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INITIAL BOUNDS FOR CERTAIN SUBCLASSES OF GENERALIZED SĂLĂGEAN TYPE BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE HORADAM POLYNOMIALS

(Batas-batas Awal untuk Sesuatu Subkelas bagi Fungsi Bi-univalen Jenis Salagean Teritlak yang Disekutukan dengan Polinomial Horadam)

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ABSTRACT

This study proposes the use of Horadam polynomials which are known as their special cases, such as the Fibonacci polynomials, the Lucas polynomials, the Pell polynomials, the Pell-Lucas polynomials, and Chebyshev polynomials of the second kind. The aim of this study is to introduce a new subclass of generalized Sălăgean type bi-univalent functions using Hadamard product and defined by Horadam polynomials $\hbar_n(x)$ in the open unit disk $U = \{z \in C : |z| < 1\}$. We obtained coefficient estimates of the Taylor-Maclaurin $|a_2|$ and $|a_3|$ for functions f belonging to this newly defined subclass. Fekete-Szegö inequalities were also studied. Moreover, we give some interesting results using the relation between Sălăgean's differential operator and generalized Sălăgean differential operator.

Keywords: univalent functions; bi-univalent functions; coefficient estimates; Horadam polynomials

ABSTRAK

Dalam kajian ini, akan digunakan polinomial Horadam, dan polinomial yang diketahui sebagai kes khas bagi polinomial tersebut, iaitu seperti polinomial Fibonacci, polinomial Lucas, polinomial Pell, polinomial Pell-Lucas dan polinomial Chebyshev jenis kedua. Tujuan kajian ini adalah memperkenalkan suatu subkelas baharu fungsi bi-univalen jenis Sălăgean teritlak dengan menggunakan hasil darab Hadamard dan ditakrif oleh polimomial Horadam $\hbar_n(x)$ dalam cakera unit terbuka $U = \{z \in C : |z| < 1\}$. Bagi fungsi f yang terkandung di dalam subkelas fungsi bi-univalen yang baharu ini, diterbitkan anggaran bagi pekali Taylor-Maclaurin $|a_2|$ dan $|a_3|$. Ketaksamaan Fekete-Szegö juga dikaji. Malah, dengan menggunakan hubungan antara pengoperasi pembeza Sălăgean dengan pengoperasi pembeza Sălăgean teritlak, diberikan beberapa hasil yang menarik.

Kata kunci: fungsi univalen; fungsi bi-univalen; anggaran bagi pekali; polinomial Horadam

1. Introduction and Preliminaries

Let $R = (-\infty, \infty)$ be the set of real numbers, C be the set of complex numbers and $N := \{1,2,3,...\} = N_0 \setminus \{0\}$ be the set of positive integers. Let A be the family of analytic functions, normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + a_2 z^2 + a_3 z^3 + ..., \tag{1}$$

defined in the open unit disk U. We also denote by S the class of all functions in A which are univalent in U (see for details Duren 1983).

From the Koebe $1/4$ Theorem (see Duren 1983), every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

In fact, the inverse function f^{-1} is given by

$$F(w) = f^{-1}(w) = w - a_2w^2 - (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \quad . \tag{2}$$

The function f is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1). For detailed information about the class of Σ, see for instance Brannan and Taha (1986), Lewin (1967), Netanyahu (1969), Srivastava *et al.* (2010) and the references cited therein. From Srivastava *et al.* (2010), we state some examples of the functions in the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

etc. On the other hand, the Koebe function does not be an element of bi-univalent function class Σ. In addition, some other examples, which are not a member of the class Σ, are as follows (see, for details, Srivastava *et al.* 2010):

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Furthermore, the class of bi-univalent functions was firstly investigated by Lewin (1967) and the coefficient estimate $|a_2| \leq 1.51$ were obtained. Afterwards, for $f \in \Sigma$, $|a_2| \leq \sqrt{2}$ was estimated by Brannan and Clunie (1980) . Later, Netanyahu (1969) showed that $\max|a_2| = 4/3$ if $f \in \Sigma$.

