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EXTENSIONS OF HERMITE-HADAMARD TYPE INEQUALITY FOR CO-ORDINATED $(\alpha, m)-$CONVEX FUNCTIONS

(Perluasan Ketaksamaan Jenis Hermite-Hadamard untuk Fungsi $(\alpha, m)-$Cembung Terkoordinat)

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ABSTRACT

In this paper, the classes of $(\alpha,m)-$convex functions in real and complex co-ordinated space have been introduced. Some new Hermite-Hadamard type inequalities have been obtained for these classes of functions.

Keywords: convex function; Hermite-Hadamard inequality; co-ordinated space

ABSTRACT

Dalam makalah ini, kelas fungsi $(\alpha,m)-$cembung dalam ruang terkoordinat nyata dan kompleks diperkenalkan. Beberapa ketaksamaan baharu jenis Hermite-Hadamard diperoleh untuk kelas-kelas fungsi tersebut.

Kata kunci: fungsi cembung; ketaksamaan Hermite-Hadamard; ruang terkoordinat

1. Introduction

The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on I if

$$f\left(tx+(1-t)y\right) \le tf(x)+(1-t)f(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$.

The class of convex finctions has been generalized by Toader (1985) as $m-$convex for $m \in [0,1]$. Later, Mihesan (1993) introduced more general class of $(\alpha, m)-$convex functions, where $(\alpha, m) \in [0,1]^2$. We state the definitions as follows.

Definition 1.1. The function $f : [0,b] \to \mathbb{R},\ b > 0$ is said to be $m-$convex if

$$f\left(tx+m(1-t)y\right) \le tf(x)+m(1-t)f(y) \tag{2}$$

for all $x, y \in [0,b]$ and $t \in [0,1]$ with $m \in [0,1]$. f is said to be $m-$concave if the inequality in (2) is reversed.

Definition 1.2. The function $f : [0,b] \to \mathbb{R},\ b > 0$ is said to be $(\alpha, m)-$convex if

$$f\left(tx+m(1-t)y\right) \le t^{\alpha} f(x)+m\left(1-t^{\alpha}\right) f(y) \tag{3}$$

for all $x, y \in [0,b]$ and $t \in [0,1]$ with $(\alpha, m) \in [0,1]^2$.

The class of all $m-$convex and $(\alpha,m)-$convex functions are denoted by $K_m(b)$ and $K_m^\alpha(b)$, respectively. The concept of convexity, $m-$convexity and $(\alpha,m)-$convexity were then extended to a real co-ordinated space by considering the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. Dragomir (2001) was first introduced a convex function on the co-ordinates as follows.

Definition 1.3. A function $f:\Delta\to\mathbb{R}$ is called convex on the co-ordinates if the partial mappings $f_y:[a,b]\to\mathbb{R},\ f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R},\ f_x(v)=f(u,v)$ are convex where defined for all $x\in[a,b]$ and $y\in[c,d]$.

Definition 1.4. The mapping $f:\Delta\to\mathbb{R}$ is convex in Δ if the following inequality:

$$f\left(\lambda x+\left(1-\lambda\right)z,\lambda y+\left(1-\lambda\right)w\right)\leq\lambda f\left(x,y\right)+\left(1-\lambda\right)f\left(z,w\right)\tag{4}$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Later, Latif and Alomari (2009) modified these two definitions to be more precise as follows.

Definition 1.5. The function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates if the partial mappings $f_y:[a,b]\to\mathbb{R},\ f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R},\ f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b]$ and $y\in[c,d]$.

A formal definition of a co-ordinated convex function is given by Latif and Alomari (2009) as follows.

Definition 1.6. $f:\Delta\to\mathbb{R}$ is said to be a co-ordinated convex function on Δ if

$$\begin{aligned}f\left(tx+\left(1-t\right)y,su+\left(1-s\right)w\right)&\leq tsf\left(x,u\right)+t\left(1-s\right)f\left(x,w\right)\\&\quad+s\left(1-t\right)f\left(y,u\right)+\left(1-t\right)\left(1-s\right)f\left(y,w\right)\end{aligned}\tag{5}$$

for all $(x,y),(z,w)\in\Delta$ and $t,s\in[0,1]$.

