On the Toral Rank Conjecture and Variants of Equivariant Formality

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)

am Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg

vorgelegt von

Leopold Zoller

unter Betreuung von

Prof. Dr. Oliver Goertsches

Philipps Universität Marburg

On the Toral Rank Conjecture and Variants of Equivariant Formality

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)

am Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg
Hochschulkennziffer: 1180

vorgelegt von

Leopold Zoller

geboren am 17.06.1994 in München

Erstgutachter: Prof. Dr. Oliver Goertsches

Zweitgutachter: Prof. Dr. Bernhard Hanke

Eingereicht: 19.07.2019

Disputation: 17.10.2019

Abstract

We investigate the topological consequences of actions of compact connected Lie groups. Our focus lies on the *Toral Rank Conjecture*, which states that a suitable space X with an almost free T^r-action has to satisfy $\dim H^*(X;\mathbb{Q}) \geq 2^r$. We investigate various refinements of formality in an equivariant setting and show that they imply the TRC in several cases. Furthermore, we study the properties of the newly developed terminology with regards to possible implications, inheritance under elementary topological constructions, and characterizations in terms of higher operations on the equivariant cohomology. We also attack the problem of finding bounds for $\dim H^*(X;\mathbb{Q})$ in the spirit of the TRC outside of the formal context. Different lower bounds are constructed and applied in particular to the case of cohomologically symplectic spaces.

Zusammenfassung

Wir untersuchen topologische Konsequenzen von Wirkungen kompakter und zusammenhängender Liegruppen. Im Vordergrund steht dabei die *Toral Rank Conjecture* (TRC), welche besagt, dass für einen geeigneten Raum X mit einer fast freien T^r-Wirkung stets $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2^r$ gilt. Dazu studieren wir unterschiedliche Begriffe von Formalität im äquivarianten Kontext. Wir zeigen, dass diese in einigen Fällen die TRC implizieren und beleuchten die neu entwickelten Konzepte unter verschiedenen Gesichtspunkten: Neben grundlegenden Eigenschaften und möglichen Implikationen, untersuchen wir Vererbarkeitseigenschaften unter elementaren topologischen Konstruktionen, sowie mögliche Charakterisierungen anhand höherer Operationen auf der äquivarianten Kohomologie. Darüber hinaus widmen wir uns dem Problem der Abschätzung von $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q})$ im Sinne der TRC in einem allgemeineren Kontext. Wir konstruieren verschiedene untere Schranken für die Summe der Bettizahlen von X, welche keine zusätzlichen Formalitätsannahmen benötigen. Diese werden insbesondere auf kohomologisch symplektische Räume angewendet.

Acknowledgements

Mathematically, I want to thank Manuel Amann for the fruitful cooperation on many of the central questions of this thesis. I also want to thank Chistopher Allday, Mark Walker, and Bernhard Keller for helpful conversations. The author was supported by the German Academic Scholarship Foundation.

On a more personal note, I am sincerely grateful for the experience of my time as a PhD student in Marburg. My special thanks go to my supervisor Oliver Goertsches for making this period of my working life as enjoyable as it was and teaching me a great deal of things about mathematics and beyond. I believe you have succeeded very well in finding the right balance between giving me the freedom to work independently and guiding me along the way. Your contagious enthusiasm about mathematics has been and will continue to be an inspiration.

Regarding the pleasant working atmosphere I of course want to extend my thanks to all inhabitants of A8. Thank you, Ilka, for being a cornerstone of the (rightfully) self proclaimed greatest research group. Thanks to Pablo for teaching me the way of the great poet Michael T. – and training my reflexes. Thanks to my fellow PhD students Stefan, Landi, and Marius for fun times and also the occasional exchange of pity (which is hard to get from the higher-ups). Additionally I want to thank Marius for being my – highly competent – personal technical support. Thank you, Taki, for an equally fun and productive relationship; maybe with a slight tendency towards the fun side, but still very productive. I also want to add that you have a great taste in music. Thanks to my office colleagues Eugenia and Max for the very enjoyable atmosphere and to Birgit for being the good soul of the research group. I was also happy to get to know the next generation on A8. As for the previous generation, I want to thank Sam, Giovanni, Reinier, and Benjamin for making my early days in Marburg a very pleasant time. You all left too early. Outside of A8, I very much enjoyed the occasional exchanges with Ben, Sönke, and Andreas which helped to preserve a welcome amount of abstractness in my life as a member of “the smooth folk”.

Last but not least I want to thank my family and my friends for their continuous support. Above all, I want to thank Hannah who has been a supporting pillar for all this time. I am eternally grateful to have you in my life.

Contents

Introduction

Taking advantage of symmetries in order to understand geometric objects is an approach which is deeply rooted within our perception. It is therefore not surprising that the study of topological consequences of certain symmetries has become a classical problem in algebraic topology. In this thesis we are particularly interested in symmetries that manifest through actions of a compact, connected Lie group G. The last century has brought forth many fundamental results such as Borel localization and Atiyah-Bott-Berline-Vergne localization that enable the study of the geometry and topology of a G-space through the fixed point data of the action. This strategy can be a powerful tool in the right situations. For example if an action of a compact torus T^r on a (sufficiently nice) space X is equivariantly formal in the sense of [29], then the sum of all Betti numbers is determined by the fixed point set as

$$\dim H^*(X;\mathbb{Q}) = \dim H^*(X^{T^r};\mathbb{Q}).$$

An opposite extreme case, where the strategy of analysing X through fixed point data is least applicable, is the case of a free action. Still one finds topological restrictions, imposed on the space by the existence of such an action. Most famously, there is the Toral Rank Conjecture (TRC), which was posed by Steve Halperin. The first printed appearance known to the author is in [31] from the year 1985. The conjecture is usually formulated, a little more generally, for almost free actions, which means that all stabilizers are finite groups. The exact topological requirements for X vary throughout the literature but we will stick to the following version:

Conjecture. *Let T^r act almost freely on a compact Hausdorff space X, then*

$$\dim H^*(X;\mathbb{Q}) \geq 2^r.$$

In other words the conjecture states that in order for a torus to be able to act almost freely on X, it is required that X has at least as many holes, in a cohomological sense, as the torus, for which $\dim H^*(T^r;\mathbb{Q}) = 2^r$. The maximal number r such that T^r acts almost freely on X is also referred to as the toral rank of X – hence the name of the conjecture. The beauty of the conjecture lies in its simplicity as well as the fact that it is easy to verify by naive geometric intuition: for example if we imagine an S^1-action on X as a rotation symmetry on X, then the centre of the rotation remains fixed under the action. Thus in order for the action to be free, the centre must be removed from X, creating cohomology.

Maybe it is due to the beautifully simple nature of the problem that the conjecture has become one of the most prominent questions in rational homotopy theory and the cohomological theory of transformation groups. While it remains unsolved until today, there has been some success in attacking the conjecture for specific kinds of spaces: it was shown to hold e.g. for Hard Lefschetz spaces [3] or, more generally, c-symplectic spaces of Lefschetz type [48]. It was also also proved for certain 2-stage spaces in [37], which generalizes in particular the case of homogeneous spaces due to [31] and 2-step nilpotent nilmanifolds from [13]. Another possible axis on which the conjecture can be approached is by proving possibly weaker lower bounds on $\dim H^*(X)$. A classical result (see [4, Theorem 4.4.3]) in this direction is the inequality

$$\dim H^*(X;\mathbb{Q}) \geq \begin{cases} 2r \\ 2(r+1) & \text{if } r \geq 3 \end{cases}.$$

which has seen slight improvement in [5], where a linear lower bound of slope 8/3 is given. The TRC is one of the guiding problems of this thesis and we will try to attack it in both of the above ways: we will prove it under specific conditions and also provide lower bounds on $\dim H^*(X;\mathbb{Q})$ in the general case. Our main result with regards to the latter aspect is

Theorem A. *Let X be a compact Hausdorff space of formal dimension n with an almost free T^r-action.*

(i) Let b be the first Betti number of X and $m = \max(b-r, 0)$. Then

$$\dim H^*(X;\mathbb{Q}) \geq \min_{k=m,\dots,b} \frac{n+r-1}{n-r+1} 2k + 2^{b-k}.$$

(ii) Let k be the degree corresponding to the first nontrivial odd Betti number. Then

$$\dim H^*(X;\mathbb{Q}) \geq \frac{n+r-1}{n-r+1} 2 \dim H^{<k}(X;\mathbb{Q}).$$

We also discuss applications of the above theorem in the specific scenario of actions on c-symplectic spaces, where it turns out to be particularly effective. We obtain bounds that are, in a sense, quadratic and actually stronger than the TRC in a certain range of values of r. In particular they imply the TRC for c-symplectic spaces of dimension ≤ 8.

With regards to the approach of proving the TRC under specific conditions, we will explore actions that satisfy certain formality properties. This may sound contradictory to the aforementioned fact that free actions and equivariantly formal actions are quite at the opposite ends of the spectrum. Indeed, we call a G-action on X equivariantly formal if the Borel fibration

$$X \to X_G \to BG$$

turns $H^*_G(X;\mathbb{Q}) := H^*(X_G;\mathbb{Q})$ into a free module over $H^*(BG;\mathbb{Q})$, which, in particular, implies the existence of a fixed point. The condition got its name in the celebrated paper [29]. However, the fact that the name stuck to the condition has to be attributed rather to the success of the tools from [29] than to its actual connection to the classical (nonequivariant) notion of formality.

The two are actually rather unrelated: we call a space X formal if there is a quasi-isomorphism $(\Lambda V, d) \to (H^*(X), 0)$ of commutative differential graded algebras, where ΛV is the minimal Sullivan model of X. This condition has far reaching consequences in that it allows us to reconstruct the rational homotopy type of a nilpotent space X directly from its cohomology. Also, formal spaces are in a certain sense less complicated because formality enforces the vanishing of higher operations on the cohomology known as Massey products. A nice example which illustrates their meaning is given by the Borromean rings: their linking can be detected by a nontrivial triple Massey product of the complement of an embedding into S^3 (see e.g. [53, 2.6 in Chapter 2]).

While some connections between the notions of formality and equivariant formality do exist (see [12] for a result in the specific case of isotropy actions), it is fair to say that the choice of name is suboptimal. It is thus not surprising that, over time, other concepts arose which tried to stay more true to the classical notion from rational homotopy theory. In [46], Lilliwhite considers actions for which the homotopy quotient X_G is a formal space. There has also been a redefinition by Scull in [59], [58] (see also [23] for the discrete case), taking into account the richer structure provided by all the isotropy groups. We study possible refinements of the existing notions and investigate implications within the new terminology. Two main concepts eventually arise from the discussion, namely those of *$\mathcal{MOD}$-formal* actions and actions with *formal core*. Both are simultaneous generalizations of all three of the previously mentioned notions and are, in particular, interesting in the realm of free actions. With regards to our original motivation, we have the following central

Theorem B. *The TRC holds for $\mathcal{MOD}$-formal actions and actions with formal core.*

This is supplemented by an extensive study of these types of actions. In particular we investigate inheritance under elementary constructions and give several criteria under which the conditions hold. Additionally, we provide classes of examples as well as many counterexamples, sharpening the terminology. More specific details are provided below where the structure of the thesis is summarized.

To conclude the discussion on equivariant formality, we want to point out that our new definitions are by no means replacements for the original notion of equivariant formality. Also we want to mention that the latter has a natural generalization in the form of Cohen–Macaulay actions (see [28]), which is seemingly more or less independent from our terminology. In the author's opinion, the most natural definition of equivariant formality would be to find an algebraic category which is equivalent to the category of G-spaces up to a certain rational notion of equivalence and then define equivariant formality through the algebraic model, mirroring the nonequivariant definition. This is the approach of [59], [58], which however comes with large technical difficulties and is not suited for our applications.

As a last point, we want to comment on the methodology throughout the thesis. In the case of free actions, the internal isotropy data is trivial so there is no obvious geometric way of exploiting the existence of a free action. For this reason, the arguments usually come down to algebraic considerations in suitable models of the equivariant cohomology $H^*_G(X)$. The choice of model, i.e. algebraic structure which computes the equivariant cohomology, is very important: although all of them extract certain information on the action from the space X_G, some models are more apt to display certain information than others. It is often beneficial to pass to a smaller model while losing information. In this spirit, many of our results stem from the realm of graded modules, only exploiting the module structures provided by the map $X_G \to BG$ even though we start with a much richer structure.

Most notably we profit from two recent developments, the first of which are the solved Boij–Söderberg conjectures (see [17], [16], [10]). They give a precise description, up to scalars, of the possible Betti numbers of graded modules over polynomial rings. The second essential ingredient is the Buchsbaum-Eisenbud-Horrocks conjecture, which is still unsolved in full generality but saw serious progress in the recent paper [61]. It predicts lower bounds on the ranks in certain free resolutions and has a strong connection to the TRC, which was used e.g. in [60]. Oftentimes, the aforementioned discussion of formality is what allows us to draw the connection to these algebraic results on graded modules.

We point out that formality is essential to the success of reducing the information to the level of module structures: in [36], an example of a differential graded module over a polynomial ring in r variables was given, such that the total rank of the module is less than 2^r while the cohomology has finite length. This essentially shows that the arising module structures alone do not contain enough information to deduce the TRC in general. However, the counterexample is not topologically realizable and is hence not a counterexample to the TRC. Still, a proof of the TRC – which the author strongly believes to exist – will need to make use of more intricate structures.