The Hadamard product or convolution of two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $\phi(z) = z + \sum_{k=2}^{\infty} \xi_k z^k \in A$, denoted by $f * \phi$, is defined by

$$(f * \phi)(z) = z + \sum_{k=2}^{\infty} a_k \xi_k z^k = (\phi * f)(z), \qquad z \in U.$$

Al-Oboudi (2004) introduced the following differential operator for $\delta \geq 1$ and $f \in A$,

$$\begin{aligned} &D_\delta^0 f(z) = f(z), \\ &D_\delta^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \\ &\vdots \\ &D_\delta^n f(z) = (1-\delta) D_\delta^{n-1} f(z) + \delta z \left(D_\delta^{n-1} f(z)\right)' = D\left(D_\delta^{n-1} f(z)\right), \quad z \in U, \quad n \in N_0 = N \cup \{0\}. \end{aligned} \tag{3}$$

It is worthy mentioning that when $\delta = 1$ in (3), we have the differential operator of Sălăgean (1983).

If f is given by (1.2), we see that

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\delta\right]^n a_k z^k$$

with $D_\delta^n f(0) = 0.$ Thus

$$D_\delta^n f(z * \phi)(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\delta\right]^n a_k \xi_k z^k .$$

For analytic functions f and g in U, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \qquad \text{and} \qquad f(z) = g(w(z)) \qquad (z \in U).$$

This subordination is denoted by $f(z) \prec g(z),\ (z \in U).$ Especially, when g is univalent in U,

$$f \prec g \iff f(0) = g(0) \qquad \text{and} \qquad f(U) \subset g(U).$$

Horzum and Gökçen Koçer (2009) considered the Horadam polynomials $\hbar_n(x, a, b; p, q)$, or briefly $\hbar_n(x)$, which is given by the following recurrence relation (see also Horadam and Mahon, 1985):

$$\hbar_n(x) = px\hbar_{n-1}(x) + q\hbar_{n-2}(x) \qquad (n \in N) \tag{4}$$

with

$$\hbar_1(x) = a, \ \hbar_2(x) = bx \qquad \text{and} \qquad \hbar_3(x) = pbx^2 + aq$$

for some real constants a, b, p and $q.$ The characteristic equation of recurrence relation (4) is $t^2 - pxt - q = 0$. This equation has two real roots as follows;

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2}, \quad \text{and} \quad \beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Some particular cases of the polynomials $\hbar_n(x)$ (Horadam & Mahon 1985; Horzum & Gökçen Koçer 2009) are given as follows:

a) If $a = b = p = q = 1,$ the Fibonacci polynomials sequence is obtained

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad F_1 = 1, \quad F_2 = x.$$

b) If $a = 2, b = p = q = 1,$ the Lucas polynomials sequence is obtained

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x), \quad L_0 = 2, \quad L_1 = x.$$

c) If $a=q=1,\ \ b=p=2,$ the Pell polynomials sequence is obtained
$P_n(x)=2xP_{n-1}(x)+P_{n-2}(x),\quad P_1=1,\ \ P_2=2x.$
d) $a=b=p=2, q=1,$ the Pell-Lucas polynomials sequence is obtained
$Q_{n-1}(x)=2xQ_{n-2}(x)+Q_{n-3}(x),\quad Q_0=2,\ \ Q_1=2x.$
e) If $a=b=p=x=1,\ \ q=2y,$ the Jacobsthal polynomials sequence is obtained
$J_n(y)=J_{n-1}(y)+2yJ_{n-2}(y),\quad J_1=J_2=1.$
f) If $a=2,\ \ b=p=x=1,\ \ q=2y,$ the Jacobsthal polynomials sequence is obtained
$j_{n-1}(y)=j_{n-2}(y)+2yj_{n-3}(y),\quad j_0=2,\ \ j_1=1.$
g) If $a=1, b=p=2,\ \ q=-1,$ the Chebyshev polynomials of second kind sequence is obtained $U_{n-1}(x)=2xU_{n-2}(x)+U_{n-3}(x),\quad U_0=1,\ \ U_1=2x.$
h) If $a=b=1,\ p=2,\ \ q=-1,$ the Chebyshev polynomials of first kind sequence is obtained
$T_{n-1}(x)=2xT_{n-2}(x)-T_{n-3}(x),\quad T_0=1,\ \ T_1=x.$
i) If $x=1,$ the Horadam numbers sequence is obtained
$\hbar_{n-1}(1)=p\hbar_{n-2}(1)+q\hbar_{n-3}(1),\quad \hbar_0=1,\ \ \hbar_1=b.$