Özdemir *et al.* (2011) then extended the definition of the co-ordinated convex functions for coordinated of $m-$convex and $(\alpha,m)-$convex functions.

Definition 1.7. The mapping $f:\Delta_{(\alpha,m)}\to\mathbb{R}$ is $m-$convex on $\Delta_{(\alpha,m)}$ if

$$f\left(tx+(1-t)z,ty+(1-t)w\right)\leq tf(x,y)+m(1-t)f(z,w)\tag{6}$$

holds for all $(x,y),(z,w)\in\Delta_{(\alpha,m)}$ with $t\in[0,1]$ and for some fixed $m\in[0,1]$.

Definition 1.8. The mapping $f:\Delta_{(\alpha,m)}\to\mathbb{R}$ is $(\alpha,m)-$convex on $\Delta_{(\alpha,m)}$ if

$$f\left(tx+(1-t)z,ty+(1-t)w\right)\le t^{\alpha}f(x,y)+m\left(1-t^{\alpha}\right)f(z,w) \tag{7}$$

holds for all $(x,y),(z,w)\in\Delta_{(\alpha,m)}$ and $(\alpha,m)\in[0,1]^2$ with $t\in[0,1]$.

Definition 1.9. A function $f:\Delta_{(\alpha,m)}\to\mathbb{R}$ is called co-ordinated $(\alpha,m)-$convex on $\Delta_{(\alpha,m)}$ if the partial mappings $f_y:[0,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[0,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are $(\alpha,m)-$convex for all $y\in[0,d]$ and $x\in[0,b]$ with some fixed $(\alpha,m)\in[0,1]^2$.

Note that, every $(\alpha,m)-$convex mapping $f:\Delta_{(\alpha,m)}\to[0,\infty)$ is $(\alpha,m)-$convex on the co-ordinates (Özdemir *et a*l. 2011).

2. Preliminaries

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and $a,b\in I$ with $a<b$. Then, the following inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \tag{8}$$

is known as Hermite-Hadamard inequality.

This inequality is very famous and is widely used in mathematical analysis. Dragomir (2001) extended this inequality for a co-ordinated convex function in a square on $\mathbb{R}^2$ as follows.

Theorem 2.1. (Dragomir 2001) *Suppose that* $f:\Delta\to\mathbb{R}$ *is a convex function on the co-ordinates on* Δ. *Then:*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\
&\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\right.\\
&\quad\left.+\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right]
\end{aligned}$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{9}$$

The above inequalities are sharp.

Set *et al.* (2012) proved Hermite-Hadamard inequality for $(\alpha, m)-$convex functions. The result is given in the following.

Theorem 2.2. (Set *et al.* 2012) *Let* $f:[0,\infty)\to\mathbb{R}$ *be an* $(\alpha,m)-$*convex function with* $(\alpha,m)\in(0,1]^2$. *If* $0\leq a<b<\infty$ *and f is an integrable function on* [*a*,*b*], *then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}\left[\frac{f(a)+f(b)+\alpha m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{\alpha+1}\right]. \tag{10}$$

Özdemir *et al.* (2011) provided Hermite-Hadamard inequality for co-ordinated $(\alpha, m)-$convex functions, which has extended the Dragomir's result (Dragomir 2001). The result is given as follows.

Theorem 2.3. (Özdemir *et al.* 2011) *Suppose that* $f:\Delta_{(\alpha,m)}=[0,b]\times[0,d]\to\mathbb{R}$ *is* $(\alpha,m)-$*convex function on the co-ordinates on* $\Delta_{(\alpha,m)}$. *If* $0\leq a<b<\infty$ *and* $0\leq c<d<\infty$ *with* $(\alpha,m)\in(0,1]^2$, $f_x\in L_1([0,d])$ *and* $f_y\in L_1([0,b])$, *then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\
&\leq \frac{1}{4(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx+\frac{m\alpha}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx\right. \\
&\quad+\frac{m\alpha}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx+\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy \\
&\quad\left.+\frac{m\alpha}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy+\frac{m\alpha}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy\right].
\end{aligned} \tag{11}$$

Some other results related to Hermite-Hadamard inequality can be found in Akkurt *et al.* (2017), Hwang *et al.* (2007), Ibrahim (2018), Noor *et al.* (2017), Pavic (2015) and others.

