Sains Malaysiana 48(8)(2019): 1777–1785
http://dx.doi.org/10.17576/jsm-2019-4808-25

A New Test for the Homogeneity of Inverse Gaussian Scale Parameters Based on Computational Approach Test

(Ujian Baru untuk Kehomogenan Parameter Skala Gaussian Songsang Berdasarkan Ujian Pendekatan Pengkomputeran)

HASAN HÜSEYIN GÜL, ESRA GÖKPINAR*, MERAL EBEGIL, YAPRAK ARZU ÖZDEMIR & FIKRI GÖKPINAR

ABSTRACT

In this paper, we focused on testing homogeneity of scale parameters of k Inverse Gaussian distributions (IGDs) since this distribution is one of the most common distribution for analyzing nonnegative right-skewed data. We have proposed a new test statistic based on the Computational Approach Test (CAT), which is a type of parametric bootstrap method, for testing homogeneity of scale parameters of k IGDs. Simulation results have been presented to compare the performances of the proposed method and existing methods such as the likelihood ratio test, modified likelihood ratio test and generalized likelihood ratio test in terms of type I error rate and power. The results showed that the proposed CAT is better than the others in terms of the type I error rates and powers in some cases.

Keywords: Computational Approach Test; generalized likelihood ratio test; inverse Gaussian distribution; maximum likelihood estimation; modified likelihood ratio test

ABSTRAK

Dalam kertas ini, tumpuan diberikan kepada ujian kehomogenan skala parameter, k, bagi Pengagihan Songsang Gaussian (IGDs) kerana pengagihan ini adalah salah satu daripada pengagihan paling kerap digunakan untuk menganalisis data non-negatif terpencong kanan. Dicadangkan ujian statistik baru berdasarkan pada Ujian Pengiraan Pengkomputeran (CAT), yang merupakan sejenis kaedah butstrap berparameter untuk ujian kehomogenan skala parameter k IGDs. Keputusan simulasi telah dibentangkan untuk membandingkan prestasi kaedah cadangan dan kaedah sedia ada seperti ujian nisbah kebolehjadian, ujian nisbah kebolehjadian terubah suai dan ujian nisbah kebolehjadian umum Jenis I untuk ralat kadar dan kuasa. Hasil kajian menunjukkan bahawa kajian (CAT) lebih baik berbanding lain daripada segi Jenis I untuk kadar ralat dan kuasa di dalam beberapa kes.

Kata kunci: Anggaran kebolehjadian maksimum; pengagihan songsang Gaussian; ujian pendekatan pengkomputeran; ujian nisbah kebolehjadian terubah suai; ujian nisbah kebolehjadian umum

INTRODUCTION

The probability density function of an Inverse Gaussian distribution (IGD) with parameters μ and λ is defined as follows:

$$f(x,\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\},\quad x>0,\quad \mu,\lambda>0$$

where μ is the mean parameter and λ is the scale parameter. The IGD was originally introduced as the first passage time distribution in Brownian motion with positive drift (Schrödinger 1915; Von Smoluchowski 1915). Tweedie (1945) has shown the inverse relationship between the cumulant generating function of the first passage time distribution and that of the normal distribution, so it is called as IGD for the first passage time distribution. Wald (1947) has derived the limiting form of IGD. Therefore, it is also called as Wald's distribution, especially in the Russian literature (Gökpınar et al. 2013).

The main appeal of IGD lies in these facts: it can accommodate a variety of shapes from highly skewed to almost normal; it is unique among the distributions for nonnegative right-skewed data such as Weibull, gamma and log-normal due to the fact that it shares many elegant and convenient properties with Gaussian models. In recent years, the IGD has been widely used in describing and analyzing right skewed data in many different fields, such as life tests, psychology, demography, linguistics, environment and finance. In generalized linear models, the distribution for a response variable can be any member of the natural exponential dispersion family such as normal, gamma and inverse Gaussian distributions. This distribution is often a reasonable representation of the distributions for many measures found in psychological and behavioral science research widely (Anderson et al. 2010). For example, Anderson et al. (2010) gave a data set consisting of a sub-set of data for 149 elderly participants in a study on cognition and aging from Stine-Morrow et al. (2008). Elderly individuals were shown words on a computer screen and the words were presented one at a time and a sequence of words made up a sentence. Each subject read multiple sentences. A word was presented and the subject hit the space bar when they were ready for the next word. Their reading time measured in ml seconds between the presentation of a word and hitting of the space bar. The reaction times are continuous and positively skewed. Given the nature of the response variable, two plausible distributions for these data are the gamma and IGD.

Comprehensive characterization properties of IGD were given by Chhikara and Folks (1989), Seshadri (1999, 1993). In this article, the testing problem of the homogeneity of k IGD scale parameters has been considered. One of the reasons of dealing with this problem is that the analysis of data is simplified when the scale parameters are equal. In such a case, if the IGD scale parameters are found to be equal, the Analysis of Reciprocals (ANORE) F test is the most widely used test for the equality of multiple IGD means. However, the type I error rates of the ANORE test may be much larger than the nominal level when the scale parameters of IGDs are non-homogeneous (Tian 2006). For this reason, the use of this test is not appropriate for the nonhomogeneous scale case. Therefore, it is important to test the assumption of homogeneous scale parameters before applying ANORE test. For testing the homogeneity of IGD scale parameters, there are several studies in the literature. Chhikara and Folks (1989) gave a modified likelihood ratio test by using Bartlett's approximation. Besides, Liu and He (2013) proposed a new exact generalized likelihood ratio test based on the idea of generalized likelihood ratio (GLR) method introduced by Weerahandi (1995). The concept of GLR has been widely applied to a variety of practical settings, where standard inference methods do not exist. In this concept, the generalized pivotal quantities are needed to construct generalized p-value. However, unlike suggested various GLR methods, Liu and He (2013) directly got the generalized p-value from a special likelihood ratio for testing the homogeneity of IGD scale parameters. They compared this test with the modified likelihood ratio test (MLRT) given by Chhikara and Folks (1989) and the GLR test in terms of type I error rates and powers. The simulation results of Liu and He (2013) demonstrate that their proposed test and MLRT have type I error rates close to the nominal level. The powers of the proposed test are at least as large as those of MLRT, except for several scenarios when the sample sizes are equal and small.

