivative?' In this section, the basic constructs of conceptual knowledge in calculus based on the definition of conceptual knowledge adapted, and the theoretical framework of the study will be presented.

Consistent concept image/ Schema

From the constructivist learning theory point of view, students should be active participants in constructing knowledge of mathematics. They build on and modify their existing cognitive structure based upon new exposure they imposed on. Since this construction is not always smooth, it follows that students can and do make construction errors of various kinds (Ernest, 1994). Those construction errors may be due to the presence of alternative conception, incorrect generalizations, and interference of past knowledge or absence in pre-requisite knowledge. When a student makes such a construction error, the cognitive structure or “concept image” the individual has, differ in various aspects from the formal mathematical concepts. It is also possible for an individual to have more than one cognitive structure or concept image of a concept that conflict with each other (Tall & Vinner, 1981).

Since such concept images consist of all experiences connected to the concept, in which there may be quite a lot of such images assembled in diverse contexts, those representations perhaps come together as the individual becomes more mathematically mature. Otherwise, such concept images can co-exist in multiple forms and make an unnecessary cognitive load. Tall and Vinner (1981, p.152) employ the term “evoked concept image” to explain the existence of an inconsistent concept image. Accordingly, based on context the same concept name may remind different concept images from the mind. Thus, if a student has a matured and stable concept image, she/he can demonstrate consistency and flexibility during problem-solving.

Since the learning of conceptual knowledge is a lot of consciousness and cognitive resources demanding (Schneider & Stern, 2005) afterwards it can be characterized as reflective and communicable for a variety of contexts. In other words, students have to minimize conflicting concept images. According to Siemon (2013), if one has conceptual knowledge she/he will be able to- generalise from particular examples, expand ideas to new situations, approach problems in different perspectives and demonstrate flexibility in the form of representations, interpret and associate ideas, and recognize the limitations of an idea. In general, consistency and flexibility are constructs of conceptual knowledge that reveal through students’ concept images.

Connection between forms of representations

Hähkiöniemi (2006) describes, “Representation” as a tool to think of something. Representations are not only tools to think with but also tools for expressing our thoughts. Thus, a representation of a certain concept consists of an invisible internal system (concept image) and of a visible external system (a visual, verbal, or symbolic reflection of the concept image) (Goldin & Shteingold, 2001). The internal representation of a concept is part of one's cognitive structure, maybe a single or several computing parts, and serves to interact with the external world, and the external system is symbolic and serves to facilitate the interaction (Dreyfus, 2002). An individual's representation of a concept is said to be strong if it incorporated many related aspects of a concept, so that the individual can manipulate it flexibly. Otherwise, it is said to be poor (Dreyfus, 2002). One means to do well in mathematics is to have such multiple representations of concepts, i.e. able to recognize or describe the same concept or idea using a different form of representations (Aspinwall & Shaw, 2002).

Describing a concept using multiple forms of representations has been strongly connected with learning advanced concepts. More particularly, with the formation of conceptual knowledge in calculus that should be adaptable to the different contexts of a problem (Aspinwall & Shaw, 2002; Herbert, 2013). Approaching a concept in multiple ways (visually, numerically or algebraically) and able to shift simply among forms of representation is one aspect of a having a conceptual knowledge (Aspinwall & Miller, 2001; Lauritzen, 2012). Hähkiöniemi (2006) expresses that while procedural knowledge often stands for the use of representations, conceptual knowledge is described by the flexibility among representations.

Underlining the significance of multiple representations in calculus Tall and Mejia-Ramos (2004) mentioned that student's exposure to numeric data, symbolic manipulation, and graph sketch or interpretation in calculus could have to be performed at an advanced level and done that way, it paves the way in for progressions. Besides addressing individuals learning style preferences, and challenges of linguistic issues, the interaction among multiple representations of the same concept helps to obtain better mathematical concept images, which in turn improve the depth of conceptual knowledge (Aspinwall & Miller, 2001; Berry & Nyman, 2003).

One of the critics on calculus teaching-learning is that the practice is more focused on symbolic manipulations according to given rules than construct mathematical knowledge by solving problems and investigating patterns (Schoenfeld, 1992). When students got exposure to multiple representations, they recognize a mathematical object in different illustrations and interpret the idea from one form of expression to another. Of course if not properly manipulated, the use of multiple representations has its own limitations. Taught the same concept with different representations, unless they are well aware how to sort out the different forms of the same concept, their cognitive load would be Junk (Dreyfus, 2002).

Abstraction

One cognitive demand for advanced mathematics like calculus is an abstraction (Dreyfus, 2002; Tall, 2002). Tall (2002) discusses “generalization” and “abstraction” as a twin mechanism in mathematical thinking which is used to denote both processes and products involved in concept formation. As in Dubinsky (2002), Piaget distinguished three types of abstraction:

- Empirical abstraction- occurs when one focuses on the general nature of objects obtained through perception. According to Piaget (as in Jojo, 2011) this abstraction leads to the mining of common possessions of objects. So, it is the means to access the general from the explicit.
- Pseudo empirical abstraction- is in the middle of empirical and reflective abstraction. It serves to extract characteristics that the actions of an individual have established into an object (Dubinsky, 2002).
- Reflexive abstraction- occurs when the focus is on reflection on perceptions or actions done by an individual on (mental) objects. Piaget (as in Jojo, 2011) emphasizes that reflexive abstraction directs us to a unique type of generalization.

As described in section 2.2.3.1, reflective abstraction is a progression through construction, and Dubinsky (2002) identified five types of construction in reflective abstraction. These are interiorization, coordination, encapsulation, generalization, and reversal.

- Interiorization- is a phase where internal processes are constructed as a result of perceived phenomena occurred. Here actions are internalized, mentally represented, and a student becomes familiar with a process (Jojo, 2011). In finding the limit of sequences (say $\lim_{n\to\infty}\frac{1}{n}$), a table of values may be constructed for exhaustive elements of the domain. Since $n \to \infty$, all the computations are not actually performed. Thus, the student can conclude not only by computation, but also by contemplation, i.e. interiorization of actions in a thinkable process (Moru, 2006).

- Coordination- in this phase, two or more processes are coordinated to form a new process. In learning the limit of a function, for instance, to determine $\lim_{x\to a} f(x) = L$, a student at the process level is able to construct the following cognitive structures:
  - ➢ accumulate input values from the premise x approaches a from either side,
  - ➢ accumulate the output values from the premise $f(x)$ approaches to L,
  - ➢ coordinate the two dynamic processes (Cottrill et al., 1996).
- Encapsulation- this is the stage of knowledge construction where a translation of "a process into an object" takes place (Dubinsky, 2002, p. 101). This translation demands being aware of the totality of the process, see it as an object such that transformations can act on it. Dubinsky comments that this is the stage with twofold nature: the most significant but challenging attaining. Students, who attained this level of construction in the learning of the limit, can differentiate the limit process (which is dynamic) from the limit value (which is a static). Thus, they can easily perform operations on the limit. On the other hand, those who lack this stage can demonstrate the different form of difficulty including the limit is unreachable, an approximation, or can put multiple limit values.

In the three worlds of mathematics, "procept" is a mode of sophistication in concept formation where one can see a symbol both as a process to do, and as a concept to think with it (Gray & Tall, 1994). Further, Gray and Tall strongly argue that; this level of conception makes the distinction between students. Those who able to manipulate symbols as thinkable concepts operate dually as a process and as a concept, and be successful, whereas, those who focus more on the actions and perform simple routine actions be fail to proceed in higher-level problems. So, one could conclude that an action level of concept formation is restricted to procedural knowledge as it is static and less conciseness demanding. The process level conception is dynamic and is the beginning of conceptual knowledge formation. In fact, process level demonstration is the necessary but not sufficient level for conceptual knowledge (Asiala *et al.*, 1997; Cotrill *et al.*, 1996). Thus, an individual who attained this level can think about mathematics symbolically and focuses on mental objects (Gray & Tall, 1994). Conceptual knowledge is secured when the student is clearly capable of encapsulating the process to object mode of conception (Cotrill *et al.*, 1996).

- Generalization- this is the phase where an individual student is aware and able to use an existing mental structure to a wider situation of problem-solving without affecting or altering the existing mental structure (Dubinsky, 2002).

As one of the basic forms of making mental objects in advanced mathematical thinking, Tall (2002) classified generalization into three based on the cognitive activities required as follows- expansive, reconstructive, and disjunctive generalization. Expansive generalization, as the name itself implies is more of expanding the existing than constructing a new one. In that sense, it resembles one of the constructivism's cognitive tools called "assimilation." On the other hand, reconstructive generalization is more similar to "accommodation" in that it involves reconstructing the existing knowledge structure to accommodate new information. According to Tall (2002), in linear algebra course the general vector space R^n where $n > 3$ for most students is an expansive generalization. Whereas the abstract vector space is reconstructive generalization. In calculus for instance, the derivative of x^n (where n is a non-negative integer) is nx^{n-1}. For an average student, this is an expansive generalization whereas anti-derivative, for most students, is a reconstructive generalization. In calculus, a reconstructive generalization is recommended to overcome students' difficulties in relating symbolic and graphical aspects (Hashemi et al., 2015; Tall, 2002).

The third type called disjunctive generalization, although it has less influence relative to the previews two forms of generalizations, can be used to solve problems (Tall, 2002). Disjunctive generalization happens when students operate in difficulties, so that “they simply engaged in memorizing the new information and put aside without any effort to incorporate it with the existing one” (Tall, 2002, p.12).

- Reversal- this construction occurs when a student is able to construct a new process based on existing internal processes but thinking conversely (Dubinsky, 2002).

Basic Evidence of conceptual knowledge- in this study having conceptual knowledge is characterized by an individual’s ability to (where C_i refers to construct number for the advantage of later reference):

C_1: Define or represent a concept in her/his own words,

C_2: Make a connection between concepts in calculus. This includes the interplay among domain, limit, continuity, and derivative,

C_3: Explain and justify the reason for major steps in problem-solving,

C_4: Perform computations and interpret the results (perform symbolic and numeric computation without major errors)

C_5: Demonstrate the construction of coordinated processes. This includes coordination of domain and range process during computation of the limit (also called thinking ability about co-variation).

C_6: Demonstrate the encapsulation of processes into objects. This includes a clear distinction between the dynamic process and static value of the limit.

C_7: Have multiple representation perspectives: work with concepts given in various representations consistently and demonstrated flexibility in the form of representations during answering a problem. In this context, representation form means either symbolic, graphical, and table or verbal description.

C_8: Have a problem-solving framework: transform a real-life problem into a mathematical expression and solve it. This includes making connections between application problems in business, kinematics, medical, etc., and mathematical representations (Limit, derivative . . .).

C_9: Demonstrate coherence and consistency in her/his work and have a consistent concept image about a concept. This refers to reliable results to the same idea given in different contexts.

For instance, let us describe the construct required to solve the following problem:

"Let $f(x) = \begin{cases} ax & , \ x \leq 1 \\ bx^2 + x + 1 \ , & x > 1 \end{cases}$ be differentiable at $x = 1$.

Then determine a and b" (taken from Areaya & Sidelil, 2012, p.26). The required constructs are from C_1 up to C_6.

Table 3: Constructs of conceptual knowledge required in finding unknown in a piecewise-defined differentiable function

	Steps	c_3
c_1 & c_2	$\lim_{x\to 1} f(x) = f(1)$ and $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$ exist	A differentiable function is continuing, For a differentiable function, the limit of the different-question exists at the limit point
c_4	$\lim_{x\to 1} f(x) = f(1)$ and $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$	The limit at a point exist provided both the one-side limits exist and are equal
c_4 c_5 & c_6	$\lim_{x\to 1^+} f(x) = f(1)$	$\lim_{x\to 1^-} \frac{ax-a}{x-1} = \lim_{x\to 1^+} \frac{(bx^2+x+1)-f(1)}{x-1}$
	$\lim_{x\to 1^+} (bx^2 + x + 1) = f(1)$	$a = \lim_{x\to 1^+} \frac{(bx^2+x+1)-(b+2)}{x-1}$
c_4	$b + 2 = a$	$2b + 1 = a$
	$a = 3$ and $b = 1$	

Conclusion: In the first section of this chapter, the literature review of student difficulties from prior research was synthesised, and summed up in seven themes (See section 2.1.4). In the second section, the theoretical framework that have be used to analyse students' difficulties in the diagnostic assessment and in identifying components of conceptual knowledge to overcome difficulties was discussed. In general, the framework was useful for identifying areas in which students display strengths and difficulties. The researcher's attempt was to identify basic constructs of conceptual knowledge via the bigger perspective of the framework i.e. constructivism. To understand properly a topic in mathematics, especially calculus and to work with it in diverse areas of its application, students should be able to make an appropriate set up of these constructs. However, most students' difficulties arise from lack of one or more of such constructs or the whole set up. Different learning strategies can be designed base on the nature of such constructs to help students overcome their learning difficulties of a topic.

CHAPTER THREE: RESEARCH METHODOLOGY

The methodology chapter of this study begins with a discussion on why a designed-based research approach was selected. Since more than one design is employed, first a summary of the design-based research approach and the implementation of this approach as per each research question (3.1) were presented followed by the detail of the discussion on the description of participants (3.2), data collection instruments (3.3), an intervention (3.4), and the data analysis employed (3.5). Finally, the context of the study (3.6) and ethical issues (3.7) were presented.

3.1. The design-based research

According to Miles *et al. (*2014), methodology in a research work emanates out of the purpose and nature of the problem of the study. In order to get possible answers for the research questions of this study, i.e. to synthesize students' difficulties, to explore common conceptual issues that are causes of those difficulties, to propose an intervention model to overcome those difficulties, to prepare an intervention based on the proposed model and to evaluate the possible effect, a design-based research approach was employed. Plomp (2007), states that a design-based research approach is the systematic study of designing, developing, and evaluating educational interventions. Plomp (2007) further asserts that design-based research contains phases such as preliminary, prototyping, and assessment.

The design-based research is in line with the research work of Schoenfeld (2007) that has educational backgrounds in mathematics and follows preliminary studies and designing experiments, studies on context, and validation phases. Design-based research is advantageous in overcoming the limitation of research designs. Creswell (2012) seems more concerned about the demand of today's educational problems for a large toolbox of research approaches. He further stated that educators are recommended to use design-based research and multiple data collection instruments to address today's complex educational problems.

In a design-based research approach, this research work was explained according to the aforementioned phases. In the preliminary phase of the study, after conducting a systematic review of relevant literature, a diagnostic assessment was conducted on students of sample schools in the study area.

In the prototyping phase, based on the themes of difficulties, the causes of those difficulties and a literature review on suggested strategies to overcome those difficulties, an intervention model was designed. After that, a team of professionals from high school teachers and university lecturers has tested the qualities of the model.

In the assessment phase, an intervention based on the proposed model had prepared and implemented on the experimental group participants. A quantitative (pre-test, post-test, non-equivalent group, quasi-experimental) design had applied to analyse the quantitative aspect and a text analysis followed to analyse the qualitative aspect of this sub-study. This part ended with interpretation of the possible effect of the model on students' conceptual knowledge and comments for further enhancement of the model. In general, the research is mixed-method in a sequential paradigm. The research design of each sub-study per research question is generalized as in Table 4.

Table 4: Research design per research question

	Preliminary phase		Prototype phase	Assessment phase
1	What does the current literature reveal about students' difficulties in learning calculus concepts?	What are the common conceptual issues that cause students' difficulties in calculus?	What are the components of an intervention model of learning calculus concepts that could be developed to enhance students' conceptual knowledge in calculus?	Is there a significant difference in the students' level of conceptual knowledge of calculus after learning with the proposed model?
2	Literature from 2002 to 2016 on students' difficulties	238 grade 12 NSS students	Literature, theoretical framework of the study, and output from research question 1&2	105 grade 12 NSS students
3	Literature-desktop review	Diagnostic test	Desktop review	Concept test (pre-test and post-test)
4	Synthesis	Frequencies and pattern coding triangulated with literature	Thematic analysis	Independent sample t-test and text analysis
5	Thematic report of students difficulties in calculus	Descriptive and thematic report on conceptual issues that are cause of students' difficulties	Intervention model that aimed to nurture students' conceptual knowledge of calculus	Possible effect of the proposed model

Where: 1= Research question (RQ), 2= Sample/data source, 3= Data collection instrument, 4= Data analysis technique and 5= Expected output

3.2. Description of participants

The study was conducted in one administrative zone[5] of Ethiopia. The zone is catchment area of a University located in the zone. There are eleven government upper secondary schools in the zone, which are located in each of the ten woredas[6] and one town administration, which constitutes the zone. Grade 12 natural science stream (NSS) students of these eleven upper secondary schools constituted the population of the study.

In the study, sample selection was based on a purposive approach. Purposeful sampling lets the researcher apply her/his decision to choose a sample which she/he thinks, based on previous data, would supply the data needed (Fraenkel & Wallen, 2009). The disadvantage of this approach is that the researcher's decision may be influenced by the knowledge the researcher possesses regarding the information needed. One way to reduce this bias is to predetermine criteria about the level to which the chosen respondents could supply to the study. Thus, the researcher has used the following criteria for sample selection:

1. Schools' voluntarily to provide conveniences for the researcher,
2. Teachers' voluntariness to participate in both (diagnosis and experimental) phase of the study,
3. Schools which are following the normal teaching-learning process, i.e. not participated in an intervention program, and
4. The availability of students.

Some schools have funding agencies to support students using tutorial programs. In this program, some outstanding students are selected and assisted for one hour per week in each of the four science subjects including mathematics. The researcher was concerned about this because it could affect the intervention. The third criterion was set to address this issue. On the other hand, at the end of grade 12, students sit for the national University entrance examination. For this purpose, the National

[5] The third top-down administrative level
[6] The fourth top-down administrative level

Organization for Examination (NOE) will conduct registration around November of each year. Experience revealed that in some schools students do not regularly attend classes after the registration. Thus, the fourth criterion was set to address this issue. Accordingly, from the eleven schools, four schools were selected.

Sample for the diagnostic assessment- four intact classroom students one from each school was randomly selected. Two hundred sixty-four students attending in those four classrooms were taken as a sample for this study. While 11 students were missed the test and 15 test papers were inadequate to be included, 238 students test scripts, were used for final analysis.

Sample for the experimental phase- in addition to the four criteria's set as in the above two additional criteria were added for this phase. These criteria are the comparability of teachers' profile and schools background history. These are factors, which influence the result of an intervention. Accordingly, only two of the schools and the two teachers in these two schools were comparable based on all the criteria set. In these two schools, there were five intact classrooms of students. A pre-test was administered to all these students (they were 295 in number). Based on the result of the pre-test (those with comparable mean scores), one from each school, two intact classrooms of students (they were 108 in number) were taken as a sample. They were assigned as the experimental and control group randomly.

3.3. Data collection

The study employed four data sources: literature, diagnostic test, pre-test (pre-calculus concept test) and post-test (calculus concept tests).

Desk top literature review: a practical systematic review focused to investigate literature on students' difficulties and strengths of learning calculus concepts among students taking the course at secondary schools or at a first-year University course was conducted (The detail was discussed in 2.1.1).

Diagnostic test: the purpose of this test was to find out how students understand the concepts in calculus, what sort of difficulties they form and to investigate conceptual issues and approaches that cause students difficulties based on students' work and justification they provided. The items were collected based on the content of grade 12 mathematics syllabuses, minimum learning competency, characteristic of conceptual knowledge assessment as suggested in the theoretical framework and empirical results from the literature review.

Pre-test- the purpose of this test was to compare the level of students in the experimental group and control group before the treatment begins. The result obtained was used to determine the data analysis tool for the post-test result. Since the two groups have no significant difference in the pre-test i.e. they are comparable, a simple independent t-test was used.

Post-test (calculus concept test) - the purpose of this test was twofold: - The first was to compare the possible effect of the intervention model based on students' performance on the test. The second was to examine the extent the model helped to reduce observed difficulties in calculus. The test items were prepared based on the diagnostic test items with only little modification. Thus, the discussion on test item below addresses both tests. Moreover, after analysis of the results in the diagnosis assessment, some modification was made on the items, so that it is more reliable and valid for the experimental analysis.

3.3.1. Test items

With regard to the type of items, both closed-ended (multiple-choices) and open-ended (or workout) items were included. Both types of items have their own advantage and disadvantage. For instance, Cai (1997) describes multiple-choice items allowing collecting a large amount of data quickly, administering more items in a short period, and score students' response quickly and reliably. However, it does not allow knowing how students arrived at the answer. Thus, the answer could be correct for the wrong reason. Open-ended items, on the other hand, are preferable as it tells not only students final answer but also how they get the answer (ibid). However, it is challenging administering more items or for large sample size in a short period, and score students' response quickly and reliably. In this study, a combination of both multiple-choice and open-ended items was used with caution to minimize their limitation as described below. The work done is influenced by the methods of analysing student succession on a performance task “analysis of reasons” and “analysis of errors” as described in Messick (1988, p. 87).

Each multiple-choice item has two parts to choose the correct answer from the given five alternatives and to give justification for the choice of an alternative. In most cases, the distractors are designed to inform a specific form of knowledge about a concept. For instance, in item 2.1, none of the first four alternatives is correct, and they indicate a specific form of conception about the limit at a point. Table 5 presents this item and the corresponding interpretation of the distractors.

Table 5: Interpretation of distractors in item 2.1

Which one of the following is true?	Interpretation
limit value is a number beyond which a function cannot attain values	limit is a boundary
limit is a number that the function value approaches but never reaches	limit is unreachable (and hence, not a static object)
limit value is an approximation that can be made as accurate as you wish	limit is an approximation
limit of a function is the value of the function at the limit point	limit is a substitution
none of these is true	Good conception. But has to be evaluated based on the explanation she/he provided
Explain why. . . .	

For quantitative interpretation of students’ performance on the test, only the correct choices were counted and have two marks each. Then, triangulation of these choices was made with “explain why” part to see the true nature of the conception reflected. This is due to the nature of calculus in which correct answers may be obtained for wrong reasons.

The workout part was scored using a rubric developed for this purpose. The items were designed to see students’ conceptual knowledge beyond just regurgitating procedures. Most of the items were taken from previously conducted research papers, books, and standard exams. But, most of them were modified through the multiple validation process- divided, merged or reshaped as per the feedback collected from experts and pilot tests.

3.3.2. Expert validation and pilot test result

Initially, 31 items (18 closed-ended, i.e. 11 multiple-choices and seven true or false and the remaining 13 open-ended/workout) have been selected. Informed by:

- literature,
- comment form panel of experts, and
- pilot tests the quality of the items was improved.

A pilot test of the items was conducted with students in a private school in the study area. The pilot test was conducted in two rounds. One intact classroom student (they were 27 in number.) in the first round and another intact classroom (they were 31 in number.) in the second round 58 students participated. The aim of the pilot tests had been to get feedback about the items before they were used in the study. The changes made on the items based on the feedback from the pilot and experts were discussed in the following paragraphs. To present the discussion in a reader-friendly format, the following categorizations were used limit of sequences, the limit of functions, continuity of functions, and derivative.

The limit of sequences part initially has six items (thee closed-ended/multiple-choice and three open-ended/workout). The three multiple choose items (item1.1-1.3) were taken with only little modification on the format and one new item was added. The added item (item 1.4) is designed to address the issue of multiple representations. Only one of the workouts items (item 1.5) was taken, and the remaining two items were removed as the other items address their purpose. For instance, one of the removed items was the item asking to find the limit of the sequence $\boldsymbol{a_n} = (\mathbf{1} - (\frac{\mathbf{-1}}{\boldsymbol{n}})^{\boldsymbol{n}})$ which was intended to address the issue of alternative sequence. Now, this purpose was addressed by item 1.4(c).

The limit of function part initially has 12 items (three multiple-choices, five true-false, and four workouts). Two of the multiple items (item 2.3 & 2.4) were taken with little modification, one item is removed, and one new item (item 2.5) was added to address the linguistic issues. As informed by the literature, students confuse the terms undefined, does not exist, indeterminate, and infinity. Thus, the new item was intended to confirm this.

The true-false items were converted to two multiple-choices (item 2.1 & 2.2). From the four workout items, one item (item 4.5) was modified so that it accommodates the purpose of one item from the limit of a function and one item from the application of derivatives (see Table 6).

Table 6: Former description and the two items that were incorporated with item 4.5

4.5. The percent of concentration of a certain drug in the bloodstream t hours after the drug is administered is given by the function $f(t) = \frac{5t}{t^2+1}$. Then 4.5a. Evaluate $\lim_{t\to\infty} f(t)$ and interpret this result. 4.5b. Find the time (in hours) at which the concentration is a maximum, and 4.5c. Find the maximum concentration.
…The concentration C of a drug in a person's bloodstream t hours after it was injected is given by $C(t) = \frac{0.15t}{t^2+75}$. Then $\lim_{t\to\infty} C(t) =$ ___________ Interpret this result
… $\lim_{x\to\infty} \frac{3x^2-4x+5}{x^2+3} =$ _________
…What is the maximum value of $f(x) = 2x^2 - x^4 - 4$ on $[0,2]$? A. -3 D. 12 B. 3 E. has no maximum value C. -4 Why do you think so? ____

The continuity part initially has six items (two multiple-choice, two true-false, and two workouts). One of the multiple-choice items (item 3.3) was taken as it is. One of the observations during the pilot test was that it was hard to analyse students' responses for open-ended items as their response was too diverse and the sample was large in number. Based on this observation instead of open-ended, options were provided so that students select the one they think is the right answer. With this consideration, an item (item 3.1) replaced one of the open-ended items with the opportunity to choose from in order to ease the process of analysis. Item 3.2 was developed based on the two true-false items since the multiple-choice items were observed better to address the intended purpose than true-false items. It is observed that the true-false item has less discrimination power[7].

One of the workouts items (item 3.4) was modified to accommodate the purpose of one remaining multiple-choice item. Table 7 presents the modified item (item 3.4) and the former version of this item and the multiple-choice item removed since the purpose is incorporated in this item respectively.

Table 7: Former version and an item incorporated with item 3.4

3.4. Consider the function $f(x) = \frac{2x^2-x-15}{x-3}$ 3.4a. Sketch the graph of f (discuss basic steps of the graph). 3.4b. What can you say about the continuity of the function exactly at $x = 3$? (say continuous or discontinuous.). 3.4c. Does the function have a limit value at $x = 3$? (yes /no) (underline your choice). 3.4d. If you answered in 3.4c above is yes, what is that limit value? 3.4e. Compute *f* at $x = 3$
… Sketch the graph of the function $f(x) = \frac{x+3}{x^2+3x}$ and answer the following questions. What happens to the graph of f at the point $x = -3$? _______________ What is the limit of f at $x = -3$? _________________ What is the value of the function at $x = -3$ i.e. $f(-3)$? _______________ Is the function (continuous/discontinuous) at the pint $x = -3$? _________
… Let $f(x) = \frac{x^2+\mathrm{x}-2}{1-\sqrt{x}}$ then $\lim_{x\to 1} f(x) =$ ________ A. 6 B. ∞ C. -6 D. does not exist E. -5

____

[7] See Karelia, Pillai and Vegada (2013)

The derivative part initially has seven items (three multiple-choices and four workouts). All the multiple items (item 4.1 to 4.3) were taken without any change. Based on the pilot test result, one of the workouts items (item 4.4), was modified to reduce the number of algebraic operations without affecting the intended purpose to be addressed (see Table 8). The item was intended to address the issue of the chain rule. The item has a low correct response due to the algebraic manipulation errors.

Table 8: Item 4.4 and its former description

Differentiate $y = tan^2(3x + e^{\sqrt{x^2+1}})$
Differentiate $y = \sin(e^{\sqrt{x+1}})$

One item (item 4.6), taken with little modification and one other item (item 4.7) was completely replaced due to its low discrimination power. Based on the comment from the panel of experts, and the literature the newly added item (item 4.7) is given in graph to address more multiple representations.

Finally, 21 items (15 multiple-choices and 6 workouts), were selected for final administration (see appendix D). All of the items were adapted from different sources. Accordingly, item 1.1 & 2.3 are adapted from (Areaya & Sidelil, 2012). Similarly, items 1.2, 1.4, & 2.5 are also adapted from (Moru, 2006). Likewise, items 1.3, 1.5, & 4.5 are adapted from (Chung, n.d.). In the same way, items 2.1 & 3.4 are adapted from (Jordaan, 2005). Alike, items 2.2, 3.1 & 3.2 are adapted from (Wangle, 2013); items 2.4 & 4.6 are also adapted from (Bezuidenhout, 2001). Correspondingly, items 3.3, 4.1 & 4.7 are adapted from (Rabadi, 2015). Again, item 4.4, item 4.2, and item 4.3 are adapted from (Jojo, 2011), (GRE, 2008), and (IER & AAU[8], 2015) respectively.

The purpose of item 1.1 was to establish students' knowledge of the definition of terms and the relations and conditions among these terms. Item 1.2 was aimed to determine students' computational ability of convergence of different types of sequences. The difference between item 1.2 and item 1.4 is form of representations.

[8] Institute of Educational Research (IER), Addis Ababa University (AAU)

Triangulation of the two-items result gave an opportunity to establish students' abilities in multiple representations and how consistent their knowledge is. The purpose of item 1.3 was to examine students' ability in visualization and coordination of processes.

The purpose of item 2.1 and 2.2 were to examine students' concept images of the limit of functions. The distractors were designed to accommodate frequently occurring alternative conceptions as described in the literature. Item 2.3 was aimed to examine students' knowledge of the non-existence of a limit at a point. Item 2.4 is also aimed to establish students' knowledge of the relationship between limit value and function value and the existence of the limit and continuity of functions. Item 2.3 and 2.4 were designed to observe if students are able to interpret the symbolic expression of limit. Item 2.5 was aimed to establish students' linguistic ambiguity in a limit. It also reveals more about students' algebraic manipulation skills.

The purpose of item 3.1 was to establish students' concept image of continuity. The item was designed to incorporate domain-continuity, limit-continuity, and continuity-connectedness interplay. Item 3.2 was also designed to establish more on the interplay between continuity and the other concepts in calculus differentiation, limit, and being defined. The purpose of item 3.3 was to establish how students understand continuity in the subject matter of limit. In addition, the item was aimed to see students' ability to compute the one-sided limits.

Item 4.1 was aimed to establish students' visualization of the derivative. Besides, it aimed to see computational ability on procedures of the derivative. Items 4.2 and 4.3 were designed to see students' knowledge of the conceptual level and how it goes beyond algebraic manipulation. Moreover, item 4.2 demanded reverse thinking, whereas, item 4.3 addressed students' ability to form networks of concepts the limit, continuity, and derivatives.

On all these multiple-choice items, besides the purpose in the objective part as explained above, was intended to establish students' ability to explain and justify the reasoning employed together with the ability to communicate in written their mathematical knowledge in a coherent, consistent, and flexible mathematical practice.

The purpose of item 1.5 was to dig students' representation of the limit (dynamic-static interplay), co-variation, and infinity (actual or potential). The main purpose of item 3.4 was to see how students treat points of discontinuity both algebraically and graphically. On the way to attain this purpose, it also helped to explore students' ability on algebraic manipulations, the existence of a limit at a point where the function is undefined, and how they relate limit and the function values. The purpose of item 4.4 was to explore how students' understand the chain rule and their computational ability on rules and procedures of the derivative.

The main purpose of item 4.5 was to see how students extend their knowledge on limit and derivative to a real-life problem. On the way to attain this purpose, it also helped to establish students': concept image of infinity, knowledge of coordination of processes, the nature of their limit conception, and knowledge on rules and procedures of the derivative. Item 4.6 is aimed to see how well students' knowledge structure is synchronized. It addresses the issue of integration among concepts in calculus, i.e. the limit, continuity, and derivatives. It also addresses the issue of representation forms and symbolic interpretation. Item 4.7 is designed to address three purposes- to see students' knowledge on continuity in a closed interval, how they interpret the meaning of derivative of a function at a point, and how they relate continuity and differentiability at the same point. All the open-ended items were labelled as object-level conception demanding of the respective concepts.

Finally, appropriateness of language, the time frame of the test, and workspace, level of difficulty, and discrimination power about each item was addressed based on the feedback from both pilot tests and expert's comment. While eighteen items were used for the diagnostic assessment (see appendix D), twelve items were used for the post-test (see appendix F).

Similarly, the pre-test items were passed through the same process of a pilot and validation. From initially identified 30 items (most of them taken from EUEE[9]), through validation and pilot test 25 items (function, sequence, geometry, algebraic computation, and application problems) had been selected and was used for final administration (see appendix E).

3.3.3. Validity and Reliability of the test

One dimension of research quality is validity and reliability of the instrument used to collect data because the conclusions draw is based on inference from the data collected. While validity points to whether a research instrument explores what is proposed to be examined, validation is the process of assuring whether the instrument really supplies such inferences (Fraenkel & Wallen, 2009). Reliability is an investigation of how consistent results are. In this study, validity covers the two types: content and construct, whereas, issues of reliability cover the two types: inter-rater reliability and internal consistency reliability.

Content validity

Content validity refers to whether the scores from the instrument show that the test's content narrates what the test is proposed to assess (Creswell, 2012). The most customary method to secure content validity is to apply expert validation (Creswell, 2012; Fraenkel & Wallen, 2009). In this study, there are two pieces of evidence of content validity. The first evidence is that the items were drawn from prior research measuring student difficulties and understanding in the limit, continuity, and derivative. The second evidence is the judgement of experts. A panel of four experts, one grade 12 mathematics teacher who has extensive experience in teaching calculus and is also recognized as the best performing mathematics teacher in the study area, one university mathematics lecturer who has been a tutor for over four years in grade12 mathematics students in a private school, and one mathematics education PhD candidate in Addis Ababa University was participated.

[9] Ethiopian University Entrance Examination

In addition to the comments from the experts, the pilot test was also used to shape content validity concerns such as appropriateness of language, the time frame of the test, and the workspace. Furthermore, items were designed to cover all specific topics in the scope of the study: limit of sequences, limit of functions at a point, limit involving infinity, non-existence case of limit of functions, rational, exponential and trigonometric and piecewise-defined functions, continuity at a point, continuous functions, derivative of simple, compound and composition functions in different forms of representations: symbolic, table, graph and verbal descriptions.

Construct Validity

This type of validity stands to check how a test evaluates the construct it intended to measure (Fraenkel & Wallen, 2009). A construct is a trait, expertise, ability, or skill that exists in the mind of an individual and is defined by recognized theories. In this study, the term “construct”, points to any form of students’ mental image (strong or weak), about concepts in calculus. Thus, construct validity is necessary for assuring that the instruments used in the study accurately measured the constructs of conceptual knowledge so that specific difficulties and strength of students’ knowledge can be identified.

To address this purpose of the test, first, the construct of conceptual knowledge was clearly defined (As described in section 2.2.5), followed by a well-defined rubric (see appendix G), that was aligned with the relevant working definitions. The rubric for each open-ended item consists of potential student responses that indicated a particular level of conceptual knowledge which in line with the working definition. Besides the two efforts, the feedback from the panel of experts and pilot test was also used to ensure contract aspect of validity.

Reliability

Reliability refers to the uniformity of scores from repeated administration of an instrument (Fraenkel & Wallen, 2009). Two different types of reliability are relevant to the study: Inter-rater reliability and internal consistency reliability. An instrument is said to have Inter-rater reliability if two or more independent scorers consistently assign the same scores to the same responses. For this study, a rubric was developed to guide rating. The rubric is designed based on the definitions for the constructs in this study and the experience obtained from the review of the literature. The rubric is also tested during the pilot study. Fifteen test papers from the pilot study participants were duplicated and rated by two individuals. The scores were compared and inconsistencies were discussed until we reached an agreement.

Internal consistency reliability stands for whether two or more items on the same instrument measuring the same construct give up reliable results. This kind of reliability was recognized during the pilot study. Participants tended to answer similar questions in the same way during the pilot study, describing that the instruments had internal consistency reliability. In addition, the triangulation done by using the items within the multiple-choice and between the multiple-choice and the closed-ended items during the two-phase pilot revealed a reliable result. Moreover, the internal consistency of the pilot test was measured using Cronbach's alpha. Accordingly, $\alpha = .763$ was obtained which is acceptable (of course less), for the diagnostic assessment and $\alpha = .766$ for the pre-test.

3.4. The intervention

Based on the proposed model (see section 4.2.2 and figure 32), an intervention was designed. The intervention includes arranging the teaching-learning environment according to the proposed criteria and working on sets of activities. The activities aimed to encourage attaining the constructs of conceptual knowledge specified in the proposed model and to lift students' knowledge to a higher-level aspect of mathematical thinking which in turn reduces observed difficulties and enhances conceptual knowledge. The term "activity" refers to an open-ended or closed-ended item of a classroom, homework and formative assessment tasks, which the students are asked to work on either on their own or in a group at the end of the teachers' conventional introduction of each concept. The activities are compiled together and quoted as an "activity sheet".

The activities are designed for these concepts- limit of sequences, the limit of functions, continuity, and derivatives. Most of the activities were selected from previous study instruments, national exams, and books and some of them were designed by the researcher. Of course, even for those taken from the literature, all of them were modified to fit the intended purpose.

The purpose of the activities was addressing observed difficulties, so that students enhance their conceptual knowledge. The items were collected based on the required constructs of conceptual knowledge and content of grade 12 mathematics syllabuses. With regard to the type of items, the activities consist of both open-ended and closed-ended. But the closed-ended items also ask not only selecting the correct answer, but justification why a certain alternative is selected. The items also include scripts from students' work. This is deliberately done so that students exercise how to "analyse errors" and think of their own thinking.

A month before the intervention, three-day training was provided to 21 selected upper secondary school mathematics teachers by the researcher, in collaboration with the researchers' employ University and the zone education department. There were 21 participants (18 males and three females). In the training entitled, "error analysis: a tool to enhance students conceptual knowledge", issues like assessment practice, common student errors, feedback as a pedagogical tool, constructs of conceptual knowledge and mathematical thinking practice, was presented. The experimental group teacher was part of the training. Besides the training, an individual orientation and subsequent discussions were conducted with the teacher, so that the intervention was implemented as intended.

The intervention was administered for eight weeks, 80 minutes per week running parallel to the normal teaching-learning program. In the intervention session, students were arranged in mixed ability groups of five to six. After the first, the sessions were arranged as group work, presentation, reflection on the presentation and stabilization, group discussion and homework for the next class meeting. A week after the intervention was terminated; the post-test that aimed to examine students' conceptual knowledge in calculus was administered.

3.4.1. Expert validation and pilot test of the items in the activity sheet

Initially, 35 activities were selected. Informed by the comments from a panel of experts and pilot tests, the qualities of the items in the activities were improved. First, a pilot test of the items was conducted with incoming first-year mathematics department students at a University. Twenty-eight students (12 males and 14 females) participated. The purpose of the pilot test had been to obtain feedback about the items before they were used in the study. During the pilot, the researcher observed students doing the activities to assess the quality of the items in the activity sheet. The researcher's observation was focused on whether the activities are appropriate for the intended method of instruction (individual work, group work, qualitative description, quantitative description), encouraging or not, helped to construct the intended components of conceptual knowledge (interiorization, encapsulation, and coordination), and whether the language of the items and the instructions of the activities are clear and understandable. Based on the experience gained adjustment was made on the time frame, work-load and level of difficulties on each item.

Besides the pilot test, the judgement of experts was also implemented to improve the quality of the activities. A panel of four experts- two grade 12 mathematics teachers, who have extensive experience teaching calculus and two university mathematics lecturers (who have masters in mathematics education) have participated. Based on the feedback collected, some of the activities were modified, some of them were removed and some new activities were added.

Finally, 30 activities were selected for final administration (see appendix H). In the development of the activity sheet, different sources were used. Although the present description of some items may not be the same as to the description in the sources, the beginning sources are the following: Areaya & Sidelil (2012); Bezuidenhout (2001); IER and AAU[10] (2013, 2014, 2015, & 2016); Jordaan (2005); Maharaj (2010); Moru (2006); Rabadi (2015); Wangle (2013).

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[10] Institute for Educational Research (IER) and Addis Ababa University (AAU)

The purpose of activity one to five is to establish students' conceptual knowledge in the limit of sequences. The activities were focused on overgeneralization, conflicting concept image due to linguistic ambiguity, knowledge of the definition of terms, and the relations and conditions among these terms. Activity six to 19 aims to address the difficulties in limits and continuity. Most items in this section will demand object-level concept formation, reconstructive generalization, and reasoning level problem-solving skills. Activity 20 to 30 is intended to address difficulties in the derivative. In this section too, most items demand object-level conception, reconstructive generalization, and multi-step reasoning level problems solving ability.

3.5. Data analysis

Desktop literature review- the data analysis technique implemented is "thematic synthesis" as suggested by Thomas and Harden (2008, p. 2). After an exhaustive and systematic literature search, the researcher treated each article as a case and analysed in the following steps: quoting of difficulties this includes mentioned errors, ways of thinking or alternative conception/misconception. Then, triangulating the quotations from each article to build initial codes followed by finding for similarity and difference among the initial codes to categorize them in a more general code called second-level codes or "descriptive themes" (ibid). Finally, the difficulties were categorized in more general and meaningful groups called analytical themes. The detail of the analysis procedure is discussed in section 2.1.2 and 2.1.3.

Diagnostic assessment- to analyse the test results in the diagnostic assessment: first, respondents scripts for each item were categorized as correct, incorrect and no response. Second, for each item, the respondent errors were identified by looking for the wrong choice or wrong working from the respondent scripts for each item. Since these wrong answers constitute ways of difficulties and origins of difficulties and approaches that they employed, the data were read repeatedly to get an overall picture of the type of difficulties that respondents have and to make themes. The result was used to answer the second research question.

Pre-test: the pre-test was aimed to examine the comparability of the students in the experimental group and the control group before the intervention was carried out. To do this, first, respondents scripts for each item was categorized as correct, incorrect, and no response. Then, by counting the frequency of correct responses for each student, the total score was recorded and an independent t-test was used to compare their mean score. The result revealed no significant difference in the pre-test between the two groups i.e. they are comparable. This result was used to determine the option of data analysis for the post-test result. Since the two groups have no significant difference in the pre-test, a simple independent t-test was used in the post-test. If that were not the case, ANOVA would have been used.

