Reduced Order Model
Proper Generalized Decomposition

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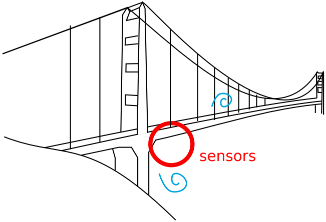
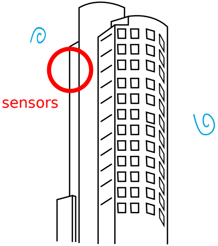
Monday 9 of December

Context : small electronic devices alimentation





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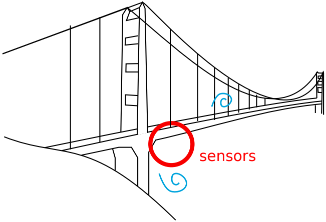

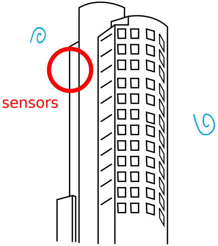

- Bridge or building sensors connected to a grid / battery ☹

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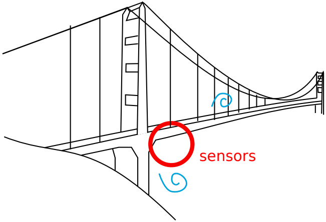

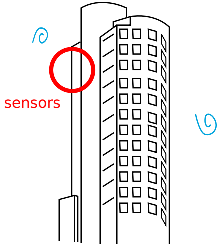

- Bridge or building sensors connected to a grid / battery ☹
- Lost source of energy : ambient fluid flow energy ☹

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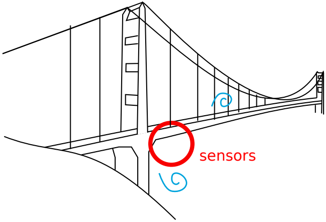

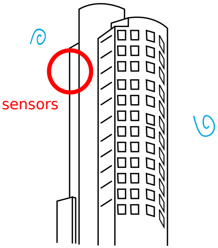

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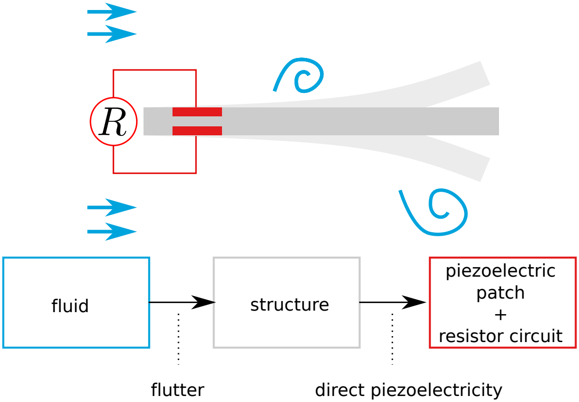
Solution : harvest the energy to design autonomous devices generators

Context : piezoelectric energy harvester



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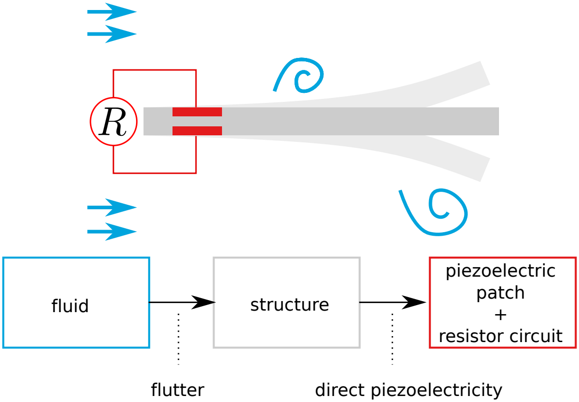
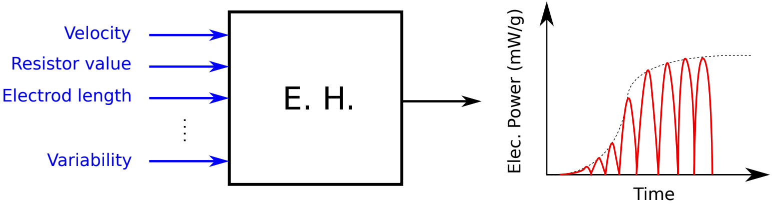


Transform ambient fluid flow energy into electrical energy

Context : parametric design with nonlinearities



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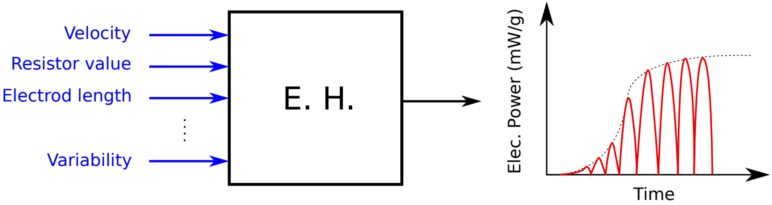