Summary and structure of the thesis. The first chapter is devoted to setting up the machinery that converts geometrical information of the group action into algebraic models. In Section 1.1 we begin with a short recollection on equivariant cohomology and central results of the theory. Subsequently, in 1.2, Sullivan models for equivariant cohomology are discussed. We assume the reader is familiar with the basic theory of Sullivan models and rational homotopy theory. Section 1.3 introduces minimal models of differential graded modules and the Hirsch–Brown model of an action. While the results in this section can mostly be found within the existing literature, we have provided simplified proofs in the generality which is suited to our applications. We also provide explicit constructions which will be useful throughout the later parts of the thesis. We aimed for a rather elementary presentation while still covering the necessary theoretical aspects. This spirit carries over to the final section 1.4 of the chapter, where we discuss the notions of A_∞- (resp. C_∞)-algebras and modules. They provide alternative models which are very well suited to display formality properties. Again, the aim is to lay the technical foundation for later chapters and introduce terminology. While the section is meant as a concise introduction, we have provided proofs only when the respective statements are elusive in the literature.

The second chapter is devoted to the discussion of formality in an equivariant setting. Section 2.1 forms the heart of the chapter. Here the terminology is developed and fundamental properties are discussed. We go on to study the behaviour of the arising concepts under certain elementary topological constructions in Section 2.2. More specifically we discuss products, certain gluing constructions, such as the equivariant connected sum, and restriction to subgroups. As a next step, in 2.3, we investigate possible characterisations of our formality properties in the language of higher operations of the minimal C_∞-model. Throughout the chapter, we have avoided to interrupt the presentation through the often rather lengthy but necessary (counter)examples and instead have collected them in the final Section 2.4. Also we show that a large class of examples of actions fulfilling the central formality related properties is provided by actions on Hard Lefschetz spaces and symplectic actions on Lefschetz type symplectic manifolds.

In the third and last chapter we focus on applications with regards to the TRC and, more generally, restrictions on the Betti numbers of G-spaces. The Chapter is divided into two sections, the first of which discusses consequences of the formality properties from the second chapter. We start Section 3.1 by discussing the connections between the TRC and the Buchsbaum-Eisenbud-Horrocks conjecture and prove the TRC for $\mathcal{MOD}$-formal actions and actions with formal core. We also obtain a structural result on the case where the bound of the TRC is sharp. This is followed by an investigation of the TRC in low dimensions. In 3.1.3, we take a step back from the dimensional bounds of the TRC and instead, after giving a short introduction to Boij–Söderberg theory, consider the topological consequences of the latter. We make the immediate observation that the vector of Betti numbers of a $\mathcal{MOD}$-formal T-space is contained in a certain rational cone, and go on to investigate the question of realizing integral points of the cone as Betti numbers of T-spaces. Our final objective in the formal realm is to prove the TRC for spaces that are elliptic and formal (in the classical nonequivariant sense). We noticed that this has already been observed in [44] and we do not claim originality of the result. However, it essentially builds upon a more general structural observation on formal elliptic spaces, which we did not find explicitly stated. We feel like there is some value to a compact display of the material and this seems like a fitting place to do so. The second section 3.2 is concerned with results on the Betti numbers outside of the formal setting. We use Boij–Söderberg theory to derive a lower bound on the sum of Betti numbers of an almost free T-space under assumptions on certain Betti numbers. This general bound is applied to the specific case of cohomologically symplectic spaces providing a quadratic lower bound on the Betti numbers.

We point out that Section 3.2 consists of results that we already published in the paper [62]. This also includes parts of sections 1.3 and 3.1.3. Parts of the remaining material, in particular of Chapter 2 and Section 3.1, were developed in cooperation with Manuel Amann.

Chapter 1

Equivariant cohomology and its models

1.1 The Borel fibration and equivariant cohomology

Throughout the thesis, G will denote a compact and connected Lie-Group and X will be a topological space with a continuous G-action. The G-spaces considered are assumed to be Hausdorff, connected, and have finite-dimensional rational cohomology. The latter always refers to singular cohomology. Coefficients will be taken in the field $\mathbb{Q}$ if not stated otherwise and will be suppressed in the notation.

Our main tool for studying topological aspects of a G-action on X is the Borel fibration

$$X \to X_G \to BG$$

where $BG = EG/G$ for some contractible space EG on which X acts freely and X_G is the orbit space of the diagonal action on $EG \times X$. The map $X_G \to BG$ is given by projection onto the first component.

The cohomology of X_G is called the equivariant cohomology of X and denoted by $H^*_G(X)$. The map $X_G \to BG$ induces a natural $H^*(BG)$-module structure on $H^*_G(X)$. The ring $H^*(BG)$ is a polynomial algebra with generators of even degree and will be denoted by R. In case G is a torus, the generators of R are all of degree 2. The assumption $\dim H^*(X) < \infty$ ensures that $H^*_G(X)$ is finitely generated as an R-module (see [4, Proposition 3.10.1]). Equivariant cohomology provides an essential link between geometry and algebra and captures many important properties of the group action. For example, the information of an action being almost free (meaning that all isotropy groups are finite) can immediately be read off from the algebraic data in case X is compact.

Theorem 1.1.1. *Assume X is compact. The G-action on X is almost free if and only if*

$$\dim H^*_G(X) < \infty.$$

This is a classical theorem due to Hsiang. In the torus case it is actually a direct consequence of the more general Lemma 1.1.6 below. We have the following supplementary proposition. Here $\mathrm{fd}(X)$ denotes the formal dimension, which is the highest integer n such that $\dim H^n(X) \neq 0$.

Proposition 1.1.2. *Assume G acts almost freely on X.*

(i) If X is compact, we have $\mathrm{fd}(X_G) = \mathrm{fd}(X) - \dim G$.

(ii) If X is a manifold, $X_G \to X/G$ induces an isomorphism in cohomology.

(iii) Suppose X is compact, $H^*(X)$ satisfies Poincaré duality, and $H^1(X) = 0$. Then $H^*_G(X)$ satisfies Poincaré duality.

Part (i) is an easy observation using the Serre spectral sequence of the fibration (up to homotopy) $G \to X \to X_G$. For (ii) see [21, Theorem 7.6]. Finally, (iii) follows by applying [19, Theorems 3.6 and 4.3] to the Borel fibration $X \to X_G \to BG$ and using that BG is a Gorenstein space by [19, Proposition 3.4]. Rather contrary to the above case, where $H^*_G(X)$ is a torsion module, we formally recall the following classical

Definition 1.1.3. The G-action on X is equivariantly formal if one of the following equivalent conditions holds:

(i) $H^*_G(X)$ is a free R-module.

(ii) $H^*_G(X) \cong R \otimes H^*(X)$ as modules.

(iii) The Serre spectral sequence of the Borel fibration is totally non-homologous to zero (TNHZ), i.e. it collapses at E_2.

(iv) The map $H^*_G(X) \to H^*(X)$ is surjective.

The equivalence of (ii), (iii), and (iv) follows from standard considerations on the Serre spectral sequence of the Borel fibration. For the equivalence of the condition (i) see [4, Cor. 4.2.3], [27, Prop. 2.3]. A central theorem in the cohomological theory of transformation groups is the Borel localization theorem. For any multiplicatively closed subset $S \subset R$, we set

$$X^S = \{x \in X \mid S^{-1}H^*_G(G \cdot x) \neq 0\}.$$

Here $S^{-1}M$ denotes the localization of an R-module M at S. The following version of the localization theorem is [4, Theorem 3.2.6], where $\overline{H}^*_G(X) = \overline{H}^*(X_G)$ denotes (equivariant) Alexander-Spanier cohomology.

Theorem 1.1.4. *Suppose X is compact, let $Y \subset X$ be a closed invariant subspace, and let $S \subset R$ be a multiplicative subset. Then the localized map*

$$S^{-1}\overline{H}^*_G(X, Y) \longrightarrow S^{-1}\overline{H}^*_G(X^S, Y^S)$$

is an isomorphism.

Remark 1.1.5. The compactness condition above can be replaced by other finiteness conditions. We point the reader towards [4, Section 3.2] for details. In order to obtain a result on singular cohomology, which is the theory we will be working with, one would need further assumptions which assure that Alexander-Spanier and singular cohomology agree on the occurring spaces. This is known to hold e.g. if the spaces are locally contractible.

For us, the important application of Borel localization is the lemma below, which only uses the case where $Y = \emptyset$ and $X^S = \emptyset$. In this case, Borel localization actually has an easy proof which works for singular cohomology, provided the existence of tubular neighbourhoods (see e.g. the first part of the proof of [35, Theorem III.1]). This is assured by working with Hausdorff spaces since a compact Hausdorff space is Tychonoff and those spaces admit tubular neighbourhoods (see [11, Theorem 5.4]). This leads to the assumption of X being compact (and Hausdorff) which appears frequently throughout the thesis. Without exception, the results carry over if we replace the assumption by suitable other conditions as mentioned above.

Finally, we point out that the lemma below is proved in [25, Proposition 5.1]. We will give a short proof nonetheless, for the sake of a compact presentation of this essential ingredient and to provide clarity on the necessary topological requirements.

Lemma 1.1.6. *Let X be a compact space with an action of a torus T. Then the codimension of $H_T^*(X)$, i.e. the height of its annihilator as an R-module, is the minimal dimension among the orbits.*

Proof. If X has an orbit $O = T/H$ of dimension c, then $\ker(R \to H_T^*(O) = H^*(BH))$ is generated by c linearly independent generators of R^2 and is thus an ideal of height c. Since this map factors through $H_T^*(X)$, it follows that $\mathrm{Ann}(H_T^*(X)) = \ker(R \to H_T^*(X))$ is contained in this ideal and is therefore of height $\leq c$.

Assume now that c is the minimal dimension among the orbits and let $\mathfrak{p}$ be a prime ideal of height $c-1$. By the previous considerations, $\ker(R \to H_T^*(O))$ is not contained in $\mathfrak{p}$ for any orbit $O \subset X$. Hence, setting $S = R\backslash\mathfrak{p}$, we have $X^S = 0$. Borel localization implies $S^{-1}H_T^*(X) = 0$ which means that every element from $H_T^*(X)$ is annihilated by some element in S. Since $H_T^*(X)$ is finitely generated, we obtain an element in $S \cap \mathrm{Ann}(H_T^*(X))$ by taking products. We have shown that $\mathrm{Ann}(H_T^*(X))$ is not contained in $\mathfrak{p}$. □

1.2 Sullivan models

While cohomology is a powerful tool to extract information from the Borel fibration, we want to go deeper to the cochain level. Our main tool is the language of rational homotopy theory and in particular commutative differential graded algebras (cdga) and Sullivan models. We assume the reader is familiar with those theories and refer to [20] for missing definitions. The point here is to preemptively sort out technical difficulties that arise in later discussions, as well as to comment on the problem of realizing algebra through geometry. We fix some notation: if V is a graded vector space, then ΛV will denote the free unital commutative graded algebra on V. If not stated otherwise, all cdgas will be assumed to be non-negatively graded. We will furthermore assume all (cd)gas to be unital which means that they come with a fixed multiplicative unit element which is preserved by morphisms.

One of the central objects in this thesis is the following construction: we fix a Sullivan minimal model of the algebra $A_{pl}(BG)$ of the piecewise linear forms on BG. It is of the form $(R, 0)$ where $R = H^*(BG)$. The fibration induces a map $(R, 0) \to A_{pl}(X_G)$ for which we choose a relative minimal model. This results in an extension sequence

$$(R, 0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$$

where the first map is the inclusion of the relative minimal model and the second one is the projection onto ΛV (with the induced differential) via the canonical augmentation of

R that sends R^+ to 0 and is the identity on $R^0 = \mathbb{Q}$. Then said projection is actually a Sullivan model for $X \to X_G$. We will refer to such an extension sequence as a (minimal) model for the Borel fibration.

Remark 1.2.1. (i) Throughout the thesis, we will work with a fixed minimal model $R \to A_{pl}(BG)$. As the minimal model of the Borel fibration will be a central object, we want to point out that it is independent of the choices made in the construction: if $(R', 0) \to A_{pl}(BG)$ is another minimal model and $(R' \otimes \Lambda V', D')$ is a relative minimal model for $R' \to A_{pl}(X_G)$, then there is an isomorphism $(R \otimes \Lambda V, D) \cong (R' \otimes \Lambda V', D')$ that restricts to an isomorphism $R \cong R'$. This follows from the basic homotopy theory of cdgas: by uniqueness of the minimal model there is an isomorphism $R \cong R'$ such that $\varphi_0 \colon R \to A_{pl}(X_G)$ and $\varphi_1 \colon R \cong R' \to A_{pl}(X_G)$ are homotopic. Let $h \colon R \to A_{pl}(X_G) \otimes (t, dt)$ be a homotopy such that $\varphi_0 = p_0 \circ h$ and $\varphi_1 = p_1 \circ h$, where p_i is evaluation at $t = i$. Consider the commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & A_{pl}(X_G) \otimes (t, dt) \\ \downarrow & \nearrow & \downarrow p_0 \\ R \otimes \Lambda V & \xrightarrow{\varphi_0} & A_{pl}(X_G) \end{array}$$

with the dashed arrow obtained by relative lifting (see e.g. [20, Lemma 14.4]). As a result we deduce that by changing the homotopy class of φ_0, through composing the dashed arrow with p_1, we can obtain a relative minimal model for φ_1. Then the claim follows from uniqueness of the relative minimal model.

(ii) The Sullivan model $(R \otimes \Lambda V, D)$ for X_G is not minimal in general (however it is always minimal for torus actions on simply-connected spaces). Still, it is usually our preferred choice of model since it comes with a fixed R-module structure. Equivariant maps induce morphisms that respect this structure: given an equivariant map between G-spaces, we obtain a strictly commutative diagram between the Borel fibrations. If minimal models of the Borel fibrations are constructed as above, then one can show through relative lifting (see [20, Prop. 14.6]) that there is a strictly commutative diagram

$$\begin{array}{ccccc} (R, 0) & \longrightarrow & (R \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, d) \\ \downarrow \mathbf{1}_R & & \downarrow & & \downarrow \\ (R, 0) & \longrightarrow & (R \otimes \Lambda W, D) & \longrightarrow & (\Lambda W, d) \end{array}$$

in which the rows are the relative minimal models and the horizontal morphisms are Sullivan representatives for the corresponding maps between the Borel fibrations. If such structure is present, we will usually assume the morphisms to be of this type.