In this paper, a new computational approach test (CAT) was proposed for testing the homogeneity of IGD scale parameters. Pal et al. (2007) first introduced the CAT, which is a particular type of parametric bootstrap method. It is simply based on simulation and numerical computations and it uses the maximum likelihood estimates (MLEs), and does not require the knowledge of any sampling distribution. The concept of CAT has been widely applied to a variety of practical settings, where standard inference methods do not exist. For example, Pal et al. (2007) showed how the CAT can be applied to Gamma and Weibull distributions for hypothesis testing and interval estimations. In cases where the variances are unknown and arbitrary, Chang and Pal (2008) developed a CAT to test the equality of two normal population means. Chang et al. (2010, 2011) suggested test procedures based on the CAT for the hypotheses testing of the Poisson and Gamma models. Under heteroscedasticity, Gökpınar and Gökpınar (2012) applied the CAT to test the equality of several normal population means.

This paper has been organized as follows. In the next section, the likelihood ratio test and the modified likelihood ratio test (MLRT) and generalized likelihood ratio test have been introduced. After that, the details of the test based on the CAT method has been obtained. In the section that follows, the simulation results have been presented to compare the tests mentioned above in terms of type I error rates and powers.

TEST STATISTICS

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ be a random sample from IG with means μ_i and scale parameters λ_i, $i = 1, \ldots, k$. We are interested in testing

$$H_0 : \lambda_1 = \lambda_2 = \ldots = \lambda_k \text{ vs } H_1 : \lambda_i \neq \lambda_j,$$
$$\textit{for at least one pair of } i \neq j. \quad (1)$$

The existing test methods are given in the following for testing the homogeneity of scale parameters of the IGDs.

LIKELIHOOD RATIO TEST AND THE MODIFIED LIKELIHOOD RATIO TEST

Likelihood ratio test was proposed by Doornbos and Dijkstra (1983). The likelihood function is given as

$$L_1 = \prod_{i=1}^{k}\prod_{j=1}^{n_i}\left(\frac{\lambda_i}{2\pi x_{ij}^3}\right)^{-1/2} \exp\left(-\frac{\lambda_i}{2\mu_i^2 x_{ij}}(x_{ij}-\mu_i)^2\right).$$

Without the restriction of the null hypothesis, the maximum likelihood estimates (MLEs) of μ_i's and λ_i's are given as follows

$$\hat{\mu}_{i(MLE)} = \bar{X}_i,\ i = 1,2,\ldots,k,\ \hat{\lambda}_{i(MLE)} = \left(\frac{1}{n_i}\sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)\right)^{-1}. \quad (2)$$

The likelihood function under the null hypothesis is as seen here

$$L_0 = \prod_{i=1}^{k}\prod_{j=1}^{n_i}\left(\frac{\lambda}{2\pi x_{ij}}\right)^{-1/2} \exp\left(-\frac{\lambda}{2\mu_i^2 x_{ij}}(x_{ij}-\mu_i)^2\right)$$

Taking derivatives L_0 with respect to μ_i and λ yields the following results:

$$\frac{\partial \ln L_0}{\partial \mu_i} = \frac{\lambda}{\mu_i^3}\sum_{j=1}^{n_i}\left(\sqrt{x_{ij}} - \frac{\mu_i}{\sqrt{x_{ij}}}\right)^2 + \frac{\lambda}{\mu_i^2}\sum_{j=1}^{n_i}\left(1 - \frac{\mu_i}{x_{ij}}\right) = 0 \qquad (3)$$

$$\frac{\partial \ln L_0}{\partial \lambda} = \frac{\sum_{i=1}^{k} n_i}{2\lambda} - \sum_{i=1}^{k}\frac{1}{2\mu_i^2}\sum_{j=1}^{n_i}\left(\sqrt{x_{ij}} - \frac{\mu_i}{\sqrt{x_{ij}}}\right)^2 = 0. \qquad (4)$$

From (3) and (4), the restricted maximum likelihood estimates (RMLEs) of the parameters under the null hypothesis are as seen here

$$\hat{\mu}_{i(RMLE)} = \bar{X}_i = \sum_{j=1}^{n_i} X_{ij}\big/ n_i, \quad i = 1, 2, ..., k,$$

$$\hat{\lambda}_{RMLE} = \left(\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)\right)^{-1}, \qquad (5)$$

where $n = \sum_{i=1}^{k} n_i$. When $\hat{\mu}_{1(RMLE)}, ..., \hat{\mu}_{k(RMLE)}$ and $\hat{\lambda}_{(RMLE)}$ are replaced in the L_0 function, and $\hat{\mu}_{1(MLE)}, ..., \hat{\mu}_{k(MLE)}$ and $\hat{\lambda}_{1(MLE)}, ..., \hat{\lambda}_{k(MLE)}$ are replaced in the L_1 function, the likelihood ratio (LR) is obtained as

$$LR = \frac{L_1\left(\hat{\mu}_{1(MLE)}, ..., \hat{\mu}_{k(MLE)}; \hat{\lambda}_{1(MLE)}, ..., \hat{\lambda}_{k(MLE)}\right)}{L_0\left(\hat{\mu}_{1(RMLE)}, ..., \hat{\mu}_{k(RMLE)}; \hat{\lambda}_{RMLE}\right)}$$