Maximize power output
Minimize the fatigue exposure

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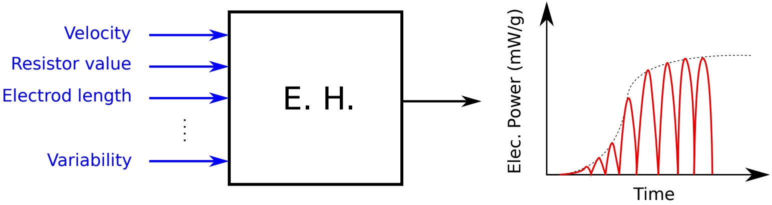

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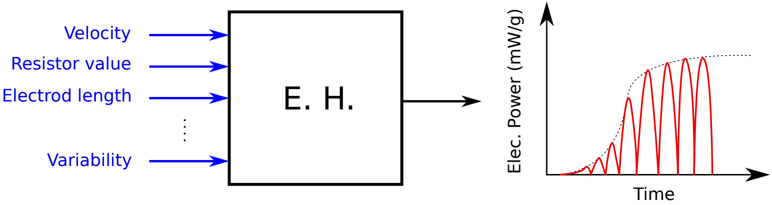

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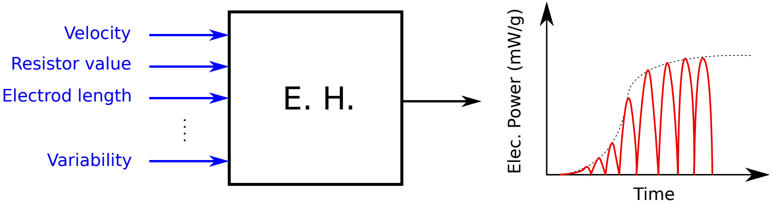

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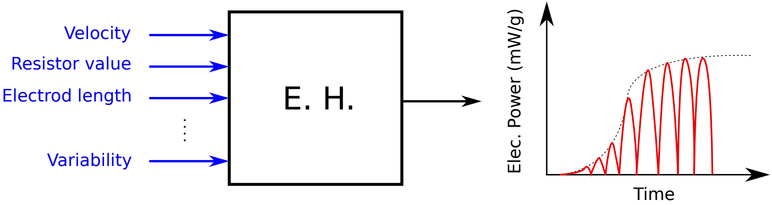

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- Predict the nonlinear coupled behavior $\longrightarrow$ complex models
- Changing conditions $\longrightarrow$ sensibility analysis

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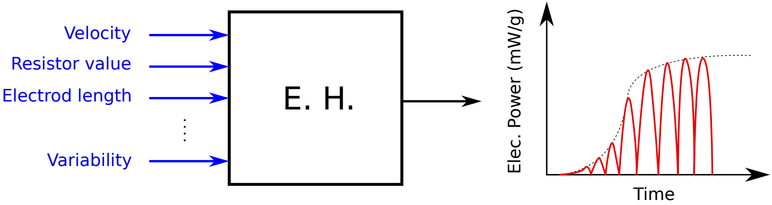

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- Changing conditions $\longrightarrow$ sensibility analysis
- Just-in-time feedback

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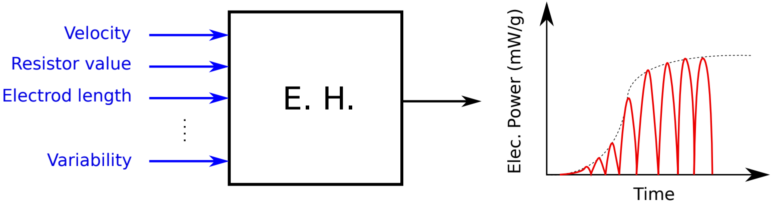

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- Changing conditions $\longrightarrow$ sensibility analysis
- Just-in-time feedback $\longrightarrow$ real-time solution approximation

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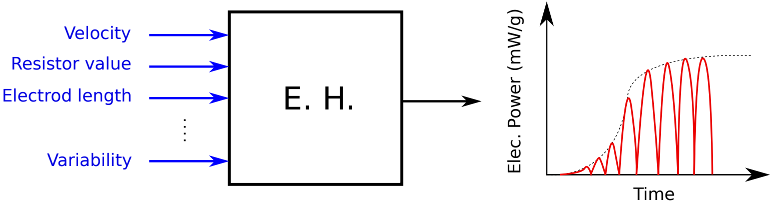