3. $(\alpha,m)-$Convex Functions in Real Co-Ordinated Spaces

By considering the bidimensional interval $\Delta_{(\alpha,m)} = [0,b]\times[0,d]$ in $[0,\infty)^2$, we modify the definition of co-ordinated $(\alpha,m)-$convex functions given by Özdemir *et al.* (2011) to be more precise. Thus, a formal definition of $(\alpha,m)-$convex functions is given as follows:

Definition 3.1. The function $f:\Delta_{(\alpha,m)}\to\mathbb{R}$ is said to be $(\alpha,m)-$convex on the co-ordinates on $\Delta_{(\alpha,m)}$ if

$$\begin{aligned} &f\left(tx+m(1-t)y, su+m(1-s)w\right) \\ &\quad\le t^{\alpha}s^{\alpha} f(x,u) + mt^{\alpha}\left(1-s^{\alpha}\right) f(x,w) + m\left(1-t^{\alpha}\right)s^{\alpha} f(y,u) \\ &\quad + m^2\left(1-t^{\alpha}\right)\left(1-s^{\alpha}\right) f(y,w) \end{aligned} \tag{12}$$

for all $t,s\in[0,1]$ and $(x,y),(u,w)\in\Delta_{(\alpha,m)}$ with $(\alpha,m)\in(0,1]^2$.

Remark 3.2. If in (12), $m=\alpha=1$, then we obtain (5) given by Latif and Alomari (2009).

Now, by considering $(\alpha,m)\in(0,1]^2$, Definition 3.1 be can modified as the following.

Definition 3.3. The function $f:\Delta_{(\alpha,m)}\to\mathbb{R}$ is said to be $(\alpha,m)-$convex on the co-ordinates on $\Delta_{(\alpha,m)}$ if

$$\begin{aligned} &f\left(tx+(1-t)y, su+(1-s)w\right) \\ &\quad\le t^{\alpha}s^{\alpha} f(x,u) + mt^{\alpha}\left(1-s^{\alpha}\right) f\left(x,\frac{w}{m}\right) + m\left(1-t^{\alpha}\right)s^{\alpha} f\left(\frac{y}{m},u\right) \\ &\quad + m^2\left(1-t^{\alpha}\right)\left(1-s^{\alpha}\right) f\left(\frac{y}{m},\frac{w}{m}\right). \end{aligned} \tag{13}$$

for all $t,s\in[0,1]$ and $(x,y),(u,w)\in\Delta_{(\alpha,m)}$ with $(\alpha,m)\in(0,1]^2$.

In this study, we use these definitions to provide new type of Hermite-Hadamard inequality for $(\alpha,m)-$convex functions, which are defined in convex subset of the real or convex co-ordinated space. First we state the following result for real co-ordinated space, which is improved (11) the result given by Özdemir *et al.* (2011).

Theorem 3.4. *Let $f:\Delta=[a,b]\times[c,d]\subset[0,\infty)^2\to\mathbb{R}$ be a co-ordinated $(\alpha,m)-$convex function with $(\alpha,m)\in(0,1]^2$. If f is an integrable function on Δ, $0\le a<b<\infty$ and $0\le c<d<\infty$, then*

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy \\
&\le \frac{1}{(\alpha+1)^2}\left[\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\right. \\
&\quad +\frac{\alpha m}{4(\alpha+1)^2}\left[f\left(\frac{a}{m},c\right)+f\left(\frac{b}{m},c\right)+f\left(\frac{a}{m},d\right)+f\left(\frac{b}{m},d\right)\right. \\
&\quad \left.\left.+f\left(a,\frac{c}{m}\right)+f\left(b,\frac{c}{m}\right)+f\left(a,\frac{d}{m}\right)+f\left(b,\frac{d}{m}\right)\right]\right. \\
&\quad \left.+\frac{\alpha^2 m^2}{4(\alpha+1)^2}\left[f\left(\frac{a}{m},\frac{c}{m}\right)+f\left(\frac{b}{m},\frac{c}{m}\right)+f\left(\frac{a}{m},\frac{d}{m}\right)+f\left(\frac{b}{m},\frac{d}{m}\right)\right]\right].
\end{aligned} \tag{14}
$$