$$= \frac{\prod_{i=1}^{k}\hat{\lambda}_{i(MLE)}^{-n_i/2}\exp\left(-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\hat{\lambda}_{i(MLE)}}{2\bar{X}_i^2 X_{ij}}\left(X_{ij} - \bar{X}_i\right)^2\right)}{\hat{\lambda}_{RMLE}^{-n/2}\exp\left(-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\hat{\lambda}_{RMLE}}{2\bar{X}_i^2 X_{ij}}\left(X_{ij} - \bar{X}_i\right)^2\right)}$$

$$= \frac{\hat{\lambda}_{RMLE}^{n/2}\exp\left(-\sum_{i=1}^{k}\frac{\hat{\lambda}_{i(MLE)}}{2}\sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)\right)}{\prod_{i=1}^{k}\hat{\lambda}_{i(MLE)}^{n_i/2}\exp\left(-\frac{\hat{\lambda}_{RMLE}}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)\right)} \qquad (6)$$

Let $V_i = \sum_{j=1}^{n_i}\left(1/X_{ij} - 1/\bar{X}_i\right)$ and $V = \sum_{i=1}^{k} V_i$, thus $\hat{\lambda}_{i(MLE)}$ in (2) and $\hat{\lambda}_{(RMLE)}$ in (5) can be written as $\hat{\lambda}_{i(MLE)} = \left(V_i/n_i\right)^{-1}$ and $\hat{\lambda}_{(RMLE)} = \left(V/n\right)^{-1}$, respectively. Therefore, the LR in (6) is obtained as

$$LR = \prod_{i=1}^{k}\left(\frac{V_i/n_i}{V/n}\right)^{n_i/2} \qquad (7)$$

(Chhikara & Folks 1989). Under the null hypothesis, likelihood ratio test (LRT) can be given as:

$$\text{LRT} = -2\ln\left(\prod_{i=1}^{k}\left(\frac{V_i/n_i}{V/n}\right)^{n_i/2}\right) = -2\left[\sum_{i=1}^{k}\frac{n_i}{2}\ln\left(\frac{V_i}{n_i}\right) - \frac{n}{2}\ln\left(\frac{V}{n}\right)\right]$$

$$= \sum_{i=1}^{k} n_i\left(\ln\left(\frac{V}{n}\right) - \ln\left(\frac{V_i}{n_i}\right)\right) \qquad (8)$$

LRT is distributed approximately chi-square with $(k - 1)$ degrees of freedom. By using Bartlett's approximation, the modified likelihood ratio test (MLRT) suggested by Chhikara and Folks (1989) is given as

$$\text{MLRT} = \frac{M}{C},$$

where $C = 1 + \frac{1}{3(k-1)}\left[\sum\frac{1}{f_i} - \frac{1}{\sum f_i}\right]$

and $M = f\ln\left(\frac{V}{f}\right) - \sum f_i \ln\left(\frac{V_i}{f_i}\right).$

Here $f = \sum\left(n_i - 1\right)$ and $f_i = n_i - 1$. Under the null hypothesis, MLRT is distributed approximately as chi-square with $(k - 1)$ degrees of freedom (Chhikara & Folks 1989).

GENERALIZED LIKELIHOOD RATIO TEST

Liu and He (2013) proposed a new test based on the idea of generalized likelihood ratio (GLR) method in Weerahandi (1995)'s study. The p-value of the GLR test is obtained as:

$$p(V_i, v_i) = P\left(\prod_{i=1}^{k} Y_i^{n_i} < \prod_{i=1}^{k}\left(\frac{v_i}{\sum_{j=1}^{k} v_j}\right)^{n_i}\right), \qquad (9)$$

where

$$Y_i = (1 - B_{i-1})B_i \dots B_{k-1}, \quad i = 2, 3, \dots, k-1$$
$$Y_1 = B_1 B_2 \dots B_{k-1},$$
$$Y_i = 1 - B_{k-1}, \qquad (10)$$

and B_i, $i = 1, 2, \dots, k - 1$ are independent beta random variables defined by

$$B_i = \frac{\sum_{k=1}^{i}\lambda_k V_k}{\sum_{k=1}^{i+1}\lambda_k V_k} \sim Beta\left(\sum_{k=1}^{i}\frac{n_k - 1}{2}, \frac{n_{i+1} - 1}{2}\right), \quad i = 1, 2, ..., k-1. \qquad (11)$$

Here, $\lambda_i V_i : \chi^2_{(n_i - 1)}$, $\sum_{j=1}^{k}\lambda_j V_j : \chi^2_{\left(\sum_{i=1}^{k} n_i - k\right)}$. The generalized p-value in Equation (9) can be computed by using the following algorithm.

For a given data set x_{ij}, $i = 1, \ldots, k, j = 1, \ldots, n_i$, compute

$$v_i = \sum_{j=1}^{n_i}\left(\frac{1}{x_{ij}} - \frac{1}{\bar{x}_i}\right), i = 1, 2, \ldots, k, \text{ and } t = \prod_{i=1}^{k}\left(\frac{v_i}{\sum_{j=1}^{k} v_j}\right)^{n_i}.$$

For $l = 1$ to m, generate random numbers from Y_i, $i = 1, 2, \ldots, k$ according to (10) - (11).

Compute $T_l = \prod_{i=1}^{k} Y_i^{n_i},\ l = 1,2,...,m.$ and

Let $M_l = 1$ if $T_l < t$ else $M_l = 0$, then $\frac{1}{m}\sum_{l=1}^{m} M_l$ is a simulated estimate of p-value for testing Equation (1) (Liu & He 2013).