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Proposition : Proper Generalized Decomposition

Proper Genearlized Decomposition (PGD)

Approximation : series of product of separated function variables

$$\mathbf{u}(\mathbf{X}, t) \simeq \sum_{i=1}^{r} \mathbf{f}_i(\mathbf{X}) g_i(t)$$

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1. PGD = reduction of dimension suited for parameters

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2. *A posteriori* PGD (with data) and *A priori* PGD (with model)

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1. PGD = reduction of dimension suited for parameters
2. *A posteriori* PGD (with data) and *A priori* PGD (with model)
3. Examples with FEniCS : steady Navier-Stokes equations

PGD = parametric and reduced approximation

Reduction of the dimension / "real time" approximated solution

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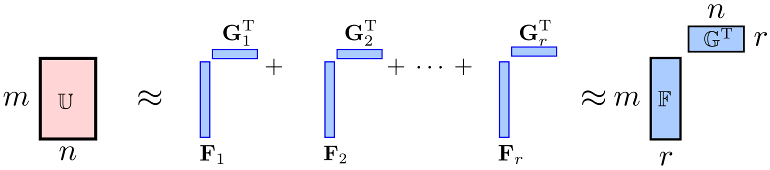

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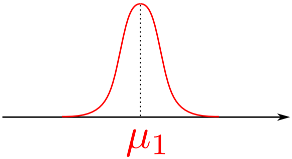
$m \times n \Longrightarrow (m+n) \times r$ where $r << m$ or n

PGD = parametric and reduced approximation

Derivation over a parameter simplified

$$\frac{\partial \mathbf{u}}{\partial \mu_1} \simeq \sum_{i=1}^{r} \mathbf{f}_i(\mathbf{X}) g_i(t) \frac{\partial h(\mu_1)}{\partial \mu_1}$$

Parameters with uncertainties (high number of simulations)



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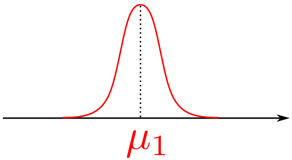

What are the limits ?

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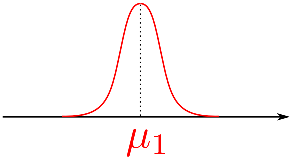

What are the limits ? No mathematical proof of convergency

A posteriori PGD and a priori PGD

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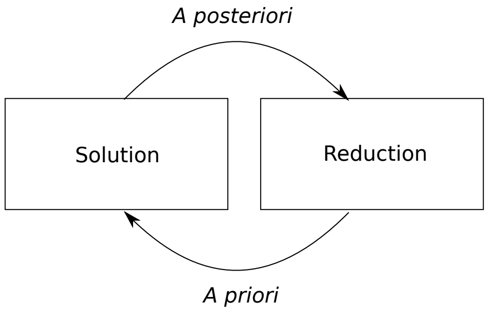

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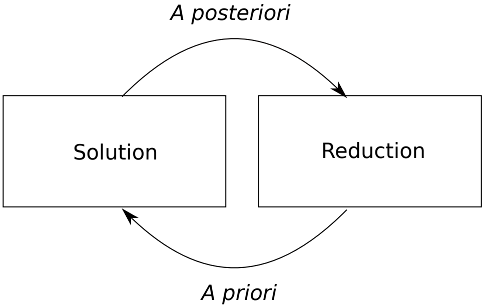
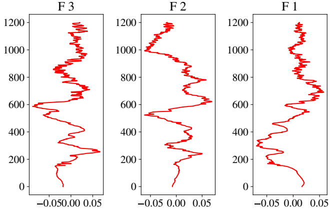


Data VS Model ? ...

A posteriori analysis : Let’s play a game



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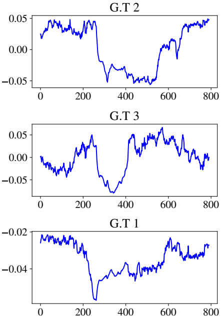

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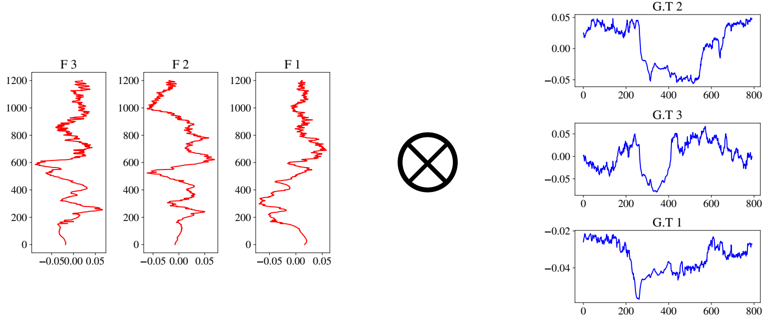
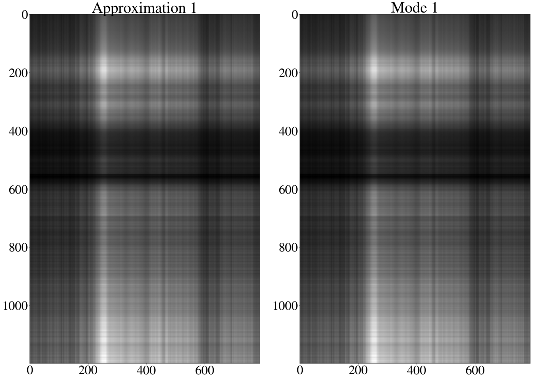