Proof. Since $f:\Delta=[a,b]\times[c,d]\subset[0,\infty)^2\to\mathbb{R}$ is co-ordinated $(\alpha,m)-$convex function, then the partial mappings $g_x:[c,d]\to\mathbb{R}$ defined by $g_x(y)=f(x,y)$ is $(\alpha,m)-$convex on $[c,d]$ for all $x\in[a,b]$. From (10), we obtain

$$
g_x\left(\frac{c+d}{2}\right)\le\frac{1}{d-c}\int_c^d g_x(y)dy\le\frac{1}{2(\alpha+1)}\left[g_x(c)+g_x(d)+\alpha m\left(g_x\left(\frac{c}{m}\right)+g_x\left(\frac{d}{m}\right)\right)\right].
$$

Thus,

$$
\begin{aligned}
f\left(x,\frac{c+d}{2}\right) &\le \frac{1}{d-c}\int_c^d f(x,y)dy \\
&\le \frac{1}{2(\alpha+1)}\left[f(x,c)+f(x,d)+\alpha m\left(f\left(x,\frac{c}{m}\right)+f\left(x,\frac{d}{m}\right)\right)\right].
\end{aligned} \tag{15}
$$

Divide (15) by $(b-a)>0$ and integrate on $[a,b]$, we have

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
&\le \frac{1}{2(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\right. \\
&\quad \left.+\alpha m\frac{1}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx+\alpha m\frac{1}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx\right].
\end{aligned} \tag{16}
$$

By applying similar arguments to the mapping $g_y:[a,b]\to\mathbb{R}$ defined by $g_y(x)=f(x,y)$, we have

$$\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy$$
$$\le \frac{1}{2(\alpha+1)}\left[\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right. \tag{17}$$
$$\left.+\alpha m\frac{1}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy+\alpha m\frac{1}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy\right].$$

By adding (16) and (17), we have

$$\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \le \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy$$
$$\le \frac{1}{2(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx+\alpha m\frac{1}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx\right.$$
$$+\alpha m\frac{1}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy \tag{18}$$
$$\left.+\alpha m\frac{1}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy+\alpha m\frac{1}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy\right].$$

From (9), we know that

$$2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy. \tag{19}$$

By combining (18) and (19), we have

$$2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$
$$\le \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy$$
$$\le \frac{1}{2(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.$$
$$+\alpha m\frac{1}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx+\alpha m\frac{1}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx \tag{20}$$
$$+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy$$
$$\left.+\alpha m\frac{1}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy+\alpha m\frac{1}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy\right]$$

$$
\begin{aligned}
&= \frac{1}{2(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\
&\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
&\quad + \frac{\alpha m}{2(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx + \frac{1}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx\right.\\
&\quad \left. + \frac{1}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy + \frac{1}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy\right].
\end{aligned}
$$

From (10), we have that

$$
\frac{1}{b-a}\int_a^b f(x,c)\,dx \le \frac{1}{2(\alpha+1)}\left[f(a,c)+f(b,c)+\alpha m\left(f\left(\frac{a}{m},c\right)+f\left(\frac{b}{m},c\right)\right)\right],
$$

$$
\frac{1}{b-a}\int_a^b f(x,d)\,dx \le \frac{1}{2(\alpha+1)}\left[f(a,d)+f(b,d)+\alpha m\left(f\left(\frac{a}{m},d\right)+f\left(\frac{b}{m},d\right)\right)\right],
$$

$$
\frac{1}{d-c}\int_c^d f(a,y)\,dy \le \frac{1}{2(\alpha+1)}\left[f(a,c)+f(a,d)+\alpha m\left(f\left(a,\frac{c}{m}\right)+f\left(a,\frac{d}{m}\right)\right)\right],
$$

$$
\frac{1}{d-c}\int_c^d f(b,y)\,dy \le \frac{1}{2(\alpha+1)}\left[f(b,c)+f(b,d)+\alpha m\left(f\left(b,\frac{c}{m}\right)+f\left(b,\frac{d}{m}\right)\right)\right],
$$