THE COMPUTATIONAL APPROACH TEST (CAT) METHOD

In this section, CAT which is a special case of parametric bootstrap test was proposed to test the equality of several scale parameters of IG distribution. Initially, before applying CAT for testing the null hypothesis given in (1), the CAT procedure was given as follows. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample having a probability density function $f(y/\theta)$, where the functional form of f is assumed to be known and $\theta = (\theta^{(1)}, \theta^{(2)})$ is known vector in parameter space Θ. $\Theta^{(1)}$ and $\Theta^{(2)}$ are the parameter of interest and nuisance parameter, respectively. The problem of interest is to test $H_0' : \theta^{(1)} = \theta_0^{(1)}$ versus a suitable alternative. To test $H_0' : \theta^{(1)} = \theta_0^{(1)}$ against H_1', H_0' was first expressed as $H_0'^{*} : \eta\left(\theta^{(1)}, \theta_0^{(1)}\right) = 0$ against $H_1'^{*}$,where η is a scalar valued function. The general methods of the CAT for testing $H_0'^{*} : \eta\left(\theta^{(1)}, \theta_0^{(1)}\right) = 0$ against a suitable alternative at a desired level α was given through the following steps (Tian 2006).

Obtain MLEs of the parameters, $\Theta^{(1)}$ and $\Theta^{(2)}$; Obtain a suitable $\eta(\theta^{(1)}, \theta_0^{(1)})$ and the MLEs of η, $\hat{\eta}_{ML} = \hat{\eta}\left(\hat{\theta}_{ML}^{(1)}, \theta_0^{(1)}\right)$ can be used as a test statistic; Under H_0, find the MLEs of $\Theta^{(2)}$ parameter, which is denoted by $\hat{\theta}^{(2)}_{RMLE}$; Generate artificial sample $Y_1^*, Y_2^*, ..., Y_n^*$ from $f\left(y/\theta_0^{(1)}, \hat{\theta}_{RML}^{(2)}\right)$ large number of times, say m times. For each of these replicated samples, recalculate the MLE of η, $\tilde{\eta}_{ML}^{(j)}$, $j = 1, ..., m$.; and Estimate the p-value as, $\hat{p} = \#\left(\tilde{\eta}_{ML}^{(j)} > \hat{\eta}_{ML}\right)/m$. In the case of $\hat{p} < \alpha$, H_0 is rejected.

According to the procedure given before, to apply the CAT procedure for testing the null hypothesis given in (1), H_0 is expressed in terms of suitable scalar η defined in (12).

$$\eta = \eta(\lambda_1, \lambda_2, ..., \lambda_k) = \sum_{i=1}^{k} n_i \left(\log \lambda_i - \frac{\sum_{i=1}^{k} n_i \log \lambda_i}{\sum_{i=1}^{k} n_i}\right)^2. \qquad (12)$$

It is clear that testing H_0 against H_1 is equivalent to testing $H_0^* : \eta = 0$ against $H_1^* : \eta > 0$. Thus, MLE of η can be used as a test statistic.

The test procedure of the proposed CAT could be given as follows:

The MLEs of the parameters are obtained as

$$\hat{\mu}_{i(MLE)} = \bar{X}_i \quad \text{and} \quad \hat{\lambda}_{i(MLE)} = \left(\frac{1}{n_i}\sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)\right)^{-1}.$$

Therefore, the test statistic is written as

$$\hat{\eta}_{MLE} = \sum_{i=1}^{k} n_i \left(\log \hat{\lambda}_{i(MLE)} - \frac{\sum_{i=1}^{k} n_i \log \hat{\lambda}_{i(MLE)}}{\sum_{i=1}^{k} n_i}\right)^2.$$

Under H_0, generate samples X_{ij} from $IG(\hat{\mu}_{i(RMLE)}, \hat{\lambda}_{RMLE})$, $(1 \le j \le n_i,\ 1 \le i \le k)$ a large number of times (m times). Hence, RMLEs of μ_i and λ are given in (5). For each of these replicated samples, recalculate the value $\tilde{\eta}_{MLE}^{(j)}$, $(j = 1,...,m)$ as follows:

$$\tilde{\eta}_{MLE}^{(j)} = \sum_{i=1}^{k} n_i \left(\log \tilde{\lambda}_{i(MLE)} - \sum_{i=1}^{k} n_i \log \tilde{\lambda}_{i(MLE)} \Big/ \sum_{i=1}^{k} n_i\right)^2.$$

Calculate the p-value as $p = \frac{\#(\tilde{\eta}_{MLE}^{(j)} > \hat{\eta}_{MLE})}{m}$. If $p < \alpha$, H_0 is rejected.

SIMULATION STUDIES

In this section, the LRT, GLRT, MLRT and CAT were compared to test the homogeneity of k IGD scale parameters in terms of type I error rates and powers in different combinations of parameters and sample sizes with different number of groups ($k = 3, 4, 5, 7$). For each parameter and sample size setting, 5000 replicate was taken to obtain power and type-I error rates. Furthermore, for GLRT and CAT procedure, m = 5000 was taken. The nominal level was taken as $\alpha = 0.05$. For estimated type-I error rates, the numerical results were given in Tables 1 - 4.

Tables 1- 4 present the estimated type I error rates of the four tests. The estimated type I error rates of the MLRT, GLRT and CAT tests are close to the nominal level in almost all cases and their type I error rates are well controlled. However, the estimated type I error rates of the LRT exceed to nominal level especially when sample sizes are small and different. Further note that the type-I error probabilities of all the four tests converge to the nominal level as the sample sizes increase.