These polynomials were theoretically studied by many authors (Horadam & Mahon 1985; Horadam 1997; Koshy 2001; Lupaş 1999). Orthogonal polynomials, other special polynomials and their generalizations are quite important, especially in mathematics, statistics and physics.

Remark 1.1 (Horzum & Gökçen Koçer 2009) Let $\Pi(x,z)$ be the generating function of the Horadam polynomials $\hbar_n(x)$. Then

$$\Pi(x,z):=\sum_{n=1}^{\infty}\hbar_n(x)z^{n-1}=\frac{a+(b-ap)xz}{1-pxz-qz^2}. \tag{5}$$

Remark 1.2 In a particular situation, if we choose $a=1, b=p=2,\ \ q=-1,$ and $x\to\ell,$ then Eq. (5) reduces to the second kind Chebyshev polynomials $U_n(\ell)$, given by the following equality (see Szegö 1975)

$$U_n(\ell)=(n+1)\,{}_2F_1\left(-n,n+2;\frac{3}{2};\frac{1-\ell}{2}\right)=\frac{\sin(n+1)\vartheta}{\sin\vartheta},\quad \ell=\cos\vartheta.$$

in respect with hypergeometric function ${}_2F_1.$

Our motivation in this study is based on the fact that Geometric Function Theory is fruitful for finding a wide variety of special interesting functions and polynomials. The main aim of our study is to introduce a new subclass $\sum_{g,h}^{t,n,\delta}(\tau,\alpha,x)$ of generalized Sălăgean differential operator type bi-univalent functions by using Hadamard product and Horadam polynomials $\hbar_n(x)$. We obtain Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for this newly defined bi-univalent function subclass, We also, found the celebrated Fekete-Szegö inequalities of functions belonging to this subclass. Furthermore, we showed some other connections in associated with those considered in Sălăgean's differential operator as some particular cases of the main results.

2. Inequalities for the Taylor-Maclaurin Coefficients

We introcude firstly, the function class $\sum_{g,h}^{t,n,\delta}(\tau,\alpha,x)$ in the following definition.

Definition 2.1 A function f given by (1) is said to be in the class $\sum_{g,h}^{t,n,\delta}(\tau,\alpha,x)$ $\left(\tau\in C\setminus\{0\},\ 0<\alpha\le 1,\ \lambda>0,\ t,n\in N_0=N\cup\{0\},\ \ t>n,\ \delta\ge 1\right)$ if the following subordination conditions are satisfied:

$$f\in\Sigma \quad\text{and}\quad 1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(f*g)(z)}{D_\delta^t(f*h)(z)}+\alpha\frac{\left(D_\delta^n(f*g)(z)\right)'}{\left(D_\delta^t(f*h)(z)\right)'}-1\right]\prec\Pi(x,z)+1-a \tag{6}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(F*g)(w)}{D_\delta^t(F*h)(w)}+\alpha\frac{\left(D_\delta^n(F*g)(w)\right)'}{\left(D_\delta^t(F*h)(w)\right)'}-1\right]\prec\Pi(x,w)+1-a, \tag{7}$$

where the real constant a and b are as mentioned in (4), $g(z)=z+\sum_{k=1}^{\infty}g_k z^k,\quad h(z)=z+\sum_{k=1}^{\infty}h_k z^k\quad (g_k>0,\ h_k>0)$ and F is an extension of f^{-1} to U given by Eq. (2).