As we will often care about R-module structures, the following lemma will be useful throughout the thesis. By an R-cdga we mean a morphism $(R, 0) \to B$ of cdgas. We will often just write B in case the specific morphism is not important or clear from the context.

Lemma 1.2.2. *(i) For any cdga B, two morphisms $(R, 0) \to B$ are homotopic if and only if they induce the same map on cohomology.*

(ii) Consider any R-cdga B and a relative Sullivan algebra $(R, 0) \to (R \otimes \Lambda V, D)$. Then any morphism $(R \otimes \Lambda V, D) \to B$ such that $\varphi^* \colon H^*(R \otimes \Lambda V) \to H^*(B)$ respects the R-module structures is homotopic to a morphism of R-cdgas.

Proof. For the proof of (i), let f, g be two such morphisms and $R = \Lambda(X_1, \ldots, X_r)$. Then, by assumption, for any X_i there is some $v_i \in B$ satisfying $D(v_i) = g(X_i) - f(X_i)$. We can define a homotopy $h \colon R \to B \otimes \Lambda(t, dt)$, with t of degree 0, by setting

$$h(X_i) = f(X_i) + (g(X_i) - f(X_i))t - v_i dt.$$

In the situation of (ii), we have a diagram

$$\begin{array}{ccc} R & & \\ \downarrow & \searrow & \\ R \otimes \Lambda V & \longrightarrow & B \end{array}$$

which commutes on the level of cohomology. By (i) it is homotopy commutative so the claim follows by extension of homotopies (see e.g. [21, Proposition 2.22]) □

Remark 1.2.3. We will frequently make use of the fact that we can obtain the model $(\Lambda V, d)$ of a space X from the (preferred) model of X_G by forming another extension sequence

$$(R \otimes \Lambda V, D) \to (R \otimes \Lambda V \otimes S, D) \to (S, 0),$$

where $S = \Lambda(s_1, \ldots, s_r)$ is generated in odd degrees and D maps the s_i bijectively to the generators of R. In fact, we have $S = H^*(G)$. Sending R and S to 0 yields a quasi-isomorphism $(R \otimes \Lambda V \otimes S, D) \simeq (\Lambda V, d)$. The extension sequence above is a model for $G \to X \to X_G$, which is a fibration up to homotopy equivalence.

For free torus actions, it is also possible to pass from algebra to geometry: given a base space Y, any choice of r classes from $H^2(Y)$ defines a morphism $R \to H^*(Y)$ which lifts (uniquely up to homotopy by Lemma 1.2.2) to a map $R \to A_{pl}(Y)$. In particular we obtain a unique relative minimal model of the form $R \to R \otimes \Lambda V$. In the proposition below, we expand on the discussion in [21, Prop. 7.17] and show that such algebraic data is always realizable by the Borel fibration of a free torus action.

Proposition 1.2.4. *Let $R = \Lambda(X_1, \ldots, X_r)$ with X_i in degree 2, $(\Lambda V, d)$ be a finite type minimal Sullivan algebra, and*

$$(R, 0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$$

be an extension sequence with maps given by canonical inclusion and projection. Then there exists a free T^r-action on some space X such that the above sequence is the minimal model of the associated Borel fibration. If the cohomology of the middle cdga is finite-dimensional, then we can take X to be compact. If additionally $H^*(\Lambda V, d)$ is simply-connected, satisfies Poincaré duality with fundamental class in degree n, and $n - r$ is not divisible by 4, then we can take X to be a compact simply-connected manifold.

In the above proposition, $(\Lambda V, d)$ is the minimal model of X. In particular, the cohomology of X is not necessarily finite-dimensional, which is an exception to the general assumption of all G-spaces having finite-dimensional cohomology. Note that even if $\dim H^*(\Lambda V, d) < \infty$, we can only choose X to be compact if also the cohomology of $(R \otimes \Lambda V, D)$ is finite-dimensional as otherwise Theorem 1.1.1 would be violated.

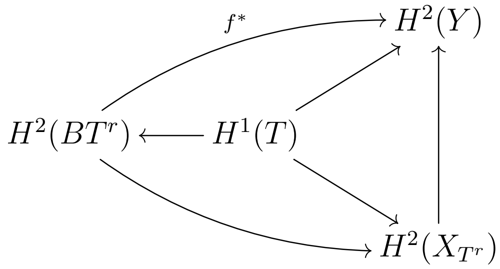
Proof. Let Y be a CW-complex with Sullivan model $(R \otimes \Lambda V, D)$. We find an integer k such that the cohomology classes of the kX_i in $H^2(Y;\mathbb{Q})$ come from classes in $H^2(Y;\mathbb{Z})$. Those uniquely determine the homotopy class of a map $f\colon Y \to K(\mathbb{Z}^r, 2) = BT^r$. The minimal model $R \to A_{pl}(BT^r)$ can be chosen in a way such that the canonical inclusion $R \to R \otimes V$ is a Sullivan model for f. Pulling back the universal principal bundle along f yields a principal bundle

$$T^r \to X \to Y.$$

We consider the following commutative diagram of principal T^r-bundles

$$\begin{array}{ccccccc}
T^r & \longleftarrow & T^r & \longrightarrow & T^r & \longrightarrow & T^r \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
ET^r & \longleftarrow & ET^r \times X & \longrightarrow & X & \longrightarrow & ET^r \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
BT^r & \longleftarrow & X_{T^r} & \longrightarrow & Y & \longrightarrow & BT^r
\end{array}$$

in which the left morphism of principal bundles is given by projection on the first component, the central one is projection on the second component, and the right one is the pullback diagram induced by f. Note that the central morphism actually consists of weak equivalences so $(R \otimes \Lambda V, D)$ is a model for X_{T^r}. By naturality of the spectral sequence, the transgressions of the associated Serre spectral sequence commute with the maps between the base spaces. Thus the inner triangles in the diagram



are commutative. But the transgression $H^1(T^r) \to H^2(BT^r)$ is actually an isomorphism so the whole diagram commutes. Since R is generated in degree 2, it follows by Lemma 1.2.2 that the canonical inclusion $R \to R \otimes \Lambda V$ is not only a model for f but also for the Borel fibration of X.

If $(R \otimes \Lambda V, D)$ has finite-dimensional cohomology we can choose Y to be a finite CW-complex and homotope f such that it has image in some compact skeleton. As a consequence, X will be compact. If (V, d) is additionally simply-connected and satisfies Poincaré duality with fundamental class of degree n, then $(R \otimes \Lambda V, D)$ satisfies Poincaré duality with fundamental class in degree $n - r$ (by the same reasoning as Proposition 1.1.2 (iii)). Then by [21, Theorem 3.2] we can choose Y as a compact simply-connected manifold. As before we can homotope f to have image in a compact skeleton which is in fact contained in some $(\mathbb{C}P^N)^r$ for N large enough. If we go on to homotope f to a smooth map, then X will be a smooth compact simply-connected manifold. □

1.3 The Hirsch–Brown model

We will frequently leave the realm of cdgas and consider only the R-module structure. While much information is lost in the process, this has the advantage of the resulting models being a lot smaller and allowing a more direct access to the (nonequivariant) cohomology of the underlying G-space.

We begin with the basic definitions leading to a notion of minimal model in the module setting and go on to prove its essential properties. The material is certainly not new: these kinds of models were introduced in [6], [20] (see in particular Exercise 8 of Chapter 6 for the notion of minimality) and were also described in [4, Appendix B]. Furthermore, the concept of minimality has been discussed in the more general framework of model categories in [55], which applies to the setting of differential graded modules (cf. [34]). More explicit applications of the general theory to differential graded modules were discussed in [56]. However, the theory needed in this thesis can be developed rather quickly, which we do for the convenience of the reader (the proofs are mainly simplified versions of those for the corresponding statements for cdgas). We feel like the available sources either do not fully cover our point of view or operate in greater generality, which makes it hard to occasionally rely on the explicit constructions. As those will be essential, we provide an alternative method of construction of the minimal model in the second part of the section.

1.3.1 Definitions and properties

Let k be some ground field and (R, d) be a cdga over k.

Definition 1.3.1.

- A graded R-module is a graded k-vector space M together with an R module structure for which the multiplication map $R \otimes M \to M$ is a graded map of degree 0.
- If a graded R-module carries a differential D of degree 1 such that $D(a \cdot m) = da \cdot m + (-1)^k a \cdot Dm$ for any $a \in R^k$, then we call (M, D) a differential graded R-module (dg$R$m).
- A morphism $f\colon (M, D_M) \to (N, D_N)$ of dg$R$ms is a degree 0 map which is R-linear and commutes with the differentials.
- A quasi-isomorphism of dg$R$ms is a morphism that induces an isomorphism in cohomology.
- Two morphisms $f, g\colon (M, D_M) \to (N, D_N)$ are homotopic if there is an R-linear map $h\colon M \to N$ of degree -1 which satisfies $f - g = D_N h + h D_M$.

In the definition above we restricted ourselves to morphisms of degree 0 to simplify the language throughout the thesis. Also, in what follows we shall work under the following assumptions: the cdga R is concentrated in non-negative degrees and is simply-connected (i.e. $R^0 = k$ and $R^1 = 0$). Furthermore all differential graded R-modules are assumed to be bounded from below (i.e. there is some $k \in \mathbb{Z}$ for which $M^{\leq k} = 0$). We denote by $\mathfrak{m} = R^+$ the maximal homogeneous ideal in R.

Definition 1.3.2. A dg$R$m (M, d) is minimal if $M = R \otimes V$ is a free module over some graded k-vector space V and $\mathrm{im}(d) \subset \mathfrak{m}M$.

Proposition 1.3.3 (Existence). *For every dgRm (M, D) there is a minimal model, i.e. a quasi-isomorphism $\varphi\colon (R \otimes V, d) \to (M, D)$ from a minimal dgRm into (M, D).*

Proof. Assume inductively that for some n we have constructed a map

$$\varphi_n\colon (R \otimes V^{\leq n}, d) \longrightarrow (M, D)$$

from a minimal dg$R$m, which induces an isomorphism on cohomology in degrees up to n and is injective in degree $n+1$. Let $A = \mathrm{coker}(H^{n+1}(\varphi_n)\colon H^{n+1}(R \otimes V^{\leq n}) \to H^{n+1}(M, D)$, set $d = 0$ on A, and extend φ_n to $R \otimes A$ by linearly choosing representatives from $(\ker D)^{n+1}$. The resulting map $\varphi'_{n+1}\colon R \otimes (V^{\leq n} \oplus A) \to (M, D)$ induces an isomorphism in degrees up to $n+1$. To achieve injectivity in degree $n+2$, take $B = \ker H(\varphi'_{n+1})^{n+2}$, considered as a vector space in degree $n+1$, and extend d to B by choosing a basis of B and mapping that basis onto representatives in $(\ker d)^{n+2}$. Setting $V^{n+1} = A \oplus B$, we can then extend φ'_{n+1} to a map $\varphi_{n+1}\colon (R \otimes V^{\leq n+1}, d) \to (M, D)$. As R is simply-connected, the introduction of B in degree $n+1$ does not generate any new cohomology in degree $n+2$. Thus φ_{n+1} is injective on degree $n+2$ cohomology. Also observe that $d(B) \subset (R \otimes (V^{\leq n} \oplus A))^{n+2}$ is contained in $\mathfrak{m}(V^{\leq n} \oplus A)$ so this inductive construction indeed yields a minimal model. □

Before discussing the fundamental properties of minimal models, we want to point out that being homotopic is an equivalence relation on the set of morphisms between two dg$R$ms and is furthermore compatible with composition of morphisms. This is easily verified and one of the points where the theory of dg$R$ms is much easier than that of cdgas. Also, we will need the following formulation of homotopy: consider the complex $I = (\langle p_0, p_1, p \rangle_k, d)$, with p_0, p_1 in degree 0, p in degree 1 and $dp_0 = p$, $dp_1 = -p$, $dp = 0$. It comes with two projections $i_j\colon I \to k$, $j = 0, 1$ defined by sending p_j to 1 and the other generators to 0. A homotopy h between two morphisms $f, g\colon M \to N$ induces a morphism

$$H\colon M \to I \otimes N$$

of dg$R$ms by setting $H(x) = p \otimes h(x) + p_0 \otimes f(x) + p_1 \otimes g(x)$. In turn any such H defines a homotopy between $(i_0 \otimes \mathbf{1}_N) \circ H$ and $(i_1 \otimes \mathbf{1}_N) \circ H$.

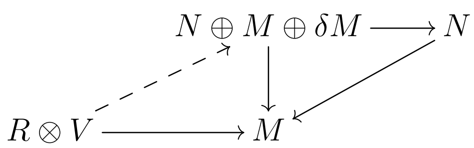
Proposition 1.3.4 (Lifting). *Let $f\colon (N, D_N) \to (M, D_M)$ be a quasi-isomorphism and $\varphi\colon (R \otimes V, d) \to (M, D_M)$ a morphism from a minimal dgRm.*

(i) If f is surjective there is $\tilde{\varphi}\colon (R \otimes V, d) \to (N, D_N)$ such that $f \circ \tilde{\varphi} = \varphi$.

(ii) There exists $\tilde{\varphi}\colon (R \otimes V, d) \to (N, D_N)$, unique up to homotopy, such that $f \circ \tilde{\varphi}$ is homotopic to φ.