The powers of all the tests for different combinations of parameters and sample sizes were calculated. The numerical results for powers of the four tests were given in Tables 5 - 8. As seen from these tables, the LRT, whose

TABLE 1. The estimated type I error rates of all tests for $k = 3$

μ	n	LRT	MLRT	GLRT	CAT
(1, 1, 1)	(10, 10, 10)	0.0734	0.0466	0.0478	0.0476
	(25, 25, 25)	0.0536	0.0450	0.0438	0.0456
	(50, 50, 50)	0.0576	0.0526	0.0530	0.0524
	(15, 10, 5)	0.0900	0.0458	0.0474	0.0508
	(30, 25, 20)	0.0592	0.0494	0.0494	0.0504
(0.5, 1, 2)	(10, 10, 10)	0.0796	0.0548	0.0546	0.0556
	(25, 25, 25)	0.0638	0.0542	0.0536	0.0538
	(50, 50, 50)	0.0578	0.0538	0.0534	0.0548
	(15, 10, 5)	0.0850	0.0448	0.0476	0.0472
	(30, 25, 20)	0.0562	0.0462	0.0478	0.0470
(0.75, 1, 1.25)	(10, 10, 10)	0.0718	0.0466	0.0462	0.0458
	(25, 25, 25)	0.0606	0.0506	0.0510	0.0504
	(50, 50, 50)	0.0598	0.0546	0.0550	0.0548
	(15, 10, 5)	0.0942	0.0502	0.0500	0.0512
	(30, 25, 20)	0.0578	0.0462	0.0480	0.0480

TABLE 2. The estimated type I error rates of all tests for $k = 4$

μ	n	LRT	MLRT	GLRT	CAT
(1, 1, 1, 1)	(10, 10, 10, 10)	0.0804	0.0508	0.0516	0.0506
	(25, 25, 25, 25)	0.0588	0.0460	0.0458	0.0472
	(50, 50, 50, 50)	0.0512	0.0470	0.0474	0.0472
	(15, 10, 10, 5)	0.0944	0.0500	0.0512	0.0526
	(30, 25, 25, 20)	0.0532	0.0430	0.0432	0.0446
(0.5, 1, 1, 2)	(10, 10, 10, 10)	0.0832	0.0502	0.0508	0.0488
	(25, 25, 25, 25)	0.0574	0.0474	0.0478	0.0466
	(50, 50, 50, 50)	0.0564	0.0522	0.0522	0.0514
	(15, 10, 10, 5)	0.0966	0.0514	0.0521	0.0502
	(30, 25, 25, 20)	0.0626	0.0512	0.0520	0.0476
(0.75, 1, 1, 1.25)	(10, 10, 10, 10)	0.0816	0.0464	0.0476	0.0496
	(25, 25, 25, 25)	0.0606	0.0528	0.0498	0.0508
	(50, 50, 50, 50)	0.0568	0.0512	0.0512	0.0528
	(15, 10, 10, 5)	0.0902	0.0466	0.0479	0.0482
	(30, 25, 25, 20)	0.0572	0.0474	0.0464	0.0482

TABLE 3. The estimated type I error rates of all tests for $k = 5$

μ	n	LRT	MLRT	GLRT	CAT
(1, 1, 1, 1, 1)	(10, 10, 10, 10, 10)	0.0884	0.0546	0.0532	0.0580
	(25, 25, 25, 25, 25)	0.0656	0.0560	0.0554	0.0558
	(50, 50, 50, 50, 50)	0.0600	0.0544	0.0540	0.0538
	(15, 15, 10, 5, 5)	0.0998	0.0462	0.0466	0.0466
	(30, 30, 25, 20, 20)	0.0592	0.0460	0.0468	0.0452
(0.5, 0.5, 1, 2, 2)	(10, 10, 10, 10, 10)	0.0876	0.0464	0.0466	0.0482
	(25, 25, 25, 25, 25)	0.0634	0.0516	0.0510	0.0548
	(50, 50, 50, 50, 50)	0.0550	0.0476	0.0472	0.0480
	(15, 15, 10, 5, 5)	0.1108	0.0488	0.0508	0.0496
	(30, 30, 25, 20, 20)	0.0670	0.0540	0.0540	0.0548
(0.75,0.75,1,1.25,1.25)	(10, 10, 10, 10, 10)	0.0936	0.0582	0.0582	0.0560
	(25, 25, 25, 25, 25)	0.0644	0.0514	0.0516	0.0494
	(50, 50, 50, 50, 50)	0.0516	0.0464	0.0458	0.0462
	(15, 15, 10, 5, 5)	0.1112	0.0518	0.0540	0.0546
	(30, 30, 25, 20, 20)	0.0590	0.0450	0.0458	0.0464

TABLE 4. The estimated type I error rates of all tests for $k = 7$

μ	n	LRT	MLRT	GLRT	CAT
(1, 1, 1, 1, 1, 1, 1)	(10, 10, 10, 10, 10, 10, 10)	0.0910	0.0488	0.0488	0.0502
	(25, 25, 25, 25, 25, 25, 25)	0.0646	0.0514	0.0518	0.0492
	(50, 50, 50, 50, 50, 50, 50)	0.0548	0.0490	0.0490	0.0508
	(15, 15, 10, 10, 10, 5, 5)	0.1152	0.0488	0.0508	0.0500
	(30, 30, 25, 25, 25, 20, 20)	0.0622	0.0496	0.0482	0.0524
(0.5, 0.5, 1, 1, 1, 2, 2)	(10, 10, 10, 10, 10, 10, 10)	0.0910	0.0468	0.0464	0.0472
	(25, 25, 25, 25, 25, 25, 25)	0.0642	0.0498	0.0508	0.0512
	(50, 50, 50, 50, 50, 50, 50)	0.0544	0.0488	0.0486	0.0500
	(15, 15, 10, 10, 10, 5, 5)	0.1146	0.0500	0.0498	0.0480
	(30, 30, 25, 25, 25, 20, 20)	0.0648	0.0492	0.0490	0.0506
(0.75, 0.75, 1, 1, 1, 1.25, 1.25)	(10, 10, 10, 10, 10, 10, 10)	0.0892	0.0478	0.0478	0.0482
	(25, 25, 25, 25, 25, 25, 25)	0.0646	0.0510	0.0510	0.0506
	(50, 50, 50, 50, 50, 50, 50)	0.0546	0.0466	0.0480	0.0500
	(15, 15, 10, 10, 10, 5, 5)	0.1142	0.0494	0.0510	0.0492
	(30, 30, 25, 25, 25, 20, 20)	0.0638	0.0484	0.0480	0.0512