We start by finding the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for the functions in the class $\sum_{g,h}^{t,n,\delta}(\tau,\alpha,x)$.

Theorem 2.2 *Let the function* f *given by (1) be in the class* $\sum_{g,h}^{t,n,\delta}(\tau,\alpha,x)$ *Then,*

$$|a_2|\le\frac{|\tau||bx|\sqrt{|bx|}}{\sqrt{|B|}}. \tag{8}$$

and

$$|a_3|\le\frac{|\tau||bx|}{(1+2\alpha)\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|}+\frac{2\tau^2b^2x^2}{(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2} \tag{9}$$

where

$$B=\tau b^2x^2\left\{(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]-(1+3\alpha)\left[(1+\delta)^{t+n}h_2g_2-(1+\delta)^{2t}h_2^2\right]\right\}$$
$$-\left(bpx^2+aq\right)(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2.$$

Proof. Let the function $\sum_{g,h}^{t,n,\delta}(\tau,\alpha,x)$ be given by Taylor-Maclaurin expansion (1). Then, there are analytic functions u and v such that

$u(0)=0;\ v(0)=0,\quad |u(z)|<1\quad$ *and* $\quad |v(z)|<1\quad (\forall z,w\in U).$

we can write

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(f*g)(z)}{D_\delta^t(f*h)(z)}+\alpha\frac{\left(D_\delta^n(f*g)(z)\right)'}{\left(D_\delta^t(f*h)(z)\right)'}-1\right]=\Pi(x,u(z))+1-a \tag{10}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(F*g)(w)}{D_\delta^t(F*h)(w)}+\alpha\frac{\left(D_\delta^n(F*g)(w)\right)'}{\left(D_\delta^t(F*h)(w)\right)'}-1\right]=\Pi\left(x,v\left(w\right)\right)+1-a. \quad (11)$$

Equivalently,

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(f*g)(z)}{D_\delta^t(f*h)(z)}+\alpha\frac{\left(D_\delta^n(f*g)(z)\right)'}{\left(D_\delta^t(f*h)(z)\right)'}-1\right]=1+\hbar_1(x)-a+\hbar_2(x)u(z)+\hbar_3(x)\left[u(z)\right]^2+\ldots \quad (12)$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(F*g)(w)}{D_\delta^t(F*h)(w)}+\alpha\frac{\left(D_\delta^n(F*g)(w)\right)'}{\left(D_\delta^t(F*h)(w)\right)'}-1\right]=1+\hbar_1(x)-a+\hbar_2(x)v(w)+\hbar_3(x)\left[v(w)\right]^2+\ldots\ . \quad (13)$$

Now, from (12) and (13), we obtain

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(f*g)(z)}{D_\delta^t(f*h)(z)}+\alpha\frac{\left(D_\delta^n(f*g)(z)\right)'}{\left(D_\delta^t(f*h)(z)\right)'}-1\right]=1+\hbar_2(x)u_1(z)+\left[\hbar_2(x)u_2+\hbar_3(x)u_1^2\right]z^2+\ldots \quad (14)$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(F*g)(w)}{D_\delta^t(F*h)(w)}+\alpha\frac{\left(D_\delta^n(F*g)(w)\right)'}{\left(D_\delta^t(F*h)(w)\right)'}-1\right] \quad (15)$$
$$=1+\hbar_2(x)v_1(w)+\left[\hbar_2(x)v_2+\hbar_3(x)v_1^2\right]w^2+\ldots\ .$$