Post-test: The purpose of the post-test was twofold. The first was to analyse the possible effect of the intervention model based on students' performance on the test. The second was to examine the extent the model helped to reduce observed difficulties in calculus. Thus, the analysis involved both quantitative and qualitative parts. In the quantitative analysis, after frequencies and pattern coding, correct response scores were added for each student. The scores were analysed using the t-test for independent groups to determine whether there is a significant difference between the mean scores of the experimental and the control groups. This analysis was aided by SPSS of version 25.

For the quantitative part, text analysis, in which one glances for the occurrence or non-occurrence of themes, was implemented (McKee, 2001). Thematic text analysis starts with pre-set themes; in this case, the themes that were identified in the first phase of the study were used.

3.6. Context and limitation of the study

In the Ethiopian educational structure, secondary education is four years in duration. Grade 9 & 10 (general secondary education) enables students to identify their area of interest in further education, specific training, and for the world of work. Grade 11 & 12 (upper secondary or preparatory program), will enable students to choose areas of training, which prepare them adequately for higher education and the world of work (FDRGE, 1994). In the preparatory program, students will be assigned to natural science and social science streams (SSS) according to their preference. While those in NS stream are allowed to join medicine, computational science, engineering, and technology fields at University, those in SSS are allowed to join social science and humanities fields. NSS students were targate population of the study.

Students' difficulties in learning and understanding concepts in calculus can be studied from different aspects. This study, however, was limited to the cognitive aspect. Grade 12 government schools NSS students were targeted. From the researcher's experience, the context of NSS and SSS is quite different as far as attitude, background, and futurity concerned. With regard to topics in grade 12 calculus, the study emphasized the limit concept, since the concepts in calculus are sequential and the limit concept is basic for the rest concepts in calculus. Specifically, the limit of a sequence, limits of functions, continuity, and derivative was included.

3.7. Ethical issue

In the study area, secondary schools are under the direct leadership of the Zone education department. The top decision-maker at the school level is the school director. The researcher has requested and got permission to conduct the study from the zone education department (see appendix I). Having the letter of permission, the researcher made a visit to all the schools and has contacted school directors and discussed the issue. After the directors, the researcher has also discussed with all grade12 mathematics teachers in each school at the department level. The researcher requested for ethical clearance and obtained approval from UNISA (see Appendix J). Then the sampling was preceded with those who volunteered to participate. Those teachers, who were selected for the study, have signed a consent letter. Students also attained the necessary orientation and have signed the approval. To protect the identity of the participants', codes (S_i) were used instead of their actual names and location. The final write-up of the thesis has been checked for similarity index using the turn-it-in software (see Appendix K for the first page of the report).

CHAPTER FOUR: RESULTS

The main purpose of the study was to synthesize students' difficulties and common conceptual issues that cause those difficulties and to design an intervention model based on those difficulties that will enhance students' conceptual knowledge. To attain these purpose multi-level studies were performed.

This chapter presents the results of the study. First, the results of the diagnostic test conducted to investigate the common conceptual issue of students' difficulties in calculus were presented in five sub-sections. Accordingly, the first section (4.1.1) presents students' level of conceptual knowledge in the limit of sequences followed by the limit of functions (4.1.2). While the next two sections present students' level of conceptual knowledge in continuity (4.1.3) and that of derivatives (4.1.4), the last section (4.1.5) presents the concluding remarks drawn from this sub-study. The result in this section, besides answering the second research question, paves the path to designing a framework of overcoming difficulties in learning calculus concepts.

Having the conclusion from the first sub-study, the next question to be answered is that "what components should be incorporated into the current practice so that students overcome observed difficulties and attain better conceptual knowledge?" Towards answering this question, the second section of this chapter contains three sub-sections. The first sub-section (4.2.1) presents constructs of conceptual knowledge that could be performed so that students enhance conceptual knowledge and consecutively overcome observed difficulties. While the second sub-section (4.2.2) presents the framework as an intervention model, the third sub-section (4.2.3) presents an intervention based on the proposed model. Finally, the third section of the chapter presents the possible effect of the proposed model in two sub-sections. While the first sub-section (4.3.1) presents a comparison of means on the two groups, the next sub-section (4.3.2) presents a text analysis of the result in order to see the possible effect beyond statistical manipulations.

4.1. Students' Level of Conceptual Knowledge

4.1.1. Students' conceptual knowledge of the limit of sequences

Section one of the test was designed to determine how students conceive the limit of sequences. The section composed of four closed-ended items and one open-ended item. On the closed-ended items, the choice of each distractor has an implication on students' concept image and level of conceptual knowledge. Each of the concept images that students possess is discussed in more detail below. Table 9 is a summary of the response for the first item on the limit of the sequence.

Table 9: Breakdown of students' choices to item 1.1

Frequency, N=238	A	B	C	D	E	None-respondent
N	24	28	116*	35	29	6
%	10.0	11.8	48.7	14.7	12.2	2.5

* correct answer of the item

In item1.1, the statement in option C is correct, whereas, options A, B, and D are distractors that were arrived at due to overgeneralizations or conflicting concept images in the limit of the sequence. Referring to Table 9, 116 (48.7%) students got the correct answer choice C, while the remaining 116 (48.7%) did not get the correct choice of this item. Six (2.5%) of them left the item unanswered. Though the item was closed-ended, students were asked to write their reason for the choice. These reasons provided students' difficulties in understanding and using technical terms and how their knowledge is disorganized. Most of them prefer to give an example than justification. Accordingly, some of the reasons given to their answer were the following:

Reasons that imply strong conceptual knowledge behind the correct choice:

- we can't find a sequence which is convergent but unbounded (six respondents),
- convergence implies being bounded but being bounded does not imply convergence (four respondents),
- because any sequence that is convergent to a number S is bounded e.g. $\boldsymbol{a_n} = \frac{\mathbf{1}}{\boldsymbol{n}}$ then $\mathbf{0} \leq \frac{\mathbf{1}}{\boldsymbol{n}} \leq \mathbf{1}$ (three respondents),

Reasons that imply weak levels of conceptual knowledge behind the correct choice:

- a convergent sequence converges to its lub or glb (the limit value is necessarily a boundary, five respondents),
- a divergent sequence may not be bounded (three respondents),
- a sequence is convergent only if it is bounded and monotonic (nine respondents),
- we have so many examples which show a convergent sequence is bounded (seven respondents; in particular, [11]S_{62}: $a_n = \frac{n}{3n+1}$, S_{09}: $a_n = 3 - \frac{1}{n}$, S_{142}: $a_n = 2 + \frac{3}{n}$, S_{211}: $a_n = 3 - \frac{2}{n}$, S_{37}: $a_n = \frac{2n+3}{2+3n}$),
- e.g. $a_n = (-1)^n$ (six respondents).

In particular, figure 5 is a direct copy of the students' test script from the correct choice followed by correct reasoning category:

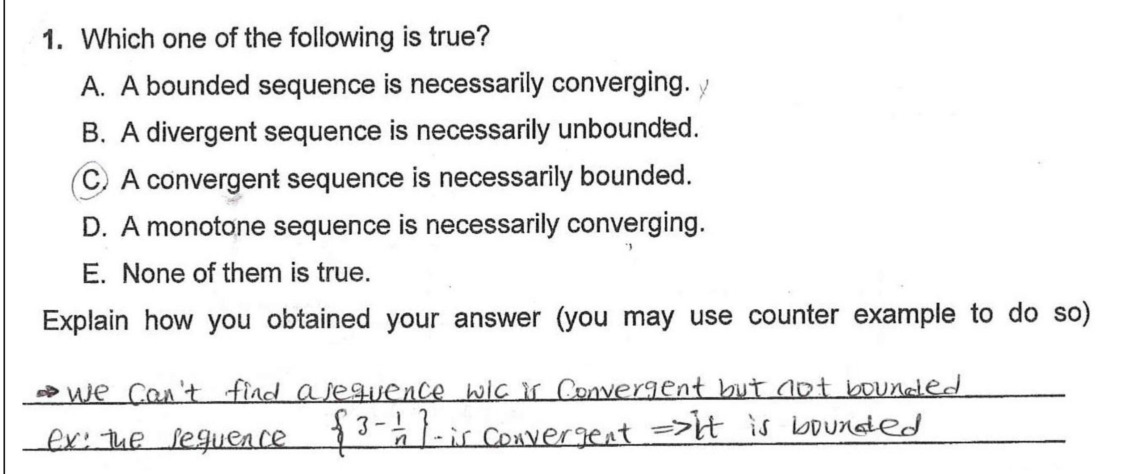
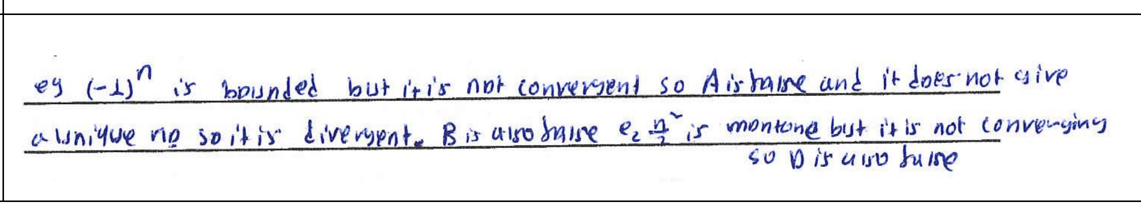
S_{58}	 1. Which one of the following is true? A. A bounded sequence is necessarily converging. B. A divergent sequence is necessarily unbounded. C. A convergent sequence is necessarily bounded. D. A monotone sequence is necessarily converging. E. None of them is true. Explain how you obtained your answer (you may use counter example to do so) we can't find a sequence wic is convergent but not bounded ex: the sequence $\{3-\frac{1}{n}\}$ is convergent => it is bounded
S_{11}	 eg $(-1)^n$ is bounded but it's not convergent so A is false and it does not give a unique no so it is divergent. B is also false eg n^2 is montone but it is not converging so D is also false

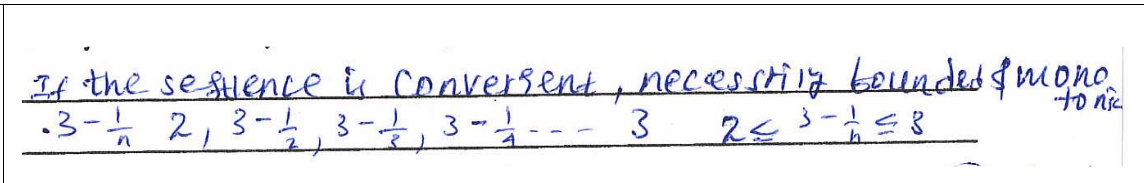
Figure 5: Correct choice of option and reasons for item 1.1

Only some students have a deep knowledge of the limit of a sequence, large example space, and are able to explain in detail (as in S_{11} in figure 5). The above-

[11] S_i where $\{i \in N/1 \leq i \leq 238\}$ refers to respondents identification code

mentioned list of reasons suggests that some other students got correct answers for the wrong reasons. For instance, S_{25} and S_{05} (see figure 6) show how students misinterpreted the “monotonic-bounded theorem”.

S_{25}



S_{05}

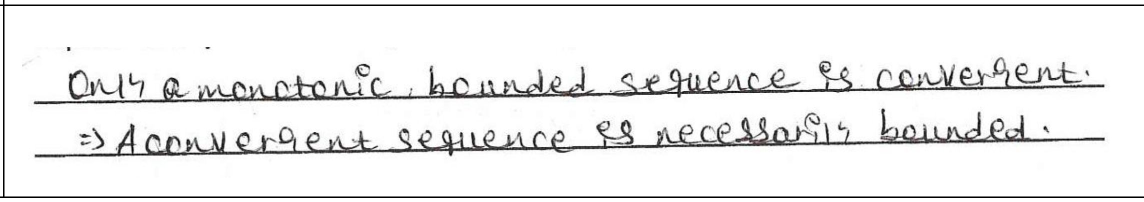
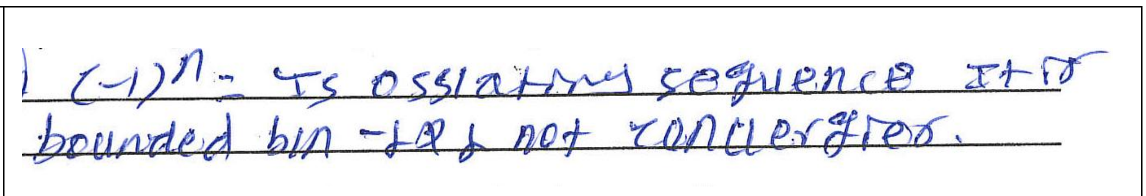


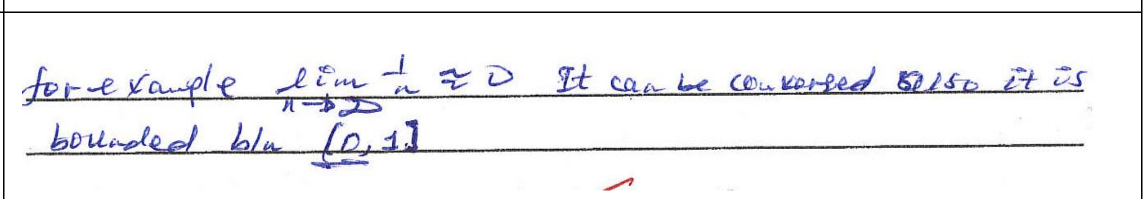
Figure 6: Unrelated reasons for option C of item 1.1

Besides misinterpretation of this theorem, as seen in the list of reasons, most students confused terms such as convergent, divergent, bounded, and unbounded. Here also most students preferred to mention a particular sequence instead of justifying the general pattern of the given statement. Figure 7 is a list of descriptions given by some students who chose option C for item 1.1, as directly taken from students’ script:

S_{23}



S_{217}



S_{27}

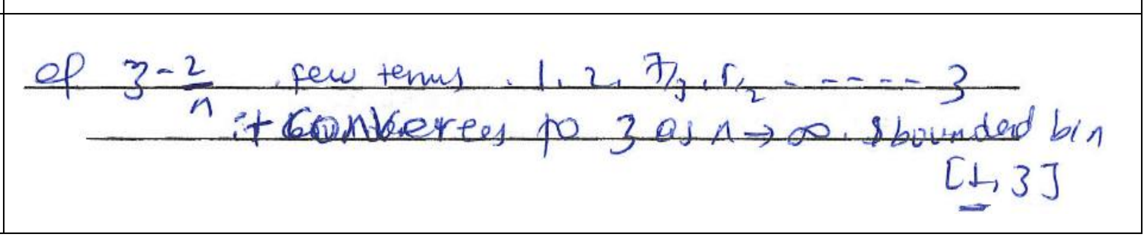


Figure 7: Unrelated reasons for option C for item 1.1

Again using the data in Table 9, the percentage of choice A suggests that 24 (10.0%) respondents concluded that a bounded sequence is necessarily converging, whereas they refused to choose C means they think that a convergent sequence may not be bounded. While figure 7 is a direct copy of reason from their scripts, most of their reasons for the choice are categorized as follows:

- a bounded sequence converges to its lub or glb (5 respondents),
- a bounded sequence is convergent e.g. $a_n = \frac{1}{n}$ then $0 \leq \frac{1}{n} \leq 1$ (three respondents).

S_{30}	b/c All boundes sequence is converging but All convergent sequence is not bounded e.g 1-1/n
S_{57}	B/t any no sequence's bounded also converges to someone of sequences.
S_{31}	when the value converge it must be bounded but it does not converges the given function cannot be bounded.

Figure 8: Some reasons for option A of item 1.1

The percentage of choice B suggests that 28 (11.7%) respondents concluded that a divergent sequence is necessarily unbounded. Here again, refusing to choose C means they think that a convergent sequence may not be bounded. Some of their reasons for the choice are categorized as follows:

- a convergent sequence is bounded (three respondents),
- there is no sequence which is divergent and bounded (five respondents),
- e.g. $\{-\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}, \ldots\}$ is not convergent because it is unbounded (two respondents),
- all bounded sequences are convergent (three respondents),
- only a monotonic and bounded sequence is convergent (three respondents).

The percentage of choice D suggests that 35 (14.7%) respondents concluded that a monotone sequence is necessarily converging whereas a convergent sequence may not be bounded. Once again, some of their reasons for the choices are categorized as in the following two statements. Figure 9 is a direct copy of reasons from their scripts:

- a monotonic sequence is always convergent (11 respondents),
- a monotonic sequence is bounded, so, it is convergent (eight respondents).

Finally, the percentage of choice E suggests that 29 (11.7%) respondents concluded all the given four statements are false.

S_{203}	The monotone sequence is the one side converged (bounded sequence) so it is necessarily convergent.
S_{226}	Monotense sequence is approaches to 0 Unbound sequence is approaches to 1
S_{14}	Because sequence is convergent it must full fill monotonic Ex $\frac{1}{n}$ => converges to 0 so it is monotonic

Figure 9: Reasons for option D of item 1.1

Generally, students' performance in the first item revealed that most of them lack conceptual knowledge in the limit of sequences, which largely originates from a misinterpretation of the "monotonic-bounded" theorem and lack of having a clear distinction between terms. According to this theorem, while a monotonic and bounded sequence is necessarily convergent, the converse may not be true. As seen from the qualitative aspect of students' responses, most students have misinterpreted this theorem. Besides, students focus on particular examples rather than a general posture of facts about a concept.

In item 1.2, the sequence in option C is not convergent as it oscillates between $\mathbf{1}$ and $-\mathbf{1}$ whereas the three sequences in option A, B, and D are convergent. Those who chose the options A, B, and D, fail to interiorize a process into an object. Table 10 is a summary of the students' answer to item 1.2.

Table 10: Breakdown of students' choices to item 1.2

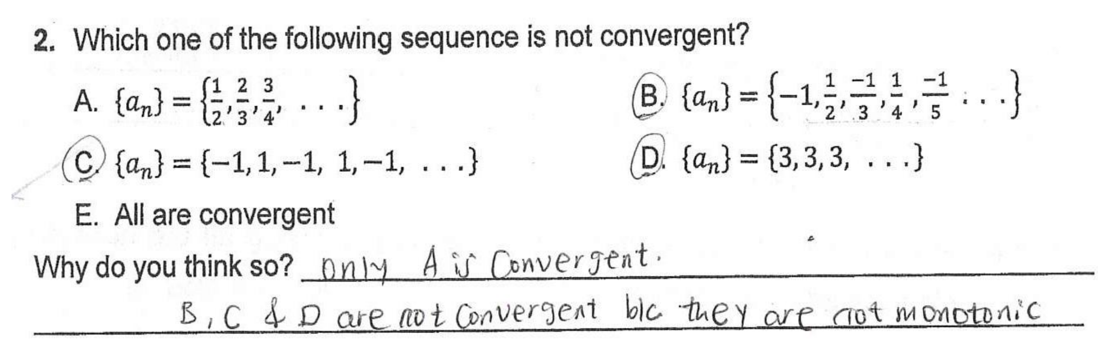
Frequency, N=238	A	B	C	D	E	Two options (bad index)	Non-respondent
N	8	18	152*	27	27	3	3
%	3.4	7.6	63.8	11.3	11.3	1.2	1.2

* correct answer of the item

Referring to Table 10, 152 (63.8%) students got the correct answer choice C while the remaining 83 (34.9%) did not get the correct choice and the remaining three (1.2%), refused to choose none of the options. Even though the item has a large number of correct respondents, the qualitative aspect has an immense implication on students' nature of conceptual knowledge. Out of the 235 (98.8%) who selected an option of the item, 123 (51.7%) of them gave clearly readable reasons for their choice of the option, and this is the highest among all the items in the test. Table 11 summarizes the five options and the corresponding reasons for the choice.

The data in Table 11 agrees with the conclusion drawn in item 1.1. Even 42 (17.7%) students have answered the item correctly with wrong reason as seen in option C. In addition, 27 (11.3%) students think that the constant sequence $\{a_n\} = \{3, 3, 3, \ldots\}$ is not convergent. The item also revealed how some students' difficulties are robust since they give the wrong answers and justification with high confidence. Figure 10 is evidence of this as directly taken from one students' test script.

S_{05}


2. Which one of the following sequence is not convergent?

A. $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

B. $\{a_n\} = \left\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5} \ldots\right\}$

C. $\{a_n\} = \{-1, 1, -1, 1, -1, \ldots\}$

D. $\{a_n\} = \{3, 3, 3, \ldots\}$

E. All are convergent

Why do you think so? only A is Convergent. B, C & D are not Convergent b/c they are not monotonic

Figure 10: An extract for the wrong answer with high confidence in item 1.2

Table 11: Students' options and corresponding reasons for item 1.2

Option	Reason	Frequency
A	It is increasing, so it is bounded	3
B	The limit does not approach to unique number	3
	It is not bounded	3
C	A convergent sequence must be bounded	3
	It is not bounded number sequence	7
	It oscillates between -1 and 1	26
	Since it is neither increasing nor decreasing	13
	It does not go to a unique number	16
	Only monotonic and bounded sequences converge	16
D	A constant sequence cannot converge	11
	A constant sequence is not bounded	6
	Since the sequence goes uniformly there is no upper and lower bound so that it is not convergent	3
	Because it is not monotonic	2
E	All are convergent because all are bounded	5
	In particular, the sequence in A converges to 0, B converges to 0, C bounded and D converges to 3	2
	They do not go to a unique number	2
B & C	In both cases limit does not approach to a unique number	3
Total		123

In item 1.3, the statement in option B is correct, whereas, options A, C, D, and E are distractors that were arrived at due to either overgeneralization or lack of encapsulating the process of the limit into an object or lack of visualization beyond action level conception. Referring to Table 12, 118 (49.6%) students got the correct answer options B while the remaining 110 (46.2%) did not get the correct option, the remaining 10 (4.2%) refused to choose any of the options.

Table 12: Breakdown of students' choices to item 1.3 and 1.4

Frequency, N=238		A	B	C	D	E	Non- respondent
1.3	N	37	118*	30	35	8	10
	%	15.5	49.6	12.6	14.7	3.4	4.2
1.4	N	55	84*	35	53	3	8
	%	23.1	35.3	14.7	22.2	1.3	3.4

* correct answer of the item

According to the data in Table 12, from option A, 37 (15.5%) students are at the action level. From option C, 30 (12.6%) students are at a process level but lack encapsulating process into an object, whereas, from option D, 43 (18.1%) students even have not attained action level conception of the limit.

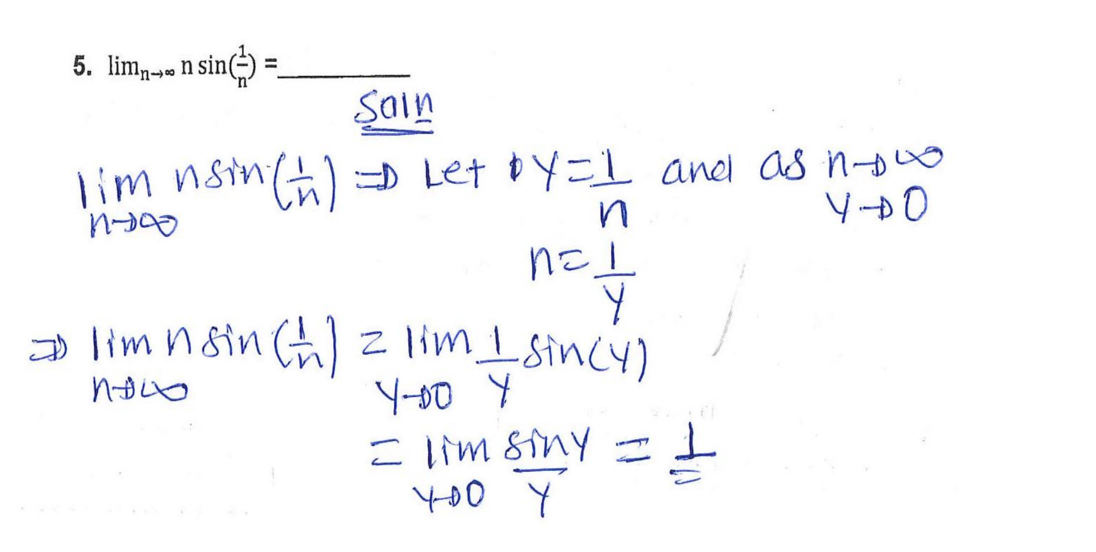
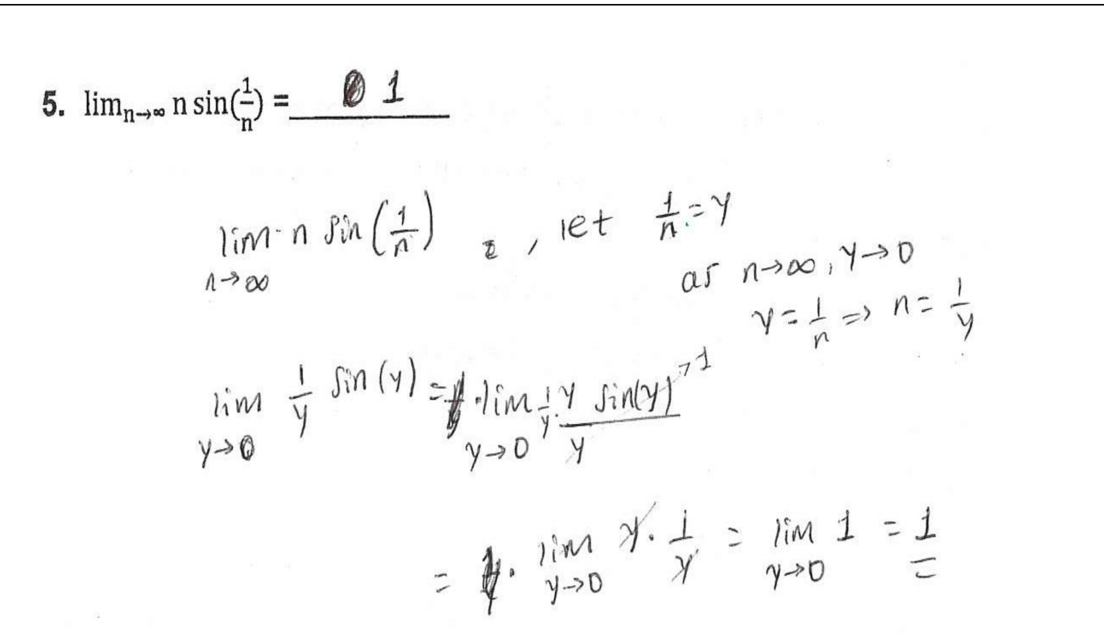
In item 1.4, the sequence in option B (which is option C in item 1.2), has no limit as it oscillates between 1 and -1 whereas the three sequences in options A, C, and D all have limits. Referring to Table 12, 84 (35.3%) students got the correct answer, choice B while the remaining 146 (61.3%) did not get the correct choice, the remaining 8 (3.3%) refused to choose none of the options. In particular, 13% indicated that the constant sequence $\{a_n\} = \{3, 3, 3, \dots\}$ have no limit. Besides, 14.7% of them have developed a generalization that a sequence that involves terms that alternate in the sign is not convergent. The difference between students' correct response to item 1.2 and item 1.4 indicated that students have a lack of using multiple forms of representation or demonstrate different levels of knowledge based on the representation used.

The last item in this section is open-ended, and students were expected to show all the steps to reach the final answer. Table 13 is a summary of the response of this item.

Table 13: Breakdown of students' answer to item 1.5

Frequency, N=238							
Correct		Incorrect		Incomplete		Non-respondent	
N	%	N	%	N	%	N	%
73	30.6	102	42.8	42	17.6	21	8.8

As the item is open-ended, there were different forms of response categories. The frequency of the correct respondents was counted for those who demonstrated all the ideas mentioned in the rubric. Those who demonstrate only partial understanding were considered as incorrect. The incomplete one points to those who started the procedure but left without clearly identified answers. As in figure 11, while S_{03} shows a correct answer with correct procedure S_{58} and S_{125} point to how the correct answer may be obtained from wrong working and S_{14} shows the wrong answers obtained from wrong work respectively.

S_{03}	
S_{58}	

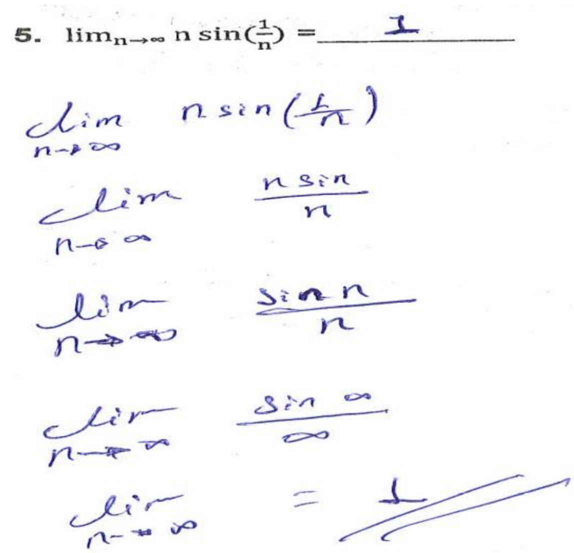
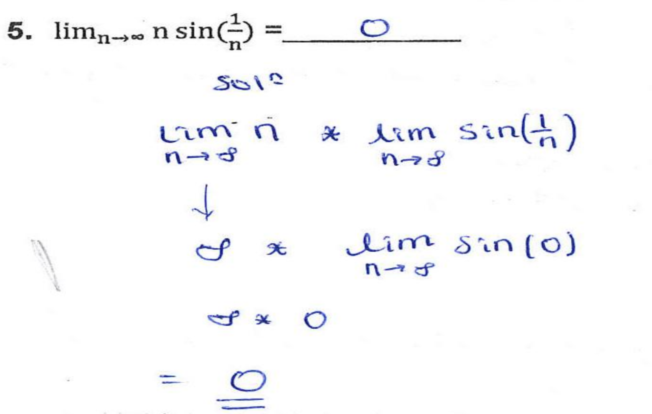
S_{125}	
S_{14}	

Figure 11: Extracts which demonstrate differs forms of difficulties

The frequency of occurrences of the incorrect answers was 0 (74 or 31.0%), ∞ (7 or 2.9%), does not exist (3 or 1.3%), and different Integer values (18 or 7.5%). The common types of difficulties observed in this item are the following (see figure 11):

- lack of symbolic manipulation (like S_{125}, 13 (4.6%) respondents),
- inappropriate interpretation of the limit rules and indeterminate forms (like S_{14}, 18 respondents),
- an action level conception of the limit and infinity (27 respondents).

In general, students' performance in the limit of sequences revealed that only a few students had strong conceptual knowledge. The observed difficulties in the limit of sequences are summarized as follows:

- Evaluate $\infty * 0 = 0$, $\frac{0}{0} = 0$, and $\frac{0}{0} = \infty$, and also consider infinity as an actual value.
- Have an action view and a static way of evaluating functions. For instance, as in item 1.5, 11.3% of participants evaluated the sequence only at the first few natural numbers.
- Think that limit value is necessarily a boundary.
- Have inconsistent concept image due to confusing terms like bounded, convergent, or divergent.
- Think that only monotonic and bounded sequences are convergent, (misinterpret the monotonic bounded theorem).
- Think that a bounded sequence is necessarily converging, and a divergent sequence is necessarily unbounded.
- Think that a monotonic sequence is necessarily convergent.
- Have concept image that a convergent sequence may not be bounded, i.e. being bounded is not a necessary condition for convergence if consecutive terms of a sequence alternate in the sign the sequence is necessarily divergent.
- Think that a constant sequence is not monotonic and hence not convergent.
- Provide the correct answer for the wrong reasons (for instance, 17 participants in the item 1.1 and 39 participants in item 1.2).
- A challenge to interiorize actions into processes or to encapsulate processes into an object.
- Demonstrate different performance based on the form of representations and display conflicting concept image that is dependent on form of representations.
- Make algebraic manipulation errors (like $\frac{siny}{y} = 1$,$\frac{sin\infty}{\infty} = 1$).

4.1.2. Students' conceptual knowledge of the limit of functions

The aim of the second section of the test was to determine how students conceive the limit of functions. The section composed of five closed-ended items. The choice of each distractor has an implication on the students' concept image and level of conceptual knowledge. Each of the concept images that students possess are discussed below in more detail. Table 14 is a summary of the response for these five items on the limit of functions.

Referring to Table 14, 69 (29.0%) of the students got the correct answer choice E for item 2.1. While 160 (67.2%) did not get the correct choice, the remaining nine (3.8%) left the item unanswered. Options A to D are distractors. These are potential to see the existence of immature conceptual structure or conflicting concept images in the limit of functions.

Table 14: Breakdown of students' choices to the items on the limit of functions

Item	Frequency, N=238										Non-respondents	
	A		B		C		D		E			
	N	%	N	%	N	%	N	%	N	%	N	%
2.1	12	5.0	67	28.2	32	13.4	49	20.6	69*	29.0	9	3.8
2.2	22	9.2	29	12.2	19	8.0	67*	28.2	98	41.2	3	1.2
2.3	10	4.2	71	29.8	37	15.5	45	18.9	69*	29.0	6	2.5
2.4	47	19.7	110	46.2	8	3.4	59*	24.8	3	1.2	11	4.6
2.5	28	11.8	73*	30.7	62	26.0	27	11.3	34	14.3	14	5.9

* correct answer of the item

Accordingly, the percentage of choice A to D on item 2.1 suggests that:

- Twelve (5%) of respondents think that limit is a boundary.
- Sixty-seven (28.2%) of respondents think that limit is not attainable.
- Thirty-two (13.4%) of respondents think that limit is an approximation (for instance, S_{03} as in figure 12), and
- Forty-nine (20.6%) of respondents think that limit at a point is the same as the value of the function at the limit point (for instance, S_{16} as in figure 12).

S_{16}	Explain why if the function has a value that value is must the limit point of the function.
S_{03}	Explain why B/se, the limit number is not the exact number but when the function goes to /approaches/ the no as the value as x goes to a specific no

Figure 12: Students' reason for their choice of options in item 2.1

Two major sources of these difficulties are clear from the students' explanations. One is common language interference and the other is the way limit is introduced (Jaffar & Dindyal, 2011; Tall, 1993). When the introduction of the limit was dominated by rational functions at the zero of the denominator (this approach is usually preferred to demonstrate the difference between function values and limit value), students, in turn, develop that the limit value is not attainable, but rather an approximation. Figure 13 is an additional explanation of the issue that suggests how the difficulty is persistent.

S_{06}	Explain why Because, as the name indicates limit is to say that approch of a number
S_{97}	Explain why limit of a function doesn't reach the exact point b/c it create undefined/∄
S_{14}	Explain why Because limits doesn't give exact value but gives the value approach to the no.

Figure 13: Extracts showing the limit value as an approximation concept image

Referring to Table 14, only 67 (28.2%) respondents recognized the dual nature of the limit and got the correct choice D for item 2.2. While 22 (9.2%) think that the limit is all about an infinite process, 29 (12.2%) think that it has a finite value and has nothing to do with the infinite process. 19 (8.0%) of participants confirmed that the limit was necessarily a boundary. Figure 14 confirms that the limit is a boundary concept image. In this item, option E has the largest response rate. This has many implications for the diversity of students' difficulties. To begin with, this misconception originates from the conception that every function is monotonic. The other is that being monotonic is a necessary condition for convergence. Most of those who select option E think that limit means a boundary, i.e. least upper bound if the function is increasing and the greatest lower bound if the function is decreasing. Since these values are unique (provided the function is monotonic), the limit is also unique or is a finite value. Figure 14 is an illustration of the limit value is boundary concept image.

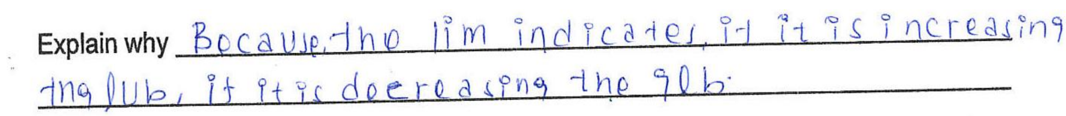
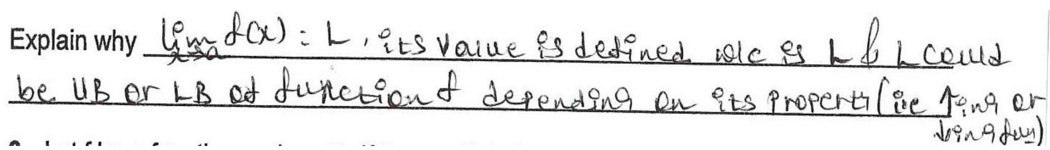
S_{06}	 Explain why Because the lim indicates, if it is increasing the lub, if it is decreasing the glb.
S_{75}	 Explain why $\lim_{x\to a} f(x) = L$, its value is defined b/c is L & L could be UB or LB of function f depending on its property (i.e. ↑ing or ↓ing func)

Figure 14: Extracts which show the limit value is a boundary concept image

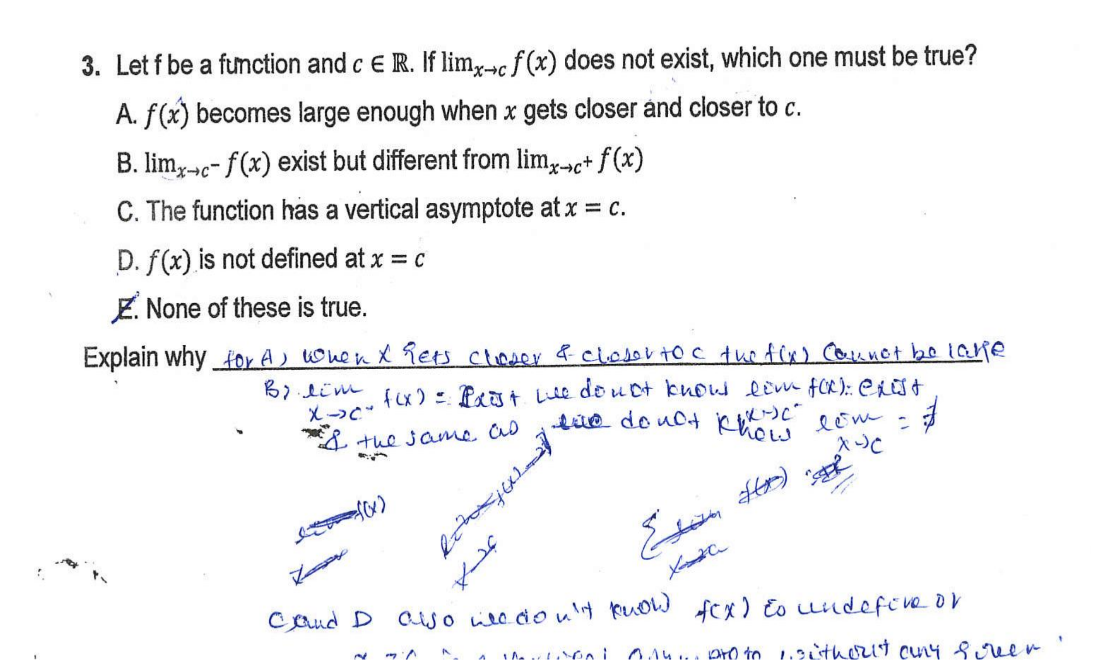
The aim of item 2.3 was to diagnose students' qualitative reasoning ability and consistency of reasoning on the non-existence of the limit at a point. Referring to the data in Table 14, only 69 (29.0%) of them have a clear symbolic interpretation as far as their response to this item is concerned. While 163 (68.5%) of them have one or the other form of difficulty, six (2.5%) of them left the item unanswered. In this item, options A to D are distractors, which were arrived at due to a lack of knowledge on limit of functions.

The percentage of choice A to D suggests that:

- Ten (4.2%) think that limit does not exist necessarily imply that the function is unbounded,
- Seventy-one (29.8%) thinks that a function will have no limit only if the two sides limits have different values.
- Thirty-seven (15.5%) think that if $\lim_{x\to c} f(x)$ does not exist, then the graph of f should have a vertical asymptote at $x = c$ (for instance, S_{30} as in figure 15),

- Forty-five (18.9%) confused existence of the limit and being defined. Figure 15 displays the correct answer with correct reason and wrong answer for the wrong reason for this item.

S_{31}



3. Let f be a function and $c \in \mathbb{R}$. If $\lim_{x \to c} f(x)$ does not exist, which one must be true?

A. $f(x)$ becomes large enough when x gets closer and closer to c.

B. $\lim_{x \to c^-} f(x)$ exist but different from $\lim_{x \to c^+} f(x)$

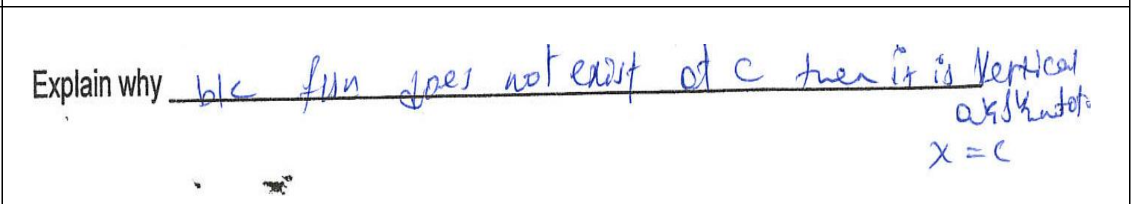
C. The function has a vertical asymptote at $x = c$.

D. $f(x)$ is not defined at $x = c$

E. None of these is true.

Explain why

S_{30}



Explain why

Figure 15: Extracts from correct and wrong answers on item 2.3

Item 2.4 was aimed at examining students' knowledge on the relationship between the limit value and the function values, the limit and continuity interplay. Regarding this, the data in Table 16 revealed that while 59 (24.8%) got the correct choice D, 168 (70.6%) selected the other options, and the remaining 11 (4.6%) refused to answer the item. Only a few students gave a satisfactory explanation and showed strong knowledge of this concept. Others got the correct option, but did not support their choice of option with an explanation. Figure 16 briefs both strong (S_{97} & S_{42}) and weak (S_{102} & S_{74}) concept images of the interaction.

Accordingly, the percentage of choice A to C suggests that:

- Forty-seven (19.7%) think that the existence of a limit is sufficient for continuity of a function at a point.

- 110 (46.2%) think that limit at a point is the same as the function value at the limit point and the existence of a limit is sufficient for being defined,
- Eight (3.4%) think that the existence of a limit is sufficient for being defined, but nothing can be said about the function value based on the limit value.