Proof. We prove the existence of $\tilde{\varphi}$ in (i). Assume inductively that we have constructed the lift $\tilde{\varphi}\colon R \otimes V^{\leq n} \to (N, D_N)$ for some n such that $\varphi|_{R\otimes V^{\leq n}} = f \circ \tilde{\varphi}$. For any x in a fixed basis of V^{n+1}, we have $dx \in R \otimes V^{\leq n}$ due to the fact that $(R \otimes V, d)$ is minimal and R is simply-connected. As a result, $\tilde{\varphi}(dx)$ is defined and closed. We have $f(\tilde{\varphi}(dx)) = \varphi(dx) = D_M\varphi(x)$ so $\tilde{\varphi}(dx)$ is exact because f is injective on cohomology. Choose $y \in N$ with $D_N y = \tilde{\varphi}(dx)$ and observe that $f(y) - \varphi(x) \in \ker D_M$. We claim that we find $z \in \ker D_N$ with $f(z) = f(y) - \varphi(x)$. If we do, we can extend $\tilde{\varphi}$ by setting $\tilde{\varphi}(x) = y - z$ which completes the induction. Since f is surjective on cohomology, we find some $a \in \ker D_N$, $b \in M$ such that $f(a) = f(y) - \varphi(x) + D_M b$. Also f is surjective so we find $c \in N$ with $f(c) = b$. Thus the element $z = a - D_N c$ has the desired properties.

The existence in (ii) follows from (i) in the following way: Consider the acyclic module $(M \oplus \delta M, \delta)$, where the differential is an isomorphism $\delta\colon M \cong \delta M$ and vanishes on δM. This maps surjectively onto (M, D_M) by $m \mapsto m$, $\delta m \mapsto D_M m$. So we obtain a surjective quasi-isomorphism $(N \oplus M \oplus \delta M, D_N \oplus \delta) \to (M, D_M)$. Now (i) yields the dashed arrow in the diagram



in which the left hand triangle commutes and the right hand triangle commutes up to homotopy. Composing with the arrow to N yields the desired lift.

Finally we argue that the lift is unique up to homotopy, which will be achieved by lifting homotopies. We start by the observation that two maps $\tilde{\varphi}_1, \tilde{\varphi}_2\colon R \otimes V \to N$ are homotopic if and only if their compositions with $N \to N \oplus M \oplus \delta M$ are homotopic. This holds because the latter map is a homotopy equivalence with the top horizontal arrow of the above diagram as a homotopy inverse. Thus in what follows, we can assume f to be surjective.

Consider the fiber product $F = (I \otimes M) \times_{M\oplus M} (N \oplus N)$, which is explicitly given by

$$F = \{(x, y, z) \in (I \otimes M) \oplus N \oplus N \mid (i_0 \otimes \mathbf{1}_M(x), i_1 \otimes \mathbf{1}_M(x)) = (f(y), f(x))\}.$$

The map $\psi = (\mathbf{1}_I \otimes f \oplus i_0 \otimes \mathbf{1}_N \oplus i_1 \otimes \mathbf{1}_N)\colon I \otimes N \to F$ defines a surjective quasi-isomorphism. This follows from a straight forward investigation of what it means to be a (exact) cocycle in the two objects, which we leave to the reader. Now if $\tilde{\varphi}_1, \tilde{\varphi}_2\colon R \otimes V \to N$ are two morphisms such that $f \circ \tilde{\varphi}_1$ and $f \circ \tilde{\varphi}_2$ are homotopic via a homotopy $H\colon R \otimes V \to I \otimes M$, then we obtain the map $H \oplus \tilde{\varphi}_1 \oplus \tilde{\varphi}_2\colon R \otimes V \to F$. By (i) we can lift this map through the surjective quasi-isomorphism ψ which yields a homotopy between $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$. □

Proposition 1.3.5 (Uniqueness)**.** *A quasi-isomorphism between minimal dgRms is an isomorphism.*

Proof. Let $\varphi\colon (R \otimes V, d) \to (R \otimes V', d')$ be a quasi-isomorphism of minimal dg$R$ms. This induces a map $\overline{\varphi}\colon V \cong (R \otimes V)/\mathfrak{m}V \to (R \otimes V')/\mathfrak{m}V' \cong V'$. We show that $\overline{\varphi}$ is an isomorphism which implies that φ is an isomorphism as well.

Lifting $\mathbf{1}_{R\otimes V'}$ through φ provides us with a morphism $\psi\colon (R \otimes V', d') \to (R \otimes V, d)$ such that $\varphi \circ \psi$ is homotopic to $\mathbf{1}_{R\otimes V'}$. By the definition of homotopy and minimality, it follows that $\varphi \circ \psi - \mathbf{1}_{R\otimes V'}$ takes values in $\mathfrak{m}V'$. In particular $\overline{\varphi} \circ \overline{\psi}$ is an isomorphism, where $\overline{\psi}\colon V' \to V$ is defined analogously.

Since $\psi \circ \varphi \circ \psi \circ \varphi \simeq \psi \circ \varphi$, the uniqueness in Proposition 1.3.4 implies that also $\psi \circ \varphi \simeq \mathbf{1}_{R\otimes V}$. Hence, as before, we deduce that $\overline{\psi} \circ \overline{\varphi}$ is an isomorphism as well. Consequently $\overline{\varphi}$ is an isomorphism. □

Together, Propositions 1.3.4 and 1.3.5 imply the following

Corollary 1.3.6. *Two dgRms are connected by a chain of quasi-isomorphisms (that is to say they have the same quasi-isomorphism-type) if and only if they have isomorphic minimal models.*

1.3.2 Construction of the Hirsch–Brown model via perturbations

The dg$R$m we are most interested in is the minimal (Sullivan) model

$$(R, 0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$$

of the Borel fibration of a G-space X, where $(\Lambda V, d)$ is the minimal Sullivan model of X.

Definition 1.3.7. The minimal Hirsch–Brown model of the action on X is the dg$R$m minimal model of $(R \otimes \Lambda V, D)$.

In this section we give an alternative method of constructing the Hirsch–Brown model of an action directly from $(R \otimes \Lambda V, D)$. This will occasionally be more useful than the construction in Proposition 1.3.3 and also provide insight to the explicit form of the Hirsch–Brown model.

We will make use of Gugenheims theory of perturbations from [30]. Consider the following situation: we have two differential graded R-modules (H, d_H) and (M, d_M) where H is a retract of M in the sense that we find $f\colon H \to M$ and $g\colon M \to H$ with $g \circ f = \mathbf{1}_H$ and $f \circ g \simeq \mathbf{1}_M$ via a homotopy ϕ of degree -1. We further assume ϕ fulfils the side conditions $\phi^2 = 0$, $\phi f = 0$, and $g\phi = 0$. We are interested in the following problem: given a new differential D_M on M, find a new differential on H such that the two new differential graded modules are again homotopy equivalent. For all $n \geq 1$ we define

$$t := D_M - d_M, \qquad t_n := (t\phi)^{n-1}t, \qquad \Sigma_n := t_1 + \ldots + t_n,$$

and

$$\delta_{n+1} := d_H + g\Sigma_n f \qquad f_{n+1} := f + \phi\Sigma_n f$$
$$g_{n+1} := g + g\Sigma_n\phi \qquad \phi_{n+1} := \phi + \phi\Sigma_n\phi.$$

Let $\mathcal{A} \subset \mathrm{End}(M)$ be the (non-commutative) algebra consisting of maps that arise as polynomials in the operators ϕ, t, and d_M. Let $J \subset \mathcal{A}$ denote the ideal generated by t.

Lemma 1.3.8 ([30])**.** *We have*

$$\delta_n\delta_n \in gJ^n f, \qquad D_M f_n - f_n\delta_n \in J^n f,$$
$$\delta_n g - gD_M \in gJ^n, \qquad g_n f_n = \mathbf{1}_H,$$
$$f_n g_n - \mathbf{1}_M - D_M\phi_n - \phi_n D_M \in J^n, \qquad \phi_n f_n = 0,$$
$$g_n\phi_n = 0, \qquad \phi_n\phi_n = 0.$$

So if, in a pointwise sense, the above sequences of maps have a limit and J^n converges to 0, the limits solve the problem described above. Now let us apply this to the case of $(R \otimes \Lambda V)$.

First we choose the data of a retract (maps and the homotopy) of differential graded $\mathbb{Q}$-modules between $(H^*(X), 0)$ and $(\Lambda V, d)$. This can be done by choosing a vector space splitting $\Lambda V = A \oplus B \oplus C$ where $B = \mathrm{im}(d)$, A is a complement of B in $\ker(d)$, and C is a complement of $\ker(d)$ in ΛV. Note that d maps C isomorphically onto B and that $A \cong H^*(X)$ via projecting $\ker(d)$ onto cohomology. Sticking to the notation above we define

$$\begin{aligned} f &\colon H^*(X) \cong A \to \Lambda V \\ g &\colon \Lambda V \to A \cong H^*(X) \\ \phi &\colon \Lambda V \to B \xrightarrow{d^{-1}} C \to \Lambda V, \end{aligned}$$

where all non-specified arrows correspond to the inclusions and projections with respect to the decomposition. This defines a retract satisfying all the assumptions above.

Now extend all the maps R-linearly to a homotopy equivalence of the differential graded R-modules $(R \otimes H^*(X), 0)$ and $(R \otimes \Lambda V, \tilde{d})$ where $\tilde{d} = \mathbf{1}_R \otimes d$. As we are not interested in $\tilde{d}$ but in the twisted differential D on $R \otimes \Lambda V$, set $t = D - \tilde{d}$ and also $\tilde{\phi} = \mathbf{1}_R \otimes \phi$. We want to show that elements in a sufficiently high power of the ideal J, which is additively generated by compositions of t, $\tilde{d}$, and $\tilde{\phi}$ with at least one t, vanish on $R \otimes (\Lambda V)^{\leq n}$ for some fixed n. Observe that since D and $\tilde{d}$ agree on the component mapping $\mathbf{1}_R \otimes \Lambda V$ to itself, t maps $R \otimes (\Lambda V)^n$ into $R^+ \otimes (\Lambda V)^{\leq n-1}$ and the same is true for $\tilde{\phi}$. Using the relations

$$\tilde{d}\tilde{\phi}\tilde{d} = \tilde{d}, \quad \tilde{\phi}\tilde{d}\tilde{\phi} = \tilde{\phi}, \quad \tilde{d}^2 = 0 = \tilde{\phi}^2,$$

we see that any monomial in J^{n+1} starts with a composition of $n+1$ factors of the form t, $t\tilde{d}$, $t\tilde{\phi}$, $t\tilde{d}\tilde{\phi}$, or $t\tilde{\phi}\tilde{d}$. All of those except for $t\tilde{d}$ reduce the degree in ΛV. Using the relation $D^2 = 0$, which implies $t\tilde{d} = t^2 - \tilde{d}t$, we can replace the $t\tilde{d}$ factors with ones that do lower the degree in ΛV. Hence the operators in J^{n+1} vanish on $R \otimes (\Lambda V)^{\leq n}$.

In particular, on this domain, we have $t_k = 0$ and $\Sigma_n = \Sigma_k$ for $k > n$. Thus all sequences of maps in Lemma 1.3.8 converge (pointwise) to respective limits, which by the lemma define a differential δ on $R \otimes H^*(X)$ and a homotopy equivalence between $(R \otimes H^*(X), \delta)$ and $(R \otimes \Lambda V, D)$. Also $(R \otimes H^*(X), \delta)$ is minimal because t takes values in $\mathfrak{m} \otimes \Lambda V$. We deduce:

Corollary 1.3.9. *The minimal Hirsch–Brown model of a G-action on X is of the form $(R \otimes H^*(X), \delta)$.*

On $R \otimes H^{\leq n}(X)$, the differential is explicitly given by

$$\delta = \tilde{g}\Sigma_n\tilde{f},$$

where $\tilde{f} = \mathbf{1}_R \otimes f$ and $\tilde{g} = \mathbf{1}_R \otimes g$.

1.4 Higher homotopy models

The last type of models that we will discuss is the one that arises when relaxing the structure of the previous models (cdga or dg$R$m) up to homotopy. In each case, no information is lost in the process in the sense that the minimal models of the respective type of structure can be constructed from one another (see Theorem 1.4.19 below). The usefulness of the relaxed structures is that we obtain another type of minimal model whose underlying algebraic object is just the cohomology, and the additional information, of say the rational homotopy type, is encoded in higher operations on the cohomology. These operations have strong connections to the classical notion of Massey products and are well suited to discuss formality properties.

We give a brief summary of some basic definitions and results surrounding A_∞-algebras and modules. The aim here is to access the results we need while staying as elementary as possible. For a more detailed introduction to the subject see e.g. [54], [45], or alternatively [47] for the operadic viewpoint which we will occasionally refer to. For a broader overview on the subject see [42]. In what follows, all vector spaces are considered over a field k of characteristic 0. We will make use of the Koszul sign convention:

$$(f \otimes g)(a \otimes b) = (-1)^{|g|\cdot|a|} f(a) \otimes g(b)$$

if f and g are graded linear maps and a, b are homogeneous elements from the respective domains of f and g. Also the general assumptions that all (commutative) differential graded algebras are non-negatively graded and unital will be suspended within this section (unless stated otherwise).

1.4.1 A_∞- and C_∞-algebras

Definition 1.4.1. An A_∞-algebra is a graded vector space A together with linear maps $m_i \colon A^{\otimes i} \to A$ of degree $2 - i$ for each $i \geq 1$, satisfying for each n the relation

$$\sum (-1)^{jk+l} m_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = 0$$

where the sum runs over all decompositions $j + k + l = n$ with $k \geq 1$ and $i = j + l + 1$.

While the equations may look complicated at first glance, they take a familiar form if n is small: for $n = 1$ we obtain $m_1^2 = 0$ so m_1 is a differential. For $n = 2$ we obtain the statement that m_1 is a derivation with respect to the binary product m_2. In general m_2 is not associative but it is so up to a homotopy given by m_3 in the equation for $n = 3$.

Definition 1.4.2. The cohomology of $(A; m_i)$ is the cohomology of the chain complex (A, m_1).

By the discussion above, the product m_2 induces a product on cohomology. This product is associative giving the cohomology the structure of a graded algebra.

Remark 1.4.3. Any ordinary differential graded algebra (A, d) can be considered as an A_∞-algebra by taking m_1 to be d, m_2 the multiplication in A, and $m_i = 0$ for $i \geq 3$.