It is fairly well known that

$$|u(z)|=|u_1z+u_2z^2+\ldots|<1 \quad and \quad |v(w)|<|v_1w+v_2w^2+\ldots|<1,$$

then

$$|u_k|\le 1 \quad and \quad |v_k|\le 1 \quad (k\in N).$$

Thus upon comparing the corresponding coefficients in (14) and (15)

$$\tau\hbar_2(x)u_1=(1+\alpha)\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]a_2, \quad (16)$$

$$\tau\left\{\hbar_2(x)u_2+\hbar_3(x)u_1^2\right\}$$
$$=(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]a_3-(1+3\alpha)\left[(1+\delta)^{t+n}h_2g_2-(1+\delta)^{2t}h_2^2\right]a_2^2, \quad (17)$$

$$\tau\hbar_2(x)v_1=-(1+\alpha)\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]a_2, \quad (18)$$

$$\tau\left\{\hbar_2(x)v_2+\hbar_3(x)v_1^2\right\}=(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\left[2a_2^2-a_3\right]$$
$$-(1+3\alpha)\left[(1+\delta)^{t+n}h_2g_2-(1+\delta)^{2t}h_2^2\right]a_2^2. \quad (19)$$

From (16) and (18), we can easily see that

$$u_1=-v_1 \quad (20)$$

$$2(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2 a_2^2=\tau^2\left[\hbar_2(x)\right]^2\left(u_1^2+v_1^2\right)$$

$$a_2^2=\frac{\tau^2\left[\hbar_2(x)\right]^2\left(u_1^2+v_1^2\right)}{2(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2} \quad (21)$$

Now by adding Eqs. (17) and (19), we obtain

$$\left\{2(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]-2(1+3\alpha)\left[(1+\delta)^{t+n} h_2 g_2-(1+\delta)^{2t} h_2^2\right]\right\}a_2^2$$
$$=\tau\left\{\hbar_2(x)(u_2+v_2)+\hbar_3(x)\left(u_1^2+v_1^2\right)\right\}. \tag{22}$$

By substituting (21) in (22), we reduce that

$$a_2^2=\frac{\tau^2\left[\hbar_2(x)\right]^3(u_2+v_2)}{A}, \tag{23}$$

where

$$A=\tau\left[\hbar_2(x)\right]^2\left\{2(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]-2(1+3\alpha)\left[(1+\delta)^{t+n} h_2 g_2-(1+\delta)^{2t} h_2^2\right]\right\}$$
$$-2\hbar_3(x)(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2.$$

Now taking the absolute value of (23) we have the following inequality

$$|a_2|\le\frac{|\tau||bx|\sqrt{|bx|}}{\sqrt{|B|}}. \tag{24}$$

$$B=\tau b^2x^2\left\{(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]-(1+3\alpha)\left[(1+\delta)^{t+n} h_2 g_2-(1+\delta)^{2t} h_2^2\right]\right\}$$
$$-\left(bpx^2+aq\right)(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2.$$

This gives the bound of $|a_2|$ as given in (8).

Next, to obtain the solution of the coefficient bound on $|a_3|$, we subtract (19) from (17). Also, in view of (20), we have,

$$2(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\left[a_3-2a_2^2\right]=\tau\left\{\hbar_2(x)(u_2-v_2)+\hbar_3(x)\left(u_1^2-v_1^2\right)\right\}$$
$$a_3=\frac{\tau\hbar_2(x)(u_2-v_2)}{2(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]}+2a_2^2. \tag{25}$$

Using (20) in (25) we obtain that

$$a_3=\frac{\tau\hbar_2(x)(u_2-v_2)}{2(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]}+\frac{\tau^2\left[\hbar_2(x)\right]^2\left(u_1^2+v_1^2\right)}{(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2} \tag{26}$$

Taking the absolute value of (26) and applying (4) we get the desired result given by (9), that is

$$|a_3|\le\frac{|\tau||bx|}{(1+2\alpha)\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|}+\frac{2\tau^2b^2x^2}{(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2}.$$

This completes the proof of Theorem 2.2. □

3. The Fekete-Szegö Problem for the Class $\Sigma_{g,h}^{t,n,\delta}(\tau,\alpha,x)$

The following classical Fekete-Szegö inequality is related to the Taylor-Maclaurin coefficients of $f\in S$ given by (1):

$$\left|a_3-\vartheta a_2^2\right|\le 1+2\exp\left(-\frac{2\vartheta}{1-\vartheta}\right)\quad(0\le\vartheta<1). \tag{27}$$