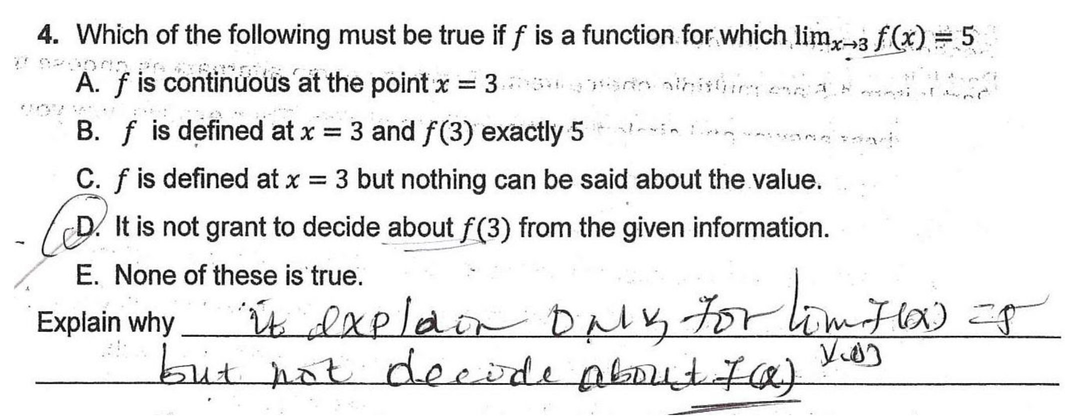
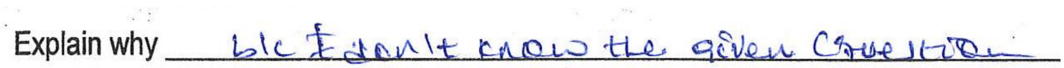
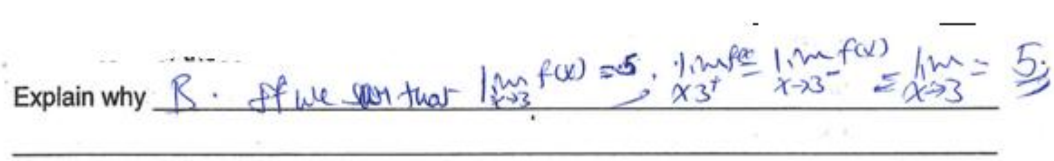
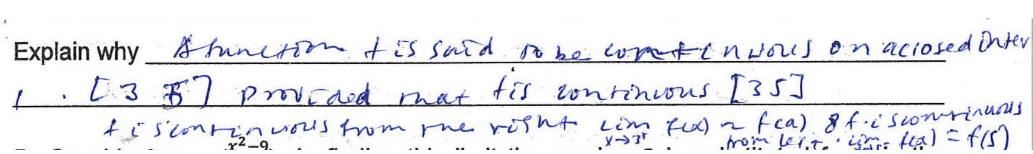
S_{97}	 4. Which of the following must be true if f is a function for which $\lim_{x\to 3} f(x) = 5$ A. f is continuous at the point $x = 3$ B. f is defined at $x = 3$ and $f(3)$ exactly 5 C. f is defined at $x = 3$ but nothing can be said about the value. D. It is not grant to decide about $f(3)$ from the given information. E. None of these is true. Explain why it explain only for lim f(x) = 5 but not decide about f(x)
S_{42}	 Explain why b/c I don't know the given question
S_{102}	 Explain why B. If we see that $\lim_{x\to 3} f(x) = 5$, $\lim_{x\to 3^+} f = \lim_{x\to 3^-} f(x) = \lim_{x\to 3} = 5$
S_{74}	 Explain why A function f is said to be continuous on a closed interval [3 5] provided that f is continuous [3 5] f is continuous from the right $\lim f(x) = f(a)$ & f is continuous from left $\lim f(x) = f(5)$

Figure 16: Extracts of strong and weak reasons for item 2.4

In this item, the option B has the highest respondent. The implication is that either many students do not differentiate the limit value from the function value or their experience is limited to continuous functions.

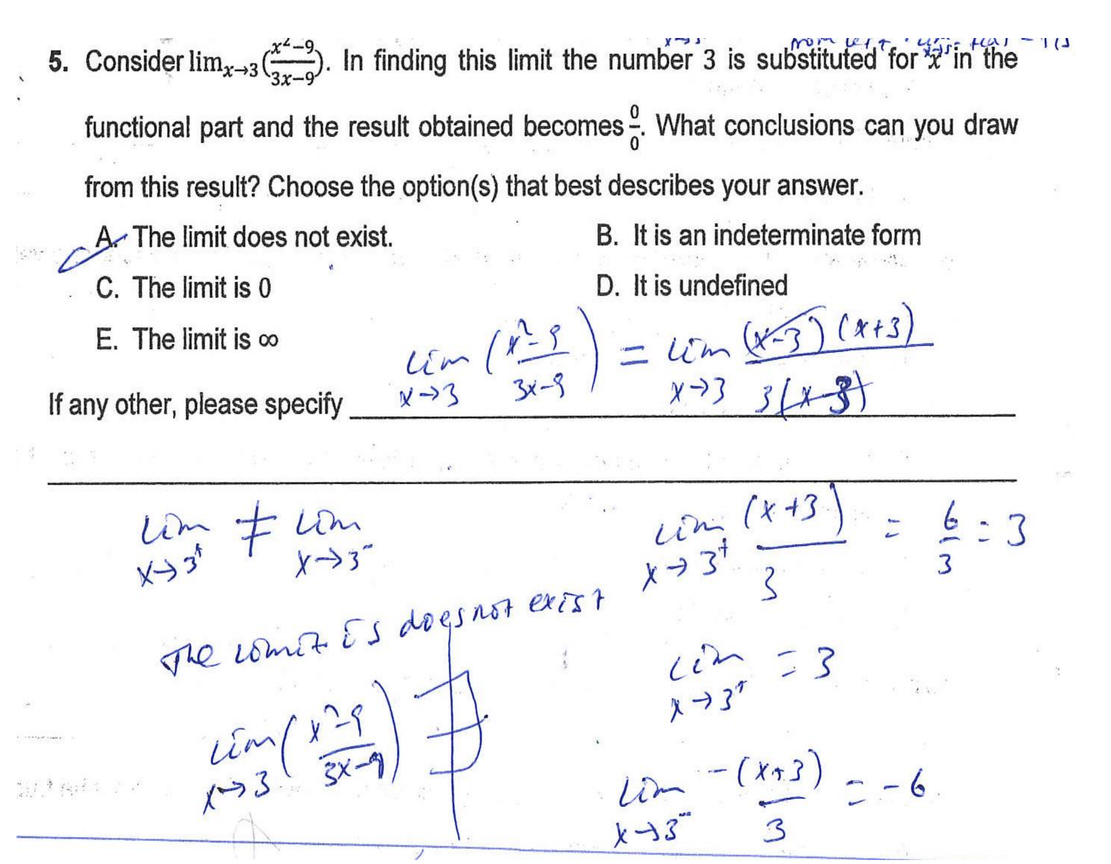
The aim of item 2.5 was to establish students' linguistic issues in the limit. It also reveals more about students' algebraic manipulation skills. All options, except B, are distractors that were arrived at due to linguistic ambiguity on the limit of functions. Accordingly, 73 (30.7%) of them got the correct answer, and 151 (63.4%) missed it.

The remaining 14 (5.9%) left it unanswered and this is the highest non-response rate among all the five items in the limit of functions. This may have its own implication on how the terms are confusing. The percentage of respondents on these incorrect options suggests that:

- Twenty-eight (11.8%) students think that limit at a point is a substitution, and if that substitution results indeterminate form the conclusion is, that limit does not exist.
- Sixty-two (26%) students think that $\frac{0}{0} = 0$.
- Twenty-seven (11.3%) students think that the indeterminate form $\frac{0}{0}$ is the same as undefined and hence the limit value does not exist.
- Thirty-four (14.3%) students think that the indeterminate form $\frac{0}{0}$ entails the limit is infinity.

Besides, students have an incorrect interpretation of symbolic notations. Figure 17 displays, how some of them incorrectly interpreted the one side limit notation.

S74



5. Consider $\lim_{x\to3}\left(\frac{x^2-9}{3x-9}\right)$. In finding this limit the number 3 is substituted for x in the functional part and the result obtained becomes $\frac{0}{0}$. What conclusions can you draw from this result? Choose the option(s) that best describes your answer.

A. The limit does not exist.
B. It is an indeterminate form
C. The limit is 0
D. It is undefined
E. The limit is ∞

If any other, please specify

Figure 17: An extract displaying a wrong interpretation of the symbolic notation

In addition to the five items in section two of the test, item 3.4c and item 4.5a in section three and four have the potential to diagnose students' algebraic manipulation skills, and how they extend their knowledge of the limit to a real-life problem. In particular, item 4.5a helped to establish students' concept images of infinity and the knowledge of coordination of processes into objects. Students' response to this item is summarized as in Table 15.

Table 15: Breakdown of students' response to item 4.5a

Frequency, N=238							
Correct		Incorrect		Incomplete		Non-respondent	
N	%	N	%	N	%	N	%
69	28.9	84	35.5	45	18.9	40	16.5

According to the data in Table 15, 40 (16.5%) students did not have an answer. While 69 (28.9%) of them described it correctly, 84 (35.5%) of them gave complete and meaning full procedures but incorrect conclusion, whereas, the remaining 45

(19%) started the procedure but interrupted without a meaningful conclusion. Some difficulties observed in the incorrect responses were summarized as follows: wrong interpretation of the limit rules, confusing limit and other concepts in calculus, treating infinity as a number and errors in symbolic manipulation (see figure 18).

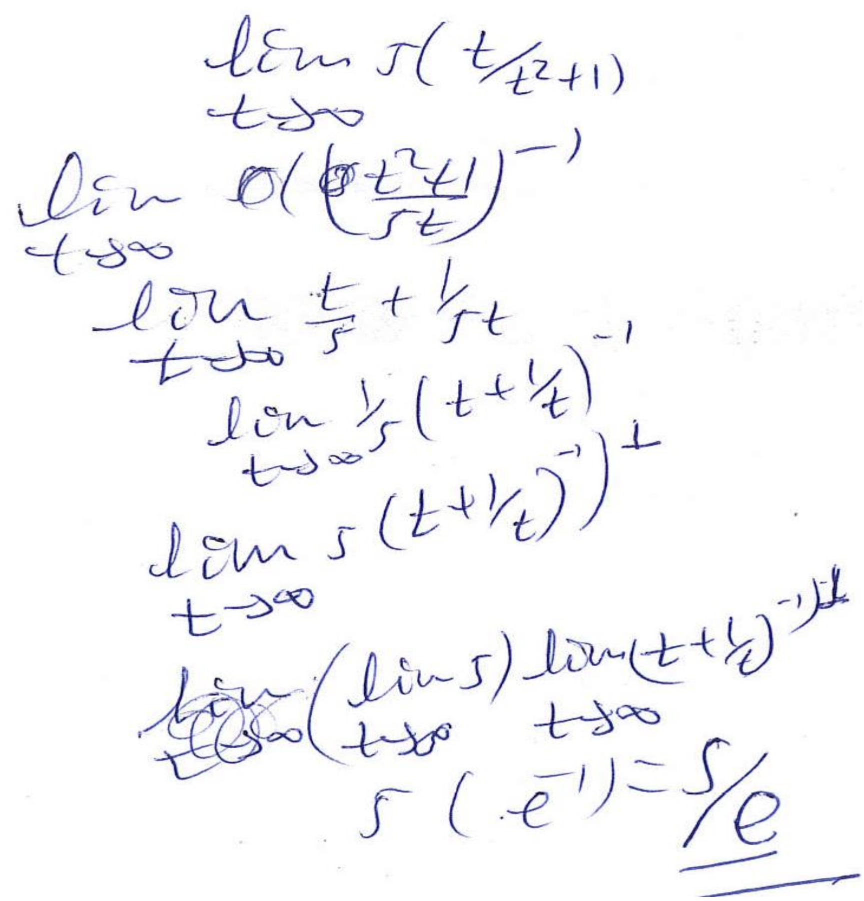
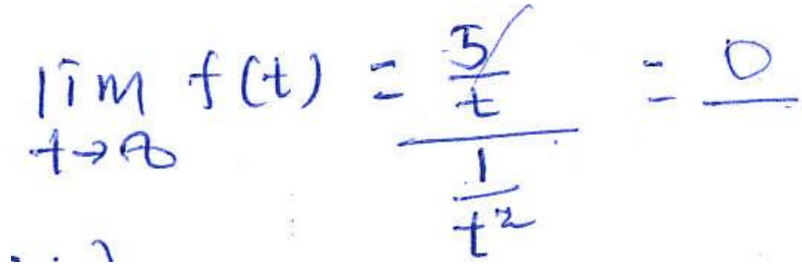
S_{22}	
S_{232}	

Figure 18: An extract that revealed the wrong working of the limit

In item 3.4c and 3.4d (see table 18), 118 (49.5%) of them correctly answered that the functions' limit exists at $\boldsymbol{x = 3}$, but only 57 (24.0%) of them computed the correct limit value. Many students missed the result due to an algebraic manipulation errors and knowledge of indeterminate forms (see figure 19).

S_{144}	c/ The limit doesn't exist b/c at x=3 the denominator will be zero
S_{181}	d, $\lim_{x\to 3} f(x) = \frac{2x^2-x-15}{x-3} = \frac{2(3)^2-3-15}{3-3} = \frac{18-3-15}{3-3} = \frac{0}{0} = D$

Figure 19: An extract of limited knowledge of the limit

In general, students' performance in the limit test items revealed that many students' knowledge on limit is limited and suitable for continuous functions. The following is a list of observed difficulties:

- Influenced by an arithmetic approach for items demanding an algebraic approach. For instance, evaluate the function just at $x = c$ instead of simplifying the rational expression to find the limit as $x \to c$.
- Hard to find the limit of a rational function at the zero of the denominator, and understand the indeterminate form $\frac{0}{0}$ as undefined.
- Think that the limit is not attainable, but is an approximation.
- While some student's think the limit is all about an infinite process and has nothing to do with finite value, others think that limit is all about a finite value and has nothing to do with an infinite process.
- Think that the non-existence of a limit necessarily implies the function is unbounded; a function will have no limit only when the two side limits have different values.
- Think that if $\lim_{x\to c} f(x)$ does not exist, then the graph of f should have a vertical asymptote at $x = c$.

- Think that the limit at a point is the same as the function value at the limit point; also confuse the existence of a limit and being defined.
- Think that the existence of the limit is sufficient for being defined, the limit at a point is a substitution, and if that substitution results in indeterminate form, the conclusion is that limit does not exist.
- Misinterpret symbolic notations and make algebraic manipulation errors.
- Misinterpret limit rules and indeterminate forms.
- Confuse the limit and other concepts in calculus.
- Treat infinity as a number.
- Have difficulty to compute the limit of piecewise-defined functions.

4.1.3. Students' conceptual knowledge of continuity

The purpose of section three of the test was to diagnose students' difficulties with continuity. The section consists of three closed-ended items and one open-ended item. In particular, the purpose of item 3.1 is to establish students' concept images of continuity. The item is designed to incorporate domain-continuity, limit-continuity, and continuity-connectedness interplay. Accordingly, students are expected, first, to decide whether the piecewise-defined algebraic form of the given function is continuous or not, then to choose a justification from the given options in one of the two categories. Surprisingly, this is the only item attempted by all the participants. While 207 (86.9%) of them correctly identified it as continuous, 25 (10.5%) of them said it is discontinuous and the remaining six (2.5%) selected an option from both categories', so that they are grouped as "bad indexed." Table 16 presents a summary of respondents in two categories.

Table 16: Breakdown of students' choices to item 3.1

N=238	The function is continuous on its domain because, N=207					The function is not continuous on its domain because, N=25					Bad-index, N=6
	A	B	C	D	E	F	G	H	I	J	
N	28	9	156*	5	9	0	2	23	0	0	6
%	11.7	3.8	65.5	2.1	3.8	0	0.8	9.7	0	0	2.5

* correct answer of the item

Among the 25 (10.5%) respondents, who said the function is not continuous in its domain, 92% (23 out of 25) said that the function is not continuous on its domain because "there is a number "a" in the domain for which $\lim_{x\to a} f(x)$ does not exist, or $\lim_{x\to a} f(x) \neq f(a)$" and the remaining two (0.8%) said that, "the function is not defined for every real number".

Generally, 179 (75.2%) respondents described continuity in the subject of limit. This is the good opportunity for progression. As observed in the literature, most students' difficulties with continuity originate from lack of describing continuity in the subject of limit. Additionally, few numbers of students have a problem of confusing continuity with the pencil metaphor. They think the existence of limit as sufficient for being continuous, which according to the literature, is a common problem for most students. On the other hand, 28 (11.7%) of the students have confused continuity with being defined. However, 156 (65.5%) of them clearly displayed a good understanding of continuity as far as their response on this item is concerned.

While item 3.2 was also designed to establish more on the interplay between continuity and the other concepts: differentiation, limit, and domain or being defined, item 3.3 is designed to discover more about how students' understand continuity in the subject of limit. In addition, the items aimed to see the students' ability to compute the one-sided limit. Table 17 presents the result of these two items.

Table 17: Breakdown of students' choices to item 3.2 and 3.3

Item	Frequency, N=238										Non-respondent	
	A		B		C		D		E			
	N	%	N	%	N	%	N	%	N	%	N	%
3.2	42	17.7	22	9.3	72*	30.2	14	5.9	83	34.8	5	2.1
3.3	52	21.8	122*	51.3	13	5.4	21	8.8	19	8.0	11	4.6

* correct answer of the item

The data in Table 17 revealed that 72 (30.2%) of students answered item 3.2 correctly. 161 (67.6%) of them selected the wrong options and the remaining five (2.1%) left the item unanswered. Among the alternatives, option C is correct and the remaining are distractors that were arrived at due to immature formation of the continuity concept and lack of recognizing the relation among concepts. Accordingly, 161 (67.6%) of students demonstrated such difficulty. Particularly, most students complicate properties of continuity with properties of derivatives of a function. Even those who know the correct definition of continuity in the subject of limit, misinterpret it when they come to a specific case. Figure 20 shows how two students misinterpreted continuity properties.

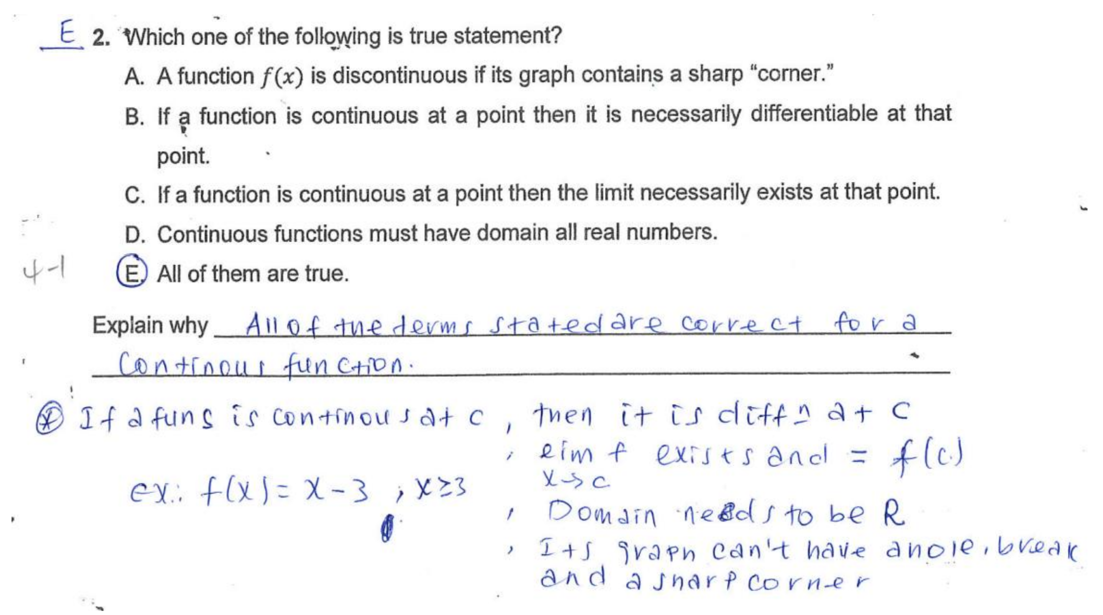
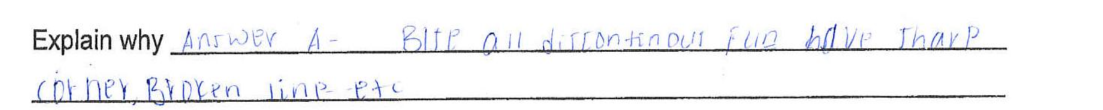
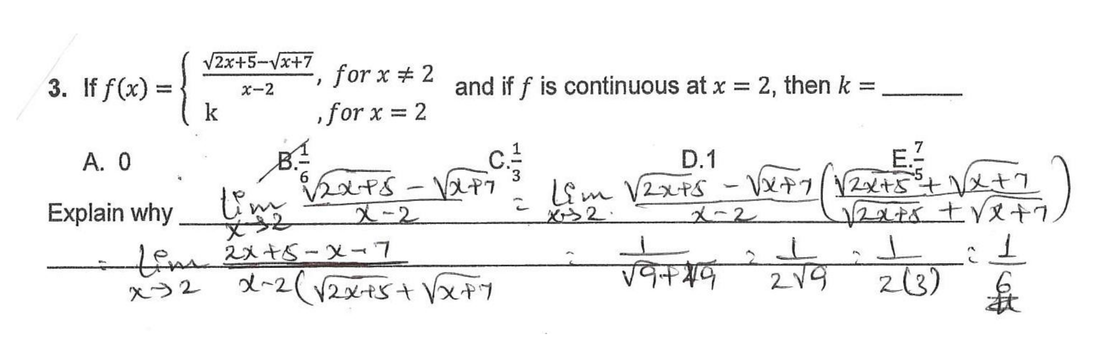
S_{38}	 E 2. Which one of the following is true statement? A. A function $f(x)$ is discontinuous if its graph contains a sharp "corner." B. If a function is continuous at a point then it is necessarily differentiable at that point. C. If a function is continuous at a point then the limit necessarily exists at that point. D. Continuous functions must have domain all real numbers. E. All of them are true. Explain why All of the derms stated are correct for a Continous function.
S_{82}	 Explain why Answer A - Bite all discontinous fun have sharp corner, Broken line etc

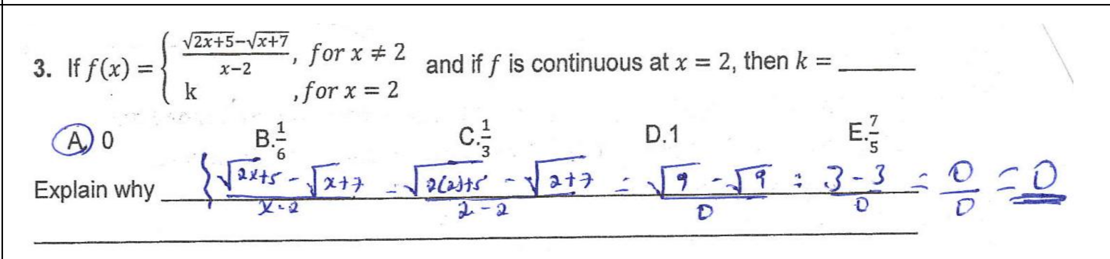
Figure 20: An extract that revealed a wrong interpretation of continuity properties

Again referring to Table 17 for item 3.3, 122 (51.3%) students answered it correctly and 105 (44.1%) of them answered it incorrectly. The remaining 11 (4.6%) left the item unanswered. While option B is the correct answer, the remaining alternatives are distractors arrived at due to either lack of knowledge or algebraic manipulation errors. As in figure 21, S_{06} is evidence for correct answer with correct reasoning and S_{79} and S_{14} are evidence for $\frac{0}{0} = 0$ and $\frac{0}{0} = \infty$ misinterpretations. This suggests that students lack the necessary pre-calculus skill, and this, in turn, affects their performance in calculus.

S_{06}



S_{79}



S_{14}

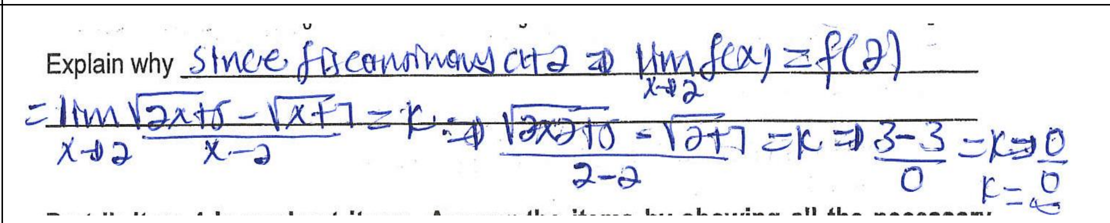
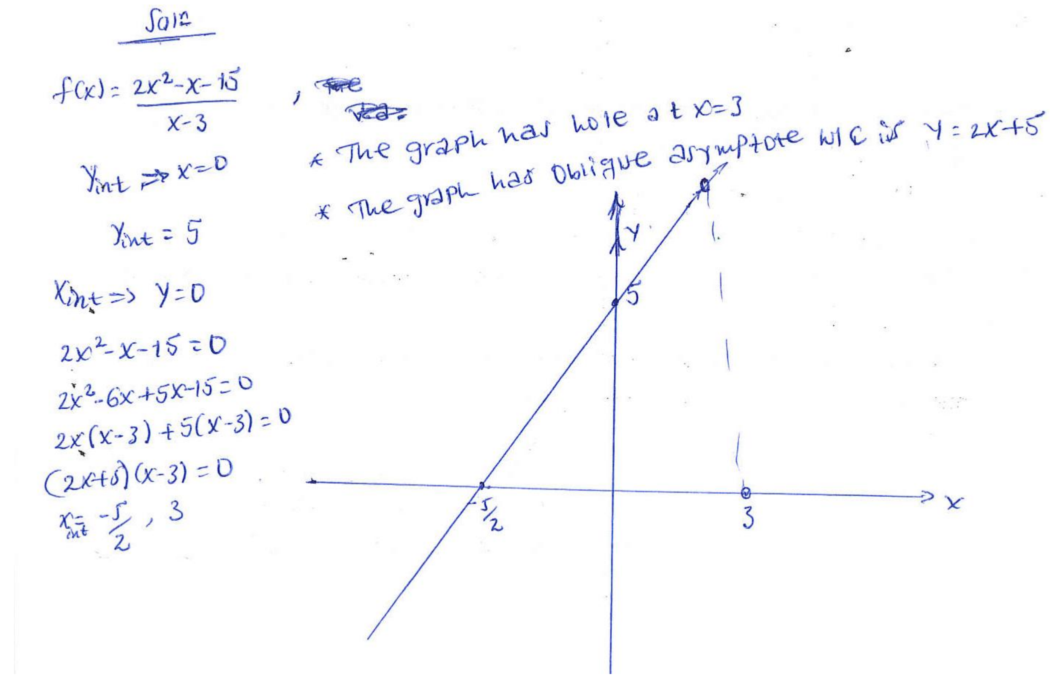


Figure 21: Some difficulties observed in continuity at a point

The main purpose of item 3.4 was to see how students treat the point of discontinuity of a function, both algebraically and graphically. The way to attain this purpose also helped to explore students' ability on algebraic manipulation, the existence of a limit at a point where the function is undefined, and how they relate a limit value and a function value. Table 18 is a summary of the response to item 3.4.

On item 3.4a, the instruction was to draw the graph of $f(x) = \frac{2x^2 - x - 15}{x - 3}$ and to answer the question that follows using the information from the graph. Referring to Table 18, only 49 (20.5%) of them sketched it correctly. While 108 (45.4%) sketched an incorrect graph, 81 (34.0%) left the item unanswered. Examples of correct (S_{211}) and incorrect (S_{14}) graphs respectively from students' scripts are shown in figure 22.

S_{211}



S_{14}

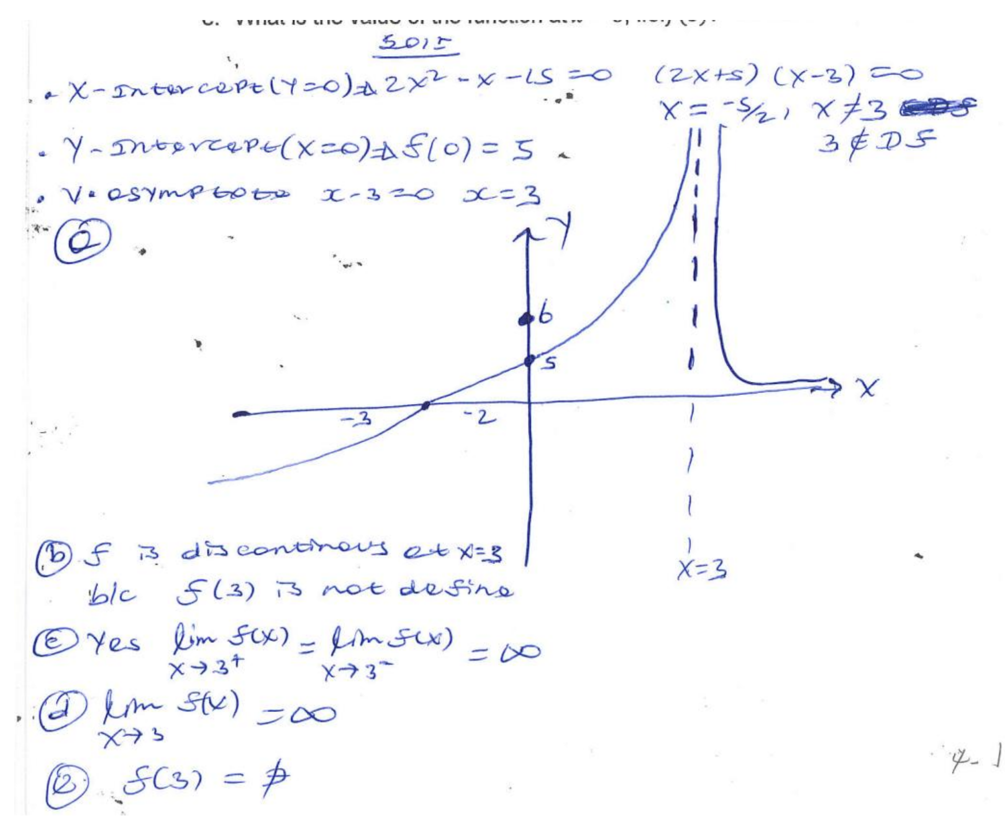


Figure 22: Extracts that show a correct and an incorrect graph respectively

The major reasons that lead them to sketch incorrect graphs were considering every point of discontinuity as an asymptote. It is also observed that many students try to draw the graph without considering sufficient points that lay on the graph. Even those who correctly specified properties of the graph, such as domain, intercepts, and point of discontinuity sketched an incorrect curve due to lack of considering sufficient points that show the pattern.

Table 18: Breakdown of students' choices to item 3.4

Sub-items	Frequency, N=238					
	Correct		Incorrect		Non-respondent	
	N	%	N	%	N	%
3.4a	49	20.5	108	45.4	81	34.0
3.4b	138	58.0	63	26.4	37	15.5
3.4c	118	49.5	85	35.7	35	14.8
3.4d[12]	57	24.0	31	13.0	30	12.6
3.4e	95	39.9	102	42.8	41	17.2

Referring to item 3.4b, 138 (58.0%) of students said the given function is discontinuous at $\boldsymbol{x} = \mathbf{3}$, while 63 (26.4%) of them said it is continuous and 37 (15.5%) of them left the item unanswered. Even some students, who draw a smooth continuous line near $\boldsymbol{x} = \mathbf{3}$, answered this item as discontinuous correctly. This shows that these students have conflicting concept images that are dependent on forms of representation. Some of them said that the given function was rational, and a rational function has a vertical asymptote at the zero of the denominator. This is a good indication of how students' conception is unsynchronized and dominated by symbolic manipulation (Luneta & Makonye, 2010).

Referring to Table 18 for item 3.4c, 118 (49.5%) students correctly answered that the function's limit exists at $\boldsymbol{x} = \mathbf{3}$ but only 57 (24.0%) computed the correct limit value, which is 11. 31 (13%) of them have incorrect values which include ∞, 0, 1 and the

[12] based on correct respondents of 3.4C

remaining 30 (12.6%) left blank. This shows that almost 75% of students face challenges either to compute the limit of rational functions, manipulate algebraic notations, or interpret indeterminate forms. Only those who said the limit exists were expected to compute the value and answer item 3.4d. What is observed was that 17 (7.1%) among these who said the function has no limit in 3.4c also computed the value in which six (2.5%) is the correct limit value. This shows that some students also lack attention to what they are thinking and doing, i.e. making connection.

With regard to 3.4e, while 95 (39.9%) of them correctly said that, the function has no value at $\boldsymbol{x = 3}$, 102 (42.8%) said the function has a value. The remaining 41 (17.2%) said nothing about the function value. Some of the incorrect values and the reasons behind these incorrect conclusions are summarized as in Table 19 (these errors are also observed in 3.4d).

Table 19: Reasons behind the incorrect responses to item 3.4e

No.	Response	Frequency	Reason
1	11	24	Ignore the restriction on the domain after simplification, i.e. they consider $\frac{2x^2-x-15}{x-3} = 2x+5, \ \forall x$
2	0	19	Most of them think that $\frac{0}{0} = 0$
3	3	9	As in 1 above and manipulation errors, i.e. simplify $\frac{2x^2-x-15}{x-3}$ as $2x-5$, $(x-3)(x-5)$, $\left(x+\frac{5}{2}\right)$
4	1	4	
5	-1	3	
6	∞	6	Think that $\frac{0}{0} = \infty$
7	Others (9, 45, 4.5, $\frac{0}{0}$ so on)	37	Different reasons

Generally, students' result in item 3.4 is a good indication of their knowledge on functions. Besides, their responses indicate that how students understand points of discontinuity as an asymptote has something to do with their misunderstanding of the concept of function. It is also observed that some students confuse terms specific to different ways of representing a function (graphic or algebraic). Their difficulties also include the belief that the existence of limit is sufficient for continuity of a function at a point and considering every point of discontinuity as an asymptote.

4.1.4. Students' conceptual knowledge of the derivatives

Section four of the test is designed to assess students' knowledge of the derivative concept. The section consists of three closed-ended and four open-ended items. Accordingly, Item 4.1 is aimed to establish students' visualization of derivative. Besides, it aimed to see computational ability on procedures of the derivative. Item 4.1 and 4.2 are also designed to see students' knowledge of conceptual level and how it goes beyond algebraic manipulation. In addition, item 4.1 demands reverse thinking, whereas; item 4.3 demands having a network of concepts: limit, continuity, and derivative. Table 20 is a summary of students' responses on these three items.

Table 20: Breakdown of students' response to item 4.1 to 4.3

Items	Frequency, N=238										Non-respondent	
	A		B		C		D		E			
	N	%	N	%	N	%	N	%	N	%	N	%
4.1	40	16.8	25	10.5	46	19.3	40	16.8	75*	31.5	12	5.0
4.2	45	18.9	21	8.8	16	6.7	129*	54.2	18	7.6	9	3.8
4.3	28	11.7	101*	42.4	57	23.9	33	13.9	15	6.3	4	1.7

Referring to the data in Table 20, 75 (31.5%) of them correctly answered item 4.1. While 151 (63.4%) missed it, the remaining 12 (5.0%) left the item unanswered. Option E is the correct answer, whereas, options A to D are distractors arrived at due to lack of knowledge or lack of visualizing the network of concepts beyond computational purposes. One major difficulty observed was that misinterpretations of the quotient rule, i.e. almost 19% students think that since $f(x) = \frac{h(x)}{g(x)} = \frac{h'(x)}{g'(x)}$ then $f(x) = e^x$. The other difficulty is that they think if $f'(x) = g'(x)$ then necessarily $f(x) = g(x)$. Figure 23 displays incorrect reasons for both a correct (S_{17}) and incorrect (S_{42}) answers respectively; S_{198} is a correct justification for the correct answer.

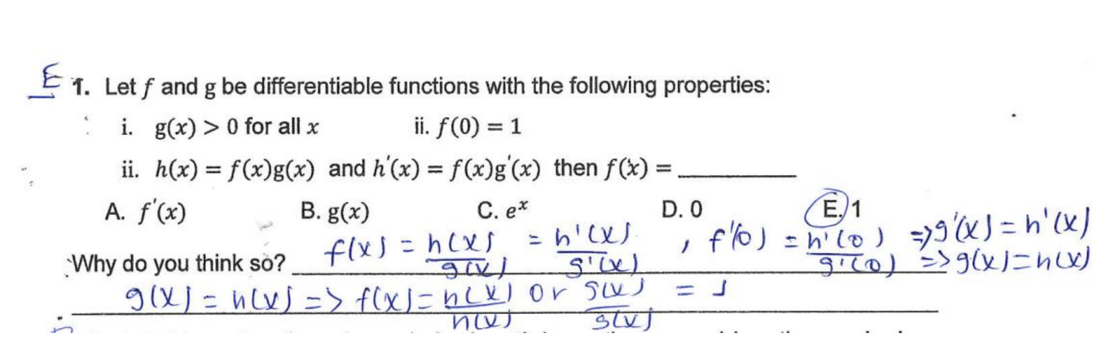
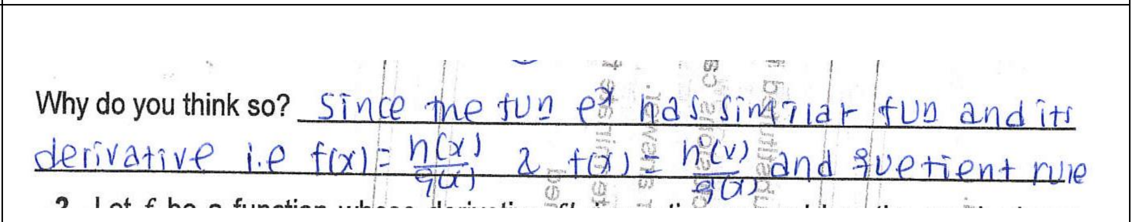
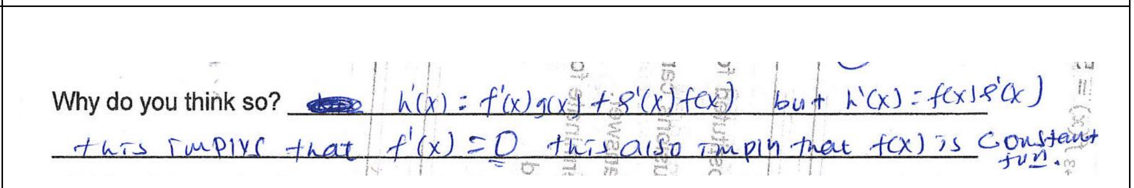
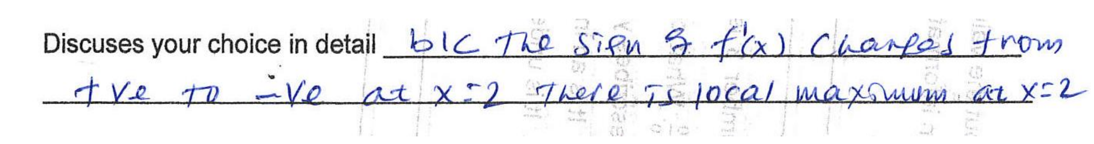
S_{17}	
S_{42}	
S_{198}	

Figure 23: Varied form of response in item 4.1

Item 4.2 is one of the items with a low-level of difficulty, but it is potential in the subject of displaying students' difficulty. Referring to the data in Table 20, 129 (54.2%) of them correctly answered it. While 90 (37.8%) of them missed it, the remaining nine (3.8%) left the item unanswered. Option D is correct, whereas A to C and E are distractors arrived at due to failure to interpret the first derivative test for extreme values graphically. One of the observed difficulties is that even those who know the statements of the first derivative test to find extreme values of a function, they do not give attention to direction, i.e. the theorem holds true when one moves from left to right along the x-axis on the graph of the given function. However, 45 (18.9%) students move from right to left. That is why the option A has a higher response rate than the other distractors. Even those who are good at finding the derivative of a function, consider the properties of the graph of the derivative function the same as properties of the graphs of the function. This shows that students lack reverse thinking. Figure 24 displays justifications given to the correct answer and an incorrect response respectively.

S_{198}



S_{57}

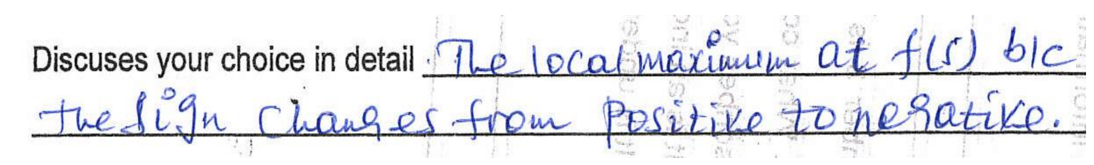


Figure 24: Justification given to the correct answer and an incorrect response respectively

Referring to the data in Table 20 again, 101 (42.4%) of students correctly answered item 4.3. While 133 (55.9%) missed it, the remaining four (1.7%) left the item unanswered. Option B is the correct answer, whereas alternative A, C, D, and E are distractors arrived at due to a lack of organizing the required schema of the derivative concept (Maharaj, 2012). In particular, the item demands information on the one-sided limit, being aware that a differentiable function is continuous and algebraic manipulation skills as well. It was observed that most students' difficulties originate from being unaware that the function is differentiable implies both the one-sided limits of the different-quotient exist and are equal. They write $\mathbf{lim}_{x\to 1}\, \boldsymbol{ax} + \boldsymbol{b} = \boldsymbol{f}(\mathbf{1}) = -\mathbf{2}$ and stuck. Possibly that is why the option C has the highest response rate. Some students also have made algebraic manipulation errors. Only a few students showed clear and neat steps on their paper. In general, the item revealed that almost 40% of the students could find the derivative of the two formulas separately but lack to coordinate them. Figure 25 presents two students' scripts in which one is labelled as strong (S_{42}) and the other with difficulties and categorised as weak (S_{69}) respectively.