Before we turn our attention to the definition of a morphism between A_∞-algebras, let us reinterpret the defining equations. We define the suspension sA of A via $(sA)^n = A^{n+1}$ and also denote by $s\colon A \to sA$ the canonical isomorphism of degree -1. Consider the reduced tensor coalgebra

$$\overline{T}sA = sA \oplus (sA \otimes sA) \oplus \dots$$

and the map $\overline{T}sA \to sA$ of degree 1 which, on n-tensors, we define to be $b_i = -s^{-1} \circ m_i \circ s^{\otimes i}$. This map extends uniquely to a coderivation b of $\overline{T}sA$ by setting

$$b|_{sA^{\otimes n}} = \sum_{i+k+l=n} \mathbf{1}_{sA}^{\otimes k} \otimes b_i \otimes \mathbf{1}_{sA}^{\otimes l}.$$

The equations in Definition 1.4.1 are equivalent to the fact that $b^2 = 0$. In particular there is a one-to-one correspondence between A_∞-structures on A and codifferentials of the coalgebra $\overline{T}sA$.

Definition 1.4.4. The differential graded coalgebra $(\overline{T}sA, b)$ is called the bar construction of $(A; m_i)$ and denoted BA.

Remark 1.4.5. The transition from the m_i to the b_i is not canonical and there exist different sign conventions in the literature, giving rise to different signs in the definition of A_∞-algebra. We stick to the ones used e.g. in [45] and [49].

A morphism between A_∞-algebras $(A; m_i)$ and $(C; m_i)$ can just be defined as a morphism of differential graded coalgebras $f\colon BA \to BC$. By the universal property of the cofree coalgebra, such a morphism is defined by the projection $\overline{f}\colon BA \to sC$. This data is equivalent to a collection of maps $f_i\colon A^{\otimes i} \to C$ of degree $i-1$ such that on i-tensors, $\overline{f}$ is given by $s^{-1} \circ f_i \circ s^{\otimes i}$. The condition of f being a morphism of differential graded coalgebras translates, for each n, to the equation

$$\sum (-1)^{jk+l} f_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = \sum (-1)^s m_k(f_{i_1} \otimes \dots \otimes f_{i_k}),$$

where the left hand sum runs over all decompositions $n = j + k + l$ with $k \geq 1$ and $i = j + 1 + l$ and the right hand sum runs over all decompositions $n = i_1 + \dots + i_k$ and

$$s = \sum_{1 \leq \alpha < \beta \leq k} (i_\alpha)(i_\beta + 1)$$

which we could take as an alternative definition of morphism. The equations show that $f_1\colon A \to B$ is always a chain map with respect to the differentials m_1 on A and C. Therefore it induces a map $H^*(A) \to H^*(C)$. We call f a quasi-isomorphism if f_1 induces an isomorphism on cohomology.

We briefly introduce the concept of homotopy for morphisms of A_∞-algebras. Consider the dg-coalgebra (I, d) where I has basis e in degree -1 and e_0, e_1 in degree 0. The differential is defined by $d(e) = e_0 - e_1$ and the coalgebra structure is defined by $\Delta e = e_0 \otimes e + e \otimes e_1$, $\Delta e_0 = e_0 \otimes e_0$, and $\Delta e_1 = e_1 \otimes e_1$. If A is an A_∞-algebra, we may form the coalgebra $BA \otimes I$ together with the two inclusions i_0 and i_1 mapping BA to $BA \otimes e_0$ and $BA \otimes e_1$ and the projection $p\colon BA \otimes I \to BA$ which maps $x \otimes e + x_0 \otimes e_0 + x_1 \otimes e_1$ to $x_0 + x_1$. Then p is a quasi-isomorphism and $p \circ i_k = \mathbf{1}_{BA}$ (see [45, 1.3.4.1]).

Definition 1.4.6. We say two morphisms $f, g\colon A \to C$ are homotopic if there is a morphism of coalgebras $h\colon BA \otimes I \to BC$ with $h \circ i_0 = f$ and $h \circ i_1 = g$.

There is also a commutative version of A_∞-algebras defined as follows:

Definition 1.4.7. Let $(A; m_i)$ be an A_∞-algebra.

(i) The shuffle product on $\overline{T}sA$ is defined by

$$(a_1 \otimes \ldots \otimes a_n) *_{sh} (a_{n+1} \otimes \ldots \otimes a_{n+k}) = \sum_{\sigma \in Sh(n,k)} (-1)^{s(\sigma)} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n+k)}$$

where $sh(n,k)$ consists of all permutations σ with $\sigma(i) < \sigma(j)$ whenever $i < j$ and either $1 \leq i, j \leq n$ or $n+1 \leq i, j \leq n+k$. The sign $s(\sigma)$ is given by the usual sign of the action of the symmetric group on the graded space $(sA)^{\otimes n+k}$.

(ii) $(A; m_i)$ is called a C_∞-algebra if the maps $m_i \circ (s^{-1})^{\otimes i}$ vanish on all shuffles in $\overline{T}sA$ (note that applying $(s^{-1})^{\otimes i}$ to elements of word length $i \geq 2$ changes signs by the Koszul sign rule).

(iii) A morphism of C_∞-algebras is a morphism f of A_∞-algebras such that the maps $f_i \circ (s^{-1})^{\otimes i}$ vanish on all shuffles in $\overline{T}sA$.

Remark 1.4.8. From an operadic viewpoint, the commutative version of an A_∞-algebra would be to consider the free Lie coalgebra instead of the free coalgebra as we did in the bar construction of an A_∞-algebra (cf. [57]). One can prove that the definition of a C_∞-algebra above is equivalent to such a structure (see [47, Prop. 13.1.14]).

We conclude this section by introducing the unital and augmented versions of the previous structures. We consider the ground field k as a cdga concentrated in degree 0. All of the following notions for C_∞-algebras have obvious analogous definitions for A_∞-algebras which we will not repeat.

Definition 1.4.9. (i) A strictly unital C_∞-algebra is a C_∞-morphism $\eta\colon k \to (A; m_i)$ such that

$$m_i(\mathbf{1}_A \otimes \ldots \otimes \mathbf{1}_A \otimes \eta \otimes \mathbf{1}_A \otimes \ldots \otimes \mathbf{1}_A) = 0$$

for $i \geq 3$ and $m_2(\mathbf{1}_A \otimes \eta) = m_2(\eta \otimes \mathbf{1}_A) = \mathbf{1}_A$. A morphism of strictly unital C_∞-algebras is a morphism of the underlying C_∞-algebras that commutes with the units.

(ii) An augmented C_∞-algebra is a strictly unital C_∞-algebra A together with a strict morphism $\varepsilon\colon A \to k$ of strictly unital C_∞-algebras. By ε being strict we mean that the only nontrivial component is ε_1. A morphism of augmented C_∞-algebras is a morphism of the underlying strictly unital C_∞-algebras that commutes with the augmentations.

We point out that there is an equivalence of categories between C_∞-algebras and augmented C_∞-algebras. In the presence of an augmentation ε, one can consider the non-augmented C_∞-algebra $\ker \varepsilon$ with the induced structure. Conversely, for any C_∞-algebra $(A; m_i)$, we can consider $A \oplus k$ with the unique strictly unital C_∞-structure that agrees with the m_i on A, where the unit and augmentation are given by the canonical inclusion and projection of k. The two constructions are naturally inverse to another. All notions such as quasi-isomorphisms or homotopy carry over to the augmented setting.

Remark 1.4.10. If A is strictly unital, concentrated in positive degrees and $A^0 = k$, then it is naturally augmented by projecting onto the degree 0 component.

1.4.2 A_∞-modules

Definition 1.4.11. Let $(A; m_i)$ be an A_∞-algebra. An A_∞-A-module structure on a graded vector space M is a collection of maps $m_i^M : M \otimes A^{\otimes i-1} \to M$ of degree $2 - i$ for each $i \geq 1$, satisfying for each n the relation

$$\sum (-1)^{jk+l} m_i^M (\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = 0$$

where the sum runs over all decompositions $j + k + l = n$ with $k \geq 1$, $i = j + l + 1$, and the expression $\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}$ is interpreted as $m_k^M \otimes \mathbf{1}^{\otimes l}$ whenever $j = 0$.

Let $TsA = k \oplus \overline{T}sA$ be the free augmented coalgebra with comultiplication

$$\Delta_{TsA}(x) = x \otimes 1 + \Delta_{\overline{T}sA}(x) + 1 \otimes x.$$

As before, the above data defines a map $sM \otimes TsA \to sM$ of degree 1 which we define to be $-s \circ m_i \circ (s^{-1})^{\otimes i}$ on $sM \otimes sA^{\otimes i-1}$. Said map can be extended uniquely to a differential d on the cofree comodule $sM \otimes TsA$ as in the following

Lemma 1.4.12. *Let V be a vector space and (C, Δ_C, d_C) a differential graded coalgebra. Then any linear map $\overline{d} : V \otimes C \to V$ can be coextended uniquely to a coderivation d of the co-free comodule $V \otimes C$ by setting*

$$d = \mathbf{1}_V \otimes d_C + (\overline{d} \otimes \mathbf{1}_C) \circ (\mathbf{1}_V \otimes \Delta_C).$$

For counitary coalgebras this yields a one-to-one correspondence of maps $\overline{d}$ and coderivations d.

Definition 1.4.13. The cofree differential graded comodule $(sM \otimes TsA, d)$ is called the bar construction of $(M; m_i^M)$ and denoted BM.

A morphism of A_∞-modules is defined as a dg-comodule map f between the respective bar constructions. Such a map $f : BM \to BN$ is determined by the projection

$$\overline{f} : sM \otimes TsA \to sN$$

and can thus be expressed by a collection of maps $f_i : M \otimes A^{\otimes i-1} \to N$ of degree $1-i$ such that $\overline{f} = s \circ f_i \circ (s^{-1})^{\otimes i}$ on word length i. The fact that f commutes with the differentials of the bar constructions translates to the equations

$$\sum (-1)^{jk+l} f_i (\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = \sum m_{k+1} (f_i \otimes \mathbf{1}^{\otimes k})$$

for every $n \geq 1$, where the left hand sum runs over all decompositions $n = j + k + l$, $k \geq 1$ and the right hand sum runs over all decompositions $n = i + k$, $i \geq 1$. From this, f is reconstructed by just setting $f = (\overline{f} \otimes \mathbf{1}_{TsA}) \circ (\mathbf{1}_{sM} \otimes \Delta_{TsA})$.

As for A_∞-algebras, the operation m_1^M of an A_∞-module $(M; m_i^M)$ is a differential and we define the cohomology of M as that of the chain complex (M, m_1^M). If $f : (M; m_i^M) \to (N; m_i^N)$ is a morphism of A_∞-modules, then the above equation for $n = 1$ implies thatf_1 is a chain map. Thus we obtain a map $H^*(M) \to H^*(N)$ on cohomology. We call f a quasi-isomorphism if the map on cohomology is an isomorphism.

Remark 1.4.14. As for A_∞-algebras, if (A, d) is a regular dga, the category of classical dg-modules over A can be seen as a subcategory of A_∞-A-modules. Just define the operation m_1^M to be the differential, m_2^M to be the multiplication map of the module, and $m_i^M = 0$ for $i \geq 3$.

Remark 1.4.15. If $(A; m_i)$ is an A_∞-algebra, it is in particular an A_∞-module over itself because $\overline{T}sA = A \otimes TsA$.

We discuss how A_∞-modules behave with respect to maps of the underlying A_∞-algebras. Analogous to the case of regular modules, a morphism $f\colon (A; m_i) \to (C; m_i)$ of A_∞-algebras yields a restriction functor from A_∞-C-modules to A_∞-A-modules. If $(M; m_i^M)$ is a C-module and $f_i\colon A^{\otimes i} \to C$ are the components of f, then the A-module f^*M is defined by

$$m_i^{f^*M} = \sum (-1)^s m_{r+1}^M(\mathbf{1}_M \otimes f_{i_1} \otimes \ldots \otimes f_{i_r})$$

where the sum runs over all decompositions $i - 1 = i_1 + \ldots + i_r$ for $r = 1, \ldots, i-1$ and

$$s = \sum_{1 \leq \alpha < \beta \leq r} i_\alpha(i_\beta + 1) + \sum_{j=1}^{r} (i_j - 1).$$

In the language of the corresponding bar constructions BM and Bf^*M, this can be understood as the pullback of the differential along the map $f\colon BA \to BC$ in the following sense: the differential on $BM = (sM \otimes TsC, d)$ is the coextension of the projection $\overline{d}\colon sM \otimes TsC \to sM$ (see Lemma 1.4.12). Then the corresponding differential of $Bf^*M = (sM \otimes TsA)$ is the coextension of $\overline{d} \circ (\mathbf{1}_{sM} \otimes f^+)$ where $f^+ = (\mathbf{1}_k, f)\colon k \oplus \overline{T}sA \to k \oplus \overline{T}sC$.

Lemma 1.4.16. *Consider a commutative triangle*

$$\begin{array}{ccc} (R; m_i) & \xrightarrow{\ f\ } & (A; m_i) \\ & \searrow^{h} & \downarrow{\scriptstyle g} \\ & & (C; m_i) \end{array}$$

of A_∞-Algebras. Then g induces an A_∞-module homomorphism between the A_∞-R-module structures induced on A and C by the morphisms f and h.

Proof. Let d_A, d_C, d_R be the differentials of the bar constructions BA, BC, BR of A_∞-algebras and let d_A^+, d_C^+, d_R^+ denote the extensions to TsA, TsC, TsR by sending k to 0. Also denote by $\pi_A\colon BA \to sA$ and $\pi_C\colon BC \to sC$ the projections and set $\overline{g} = \pi_C \circ g$ as well as $\overline{d_A} = \pi_A \circ d_A$, $\overline{d_C} = \pi_C \circ d_C$.