In the case of $\vartheta \to 1^-$, it is clear that $\left|a_3 - a_2^2\right| \le 1.$

The coefficient functional $\Psi_\vartheta\left(f\right) = a_3 - \vartheta a_2^2$ has a significant role in function theory for normalized analytic functions f in the open unit disk U. The Fekete-Szegö problem is known to maximize the modulus of the functional $\Psi_\vartheta\left(f\right)$ (Fekete & Szegö 1933).

In this part, we obtain inequalities of Fekete-Szegö for functions in the class $\sum_{g,h}^{t,n,\delta}\left(\tau,\alpha,x\right)$, which is given by Definition 2.1.

Teorem 3.1. *Let the function* f *given by* (1) *be in the class* $\sum_{g,h}^{t,n,\delta}\left(\tau,\alpha,x\right)$. *Suppose also that* $\vartheta \in R$, $\left(\tau \in C\backslash\{0\},\ 0<\alpha\le 1,\ \lambda>0,\ t,n\in N_0 = N\cup\{0\},\ \ t>n,\ \delta\ge 1,\ g_k,h_k>0\right)$. *Then*

$$\left|a_3 - \vartheta a_2^2\right| \le \begin{cases} \dfrac{|\tau||bx|}{(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]} & 0 \le |2-\vartheta| \le \dfrac{2|B|}{2\tau^2(1+2\alpha)\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right| b^2x^2} \\ \dfrac{2\tau^2\left|(2-\vartheta)\right||bx|^3}{2|B|} & |2-\vartheta| \ge \dfrac{2|B|}{2\tau^2(1+2\alpha)\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right| b^2x^2} \end{cases} \tag{28}$$

where

$$B = \tau b^2x^2\left\{(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right] - (1+3\alpha)\left[(1+\delta)^{t+n} h_2 g_2 - (1+\delta)^{2t} h_2^2\right]\right\}$$
$$-\left(bpx^2 + aq\right)(1+\alpha)^2\left[(1+\delta)^n g_2 - (1+\delta)^t h_2\right]^2.$$

Proof. From (25), for $\vartheta \in \mathbb{R}$, we write

$$a_3 - \vartheta a_2^2 = \frac{\tau\hbar_2(x)(u_2 - v_2)}{2(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]} + (2-\vartheta)a_2^2.$$

By substituting (23) in (29), we have

$$a_3 - \vartheta a_2^2 = \frac{\tau\hbar_2(x)(u_2 - v_2)}{2(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]} + \frac{(2-\vartheta)\tau^2\left[\hbar_2(x)\right]^3(u_2+v_2)}{A},$$
$$= \tau\hbar_2(x)\left[\left(\Omega(\vartheta,x) + \frac{1}{2(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]}\right)u_2\right.$$
$$\left. + \left(\Omega(\vartheta,x) - \frac{1}{2(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]}\right)v_2\right],$$

where

$$A = \tau\left[\hbar_2(x)\right]^2\left\{2(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right] - 2(1+3\alpha)\left[(1+\delta)^{t+n} h_2 g_2 - (1+\delta)^{2t} h_2^2\right]\right\}$$
$$-2\hbar_3(x)(1+\alpha)^2\left[(1+\delta)^n g_2 - (1+\delta)^t h_2\right]^2,$$

and

$$\Omega(\vartheta,x) = \frac{(2-\vartheta)\tau^2\left[\hbar_2(x)\right]^2}{A}.$$

Hence, in view of (4), we conclude that

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{|\tau||\hbar_2(x)|}{(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]} & 0\le|\Omega(\vartheta,x)|\le\dfrac{1}{2(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]},\\[2mm] 2|\tau||\hbar_2(x)||\Omega(\vartheta,x)| & |\Omega(\vartheta,x)|\ge\dfrac{1}{2(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]}\end{cases}$$

which evidently completes the proof of Theorem 4.3. □

4. Some Particular Cases of the Main Results

For $\tau=1$, Theorem 2.2 and Theorem 3.1 readily yields the following results respectively.