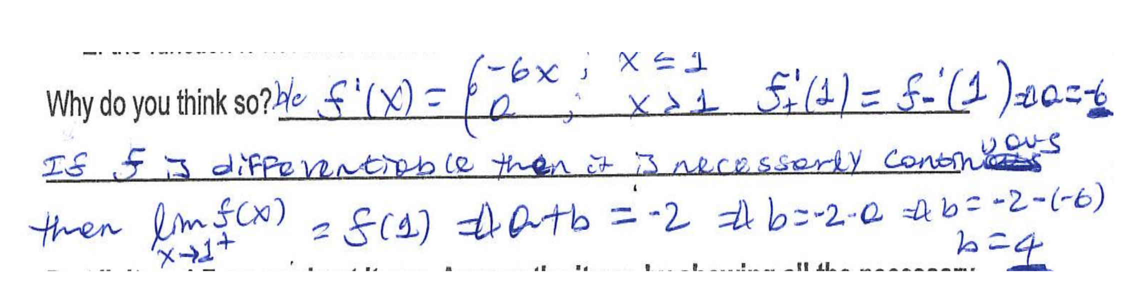
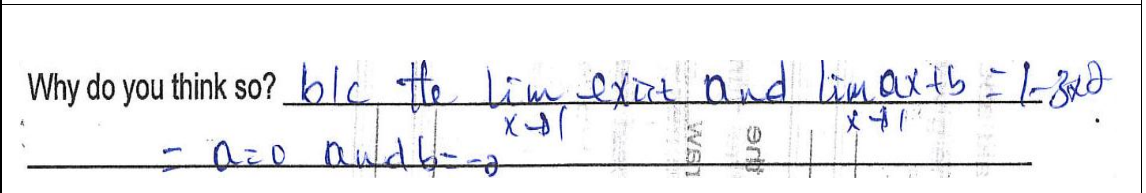
S_{42}	
S_{69}	

Figure 25: Strong and weak students' scripts respectively on item 4.3

The purpose of item 4.4 is to explore how students' understand chain rule and their computational ability on rules and procedures of the derivative. The item requires a derivative schema, which includes a process of repeated actions and an object conception, which enables the bearing in mind of strings of processes as a totality. Table 21 summarizes the response to this item.

Table 21: Breakdown of students' response to item 4.4

Frequency, N=238					
Correct		Incorrect		Non-respondent	
N	%	N	%	N	%
89	37.4	107	45.0	42	17.6

According to the data in Table 21, 89 (37.4%) got the correct answer and this shows that they have the appropriate schema for the derivative of composition functions. However, the script from the remaining 107 (45%) who missed the answer, suggest that they are at action level conception of differentiating composition functions. Confuse rules of differentiation, like $(\boldsymbol{e^x})' = \boldsymbol{xe^x}$ (as S_{54} in figure 26), interchanging derivative of combination function, and composition function rules (for instance, S_{70} as in figure 26), were some of the observed difficulties.

The main purpose of item 4.5 is to see how students extend their knowledge on the limit and derivative to solve optimization problems. On the way to attain this purpose, it also helps to establish students': concept image of infinity, knowledge of coordination of processes, the nature of their limit conception, and knowledge of rules and procedures of the derivative. The item demands to be aware of techniques of differentiating rational functions, application of the first derivative test, and algebraic manipulation. Table 22 presents a summary of responses to items 4.5b and 4.5c (see Table 15 for a response rate of 4.5a).

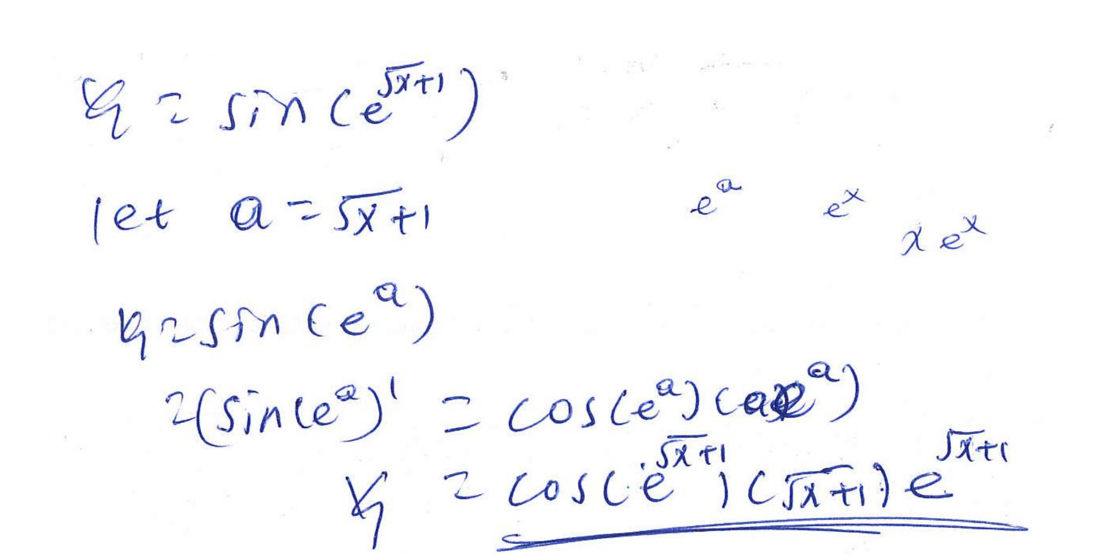
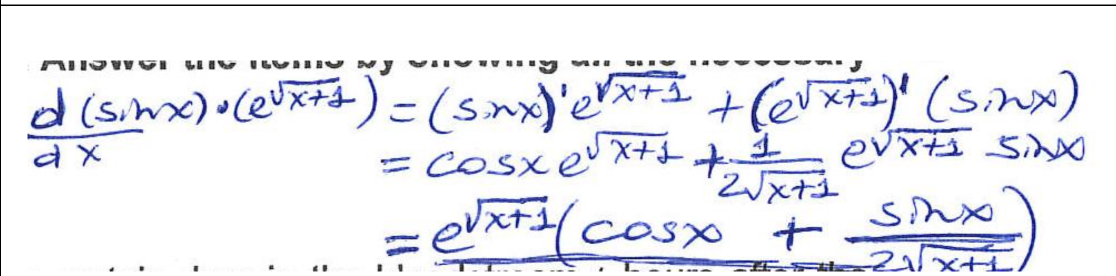
S_{54}	
S_{70}	

Figure 26: Weak students' script on item 4.4

Referring to Table 22 for item 4.5b, 78 (32.7%) of them got the correct answer. While 123 (51.7%) used incorrect methods or left it incomplete, the remaining 37 (15.5%) left the item unanswered. Generally, from this item the following difficulties of understanding are observed:

- Begin the process of solving the problem correctly and end with an incorrect result. This is due to a problem with algebraic manipulation.
- Confuse critical numbers with extreme value.
- Fails to recognize restrictions (whereas the domain $\boldsymbol{t} > \mathbf{0}$, they consider both $\boldsymbol{t} = \pm\mathbf{1}$ as critical points).

Table 22: Breakdown of students' choices to item 4.5b and 4.5c

Item	Frequency, N=238					
	Correct		Incorrect		Non- respondent	
	N	%	N	%	N	%
4.5B	78	32.7	123	44.9	37	22.3
4.5C	59	24.8	129	55.0	48	20.1

Even those who answered the item correctly demonstrated some sort of deficiency in their conceptual knowledge. As in figure 27, S22 did not recognize the functions' domain so he computed the value of the function at both $\boldsymbol{t} = \pm\mathbf{1}$ and then compared which is unnecessary.

S22

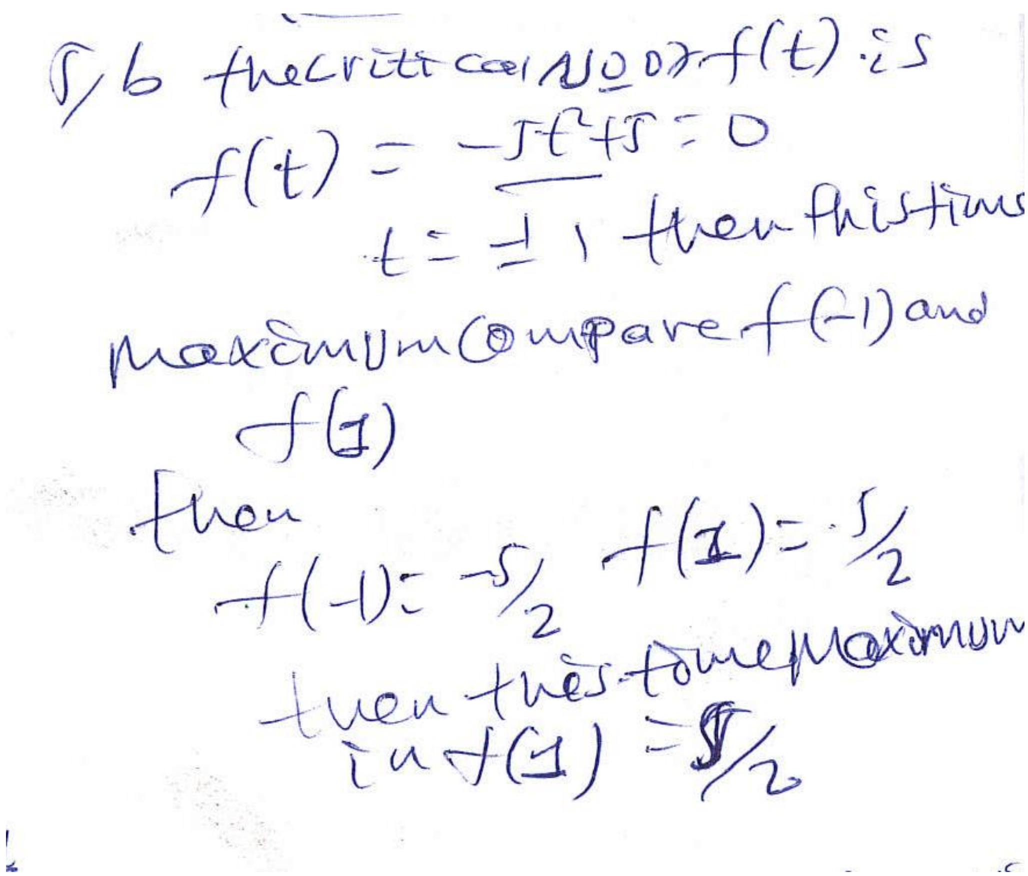


Figure 27: Scripts that display difficulty in item 4.5b

Referring to Table 22 again for item 4.5c, 59 (24.8%) of the students obtained the right response. While 129 (55.0%) used incorrect methods or left incomplete, the remaining 48 (20.1%) left the item unanswered. Most of the students who got the correct answer for item 4.5b procedurally done. The problem is a lack of recognizing the domain of the given function. Thus, the item suggests that most students' knowledge is procedural and more rigid than conceptual and flexible. Figure 28 is a display of an extract to demonstrate how students answer deviate from the one expected due to this lack of being aware of what they are doing or over-dependence on the procedural knowledge.

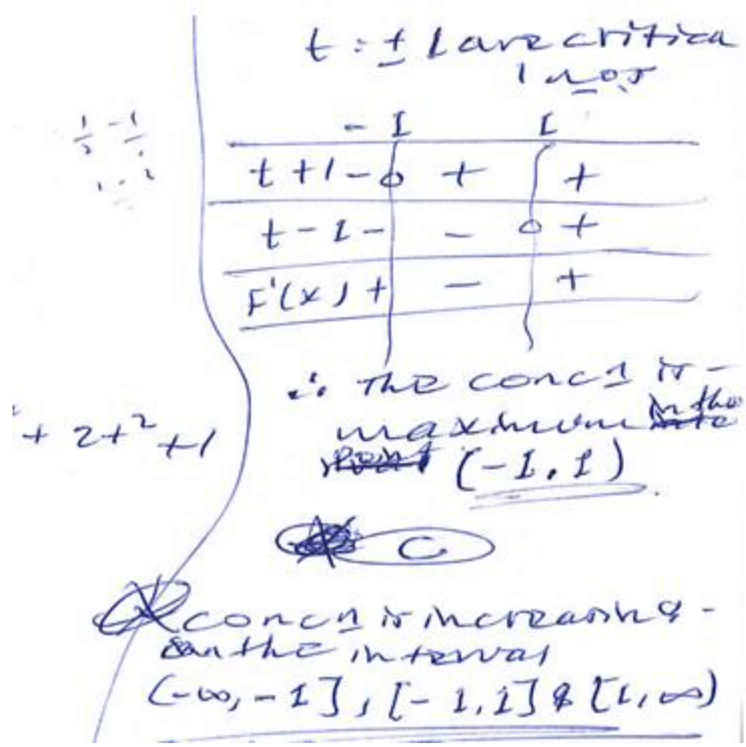
S_{164}	

Figure 28: Scripts showing the diversity of response on item 4.5c

Item 4.6 is aimed to see how well students' knowledge structure is synchronized. Table 23 presented a summary of the response to this item.

Table 23: Breakdown of students' response to item 4.6

Sub-items	Frequency, N=238					
	Correct		Incorrect		Non-respondent	
	N	%	N	%	N	%
4.6a	67	28.1	136	57.1	35	14.7
4.6b[13]	51	21.4	13	5.4	3	1.3
4.6c	59	24.8	144	60.5	35	14.7

The item demands a relational understanding of the limit, continuity, and derivative concepts. Besides, the information is given numerically in tabular form. This is done deliberately to address the issue of representation. To answer this item, a student has to know that a differentiable function is continuous but for a continuous function $\lim_{x \to a} f(x) = f(a)$. Now, given that $f'(2) = -2$ and $f(2) = 1$, then one can conclude that $\lim_{x \to 2} f(x) = 1$. Accordingly, the item requires to determine, if possible, $\lim_{x \to 2} f(x)$, from the given information and to justify the reason why. 141 (59.2%) students said, "Yes" but only 67 (28.1%) tried to justify why and among them 51 (21.4%) gave correct justification and have computed the correct value. Thus, the item suggests that many students are over-dependent on the symbolic representation. Some of the observed difficulties were (see figure 29): because the two sides limits are not equal (e.g. S_{174}), use the concept of slope of a straight line (e.g. S_{137}).

[13] Based on correct respondents of 4.6a

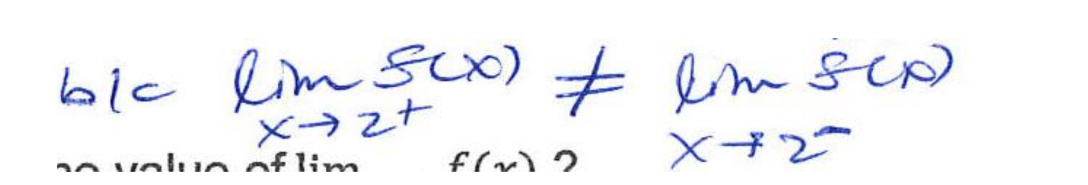
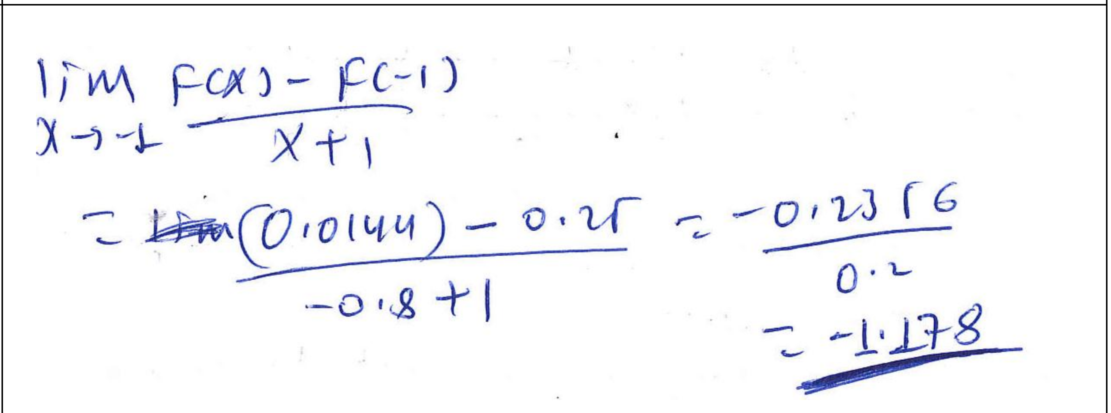
S_{174}	
S_{137}	

Figure 29: Scripts showing the diversity of response on item 4.6b

With regard to item 4.6c, only 59 (24.8%) of them were aware that the required value is $f'(-1)$ and identified the correct value. While 144 (64.7%) tried, but in the wrong ways and 35 (14.7%) of them refused to answer the item. Figure 30 displays the correct answers (S_{174}) and wrong answers (S_{22} & S_{106}). After all students' performance in this item suggests their knowledge is dominated by the action view of the limit of functions (like S_{106}) and lack of understanding definitions (like S_{22}).

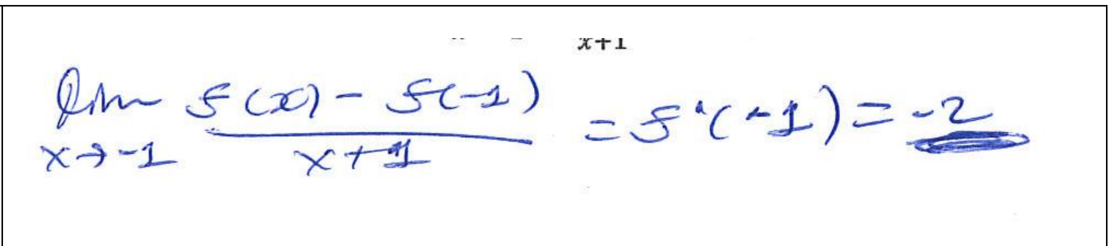
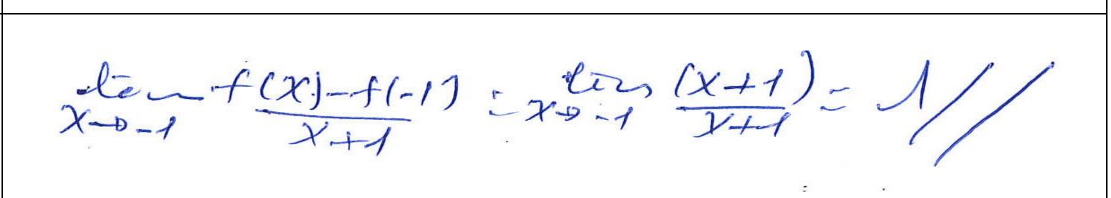
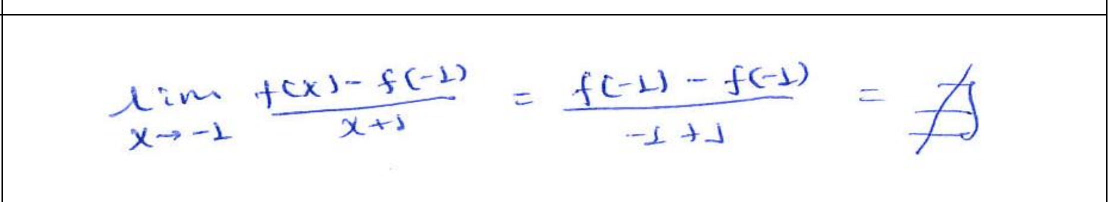
S_{174}	
S_{22}	
S_{106}	

Figure 30: Scripts showing the diversity of response on item 4.6c

Item 4.7 is designed to address three purposes: to see students' knowledge of continuity in a closed interval, how they interpret the meaning of derivative of a function at a point, and how they relate continuity and differentiability at a point. The student response to this item is summarized as in Table 24.

Table 24: Breakdown of students' response to item 4.7

Sub-items	Frequency, N=238					
	Correct		Incorrect		Non-respondent	
	N	%	N	%	N	%
4.7A	72	30.2	130	54.6	36	15.1
4.7B	64	26.9	133	55.9	41	17.2
4.7C	59	24.8	132	55.4	47	19.7

Referring to Table 24 for item 4.7a, 72 (30.2%) of them got the correct answer. While 130 (54.6%) used incorrect methods or left incomplete, the remaining 36 (15.1%) left the item unanswered. For those who said the function is discontinuous, the three most frequently occurring reasons were- the graph has a sharp corner, the domain is not all real numbers, and the function has a point of discontinuity respectively.

Referring to Table 24 for item 4.7b, 64 (26.9%) of them got the correct answer. While 133 (55.9%) used incorrect methods or left incomplete, the remaining 41 (17.2%) left the item unanswered. The item requires knowing that the derivative at a point can be computed as the slope of a line tangent to the graph of the function at the given point. This value can be obtained from the limit of the difference-quotient. When the graph is a straight line, the limit of the difference-quotient (slope of the tangent) becomes the same as the value of difference-quotient (slope of secant). Students' test scripts revealed that most of them lacked a geometric interpretation of the derivative value.

Referring to Table 24 for item 4.7c, 59 (24.8%) students got the correct answer. While 132 (55.4%) used incorrect methods or left incomplete, the remaining 47 (19.7%) left the item unanswered. Of course, this is the highest non-response rate among all the items in the test. While 22, 7 and 24 students pointed out $\mathbf{-2, -1}$ and $\mathbf{1}$ respectively as points where the function is continuous but not differentiable, the remaining 6 indicated two of these three points. That most students challenge was confusing continuity implies differentiability than differentiability implies continuity.

In general, students' performance on derivative items revealed that many students' are at action level conception. Besides, their response indicates that students lack reverse thinking, perform diverse algebraic manipulation errors, confusing rules of differentiation, and confuse critical numbers with extreme value. Most students' knowledge is procedural and ridged than conceptual and flexible. They think that a function is discontinuous if the graph has a sharp corner, or the domain is not all real numbers. Seventy percent of the students' failed to interpret derivative values as a slope of the line tangent to the graph of the given function. In particular, the following was the most frequently observed difficulties in dealing with the derivative concept:

- Misinterpreting the derivative rules.
- Overgeneralize that if $f'(x) = g'(x)$ then necessarily $f(x) = g(x)$.
- Hard to interpret properties of a function from the graph or reverse thinking.
- Hard to interpret results obtained from computations.
- Failure to coordinate two processes.
- Interchange derivative of combination and composition function rules.
- Confuse the critical numbers with extreme value.
- Fail to recognize restrictions of domain values.
- Over-dependence on procedural knowledge.
- Over-dependence on symbolic representation than another form of representation

4.1.5. Conclusion

Based on the analysis made on the data gathered through the test in the specified area, students approach to those conceptual issues and observed difficulties were summarized as follows:

- Many students thinking is influence by an arithmetic approach for items demanding an algebraic approach (for instance, in item 3.4d, 11.7% students evaluate the function just at $\boldsymbol{x = 3}$ instead of simplifying the rational expression), whereas, in item 1.5, 11.3% students evaluate the sequence at the first three or four integers. This practice of “point-by-point or a static way” of evaluating an independent variable of a function is termed as “an action view of function” (Carlson *et al.,* 2010) and this action view of a function than a process-based view is the main challenge to progress in calculus (Maharaj, 2013).

- Most students (11.3% and 9.6% as observed in item 1.5 and 4.5a respectively), have an actual value image of infinity than potential. Nevertheless, the potential infinity conception has to do more to compute the limit at infinity. According to Jones (2015, p.108), “potential infinity is more in line with a process, so valuable to limit at infinity ($\lim_{x\to\infty} f(x)$), but actual infinity has more in common with an object”, so valuable for the infinite limit.
- Different types of algebraic manipulation errors, which rooted from a lack of conceptual knowledge of pre-calculus algebra. It is common to see errors such as $\frac{siny}{y} = 1, \frac{sin\infty}{\infty} = 1,$ simplifying $\frac{2x^2 - x - 15}{x-3} = 2x+5 \ \ \forall x,$ or $\frac{2x^2 - x - 15}{x-3}$ as $2x - 5, (x-3)(x-5), \left(x + \frac{5}{2}\right)$. The literature (For instance, Siyepu, 2015; Maharaj, 2010; Pillay, 2008; Juter, 2006; Jordaan, 2005), has also documented that most students’ gap in computational abilities or algebraic manipulation skill from pre-calculus algebra restrict their performance in calculus concepts. According to Siyepu (2015, p.15), the difficulty roots from focus of prior learning, i.e. “prior learning subject to surface learning of familiar exercises.”
- Besides, some students lack proper handling of symbolic notation (for instance, $\lim_{x\to 3^+} = 3$, or $\lim_{t\to\infty} = \frac{5}{\infty} = 0$), which display their knowledge is based on symbolic manipulations that do not give attention to imbedded concepts.
- Thirty-five percent of participants demonstrated misinterpretation of the indeterminate form (Evaluate $\infty * 0 = 0, \frac{0}{0} = 0$, and $\frac{0}{0} = \infty$)**.** This agrees with the finding in the literature (Elia et al., 2009; Jaffar & Dindyal, 2011; Jordaan, 2005; Moru, 2006; Nair, 2010). The literature has found that most students are not aware of when to use these terms. According to Jaffar and Dindyal (2011), these difficulties rooted in the introduction of operations on real numbers. These misinterpretations together action views of the function are the main sources of

difficulties, in particular, to the limit of rational functions. Because, as students' test scripts revealed, after substitution when they get in indeterminate form $\frac{0}{0}$, they conclude that, either the limit is zero or the limit does not exist.

- Students also face challenges due to linguistic ambiguity in the contextual meaning of terms and their common language use: inconsistent concept image due to confusing terms like bounded and convergent, convergent and has a limit, bounded and monotonic, convergent or has a limit and monotonic.

- As noted from the students' qualitative description, besides the linguistic issue, misinterpretation of the monotonic-convergence theorem has its own share of blame the formation of these misconceptions. According to this theorem, while a sequence which is both monotonic and bounded, is necessarily convergent, the converse may not be true. What was observed is that most students interpret the converse as true. Because of this, many of them conclude that only monotonic and bounded sequences are convergent. It is also observed that 22% of participants think (as in item 1.3D) a constant sequence is not bounded; a constant sequence is not monotonic and hence not convergent. This difficulty is also observed in the literature, but the difference is the percentage, i.e. 22% is too much as compared to the figures in the literature. It has also been noted that those who interiorized actions into processes and able to coordinate processes have less of these linguistics concerning difficulties. Within the linguistic issue in the limit of functions: a limit is a boundary, a limit is never attainable, and a limit is approximation generalization was also observed. In particular, a limit is a boundary, and a limit does not exist necessarily imply that the function is unbounded were noticed from students' qualitative description.

- Most students have no coherence and consistency in their work and have conflicting concept images about a concept. They have a limited concept image of the limit of functions, as a result, their concept image of limit fails in to either all about an infinite process and nothing to do with finite value, or limit is all about a finite value and nothing to do with an infinite process. Only 28.2% of participants recognize the dual nature of the limit, i.e. limit involves an infinite process and has a finite value, provided it exists.

- Most students overgeneralize that the limit at a point is a substitution. If $\lim_{x \to c} f(x)$ does not exist, then the graph of f should have a vertical asymptote at $x = c$. A function will have no limit only if when the two-side limits have different values, the existence of the limit is sufficient for continuity of a function at a point, and every point of discontinuity is an asymptote. Most students' knowledge is limited and seems fair only for continuous functions. Most of these overgeneralizations rooted in the introduction of the limit (Tall, 1993). When the introduction of the limit was dominated by continuous functions, students, in turn, develop that limit is nothing but the same as the function's value at the limit point.

- Most students can compute a limit or differentiate a function, but they face a challenge to attach a meaning to the calculated value. For instance, in item 4.5a, only 5% of participants interpret the result of the computation of limit. Of course, some students also fail to demonstrate correct symbolic manipulation and computations. Some of the observed difficulties are misinterpretation of the limit rules and indeterminate forms, confusing properties of continuity with properties of derivatives, and confusing continuity and differentiability relationship. Besides, misinterpretation of the quotient rule for the derivative, over generalize like if $f'(x) = g'(x)$ then necessarily $f(x) = g(x)$, and low response rate for an item demanding interpretation of properties of a function from the graph or reverse thinking were observed. Moreover, confuse critical number with extreme value, unconscious about restrictions on domain values and low response rate for application problems and items in non-algebraic representation were observed. Almost certainly, students lack consistency, flexibility, and framework to solve a problem. Only a few participants demonstrated consistent beyond an action level conception and coordination of processes on the limit, continuity, and the derivative.

- Most students lack knowledge of representing function using different methods, lack knowledge of algebraic manipulation and their mental image of functions is restricted. Most students' knowledge is dominated by the symbolic world. Even for the same concept represented graphically and algebraically, the response is different in favour of the algebraic one. They have faced more challenges due to the lack of a problem-solving framework, to convert a given problem into a mathematical expression and solve. Thus, students' focus can be generalized as over dependence on procedural knowledge and over dependence on symbolic representation than another form of representation.

- Students seem more convinced procedurally explaining their ideas and giving particular examples than explaining quantitatively and justifying reasons. This may be that they are unfamiliar with such type of reasoning in their exercise and assessment. Even those who had some conceptual knowledge could lack a making connection between concepts. Students' knowledge is procedural and ridged than conceptual and flexible. Rational functions, pricewise-defined functions, and composition functions are more areas of attention. Students got the correct answer for a wrong reason. This shows that some of these difficulties may be persistent to overcome.

In general, the data obtained revealed that most students' level of conceptual knowledge is less than expected and their mean score on the test is below 50% of the total. Even those, who are classified as average in their performance, are good at symbolic manipulation and their knowledge is procedure dominated. Some students, who are classified as active, these are not more than 3.8% of all the participants, demonstrate: large example space, express continuity in the subject of limit, consistent concept image (including multiple representations), interiorize actions into processes, construct coordinated processes; and encapsulate processes into objects, have a problem-solving framework and a coherent framework of reasoning. These observed difficulties are categorised into themes as follows**:** a static view of the dynamic process, lack of describing definitions and relationship of terms, overgeneralization and inconsistent cognitive structure, over-dependence on procedural learning, lack of making a logical connection between conceptual aspects, a lack of a coherent and a flexible way of reasoning, and lack of procedural fluency and wrong interpretation of symbolic notations. Ways of thinking and approaches that caused these difficulties are also synthesized as: arithmetic thinking rather than algebraic, linguistic ambiguity, compartmentalized learning, a dependence on concept image than concept definition, obtain correct answers for wrong reasons, focuses only on the algebraic form of representation, and focuses on lower-level cognitive demanding exercises.

4.2. A Framework to Overcome Difficulties

This section encompasses the attempt made to answer the third research question. The components of an intervention model of learning calculus concepts that could be developed to enhance students' conceptual knowledge in calculus were extracted from the result of the literature in chapter two and the diagnostic assessment results in the preceding sections of this chapter.

The synthesis from literature and the diagnostic assessment revealed that students in the study area have difficulties that are not far from those in the literature with regard to analytical themes. In general, triangulated themes of difficulties and the approach or conceptual issues that are causes of these synthesized difficulties are summarized as in Table 25.

Table 25: Observed difficulties and their causes

Synthesized difficulties	Causes of these difficulties
• a static view of the dynamic process • lack of describing definitions and relationship of terms • overgeneralization and inconsistent cognitive structure • over-dependence on procedural learning • lack of making a logical connection between conceptual aspects • lack of a coherent framework of reasoning • lack of computational ability	• arithmetic thinking rather than algebraic • linguistic ambiguity • compartmentalized and surface learning • dependence on concept image, rather than concept definition • obtain the correct answer for the wrong reasons • focus only on the algebraic form of representation • focus on lower-level cognitive demanding exercises

4.2.1. Basic constructs of conceptual knowledge that should be addressed

It is true that teaching-learning occurs in a multifaceted system of interaction. Many educators use the triangle of interaction to describe a particular classroom culture. In this interaction, the students, the teacher, and the subject matter are the players and the classroom environment is the play-station. In this interaction, the traditional teaching-learning process of calculus is in general characterized as:

- The subject matter is just action on objects, algebraic manipulation (less on a graph), quantitative and objective description, dominated by familiar and routine type exercises.
- The teacher emphasizes how much knowledge has been acquired, focuses on the quantitative part of doing exercise, symbolic manipulations according to given rules, and first skill then concepts approaches.
- The classroom environment focuses on teachers' idea and whole-class lecture; as a result, students focus on memorizing rules and procedures, spot success and satisfaction, stack for items that are different from the textbook and

teachers made items (Hähkiöniemi, 2006; Aspinwall & Miller, 2001; Ferrini-Mundy & Gaudard, 1992).

The output of this process is characterized as rule-based thinking and procedural knowledge. Students learn the symbolic manipulations, but lack a sound conceptual knowledge of calculus (Bezuidenhout, 2001; Kinley, 2016; Lauritzen, 2012; Abbey, 2008). Figure 31 is pictorial design of this current practice. Now, the identified difficulties have occurred due to the limitation of this model, thus all the parts of the interaction are potentials for intervention.

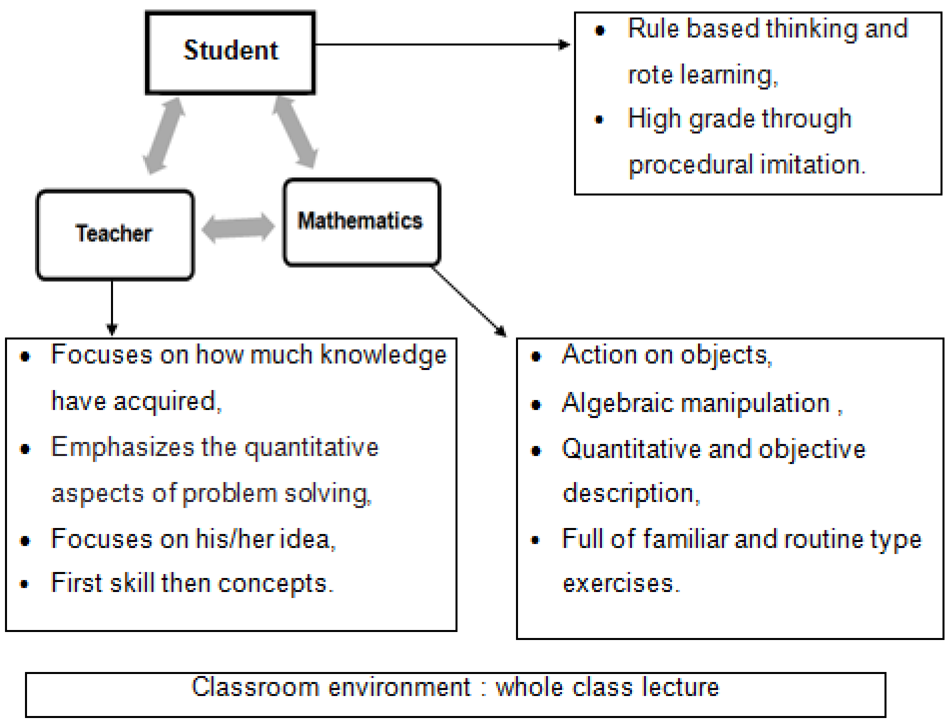


Figure 31: Model of traditional calculus classroom components

Nature and role of classroom tasks

The role of tasks presented by classroom teachers, textbooks, and reference books has an important influence on the resulting nature of students' knowledge (Aspinwall & Shaw, 2002**;** Berry & Nyman, 2003, Roble, 2017). Conventional teachers made exercises, test items, and textbook items are mostly lower-level (action level) cognitive demanding. In particular, for calculus, the literature revealed that current assessment tasks are procedural demanding than conceptual knowledge demanding (ibid). Teachers, students, and the textbook approaches contribute a share to the observed difficulties as their focus are largely on the manipulation of symbolic aspect on routine exercises as compared to problems or reasoning level exercises (Breen & O'Shea, 2010; Cangelosi, 2003; Keri et al., 2010; Kinely, 2016). Teachers in the usual approach do not prepare multi-step problems or activities that enhance such a conceptual knowledge and preparing such kind of activities are opportunities for intervention (Bransford *et at*., 2000; Bezuidenhout, 2001; Hiebert *et al.,* 2000). In particular, if tasks are designed to meet the constructs of conceptual knowledge, it will be appropriate to overcome the identified difficulties. Now, if tasks have to be designed in such ways that incorporate the components of conceptual knowledge identified as above, the next question is what should the teachers' role be?

Factors related to the teacher's role

Mostly teachers' knowledge can be categorised as content-knowledge (the content that the teacher knows about a specific subject) and pedagogical knowledge (the knowledge of teachers about teaching). With regard to classroom action, however, there is a very important third type. Shulman (1986 as cited in Bransford et al., 2000) describes the three categories of teachers' knowledge as follows: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge which are intertwined in practice. According to Shulman, pedagogical content knowledge comprises- the ability to present a specific concept in an uncomplicated approach and the awareness about students' hypothetical concept image of a specific concept. If those supposed concept images are difficulties, teachers need an understanding of an alternative approach to enhance students' learning. Specifically, this pedagogical content knowledge of teachers is influential in calculus. Currently, there is also a fourth component of these teachers' knowledge known as educational-technological knowledge (Koehler, Mishra, Kereluik, Shin, Graham, 2014).

Teachers with an integrated knowledge of these components of teachers' knowledge have the tendency to arrange their teaching platform and learning activities, so that their students initiate and cooperate to focus on the conceptual and embedded aspects of learning mathematics. From a pedagogical content knowledge perspective, the responsibility of the teacher is to be aware of students' difficulties, and to think of how to derive students differently towards conceptual knowledge approach. From the subject matter-content knowledge perspective, the teacher is responsible to established tasks that are genuine for students to reflect and communicate about the content they are learning including selecting and sequencing those tasks. From a curricular knowledge perspective, she/he has to know the prerequisites of the current topic, integration of topics within a subject, integration among subjects and real-life. From an educational technology perspective, if it is available, (s)he has to know how to handle the technology, how to integrate the topic to the technology and how to introduce it to the students without adding extra cognitive load to them.

Factors related to nature of the classroom environment

One of the constructs of conceptual knowledge is to reflect and communicate. To make a reflection and communication effect, the classroom culture should be the social constructive in nature. In particular, the classroom should be student-centred so that their preconception could be revealed and their voice is heard. They have to think, pair and share their thinking so that they convince themselves, convince friends and their concept image should be revealed and to be adjusted if necessary.

From all these parts of classroom interaction, observed difficulties, and causes of difficulties as identified in the previous sub-study, the following components of conceptual knowledge were significant to overcome observed difficulties and enhance students' conceptual knowledge.

4.2.1.1. Dual nature of concept development

Some of the difficulties in learning calculus emanate from a lack of mental structure developed to the required cognitive level of function and limit (Brijlall & Ndlovu, 2013; Çetin, 2009; Luneta & Makonye, 2010; Maharaj, 2013; Siyepu, 2015). The empirical study also revealed that most difficulties rooted due to the action view of functions, infinity, and limit. Thus, supporting the process-object development in general and providing students with activities that give them exposure to the interiorization of actions into process, coordination of processes, and the encapsulation of a process into an object would be valuable (Hähkiöniemi, 2006; Maharaj, 2010). Interiorization, encapsulation, and coordination are among the constructs in reflective abstraction used to describe how process and object-level conception are constructed and formulate APOS theory (Dubnisky, 2010). The detail of APOS theory is given in section 2.2.3.1. Thus, one room for intervention is to prepare activities that demand cognitive gymnastic on the duality of concepts.

4.2.1.2. Connection between forms of representations

Most researchers mentioned that lack of relationships among concepts and making logical connections between conceptual aspects and representations occurs due to “compartmentalized learning” and set as one of the major blocks for the construction of conceptual knowledge (Berry & Nyman, 2003; Kinely, 2016; Lauritzen, 2012). Hähkiöniemi (2006), describes, “Representation” as a tool to think of something. Representations are not only tools to think with but also tools for expressing our thoughts. Thus, a representation of a certain concept consists of an invisible internal system (concept image), and of a visible external system (a visual, verbal or symbolic reflection of the concept image) (Goldin & Shteingold, 2001). The internal representation of a concept is part of students’ cognitive structure, maybe a single or several computing parts, and serves to interact with the external world and the external system is symbolic and serves to facilitate the interaction (Dreyfus, 2002).

An individual’s representation about a concept is said to be rich if it includes various related features of the concept so that she/he demonstrates flexibility in solving a problem, otherwise it is said to be poor (Dreyfus, 2002). Such a rich mental representation, i.e. able to recognize or describing the same concept or idea using different forms of representation is necessary to be successful in mathematics (Tall & Mejia-Ramos, 2004).

Describing a concept using multiple forms of representations have been strongly connected with learning advanced concepts (Herbert, 2013), and more particularly, with the formation of conceptual knowledge in calculus that should be adaptable to a different context of a problem (Aspinwall & Shaw, 2002). Approaching a concept in multiple ways (visually, numerically, or algebraically) and convert easily from one form of representation to another is one aspect of having a conceptual knowledge (Lauritzen, 2012; Aspinwall & Miller, 2001; Zollman, 2014). Hähkiöniemi (2006) expresses that while procedural knowledge often stands for the ability to use representations; conceptual knowledge is described by the flexibility among representations.

One of the critics on the traditional calculus teaching-learning is that the practice is more focused on symbolic manipulations according to given rules than construct mathematical knowledge by solving problems and investigating patterns (Kinely, 2016; Schoenfeld, 1992). The advantage of being familiar with multiple forms of representation of a concept is that students turn out to be confident with a variety of algebraic, graph, table, numeric, and word descriptions of data.

4.2.1.3. Solve reasoning level problems

As calculus is a prerequisite to learn other concepts, the quality of students benefit from this course depends on their ability to solve problems beyond the calculus classroom. Conceptual knowledge, on the other hand, is characterized by students' ability to make logical connections between concepts, concepts, and procedures, flexibly solve problems given in various representations (Rittle-Johnson *et al.*, 2001). Thus, conceptual knowledge and problem-solving are inseparable components of the learning processes. Problem-solving can be used as a tool to enhance students' level of conceptual knowledge and conceptual knowledge, in return, is a tool to be successful in problem-solving (Tall *et al.,* 2000).

To consider problem-solving ability as a construct of conceptual knowledge, the context in which the term "problem" has defined is very important. An item is said to be a problem if it is non-routine in the sense that it is different from exercises in the textbook or used in the classroom by the teachers. It should be conceptual and subjective in nature rather than procedural and objective, open-ended, and qualitative rather than closed-ended and quantitative.

Incorporation of problem-solving in calculus teaching-learning assist students to move from routine exercises that most frequently focus on algorithmic skills, to non-routine exercises or problems that encourage conceptual thinking and demonstration of underpinning concepts and their connection in different ways. Thus, students' ability that they demonstrate- in a new situation beyond classroom exercises, in making connections among concepts in a variety of representations, and flexibility that lets them adapt adequately concepts, via problem-solving are basic constructs to attain conceptual knowledge.