We want to define a map φ between the bar constructions of A and C when considered as A_∞-modules over $(R; m_i)$. Explicitly, the bar constructions are given as $(sA \otimes TsR, D_A)$, $(sC \otimes TsR, D_C)$, where D_A and D_C are the coextensions (in the sense of Lemma 1.4.12) of

$$\overline{D_A} = \overline{d_A} \circ (\mathbf{1}_{sA} \otimes f^+) \quad \text{and} \quad \overline{D_C} = \overline{d_C} \circ (\mathbf{1}_{sC} \otimes h^+).$$

We define $\overline{\varphi}\colon sA \otimes TsR \to sC$ to be the composition $\overline{g} \circ (\mathbf{1}_{sA} \otimes f^+)$ and define φ to be the coextension of $\overline{\varphi}$, that is $\varphi = (\overline{\varphi} \otimes \mathbf{1}_{TsR}) \circ (\mathbf{1}_{sA} \otimes \Delta_{TsR})$. It remains to check that

φ commutes with the differentials, which can be confirmed on cogenerators: we need to prove that $\overline{D_C} \circ \varphi = \overline{\varphi} \circ D_A$ and calculate

$$\begin{aligned}
\overline{\varphi} \circ D_A &= \overline{g} \circ (\mathbf{1}_{sA} \otimes f^+) \circ (\mathbf{1}_{sA} \otimes d_R^+ + (\overline{D_A} \otimes \mathbf{1}_{TsR}) \circ (\mathbf{1}_{sA} \otimes \Delta_{TsR})) \\
&= \overline{g} \circ (\mathbf{1}_{sA} \otimes (f^+ \circ d_R^+) + (\overline{d_A} \otimes \mathbf{1}_{TsA}) \circ (\mathbf{1}_{sA} \otimes f^+ \otimes f^+) \circ (\mathbf{1}_{sA} \otimes \Delta_{TsR}) \\
&= \overline{g} \circ (\mathbf{1}_{sA} \otimes d_A^+ + (\overline{d_A} \otimes \mathbf{1}_{TsA}) \circ (\mathbf{1}_{sA} \otimes \Delta_{TsA})) \circ (\mathbf{1}_{sA} \otimes f^+) \\
&= \overline{g} \circ d_A \circ (\mathbf{1}_{sA} \otimes f^+) \\
&= \overline{d_C} \circ g \circ (\mathbf{1}_{sA} \otimes f^+) \\
&= \overline{d_C} \circ (\overline{g} \otimes g^+) \circ (\mathbf{1}_{sA} \otimes \Delta_{TsA}) \circ (\mathbf{1}_{sA} \otimes f^+) \\
&= \overline{d_C} \circ (\overline{g} \otimes g^+) \circ (\mathbf{1}_{sA} \otimes f^+ \otimes f^+) \circ (\mathbf{1}_{sA} \otimes \Delta_{TsR}) \\
&= \overline{d_C} \circ (\mathbf{1}_{sC} \otimes h^+) \circ (\overline{\varphi} \otimes \mathbf{1}_{TsR}) \circ (\mathbf{1}_{sA} \otimes \Delta_{TsR}) \\
&= \overline{D_C} \circ \varphi,
\end{aligned}$$

where we have identified $\overline{T}sA = sA \otimes TsA$ and $\overline{T}sC = sC \otimes TsC$. □

Lemma 1.4.17. *Let $f, g\colon (A; m_i) \to (C; m_i)$ be two homotopic morphisms between A_∞-algebras. Then the two induced A-module structures on C are quasi-isomorphic.*

For the proof we will need the following result ([42, Section 6.2], [45, Théorème 4.1.2.4]). The notion of homotopy category is recalled in the next section.

Theorem 1.4.18. *Let $f\colon (A; m_i) \to (C; m_i)$ be a quasi-isomorphism of A_∞-algebras. Then the restriction functor defines an equivalence between the homotopy categories $\mathcal{D}_\infty A$ and $\mathcal{D}_\infty C$ of the categories of A_∞-modules over A and C.*

Proof of the lemma. With the terminology of Definition 1.4.6, it suffices to show that the restrictions of the $BA \otimes I$-module h^*C along i_0 and i_1 are quasi-isomorphic, where h is a homotopy between f and g. Since p is a quasi-isomorphism, restriction along p defines an equivalence $\mathcal{D}_\infty BA \to \mathcal{D}_\infty BA \otimes I$ so h^*C is quasi-isomorphic to p^*M for some dg-BA-comodule M. But this means that $i_0^*(h^*C)$ is quasi-isomorphic to $i_0^*(p^*M) = M$ and the same holds for i_1. □

1.4.3 On the homotopy categories

Let $\mathcal{C}$ be one of the categories of augmented dgas, augmented cdgas, or dg-A-modules, where A is a fixed dga with unit. Also let $\mathcal{C}_\infty$ be the corresponding category of either augmented A_∞-algebras, augmented C_∞-algebras, or strictly unital A_∞-modules over A. The latter is defined as the full subcategory of A_∞-modules $(M; m_i^M)$ over A such that $m_i(\mathbf{1}_M \otimes \mathbf{1}_A \otimes \ldots \otimes \mathbf{1}_A \otimes \eta \otimes \mathbf{1}_A \otimes \ldots \otimes \mathbf{1}_A) = 0$ for $i \geq 3$ and $m_2(\mathbf{1}_M \otimes \eta) = \mathbf{1}_M$, where $\eta\colon k \to A$ is the unit of A. We have seen that there is an inclusion of categories $\mathcal{C} \to \mathcal{C}_\infty$ (see Remarks 1.4.3 and 1.4.14). The category $\mathcal{C}_\infty$ is much larger in the sense that it contains many new objects and morphisms. However, it does not introduce new quasi-isomorphism types. Hence $\mathcal{C}_\infty$ can be very useful when studying the category $\mathcal{C}$ up to quasi-isomorphism.

Let $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{C}_\infty)$ denote the homotopy categories of $\mathcal{C}$ and $\mathcal{C}_\infty$ which are defined as the localizations of the respective category at the set of quasi-isomorphisms. The categories $\mathcal{C}$ and $\mathrm{Ho}(\mathcal{C})$ have the same objects and two of them are isomorphic in $\mathrm{Ho}(\mathcal{C})$ if and only if they are connected by a zigzag of quasi-isomorphisms in $\mathcal{C}$ (analogous for $\mathcal{C}_\infty$).

Theorem 1.4.19. *The inclusion $\mathcal{C} \to \mathcal{C}_\infty$ induces an equivalence of categories $\mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho}(\mathcal{C}_\infty)$.*

For A_∞- and C_∞-algebras this follows from [47, Thm. 11.4.12] which covers the general setting of homotopy algebras over Koszul operads and naturally carries over to the augmented case through the equivalence at the end of Section 1.4.1. The case of A_∞-modules over A was proved in [45, Lemma 4.1.3.8, Prop. 3.3.1.8]. A similar result on A_∞-modules is more explicitly stated in [42, Section 4.3], however referring to the previous reference for proofs.

1.4.4 Minimal models and formality

It has long been known that Massey products form an obstruction to formality. Conversely, the uniform vanishing of all Massey products implies formality. This idea of uniform vanishing is best captured by seeing Massey products as the higher operations in an A_∞-structure on the cohomology. The following theorem goes back to [39] and [40]. It has since been generalized in the language of operads [47, Theorem 10.3.15].

Theorem 1.4.20. *Let $(A; m_i^A)$ be an A_∞-Algebra (resp. C_∞-algebra). Then there is an A_∞- (resp. C_∞-)algebra structure $(H^*(A); m_i)$ on the cohomology such that*

- *$m_1 = 0$ and m_2 is the product induced by m_2^A.*
- *there is a quasi-isomorphism $(H^*(A); m_i) \to (A; m_i)$ of A_∞- (resp. C_∞-)algebras lifting the identity on cohomology.*

This structure is unique in the sense that two of them are isomorphic via an isomorphism of A_∞- (resp. C_∞-)algebras whose first component is the identity.

We will refer to the quasi-isomorphism $H^*(A) \to A$ as a minimal model for A. More generally we will call an A_∞-(resp. C_∞-)algebra minimal if the operation m_1 vanishes. The operations m_i, $i \geq 3$ of the minimal model are also referred to as the higher Massey products. There are several known formulas which compute these operations from the A_∞-structure on A (see e.g. [51], [47]). We quickly recall the original construction by Kadeishvili ([41, Theorem 1]) of how to compute the minimal A_∞-model of a dga. It was shown in [49] that the same construction yields a minimal C_∞-model when applied to a cdga.

Let (A, d) be a (c)dga. Set $m_1 = 0$ and let $f_1 \colon H^*(A) \to A$ be a cycle choosing homomorphism. Assume inductively that f_i and m_i have been constructed until $i = n-1$. Define the operator $U_n \colon H^*(A)^{\otimes n} \to A$ as $U_n^1 + U_n^2$ where

$$U_n^1 = \sum_{k=1}^{n-1} (-1)^{k(n+k+1)} f_k \cdot f_{n-k}$$

$$U_n^2 = -\sum_{k=2}^{n-1} \sum_{i=0}^{n-k} (-1)^{ik+n+k+i} f_{n-k+1}(\mathbf{1}_A^{\otimes i} \otimes m_k \otimes \mathbf{1}_A^{\otimes n-i-k}).$$

One can check that U_n maps to cocycles and we define $m_n := [U_n]$. Now choose f_n in a way that $df_n = f_1 m_n - U_n$.

Remark 1.4.21. If (A, d) is a formal cdga, then it has a minimal Sullivan model $(\Lambda V, d)$ with a splitting $V = W_1 \oplus W_2$ such that $d(W_1) = 0$ and any closed element in the ideal generated by W_2 is exact. In the construction of the minimal C_∞-model, f_n $(n \geq 2)$ can be chosen to have image in the ideal generated by W_2 which yields $m_n = 0$ for $n \geq 3$.

The converse statement of the remark is also true (see e.g. [38, Theorem 8]):

Theorem 1.4.22. *A cdga (A, d) is formal if and only if it has a minimal C_∞-model of the form $(H^*(A); m_i)$ with $m_i = 0$ for $i \neq 2$.*

We will also make use of the following observations. A minimal model as in (ii) will be called a unital minimal model.

Lemma 1.4.23. *(i) Let $\varphi\colon A \to B$ be a cohomologically injective morphism of (c)dgas. Then for arbitrary choices of the f_n^A in the construction of the minimal model $(H^*(A); m_i^A)$ above, we can choose the maps f_n^B for the construction of $(H^*(B); m_i^B)$ in a way that $\varphi^* \circ m_n^A = m_n^B \circ (\varphi^*)^{\otimes n}$ and $\varphi \circ f_n^A = f_n^B \circ (\varphi^*)^{\otimes n}$.*

(ii) The A_∞- (C_∞-)minimal model of a unital (c)dga A can be constructed in a way that $H^*(A) \to A$ is a morphism of strictly unital A_∞- (C_∞-)algebras. If A is formal, then the construction of the unital minimal model is compatible with Remark 1.4.21.

(iii) If $\varphi\colon A \to B$ is a cohomologically injective morphism of unital (c)dgas and A is formal, then the unital minimal model of B can be constructed such that $m_n^B \circ (\varphi^*)^{\otimes n}$ vanishes for $n \geq 3$.

Proof. In the situation of (i) we choose any cycle choosing homomorphisms f_1^A. Due to the injectivity of φ^* we can define f_1^B in a way that $\varphi \circ f_1^A = f_1^B \circ \varphi^*$. Assume inductively that we have constructed m_{n-1}^A, m_{n-1}^B, f_{n-1}^A, and f_{n-1}^B such that the statement of the lemma holds. By the formulas in the construction we also have $\varphi \circ U_n^A = U_n^B \circ (\varphi^*)^{\otimes n}$. As a result

$$\varphi(f_1^A m_n^A - U_n^A) = (f_1^B m_n^B - U_n^B) \circ (\varphi^*)^{\otimes n}.$$

Now for any choice of f_n^A we can define f_n^B on $\operatorname{im}(\varphi^*)^{\otimes n}$ to be $\varphi \circ f_n^A \circ ((\varphi^*)^{\otimes n})^{-1}$ and complete the definition arbitrarily on a complement. Then f_n^A, f_n^B, m_n^A, and m_n^B satisfy the desired properties.

For the proof of (ii) we can choose f_1 such that it maps $1 \in H^*(A)$ to $1 \in A$. The map f_2 can be chosen such that $f_2(1 \otimes a) = f_2(a \otimes 1) = 0$ for all $a \in H^*(A)$. From there one proves inductively that U_n vanishes on $a_1 \otimes \ldots \otimes a_n$, whenever $n \geq 3$ and $a_j = 1$ for some $1 \leq j \leq n$, and thus also f_n can be chosen equal to 0 on this kind of tensor. If A is formal and has a minimal Sullivan model $(\Lambda V, d)$ as in Remark 1.4.21, then we can choose f_n to have image in the ideal generated by W_2 whenever we want f_n to be nontrivial.

Finally, (iii) results from the fact that the choices made in (i) and (ii) can be performed in a compatible way: as in (ii), we construct the unital minimal model of A such that the higher operations vanish. Then the condition $f_n^B|_{\operatorname{im}(\varphi^*)^{\otimes n}} = \varphi \circ f_n^A \circ ((\varphi^*)^{\otimes n})^{-1}$ already implies that f_n^B vanishes on pure tensors in $\operatorname{im}(\varphi^*)^{\otimes n}$ which have the unit as a factor. Thus we can extend this partial definition of f_n^B in a way that produces a unital minimal model of B as in (ii). ∎

Remark 1.4.24. We have only discussed minimal models of A_∞- and C_∞-algebras but analogous concepts do exists for A_∞-modules.