Can you guess the surface response ?

A posteriori analysis : Let's play a game



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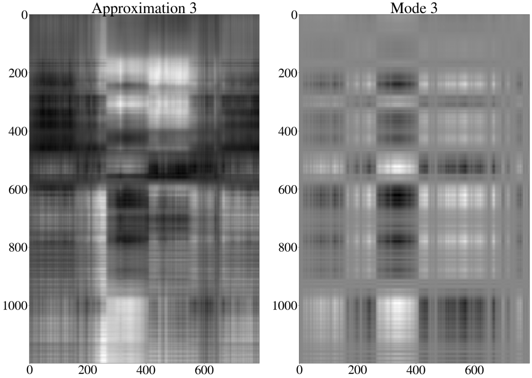

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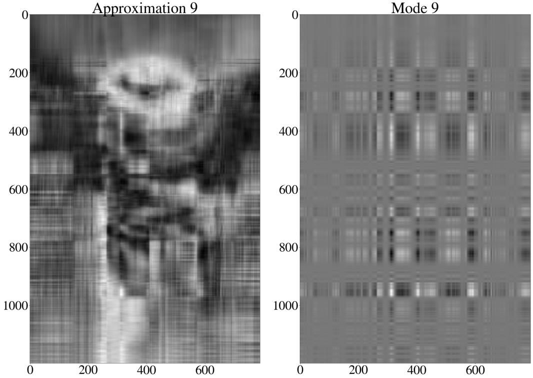

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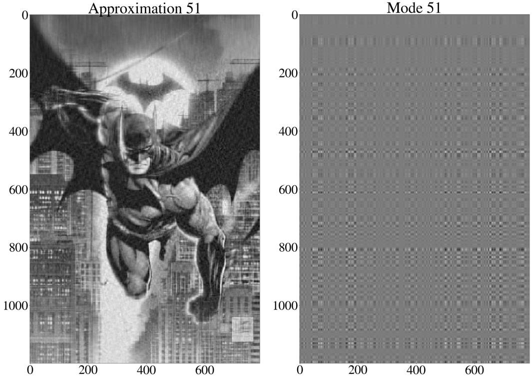

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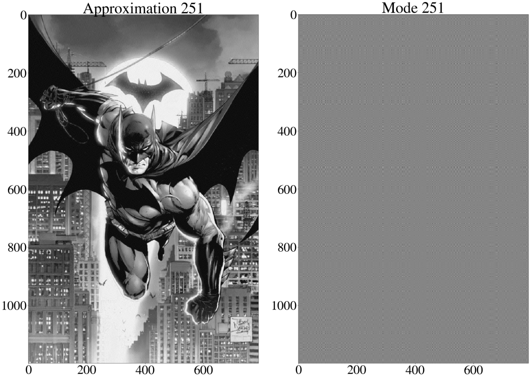
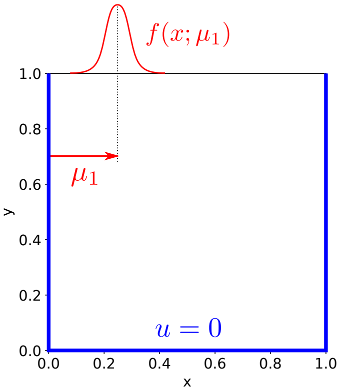
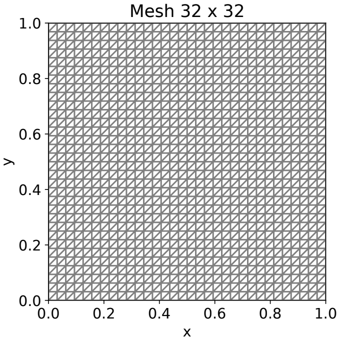


I am batman

A posteriori analysis : Poisson with a moving source

$$\int_\Omega \nabla u \cdot \nabla \delta u \, dv = \int_\Sigma f(\mu_1) \delta u \, ds, \quad \forall \delta u \in \mathcal{C}_\mathsf{u}$$