TABLE 5. The powers of all tests for $k = 3$

μ	n	μ = (1, 1, 1)				μ = (0.5, 1, 2)			
		LRT	MLRT	GLRT	CAT	LRT	MLRT	GLRT	CAT
(1, 1, 2)	(10, 10, 10)	****	0.1414	0.1396	0.1418	****	0.1424	0.1432	0.1420
	(25, 25, 25)	0.3972	0.3656	0.3670	0.3780	****	0.3654	0.3676	0.3800
	(50, 50, 50)	0.6950	0.6828	0.6840	0.6958	0.6938	0.6800	0.6804	0.6932
	(15, 10, 5)	****	0.0884	0.1138	0.1344	****	0.0896	0.1140	0.1364
	(30, 25, 20)	0.3574	0.3114	0.3282	0.3538	0.3592	0.3144	0.3272	0.3542
(1, 1, 3)	(10, 10, 10)	****	0.3058	0.3050	0.3206	****	0.3090	0.3080	0.3288
	(25, 25, 25)	0.7884	0.7616	0.7578	0.7804	****	0.7624	0.7622	0.7786
	(50, 50, 50)	0.9836	0.9820	0.9818	0.9854	0.9836	0.9822	0.9818	0.9850
	(15, 10, 5)	****	0.1462	0.1830	0.2306	****	0.1486	0.1854	0.2318
	(30, 25, 20)	0.7500	0.7028	0.7210	0.7538	0.7496	0.7048	0.7200	0.7542
(1, 2, 4)	(10, 10, 10)	****	0.3998	0.3992	0.3850	****	0.3986	0.3968	0.3856
	(25, 25, 25)	0.8736	0.8514	0.8522	0.8542	****	0.8534	0.8534	0.8580
	(50, 50, 50)	0.9954	0.9946	0.9948	0.9946	0.9952	0.9948	0.9948	0.9944
	(15, 10, 5)	****	0.2908	0.3428	0.3506	****	0.2964	0.3386	0.3452
	(30, 25, 20)	0.8652	0.8364	0.8458	0.8564	0.8716	0.8382	0.8488	0.8592
(1, 4, 8)	(10, 10, 10)	****	0.7752	0.7742	0.7486	****	0.7782	0.7792	0.7536
	(25, 25, 25)	0.9968	0.9964	0.9964	0.9956	****	0.9960	0.9960	0.9960
	(50, 50, 50)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	(15, 10, 5)	****	0.7272	0.7674	0.7482	****	0.7238	0.7624	0.7414
	(30, 25, 20)	0.9982	0.9962	0.9972	0.9968	0.9978	0.9962	0.9968	0.9966

type I error rates are above 0.06, was ignored for the power comparison. For this reason, this situation was expressed by denoting with "****" in power tables.

It has been observed that when the differences among the scale parameters are small, for example λ=1, 1, 2, the CAT appears to be more powerful than the other tests for all group values and sample sizes. For example, for $\mu = 1, 1, 1$, $\lambda = 1, 1, 3$, $n = 15, 10, 5$, the power values of MLRT, GLRT, CAT are 0.1462, 0.1830 and 0.2306, respectively. The thing which is particularly worth mentioning is that the CAT detects the small differences between the scale parameters more easily than the other tests. In other cases where the increases in the differences between the scale parameters are large, for example λ = 1, 4, 8, the power values of the CAT are very close to those of other tests, especially as the sample sizes increase. Additionally, the powers of all the tests are positively affected from the increases in the differences between scale parameters and sample size, the powers of all the tests increase for these cases.

CONCLUSION

In this paper, a new test statistic based on the CAT method has been introduced to test the homogeneity of k IGD scale parameters. The proposed test was also compared with LRT, MLRT and GLRT in terms of type I error rates and powers. The simulation results show that the proposed test performs quite well in terms of the estimated type I