Chapter 2

Formality in an equivariant setting

2.1 Refinements of equivariant formality

2.1.1 $\mathcal{MOD}$-formality and consequences

Let X be a G-space and $(R,0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$ be a model of the Borel fibration. In particular $(R \otimes \Lambda V, D)$ is a Sullivan model for X_G. When it comes to common generalizations of equivariantly formal actions and actions with formal homotopy quotient, we have the following natural

Definition 2.1.1. The action is called $\mathcal{MOD}$-formal if $(R \otimes \Lambda V, D)$ is formal as a differential graded R-module (dg$R$m), i.e. it is connected to $(H_G^*(X), 0)$ via quasi-isomorphisms of dg$R$ms.

Remark 2.1.2. In the definition above, as opposed to the condition of X_G being a formal space, we only require the quasi-isomorphisms to be multiplicative with respect to the classes coming from BG. In this way one gets rid of formality obstructions that exist within X independently of the action.

Lemma 2.1.3. *Equivariantly formal actions and actions with formal homotopy quotient are $\mathcal{MOD}$-formal.*

Proof. If X_G is formal, there is a quasi-isomorphism of cdgas $(R \otimes \Lambda V, D) \to (H_G^*(X), 0)$ which covers the canonical projection on closed elements. This is in particular a morphism of dg$R$ms so the statement follows.

Now if $H_G^*(X)$ is free, let $b_1, \ldots, b_k \in R \otimes \Lambda V$ be representatives of an R-basis. Then the inclusion $H_G^*(X) \cong R \otimes \langle b_1, \ldots, b_k \rangle_{\mathbb{Q}} \to R \otimes \Lambda V$ induces an isomorphism on cohomology if we take the differential on the left hand side to be trivial. □

Example 2.1.4. When it comes to examples of actions that are neither equivariantly formal nor have a formal homotopy quotient but satisfy $\mathcal{MOD}$-formality, there are a few trivial candidates: For example every S^1-action can be seen to automatically be $\mathcal{MOD}$-formal (see Remark 2.1.7). Also take any G-action on X such that X_G is formal and let Y be a non-formal space. Then $(X \times Y)_G = X_G \times Y$ is not formal (see [9, Prop. 5]). The Hirsch–Brown model of $X \times Y$ however arises from the Hirsch–Brown model $(R \otimes H^*(X), D)$ of X by tensoring with $H^*(Y)$ and extending the differential to $R \otimes$

$H^*(X) \otimes H^*(Y)$ in the obvious way. A quasi-isomorphism $(R \otimes H^*(X), D) \simeq (H_G^*(X), 0)$ thus induces a quasi-isomorphism $(R \otimes H^*(X) \otimes H^*(Y), D) \to (H_G^*(X) \otimes H^*(Y), 0)$ so the action is $\mathcal{MOD}$-formal. The discussion in Section 2.3 is helpful for constructing more interesting examples as Example 2.4.9.

One of the defining traits of $\mathcal{MOD}$-formal actions is given by the following observation: Let X be a $\mathcal{MOD}$-formal G-space. As minimal models of dg$R$ms are unique among a quasi-isomorphism type, it follows that the Hirsch–Brown model of the action is the minimal model of the dg$R$m $(H_G^*(X), 0)$. To construct the latter, recall the notion of minimal graded free resolution from commutative algebra: a graded free resolution of $H^*(X_G)$ is an exact complex

$$0 \leftarrow H_G^*(X) \leftarrow F_0 \overset{d}{\leftarrow} F_1 \overset{d}{\leftarrow} \ldots \overset{d}{\leftarrow} F_r \leftarrow 0$$

consisting of free graded R-modules and graded maps. In the usual conventions, those maps are of degree 0 but we apply suitable degree shifts to consider them to be of degree 1. The resolution is said to be minimal if $d(F_i) \subset \mathfrak{m}F_{i-1}$, where $\mathfrak{m} = R^+$ is the maximal homogeneous ideal. Thus for a minimal resolution, the projection map

$$\bigoplus_{i=0}^{r} F_i \to F_0/d(F_1) \cong H_G^*(X)$$

defines a minimal dg$R$m-model for $(H_G^*(X), 0)$, where we equip $\bigoplus_i F_i$ with the differential that sends F_0 to 0 and equals d on F_i for $i \geq 1$. We have shown

Theorem 2.1.5. *An action is $\mathcal{MOD}$-formal if and only if the minimal Hirsch–Brown model is isomorphic to the minimal graded free resolution of the R-module $H_G^*(X)$ as a differential graded R-module.*

Remark 2.1.6. It is, of course, not true that the rational homotopy type of X and X_T is "a formal consequence" of the R-Algebra structure on $H_G^*(X)$, as is the case for actions with formal homotopy quotient. However for $\mathcal{MOD}$-formal actions, the spirit of formality still lives on in the fact that the (additive) cohomology of X can be retrieved from the R-module $H_G^*(X)$. Also, the algebra structure on the image of $H_G^*(X) \to H^*(X)$ can be reconstructed from the R-algebra structure of $H_G^*(X)$, which holds more generally for spherical actions which we define below. Note however that this map is usually not surjective as its surjectivity is equivalent to classical equivariant formality.

Remark 2.1.7. In view of the characterization of Theorem 2.1.5, one can see from the explicit construction of the minimal Hirsch–Brown model in Proposition 1.3.3 that any S^1-action is $\mathcal{MOD}$-formal. Assume that we have constructed the model up until a certain degree and that it has the structure of a free resolution

$$0 \leftarrow F_0 \overset{D}{\leftarrow} F_1 \leftarrow 0$$

with cohomology concentrated in F_0. Then when adding another generator to generate cohomology, we can obviously add it to F_0 and keep the structure of a free resolution. When adding a generator α in order to kill cohomology, we can choose $D(\alpha)$ to be some (non-exact) element of F_0. Also D is injective on $F_1 \oplus R\alpha$, which means the free resolution structure is preserved. To see this, we write $R = \mathbb{Q}[X_1]$ and assume the existence of $v \in F_1$ with $D(v + X_1^k\alpha) = 0$ for some $k \geq 1$. Then, since α is of maximal degree among the generators, v is divisible by X_1^k so $D\alpha = -D(vX_1^{-k}) \in D(F_1)$ was already exact, which is a contradiction.

Before investigating further aspects of formality and their relations, we give some reformulations of $\mathcal{MOD}$-formality which will be useful throughout the thesis.

Lemma 2.1.8. *Let $(R \otimes H^*(X), D)$ be the Hirsch–Brown model of a G-action on a space X. The following are equivalent:*

(i) The action is $\mathcal{MOD}$-formal.

(ii) There exists a splitting $R \otimes H^*(X) = V \oplus W$ of the Hirsch–Brown model into free submodules such that $D(V) = 0$ and every closed element in W is exact.

(iii) There is a vector space splitting $R \otimes H^*(X) = \ker D \oplus C$ such that $C \oplus \mathrm{im} D$ is an R-submodule.

Proof. As we have seen, in case the action is $\mathcal{MOD}$-formal, we can choose the minimal Hirsch–Brown model to be the minimal free resolution $\bigoplus F_i$ of $H_G^*(X)$. The desired decomposition in (ii) is given by $V = F_0$ and $W = \bigoplus_{i \geq 2} F_i$.

In the situation of (ii) we can choose a vector space splitting $R \otimes H^*(X) = \ker D \oplus C$ such that $C \subset W$. Now if α is any R-linear combination of elements in $C \oplus \mathrm{im} D$, then, since $\mathrm{im} D = \mathrm{im} D|_C$, we find some $c \in C$ with $D\alpha = Dc$. Thus $\alpha - c$ is the sum of exact elements and closed elements in W which are exact by assumption. Consequently we have $\alpha - c \in \mathrm{im} D$ and $\alpha \in C \oplus \mathrm{im} D$.

Finally, assume (iii) holds. We define a map φ from $R \otimes H^*(X)$ to its cohomology such that it is the canonical projection on $\ker D$ and trivial on C. This obviously commutes with the differential (which is trivial on $H_G^*(X)$) and all that remains to show is the R-linearity of φ. This clearly holds on $\ker D$ and by assumption, for any $c \in C$, $r \in R$ we have $rc \in C \oplus \mathrm{im} D = \ker \varphi$ which proves the claim. □

Definition 2.1.9. We say that an action is *almost $\mathcal{MOD}$-formal* if $\dim H^*(X)$ is equal to the rank of the minimal free resolution of $H_G^*(X)$ as an R-module.

Remark 2.1.10. The Eilenberg-Moore spectral sequence of the Borel fibration converges to $H^*(X)$ and has $E_2^{*,*} = \mathrm{Tor}_R^{*,*}(H_G^*(X), R/\mathfrak{m})$. By the definition of Tor, the $\mathbb{Q}$-dimension of the right hand expression is precisely the total rank of the minimal free resolution of $H_G^*(X)$. We deduce that almost $\mathcal{MOD}$-formality is equivalent to the E_2-degeneration of the Eilenberg-Moore spectral sequence.

In case $G = T$ is a torus, almost $\mathcal{MOD}$-formality has the following interpretation in terms of another spectral sequence (see also [60, Lemma 1.4] and [52, Remark 7]): taking one step back in the Barratt-Puppe sequence of the Borel fibration, we obtain the fibration (up to homotopy equivalence)

$$T \to X \to X_T,$$

which in the free case is equivalent to $T \to X \to X/T$.

Proposition 2.1.11. *The T-action is almost $\mathcal{MOD}$-formal if and only if the Serre spectral sequence of*

$$T \to X \to X_T$$

degenerates at the E_3 term.

Proof. The second page of the spectral sequence is the Koszul complex $H_T(X) \otimes S$ where $S = \Lambda(s_i)$ consists of a degree 1 generator for each variable X_i of R and the differential maps s_i to the image of X_i in $H_T^2(X)$ (see Remark 1.2.3). Thus the E_3-page is precisely $\mathrm{Tor}_R(H_T^*(X), R/\mathfrak{m})$. By the commutativity of Tor, we deduce that $\dim_{\mathbb{Q}} E_3$ is the rank of the minimal free resolution of $H_T^*(X)$. As the spectral sequence converges to $H^*(X)$, this is equal to $\dim H^*(X)$ if and only if there are no more nontrivial differentials starting from E_3. □

Before we study implications, let us discuss one more closely related property. The choice of name in the following definition is motivated by [32, Section 8], which discusses similar properties in the context of path space fibrations.

Definition 2.1.12. We call the action *spherical* if $\ker(H_G^*(X) \to H^*(X))$ is equal to $\mathfrak{m}H_G^*(X)$.

Of course, $\mathfrak{m}H_G^*(X)$ is always contained in the kernel because on the level of Sullivan models the map is just the projection

$$R \otimes \Lambda V \to \Lambda V$$

obtained by sending $\mathfrak{m}$ to 0. However the kernel on the level of cohomology may be bigger for there may be Massey products represented in $\mathfrak{m} \otimes \Lambda V$ which on the level of cohomology might not lie in the multiples of $\mathfrak{m}$, see e.g. Example 2.4.5. In this light, being spherical is a restriction on the existence of nontrivial Massey products and therefore related to formality properties. Again, for $G = T$ we can express this as degeneracy in a spectral sequence.

Proposition 2.1.13. *The action of a torus T on X is spherical if and only if in the Serre spectral sequence of*

$$T \to X \to X_T$$

no nontrivial differentials enter $E_r^{*,0}$ for $r \geq 3$.

Proof. Since the map $H_T^*(X) \to H^*(X)$ factors as $H^*(X_T) \cong E_2^{*,0} \to E_\infty^{*,0} \subset H^*(X)$, it suffices to analyse the spectral sequence. From the description of the E_2-page in the proof of Proposition 2.1.11 we deduce that the image of d_2 in $E_2^{*,0}$ corresponds exactly to $\mathfrak{m}H_T^*(X)$. Thus there is more in the kernel if and only if there is a nontrivial differential mapping to the bottom row after the E_2 page. □

Of course, the above proposition as well as Proposition 2.1.11 can be (less elegantly) formulated for arbitrary compact G. The problem however is that the cohomological generators of $H^*(G)$ will not transgress on the E_2-page for degree reasons. Still, for a general G-action we have the following

Theorem 2.1.14. *For any G-action we have the implications*

$$\mathcal{MOD}\text{-formal} \Rightarrow \text{almost } \mathcal{MOD}\text{-formal} \Rightarrow \text{spherical}.$$

Proof. The first implication is a consequence of Theorem 2.1.5. For torus actions the second implication is a consequence of the above propositions. In the general setting we instead consider the Eilenberg-Moore spectral sequence of the Borel-fibration. The column $E_\infty^{0,*}$ can be identified with the image of the map $H_G^*(X) \to H^*(X)$ (see [50, Exercise 7.5]). This is a subspace of $E_2^{0,*} = \mathrm{Tor}_R^{0,*}(H_G^*(X), R/\mathfrak{m}) = H_G^*(X)/\mathfrak{m}H_G^*(X)$. Hence for dimensional reasons (all objects are degree wise finite-dimensional), being spherical is equivalent to the vanishing of all differentials starting in this column. By Remark 2.1.10 this holds if the action is almost $\mathcal{MOD}$-formal. □

In the above theorem, none of the converse implications hold, not even for simply-connected compact manifolds. This is demonstrated by Examples 2.4.6 and 2.4.7 as well as Remark 2.4.8. In the discussion, the following description of spherical actions will be helpful.

Lemma 2.1.15. *The G-action on X is spherical if and only if there is a generating set of the R-module $H_G^*(X)$ such that the restriction $H_G^*(X) \to H^*(X)$ is injective on its $\mathbb{Q}$-span.*

Proof. The condition is equivalent to the existence of a $\mathbb{Q}$-subspace $V \subset H_G^*(X)$ such that the projection $V \to H_G^*(X)/\mathfrak{m}H_G^*(X)$ is surjective and the projection $V \to H_G^*(X)/\ker(r)$ is injective. Since all spaces are degreewise finite-dimensional and $\mathfrak{m}H_G^*(X) \subset \ker(r)$, the condition is equivalent to equality in the last inclusion. □

2.1.2 Actions with formal core

As for the previously introduced notions, we search for ways to impose the triviality of certain Massey products in the equivariant cohomology and thus create a more local version of formality.