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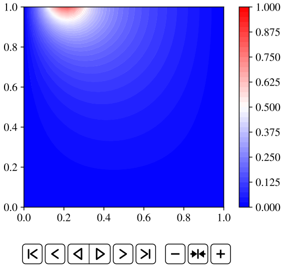
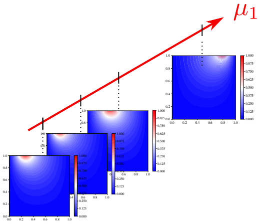
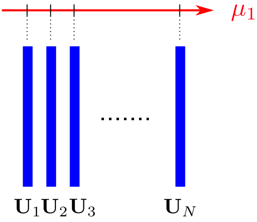


A posteriori analysis : FEM solutions



snapshots



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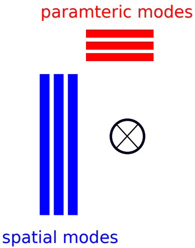
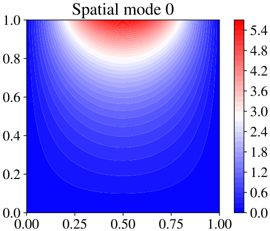


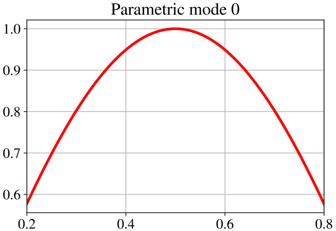
PGD modes

Can we compute / visualize the spatial and parametric modes ?

Knowing the solution over the parametric subspace

$$(f_i, g_i)_{i=1\ldots r} = \text{argmin}\|u(\mathbf{X}, t) - \sum_{i=1}^{r} \mathbf{f}_i(\mathsf{X}) g_i(t)\|_{\mathcal{L}_2}$$





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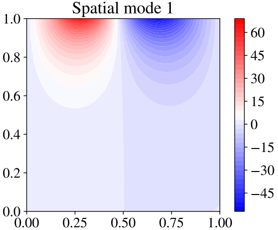

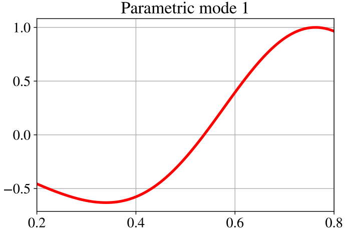

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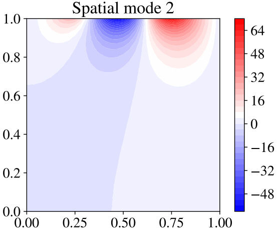

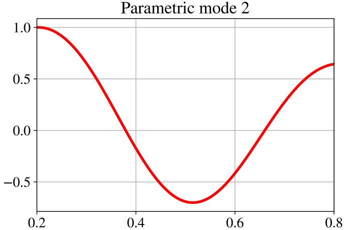

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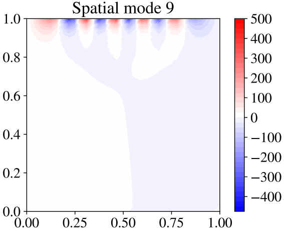
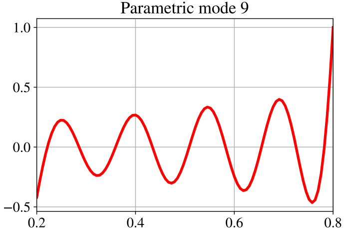
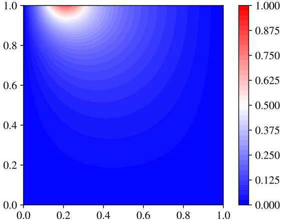
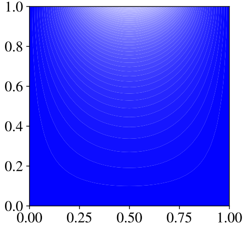




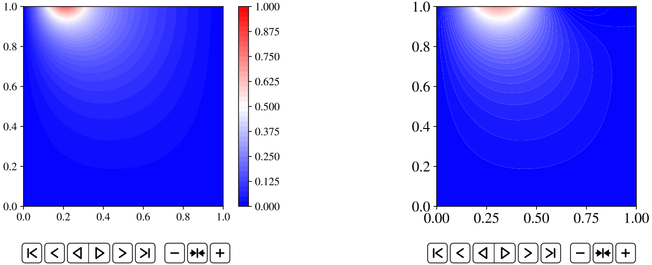
A posteriori analysis : Solution reconstruction