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx \le \frac{1}{2(\alpha+1)}&\left[f\left(a,\frac{c}{m}\right)+f\left(b,\frac{c}{m}\right)\right.\\
&\left. +\alpha m\left(f\left(\frac{a}{m},\frac{c}{m}\right)+f\left(\frac{b}{m},\frac{c}{m}\right)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx \le \frac{1}{2(\alpha+1)}&\left[f\left(a,\frac{d}{m}\right)+f\left(b,\frac{d}{m}\right)\right.\\
&\left. +\alpha m\left(f\left(\frac{a}{m},\frac{d}{m}\right)+f\left(\frac{b}{m},\frac{d}{m}\right)\right)\right],
\end{aligned}
$$

$$\frac{1}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy \le \frac{1}{2(\alpha+1)}\left[f\left(\frac{a}{m},c\right)+f\left(\frac{a}{m},d\right)\right.$$
$$\left.+\alpha m\left(f\left(\frac{a}{m},\frac{c}{m}\right)+f\left(\frac{a}{m},\frac{d}{m}\right)\right)\right],$$

$$\frac{1}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy \le \frac{1}{2(\alpha+1)}\left[f\left(\frac{b}{m},c\right)+f\left(\frac{b}{m},d\right)\right.$$
$$\left.+\alpha m\left(f\left(\frac{b}{m},\frac{c}{m}\right)+f\left(\frac{b}{m},\frac{d}{m}\right)\right)\right].$$

By substituting the above inequalities into (20) and dividing by 2, we obtain the desired result. □

4. $(\alpha,m)-$ Convex Functions in Real or Complex Co-Ordinated Spaces

The concept of $(\alpha,m)-$convexity can be extended from real co-ordinated space to complex co-ordinated space. Let C be a convex subset of real or complex co-ordinated space $X\times X$. However, only the first quadrant of the real or complex co-ordinated space is considered. The definition of $(\alpha,m)-$convex function can be extended for functions defined on C.

Definition 4.1. The function $f:C\subseteq X\times X\to\mathbb{R}$ is said to be co-ordinated $(\alpha,m)-$convex if

$$\begin{aligned} &f\left(tx+m(1-t)y,su+m(1-s)w\right)\\ &\quad\le t^\alpha s^\alpha f(x,u)+mt^\alpha\left(1-s^\alpha\right)f(x,w)+m\left(1-t^\alpha\right)s^\alpha f(y,u)\\ &\qquad+m^2\left(1-t^\alpha\right)\left(1-s^\alpha\right)f(y,w)\end{aligned} \tag{21}$$

for all $t,s\in[0,1]$ and $(x,y),(u,w)\in C$ with $(\alpha,m)\in[0,1]^2$.

Definition 4.1 can be modified as the following definition by considering $(\alpha,m)\in(0,1]^2$.

Definition 4.2. The function $f:C\subseteq X\times X\to\mathbb{R}$ is said to be co-ordinated $(\alpha,m)-$convex if

$$\begin{aligned} &f\left(tx+(1-t)y,su+(1-s)w\right)\\ &\quad\le t^\alpha s^\alpha f(x,u)+mt^\alpha\left(1-s^\alpha\right)f\left(x,\frac{w}{m}\right)+m\left(1-t^\alpha\right)s^\alpha f\left(\frac{y}{m},u\right)\\ &\qquad+m^2\left(1-t^\alpha\right)\left(1-s^\alpha\right)f\left(\frac{y}{m},\frac{w}{m}\right).\end{aligned} \tag{22}$$

for all $t, s \in [0,1]$ and $(x, y), (u, w) \in C$ with $(\alpha, m) \in (0,1]^2$.