Over-dependence on procedural learning and lack of recalling previous knowledge are the other aspects of difficulties. However, these difficulties are supposed to overcome through a shift of attention to reasoning level problems (Cangelosi, 2003) or non-routine exercises. Set of such reasoning level activities (include realistic problems combining more than one concept at a time) supposed to be valuable.

4.2.1.4. Mathematical thinking practice

Mathematical thinking is a thinking practice in learning mathematics developed based on the belief that students at all levels of schooling should be pass-through a situation that is similar to that of mathematicians are involved (Cuoco, Goldenberg & Mark, 1996). Cuoco et al, (1996 p.376) further mentioned, "The goal is not to train all students as a mathematician rather, to assist students to be trained and if possible adapt, the problem-solving approach and techniques that mathematicians used."

For Stacey (2006), mathematical thought often proceeds via two pair of processes: specializing and generalizing; conjecturing and convincing. On the other hand, based on an exhaustive literature exploration, Breen and O'Shea (2010) suggested five strands of mathematical thinking. These are conjecturing, reasoning and proving, abstraction, generalization, and specialization.

According to Stacey's investigation, the four aspects of mathematical thinking are defined as follows: specializing - trying special cases of a given condition, glance at specific examples; generalizing - searching for relationships and patterns; conjecturing - predicting relationships and results; and convincing - finding and communicating reasons why something is true.

Specializing is the process of learning through particular examples of a more general situation (Mason et al., 2010). Generalizing, on the other hand, is the process of extending a pattern from specific and a few cases to wide and vague cases (ibid). Thus, specializing can be considered as the foundation of generalization. Organizing the model that has been figured yields a conjecture. Additional specializing can maintain or disproof the created model or pattern. The process of validating the conjecture requires not only added generalization but also redirects in attention from supposition what might hold true, to looking why might it supposed to be true (ibid, p. 9).

While specialization refers to working on a number of specific illustrations which are particular instances of a broad situation in the concept to be taught (Mason *et al.,* 2010), conjecturing facilitate the learning by anticipating relationships among elements of these instances (Hashemi, *et al.,* 2015). However, these two aspects are not an end, rather they are a means to an end, which is the generalization drawn about the learned concept. That is why Hashemi *et al.* (2015, p. 233) wrote, "Specialization and conjecturing are pre-processing of generalization." Mason *et al.* (2010) also mentioned that successful specialization followed by constructive conjecturing facilitates generalization. They further mentioned that while "generalizations are the life-blood of mathematics" (p. 8), the whole development is "the essence of mathematical thinking" (p. 21).

Most students' difficulties in calculus emanate from a lack of generalization or making overgeneralization (Tall, 2002). For instance, the most common difficulty in calculus is that limit at a point is the value of the function at the limit point, providing the function is defined at that point, otherwise limit does not exist (Çiten, 2009). Thus, learning strategies' for calculus that aimed to improve generalization was suggested as being helpful (Tall, 2002; Mason *et al*., 2010; Hashemi *et al.,* 2015).

The other important aspect of mathematical thinking, according to Mason *et al*. (2010) is justifying and convincing, which corresponds to what Breen and O'Shea (2010), called reasoning and proving. This aspect of mathematical thinking corresponds to the task of "finding and communicating reasons why something is true." It is conceptual than procedural, deeper than the surface in that it requires one to think beyond ones' self-perspective. Reasoning and proving should pass through the three levels of convincing: convince self, convince a friend, and convince an enemy (Mason *et al.,* 2010). To attain this kind of thinking level, the teaching-learning environment should incorporate activities that lead to such practice including "questioning, challenging, and reflecting with ample space and time" (Mason *et al.,* 2010, p.144).

4.2.1.5. Reflection and communication via cooperative learning

The conventional teaching-learning is one-way communication and students have no chance to notice conflicting concept images. They even ignore the contextual meaning of a term, which is different from the common language use as they work on their concept images that may be different from the required concept definition (Juter, 2006). Thus, giving students a chance to communicate with their classmates in some sort of cooperative learning and allow them to reflect on reasoning level problems (Cangelosi, 2003), were supposed to be valuable to overcome these difficulties.

While reflection facilitates the cognitive aspect, communication will facilitate the affective aspect of learning (Hiebert *et al.*, 2000). Experience revealed student's communication in a small mixed ability group trouble their concept images. This is the starting point for progression. Thus, let students think of their conflicting concept images, give them exposure to comment most commonly occurring algebraic errors, misinterpretation of symbolic notations, and letting them comment on their own work. This supposed to be valuable to adjust conflicting concept images and overcome algebraic manipulation errors that they form intentionally or unintentionally.

In addition, the wrong answers for wrong reasons and wrong answers with high confidence often observed on students' performance. One way to avoid this is through students' exposure to thinking about their own thinking or "meta-cognition" (Schoenfeld, 1992). It is taught that through students' group work and allow reflection and reaction to their own answers or to others wrong answers and wrong workings, negotiate meaning to technical words and symbols, reason and justify to major steps in problem-solving are good scaffolding tool to overcome difficulties (Keri et al., 2010). Thus, designing activities that possess these constructs and implementing it in a social constructivism-learning environment was suggested. On the other hand, lack of computational ability that emanates from arithmetic thinking while algebraic thinking is demand was also observed. One tool to avoid this is using an inquiry approach (analysis of errors) and providing feedback accordingly.

4.2.1.6. Reconstructive generalization vis-à-vis cognitive conflict strategies

Overgeneralization occurs due to surface learning and the way concepts are introduced. This is one of the hidden difficulties of students because students or teachers in the usual ways of assessment do not notice it. Thus, most researchers suggest a qualitative analysis of students' answers and reasoning to analyse the true nature of students' knowledge. In particular, the literature (For instance, Luneta & Makonye, 2010) documented that students' performance indicates correct answers for wrong reasons and wrong answer with high confidence. This is noticed in the empirical study too.

As discussed in the theoretical part of the study, different types of improvement may take place in the cognitive structure of students' when they develop more experience about a concept. Such mental improvements are not always smooth, and some of them may cause cognitive conflict. Cognitive equilibrium is a process of resolving contradictions in once mental structure (Glasersfeld, 1995). Learning occurs when such conflicts are resolved through some sort of strategy. One of such a strategy is concept change or reconstructive strategy (Tall, 1993; Tall, 2002; Berry & Nyman, 2003).

A conceptual change strategy is based on the constructivist perspective of learning that learners have an active role in building and restructuring their cognitive structure and error and alternative conceptions are expected as part of the construction process. Thus, through activities allowing students to test special cases, identify examples and non-examples that contradict their overgeneralization and look for a pattern is a potential strategy to overcome these difficulties or to make an adjustment on their concept images.

In general, if the above-mentioned components are integrated into the present practice, it will be what it should have to be and allows students to overcome observed difficulties and enhance their conceptual knowledge. In particular, activities that demand "proceptual thinking" should be prepared. Additionally, students have to be assisted to make a connection between representations, provide qualitative and subjective descriptions, and develop skills and concepts parallel. Moreover, they have to focus on- how to use their knowledge (quality of knowledge), conceptual learning, and their own ideas. Further, they should have to be familiar with open problem-solving practices, exposed to unfamiliar and non-routine type problems. This all together gives students the opportunity to gain the batter level of conceptual knowledge and hence conceptual knowledge that can be extended beyond success in teachers made test items.

4.2.2. The proposed intervention model

Students' difficulties and the causes of these difficulties can be expressed in terms of an integrated theoretical background than a single theory. Thus, the overcoming framework is also best expressed in terms of a combined theoretical framework than a single theory. Accordingly, the constructs to overcome students' difficulties can be picked from different theoretical frameworks, and a combined intervention model could be designed. In particular, observed difficulties, causes of these difficulties and the identified components to overcome the difficulties are summarized as in Table 26.

Table 26: Observed difficulties, causes of these difficulties, and identified components to overcome the difficulties

Synthesized difficulties	Causes of these difficulties	Components for an intervention model
• a static view of the dynamic process • Lack of describing definitions & relationship of terms • Over-generalization and inconsistent cognitive structure • Over depend on procedural learning • Lack of making a logical connection between conceptual aspects • Lack of a coherent framework of reasoning • Lack of computational ability	• Arithmetic thinking than algebraic • Linguistic ambiguity • Compartmentalized and surface learning • Dependence on concept image than concept definition • Obtain correct answers for the wrong reasons • Focus only on the algebraic form of representation • Focus on lower-level cognitive demanding exercises	• Mathematical thinking practice: conjecturing and convincing • Reflection and communication via think-pair-share technique • Error analysis and reconstructive generalization vis-à-vis cognitive conflict strategies • Duality of concepts • Reasoning level and real-life problems • Widened their thinking through counterexamples and items that demand to conjecturing and convincing

Finally, to overcome students' difficulties in calculus, the study proposed an intervention that infuses a set of activities (hereafter called activity sheet) based on the identified components and adaption of the classroom environment accordingly. The infusion of activities (both for the class presentation and assessment) gives an opportunity to students, so that, they get exposure to: dual nature (proceptual) of thinking, make connections between representations, qualitative and subjective description as part of response to items, focus on quality of knowledge and conceptual learning, open problem-solving, making skill and concept parallel, and exposure to unfamiliar and non-routine type problems.

On the other hand, the classroom environment should be reform-oriented and characterized by: student-centred, effective communication, constructive approach, involving real-life, and reasoning level problems, students are allowed to explore and verbalize their mathematical ideas. Figure 32 illustrates the suggested intervention model.

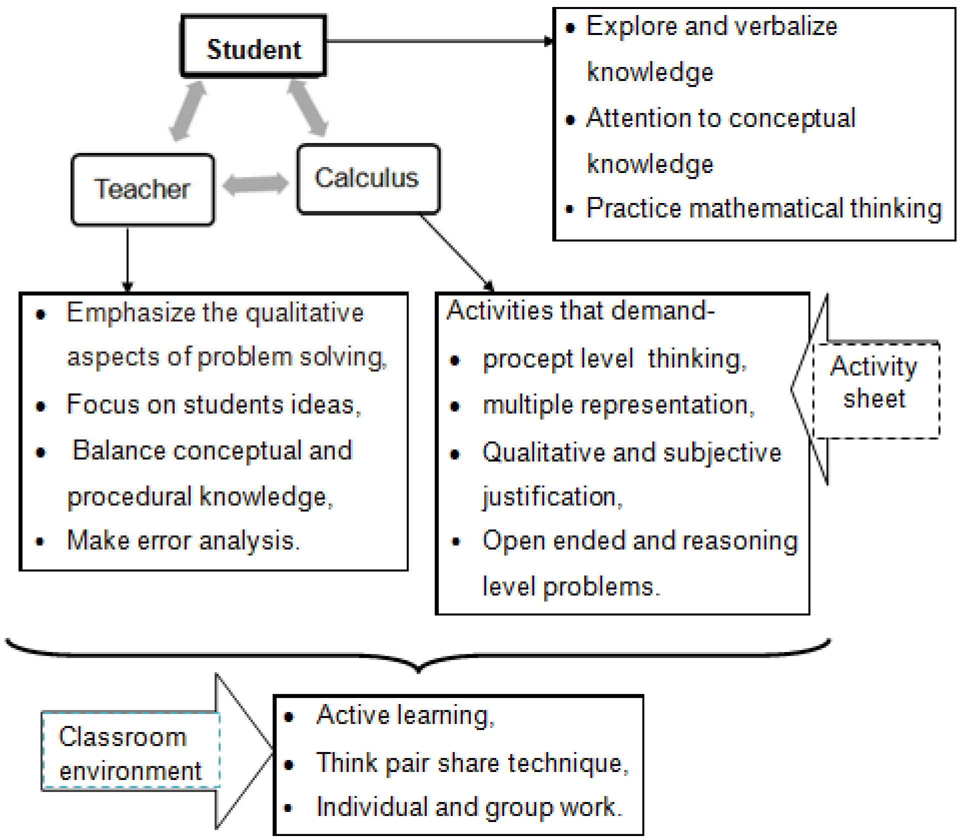


Figure 32: The intervention model

4.2.3. Intervention based on the proposed model

Based on the proposed model, an intervention was designed. The intervention includes arranging the teaching-learning environment according to the proposed criteria and working on the set of activities that aim to encourage attaining the constructs of conceptual knowledge specified in the proposed model and to lift students' knowledge to a higher-level aspect of mathematical thinking, which in turn reduce observed difficulties and enhance conceptual knowledge. The term "activity" refers to an open-ended or closed-ended item of classrooms, homework and formative assessment tasks that the students are asked to work on either on their own or in a group at the end of teachers' conventional introduction of each concept. The activities are compiled together and quoted as "activity sheet".

The activities are designed for the concepts: limit of sequences, the limit of functions, continuity, and derivatives. Most of the activities were selected from previous studies instruments, national exams, books, and the researcher designed some of them. Of course, even for those taken from the literature, all of them were modified to fit the intended purpose. The activities were pilot tested and some modifications were made based on the feedback collected. Finally, 30 items in which eight from the limit of sequences, 12 from the limit of functions and continuity, and 10 from the derivatives were selected (see appendix H).

The purpose of the activities is addressing observed difficulties, so that, students enhance their conceptual knowledge. The activities were collected based on the required constructs of conceptual knowledge and content of the grade12 mathematics syllabi. With regard to the type of items, the activities consist of both open-ended and closed-ended. Nevertheless, the closed-ended items also ask not only selecting the correct answer, but justification why a certain alternative is selected. The items also include scripts from students' work. This is deliberately done to give students an opportunity on how to "analyse errors" and think of their own thinking (see section 3.4 for the detail of the intervention activities).

4.3. Possible Effect of the Proposed Model

The section is aimed to answer the fourth research question that states:

> Is there a significant difference in the students' conceptual knowledge of calculus concepts after learning with the proposed model?

The experimental design tried to examine the cause-effect relationship between the use of the model and students' test scores on the concept test. Explicitly, the equation has the following null hypothesis:

I. Ho: There is no significant difference between the mean scores of students in the experimental group and the control group during the pre-test.
II. Ho: There is no significant difference between the mean scores of students in the experimental group and the control group during the post-test.

While the students' exposure to the proposed intervention model is the cause, the students' scores on the test is the effect. The proposed analysis technique was an independent t-test using SPSS version 25. But this technique has four assumptions (Field, 2009). Data distribution (it should be normal), measurement scale (at least interval level), homogeneity of variance and scores are independent. Accordingly, all the assumptions are assured based on the following facts:

1. The sample size is big enough to tolerate the violation of normality (Field, 2009).
2. Data is measured at a ratio scale.
3. The Levine's test (see Tables 27 & 28), assured that the variance has no significant difference hence the distribution is homogeneous.
4. The scores are independent as the two groups are different.

4.3.1. Comparison of mean scores

To determine the level of students' knowledge in the control group, and the experimental group before the intervention, a statistical test was computed for the pre-test results. The statistical test computed is a *t*-test analysis for an independent group using SPSS version 25 and it indicates that the 53 in the experimental group have a mean score of 32.19 and the 55 in the control group have a mean of 31.29. The two-tailed significance test indicates a $t = 0.502$ with 106 degrees of freedom, resulting in a two-tailed *p*- value of 0 .617. This *p* -value is not statistically significant because it is greater than alpha = .05. The result indicated that there was no statistically significant difference between the control group and the experimental group with respect to the pre-test scores. Accordingly, the null hypothesis is accepted, and the researcher concluded that the two groups were comparable before the intervention. See Table 27 for the display from SPSS version 25.

Table 27: Independent t-test statistics for pre-test result

			Score	
			Equal variances assumed	Equal variances not assumed
Levene's Test for Equality of Variances	F		3.498	
	Sig.		.064	
t-test for Equality of Means	t		.502	.504
	df		106	104.439
	Sig. (2-tailed)		.617	.616
	Mean Difference		.898	.898
	Std. Error Difference		1.788	1.782
	95% Confidence Interval of the Difference	Lower	-2.647	-2.637
		Upper	4.442	4.432

During the post-test, the 52 in the experimental group have a mean score of 28.10 with a standard deviation of 9.680 and the 53 in the control group have a mean of 20.26 with a standard deviation of 9.451. It has to be noted that three missed values were obtained. The two-tailed significance test indicates a $t = 4.195$ with 103 degrees of freedom, resulting in a two-tailed p -value of 0 .000. This p- value is statistically significant because it is less than alpha = .05 (see Table 2 for the display from SPSS version 25). Hence, the null hypothesis is rejected. The use of the proposed model had a significant effect on students' performances on conceptual items. Thus, it was found that those students in the experimental group had developed a more conceptual knowledge of calculus concepts as a result of the intervention.

Table 28: Independent t-test statistics for the post-test result

			Score	
			Equal variances assumed	Equal variances not assumed
Levene's Test for Equality of Variances	F		.382	
	Sig.		.538	
t-test for Equality of Means	t		4.195	4.194
	df		103	102.808
	Sig. (2-tailed)		.000	.000
	Mean Difference		7.832	7.832
	Std. Error Difference		1.867	1.867
	95% Confidence Interval of the Difference	Lower	4.129	4.128
		Upper	11.535	11.536

Effect size

In quantitative research, after testing a hypothesis, it is advisable to support the result by the magnitude of the effect (Green & Salkind, 2005). Accordingly, the Effect size was determined using the formula:

$d = t\sqrt{\frac{N_1+N_2}{N_1*N_2}}$ Where N_1 and N_2 are number of participants in the two groups (Green & Salkind, 2005). For $t = 4.195$ (as in the data in Table 28 from the SPSS), $N_1 = 52$ and $N_2 = 53$, $d = 4.195\sqrt{\frac{52+53}{52*53}} = .818$. This value indicates the effect is influential (Cohen et al., 2007). Nevertheless, is suggested to examine prior relevant research magnitude obtained on similar types of intervention so that current findings can be placed into an appropriate context about its practical value. Accordingly, on a study aimed to increase students' achievement in a calculus course, Pilgrim (2010) administered an intervention. The result was analysed into two different categories and found an effect size of 0.909 and 0.776 respectively. On the other hand, Fayowski (2005) in a study aimed to evaluate the effect of supplementary instructional programmes in first-year calculus found an effect size of 0.48. Thus, the comparison shows that this effect has practical significance.

4.3.2. Text analysis

The purpose of this section is to present the possible effects of the proposed intervention model on students' conceptual knowledge of calculus concepts and to examine whether students overcome their difficulties in a calculus. Since the quantitative analysis is necessary but not sufficient to conclude whether students overcome their difficulties, ways of thinking, justifications, and steps were analysed as per the considered concepts (the limit of sequences, the limit of functions, continuity, and derivative) qualitatively. The analysis is a form of text analysis via frequency coding and pattern analysis of the items to see whether the statistical significance has an implication for practical significance. It has to be noted that a result is statistical significance (not by chance) that does not mean that it has practical or educational significance (Fraenkel & Wallen, 2009). Actually, the practical significance is supported by the effect size. The attempt here is to make things more tangible by looking at the detailed effect of the students' test script. The following sections present the respective differences in the reasoning and procedures used to answer the given items in the two groups.

4.3.2.1. Students' conceptual knowledge of the limit of sequences

Among the items designed to assess students' knowledge of the limit of sequences, the average difficulty level of the items in the experimental group is 64.23% and that of the control group is 45.28%. The experimental group's mean score (6.98), is greater than that of the control group (5.13). Table 29 presents the first four items in the limit of a sequence and the compared results of the two groups.

Table 29: Breakdown of students' choices to item 1 to 4

Item		A		B		C		D		E		NR[14]	
		N	%	N	%	N	%	N	%	N	%	N	%
Exp. (N=52)	1	5	9.6	2	3.8	42*	80.8	2	3.8	1	1.9	0	0
	2	3	5.8	1	2	41*	78.8	5	9.6	2	3.8	0	0
	3	9	17.3	27*	51.9	2	3.8	8	15.4	6	11.5	0	0
	4	1	1.9	41*	78.8	6	11.5	2	3.8	1	1.9	1	1.9
Con. (N=53)	1	9	17	4	7.5	34*	64.1	3	5.7	2	3.8	1	1.9
	2	3	5.7	2	3.8	28*	52.8	6	11.3	12	22.6	2	3.8
	3	19	35.9	20*	37.7	7	13.2	6	11.3	0	0	1	1.9
	4	7	13.2	25*	47.1	2	3.8	16	30.2	0	0	3	5.7

* correct answer of the item

The data in Table 29, together with students' test script revealed that besides the difference in the correct answer the experimental group has developed better reasoning and justification habits. For instance, in item one in the experimental group, while 30 students provided reasons for their choice in which only 6 are wrong, 23 of them in the control group provided reasons in which 14 of them are wrong. From these wrong reasons, the following two difficulties were extracted:

Experimental group- a divergent sequence is neither increasing nor decreasing (2 respondents); a divergent sequence never bounded (2 respondents) and unrelated reasons (2 respondents).

Control group- only convergent sequence is bounded (4 respondents), a bounded sequence is necessarily convergent (3 respondents), a sequence is convergent only if it is bounded and monotonic (2 respondents), a divergent sequence is neither increasing nor decreasing (2 respondents), and unrelated reason (3 respondents). These reasons also revealed that students in the experimental group are able to overcome some of the difficulties in the interplay between terms.

[14] Non-respondents

In item 2, the students in the experimental group were able to overcome confusing the monotonic and bounded sequence theorem. In item 4, a similar difference in results was observed. In item 3, a significant gap was observed among those at action view and those reached a process level of concept formation.

In item 10, from incorrect workouts in the experimental group, the following two difficulties were observed: $\lim_{n\to\infty} nsin\frac{1}{n} = \lim_{n\to\infty} nsin(0) = 0$ (three respondents) and $\infty.0 = 0$ (two respondents). Whereas in the control group from the 13 incorrect workouts, the following difficulties were observed (comparison of the result is given in Table 30):

- Symbolic manipulation problems (for instance, $nsin\left(\frac{1}{n}\right) = n.sin1 = 1$) (two respondents).
- An action view of the limit and infinity, i.e. just substituting infinity instead of n (three respondents),
- limit as a boundary (three respondents),
- $\lim_{n\to\infty} nsin\frac{1}{n} = \lim_{n\to\infty} nsin(0) = 0$ (three respondents),
- $\lim_{n\to\infty} nsin\frac{1}{n} = \lim_{n\to\infty} \infty sin\left(\frac{1}{\infty}\right) = \infty$ (two respondents).

Table 30: Breakdown of students' choices to item 10

Group	Correct		Partially correct		Incorrect		Non-respondent	
	N	%	N	%	N	%	N	%
Experimental (N=52)	16	30.7	15	28.8	11	21.1	10	19.2
Control (N=53)	13	24.5	13	24.5	11	20.7	16	30.1

Figure 33 presents one student script from the control group that shows an action view of the limit.

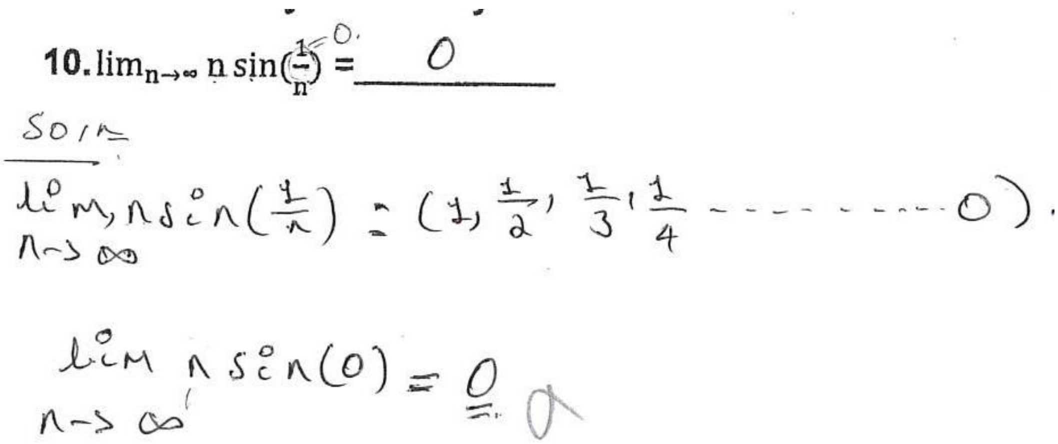
CS_{66}	

Figure 33: An extract of a student at action view of the limit from the control group

In general, the result in the limit of sequences revealed that the model had a practical significance in students' conceptual knowledge. In particular, above 51% of students in the experimental group attained- process view of the limit of sequences, potential view of infinity, able to qualitatively justify their answer, and overcome confusing definitions of terms. For some others, although the difficulties are not completely prevailed, the model is helpful in narrowing the diversity of the difficulties as compared to their counterparts in the control group.

4.3.2.2. Students' conceptual knowledge of the limit of functions

Among the items designed to assess students' knowledge of the limit of functions, the average difficulty level in the experimental group is 58.84% and that of the control group is 38.11%. The experimental group's mean score (5.88), is greater than that of the control group (3.81). Table 31 presents the five items in the limit of functions and compared results of the two groups.

The data in Table 31 revealed that the experimental group students performed higher than the control group students in all the five items did. From the reason for correct answers, the experimental group students have demonstrated fewer difficulties than the control group. For instance, in item 5 of the experimental group, from the 19 correct respondents, 16 of them provided reasons in which 10 of them are correct, and the three are unrelated. In contrast, in the control group out of the 16 correct respondents, only 10 of them provided reasons in which only four of them are correct.

Table 31: Breakdown of students' choices to items in the limit of functions

Item		Experimental (N=52)						Control (N=53)					
		A	B	C	D	E	NR	A	B	C	D	E	NR
5	N	6	14	3	6	23*	0	6	11	3	14	16*	3
	%	11.5	26.9	5.8	11.5	44.2	0	11.3	20.7	5.7	26.4	30.2	5.7
6	N	2	6	28*	2	12	2	3	5	26*	4	14	1
	%	3.8	11.5	53.8	3.8	23.1	3.8	5.7	9.4	49.1	7.5	26.4	1.9
7	N	2	6	2	4	36*	2	2	21	2	8	20*	0
	%	3.8	11.5	3.8	7.7	69.2	3.8	3.8	39.6	3.8	15.1	37.7	0
8	N	6	8	2	34*	1	1	8	11	4	23*	4	3
	%	11.5	15.4	3.8	65.4	1.9	1.8	15.1	20.8	7.5	43.4	7.5	5.7
9	N	2	32*	5	10	3	0	4	16*	11	12	9	1
	%	3.8	61.5	9.6	19.3	5.8	0	7.5	30.2	20.8	22.6	17	1.9

From item 6, it is observed that in terms of the correct choice, the difficulty is persistent, but the reasons revealed that students in the experimental group has developed a process view of function, but still lack to consider it as an object. In the control group, most of them explained it as an action. In item 7, students in the experimental groups clearly able to differentiate the case where limit fails to exist but in the control group, most of them still lack clarity. In particular, 21 (39.6%) of the students in the control group think that limit fails to exist only at the point of discontinuity and that is why alternative B got the high response rate.

In item 8, similar types of difficulties were observed in both groups, but very different in frequency. The difficulties were- we do not know the function (since the algebraic expression is not given), the limit value is the same as the function value, and limit is sufficient for continuity. In item 9, while $\frac{\mathbf{0}}{\mathbf{0}} = \mathbf{1}$ is mentioned only by one student in the experimental group it is mentioned by three students in the control group. While $\frac{\mathbf{0}}{\mathbf{0}} = \mathbf{0}$

is not mentioned in the experimental group it is mentioned by three students in the control group.

In general, the model has a significant effect on the experimental group in that it helps to overcome most of the difficulties in limit of functions. In particular, the experimental group students were better in terms of going beyond the action view of the limit of functions (reached a process view, but still lack to encapsulate the process into object/there is a limitation), differentiate the meaning of terms (limit does not exist, indeterminate, and infinity), able to manage overgeneralizations, identifying cases where limit of a function fails to exist, i.e. it may fail to exist due to discontinuity or being an unbounded. In addition, the frequency of correct answer for the wrong reason was reduced in the experimental group students as compared to those in the control group.

4.3.2.3. Students' conceptual knowledge of continuity

As seen in the diagnostic assessment, the interplay between the existence of the limit, continuity, and derivative was controversial for most students. The data in Table 32 revealed that students in the experimental group are able to overcome their confusion. It is observed that many students in this group, reason out by writing the statement, and the backwards implication on the interplay between the limits and continuity. In item 11, even if 23 (43.3%) of students in the control group got the correct answer, no one qualitatively explains the reason behind the procedures used to arrive at the solution.

Table 32: Breakdown of students' choices to item 11 and 12

Item		Experimental (N=52)						Control (N=53)					
		A	B	C	D	E	NR	A	B	C	D	E	NR
11	N	11	3	34*	2	1	1	3	2	23*	2	21	2
	%	21.2	5.8	65.4	3.8	1.9	1.9	5.7	3.8	43.3	3.8	39.6	3.8
12	N	4	40*	3	1	1	3	22	24*	2	1	2	2
	%	7.7	76.9	5.8	1.9	1.9	5.8	41.5	45.2	3.8	1.9	3.8	3.8

In item 16a, there is a big difference between non-respondents in the two groups. While in the experimental group only five (9.6%) students left blank in the control group 15 (28.3%) left the item unanswered. This indicates that the model somehow has a positive effect on students' motivation to think of alternative representations. Table 33 presents the breakdown of students' choices to item 16.

Table 33: Breakdown of students' choices to item 16

Item		Experimental (N=52)				Control (N=53)			
		Correct	PC[15]	Incorrect	NR	Correct	PC	Incorrect	NR
16a	N	17	6	24	5	14	2	22	15
	%	32.7	11.5	46.1	9.6	26.4	3.7	41.6	28.3
16b	N	38	0	12	2	30	0	16	7
	%	73	0	23	3.8	56.6	0	30.2	13.2
16c	N	29	5	13	5	21	5	18	9
	%	55.7	9.6	25	9.6	39.6	9.4	34	17

In general, on these items of continuity, the mean score in the experimental group is 5.84 and that of the control group is 4.05. While the average difficulty level of the experimental group is 69.23% that of the control group is 42.26%. Thus, students in the experimental group have improved their level of conceptual knowledge in the continuity of functions. In particular, their pre-calculus misconception (confusing continuity and connectedness), algebraic manipulation of rational functions and limit continuity interplay were improved.

4.3.2.4. Students' Conceptual Knowledge of Derivatives

In the derivative items, the mean score of students in the experimental group is 9.67 and that of the control group is 6.96. While the average difficulty level of students in the experimental group is 43.58% that of the control group is 33.75%. Table 34 presents the first three items in the derivative of functions and compared results in the two groups. The data in the table and students' test script revealed that the number of students in the experimental group who able to overcome their difficulties in the derivative is more than those who are able to overcome their difficulties in the control group. Moreover, it is observed that students in the experimental group

[15] Partially correct

familiarized themselves with writing a sequence of statements that justify the reasons behind the procedures. For instance, in item 14, many students noticed that since the function is differentiable the limit of the different-quotient exists both from the right and from the left of the limit point, i.e. two, which is a very short method to compute one of the required. Figure 34 is one of these students' scripts from the experimental group.

ES35

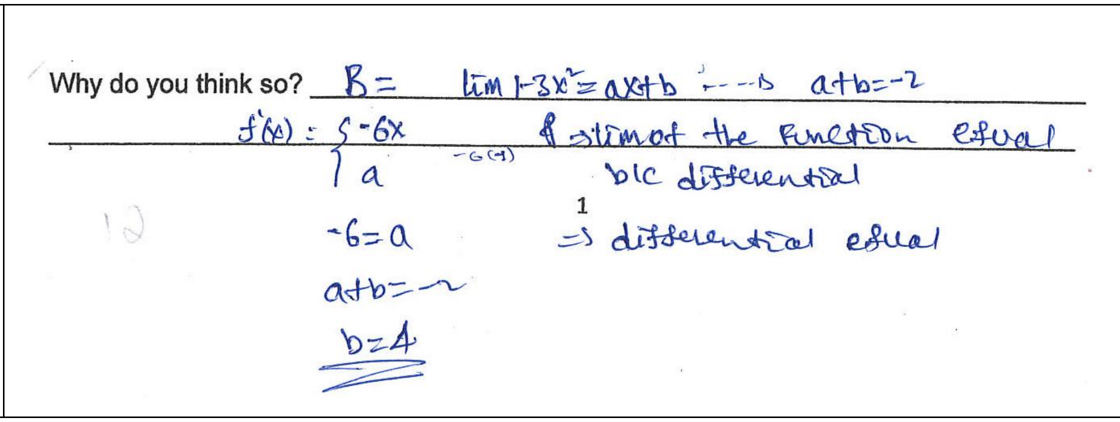


Figure 34: An extract of correct answer with correct procedure and reasoning

Table 34: Breakdown of students' choices to item 13 to 15

Item		Experimental (N=52)						Control (N=53)					
		A	B	C	D	E	NR	A	B	C	D	E	NR
13	N	6	2	11	8	25*	0	9	6	20	2	13*	3
	%	11.5	3.8	21.2	15.4	48.1	0	17	11.3	37.7	3.8	24.5	5.7
14	N	0	41*	6	2	2	1	7	29*	3	5	6	3
	%	0	78.8	11.5	3.8	3.8	1.9	13.2	54.7	5.7	9.4	11.3	5.7
15	N	5	2	3	42*	0	0	8	2	5	32*	4	2
	%	9.6	3.8	5.7	80.7	0	0	15.1	3.8	9.4	60.4	7.5	3.8

In item 13, misinterpretation of the quotient rule is still a source of confusion for most students in both group i.e. many students had thought that since $f(x) = \frac{h(x)}{g(x)} = \frac{h'(x)}{g'(x)} = \left(\frac{h(x)}{g(x)}\right)'$ then $f(x) = e^x$. That is why choice C has a high response rate in both groups as compared to the other distractors. Figure 35 is one of the students test script from the control group.

CS$_{85}$

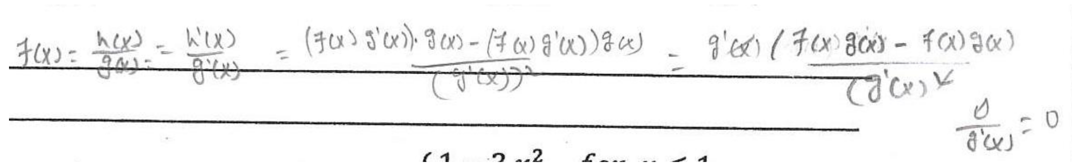


Figure 35: An extract of the misinterpretation of the quotient rule

In item 17, while both groups are ignorant of the mathematical procedure, and the contextual restriction (whereas the domain $t > 0$, they consider both $t = \pm 1$ as critical points), the experimental group is better in terms of algebraic manipulation, confuse a critical number with an extreme value, and in terms of infinity image. In item 17a, only a few students from the experimental group gave an interpretation for the given quantity, and one is as shown in figure 36.

ES$_{52}$

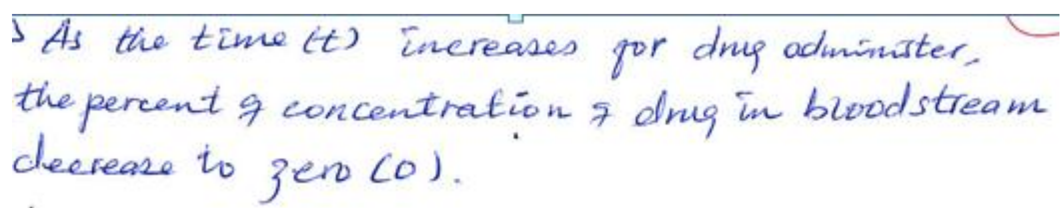


Figure 36: An extract of reasoning ability from the experimental group students

Table 35 summarises both group students' responses to items 17 and 18. The result obtained from item 18a and 18b revealed that in making the connection among concepts, both groups have a comparable result, but the difference is the interpretation of the result.

Table 35: Breakdown of students' response to item 17 and 18

Item		Experimental (N=52)				Control (N=53)			
		Correct	PC	Incorrect	NR	Correct	PC	Incorrect	NR
17a	N	21	10	13	8	12	13	13	15
	%	40.3	19.2	25	15.3	22.6	24.5	24.5	28.3
17b	N	22	8	16	6	20	12	12	9
	%	42.3	15.3	30.7	11.5	37.7	22.6	22.6	17
17c	N	22	0	26	4	22	0	17	14
	%	42.3	0	50	7.7	41.5	0	32	26.4
17d	N	28	0	14	10	20	2	15	16
	%	53.8	0	26.9	19.2	37.7	3.7	28.3	30.2
18a	N	16	8	17	11	15	0	22	16
	%	30.7	15.3	32.7	21.1	28.3	0	41.5	30.2
18b	N	11	0	27	14	7	0	28	18
	%	21.1	0	51.9	26.9	13.2	0	52.8	34

From what is presented above it is enough to conclude that students in the experimental group performed better than the students in the control group on the test items. Their ways of thinking, reasoning, and justification are also improved. Their concept images were adjusted, and they were able to even answer items that were left blank in by all students in the control group. Thus, the model was helpful to overcome most of the difficulties, and even to narrow the diversity of difficulties that are persistent. In general, the average difficulty level of the items in the experimental group is 56.41% and that of the control group is 38.83%. The experimental group mean score (mean=28.10, and SD=9.680), is greater than that of the control group (mean=20.26 and SD=9.451).

4.3.3 The possible effect of the model via the theme of difficulties

Since the theme of difficulties are the major areas of concern, the result of the intervention for each theme of difficulty is described as follows.

A static view of the dynamic process: In the intervention activities 4, 5, 7, 17, 18, and 19 aimed to address this theme of difficulty. At the end (in particular as revealed by the result from post-test items 3, 5, 6 and 17a) 50% of students attained process view of limit. Moreover, the result from item10 and 17a revealed that many students (44.2%) avoided plugging infinity as a number to calculate limit at infinity, minimized confusing terms like undefined, indeterminate, and infinity.

Lack of describing definitions and relationships of terms: In the intervention, this difficulty was addressed in two ways. The first is allowing students to work on activities 1, 8, 21, 22 and 27. The second is to use reflection and communication via think-pair-share technique. After students work individually for a few minutes, they were allowed to work in pairs and share what they thought individually and comment to each other. As revealed from the result in the post-test (Item 1, 5, 8, 9 and 11), above 60% of students were able to overcome such difficulties related to definitions and terms. In particular, the number of correct answers for wrong the reasons was significantly different on these items. As the discussion proceeded, students got the chance to notice conflicting concept images and even some of them were able to notice that their working is correct answers for wrong the reasons. That helped them a lot in terms of developing meta-cognition.

Overgeneralization and inconsistent cognitive structure: This theme of difficulty was also addressed through activities that evoke concept change (conflict teaching), including items that ask conjecturing and convincing, reconstructive generalization vis-à-vis cognitive conflict strategies. For this purpose, activities 1, 2, 3, 7, 8, 9, 13, 16, 17, 22 and 27 were included. These resources were proposed to create cognitive trouble of students' concept images. Group discussions and questions asked during the discussions have promoted students to analyse and reflect on their methods and reconfigure their conceptions.

It is observed that (in particular from post-test item 1, 2, 3, 5, 7) many of the students are able to defeat the formation of such overgeneralizations. In addition, the number of students who got correct answers for wrong reasons in the experimental group is relatively less than that of the control group per each item. For instance, in item 5, in the experimental group from the 19 correct respondents, 16 of them provided reason in which 10 of them are correct and the three are unrelated. On the other hand, in the control group out of the 16 correct respondents, only 10 of them provided reasons in which only four of them are correct. The result from both the distractors and the justification provided by the students for the multiple-choice items revealed that the model helped them to narrow the diversity of their inconsistency.

Lack of making a logical connection between conceptual aspects, and a coherent framework of reasoning: Most of the activities in the intervention (3, 9, 10, 11, 12, 13, 14, 15, 16, 17, 22, 27, and 28) address this theme of difficulty. The result of the post-test (item 9, 11, 12, 13, 14, and 18) revealed that many students were able to defeat difficulties of making connections between conceptual aspects (domain, limit, continuity, and derivative) and qualitatively describing their knowledge.

Although some students' ability to solve problems in different representations (as observed in items 2, 4, 16, 15, 17 and 18), most of them keep on using only one representation, and find it hard to include multiple representation in their reason and justification. For instance, only two students try to demonstrate item 17 using a graph. In particular, item 18 is unique in that students are not familiar in terms of such representation and only active students are able to interpret the given data from the table.

Over-dependence on procedural learning: In the intervention, activity 5, 7, 15, 16, 17, 18, 19, 24, 25, 26, 29, and 30 were planned to address this difficulty. In item 17 from the post-test, the experimental group has 44.71% and the control group has 34.90% correct response rate. Although the experimental group students' score is better than those in the control group, the problem still persists and needs attention.

Some studies which report positive effects of an intervention lack to assure whether that positive effect is due to the presence of conceptual knowledge or memorization of procedures (Çetin, 2009). According to the literature, some studies found success to enhance conceptual knowledge using APOS and computer programs (ibid). However, in this study area, students at grade 12 level have no access to this educational technology.

Poor computational ability and algebraic errors: In the activities, attention was given to incorporate most frequently occurring algebraic errors (as in activity 6, 20, and 23). In the reflection, and error analysis part, most students start revising their own procedures, and able to notice and correct their algebraic errors. In particular, in items 10, 16, and 17 both in the diagnostic assessment, and in the control group, many students' were observed who start the procedure correctly and got a wrong answer due to algebraic manipulation errors. In the contrary, in the experimental group, many of them able to notice the errors and tried to correct it. The challenge still is that most of these algebraic difficulties originate from pre-calculus and need time to be avoided.

In general, both the quantitative and qualitative result in section 4.3 has shown that the proposed model was valuable to overcome observed difficulties.

CHAPTER FIVE: DISCUSSION, CONCLUSIONS, AND RECOMMENDATIONS

The main purpose of the study was to explore how to enhance students' conceptual knowledge of calculus concepts by developing a literature informed intervention model. To meet this purpose, three sub-studies were accomplished. This chapter presents a summary of the study (5.1), discussion of the results (5.2) followed by a conclusion of the study (5.3). The chapter ends with recommendations for practice and further study (5.4).

5.1. Summary of the study

The study begins with a systematic review of existing literature on students' difficulties in understanding calculus concepts. A literature search from international and local sources was conducted. Using eight set criteria 43 articles that range from 2002 to 2016 were selected for the last analysis. The review concluded with seven themes of difficulties.