1 spatial mode and 1 parametric mode

A posteriori analysis : Solution reconstruction



2 spatial modes and 2 parametric modes

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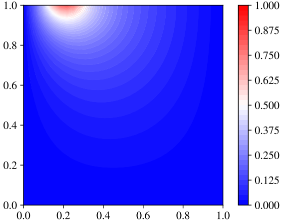
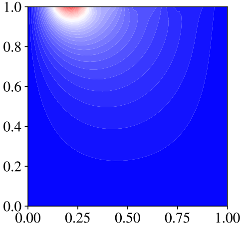
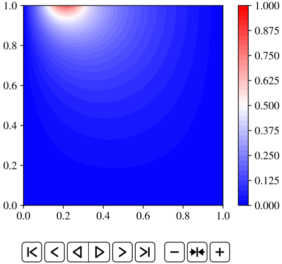
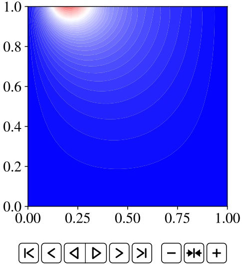




5 spatial modes and 5 parametric modes

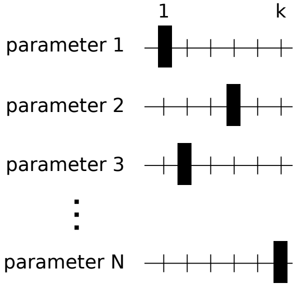
A posteriori analysis : Solution reconstruction





10 spatial modes and 10 parametric modes

A posteriori analysis : Limits ?



The curse of dimensionality : k^N simulations

A priori analysis : Find the reduced basis from the model

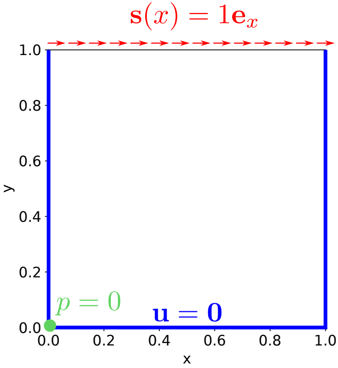
$$\mathcal{L}(\mathbf{u}) = 0, \quad \forall \delta \mathbf{u} \in \mathcal{C}_{\mathsf{u}}$$

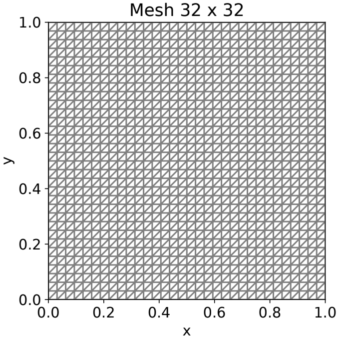
The form of the solution is postulate *a priori*

$$\mathbf{u}(\mathbf{X}, t, \mu_1, \dots, \mu_N) \simeq \sum_{i=1}^{r} \mathbf{f}_i(\mathbf{X}) g_i(t) h_i(\mu_1) \dots s_i(\mu_N)$$

The curse of dimensionality overcome : $r \times k \times N \times$ **simulations**

Apriori analysis : Steady Navier Stokes (Neumann BC)





Modified lid-driven cavity with a source term

A priori analysis : Find the reduced basis from the model

Find the velocity u and pressure p :

$$c(\mathbf{u}, \mathbf{u}, \delta\mathbf{u}) - b(p, \delta\mathbf{u}) + \frac{1}{\mathsf{Re}} a(\mathbf{u}, \delta\mathbf{u}) = f(\delta\mathbf{u}) \qquad \forall \delta\mathbf{u} \in \mathcal{C}_{\mathsf{u}}$$

$$b(\delta p, \mathbf{u}) = 0 \qquad \forall \delta p \in \mathcal{C}_{\mathsf{p}}$$

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$$b(\delta p,\mathbf{u}) = 0 \qquad \forall\delta p \in \mathcal{C}_{\mathsf{p}}$$

$$a(\mathbf{u},\delta\mathbf{u}) = \int_\Omega \nabla\mathbf{u} : \nabla\delta\mathbf{u}\, d\Omega$$

$$b(p,\delta\mathbf{u}) = \int_\Omega p\nabla\cdot\delta\mathbf{u}\, d\Omega$$

$$c(\mathbf{u},\mathbf{u},\delta\mathbf{u}) = \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\delta\mathbf{u}\, d\Omega$$

$$f(\delta\mathbf{u}) = \int_\Omega 1\mathbf{e}_{\mathsf{x}}\cdot\delta\mathbf{u}\, dS$$

A priori analysis : Find the reduced basis from the model

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$$f(\delta\mathbf{u}) = \int_\Omega 1\mathbf{e}_{\mathsf{x}}\cdot\delta\mathbf{u}\, dS$$