We prove the Hermite-Hadamard type inequalities for $(\alpha, m)-$convex functions defined on C by using the above definition. First we prove the following lemma.

Lemma 4.3. *Let $f : C \subseteq X \times X \to \mathbb{R}$ be a co-ordinated $(\alpha, m)-$convex function. Then,*

$$\begin{aligned}
&\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+(1-s)d\right)dt\,ds \\
&\quad=\int_0^1\int_0^1 f\left(ta+(1-t)b,(1-s)c+sd\right)dt\,ds \\
&\quad=\int_0^1\int_0^1 f\left((1-t)a+tb, sc+(1-s)d\right)dt\,ds \\
&\quad=\int_0^1\int_0^1 f\left((1-t)a+tb,(1-s)c+sd\right)dt\,ds.
\end{aligned} \tag{23}$$

Proof. Consider the following integrals:

$$\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+(1-s)d\right)dt\,ds\,, \tag{24}$$

$$\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+(1-s)d\right)dt\,ds\,, \tag{25}$$

$$\int_0^1\int_0^1 f\left((1-t)a+tb, sc+(1-s)d\right)dt\,ds\,, \tag{26}$$

$$\int_0^1\int_0^1 f\left((1-t)a+tb,(1-s)c+sd\right)dt\,ds. \tag{27}$$

By substituting $x = ta + (1-t)b$ and $y = sc + (1-s)d$, $x = ta + (1-t)b$ and $y = (1-s)c + sd$, $x = (1-t)a + tb$ and $y = sc + (1-s)d$, $x = (1-t)a + tb$ and $y = (1-s)c + sd$ into (24), (25), (26) and (27) respectively, we complete the proof of Lemma 4.3. □

Theorem 4.4. *Suppose $f : C \subseteq X \times X \to \mathbb{R}$ is a co-ordinated $(\alpha, m)-$convex function with $(\alpha, m) \in (0,1]^2$. Let $(a,b),(c,d) \in C$ with $a \neq b$ and $c \neq d$. Suppose the mapping $(t,s) \mapsto f\left[ta + (1-t)b, sc + (1-s)d\right]$ is Lebesgue integrable on $[0,1]$ with $(t,s) \in [0,1]^2$. If $f \in L_2(\Delta)$, $0 \le a < b < \infty$ and $0 \le c < d < \infty$, then*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \int_0^1\int_0^1 f\left(ta+(1-t)b, sc+(1-s)d\right)dt\,ds \\ &\le \frac{1}{(\alpha+1)^2}\left[\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\right] \\ &+\frac{\alpha m}{4(\alpha+1)^2}\left[f\left(\frac{a}{m},c\right)+f\left(\frac{b}{m},c\right)+f\left(\frac{a}{m},d\right)+f\left(\frac{b}{m},d\right)\right. \\ &\left.+f\left(a,\frac{c}{m}\right)+f\left(b,\frac{c}{m}\right)+f\left(a,\frac{d}{m}\right)+f\left(b,\frac{d}{m}\right)\right] \\ &+\frac{\alpha^2 m^2}{4(\alpha+1)^2}\left[f\left(\frac{a}{m},\frac{c}{m}\right)+f\left(\frac{b}{m},\frac{c}{m}\right)+f\left(\frac{a}{m},\frac{d}{m}\right)+f\left(\frac{b}{m},\frac{d}{m}\right)\right]. \end{aligned} \tag{28}$$