TABLE 6. The power values of all tests for $k = 4$

μ	n	μ = (1, 1, 1, 1)				μ = (0.5, 1, 1, 2)			
		LRT	MLRT	GLRT	CAT	LRT	MLRT	GLRT	CAT
(1, 1, 1, 2)	(10, 10, 10, 10)	****	0.1262	0.1272	0.1262	****	0.1258	0.1262	0.1290
	(25, 25, 25, 25)	0.3460	0.3088	0.3076	0.3316	0.3466	0.3144	0.3112	0.3334
	(50, 50, 50, 50)	0.6928	0.6740	0.6740	0.6980	0.6932	0.6746	0.6740	0.7000
	(15, 10, 10, 5)	****	0.0854	0.1026	0.1252	****	0.0834	0.1032	0.1242
	(30, 25, 25, 20)	0.3154	0.2732	0.2826	0.3208	****	0.2722	0.2826	0.3220
(1, 1, 1, 3)	(10, 10, 10, 10)	****	0.2710	0.2740	0.2906	****	0.2720	0.2738	0.2892
	(25, 25, 25, 25)	0.7806	0.7518	0.7520	0.7792	0.7802	0.7514	0.7506	0.7806
	(50, 50, 50, 50)	0.9868	0.9860	0.9856	0.9888	0.9870	0.9856	0.9862	0.9886
	(15, 10, 10, 5)	****	0.1322	0.1666	0.2114	****	0.1306	0.1682	0.2132
	(30, 25, 25, 20)	0.7042	0.6432	0.6642	0.7226	****	0.6472	0.6618	0.7214
(1, 2, 2, 4)	(10, 10, 10, 10)	****	0.3394	0.3418	0.3148	****	0.3400	0.3416	0.3126
	(25, 25, 25, 25)	0.8300	0.8060	0.8068	0.7970	0.8300	0.8098	0.8082	0.8010
	(50, 50, 50, 50)	0.9890	0.9878	0.9878	0.9876	0.9878	0.9872	0.9874	0.9878
	(15, 10, 10, 5)	****	0.2780	0.3222	0.3196	****	0.2748	0.3234	0.3146
	(30, 25, 25, 20)	0.8326	0.7982	0.8070	0.8132	****	0.7954	0.8058	0.8138
(1, 4, 4, 8)	(10, 10, 10, 10)	****	0.7512	0.7510	0.7014	****	0.7520	0.7528	0.6984
	(25, 25, 25, 25)	0.9960	0.9952	0.9954	0.9942	0.9960	0.9956	0.9956	0.9954
	(50, 50, 50, 50)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	(15, 10, 10, 5)	****	0.7492	0.7842	0.7348	****	0.7504	0.7810	0.7330
	(30, 25, 25, 20)	0.9956	0.9944	0.9946	0.9938	****	0.9948	0.9948	0.9940

TABLE 7. The power values of all tests for $k = 5$

μ	n	m = (1, 1, 1, 1, 1)				m = (0.5, 0.5, 1, 2, 2)			
		LRT	MLRT	GLRT	CAT	LRT	MLRT	GLRT	CAT
(1, 1, 1, 1, 2)	(10, 10, 10, 10, 10)	****	0.1196	0.1200	0.1256	****	0.1230	0.1240	0.1280
	(25, 25, 25, 25, 25)	****	0.3098	0.3106	0.3378	****	0.3114	0.3132	0.3382
	(50, 50, 50, 50, 50)	****	0.6464	0.6450	0.6788	0.6620	0.6418	0.6410	0.6778
	(15, 15, 10, 5, 5)	****	0.0718	0.0842	0.0990	****	0.0740	0.0856	0.0998
	(30, 30, 25, 20, 20)	0.2982	0.2542	0.2630	0.2948	****	0.2516	0.2616	0.2970
(1, 1, 1, 1, 3)	(10, 10, 10, 10, 10)	****	0.2542	0.2546	0.2720	****	0.2506	0.2518	0.2718
	(25, 25, 25, 25, 25)	****	0.7178	0.7148	0.7624	****	0.7174	0.7170	0.7638
	(50, 50, 50, 50, 50)	****	0.9818	0.9818	0.9870	0.9850	0.9824	0.9820	0.9874
	(15, 15, 10, 5, 5)	****	0.1242	0.1492	0.1798	****	0.1206	0.1486	0.1792
	(30, 30, 25, 20, 20)	0.6738	0.6140	0.6296	0.6974	****	0.6068	0.6220	0.6934
(1, 1, 2, 4, 4)	(10, 10, 10, 10, 10)	****	0.6036	0.6034	0.5556	****	0.6036	0.6020	0.5568
	(25, 25, 25, 25, 25)	****	0.9798	0.9806	0.9778	****	0.9796	0.9798	0.9774
	(50, 50, 50, 50, 50)	****	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	(15, 15, 10, 5, 5)	****	0.3940	0.4732	0.4710	****	0.3904	0.4728	0.4678
	(30, 30, 25, 20, 20)	0.9846	0.9758	0.9796	0.9814	****	0.9750	0.9790	0.9806
(1, 1, 4, 8, 8)	(10, 10, 10, 10, 10)	****	0.9442	0.9434	0.9280	****	0.9414	0.9412	0.9254
	(25, 25, 25, 25, 25)	****	1.0000	1.0000	1.0000	****	1.0000	1.0000	1.0000
	(50, 50, 50, 50, 50)	****	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	(15, 15, 10, 5, 5)	****	0.8712	0.9122	0.8940	****	0.8676	0.9102	0.8916
	(30, 30, 25, 20, 20)	1.0000	1.0000	1.0000	1.0000	****	1.0000	1.0000	1.0000