Reynolds number is the parameters Re $\in I = [1, 100]$

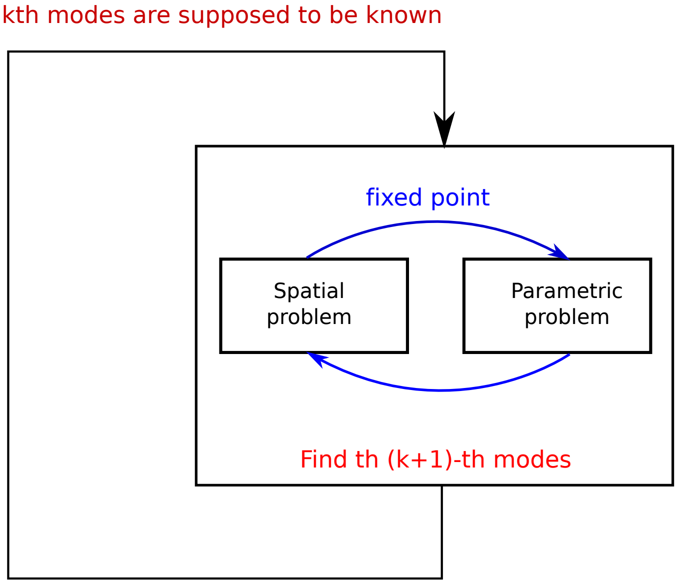
A priori : Key idea

Instead of finding $\mathbf{u}$ and p
The form of the solution is supposed to be separated :

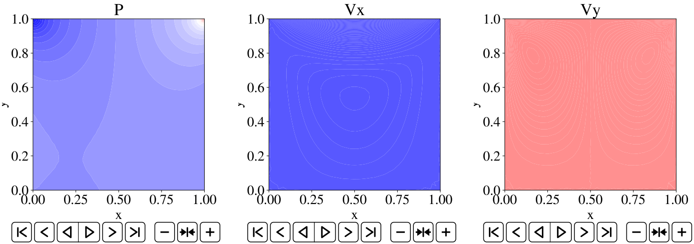
$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \sum_{i=1}^{r} \begin{pmatrix} \mathbf{f}_i(\mathbf{X}) \\ h_i(\mathbf{X}) \end{pmatrix} g_i(Re)$$

Objective : compute directly the modes $(\mathbf{f}_i(\mathbf{X}), h_i(\mathbf{X}))$ and $g(Re)$

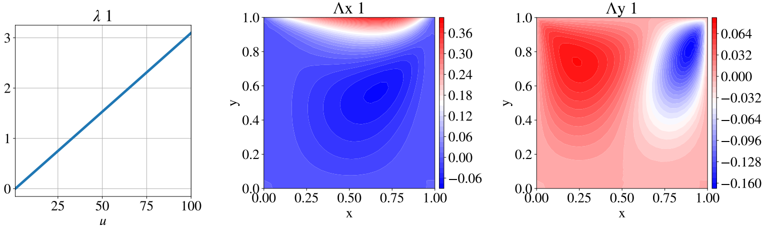
A priori PGD : Algorithm with fixed points