Proof. Since f is a co-ordinated $(\alpha,m)-$convex function, then we have

$$\begin{aligned} &f\left(tx+(1-t)y, su+(1-s)w\right) \\ &\quad\le t^\alpha s^\alpha f(x,u)+mt^\alpha(1-s^\alpha)f\left(x,\frac{w}{m}\right)+m(1-t^\alpha)s^\alpha f\left(\frac{y}{m},u\right) \\ &\quad+m^2(1-t^\alpha)(1-s^\alpha)f\left(\frac{y}{m},\frac{w}{m}\right), \end{aligned} \tag{29}$$

$$\begin{aligned} &f\left(tx+(1-t)y, sw+(1-s)u\right) \\ &\quad\le t^\alpha s^\alpha f(x,w)+mt^\alpha(1-s^\alpha)f\left(x,\frac{u}{m}\right)+m(1-t^\alpha)s^\alpha f\left(\frac{y}{m},w\right) \\ &\quad+m^2(1-t^\alpha)(1-s^\alpha)f\left(\frac{y}{m},\frac{u}{m}\right), \end{aligned} \tag{30}$$

$$\begin{aligned} &f\left(ty+(1-t)x, su+(1-s)w\right) \\ &\quad\le t^\alpha s^\alpha f(y,u)+mt^\alpha(1-s^\alpha)f\left(y,\frac{w}{m}\right)+m(1-t^\alpha)s^\alpha f\left(\frac{x}{m},u\right) \\ &\quad+m^2(1-t^\alpha)(1-s^\alpha)f\left(\frac{x}{m},\frac{w}{m}\right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} &f\left(ty+(1-t)x, sw+(1-s)u\right) \\ &\quad\le t^\alpha s^\alpha f(y,w)+mt^\alpha(1-s^\alpha)f\left(y,\frac{u}{m}\right)+m(1-t^\alpha)s^\alpha f\left(\frac{x}{m},w\right) \\ &\quad+m^2(1-t^\alpha)(1-s^\alpha)f\left(\frac{x}{m},\frac{u}{m}\right). \end{aligned} \tag{32}$$

By adding (29), (30), (31) and (32) and integrating with respect to t and s on [0,1], we get

$$\int_0^1\int_0^1 f\left(tx+(1-t)y, su+(1-s)w\right)dtds+\int_0^1\int_0^1 f\left(tx+(1-t)y, sw+(1-s)u\right)dtds$$
$$+\int_0^1\int_0^1 f\left(tx+(1-t)y, sw+(1-s)u\right)dtds+\int_0^1\int_0^1 f\left(ty+(1-t)x, sw+(1-s)u\right)dtds$$
$$\leq \frac{1}{(\alpha+1)^2}\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right]$$
$$+\frac{\alpha m}{(\alpha+1)^2}\left[f\left(\frac{a}{m},c\right)+f\left(\frac{b}{m},c\right)+f\left(\frac{a}{m},d\right)+f\left(\frac{b}{m},d\right)+f\left(a,\frac{c}{m}\right)\right.$$
$$\left.+f\left(b,\frac{c}{m}\right)+f\left(a,\frac{d}{m}\right)+f\left(b,\frac{d}{m}\right)\right]+\frac{\alpha^2m^2}{(\alpha+1)^2}\left[f\left(\frac{a}{m},\frac{c}{m}\right)+f\left(\frac{b}{m},\frac{c}{m}\right)\right.$$
$$\left.+f\left(\frac{a}{m},\frac{d}{m}\right)+f\left(\frac{b}{m},\frac{d}{m}\right)\right].$$

Utilising Lemma 4.3, we get the second inequality in (28). To prove the first inequality in (28), we substitute $t=s=\frac{1}{2}$ and $\alpha=m=1$ into (21) to obtain

$$f\left(\frac{x+y}{2},\frac{u+w}{2}\right)\leq \frac{1}{4}\left[f(x,u)+f(x,w)+f(y,u)+f(y,w)\right]. \tag{33}$$

By substituting $x=ta+(1-t)b$, $y=(1-t)a+tb$, $u=sc+(1-s)d$ and $w=(1-s)c+sd$ into (33) and integrating with respect to s and t on [0,1], we obtain

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq \frac{1}{4}\int_0^1\int_0^1\left[f\left(tx+(1-t)y, su+(1-s)w\right)+f\left(tx+(1-t)y,(1-s)u+sw\right)\right.$$
$$\left.+f\left((1-t)x+ty, su+(1-s)w\right)+f\left((1-t)x+ty,(1-s)u+sw\right)\right]dtds.$$

Again, by Lemma 4.3, we complete the proof of the first inequality in (28). □

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