A diagnostic assessment aimed to explore students' difficulties in calculus and the causes of those difficulties in the study area was also conducted. For this, a diagnostic assessment (a concept test of 18 items) was prepared. Informed by the literature and experience, the items were selected, piloted, and evaluated to improve the validity and reliability of the test. Finally, the test was administered to 238 grade 12 natural science stream students selected purposefully from four different schools in one administrative zone of Ethiopia. To analyse the test results, first respondent scripts for each item were categorized as correct, incorrect and no response. Second, for each item, the respondents' errors were identified by looking for the wrong choice or wrong working from the respondents' scripts. Since these wrong answers constitute difficulties, ways of thinking, and origins of difficulties that students have, the data was read over and over to get an overall picture of the type of difficulties students have and to look at how they approach these conceptual issues. Finally, from the test scripts, components of students' difficulties and supposed causes of those difficulties were identified.

A desktop review of those identified difficulties and the supposed causes of those from different perspectives were analysed. By comparing the limitation of the traditional approach and by incorporating those with basic constructs of conceptual knowledge (as identified in the theoretical framework), components of an intervention model was identified and a new model of intervention was proposed.

Based on the model an intervention was developed and administered. The intervention was a set of activities that aimed to enhance students' conceptual knowledge in calculus. The activities compiled together and named the activity sheet. The experimental group teacher had received training and orientation on how to carry out the proposed model. A copy of the activity sheet was given to students in the experimental group. The classroom environment was also adjusted as specified in the model that includes students' active participation, group work, error analysis, and reflection. For the intervention, two intact classrooms (108 in number) were selected and assigned randomly to control and experimental groups. Earlier to the intervention, a pre-test of 25 items from pre-calculus concepts (sequence, polynomial, rational, exponential, and trigonometric functions, the graph of functions and coordinate geometry) aimed to assess the students' level of knowledge was administered. Then, the intervention was administered for eight weeks, 80 minutes per week parallel to the normal teaching-learning program for the experimental group students. In the intervention session, the students were arranged in a mixed ability group of five to six.

A week after the intervention was terminated, a post-test that aimed to examine the students' conceptual knowledge in calculus was administered. The test items were selected from the items in the diagnostic assessment. The result was analysed using a t-test for an independent sample with the help of SPSS version 25. A textual analysis of the test result also made to see the possible effect of the intervention.

5.2. Discussion of the results

The main purpose of the study was to overcome students' difficulties and enhance their conceptual knowledge of calculus by developing a literature informed intervention model. The discussion is presented in the order in which the research questions were asked and listed under separate subheadings.

5.2.1. What does the current literature reveal about students' difficulties in learning calculus concepts?

The results from forty-three systematically selected articles (see appendix A for the list of articles) indicated that students' knowledge gap is manifested in the following ways.

- a static view of the dynamic process,
- lack of describing definitions and relationship of terms,
- overgeneralization and inconsistent cognitive structure,
- over depending on procedural learning,
- failure to make a logical connection between conceptual aspects,
- lack of a coherent framework of reasoning,
- a lack of computational ability.

The literature also revealed that only a few students demonstrated strength in calculus that evidences through avoidance of these synthesized difficulties. In addition, the strength can be manifested through large example space, consistency in concept images (including multiple representations), express continuity in terms of the limit, interiorize actions into processes, construct coordinated processes; encapsulate processes into objects, have a problem-solving framework and having a coherent framework of reasoning.

5.2.2. What are the common conceptual issues that cause students' difficulties in calculus?

The diagnostic assessment revealed that students of the study area have difficulties that are not far from those in the literature in terms of analytical themes. Besides, the diagnosis assessment revealed the causes of these difficulties in terms of the following points.

- arithmetic thinking than algebraic,
- linguistic ambiguity,
- compartmentalized and surface learning,

- more dependence on concept image than concept definition,
- get the correct answer for the wrong reasons.
- focuses only on an algebraic form of representation,
- focuses on lower-level cognitive demanding exercises were explored as causes of those difficulties from students' part whereas, the focus of attention (activities, tasks and assessment items) in which all are more procedural than conceptual and lack of working on real-life problems were identified as factors that contribute to those difficulties from the curriculum and teachers part.

The finding of the study stated under the research question that reads, what are common conceptual issues that cause students' difficulties in calculus, are in line with some other studies (Blaisdell, 2012; Çetin, 2009; Duru, 2011; Jaffar & Dindyal, 2011; Jones, 2015). As Çetin (2009) and Duru (2011), teachers focus on information transition and surface learning while the subject demands deep approach to learning. Moreover, students view calculus as a collection of procedures to memorize. As Blaisdell (2012) said, representation and question formats influence students' concept images. Likewise, Maharj (2010) and Jaffar and Dindyal (2011) argue that students' difficulties are an effect of not having the proper mental structure. While Jones (2015) and Elia et al. (2009) suggest the infusion of realistic problems as opposed to routine and lower-level cognitive ability demanding to overcome the difficulties, Jayakody (2012) suggest the inclusion of the cognitive conflict strategy. Generally, major attention of researchers to enhance students' conceptual knowledge is to do well on the nature of activities used in teaching-learning.

5.2.3. What are the components of an intervention model of learning calculus concepts that could be developed to enhance students' conceptual knowledge in calculus?

The third research question focuses on intervention. Thus, the researcher guided by all these data (i.e. the literature, the empirical evidence, and his experience) developed an intervention model (see figure 32). The model was intended to enhance conceptual knowledge by focusing on:

- the duality of concepts,

- reasoning level and real-life problems,
- error analysis,
- mathematical thinking practice: conjecturing and convincing,
- reflection and communication via think-pair-share technique,
- reconstructive generalization vis-à-vis cognitive conflict strategies,
- widening students' thinking and example space through counter-examples & items that demand to conjecture and convincing, include activities that demand make a connection between forms of representations.

Finally, these identified components and the description of their interaction is pictorially presented as in see figure 32.

5.2.4. Is there a significant difference in the students' level of conceptual knowledge of calculus after learning with the proposed model?

After the implementation of the model, a post-test was administered to both experimental and control group students and the result was analysed both quantitatively and qualitatively. The quantitative analysis revealed that the intervention had a positive effect. The experimental group score (mean=28.10, SD=9.680) is better than the controlled group score (mean=20.26, SD=9.451) with independent t-statistics, $t = 4.195$ with alpha =.05. This result suggests that students in the experimental group performed significantly better than the control group. This result has also practical significance (Effect size .818). The qualitative analysis revealed that students in the experimental group are able to overcome many of the difficulties and misconceptions observed in the literature and the diagnostic assessment.

The result of the study provided an understanding and insight of the stipulated research questions. Some of the difficulties in learning calculus emanate from lack of mental structure developed to the required cognitive level (process and object level) of function and limit. In particular, the result revealed that most difficulties rooted due to an action view of function, infinity, and the limit process. It is claimed that, arithmetic thinking than algebraic is the cause of these difficulties and supporting the process-object development in general and providing students with activities that give them exposure to interiorization of actions into a process, coordination of processes, and encapsulation of the process into an object is valuable to overcome the difficulties (Hähkiöniemi, 2006; Maharaj, 2010). The result from the intervention showed that 50% of students are able to attain the process view of the limit. Likewise, the work of Maharaj (2010) asserts that attaining the process level is the most challenging in the process of concept formation. Besides, the result revealed that 44.2% of students avoided plugging infinity as a number to calculate the limit value at infinity, minimized confusing terms like undefined, indeterminate, and infinity.

The literature is full of evidence that confirms most students' reasoning lacks process view of the limit (Wangle, 2013; Jones, 2015; Oehrtman, 2002; Roh, 2005; Takaci et al., 2006). One difficulty with the concept "infinity" is considering it as an object or plugged in infinity as a number to calculate the limit at infinity while process view is required (Jones, 2015; Moru, 2006; Oehrtman, 2002; Parameswaran, 2007; Roh, 2005). The model seems adequate in terms of assisting students to attain a process view, dynamic reasoning and process view of infinity. However, still many students lack to encapsulate the process as an object. Thus, more time needs to be devoted to plan more activities and help students develop the required mental structure.

Failure to describe definitions and the interplay of concepts was one of the themes of difficulties identified. Others mentioned that failure to make a logical connection between conceptual aspects occurs due to compartmentalize and surface learning and set it as one of the major blocks for students' progression (Berry & Nyman, 2003; Kinely, 2016; Lauritzen, 2012). The empirical data also revealed that students face a challenge due to linguistic ambiguity in a contextual meaning of terms and their common language uses inconsistent concept image due to confusing terms like bounded and convergent, convergent and has a limit, bounded and monotonic, convergent or has a limit and monotonic. Due to the linguistic ambiguity, most students show difficulties in the limit of sequences. Like the limit value is necessarily a boundary, a bounded sequence is necessarily convergent, a divergent sequence is necessarily unbounded, a monotonic sequence is necessarily convergent, a convergent sequence may not be bounded, and if consecutive terms of a sequence alternate in sign then the sequence is necessarily divergent are the difficulties.

The researcher supposed that incorporating reflection and communication via think-pair-share technique in the learning process reduces such problems related to linguistic ambiguity, compartmentalized and surface learning. In the intervention, after students had worked individually for few minutes, they allowed working in a group, share what they thought individually, and comment with each other. The result revealed that 60% of students were able to overcome such difficulties related to definitions and terms. As the discussion proceeded, students got a chance to notice conflicting concept images and even some of them were able to notice that their working is correct answers for wrong reasons so that the discussion helped them a lot in terms of developing meta-cognition. In fact, the difficulties in calculus are systematic and hidden to students. In the usual teaching and assessments practice, students have no chance to notice conflicting concept images, even if it is a common practice that they give a correct answer for an incorrect reason. Thus, giving students a chance to communicate with their classmates in some sorts of cooperative learning and allow them to reflect on reasoning level problems (Cangelosi, 2003), were supposed to be valuable to overcome these difficulties.

From a constructivist-learning point of view, learning is an adaptive activity; learning depends on a context where it occurs. Meaningful learning occurs when students are in a context where it occurs and builds up ways to come out of a difficult situation. Such circumstances encourage the students, as it is an exposure to hardship, pleasure, and satisfaction inherent in solving problems. Problem-solving ability helped students to extend what they learned in the classroom to situations that they meet in real-life (keri et al., 2010).

The set of articles (Such as Jaffar & Dindyal, 2011; Jayakody, 2012; Maharajh et al., 2008; Denbel, 2015) documented that students do not pay attention to the contextual meaning of terms in problem-solving; that is, the calculation is based on their concept image and not the concept definition. As a result, most students conclude that a function does not attain its limit value, the limit value is unreachable or the limit is an approximation and confuses continuity with connectedness (Cetin, 2009; Jordaan, 2005; Vela, 2011; Wangle, 2013). The result in the intervention has shown improvement as compared to the result of the study by Denbel (2015). Besides, in terms of the application problems, the result is in line with the finding of Cetin (2009).

The experiences gained during the pilot test of the concept test revealed that the type of items provided to students affect their orientation and performance. The observation was that students appreciated the existence of such type of conceptual items in calculus. The researcher strongly believes that one of the ways to come out of the current practice, which is characterized as procedure oriented teaching-learning, is to use the component of mathematical thinking, i.e. providing students with the exposure to justify, reason, interpret or prove what they are manipulating. In mathematical thinking, this is termed as “convincing” (Mason, Burton & Stacey, 2010). However, to do so the teachers have to provide such activities to the teaching-learning environment. In particular, problem-solving change students’ focus from purely computational in nature to computation correlated to real-life (Kelley, 2006). Above all, the construct of mathematical thinking: convincing, also called reasoning and proving (Mason *et al.,* 2010) is supposed to be good as it encourages students to explore and to visualize their mathematical ideas.

Irrespective of whether it is the limit, continuity or derivative, both the literature and the empirical data documented that the majority of students demonstrated over generalisation or immature conception (Duru, 2011; Jordaan, 2005; Maharajh et al*.,* 2008; Nair, 2010; Vela, 2011; Wangle, 2013). Due to overgeneralisations, students sometimes show correct answers for wrong reasons and wrong answers with high confidence (Çetin, 2009; Luneta & Makonye, 2010; Moru, 2006; Przenioslo, 2003). Thus, overgeneralised or immature conceptions, but not noticed by students accordingly hence conflicting concept images (Juter, 2006) seems troublesome when learning calculus concepts. In the intervention, such difficulties were addressed by incorporating activities that evoke a concept change (conflict teaching), including items that ask conjecturing and convincing, reconstructive generalization vis-à-vis cognitive conflict strategies. The resources were planned to create cognitive trouble in students’ concept images. Group discussions and questions asked during the discussions have promoted students to analyse and reflect on the methods they are using and reconfigure their conceptions and hence able to adjust their difficulties.

It is observed that many of the students are able to defeat the formation of such overgeneralizations. In addition, the number of students who got a correct answer for the wrong reasons in the experimental group is relatively less than that of the control group per each item. For instance, in item 5, in the experimental group from the 19 correct respondents, 16 of them provided reasons in which 10 of them are correct, and three are unrelated. On the other hand, in the control group out of 16 correct respondents, only 10 of them provided reasons in which only four of them are correct. The result from both the distractors and the justifications provided by the students for the multiple-choice items revealed that the model helped them to narrow the diversity of their inconsistency. Students build generalization inductively through time from their learning experience. Nevertheless, when solving a problem, they use deductive reasoning (Cangelosi, 2003). This argument implies that students' achievement in problem-solving depends on their generalization schema. Thus, working on the students' ability to decrease their overgeneralization will improve problem-solving ability.

The literature has also revealed that most students either do not respond at all, or they show low success for unfamiliar items or items demanding higher levels of cognitive thinking (Horvath, 2008; Juter, 2006; Makonye, 2012; Roh, 2005; Usman, 2012). In item 17 from post-test, the experimental group has 44.71%, and the control group has a 34.90% correct response rate. Although the result seems promising, the ability to extend or apply their knowledge to unfamiliar items still persistent and needs attention.

One other construct of conceptual knowledge in calculus is being familiar with multiple forms of representation of a concept. When students develop multiple form of representing a concept, i.e. algebraic, graph, table, numeric, and word descriptions of data they turn out to be confident and flexible in their reasoning. Of course, if not properly manipulated, the use of multiple representations has its own limitations. Taught the same concept with different representations, unless they master sorting out the different forms of the same concept their cognitive load would be junk (Dreyfus, 2002).

Although some students are able to solve problems in different representations, most of them keep on using only one representation and find it is hard to include multiple representations in their reason and justification. For instance, only two students try to show item 17 using a graph. In particular, item 18 is unique in that students are not familiar in terms of such representation and only active students are able to interpret the given data from the table. This finding aligns with what Blaisdell (2012), Moru (2006), and Wangle (2013) documented. Likewise, Abbey (2008) found that students' knowledge and attitude to graphical form in calculus is deficient for different reasons. The literature has strong evidence that the use of technology allows multiple representation perspectives. In addition, some others reported success in improving students' conceptual knowledge in calculus using APOS and computer programs (e.g. Çetin, 2009). Thus, incorporating the model with technology may avoid the limitation. However, the problem is that in this study area student at grade 12 level have no access to educational technology.

In general, the model has a practical significance in terms of enhancing many students to attain process view and dynamic reasoning, reducing difficulties related to definitions and terms, reduces a correct answer for wrong reasons, narrowing the diversity of inconsistent concept images, facilitate making a connection between conceptual aspects, and reducing algebraic errors. On the other hand, the model has limitations in terms of contributing to attain encapsulation, apply multiple representations, and establish a strong problem-solving framework.

Still, the proposed model needs modification. Initially, the model was developed for the context where the practice of educational technology is absent. Nevertheless, if available it is possible to integrate with the proposed model, as it is generic. One weakness of the model is that many of the students still lack to encapsulate the process into an object and lack to focus on embedded concepts. However, by adding activities that can be done through computer programs, the weakness may be resolved. Thus, the model with all its strength and its limitation, if integrated with computer-assisted activities, students can be better equipped with the required conceptual knowledge. After all, to keep its strengths and avoid its weaknesses, the researcher suggests the modified version of the model (see figure 37).

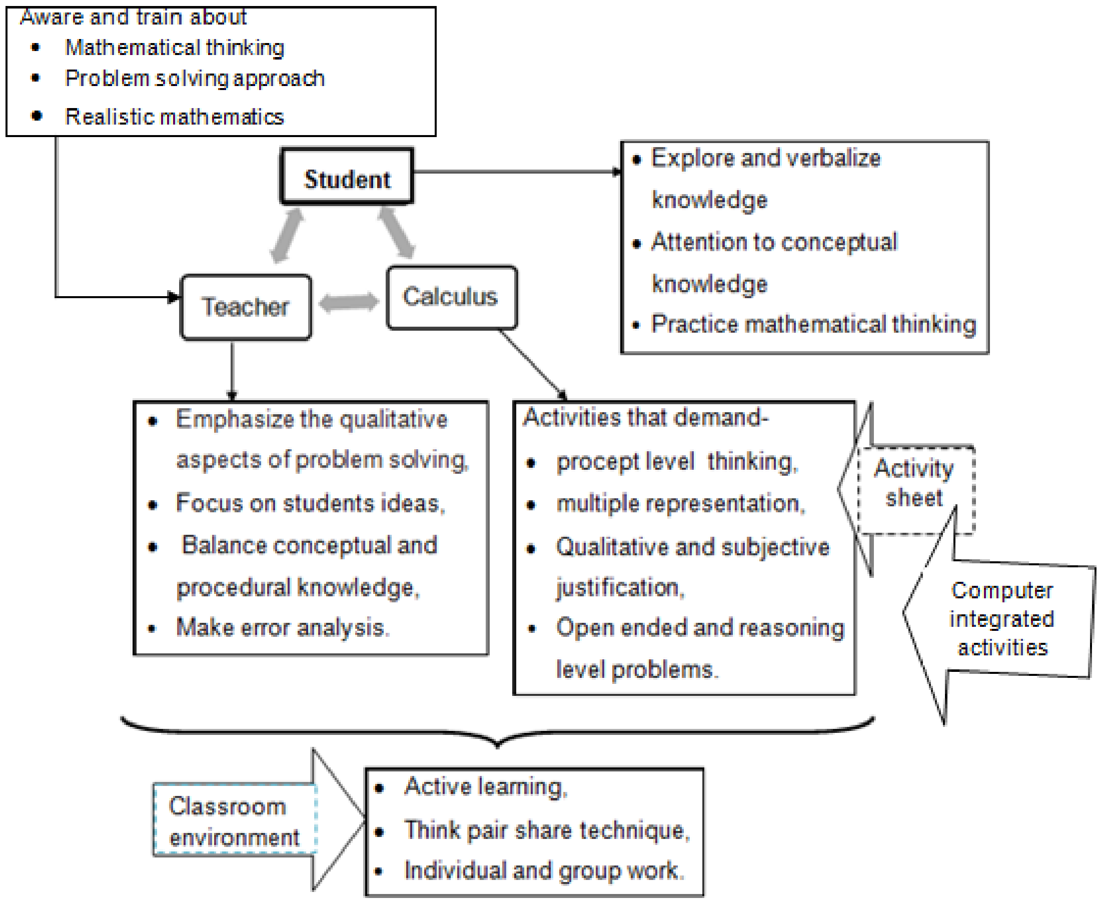


Figure 37: The modified intervention model

5.3. Conclusions

Calculus concepts are the preconditions for most science, engineering, and technology fields of undergraduate programs. Students' understanding of these concepts affects not only their performance and involvement in mathematics but also in these fields. Thus, it is critical that this topic has to be learned carefully for the goods of it. Despite the consequences of comparative importance, it is very unsatisfactory that students' performance in calculus is deprived and there are many difficulties that were investigated in the past and are still today. It is well recognized that the traditional approach to calculus is not effective in reducing these difficulties and misconceptions. Thus, the main purpose of the study was to overcome students' difficulties and enhance their conceptual knowledge of calculus by developing a literature informed intervention model.

One of the most important findings of this study is the synthesized difficulties that students encounter in coming to understand calculus concepts. Accordingly, one of the most distinguishing features of the traditional approach to calculus is that procedural approach, surface learning, and lack of feasibility dominate it. The diagnosis assessment revealed that students of the study area have difficulties that are not far from those in the literature. Triangulated themes of difficulties revealed that students' learning involves a static view of the dynamic process. Additionally, a lack of describing definitions and relationships of terms was investigated as difficulties. Moreover, overgeneralization and inconsistent cognitive structure, over-dependence on procedural learning, and lack of making a logical connection between conceptual aspects were found as students' difficulties. Further, the lack of a coherent framework of reasoning and lack of computational skill were found as students' difficulties.

This study showed that even active students (according to a teacher-made test), knowledge is questionable when screened through items that are designed to identify those misconceptions (systematic errors). Many students get the correct answer for the wrong reason and a wrong answer with high confidence.

Besides, the diagnosis assessment revealed student's approaches to the conceptual issues and causes of those difficulties. In particular, arithmetic thinking than algebraic, linguistic ambiguity, compartmentalized learning, dependence on concept image than concept definition, obtain a correct answer for a wrong reason, focuses only on an algebraic form of representations, and focuses on lower-level cognitive demanding exercises and in general surface learning approaches were identified as conceptual issues behind the difficulties. Thus, the researcher guided by all these data developed on an intervention model. The model was intended to enhance conceptual knowledge through focusing on mathematical thinking practice: conjecturing and convincing, reflection and communication via think-pair-share technique and on the dual nature of concepts, reconstructive generalization vis-à-vis cognitive conflict strategies. In addition, incorporating reasoning level and real-life problems, widening students thinking through counterexamples, and error analyses were included.

The result suggests that students in the experimental group performed significantly better than the control group. The text analysis on the students' test script showed that many students in the experimental group were able to overcome most of the observed difficulties. In particular, most students demonstrated process level conception, conceptual reasoning, qualitative justification, consistency in reasoning, and less algebraic and symbolic manipulation errors.

Another prominent finding is the error analyses by itself have an implication on how to design an alternative approach to the teaching-learning of calculus and the beginning level of learning calculus is the best junction for an intervention. By properly designing activities and shifting the classroom approach to student-centred, it is possible to reduce the incidence of those difficulties and change students' focus of attention to conceptual issues than rules and procedures. In particular, assessment items are potential areas of attention in terms of exploring existing difficulties and indicating point of intervention so that students focus the required conceptual knowledge.

One of the main challenges faced by students who join the science and technology fields of study is their knowledge of calculus concepts. To understand calculus properly, and to work with it in diverse areas of its application, students should be able to make a proper set up of conceptual constructs. However, most students' difficulties arise from lack of one or more of such constructs or the whole set up. Different learning strategies can be designed based on the nature of such constructs to help students overcome their learning difficulties of a topic. In the proposed model, many of such constructs are incorporated. Perhaps students may need more time to let go of their difficulties.

One limitation of the study is the scope of the literature search for the systematic review. The review considered the starting of the new curriculum implementation year in the study area as a benchmark for the inclusion of studies for the review. It would have had an effect on the themes of difficulties if it had been extended beyond the side period. The diagnosis assessment and the experimental phase was also bound to the assumption that students written scripts are genuine enough to reveal the knowledge and understanding that students possess about a learned mathematical concept. It would be better if an interview had been incorporated. Moreover, the students' in the experimental phase are intact classroom students to be ethical and to resolve the administrative issues. The intervention time, to significantly change students' understanding is not much enough.

5.4. Recommendation

The purpose of the study was to assist students at the early stage of learning calculus (grade 12 in this context) overcome difficulties and get better conceptual knowledge. The assumption is that if students overcome their difficulties and develop a better conceptual knowledge and understanding then they better perform in the university entrance examination and will join university courses with the prerequisite. The result indicated that the study had accomplished as intended. The study is valuable to policymakers, researchers, teachers, and students. In particular, the themes of difficulties, the assessment items, the proposed model, and the activity sheet are valuable for practitioners as they can be used as a springboard for further inquiry and progression.

First of all, practitioners (particularly university lecturers), have to be aware of those difficulties that the students bearing into a University. This is valuable to come from an expectation crisis. Besides, they can take those themes of difficulties into consideration during planning a lesson. They can also plan alternative intervention model or implement the suggested model. They can also do more on designing of further activities for assessment or for practices. It is time to shift the trend of teaching-learning from procedural and algebraic manipulation of exercises to conceptual and reasoning level problems.

Practitioners also could make use of the synthesized difficulties as a springboard for further inquiry. They have to shift their practice of providing feedback to assessment items. Instead of simply making right or wrong of students test scripts, making error analysis (looking for patterns of error in interpreting, approaching to conceptual issues and ways of thinking and applying the concepts in problem-solving), then use the result as feedback to prepare subsequent lessons or intervention in the form of tutorials.

Practitioners also have to take into consideration that a correct answer does not guarantee the required conceptual knowledge. Thus, they have to think of their assessment habit, i.e. the nature of items and feedback providing strategies. Due to the constructive nature of knowledge formation, most difficulties emanate from early definitions and introductions of a concept. Therefore, in the early stage, teachers' awareness about students' difficulties and the subsequent effect could be valuable to students learning. Teachers also should have to open their eyes and look around to generate a practical example, so that the students make sense of the concepts instead of the dogmatic approach that stick within a textbook and reference book exercises. Since a correct answer for a wrong reason and a wrong answer with high confidence are also frequently occur as the part of challenges in calculus, it is recommended to incorporate the Certainty of Response Index (CRI) in a diagnostic test or continuous assessment items.

Last, but not least is the implication of the study for policymakers about the issue of teachers' training. One focus of the proposed model is to incorporate activities that are somewhat different from the usual teachers made or those in textbooks. However, the question remains to be raised is whether teachers' themselves are competent enough to prepare such activities or manage their classes in a problem-solving approach. One suggestion to overcome the problem may be to include "problem-solving and mathematical thinking practice" in teachers' training or to provide it as on job training. The observation made during the training provided to teachers revealed that most teachers are naive to the practices like "error analysis," "using feedback as a pedagogical tool" and are unaware of how to prepare real-life and context-laden problems, so that students make sense about calculus. For instance, piece-wise defined function is one of the concepts that are abstract and ideal to students. During the training, the researcher gave the participants to describe their monthly salary tax or monthly water bill in an algebraic expression. Most of them were surprised that it was as simple as this to make students “make sense” of what they are learning. Thus, policymakers have to do well in teachers’ competence and awareness of the emerging approaches. Refreshing teachers may include how to reflect on their own thinking, meta-cognition, and reflection on others’ work (most probably their students), think about realistic mathematics and using errors as a springboard for further progression. Additionally, assessing teachers’ awareness and opinions about emerging pedagogical and theoretical frameworks are points that seek further attention and research.

Based on the result of the study, the researcher suggested the following recommendations for further study:

- Assess the attitude of students’ towards calculus after learning with the model.
- Investigate students’ retention of conceptual knowledge after learning with the model.
- Replicate the study in a different context to assure generalization of the results.
- Compare the effectiveness of the intervention used in this study with an intervention based on computer programs.

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Appendix A: Detail of the studies used for the systematic review

Author	Country	Level	Focused area	Data collection[16]	Sample[17]	Type
Abbey (2008)	USA	1st year University	Derivative	Test, interview-	235,11(not mentioned)	Master's Thesis
Areaya and Sidelil (2012)	Ethiopia	Secondary school	Limit, continuity, and derivative	Test-adapted, prepared	135-random	Journal article
Bergsten (2006)	Spain	1st year University	Limit of functions	Observation-interview	6-convenient	Conference Proceeding
Blaisdell (2012)	USA	1st year university	Limit	Questionnaire-adapted & prepared	111-purposive	Journal article
Brijlall and Ndlovu (2013)	South Africa	Grade 12	Optimization problems	Questionnaire, interview-	10,3-availability	Journal article
Çetin, I. (2009)	Turkey		Limit of functions	Questionnaire, interview-adopted	25-convenient	Doctoral thesis
Cetin, N. (2009)	Turkey	1st year University	Limit of functions	Test-prepared	63-availability	Journal article
Denbel (2015)	Ethiopia	1st year University	Derivative	Questionnaire-adapted	60-availability	Journal article

[16] Data collection instrument and source of the instrument
[17] Number of participants and sampling method implemented to select the sample

Duru (2011)	Turkey	1st year University	Limit	Test, interview-adapted	95,8-convenient	Journal article
Elia *et.al.,(*2009	Greece	Grade 12	Limit of functions	Questionnaire-	222(not mentioned)	Journal article
Fernandez-Plaza *et al.* (2013)	Spain	2nd ed. Level	Finite limit at a point	Questionnaire-adapted	36-purposive	Journal article
Gray *et al.* (2009)	New England	1st year University	Variables	Test-adopted	174-(not mentioned)	Journal article
Hashemi *et al. (*2014)	Iran	1st year University	Derivative	Questionnaire-designed	63-availability	Journal article
Horvath (2008)	USA	1st year University	Chain rule	Interview, observation	18 (not mentioned)	Journal article
Jaffar and Dindyal (2011)	Singapore	1st year University	Limit at a point	Test, interview-adapted	50, 10-convenient	Journal article
Jayakody 2012	Canada	1st year University	Continuity	Test-	37- convenient	Journal article
Jones (2015)	USA	1styear university	Limit involving infinity	Interview-prepared	7-purposive	Journal article
Jordaan (2005)	South Africa	1st year University	Limit of functions	Questionnaire-interview	47, 6-availability	Master's thesis

Juter (2005a)	Sweden	1st year University	Limit of functions	Questionnaire, interview-prepared, adapted	112,15-availability	Journal article
Juter (2005b)	Sweden	1st year University	Limit of functions	Questionnaire, interview-prepared, adapted	112,15-availability	Journal article
Juter (2006)	Sweden	1st year University	Limit of functions	Questionnaire, interview-adopted	111,15-availability	Journal article
Ko and Knuth (2009)	Taiwan	1st year University	Continuity	Test, interview-adapted	11, convenient	Journal article
Luneta and Makonye (2012)	South Africa	Grade 12	Derivatives	Test, selective interviewed	45 (not mentioned)	Journal article
Maharaj (2010)	South Africa	1st year University	Limit of functions	Test-adopted	891	Journal article
Maharaj (2013)	South Africa	1st year University	Derivatives	Test-designed	857-convenient	Journal article
Maharajh *et al.* (2008)	South Africa	Second year	Continuity	Questionnaire, interview-	12-(not mentioned)	Journal article
Makgakga & Makwakwa (2016)	South Africa	Grade 12	Derivatives	Test, interview	37-convenient	Conference Proceeding

Makonye (2012)	South Africa	Grade 12	Derivatives	Exam	1000 (not mentioned)	Journal article
Moru (2006)	Lesotho	First & sec. year	Limit of functions	Questionnaire-interview	307,33-convenient	Doctoral thesis
Nair (2010)	USA	1st year University	Limit of rational functions	Interview-	19- convenient	PhD thesis
Oehrtman (2002)	USA	1st year University	Limit of functions	Interview, questionnaire-prepared, adapted	9,120-	PhD Thesis
Orhun (2012)	not mentioned	Grade 12	Graph of derivative functions	Test-designed	102 (not mentioned)	Journal article
Parameswan (2007)	India	Grade 12	Limit of functions	Test, interview-	79,16-availability	Journal article
Pillay (2008)	South Africa	Grade 12	Derivatives	Exam, interview	27,4-convenient	Journal article
Przenioslo (2003)	Poland	Secondary school	Limit of functions	Test, observation, interview-prepared	512-	Journal article
Roh (2005)	USA	1st year University	Limit of sequences	Interview-prepared	12-purposive	PhD dissertation
Siyepu (2013)	South	1st year	Derivatives	Observations	30-purposive	*Proceeding*

	Africa	University				
Siyepu (2015)	South Africa	1st year University	Derivatives	Test-	30-purposive	Journal article
Takaci *et al.* *(*2006)	not mentioned	Grade 12	Continuity	Questionnaire-adapted	41-availability	Journal article
Usman (2012)	Nigeria	1st year University	Optimization problems	Test-designed	156-convenient	Conference Proceeding
Vandebrouck & Leidwanger (2016)	France	1st year University	Limit of functions	Test, interview	513-availability	Conference Proceeding
Vela (2011)	USA	Grade 12, first year	Continuity	Test-adapted	23- convenient	Master's thesis
Wangle (2013)	USA	1st year University	Continuity	Test, interview-	19- convenient	Doctoral dissertation

Appendix B: Quotations

No.	Item	Identified students conceptions
1	Abbey (2008)	Lack to interpret critical points
		Focus on surface learning and memorized rules
		Manage procedures inappropriately
2	Areaya and Sidelil (2012)	Belief that a constant sequence is not monotonic
		Correct answer for wrong reason
		Infinity as a number
		Limit is a substitution
		Being monotonic is sufficient for limit of a sequence
		Over generalize limit procedures
		Limit is unreachable
		Limit is an approximation
		Limit is a value which exist at any point
		Confusing notation or symbol and meaning or definition
		Being defined is necessary for existence of limit
		If a function has no limit at a point then it must have a vertical asymptote
		Confusing critical points and extreme values
		Face additional difficulty to deal with split half functions
		Belief that rationalization is a must to do when a radical is involved in a limit
3	Bergsten (2006)	Have wrong image of infinity
		More focused on random algebraic manipulations than conceptual understanding
		Lack of establishing link between calculus concepts and procedures and pre-calculus concepts and procedures
4	Blaisdell (2012)	Stimulate different concept images of the same idea based on type of representation
		Less difficulty in graphical form of limit

5	Brijlall and Ndlovu (2013)	Not understand the importance of dimensions in problems
		Apply memorized rules without attention to the context provided by the task
		Limited understanding of algebraic expressions
		Relied mostly on procedural thinking
		Unsynchronized knowledge structure of derivative
		Instrumental understanding of the notation dy/dx
		Do well with routine type questions i.e. functioning at an action level
		Hard to model problems in to mathematical expression
6	Çetin, I. (2009)	Limit of $\boldsymbol{f}$ at $\boldsymbol{a}$ is $\boldsymbol{f}(\boldsymbol{a})$
		Lack coordinated process schema of limit
		Correct answer for wrong reasons
7	Cetin, N. (2009)	Lack the meaning of the limit concept
		Unable to apply limit concept to solve unfamiliar exercises
		Recognize limit value only as a number and lack to interpret results
8	Denbel (2015)	Unable to make connection between meaning of terms in common language use and interpretations in calculus
		Problem of visual /graphical representation of concept like turning point
		Restricted mental image of derivative
		Do not pay attention to contextual meaning of terms
9	Duru (2011)	Confusing limit and continuity definitions
		Being defined at a is necessary condition to compute limit at $\boldsymbol{x} = \boldsymbol{a}$
		Algebraic manipulation errors
		Limit and function values are the same
10	Elia *et al.* (2009)	Hard to describe what limit is
		Limit as a point that cannot be attained
		Limit of $\boldsymbol{f}$ at a is $\boldsymbol{f}(\boldsymbol{a})$

		Being defined at a is necessary to compute limit at $x = a$
		Non-existence of limit at a point occurs only when the limits from both sides is different
		Lack of flexibility among different modes of representation
		Great difficulties in non-routine problems of limit
		More difficulty of tasks involving infinity
11	Fernandez-Plaza *et al.* (2013)	Limit value is unreachable
		Limit is an upper bound
		Limit value is an approximation
		Limit is non-exceedable
		Convergence is strictly monotone
12	Gray *et al.* (2009)	Lack to recognize variables as generalized numbers and varying quantities
		Unsuccessfully used symbolic manipulation, inappropriate use of the inequality sign
		Sequentially based thinking of variables than real number domain of equations and inequalities
		Inability to recognize co-variation among variables
		Arithmetic approach for items demanding an algebraic approach
13	Hashemi *et al.* (2014)	Challenge to characterize a function based on information from graph
		Lack to use both geometric and algebraic aspects together
		Influenced by algebraic notation more than geometric form of derivatives
14	Horvath (2008)	Unsuccessful on problems involving unfamiliar functions
		Confusing function composition and combination
15	Jaffar and Dindyal (2011)	Confusing terms- infinite, not exist, and indeterminate in computing limit
		Confusing infinity and undefined
		Limit value is an approximation

16	Jayakody (2012)	Limit as plugging a value into the function
		To be continuous, a function should be in one piece
		Concept image different from concept definition
17	Jones (2015)	Insert infinity in for $\boldsymbol{x}$,
		Inappropriate use of L'Hopital's rule
		Focused on what happens at the point infinity than as $\boldsymbol{x}$ approaches to infinity
		Point-by-point or static image of change
18	Jordaan (2005)	Limit as a boundary
		Limit of $\boldsymbol{f}$ at a is $\boldsymbol{f}(\boldsymbol{a})$
		Limit as unreachable
		Limit is an approximation
		There should be a limit value of a function at any given point
		Lack to describe what limit is in their own words
		Limit as a substitution process
		Continuous is necessary for existence of limit
		Being defined is necessary for existence of limit
		Difficulty in sketching the graph of rational functions
		Difficulty of indeterminate forms
19	Juter (2005a)	Hard to describe limit in their own words
		Limit is never attained
20	Juter (2005b)	Functions cannot attain limit values
		Limit as approximation of function values
		Limit as border
		Limit value as function value
		Function value as limit value
		Difficulty to compute limit at point of discontinuity
		Perceive limit as object and as process, as unreachable or reachable based on context
21	Juter (2006)	Algebraic manipulation errors

		Wrong concept image of indeterminate form
		Low success for unfamiliar task items
		Difficulty with image of infinity
		Correct answer for wrong reason
22	Ko and Knuth (2009)	If f is discontinuous at a, then f is not defined at a
		Incomplete mental image of limit notations
		Hard to producing proofs and counterexamples of continuity and discontinuity
23	Luneta and Makonye (2010)	Difficulty in using the functional notation
		Error of limit notation
		Ignorance of rule restrictions in algebraic expressions
		Incomplete application of differentiation rules
		Wrong answer with high confidence
		False concepts hypothesized to form new concepts
		Unsynchronized conceptual and procedural knowledge in calculus
24	Maharaj (2010)	Limit as one of the one-sided limits only
		Algebraic manipulation errors of rational expressions
25	Maharaj (2013)	Difficulty in applying the rules for derivatives
		Not having appropriate mental structures of derivative
		Inadequate schema for composition of functions
		Difficulty of decomposing a compound function
		Inadequate schema for graph of the derivative function
		Difficult to relate function and its derivative geometrically
26	Maharajh *et al.* (2008)	Lack ability to integrate given and required that satisfy the conditions In a problem /problem-solving framework
		A piecewise defined function is not one function
		Existence of limit is an essential premise to compute continuity
		Confusing limit value and function value/ inadequate generalization
		Confusing continuity with connectedness
		Language issue/linguistic ambiguity

		Confuse connectedness of graph with smoothness of graph
27	Makgakga and Makwakwa (2016)	Incorrect substitution into a function to compute derivative
		Difficulties in relating symbols and the use of variables
		Difficulties in mathematical operations
		Procedural knowledge which is suitable for simple functions
28	Makonye (2012)	Inadequate concept image of functions
		Misinterpret derivative rules
		Overgeneralization of rules
		Confusing terms
		Lack of meta-cognition
		Low response rate for problems demanding higher levels of cognitive thinking
29	Moru (2006)	Limit of $\boldsymbol{f}$ at a is $\boldsymbol{f}(\boldsymbol{a})$
		The limit value is the function value
		Correct answer using inappropriate method
		Infinity as one big number
		Limit value is unreachable
		Limit is a boundary
		Limit value is an approximation
		Being defined at a is necessary to compute limit at $\boldsymbol{x} = \boldsymbol{a}$
		Difficulty to translate among representations
		Lack of symbolic interpretation
		A piece-wise defined function has two limits
		Shortage of co-variational reasoning
		Limit values are whole numbers
		The limit value is a dynamic object
		A well-defined sequence should be monotonic
		Divergent means tends to infinity
		Improper simplification
		Problem with the chain rule

		Problem with indeterminate form
		An alternating sequence is not one but two sequences
		A well-defined sequence has a single formula
		Different modes of representation represent different sequences
30	Nair (2010)	Being defined is necessary condition for existence of limit
		Inability to discriminate between indeterminate and undefined forms
		Belief that a function could not be continuous at cusps or sharp corners
		Difficulty to identify vertical asymptote of a rational function
		The believe that every point of discontinuity is a vertical asymptote
		Being defined is necessary for limit
		Limit is the same as function value
		Face more challenge to compute limit involving infinity
		If a functions domain is all real numbers, then it is necessarily continuous
		Existence of limit is sufficient for continuity at a point
31	Oehrtman (2002)	Limit exist if the terms collapse to zero
		Limit as the value being approximated
		Plugged infinity as a number
		Interpretation of “approaches” as chunky images of change
32	Orhun (2012)	Difficult to make connections between function and its derivative
		Lack to use correct terms to describe graph of derived function
		Unable to interpret function properties from graph of the derivative function
33	Parameswan (2007)	Confused by jump discontinuity
		Recognize limit computation as an approximation
		Equating quantities that they perceive as small to zero
		Infinity as a large number
34	Pillay (2008)	Used inappropriate algorithms

		Carried out incorrect algebraic simplification
		Present partial solution of derivatives
		Incorrect representation of limit notation
		Incorrect use of derivative notation
35	Przenioslo (2003)	Being defined is necessary to compute limit
		Continuous at a point is necessary to compute limit at that point
		If a is in the domain of $\boldsymbol{f(x)}$ then $\boldsymbol{lim_{x \to a} f(x)}$ must exist
36	Roh (2005)	A number sequence continue endlessly, hence no limit/ associating convergence with only the index process
		Limit value is unreachable
		Lack of recognizing constant sequences as sequences
		Oscillating behavior always leads to divergence
		Graph of a sequence is continuous
		Lack of recognizing uniqueness of limit value/multiple value
		No reaction to unfamiliar sequences
		Plugged in infinity for n
		Limit value is unreachable
37	Siyepu (2013)	Confusing rules of differentiations
		Inadequate interpretation of the derivative concepts
		Fall to choose appropriate procedures to a given problem
		Interference i.e. incorrect understanding of a concept because of an existing overgeneralization
38	Siyepu (2015)	Lack to capture set of idea in a given problem
		Lack of a well-developed composite function schema
		Perform algebraic manipulation errors
39	Takaci *et al.* (2006)	Being defined is sufficient for continuity
		Confusing continuity with connectedness
		Point of discontinuity means asymptote
		A piecewise defined function is discontinuous
40	Usman	Low response rate for problems demanding higher levels of