Corollary 4.4 Let $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, be in the class $\Sigma_{g_k,h_k}^{t,n,\delta}(1,\alpha,x)=\Sigma_{g_k,h_k}^{t,n,\delta}(\alpha,x)$. Then,

$$|a_2|\le\frac{|bx|\sqrt{|bx|}}{\sqrt{|B|}},\qquad |a_3|\le\frac{|bx|}{(1+2\alpha)\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|}+\frac{2b^2x^2}{(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2}$$

and

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{|bx|}{(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]} & 0\le|2-\vartheta|\le\dfrac{2|B|}{2(1+2\alpha)\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|b^2x^2},\\[2mm] \dfrac{2|(2-\vartheta)||bx|^3}{2|B|} & |2-\vartheta|\ge\dfrac{2|B|}{2(1+2\alpha)\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|b^2x^2}\end{cases}$$

where

$$B=b^2x^2\left\{(1+2\alpha)\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]-(1+3\alpha)\left[(1+\delta)^{t+n}h_2g_2-(1+\delta)^{2t}h_2^2\right]\right\}$$
$$-\left(bpx^2+aq\right)(1+\alpha)^2\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2.$$

For $\alpha=0$, Theorem 2.2 and Theorem 3.1 readily yields the following results respectively.

Corollary 4.5 Let $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, be in the class $\Sigma_{g_k,h_k}^{t,n,\delta}(\tau,0,x)=\Sigma_{g_k,h_k}^{t,n,\delta}(\tau,x)$. Then,

$$|a_2|\le\frac{|\tau||bx|\sqrt{|bx|}}{\sqrt{|B|}},\qquad |a_3|\le\frac{|\tau||bx|}{\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|}+\frac{2\tau^2b^2x^2}{\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2},$$

and

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{|\tau||bx|}{\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]} & 0\le|2-\vartheta|\le\dfrac{2|B|}{2\tau^2\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|b^2x^2}\\[2mm] \dfrac{2\tau^2|(2-\vartheta)||bx|^3}{2|B|} & |2-\vartheta|\ge\dfrac{2|B|}{2\tau^2\left|\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]\right|b^2x^2}\end{cases}$$

where

$$B=\tau b^2x^2\left\{\left[(1+2\delta)^n g_3-(1+2\delta)^t h_3\right]-\left[(1+\delta)^{t+n}h_2g_2-(1+\delta)^{2t}h_2^2\right]\right\}$$
$$-\left(bpx^2+aq\right)\left[(1+\delta)^n g_2-(1+\delta)^t h_2\right]^2.$$

For $\tau = 1$, Corollary 4.5 yields the following results.

Corollary 4.6 Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, be in the class $\Sigma_{g_k,h_k}^{t,n,\delta}(\tau, x)$. Then,

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|B|}}, \qquad |a_3| \le \frac{|bx|}{\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right|} + \frac{2b^2x^2}{\left[(1+\delta)^n g_2 - (1+\delta)^t h_2\right]^2},$$

and

$$|a_3 - \vartheta a_2^2| \le \begin{cases} \dfrac{|bx|}{\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]} & 0 \le |2-\vartheta| \le \dfrac{2|B|}{2\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right| b^2x^2} \\ \dfrac{2|(2-\vartheta)||bx|^3}{2|B|} & |2-\vartheta| \ge \dfrac{2|B|}{2\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right| b^2x^2} \end{cases},$$

where

$$B = b^2x^2\left\{\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right] - \left[(1+\delta)^{t+n} h_2 g_2 - (1+\delta)^{2t} h_2^2\right]\right\} - \left(bpx^2 + aq\right)\left[(1+\delta)^n g_2 - (1+\delta)^t h_2\right]^2.$$

In view of Remark 1.2, Theorem 2.2 and Theorem 3.1, can be shown to yield the following results.