TABLE 8. The power values of all tests for $k = 7$

μ	n	μ = (1, 1, 1, 1, 1, 1, 1)				μ = (0.5, 0.5, 1, 1, 1, 2, 2)			
		LRT	MLRT	GLRT	CAT	LRT	MLRT	GLRT	CAT
(1, 1, 1, 1, 1, 1, 2)	(10, 10, 10, 10, 10, 10, 10)	***	0.1088	0.1082	0.1126	***	0.1052	0.1058	0.1136
	(25, 25, 25, 25, 25, 25, 25)	***	0.2662	0.2642	0.2924	***	0.2640	0.2644	0.2904
	(50, 50, 50, 50, 50, 50, 50)	0.5984	0.5738	0.5756	0.6196	0.5996	0.5752	0.5766	0.6228
	(15, 15, 10, 10, 10, 5, 5)	***	0.0652	0.0766	0.0948	***	0.0676	0.0780	0.0958
	(30, 30, 25, 25, 25, 20, 20)	***	0.2122	0.2224	0.2638	***	0.2160	0.2230	0.2632
(1, 1, 1, 1, 1, 1, 3)	(10, 10, 10, 10, 10, 10, 10)	****	0.2086	0.2106	0.2372	****	0.2074	0.2076	0.2268
	(25, 25, 25, 25, 25, 25, 25)	****	0.6476	0.6462	0.7228	****	0.6478	0.6512	0.7170
	(50, 50, 50, 50, 50, 50, 50)	0.9742	0.9698	0.9704	0.9816	0.9712	0.9670	0.9666	0.9800
	(15, 15, 10, 10, 10, 5, 5)	***	0.1092	0.1276	0.1562	***	0.1062	0.1270	0.1504
	(30, 30, 25, 25, 25, 20, 20)	***	0.5266	0.5428	0.6286	***	0.5222	0.5372	0.6318
(1, 1, 2, 2, 2, 4, 4)	(10, 10, 10, 10, 10, 10, 10)	***	0.5278	0.5296	0.4664	***	0.5296	0.5270	0.4658
	(25, 25, 25, 25, 25, 25, 25)	***	0.9648	0.9638	0.9600	***	0.9650	0.9642	0.9602
	(50, 50, 50, 50, 50, 50, 50)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	(15, 15, 10, 10, 10, 5, 5)	***	0.4098	0.4818	0.4480	***	0.4088	0.4820	0.4464
	(30, 30, 25, 25, 25, 20, 20)	***	0.9608	0.9658	0.9646	***	0.9594	0.9634	0.9626
(1, 1, 4, 4, 4, 8, 8)	(10, 10, 10, 10, 10, 10, 10)	***	0.9402	0.9398	0.8990	***	0.9402	0.9400	0.9016
	(25, 25, 25, 25, 25, 25, 25)	***	1.0000	1.0000	1.0000	***	1.0000	1.0000	1.0000
	(50, 50, 50, 50, 50, 50, 50)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	(15, 15, 10, 10, 10, 5, 5)	***	0.9236	0.9420	0.9074	***	0.9196	0.9404	0.9062
	(30, 30, 25, 25, 25, 20, 20)	***	0.9608	0.9658	0.9646	***	1.0000	1.0000	1.0000

error rates, and the powers of this test are also larger than the other tests, especially when the scale parameters differ greatly. Therefore, it can be said that the proposed test is a good alternative test to test the homogeneity of k IG scale parameters.

ACKNOWLEDGEMENTS

The authors are grateful to reviewer for providing useful comments and suggestions which improved the earlier version of the paper.

REFERENCES

Anderson, C.J., Verkuilen, J. & Johnson, T. 2010. *Applied Generalized Linear Mixed Models: Continuous and Discrete Data*. New York: Springer.

Chang, C.H. & Pal, N. 2008. A revisit to the Behrens-Fisher problem: Comparison of five test methods. *Communications in Statistics-Simulation and Computation* 37(6): 1064-1085.

Chang, C.H., Pal, N. & Lin, J.J. 2010. A note on comparing several Poisson means. *Communications in Statistics-Simulation and Computation* 39(8): 1605-1627.

Chang, C.H., Lin, J.J. & Pal, N. 2011. Testing the equality of several gamma means: A parametric bootstrap method with applications. *Computational Statistics* 26(1): 55-76.

Chhikara, R.S. & Folks, J.L. 1989. *The Inverse Gaussian Distribution*. New York: Marcel Decker. Inc.

Doornbos, R. & Dijkstra, J.B. 1983. A multi sample test for the equality of coefficients of variation in normal populations. *Communications in Statistics-Simulation and Computation* 12(2): 147-158.

Gökpınar, E.Y. & Gökpınar, F. 2012. A test based on the computational approach for equality of means under the unequal variance assumption. *Hacettepe Journal of Mathematics and Statistics* 41(4): 605-613.

Gökpınar, E.Y., Polat, E., Gökpınar, F. & Günay, S. 2013. A new computational approach for testing equality of inverse Gaussian means under heterogeneity. *Hacettepe Journal of Mathematics and Statistics* 42(5): 581-590.

Liu, X. & He, D. 2013. Testing homogeneity of inverse Gaussian scale parameters based on generalized likelihood ratio. *Communications in Statistics-Simulation and Computation* 42(2): 382-392.

Pal, N., Lim, W.K. & Ling, C.H. 2007. A computational approach to statistical inferences. *Journal of Applied Probability and Statistics* 2(1): 13-35.

Schrödinger, E. 1915. Zür theorie der fall-und steigversuche an teilchen mit brownscher bewegung. *Physikalische Zeitschrift* 16: 289-295.

Seshadri, V. 1993. *The Inverse Gaussian Distribution: A Case Study in Exponential Families*. New York: Oxford University Press.

Seshadri, V. 1999. *The Inverse Gaussian Distribution: Statistical Theory and Applications*. New York: Springer.

Stine-Morrow, E.A.L., Miller, L.M.S., Gagne, D.D. & Hertzog, C. 2008. Self-regulated reading in adulthood. *Psychology and Aging* 23: 131-153.

Tian, L. 2006. Testing equality of inverse Gaussian means under heterogeneity, based on generalized test variable. *Computational Statistics & Data Analysis* 51(2): 1156-1162.

Tweedie, M.C.K. 1945. Inverse statistical variates. *Nature* 155(3937): 453-453.

Von Smoluchowski, M. 1915. Notiz uiber die Berechnung der Brownschen Molekularbewegung bei der Ehrenhaft-Millikanschen Versuchsanordning. *Phys. Z.* 16: 318-321.
Wald, A. 1947. *Sequential Analysis*. New York: Wiley.
Weerahandi, S. 1995. *Generalized Confidence Intervals*: *In Exact Statistical Methods for Data Analysis*. New York: Springer.

Department of Statistics
Gazi University
06500 Teknikokullar, Ankara
Turkey

*Corresponding author; email: eyigit@gazi.edu.tr

Received: 14 July 20117
Accepted: 3 May 2019