	(2012)	cognitive thinking
		Shortage of making network of concepts toward solving a problem
		Lack of meta-cognition
		Inability to manipulate trigonometric fractions
		Unable to manipulate trigonometric identity,
41	Vandebrouck & Leidwanger (2016)	Poor skills about algebraic rules of limit
		Inconsistency in computing limit value
		Think that x always takes positive value to compute limit
		Evaluate $\frac{\infty}{\infty} = \mathbf{0}$ or $\frac{\infty}{\infty} = \infty$
		Difficult to identify the kind of indeterminate form
		Indifferently use “it is” and “it tends”
		Focus on qualitative rules of limit
42	Vela (2011)	Being defined is sufficient for continuity
		Existence of limit is sufficient for continuity at a point
		Confusing continuity with connectedness
		Confusing the relation between continuity and differentiability
		Hard to identify point of discontinuity
		point of discontinuity means asymptote
		Look only for breaks, holes, cusps or corners on the graph than limit
43	Wangle (2013)	Limited conception of functions as chunky, not smooth
		Very dependent only one form of representation
		Confuse the notion of continuity and differentiability i.e. connectedness vs. smoothness
		Being defined is sufficient for continuity
		Confusing continuity with connectedness
		The believe that continuity meant smoothness
		Did not associate limit with continuity
		Continuity is an issue only for functions defined for all real numbers
		Existence of limit is sufficient for continuity at a point

Appendix C: Initial codes

No.	Initial Code	Quotations
1	Co-variational reasoning	Lack of co-variational reasoning
		Inability to recognize co-variation among variables
		Hard to handle variables as generalized numbers and varying quantities
		A number sequence continue endlessly, hence no limit i.e. associating convergence with only the index process
		Arithmetic approach for items demanding an algebraic approach
		Limit exist if the terms collapse to zero
2	Function image	Difficulty in using the functional notation
		Difficulty with split half function
		A piecewise defined function is not one function (2)
		Inadequate concept image of functions
		Inadequate schema for composition of functions
		Difficulty of decomposing a compound function
		A well-defined sequence should be monotonic
		Lack of recognizing constant sequence as a sequence
		An alternating sequence is not one but two sequences
		A well-defined sequence has a single formula
		Different modes of representation represent different sequences
3	Algebraic manipulation errors (computational and manipulation skill)	Algebraic manipulation errors (3)
		Algebraic manipulation errors of rational expressions
		The believe that rationalization is a must to do when a radical is involved in a limit
		Unsuccessfully used symbolic manipulation, inappropriate use of the inequality sign
		Incorrect substitution into a function to compute derivative
		Fail to carry out manipulations or algorithms
		Improper simplification
		Poor skills about algebraic rules of limit

		Think that x always takes positive value to compute limit
4	Infinity image	Insert infinity in for $\boldsymbol{x}$
		Infinity as one big number
		Plugged in Infinity as a number (2)
		Infinity as a large number (2)
		Face additional difficulty in limit involving infinity (3)
5	Infinity, undefined and indeterminate interplay	Confuse use of terms infinite, non-existence of limit, and indeterminate
		Confusing infinity and undefined, difficult to identify the kind of indeterminate form
		Evaluate $\frac{\infty}{\infty} = \mathbf{0}$ or $\frac{\infty}{\infty} = \infty$
		Wrong concept image of indeterminate form (3)
		Inability to discriminate between indeterminate and undefined forms
6	Concept definition	Lack the meaning of the limit concept
		Hard to state definition of limit of a function (3)
		Focus on qualitative rules of limit
		Inadequate interpretation of the derivative concepts
7	Linguistic ambiguity	Unable to make connection between the meaning of terms in common language use & interpretations in calculus (3)
		Do not pay attention to contextual meaning of terms
		The belief that a constant sequence is not monotonic
8	Limit as unreachable	The limit value is a dynamic object
		Limit value as/is unreachable (8)
		Lack of recognizing uniqueness of limit value/multiple value
		Functions cannot attain limit value
		Limit as a number that cannot be reached
9	Limit value is a boundary	Limit is an upper bound
		Limit is non-exceedable
		Limit as a boundary (2)

		Limit as border
10	Limit value is an approximation	Limit value is an approximation (6)
		Limit as the value being approximated
		View limiting as a process of approximation
11	Conflicting concept image	Perceive limit as objects and as process, as unreachable, reachable based on context
		Concept image different from concept definition
		Indifferently use “it is” and “it tends”
		Inconsistency in computing limit value
12	A static view of the limit process	Focused on what happens at the point infinity than as x approaches to infinity
13	Discrete thinking of continuous idea	Point-by-point or static image of change
		Limited conception of functions as chunky, not smooth
		Sequentially based thinking of variables than real number domain of equations and inequalities
		Limit values are whole numbers
		Interpretations of “approaches” as chunky images of change, motion on the graph, static closeness
14	Continuous view of discrete idea	Graph of a sequence is continuous
15	Alternative conception	Correct answer for wrong reasons (3)
		Wrong answer with high confidence
		False concepts hypothesized to form new concepts
		Correct answer using inappropriate method
16	Monotonic-convergence interplay	Convergence is strictly monotone
		Being monotonic is sufficient for limit of a sequence
17	Domain-limit	Being defined at a is essential to compute limit at $\mathbf{x} = \mathbf{a}$ (4)

	interplay	Being defined is essential for existence of limit (4)
		Belongingness of a to the domain of f is an essential and enough to compute limit at a
18	Limit and function values are the same	Limit of f at a is $f(a)$ (4)
		Limit and function value are the same (4)
		Limit is a substitution (2)
		Function value as limit value
		Confusing limit value and function value/ Inadequate generalization
		Limit as plugging a value into the function
19	Non-existence case of limit	Non-existence of limit at a point occurs only when the limits from both sides are different
		Limit as one of the one-sided limits only
		Divergent means tends to infinity
		Oscillating behaviour always leads to divergence
20	Point wise thinking of limit	There should be a limit value of a function at any given point
		Limit is a value which exist at any point
21	Domain-continuity interplay	Being defined is sufficient for continuity (3)
		Continuity is an issue only for functions defined for all real numbers
22	Limit-continuity interplay	Continuity is necessary to compute limit at a point
		Continuous at a point is necessary for existence of limit at that point
		Confusing limit and continuity definitions
		Existence of limit is sufficient for continuity at a point (3)
		If f is discontinuous at a, then f is not defined at a
23	Confusing continuity with connectedness	Confusing continuity with connectedness (4)
		A piecewise defined function is discontinuous
		Did not associate limit with continuity
		To be continuous, a function should be in one piece

		If a functions' domain is all real numbers, then it is necessarily continuous
24	Continuity concept image	Difficulty producing proofs and counter examples of continuity and discontinuity
		Hard to identify point of discontinuity
25	Continuity-asymptote interplay	If a function has no limit at a point then it must have a vertical asymptote
		Difficult to compute limit at point of discontinuity
		Confused by jump discontinuity
		Point of discontinuity means asymptote
		Non-existence of vertical asymptote is sufficient condition for continuity /point of discontinuity means asymptote
		Difficulty to identify vertical asymptote of a rational function
		The believe that every point of discontinuity is a vertical asymptote
26	Definition of terminology	Confusing notation or symbol, name and meaning or definition
		Confusing critical points and extreme values
27	Difficulties with rules and procedures of derivatives	Inappropriate use of l'Hopital's rule
		Confusing rules of differentiations (2)
		Interference i.e. incorrect understanding of a concept because of an existing overgeneralization
		Problem with the chain rule
		Ignorance of rule restrictions in algebraic expressions
		Incomplete application of differentiation rules
		Used inappropriate algorithms
		Carried out incorrect algebraic simplification
		Difficulties in relating symbols and the use of variables
		Confusing function composition and combination
		Misinterpret derivative rules
		Overgeneralization of rules and procedures (2)

		Inability to manipulate trigonometric fractions
		Unable to manipulate trigonometric identity
28	Symbolic interpretation	Lack of symbolic interpretation
		Incomplete mental image of limit notations and absolute values associated to continuous functions
		Instrumental understanding of the notation dy/dx
		Error of limit notation
		Incorrect representation of limit notation (2)
		Incorrect use of derivative notation
29	Infinity small	Equating quantities that they perceive as small to zero
30	Continuity-differentiability interplay	Confuse connectedness of graph with smoothness of graph
		Look only for breaks, holes, cusps or corners on the graph than using limit
		The believe that continuity meant smoothness
		Belief that a function could not be continuous at cusps or sharp corners
		Confusing the relation between continuity and differentiability
		Confused the notion of continuity and differentiability-connectedness-smoothness
31	Procedural learning	Apply memorized rules without attention to the context provided by the task
		Rely mostly on procedural thinking
		More focused on random algebraic manipulations than conceptual understanding
		Focusing on memorized procedures
		Failed in using limit concept to solve unfamiliar problems
		Great difficulties in non-routine problems of limit
		Recognize limit value only as a number and lack to interpret results
		Procedural knowledge which is suitable for simple functions

32	Unsynchronized knowledge structure	Lack coordinated process schema of limit
		Lack of a matured composite function mental structure
		Unsynchronized knowledge structure of derivative
		Not having appropriate mental structures of derivative (2)
		Unsynchronized conceptual and procedural knowledge in calculus
		Present partial solution of derivatives
		Lack of establishing link between calculus concepts and procedures and pre-calculus concepts and procedures
33	Lack of conceptual learning	No reaction to unfamiliar sequences
		Low success for unfamiliar task items
		Do well with routine-type questions i.e. functioning at an action level
		Unsuccessful on problems involving unfamiliar functions
		Low response rate for problems demanding higher levels of cognitive thinking "non-isolated tasks" (2)
		Lack of meta-cognition (2)
34	Representation	Difficulty to translate among representations
		Conflicting concept images of sequence that evoked based on representation forms
		Problem of visual /graphical representation of concept like turning point
		Very dependent only one form of representation
		Difficulty in sketching the graph of rational functions
		Stimulate different concept images of the same idea based on type of representation
		Less difficulty in graphical form of limit
35	Visualization	Challenge to characterize a function based on information from graph
		Lack to use geometric and algebraic representation together

		Influenced by algebraic form more than geometric
		Inadequate schema of interpreting the graph of the derivative function
		Difficult to describe derivative represented geometrically
		Difficult to make connection between function and its derivative geometrically
		Lack to use correct mathematical terms to describe graph of derived function
		Unable to interpret function properties from the graph of the derivative function
		Difficulty interpreting critical points of a function's graph
36	Problem-solving	Lack ability to integrate given and required that satisfy the conditions In a problem /problem-solving framework
		Failure to choose appropriate procedures to be applied for a given problem involving derivative
		Lack to capture the set of ideas in a given problem
		Not understand the importance of dimensions in problems
		Hard to model problems in mathematical form
		Try to manage procedure inappropriately
		Shortage of making network of concepts toward solving a problem

Appendix D: Diagnostic test items

Section I: Limit of Sequences

Part I: Item 1.1 – 1.4 are multiple-choice items. From the given alternatives choose the best answer and circle the letter of your choice. Then explain how you arrived at your answer.

1.1. Which one of the followings is true?

A. A bounded sequence is necessarily converging.

B. A divergent sequence is necessarily unbounded.

C. A convergent sequence is necessarily bounded.

D. A monotone sequence is necessarily converging.

E. None of them is true.

Explain how you obtained your answer (you may use counter examples to do so)

____

____

1.2. Which one of the followings sequence is not convergent?

A. $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ . \ . \ .\right\}$

B. $\{a_n\} = \left\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5} \ . \ . \ .\right\}$

C. $\{a_n\} = \{-1, 1, -1, \ 1, -1, \ . \ . \ .\}$

D. $\{a_n\} = \{3, 3, 3, \ . \ . \ .\}$

E. All are convergent

Why do you think so? ____

____

1.3. Suppose $\{a_n\}$ is a sequence of positive terms (i.e. $0 < a_n$ for all n) and $a_1 > a_2 > a_3 > \cdots > a_n. \ . \ . \ .$ Does $\lim_{n\to\infty} a_n$ exist? What can you tell about the limit?

A. Yes, limit exists and the value is zero.

B. Yes, limit exists and the value is non-negative.

C. Yes, limit exists but nothing can be said about the value.

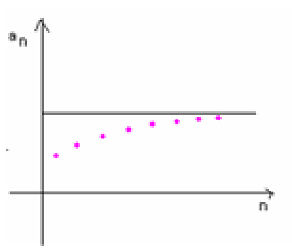
D. No, limit does not exist.

E. It is not possible to decide.

Explain your answer ____

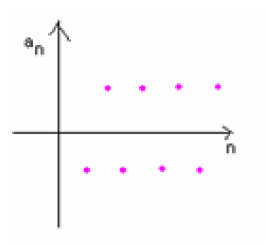
1.4. Which one of the followings sequence has no limit?

A.



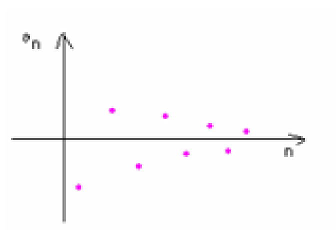
$a_n = \frac{n}{n+1}$, $n = 1, 2, 3, \ldots$

B.



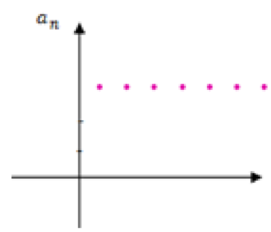
$a_n = (-1)^n$, $n = 1, 2, 3, \ldots$

C.



$a_n = \frac{(-1)^n}{n}$, $n = 1, 2, 3, \ldots$

D.



$a_n = 3$, $n = 1, 2, 3, \ldots$

.

Explain your answer ____

____

Part II: Item 1.5 is a workout. Answer the item by showing all the necessary steps clearly and neatly.

1.5. $\lim_{n\to\infty} n\sin(\frac{1}{n}) =$____________

Section II: limit of functions

Part I: Item 2.1 – 2.5 are multiple-choice items. From the given alternatives choose the best answer and circle the letter of your choice. Then explain how you arrived at your answer.

2.1. Which one of the following is true?

A. Limit value is a number beyond which a function cannot attain values.

B. Limit value is a number that the function value approaches but never reaches.

C. Limit is an approximation that can be made as accurate as you wish.

D. Limit of a function is value of the function at the limit point.

E. None of these is true.

Explain why ____

____

2.2. Which one of the following is true about the notation $\lim_{x \to a} f(x)$, provided the value is a real number?

A. It represents an infinite process.

B. It represents a finite value.

C. It is necessarily an upper boundary or a lower boundary on the range of the function f.

D. Both A & B are true.

E. Both B & C are true.

Explain why ____

____

2.3. Let f be a function and c ϵ IR. If $\lim_{x \to c} f(x)$ does not exist, which one must be true?

A. $f(x)$ becomes large enough when x gets closer and closer to c.

B. $\lim_{x \to c^-} f(x)$ exists but different from $\lim_{x \to c^+} f(x)$.

C. The function has a vertical asymptote at $x = c$.

D. $f(x)$ is not defined at $x = c$.

E. None of these is true.

Explain why ____

____

2.4. Which one of the following must be true if f is a function for which $\lim_{x\to 3} f(x) = 5$

A. f is continuous at the point $x = 3$.

B. f is defined at $x = 3$ and $f(3)$ exactly 5.

C. f is defined at $x = 3$ but nothing can be said about the value.

D. It is not grant to decide about *f(3)* from the given information.

E. None of these is true.

Explain why ____

____

2.5. Consider $\lim_{x\to 3}(\frac{x^2-9}{3x-9})$. In finding this limit the number 3 is substituted for x in the functional part and the result obtained becomes 0/0. What conclusions can you draw from this result?

A. The limit does not exist.

B. It is an indeterminate form.

C. The limit is 0.

D. It is undefined.

E. The limit is ∞.

If any other, please specify ____

____

Section III: Continuity

Part I: Item 3.1 – 3.3 are multiple-choice items. From the given alternatives choose the best answer and circle the letter of your choice. Then explain how you arrived at your answer.

3.1. Think about the function f given algebraically as follows-

$$f(x) = \begin{cases} \frac{1}{3}x + 2 & \text{if } x < 3 \\ \frac{-1}{9}x^2 + 2\text{x} - 2 & \text{if } x \geq 3 \end{cases}$$

. Is the function continuous? Why?

The function is continuous on its domain because-

A. The function is defined for every real number.
B. The limit exists for every real number.
C. For every real number "a" in the domain, $\lim_{x \to a} f(x) = f(\text{a})$.
D. I can draw the graph without lifting my pencil.
E. The graph is smooth.

The function is not continuous on its domain because:

F. The function is given by more than one formula.
G. The function is not defined for every real number.
H. There is a number "a" in the domain for which $\lim_{x \to a} f(x)$ does not exist, or $\lim_{x \to a} f(x) \neq f(a)$.
I. I cannot sketch the graph of the function without lifting my pencil.
J. The graph contains a cusp or corner.

3.2. Which one of the following is true statement?

A. A function $f(x)$ is discontinuous if its graph contains a sharp "corner."
B. If a function is continuous at a point then it is necessarily differentiable at that point.
C. If a function is continuous at a point then the limit necessarily exists at that point.
D. Continuous functions must have domain all real numbers.
E. All of these are true.

Explain why ____

____

3.3. If $f(x) = \begin{cases} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, for\ x \neq 2 \\ k \qquad\qquad , for\ x = 2 \end{cases}$ and if f is continuous at $x = 2$, then $k =$ _____

A. 0 B. $\frac{1}{6}$ C. $\frac{1}{3}$ D. 1 E. $\frac{7}{5}$

Explain why ____

____

Part II: Item 3.4 is workout item. Answer the items (3.4a to 3.4e) by showing all the necessary steps clearly and neatly.

3.4. Consider the function $f(x) = \frac{2x^2-x-15}{x-3}$

3.4a. Sketch the graph of f (discuss basic steps of graph).

3.4b. What can you say about the continuity of the function exactly at $x = 3$? (Say continuous or discontinuous).

3.4c. Does the function have limit value at $x = 3$? (yes / no) (Underline your choice).

3.4d. If your answer in 3.4c above is yes, what is that limit value?

3.4e. Compute *f* at $x = 3$

Section IV: Derivatives

Part I: Item 4.1 – 4.3 are multiple-choice items. From the given alternatives choose the best answer and circle the letter of your choice. Then explain how you arrived at your answer.

4.1. Let f and g be differentiable functions with the following properties:

i. $\mathrm{g}(x) > 0$ for all x ii. $f(0) = 1$

ii. $h(x) = f(x)\mathrm{g}(x)$ and $h'(x) = f(x)\mathrm{g}'(x)$ then $f(x) =$ __________

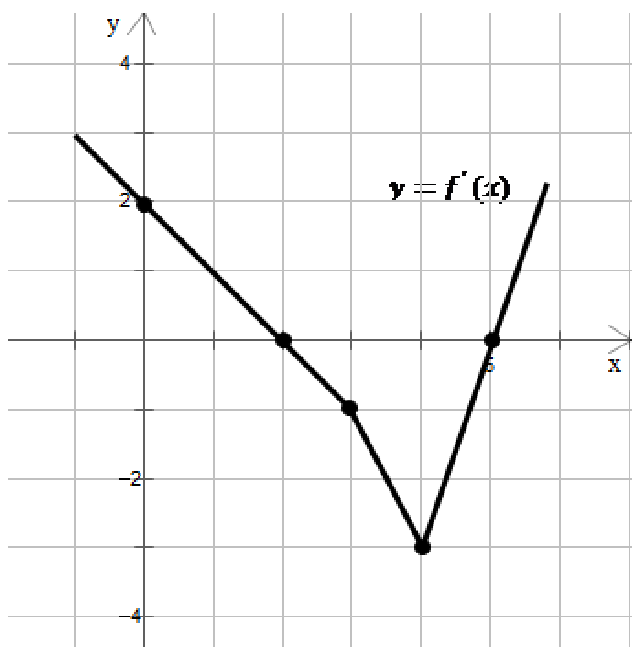
A. $f'(x)$ B. $\mathrm{g}(x)$ C. e^{x} D. 0 E. 1

Why do you think so? ____

____

4.2. Let f be a function whose derivative f' is graphically given below. Which one of the following values is a local maximum value of f?

A. f(5)
B. f(4)
C. f(3)
D. f(2)
E. f(1)



Discuss your choice in detail ____

____

4.3. For what values of a and b is the function $f(x) = \begin{cases} 1 - 3x^2, & for\ x \leq 1 \\ ax + b, & for\ x > 1 \end{cases}$

differentiable at $x = 1$.

A. $a = 6$ and $b = 0$

B. $a = -6$ and $b = 4$

C. $a = -3$ and $b = 1$

D. $a = 0$ and $b = -2$

E. the function is not differentiable at $x = 1$

Why do you think so? ____

____

Part II: Item 4.4- 4.7 are workout. Answer the items by showing all the necessary steps clearly and neatly.

4.4. Differentiate $y = \sin\left(e^{\sqrt{x+1}}\right)$

4.5. The percent of concentration of a certain drug in the bloodstream t hours after the drug is administered is given by $f(t) = \frac{5t}{t^2+1}$. Then

4.5a. Evaluate $\lim_{t\to\infty} f(t)$ and interpret this result,

4.5b. Find the time at which the concentration is a maximum, and

4.5c. Compute the maximum concentration.

4.6. The following table shows some x values and the corresponding function values of a function f and its derivative f'

x	-1.4	-1	-0.8	-0.4	0	0.8	1	1.4	2	2.8
$f(x)$	1.9044	0.25	0.0144	0.2704	1	2.1904	2.25	2.0164	1	0.0144
$f'(x)$	-6.624	-2	-0.432	1.456	2	0.592	0	-1.136	-2	0.432

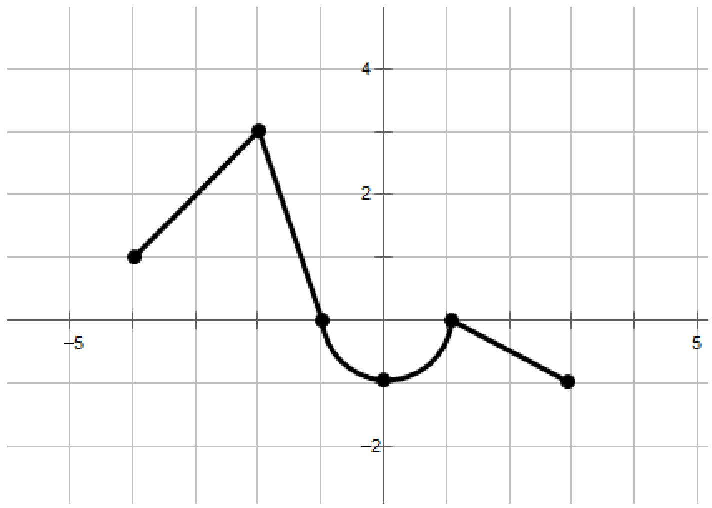
Answer the following questions based on the information given in the table.

4.6a. Is it possible to find the value of $\lim_{x\to 2} f(x)$ from the given information (yes/no). Underline your choice and justify why.

4.6b. If you have answered yes in 4.6a, what is the value of $\lim_{x\to 2} f(x)$?

4.6c. What is the value of $\lim_{x\to -1} \frac{f(x)-f(-1)}{x+1}$? Explain your answer.

4.7. Let f be a function defined on [-4, 3] whose graph consisting of three-line segments and a semicircle centred at the origin as given below. Answer 4.7a to 4.7c based on the information on the graph.



4.7a. The function is (continuous / discontinuous) in its domain. Underline your choice and explain why.

4.7b. Is it possible to find the value of $f'(2)$? (yes/ no). Underline your choice and if yes find the value otherwise explain why not.

4.7c. Identify at least one point where the function is continuous but not differentiable and explain why.

Appendix E: Pre-test items

Part I: Item 1-18 are multiple-choice items. From the given alternatives choose the best answer and circle the letter of your choice.

1. Which one of the followings relation holds for the sequence: $-10, -3, 4, 11, \ldots$?

   A. $a_n = a_{n-1} - 8$ B. $a_n = a_{n-1} + 7$
   C. $a_n = a_{n-1} - 7$ D. $a_n = a_{n-1} + 8$ E. None

2. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively by $a_1 = 1$ and $a_{n+1} = \left(\frac{n}{2} + 1\right) a_n$ for $n \geq 1$.

   Then a_{30}=_____

   A. 465 B. 930 C. $\frac{31!}{2^{30}}$ D. $\frac{32}{30}$ E. $\frac{32!}{30!2!}$

3. If $f(n) = \frac{5n+3}{2n+3}$ then $f(n+1) =$__________

   A. $\frac{8}{5}$ B. $\frac{5n+3}{2n+1} + 1$ C. $\frac{5n+8}{2n+5}$ D. $\frac{5n+4}{2n+4}$ E. None

4. The solution set of the inequality $\frac{2x-3}{x+1} \leq 1$ is __________

   A. [-4, 3] B. $[-2,0] \cup (1, \infty)$
   C. $(-\infty, -1) \cup (4, \infty)$ D. $(-1, \infty)$ E. $(-1,4]$

5. For $x, y \in \mathcal{R}$, which one of the followings is true?

   A. $\frac{1}{x-y} = \frac{1}{x} - \frac{1}{y}$ B. $(x+y)^2 = x^2 + y^2$
   C. $\sqrt{(x^2 + y^2)} = (x+y)$ D. $\frac{\frac{1}{x}}{\frac{a}{x} - \frac{b}{x}} = \frac{1}{a-b}$ E. $\frac{1+xy}{x} = 1 + y$

6. What is the solution set of $\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}} = 3x^2 - \frac{x}{1+\frac{1}{x}}$?

   A. $\{-1, \frac{1}{3}\}$ B. $\left\{\frac{1}{3}\right\}$ C. $\left\{3, \frac{-1}{3}\right\}$ D. $\left\{\frac{-1}{3}\right\}$ E. None

7. If $f(t) = t^2$ and $g(t) = t^3 + 2t$, what is the composition $(fog)(t)$?

   A. $t^5 + 2t^3$ B. $(t^3 + 2t)^2$ C. $t^6 + 4t^2$ D. $\frac{t^3+2t}{t^4}$ E. None

8. If $f(x) = \sqrt{x^3}$ and $(fog)(x) = \sqrt[4]{x}$, then what is the value of $g(8)$?

   A. $\sqrt[3]{2}$ B. 2 C. $\sqrt{2}$ D. $2\sqrt{2}$ E. None

9. Which one of the followings is a one-to-one correspondence function from $A = [0,1]$ to$B = [1,2]$?

A. $f(x) = tanx$ B. $g(x) = 2^x$

C. $h(x) = x^2$ D. $k(x) = x + 5$ E. None

10. If $p(x) = 3x^2$ and $q(x) = x^2 + x$, then what is the solution set of $\frac{p(x)}{3q(x)} - \frac{1}{x} = \frac{1}{q(x)}$?

A. $\{-1,2\}$ B. $\{2\}$

C. $\{-2,3\}$ D.$\{-3\}$ E. None

11. Which one of the followings is true?

A. A polynomial can have infinitely many vertical asymptotes.

B. The graph of a rational function can never cross its horizontal asymptote.

C. The graph of $f(x) = \frac{3x-1}{x-1}$ has no horizontal asymptote.

D. The graph of $f(x) = \frac{x^3-x}{x^2-x}$ has no vertical asymptote.

E. None

12. If $k(x) = \ln\left(\frac{x}{x-1} + 2\right)$, for $x > 1$, then which one of the followings is the inverse of k?

A. $f(x) = \frac{e^x-2}{e^x-3}$ B. $f(x) = \frac{e^x-2}{e^x+1}$

C. $f(x) = \frac{e^x}{e^x+1} - 2$ D. $f(x) = e^{\frac{x}{x-1}} - 2$ E. None

13. Which one of the followings is a simplified from of $\cos\left(\frac{\pi}{2} - x\right) cosx - sin^2x$?

A. $2cosx$ B. cos^2x

C. $2sinx$ D. $sin2x$ E. None

14. Which one of the following is true about the graph of $f(x) = \frac{x^3-x}{x^3(x-1)}$?

A. The vertical asymptotes of the graph are $x = 0$ and $x = 1$.

B. A horizontal asymptote of the graph is $y = 1$.

C. The graph intersects its horizontal asymptote at the point $(-1,0)$.

D. The graph intersects the vertical line at the point $(1,2)$.

E. None

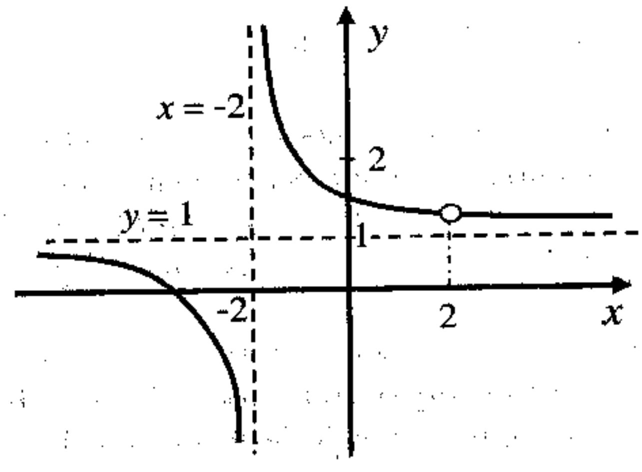
15. Which of the followings functions could most likely be drawn as in the figure below?

A. $\frac{x+3}{x+2}$

B. $\frac{x^2-2x}{x^2-4}$

C. $\frac{-x^2-x+6}{x^2-4}$

D. $\frac{x^2+x-6}{x^2-4}$



E. None

16. The graph of which of the followings equation has $y = 1$ as an asymptote?

A. $y = lnx$ B. $y = \frac{x}{x+1}$

C. $y = sinx$ D. $y = \frac{x^2}{x-1}$ E. $y = e^{-x}$

17. The point of intersection of the lines $l_1: 3x - 4y + 8 = 0$ and $l_2: 12x - 5y - 12 = 0$ is:

A. $\left(4, \frac{8}{3}\right)$ B. $(8,4)$

C. $(\frac{8}{3}, 4)$ D. $(5, -3)$ E. None

18. Which of the followings formula defines the area, A, of a circle as a function of its circumference, C?

$A = \frac{C^2}{4\pi}$ B. $A = \frac{C^2}{2}$

C. $A = (2\pi r)^2$ D. $A = \pi r^2$ E. $A = \pi(\frac{1}{4}C^2)$

Part II: Item 19- 22 are multiple-choice items. From the given alternatives choose the best answer and circle the letter of your choice. Then explain how you arrived at your answer.

19. Which one of the followings is a convergent sequence?

A. $\left\{\left(\frac{5}{3}\right)^n\right\}$ B. $\left\{\frac{2n}{n+1}\right\}$ C. $\left\{\frac{n^2}{n+1}\right\}$ D. $\left\{\frac{(-1)^n}{3}\right\}$ E. None

Why do you think so? ____

____

20. Which one of the followings relation is a function?

A. $\{(5,-7),(-7,5),(5,0)\}$ B. $x^4 = y^4$

C. $y = \begin{cases} 1, & if\ x\ \in \mathbb{Z} \\ 0, & if\ x\ \notin \mathbb{Z} \end{cases}$ D. $x^2 + y^2 = 25$ E. All

Why do you think so? ____

____

21. Suppose $f(x) = \frac{Q(x)}{x(x^2-1)}$ where $Q(x)$ a quadratic function. Which one of the followings is necessarily true about the graph of f?

A. $x = 0$, $x = 1$, and $x = -1$ are the vertical asymptote of the graph of f.

B. the graph of f does not intersect with its horizontal asymptote.

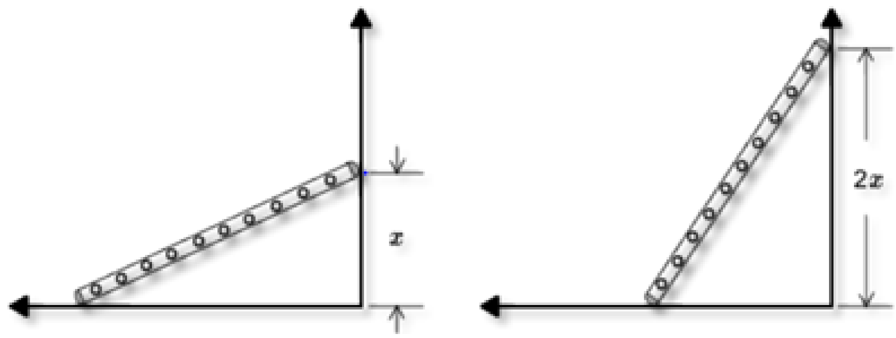
C. the vertical asymptote of the graph of f is only $x = -1$ if $Q(x) = x^2 - x$

D. the vertical asymptote of the graph of f is only $x = 1$ if $Q(x) = x^2$

Why do you think so? ____

____

22. A ladder that is leaning against a wall is adjusted so that the distance of the top of the ladder from the floor is twice as high as it was before it was adjusted. The slope of the adjusted ladder is:



Before After

A. Exactly twice what it was
B. Less than twice what it was
C. More than twice what it was
D. The same as what it was before
E. There is not enough information to determine if any of the alternatives A through D is correct.

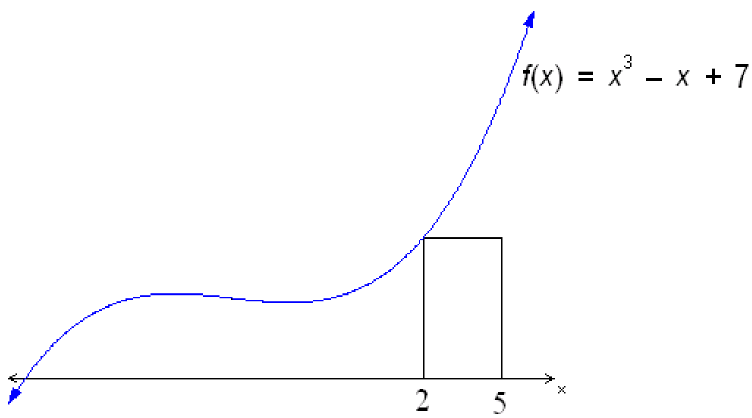
Why do you think so? ____

____

Part III: Item 23-25 are workouts. Answer the items by showing all the necessary steps clearly and neatly.

23. If $f(x) = 4x - 8$ then what is the value of $\frac{1}{20}f(x + f(x))$?

24. What is the area of the rectangle shown in the figure below? (Note that the figure is not drawn to scale)



25. Let f be a function given by $f(x) = \frac{2x}{\sqrt{9-x^2}}$ then

a. find the domain of f

b. write the equation for each vertical asymptote to the graph of f

c. write an equation for each horizontal asymptote to the graph f

d. sketch a graph of the function

(Show all the necessary steps clearly on the next paper)

Appendix F: Post- test items

Part I: Item 1-9 are multiple-choices. From the given alternatives choose the best answer and circle the letter of your choice. Then explain how you arrived at your answer.

1. Which one of the followings is true?
   A. A bounded sequence is necessarily converging.
   B. A divergent sequence is necessarily unbounded.
   C. A convergent sequence is necessarily bounded.
   D. A monotone sequence is necessarily converging.
   E. None of them is true.

Explain how you obtained your answer ____

____

2. Which one of the followings sequence is not convergent?
   A. $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ . \ . \ .\right\}$
   B. $\{a_n\} = \left\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5} \ . \ . \ .\right\}$
   C. $\{a_n\} = \{-1, 1, -1, \ 1, -1, \ . \ . \ .\}$
   D. $\{a_n\} = \{3, 3, 3, \ . \ . \ .\}$
   E. All are convergent

Why do you think so? ____

____

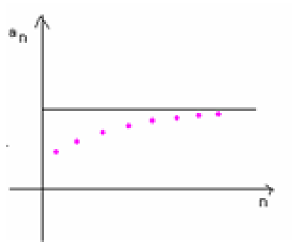
3. Suppose $\{a_n\}$ is a sequence of positive terms (i.e. $0 < a_n$ for all *n)* and $a_1 > a_2 > a_3 > \cdots > a_n . \ . \ .$. Does $\lim_{n\to\infty} a_n$ exist? What can you tell about the limit?
   A. Yes, limit exists and the value is zero.
   B. Yes, limit exists and the value is non-negative.
   C. Yes, limit exists but nothing can be said about the value.
   D. No, limit does not exist.
   E. It is not possible to decide.

Explain your answer: ____

____

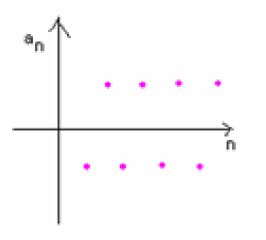
4. Which one of the followings graph of sequence has no limits?

A.



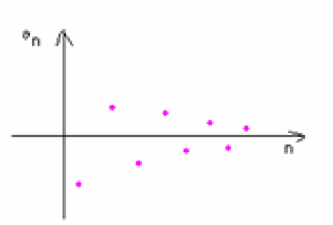
$a_n = \frac{n}{n+1}$, $n = 1, 2, 3, \ldots$

B.



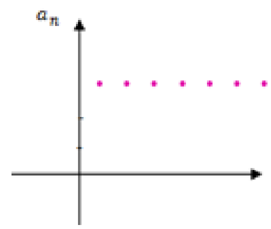
$a_n = (-1)^n$, $n = 1, 2, 3, \ldots$

C.



$a_n = \frac{(-1)^n}{n}$, $n = 1, 2, 3, \ldots$

D.



$a_n = 3$, $n = 1, 2, 3, \ldots$

.

Explain how you obtained your answer ____

____

5. Which one of the followings is true?
   A. A limit value is a number beyond which a function cannot attain values.
   B. A limit is a number that the function value approaches but never reaches.
   C. Limit is an approximation that can be made as accurate as you wish.
   D. Limit of a function is value of the function at the limit point.
   E. None of these is true.

Explain why ____

____

6. Which one of the followings is true about the notation $\lim_{x \to a} f(x)$, provided the value is a real number?
   A. It represents an infinite process.
   B. It represents a finite value.
   C. Both A & B are true.
   D. It is necessarily an upper or lower boundary on the range of the function f.
   E. Both B & D are true.

Explain why ____

____

7. Let f be a function and $c \in \mathbb{R}$. If $\lim_{x \to c} f(x)$ does not exist, which one must be true?
   A. $f(x)$ becomes large enough when x gets closer and closer to c.
   B. $\lim_{x \to c^-} f(x)$ exists but different from $\lim_{x \to c^+} f(x)$.
   C. The function has a vertical asymptote at $x = c$.
   D. $f(x)$ is not defined at $x = c$.
   E. None of these is true.

Explain why ____

____

8. Which of the followings must be true if f is a function for which $\lim_{x \to 3} f(x) = 5$
   A. f is continuous at the point $x = 3$.
   B. f is defined at $x = 3$ and $f(3)$ exactly 5.
   C. f is defined at $x = 3$ but nothing can be said about the value.
   D. It is not grant to decide about $f(3)$ from the given information.
   E. None of these is true.

Explain why ____

____

9. Consider $\lim_{x \to 3}(\frac{x^2-9}{3x-9})$. In finding this limit the number 3 is substituted for x in the functional part and the result obtained becomes $\frac{0}{0}$. What conclusions can you draw from this result? Choose the option(s) that best describes your answer.

A. The limit does not exist.

B. It is an indeterminate form.

C. The limit is 0.

D. It is undefined.

E. The limit is ∞.

If any other, please specify ____

____

Part II: Item 10 is workouts. Answer the item by showing all the necessary steps clearly and neatly.

10. $\lim_{n \to \infty} n \sin(\frac{1}{n})$ = ___________

Part III: Item 11- 15 are multiple-choice items. From the given alternatives choose the best answer and circle the letter of your choice. Then explain how you arrived at your answer.

11. Which one of the followings is true statement?

A. A function $f(x)$ is discontinuous if its graph contains a sharp "corner".

B. If a function is continuous at a point then it is necessarily differentiable at that point.

C. If a function is continuous at a point then the limit necessarily exists at that point.

D. Continuous functions must have domain all real numbers.

E. All of them are true.

Explain why ____

____

12. If $f(x) = \begin{cases} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & for\ x \neq 2 \\ \mathrm{k} & ,for\ x = 2 \end{cases}$ and if f is continuous at $x = 2$, then $k =$ ____

A. 0 B. $\frac{1}{6}$ C. $\frac{1}{3}$ D. 1 E. $\frac{7}{5}$

Explain why ____

____

13. Let f and g be differentiable functions with the following properties:

iii. $\mathrm{g}(x) > 0$ for all x ii. $f(0) = 1$

iv. $h(x) = f(x)\mathrm{g}(x)$ and $h'(x) = f(x)\mathrm{g}'(x)$ then $f(x) =$ __________

$A. f^{'}(x)$ B. $\mathrm{g}(x)$ C. e^x D. 0 E. 1

Why do you think so? ____

____

14. For what values of a and b is the function $f(x) = \begin{cases} 1 - 3x^2, & for \ x \leq 1 \\ ax + b, & for \ x > 1 \end{cases}$

differentiable at $x = 1$.

A. $a = 6$ and $b = 0$

B. $a = -6$ and $b = 4$

C. $a = -3$ and $b = 1$

D. $a = 0$ and $b = -2$

E. the function is not differentiable at $x = 1$

Why do you think so? ____

____

15. Let f be a function whose derivative f' is graphically given below. Which one of the following values is a local maximum value of f?

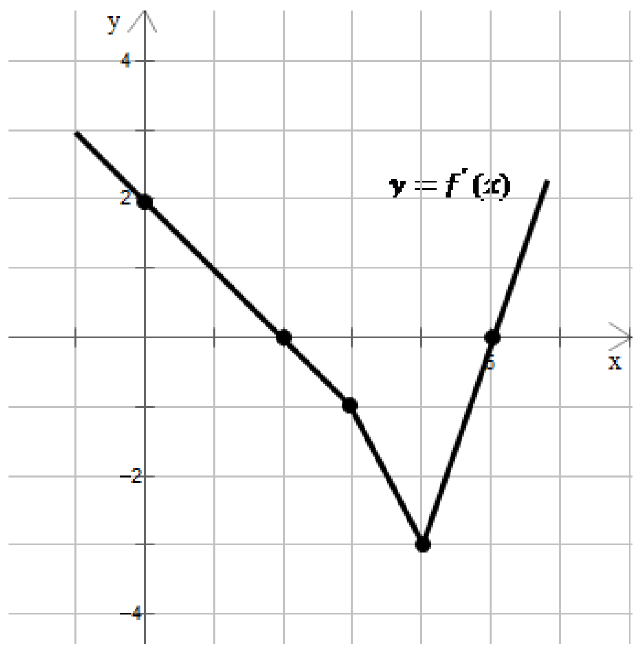
A. $f(5)$

B. $f(4)$

C. $f(3)$

D. $f(2)$

E. $f(1)$



Discuss your choice in detail ____

____

Part IV: Item 16-18 are workouts. Answer the items by showing all the necessary steps clearly and neatly.