A priori analysis : Reference for error analysis

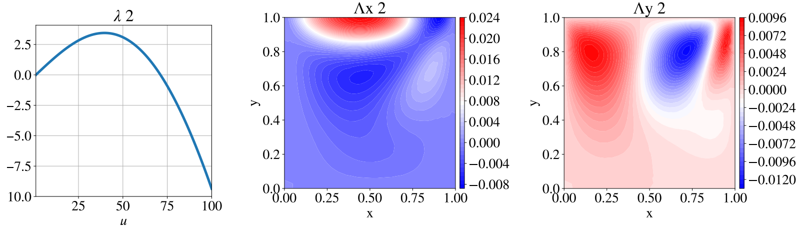


Modes computed *a priori*



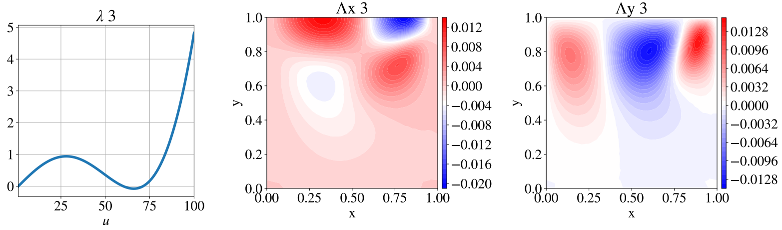
Reynolds modes and velocity modes 1

Modes computed *a priori*



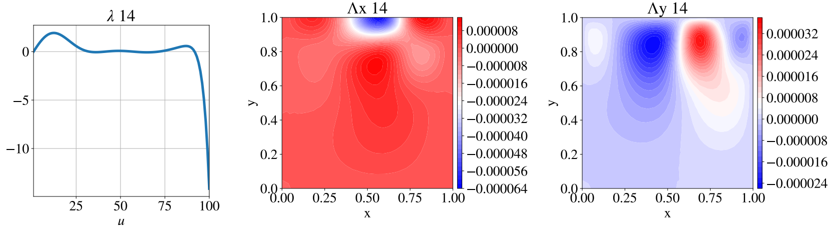
Reynolds modes and velocity modes 2

Modes computed *a priori*



Reynolds modes and velocity modes 3

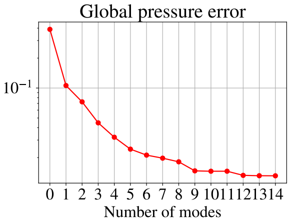
Modes computed *a priori*

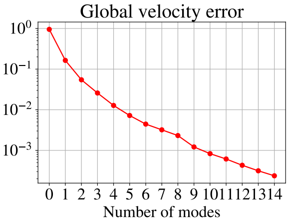


Reynolds modes and velocity modes 15

Error in velocity and pressure

$$Err_v = \int_I \int_\Omega \frac{\| \mathbf{u} - \mathbf{u}_{\text{approx}}(k) \|_{H_1}}{\| \mathbf{u} \|_{H_1}} d\mu \qquad Err_p = \int_I \int_\Omega \frac{\| p - p_{\text{approx}}(k) \|_{L_2}}{\| p \|_{L_2}} d\mu$$





Convergence of the approximation (velocity and pressure)

To sum-up

1. PGD = reduction of dimension suited for parameters
2. *A posteriori* PGD (with data) and *A priori* PGD (with model)
3. Examples with FEniCS : steady Navier-Stokes equations

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Behind the scene

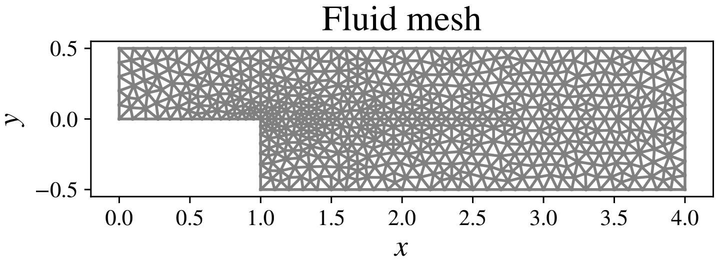
To sum-up

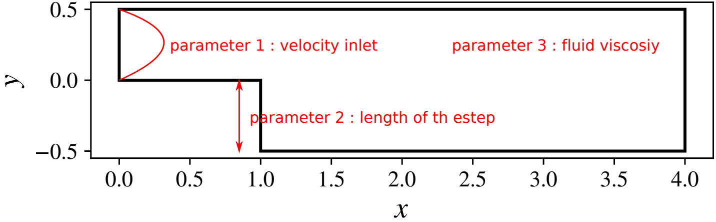
1. PGD = reduction of dimension suited for parameters
2. *A posteriori* PGD (with data) and *A priori* PGD (with model)
3. Examples with FEniCS : steady Navier-Stokes equations

Behind the scene

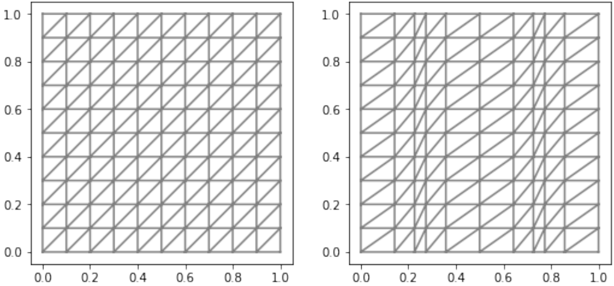
1. Tedious calculus to express the spatial and the parametric problems
2. PGD : How to turn nonlinear problem into heavily nonlinear problems
3. How do we deal with non-separable problems ?

Outlooks 1 : more parameters





Outlooks 2 : moving mesh (ALE)



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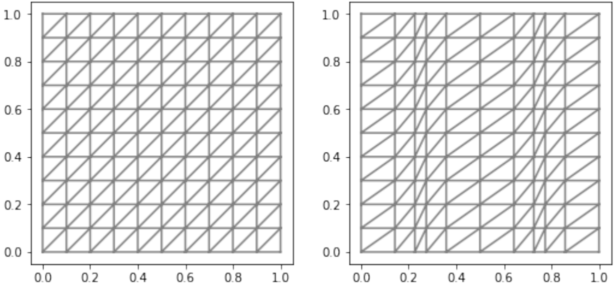

Thank you !