Corollary 4.7 Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, be in the class $\Sigma_{g_k,h_k}^{t,n,\delta}(\tau, \alpha, \ell)$. Then,

$$|a_2| \le \frac{|\tau||2\ell|\sqrt{|2\ell|}}{\sqrt{|B|}}, \quad |a_3| \le \frac{|\tau||2\ell|}{(1+2\alpha)\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right|} + \frac{2\tau^2 4\ell^2}{(1+\alpha)^2\left[(1+\delta)^n g_2 - (1+\delta)^t h_2\right]^2}$$

and

$$|a_3 - \vartheta a_2^2| \le \begin{cases} \dfrac{|\tau||2\ell|}{(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]} & 0 \le |2-\vartheta| \le \dfrac{2|B|}{2\tau^2(1+2\alpha)\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right| 4\ell^2} \\ \dfrac{2\tau^2|(2-\vartheta)||2\ell|^3}{2|B|} & |2-\vartheta| \ge \dfrac{2|B|}{2\tau^2(1+2\alpha)\left|\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]\right| 4\ell^2} \end{cases}$$

where

$$B = 4\tau\ell^2\left\{(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right] - (1+3\alpha)\left[(1+\delta)^{t+n} h_2 g_2 - (1+\delta)^{2t} h_2^2\right]\right\} - \left(4\ell^2 - 1\right)(1+\alpha)^2\left[(1+\delta)^n g_2 - (1+\delta)^t h_2\right]^2.$$

Taking $\vartheta = 2$ in Corollary 4.7, we get the following consequence.

Corollary 4.8 For $\ell(1/2,1)$, let the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, be in the class $\Sigma_{g_k,h_k}^{t,n,\delta}(\tau, \alpha, \ell)$. Then,

$$|a_3 - \vartheta a_2^2| \le \frac{|\tau||2\ell|}{(1+2\alpha)\left[(1+2\delta)^n g_3 - (1+2\delta)^t h_3\right]}.$$

Setting $\delta = 1$ in Theorem 2.2 and Theorem 3.1, we arrive at the following result.

Corollary 4.8 Let the function f given by (1) be in the class $\Sigma_{g_k,h_k}^{t,n,1}(\tau,\alpha,x) = \Sigma_{g_k,h_k}^{t,n,\delta}(\tau,\alpha,x)$. Then

$$|a_2| \le \frac{|\tau||bx|\sqrt{|bx|}}{\sqrt{|B|}}, \quad |a_3| \le \frac{|\tau||bx|}{(1+2\alpha)\left|\left[3^n g_3 - 3^t h_3\right]\right|} + \frac{2\tau^2 b^2 x^2}{(1+\alpha)^2\left[2^n g_2 - 2^t h_2\right]^2},$$

and

$$|a_3 - \vartheta a_2^2| \le \begin{cases} \dfrac{|\tau||bx|}{(1+2\alpha)\left[3^n g_3 - 3^t h_3\right]} & 0 \le |2-\vartheta| \le \dfrac{2|B|}{2\tau^2(1+2\alpha)\left|\left[3^n g_3 - 3^t h_3\right]\right| b^2 x^2} \\ \dfrac{2\tau^2 |(2-\vartheta)||bx|^3}{2|B|} & |2-\vartheta| \ge \dfrac{2|B|}{2\tau^2(1+2\alpha)\left|\left[3^n g_3 - 3^t h_3\right]\right| b^2 x^2} \end{cases}$$

where

$$B = \tau b^2 x^2 \left\{(1+2\alpha)\left[3^n g_3 - 3^t h_3\right] - (1+3\alpha)\left[2^{t+n} h_2 g_2 - 2^{2t} h_2^2\right]\right\} - \left(bpx^2 + aq\right)(1+\alpha)^2\left[2^n g_2 - 2^t h_2\right]^2.$$

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