16. Consider the function $f(x) = \frac{2x^2 - x - 15}{x-3}$

 a. Sketch the graph of f.

 b. The function is (continuous / discontinuous) at the point $x = 3$. (Underline your choice).

 c. Does the limit of $f(x)$ exist at $x = 3$? (yes / no) (Underline your choice) and explain how.

17. The percent of concentration of a certain drug in the bloodstream t hours after the drug is administered is given by $f(t) = \frac{5t}{t^2+1}$. Then:

 a. Evaluate $\lim_{t\to\infty} f(t)$ and interpret this result.

 b. Find the time at which the concentration is a maximum, and

 c. Find the maximum concentration.

 d. On what intervals is the concentration increasing? Explain why.

18. The following table shows some x-values and the corresponding function values of a function f and its derivative f'

x	-1.4	-1	-0.8	-0.4	0	0.8	1	1.4	2	2.8
$f(x)$	1.9044	0.25	0.0144	0.2704	1	2.1904	2.25	2.0164	1	0.0144
$f'(x)$	-6.624	-2	-0.432	1.456	2	0.592	0	-1.136	-2	0.432

Based on the information given in the table, compute the following values:

 a. $\lim_{x\to 2} f(x)$

 b. $\lim_{x\to -1} \frac{f(x)-f(-1)}{x+1}$?

Appendix G: Scoring rubric for open-ended/ workout items

Item	Description of response	Score
1.5/ 10	If she/he is • aware that as $n \to \infty, \frac{1}{n} \to 0$ • $nsin\left(\frac{1}{n}\right) = \frac{sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$ and hence • $\lim_{n\to\infty} nsin\left(\frac{1}{n}\right) = 1$	3
	• aware that as $n \to \infty$, $\frac{1}{n} \to 0$ but the other steps are missed or not correct	2
	• insert infinity as a number	1
	otherwise (i.e. no answer, incorrect answer or the correct answer for the wrong reason)	0
3.4a/ 16a	If she/he aware that • for $x \neq 3$ the function is equal to the linear function $2x + 5$, • the graph has a whole at $x = 3$ and plotted the graph correctly	3
	• aware of the conditions but plotted the graph incorrectly	2
	• a correct graph without saying nothing	1
	• otherwise (i.e. no graph or incorrect graph)	0
3.4b/ 16b	If she/he • underlined discontinuous	1
	• underlined continuous or not responded	0
3.4c/ 16c	If she/he • answered yes	1
	• answered no or left unanswered	0
3.4d	If she/he • answered 11	1
	• answered any other number or left unanswered	0
3.4e	If she/he • answered does not have value/undefined/	1
	• any other number or left unanswered	0

4.4	If she/he demonstrate the correct application of • chain rule, • the derivative of a trigonometric function, • the derivative of an exponential function, • procedure of combination function and found an expression equivalent to $\frac{1}{2\sqrt{x+1}}\cos\left(e^{\sqrt{x+1}}\right)e^{\sqrt{x+1}}$	3
	• a minor error such as sign the errors; or applied chain rule but an error with the derivative rules,	2
	• no evidence of considering chain rule or the correct answer without showing the necessary steps,	1
	• Otherwise (i.e. no answer, incorrect answer or the correct answer for wrong rules or procedures).	0
4.5a/ 17a	If she/he demonstrate • focuses on what happens to f as t tends to infinity, • wrote $\lim_{t\to\infty} f(t) = 0$ and explained that the concentration is null,	3
	• found $\lim_{t\to\infty} f(t) = 0$ correctly and explained that the concentration is null but lack justification,	2
	• found $\lim_{t\to\infty} f(t) = 0$ using a wrong method, like replacing infinity instead of $\mathbf{t}$ and not interpret the result,	1
	• otherwise (i.e. no answer or incorrect answer).	0
4.5b & 4.5c/ 17b & 17c	If she/he demonstrate • application of the first derivative test to find extreme value and found that $x = 1$ and, • correctly evaluate f at $t = 1$ and found $\frac{5}{2}$,	3
	• application of the first derivative test to find extreme value but one or both values are not correct due to some algebraic errors,	2
	• both answers are correct but lack justification or clarity or only one answer is given,	1
	• Otherwise (i.e. no answer, both incorrect answer or one or both correct answer for the wrong reason).	0
17d/	If she/he used the first derivative test for monotonic and stated the intervals where the function is	3

	• increasing on $(\mathbf{0}, \mathbf{1})$,	
	• used the first derivative test to monotonic but the value is not correct due to some algebraic errors,	2
	• the answers is correct but lack justification or clarity,	1
	• otherwise (i.e. no answer, correct answer for wrong reason).	0
4.6a (b)/ 18a	If she/he aware that • a differentiable function is continued, • continuity implies the limit exist and hence $\mathbf{lim}_{x\to 2} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\mathbf{2}) = -\mathbf{2}$,	3
	• a differentiable function is continues, continuity imply limit exist but interpret $\mathbf{lim}_{x\to 2} \boldsymbol{f}(\boldsymbol{x})$ as a number $\neq -\mathbf{2}$,	2
	• got $\mathbf{lim}_{x\to 2} \boldsymbol{f}(\boldsymbol{x}) = -\mathbf{2}$ but gave no reason or justification,	1
	• no or impossible to find.	0
4.6c/ 18b	If she/he aware that • $\mathbf{lim}_{x\to -1} \frac{f(x)-f(-1)}{x+1} = \boldsymbol{f}'(-\mathbf{1})$ and picked $\boldsymbol{f}'(-\mathbf{1}) = -\mathbf{2}$	1
	• otherwise (i.e. no answer, incorrect answer or the correct answer for wrong rules or procedures)	0
4.7a	If she/he answered • yes and justified continuity in terms of limit	1
	• otherwise (i.e. no answer, incorrect answer or correct answer for wrong reason)	0
4.7b	If she/he recognizes derivative as the slope of the tangent line to the graph of f and found find $\boldsymbol{f}'(\mathbf{2}) = \frac{-\mathbf{1}}{\mathbf{2}}$	2
	If she/he recognizes derivative as the slope of the tangent line but failed to found find $\boldsymbol{f}'(\mathbf{2}) = \frac{-\mathbf{1}}{\mathbf{2}}$	1
	otherwise (i.e. no answer or the correct answer for the wrong reason)	0
4.7c	If she/he identified one point correctly and explained why	1
	otherwise (i.e. no answer, incorrect answer or the correct answer for the wrong reason)	0
17d	If she/he used the first derivative test for monotonic and stated the intervals where the function is	3

	• increasing on $(\mathbf{0}, \mathbf{1})$,	
	• used the first derivative test to monotonic but the value is not correct due to some algebraic errors,	2
	• the answer is correct but lack justification or clarity,	1
	• Otherwise (i.e. no answer, incorrect answer or the correct answer for the wrong reason).	0

Appendix H: Intervention activities

Activity 1

1.1. Sort the following sequences as bounded, unbounded, monotonic, convergent or divergent

a. $a_n = (-1)^n$

b. $\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ . \ . \ . \ \}$

c. $a_n = \begin{cases} 1\ , \ if\ n\ is\ odd \\ 1 - \frac{1}{n}, if\ n\ is\ even \end{cases}$

d. $\left\{\frac{1}{3n}\right\}_{n=1}^{\infty}$

e. $a_n = \begin{cases} n\ , \ if\ n\ is\ odd \\ \frac{1}{n}, if\ n\ is\ even \end{cases}$

f. $\left\{\left(\frac{5}{3}\right)^n\right\}_{n=1}^{\infty}$

g. $a_n = \frac{1}{2}$

h. $\left\{\frac{(-1)^n}{5n}\right\}_{n=1}^{\infty}$

1.2. Based on the sequences in 1.1, which one of the followings is true:

a. one can find a sequence which is bounded and convergent,
b. one can find a sequence which is bounded and divergent,
c. one can find a sequence which is bounded and monotonic,
d. one can find a sequence which is unbounded and monotonic,
e. one can find a sequence which is monotonic and convergent,
f. one can find a sequence which is monotonic and divergent,
g. one can find a sequence which is unbounded and convergent.

Activity 2

2.1. What is the main property of a sequence that is convergent i.e.

a. what is the necessary condition to say a sequence is convergent?
b. what is the sufficient condition to say a sequence is convergent?

2.2. Classify each of the following statements as being true or being false. Give a counter example for those which are false to justify why it is false.

________ a. Every bounded sequence is convergent.

________ b. Every convergent sequence is bounded.

________ c. Every increasing sequence is convergent.

________ d. Every divergent sequence is unbounded.

________ e. Every unbounded sequence is divergent.

Activity 3

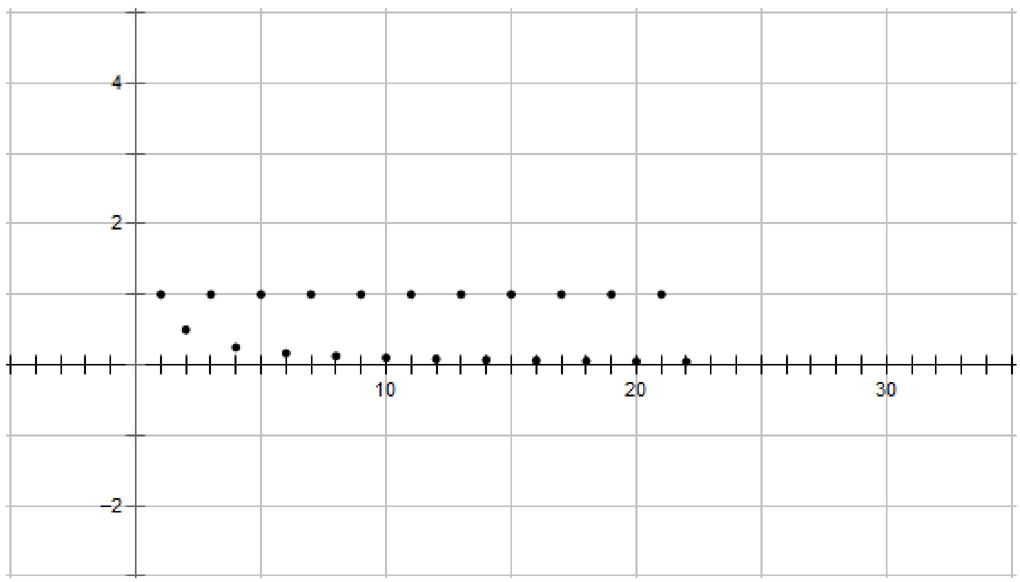
3.1 Each of the following sequences has no limit. State the reason why not the limit exists.

a. $\left\{\frac{(-1)^n}{3}\right\}$,

b. $\left\{\left(\frac{3}{2}\right)^n\right\} = \left\{\frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}, \ . \ . \ .\right\}$

c. $a_n = (-1)^n + \frac{1}{n}$

d. the sequence whose graph is given below



3.2 Each of the following sequences, as given in figure 1 and figure 2 below, has limit.

a. In each case identify the limit value and explain why the limit exists.

b. In each case, how many of the terms are at:

- $|a_n - L| \leq \frac{1}{2}$,
- $|a_n - L| \leq \frac{1}{10}$,
- $|a_n - L| \leq \frac{1}{100}$,
- $|a_n - L| \leq \frac{1}{10^3}$,
- $|a_n - L| \leq \frac{1}{10^4}$.

c. Write an algebraic formula that describes each of the two given sequence.

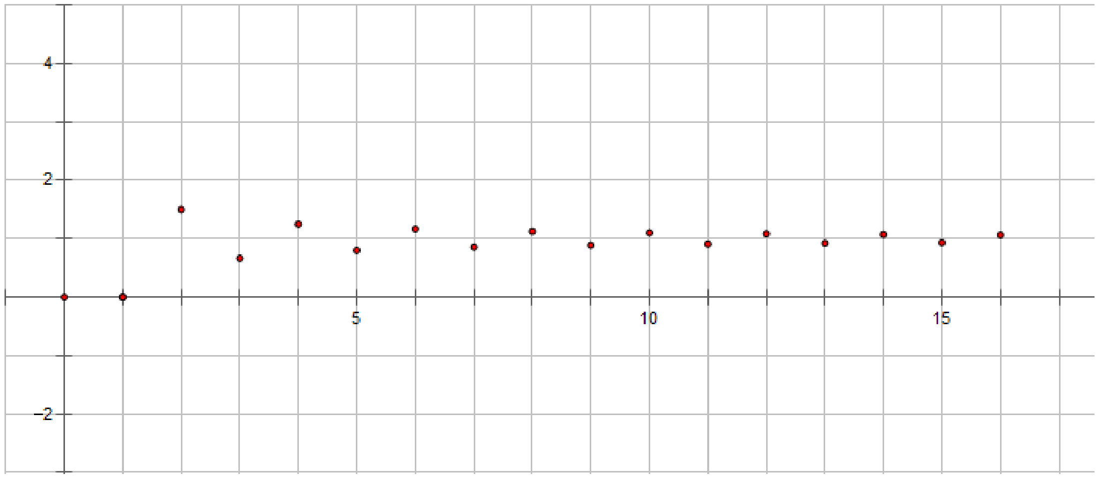


Figure 1

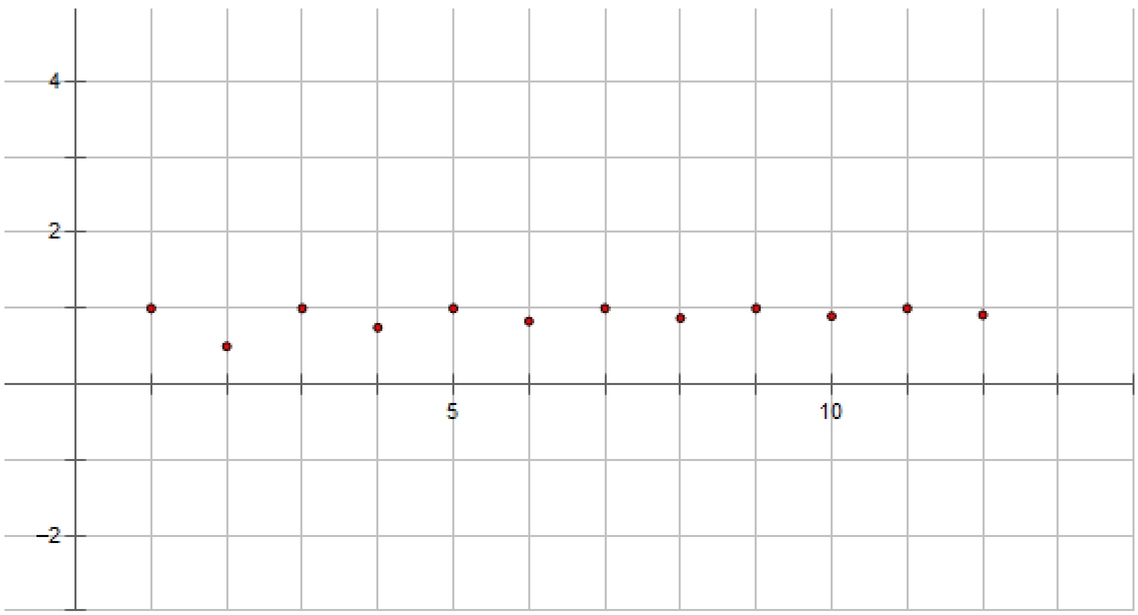


Figure 2

Activity 4

Look at the following exercise, and their solution given by someone. Is the solution correct? If you say it is wrong, identify the wrong working and give correction.

a. $\lim_{n\to\infty}\left(1+\left(\frac{1}{n}\right)^{n}\right) = 1$

Because- $\lim_{n\to\infty}\left(1+\left(\frac{1}{n}\right)^{n}\right)=\left(1+\frac{(1)^{\infty}}{\infty}\right)=\left(1+\frac{1}{\infty}\right)=(1+0)=1$

b. $\lim_{n\to\infty}(\frac{1-\sqrt{n}}{2n+3}) = \frac{1}{2}$

Because- $\lim_{n\to\infty}(\frac{1-\sqrt{n}}{2n+3}) = \lim_{n\to\infty}\left(\frac{1-\sqrt{n}}{2n+3}\right)\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) = \frac{1}{2}$

c. $\lim_{n\to\infty}(\frac{2n^2+5}{3n^2-100,000,000}) = 1$

Since

$$\lim_{n\to\infty}(\frac{2n^2+5}{3n^2-100,000,000}) = \left(\frac{2n^2+5}{3n^2-100,000,000}\right)\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1$$

Activity 5

For each of the following, choose the letter of the correct answer and write the reason of your choice on the space provided.

5.1. Which of the following is equal to $\lim_{n\to\infty}\frac{3\sqrt{n}-6n+5}{4n+1}$?

A. $\frac{-3}{2}$ B. $\frac{-3}{4}$ C. $\frac{3}{2}$ D. $\frac{5}{4}$ E. -3

Why do you think so? ____

____

5.2. The sequence $\left\{\frac{(n-1)(2n+1)}{1-n^2}\right\}_{n=1}^{\infty}$ converges to:

A. $-\infty$ B. – 2 C. 0 D. $\frac{-5}{3}$ E. None

Why do you think so? ____

____

5.3. If $a_n = \left(\frac{n+3}{n+1}\right)^n$, then the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ is equal to:

A. 1 B. $\frac{1}{2}e^2$ C. e^2 D. $+\infty$ E. None

Why do you think so? ____

____

5.4 . Which one of the following sequences is a convergent sequence?

A. $\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ . \ . \ . \ \}$

B. $\{(-1)^n\}_{n=1}^{\infty}$

C. $\left\{10^{10} - \frac{1}{100}n\right\}_{n=1}^{\infty}$

D. $\left\{\sin(\frac{1}{n})\right\}_{n=1}^{\infty}$

Why do you think so? ____

____

5.5. Which one of the followings is a convergent sequence?

A. $\left\{\left(\frac{7}{4}\right)^n\right\}$

B. $\left\{\frac{n^2}{n+1}\right\}$

C. $\left\{\frac{(-1)^n}{3}\right\}$

D. $a_n = \begin{cases} 1, & \text{if } 1{,}000{,}000 < n < 2{,}000{,}000 \\ \frac{1}{\text{n}}, & \text{if n is other natural numbers} \end{cases}$

Why do you think so? ____

____

Activity 6

6.1. Compute the limit of the following functions at the given value of the domain.

a. $f(x) = \frac{\sqrt{x+1}-1}{x}$ at $x = 0$ and $x = 1$

b. $k(t) = \frac{t^2-4t+3}{t^2-1}$ at $t = 1$

c. $h(x) = \frac{\sqrt{x^2+4}+2}{x^2}$ at $x = 0$

d. $\text{g}(x) = \frac{x^{3/2}-1}{x-1}$ at $x = 1$

e. $f(x) = xsinx$ at $x = 0$

f. $h(x) = \frac{(2-x)^2}{x-2}$ at $x = 2$ and $x = 2c$ for $c \in \mathcal{R}$

g. $f(x) = \frac{e^x-1}{x}$ at $x = 0$

h. $\text{g}(t) = \frac{t^4-1}{|t-1|}$ at $t = 1$

6.2. Compute the limit of the following functions at infinity (write the notation and compute the value)

a. $f(t) = \sqrt{t+1} - \sqrt{t}$

b. $k(x) = \frac{x^{99}}{2^x}$

c. $k(x) = \left(1 - \frac{1}{10^x}\right)$

d. $k(x) = \frac{\cos(x)}{x}$

Activity 7

7.1. Justify that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = 2.71828\ldots$ Well, if you try to use direct substitution, what will happen?

7.2. Consider the function $f(x) = \frac{tanx}{x}$. How can you find the limit of f at $x = \frac{\pi}{2}$? Well, if you try to use direct substitution, what will happen?

7.3. Notice that, in finding the limit, the three most common methods are substitution, rationalization and conjugate. Now, if any of these methods do not work what will be your conclusion? Could it be necessarily the limit does not exist? (What you did in c and g in activity 6.1?).

Activity 8

8.1. When you use words like "approach to" and "tends to", what do you actually mean? Do you think they seem to imply actual value or do you think of something in a process? Justify your answer.

8.2. Given a function f and a number c. Describe in your own words what it means to say that the limit of the function f as $x \to c$ is some number L?

8.3. Describe cases where limit of functions at a point fails to exist? Discuss all the cases exhaustively.

8.4. Explain the procedure to find the limit, $\lim_{x\to a} f(x)$, where $f(x)$ is a split-function given in symbolic or algebraic form.

Activity 9

Consider the function $f(x) = \frac{x^3-1}{x-1}$

a. what is domain of f?

b. what is limit of f at $x = 3$?

c. the only place where $\frac{x^3-1}{x-1}$ and $x^2 + x + 1$ differ is $x = 1$. Why is it acceptable to interchange these two functions even though we are trying to find limit at $x = 1$?

Activity 10

Let $f(x) = \begin{cases} 1 - x\,, & if\ x \leq 1 \\ x^2\,, & if\ x > 1 \end{cases}$ then

a. is f continuous at $x = 1$ (explain it using the given algebraic formula)
b. sketch the graph of f and describe continuity of f at $x = 1$

Activity 11

Let $f(x) = \frac{x^2+x-2}{1-\sqrt{x}}$ then

a. $\lim_{x\to 1} f(x)$ = ________
b. $\lim_{x\to \infty} f(x)$ = ________
c. Is f continuous at $x = 1$? If yes, justify how it is continuous. Otherwise define $f(1)$ to make *f* continuous at 1.

Activity 12

A function f behaves in the following way near $x = 3$:

As x approaches 3 from the left, $f(x)$ approaches 2.

As x approaches 3 from the right, $f(x)$ approaches 1.

For the above situation you are required to:

a. Draw a sketch to illustrate the behaviour of f near $x = 3$.
b. write the two sentences in symbolic form.
c. determine with reasons if $\lim_{x\to 3} f(x)$ exists.

Activity 13

Consider the split function $g(x) = \begin{cases} x + 2, if\ x \leq 3 \\ 6 - x, if\ x > 3 \end{cases}$.

For this function you are required to:

a. Use the symbolic form to explain in your own words the behaviour of g near $x = 3$.
b. Use the algebraic form to draw the graph of g.
c. Evaluate $\lim_{x\to 3^-} g(x)$and $\lim_{x\to 3^+} g(x)$.
d. Determine with reason if $\lim_{x\to 3} g(x)$ exists.

Activity 14

14.1 . Consider the function $f(x) = \begin{cases} \frac{a}{x^2+3}, if\ x \leq 1 \\ \frac{x-1}{\sqrt{x}-1}\ , if\ x > 1 \end{cases}$

If f is continuous at $x = 1$, then what should be the value of a ?

14.2 . Find a and b that will make the function f continuous in $(-\infty, \infty)$ if

$$f(x) = \begin{cases} 3x + 1\ , & x < 2 \\ ax + b\ , & 2 \leq x < 5 \\ x^2\ , & x \geq 5 \end{cases}$$

Activity 15

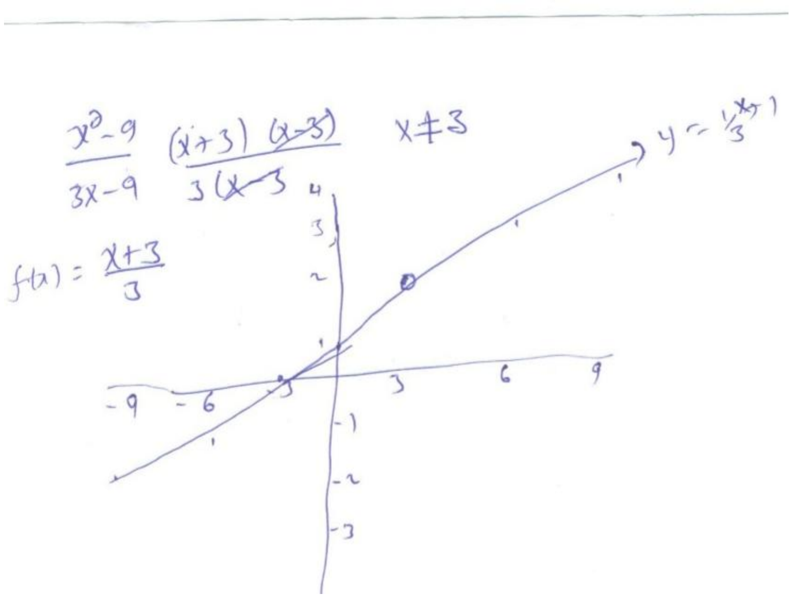
Let $f(x) = \begin{cases} a\frac{sinx}{x-|x|}, & if\ x < 0 \\ e^{-x} + cosx\ , & if\ x \geq 0 \end{cases}$

You are told that the function f is continuous at $x = 0$. The question remains to be answered is value of a. The following steps are part of the procedure to answer this question. Give reason why each of these steps is logical.

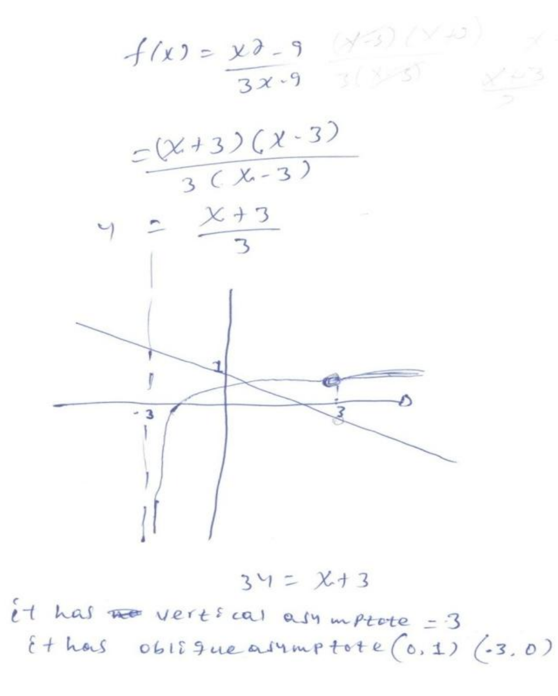
Step	Reason
$\lim_{x\to 0^-} \frac{asinx}{x-\lvert x\rvert} = \lim_{x\to 0^+}(e^{-x} + cosx)$	
$\lim_{x\to 0^-} \frac{asinx}{x-\lvert x\rvert} = \lim_{x\to 0^-} \frac{asinx}{2x}$	
$\lim_{x\to 0^-} \frac{asinx}{2x} = \frac{a}{2}$	
$\lim_{x\to 0^+}(e^{-x} + cosx) = 2$	
Hence, $a = 4$	

Activity 16

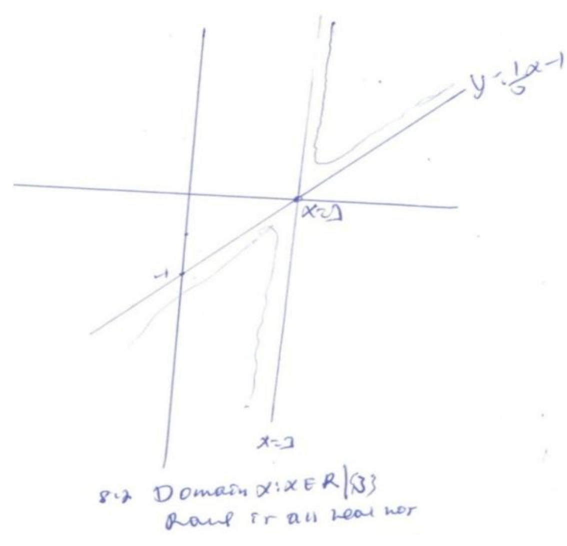
The following three figures are graphs of a function drawn by three different students as a response to the question “draw graph of $f(x) = \frac{x^2-9}{3x-3}$ ” identify the one which is correct and give your comment on the wrong ones.



a)



b)



c)

Activity 17

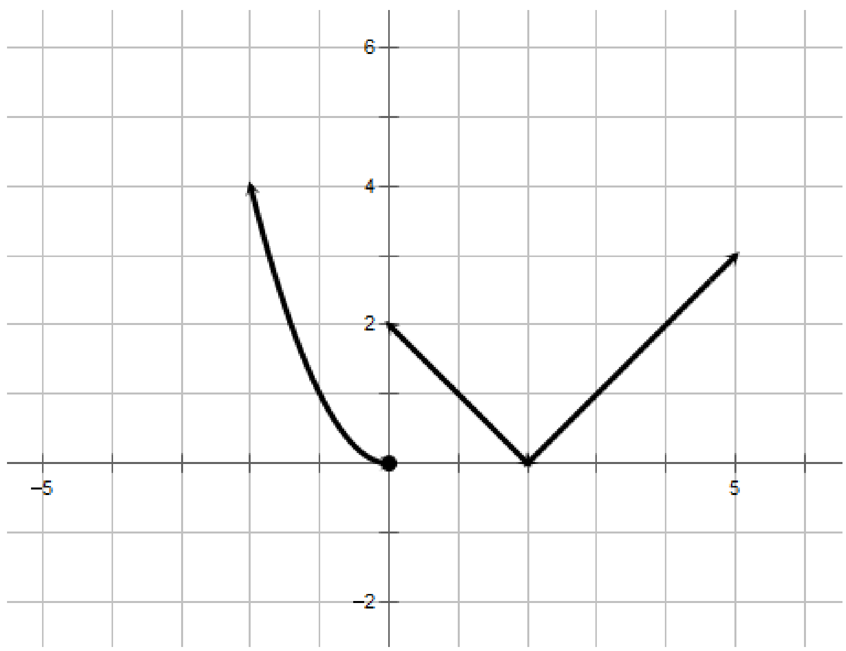
A glasshouse in horticulture has a height of two meters. The progress of a flower days after its half-life of growth was recorded. The height as a function of days is given by the function $f(x) = 2(1 - \frac{1}{2^x})$.

a. Does the height of the flower have a limit? If yes, what is the limit?
b. Will the flower reach the ceiling of the glasshouse? Justify your answer.

Activity 18

18.1**.** Consider the function whose graph is given below

a. Find a function (algebraic expression) that would be pictured by this graph.
b. Is this function continues in its domain?



18.2. Describe properties of a continuous function that should be observed on its graph i.e. based on the shape of graph of a function, how can you say that a function is continuous or discontinuous?

Activity 19

For each of the following items, choose the letter of the correct answer and write the reason of your choice on the space provided.

19.1 . What is between $0.999\ldots$ (The nines repeat.) and 1?

A. Nothing because $0.999\ldots = 1$.

B. An infinitely small distance because $0.999\ldots < 1$.

C. 0.001.

D. You cannot really answer as $0.999\ldots$ keeps on going forever and never finishes.

If you do not agree with any of the above, give your own answer and justify why?

____

____

19.2 . If $f(x) = 2x^3 - 3x$, then $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ is equal to ______________.

A. 1 B. -1 C. 3 D. ∞

Why do you think so? ____

____

19.3 . Which one of the following is equal to $\lim_{x \to \infty} \left(\frac{3x}{3x+2}\right)^{-3x}$?

A. e^2 B. e^{-3} C. e^{-2} D. e^3

Why do you think so? ____

____

19.4 . If f is continuous at $x = 0$ and $g(x) = \sqrt{x}(2f(x) + \frac{3}{\sqrt{\ }})$ for all $x > 0$, then what is the value of $\lim_{x \to 0^+} g(x)$?

A. 0 B. 2 C. 3 D. 5

Why do you think so? ____

____

19.5 . Given that $\lim_{x\to 3} f(x) = 3$ and $\lim_{x\to 3} \text{g}(x) = 5$, what is the value of

$\lim_{x\to 3}\left(\frac{(f(x)-\text{g}(x))(\text{g}(x))-2f(x))}{\text{g}(\text{x})^2-f(x)^2}\right)$?

A. $-\frac{10}{8}$ B. $-\frac{1}{8}$ C. 0 D. does not exist

Why do you think so? ____

____

19.6. What is the value of k so that $f(x) = \begin{cases} \frac{tan2x}{x}, & x > 0 \\ k - e^{2x}, x \leq 0 \end{cases}$ is continuous at $x = 0$?

A. 2 B. 3 C. 1 D. 0

Why do you think so? ____

____

19.7.The left hand limit $\lim_{x\to 0} \frac{xe^x - |x|}{x}$ is equal to ________________.

A. 0 B. 2 C. 1 D. does not exist

Why do you think so? ____

____

19.8.$\lim_{x\to\infty} xsin\frac{1}{x}$ is equal to:

A. 0 B. 1 C. ∞ D. -1

Why do you think so? ____

____

19.9. Given an arbitrary function *f*, if $\lim_{\text{x}\to 3} \text{f}(\text{x}) = 4$ what is f(3)?

A. 4 B. it must be closed to 4

C. 3 D. it is not defined

E. not enough information is given to determine $f(3)$

Why do you think so? ____

____

19.10. Let f be a continuous real-valued function defined on the closed interval $[-2,3]$. Which of the following is NOT necessarily true?

A. f is bounded.

B. For each c between $f(-2)$ and $f(3)$, there is an $x \epsilon [-2,3]$ such that $f(x) = c$.

C. $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ exists.

D. There exist a number m between $f(-2)$ and $f(3)$ which is maximum of f on $[-2,3]$.

E. None.

Why do you think so? ____

____

Activity 20

Compute derivative of each of the following functions

a. $g(x) = \pi^2$

b. $k(x) = \frac{1}{3}x^3 - \frac{9}{2}x^2 + 18x - 5$

c. $h(x) = (x-1)(x+2)^3$

d. $f(x) = \ln\left(\sqrt{x^2+1}\right)$

e. $f(x) = xe^{3x} - \cos(2x)$

f. $h(x) = \sqrt[3]{1+e^{-x}}$

g. $k(x) = \frac{1+\sin x}{\cos x}$

h. $g(x) = \ln e^{2x}$

Activity 21

Given a function f and a number a in the domain of f. Consider the expression $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, provide the limit exist.

a. What symbol we use to represent this quantity?

b. What is the name of the symbol we use to represent this quantity?

c. What is the meaning of this quantity?

d. What do you really think about the terms “symbol”, “name”, and “meaning” of mathematical notions? Discuss with the help of examples.

e. $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ do they the same value or different ? Explain.

Activity 22

From the following list of statements, choose those which are false and justify why they are false.

a. If a function is not continuous at a certain point, then that function is not differentiable there.

b. Let $f: A \to A$ be a continuous function where $A = [0,1]$. Then there exists a point $a \in [0,1]$ such that $f(a) = a$.

c. If f is differentiable function then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists $\forall x$ in the domain of f.

d. If f is differentiable function then $\lim_{x\to a} f(x)$ exists $\forall x$ in the domain of f.

Activity 23

23.1 Let f be differentiable function with $f(1) = -1$ and $f'(1) = 1$.

If $\mathrm{g(x)} = [f(2\mathrm{x}+1)+2]^2$,then what is the value of $\mathrm{g}'(0)$?

23.2 Find a if $f(x) = \begin{cases} x^3+1, & x \geq 2 \\ ax+5, & x < 2 \end{cases}$ is differentiable at $x = 2$.

23.3 A student is asked to answer the problem "For what values of a and b is the function $f(x) = \begin{cases} ax, & x \leq 1 \\ bx^2+x+1, & x > 1 \end{cases}$ differentiable at $x = 1$?"

The following steps are part of the procedure to answer this problem. Give reason why each of these steps is logical

Step	Reason
$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$	
$2b+1 = a$	
$\lim_{x\to 1} f(x) = f(1)$	
$b+2 = a$	
Hence, $a = 3$ and $b = 1$	

Activity 24

A farmer claims that the productivity (P) of the coffee stem in his fixed size farmland is given by $P(s) = \frac{3s}{s^2+2s+841}$ where s represent the number of coffee stem planted. Calculate:

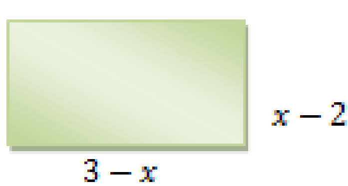
a. The rate of change of production when he plants 30 pieces of the coffee stem?
b. Was the production increasing or decreasing at $s = 30$?
c. Find the number of coffee stem that should be planted to maximize the production, and compute the maximum product.
d. For what values of s is the production increasing?
e. For what values of s is the production decreasing? Explain why.
f. Evaluate $\lim_{s\to\infty} P(s)$ and interpret this result

Activity 25

Water is poured into a cylindrical tanker of radius 5 meters at a rate of 10 meter cube/min. what is the rate of change of the height of the level of water when it rises to 3 meters?

Activity 26

The Hosanna municipality has a plan to fix the damp plot for the town residents. However, one of the identified rectangular areas is a plot of land, which is surrounded by privet landowner. The city needs to maximize the area at the same time to minimize the cost that will be paid to the landowner surrounding the area. If the plot has the following diamension:

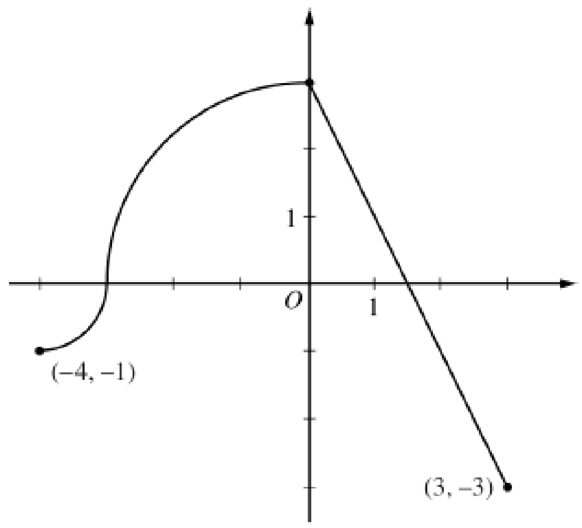


a. Write down the formula for finding the area of the land.

b. Discuss the possible value of x to get maximum area with minimum cost.

Activity 27

Let f be a continuous function defined on [-4, 3] which graph is shown below:



Graph of f

a. Find the value of $f'(2)$.

b. At what x-values (if any), is the function continuous, but not differentiable? Use the definition of derivative to justify your answer.

c. On what intervals is the function increasing?

d. On what interval is the function decreasing? Explain.

Activity 28

Let $f(t) = \begin{cases} 1 + x^2, & if\ x < 0 \\ 2 - x, & if\ 0 \leq x \leq 2 \\ (x-2)^2, & if\ x > 2 \end{cases}$

28.1. The function is (continuous / discontinuous) in its domain? Underline your choice and and explain why.

28.2. Is it possible to find the value of $f'(2)$? (yes or no). Underline your choice and if yes find the value otherwise explain why not.

28.3. Identify at least one point where the function continuous but not differentiable and explain why.

Activity 29

A rectangular field of length l and width w meters, where $\mathrm{w} < l$, has perimeter 400 meters. If a circular region of area w^2 is to be reserved for office purpose, what should be the length of the field so that the area of the remaining region is maximum?

Activity 30

For each of the following items, choose the correct answer. Discussed how you attained your choice.

30.1. Let $f(x) = \frac{6x}{x+a}$. for what value of a is $f'(a) = 1$

A. $\frac{1}{3}$ B. $\frac{2}{3}$ C. $\frac{3}{2}$ D. 3

30.2. Which one of the followings is necessarily true about a function $f(x)$?

A. If f is continuous at $x = a$, then f is differentiable at $x = a$.

B. If f is not differentiable at $x = a$, then $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.

C. If f is differentiable at $x = a$, then $\lim_{x\to a^-} f(x) = f(x) = \lim_{x\to a^+} f(x)$.

D. If the derivative $f'(a) = 0$, then f attains its maximum value at $x = a$.

30.3. If $g(x) = \frac{f(x)}{x+1} + (f(x))^2$, $f(1) = 8$ and $\mathrm{f}'(\mathrm{x}) = 2$, then $\mathrm{g}'(1)$ is equal to:

A. 36 B. 31 C. 25 D.16

30.4. If $\mathrm{f(x)} = \frac{\mathrm{x}^2}{1+\mathrm{xg(x)}}$, $\mathrm{g}(2) = 1$ and $\mathrm{g}'(2) = 10$, then which one of the following is equal to $\mathrm{f}'(2)$?

A. -8 B. $-\frac{8}{9}$ C. $\frac{4}{3}$ D. $\frac{8}{9}$

30.5. Which one of the followings is the set of all critical numbers of $\mathrm{f(x)} = \frac{1}{3}\mathrm{x}^3 - |4\mathrm{x} - 1|$?

A. $\{\frac{1}{4}, 2\}$ B. $\{-2, \frac{1}{4}, 2\}$ C. $\{-2, 2\}$ D. $\{\frac{1}{4}\}$

30.6. If $f(x) = x^2\sqrt{2x + 12}$, what is the slope of the tangent line to the graph of f at $x = 2$?

A. -4 B. 2 C. 18 D. 17

30.7. What is the equation of the tangent line to the graph of $f(x) = 3x^2 + 4x - 5$ at $(1,2)$?

A. $10x - y - 8 = 0$ C. $-10x + y - 8 = 0$

B. $-10x - y - 8 = 0$ D. $10x + y - 8 = 0$

30.8. If $h(x) = f(2x + 2).g(1 - x^2)$, with $f(2) = -3$, $f'(2) = 4$, $g(1) = -5$, and $g'(1) = 1$, then what is the actual value of $h'(0)$?

A. -40 B. -20 C. 0 D. 19

30.9. If $f(x) = e^{3x}\cos x - \frac{x+\pi}{x^2+2}$, then $f'(0)$ is equal to ______________.

A. $3 - \frac{\pi}{2}$ B. $\frac{3}{2}$ C. $\frac{7}{4}$ D. $\frac{5}{2}$

30.10. If $f(x) = 2 + |x - 3|$ for all x, then the value of the derivative $f'(x)$ at $x = 3$ is ____.

A. -1 B. does not exist C. 1 D. 2