One of the more direct ways to do this is to demand that, rationally, the map $X_G \to BG$ factors cohomologically injectively through a formal space. On models this is equivalent to the following condition: let

$$(R, 0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$$

be a model of the Borel fibration of the action. Let $A \subset H_G^*(X)$ be a subalgebra that contains $\mathrm{im}(R \to H_G^*(X))$ and let $(C, d) \simeq (A, 0)$ be a relative minimal model of the canonical morphism $(R, 0) \to (A, 0)$.

Definition 2.1.16. We call the action *formally based with respect to A* if there exists a morphism $(C, d) \to (R \otimes \Lambda V, D)$ of R-cdgas for which the induced map

$$A = H^*(A, 0) \cong H^*(C, d) \to H^*(R \otimes \Lambda V, D) \cong H_G^*(X)$$

is the inclusion. If the action is formally based with respect to $\mathrm{im}(R \to H_G^*(X))$, we also just refer to it as being *formally based.*

Remark 2.1.17. (i) The existence of the morphism $(C, d) \to (R \otimes \Lambda V, D)$ in the definition does not depend on the particular choice of relative minimal models (see Remark 1.2.1). However, note that its homotopy class may not be unique.

(ii) If an action is formally based with respect to some A, then it is automatically formally based with respect to any A' satisfying $\operatorname{im}(R \to H^*_G(X)) \subset A' \subset A$ as the inclusion $A' \subset A$ lifts to the Sullivan models. In particular the action is formally based if the homotopy quotient X_G is formal.

(iii) The existence of an orbit with isotropy of maximal rank implies the injectivity of $R \to H^*_G(X)$ (the converse follows by Borel localization in case X is compact). In this case, for $A = \operatorname{im}(R \to H^*_G(X))$, one may choose $(C, d) = (R, 0)$ so the action is formally based.

A special case of this is given by the following

Definition 2.1.18. An action is *hyperformal* if the kernel of $R \to H^*_G(X)$ is generated by a homogeneous regular sequence.

Lemma 2.1.19. *Hyperformal actions are formally based.*

Proof. In the notation above we have $A = R/(a_1, \ldots, a_k)$ for homogeneous a_i which form a regular sequence in R. A relative minimal model of $(R, 0) \to (A, 0)$ is given by $(C, d) = (R \otimes \Lambda(s_1, \ldots, s_k), d)$ with $d|_R = 0$ and $d(s_i) = a_i$. The fact that the a_i form a regular sequence implies that the map $C \to A$, defined by sending the s_i to 0 and projecting R canonically to A, is a quasi-isomorphism. Also the a_i are exact in $R \otimes \Lambda V$ by definition. We choose $z_i \in R \otimes \Lambda V$ with $D(z_i) = a_i$. The desired lift $C \to R \otimes \Lambda V$ is now obtained by sending s_i to z_i and R identically to R. □

It is natural to not only demand "formality of the image" of $H^*(BG) \to H^*_G(X)$ through the formally based condition but to also pay attention to how said image is embedded in the ambient space. Especially with regards to the TRC, it is interesting to impose additional degeneracy conditions. For example, this shows up in the requirements of Theorem 2.3.1 compared to those of Theorem 2.3.5.

We will investigate another such condition: given an action that is formally based with respect to some $A \subset H^*_G(X)$, we define the algebra (C, d) as above. We may form the cdga $(\overline{C}, \overline{d})$ where $\overline{C} = C/(R^+)$, which is again a Sullivan algebra. Now any morphism $(C, d) \to (R \otimes \Lambda V, D)$ of R-cdgas induces a morphism $(\overline{C}, \overline{d}) \to (\Lambda V, d)$ into the model of X because $\Lambda V = (R \otimes \Lambda V)/(R^+)$.

Definition 2.1.20. Let A be an R-subalgebra of $H^*_G(X)$. We say an action has *formal core with respect to* A if there is a morphism $(C, d) \to (R \otimes \Lambda V, D)$ as in Definition 2.1.16 such that additionally the induced morphism $(\overline{C}, \overline{d}) \to (\Lambda V, d)$, obtained by dividing out R^+, is cohomologically injective. If such an A exists, we also just refer to the action as having *formal core*.

Remark 2.1.21. (i) If X_G is formal, then we can choose $A = H^*_G(X)$, $C = R \otimes \Lambda V$. The map $\overline{C} \to \Lambda V$ is just the identity and the action has formal core.

(ii) Other than for the notion of being formally based, having formal core with respect to A does not imply formal core with respect to any $A' \subset A$: the injectivity of $H^*(\overline{C}, \overline{d}) \to H^*_G(X)$ depends on the particular choice of A. This can be observed in Example 2.4.12.

(iii) In case the action has an orbit with isotropy of maximal rank, as in Remark 2.1.17, the action is formally based with $(C, d) = (R, 0)$. Thus $\overline{C} = \mathbb{Q}$ and the action automatically has formal core.

(iv) Any S^1-action has formal core: we are either in case (iii) or $\operatorname{im}(R \to H_{S^1}(X))$ is isomorphic to $R/(X_1^n)$ for some n, where $R = \mathbb{Q}[X_1]$. We may thus set $C = (R \otimes \Lambda(\alpha), D)$ with $D\alpha = X_1^n$ and obtain a lift $C \to R \otimes \Lambda V$ by mapping α to some β with $D\beta = X_1^n$. It suffices to argue that the projection $\overline{\beta} \in \Lambda V$ is not exact. If $d\gamma = \overline{\beta}$, then $\beta - D\gamma$ is divisible by X_1 and thus $D(X_1^{-1}(\beta - D\gamma)) = X_1^{n-1}$ which is a contradiction.

Proposition 2.1.22. *Let $f\colon X \to Y$ be an equivariant map of G-spaces that induces injections on both regular and equivariant cohomology (e.g. an equivariant retract). If the action on Y has formal core, then so does the action on X.*

Proof. The equivariant map f induces a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & \mathcal{M}_G^X & \longrightarrow & \mathcal{M}^X \\ \uparrow{\scriptstyle \mathbf{1}_R} & & \uparrow{\scriptstyle f_G^*} & & \uparrow{\scriptstyle f^*} \\ R & \longrightarrow & \mathcal{M}_G^Y & \longrightarrow & \mathcal{M}^Y \end{array}$$

in which the rows are relative minimal models for the Borel fibrations of X and Y. Suppose Y has formal core with respect to $A \subset H^*_G(Y)$ and let $\iota\colon (C, d) \to \mathcal{M}_G^Y$ and $\overline{\iota}\colon (\overline{C}, \overline{d}) \to \mathcal{M}^Y$ be constructed as above. By assumption the maps $f_G^* \circ \iota$ and $f \circ \overline{\iota}$ are both injective on cohomology. Consequently the action on X has formal core with respect to $f_G^*(A) \subset H^*_G(X)$. □

Theorem 2.1.23. *(i) Let G act on X such that the action is almost free and formally based with respect to $A \subset H^*_G(X)$. If A satisfies Poincaré duality, then the action has formal core with respect to A.*

(ii) Any hyperformal action has formal core.

Proof. Let C and $\overline{C}$ be defined as above. By assumption we have a commutative diagram

$$\begin{array}{ccc} (C \otimes S, d) & \longrightarrow & (R \otimes \Lambda V \otimes S, D) \\ \downarrow & & \downarrow \\ (\overline{C}, \overline{d}) & \longrightarrow & (\Lambda V, d) \end{array}$$

where the differentials in the top row map the generators of $S = \Lambda(s_1, \ldots, s_r)$ bijectively onto the generators of $R = \Lambda(X_1, \ldots, X_r)$ and the vertical maps are quasi-isomorphisms defined by sending R^+ and S^+ to 0. It suffices to show cohomological injectivity of the top horizontal map which we denote by φ.

As $A = H^*(C)$ is a Poincaré duality algebra, it follows that the cohomology of $C \otimes S$ satisfies Poincaré duality as well (see [20, Lemma 38.2]). The fundamental class of $C \otimes S$ is contained in every nontrivial ideal of $H^*(C \otimes S)$. Hence it maps to 0 under

$$\varphi^*\colon H^*(C \otimes S) \to H^*(R \otimes \Lambda V \otimes S)$$

if and only if $\ker \varphi^* \neq 0$. As a consequence, we only need to prove injectivity of φ^* on the top degree cohomology.

To see this, filter $C \otimes S$ by degree in C and $R \otimes \Lambda V \otimes S$ by degree in $R \otimes \Lambda V$. The morphism φ respects this filtration and consequently induces a map between the spectral sequences. On the E_2 page, this morphism is given by the inclusion

$$A \otimes S \to H_G^*(X) \otimes S.$$

Since the top degree cohomology of $A \otimes S$ is located in the top row, its image on the right cannot be killed by any differential and hence induces a nontrivial element on the ∞-page. This shows that φ^* is not 0 in the top degree which concludes the proof of (i).

In the situation of (ii), if the kernel of $R \to H_G^*(X)$ is generated by a homogeneous regular sequence $f_1, \ldots, f_k$, then the action is formally based by Lemma 2.1.19 and we have a morphism of R-cdgas $\psi\colon (C, d) \to (R \otimes \Lambda V, D)$, where $C = R \otimes \Lambda(a_1, \ldots, a_k)$ and $da_i = f_i$. We want to prove that the map $\overline{\psi}\colon \overline{C} \to \Lambda V$ is cohomologically injective. Observe that if the regular sequence was maximal in $R = \Lambda(X_1, \ldots, X_r)$, meaning $k = r$, then $H^*(C)$ would be a Poincaré duality algebra and we could use (i) to finish the proof.

If $k < r$ we may complete the f_i to a maximal regular sequence and extend ψ to a map $(C \otimes \Lambda(a_{k+1}, \ldots, a_r), d) \to (R \otimes \Lambda V \otimes \Lambda(a_{k+1}, \ldots, a_r), D)$ where the differentials map the additional a_i onto the additional f_i. Cohomological injectivity of

$$\overline{\psi} \otimes \mathbf{1}\colon \overline{C} \otimes \Lambda(a_{k+1}, \ldots, a_r) \to \Lambda V \otimes \Lambda(a_{k+1}, \ldots, a_r)$$

is equivalent to the original map $\overline{\psi}$ being injective. Applying (i) yields the claim. □

Remark 2.1.24. We want to investigate how all of the previously defined notions interact. As it turns out, the concepts introduced in this section seem to be rather independent from the notion of $\mathcal{MOD}$-formality: Example 2.4.10 is a hyperformal action (with formal core), which is not spherical and thus in particular not (almost) $\mathcal{MOD}$-formal. Furthermore, in 2.4.11, we construct an example of a $\mathcal{MOD}$-formal action which is not formally based. However, we want to point out that while no direct implications exist, common roots can be found in the vanishing of certain Massey products, which is best captured in the degeneracy of minimal C_∞-structures: in Theorem 2.3.1 we give a sufficient condition for $\mathcal{MOD}$-formality that builds upon the notion of being formally based (see also Theorem 2.3.5).

2.2 Inheritance under elementary constructions

2.2.1 Products

If X is a G-space and Y is a G' space, then $X \times Y$ is naturally a $G \times G'$ space.

Proposition 2.2.1. *Let X be a G-space, Y be a G' space. If both, the G- and the G'-action, satisfy one of the conditions of being spherical, (almost) $\mathcal{MOD}$-formal, formally based or having formal core, then the same is true for the $G \times G'$-action on $X \times Y$.*

Proof. Let $R \to \mathcal{M}_G^X \to \mathcal{M}^X$ and $R' \to \mathcal{M}_{G'}^Y \to \mathcal{M}^Y$ be relative minimal models for the respective Borel fibrations. Then

$$R \otimes R' \to \mathcal{M}_G^X \otimes \mathcal{M}_{G'}^Y \to \mathcal{M}^X \otimes \mathcal{M}^Y$$

is a relative minimal model for the Borel fibration of the $G \times G'$-action. The associated Hirsch–Brown model is given by the tensor product of the minimal Hirsch–Brown models of the G- and the G'-action hence the statement for almost $\mathcal{MOD}$-formal actions follows. Furthermore, formality is compatible with the tensor product and we obtain the proposition under the assumption of $\mathcal{MOD}$-formality. In the case of spherical actions, the kernel of $H_G^*(X) \otimes H_{G'}^*(Y) \to H^*(X) \otimes H^*(Y)$ is given by

$$(R^+ \cdot H_G^*(X)) \otimes H_{G'}^*(Y) + H_G^*(X) \otimes (R'^+ \cdot H_{G'}^*(Y)) = (R \otimes R')^+ \cdot H_{G\times G'}^*(X \times Y).$$

Finally, if the G-action is formally based with respect to $A \subset H_G^*(X)$ and the G'-action is formally based with respect to $A' \subset H_{G'}^*(Y)$, then, by considering the tensor product of the occurring maps, we see that the $G \times G'$-action is formally based with respect to $A \otimes A' \subset H_G^*(X) \otimes H_{G'}^*(Y) \cong H_{G\times G'}^*(X \times Y)$. In the same way one obtains the statement for actions with formal core. ☐

2.2.2 Gluing

In this section we assume all G-spaces to be Tychonoff spaces (so e.g. CW-complexes) in order to ensure the existence of tubular neighbourhoods (see [11, Theorem 5.4]). Let X and Y be two G-spaces. If $G \to X$ and $G \to Y$ are equivariant maps onto orbits of X and Y, we may construct their pushout $X \vee_G Y$, which is naturally a G-space. If the stabilizers of the image of $1 \in G$ in X and Y agree, then $X \vee_G Y$ is just X and Y glued together at these orbits.

Proposition 2.2.2. *Suppose that X and Y are $\mathcal{MOD}$-formal (resp. have formal homotopy quotients X_G and Y_G) and have an almost free G-orbit of the same orbit type. Then the G-space $X \vee_G Y$ obtained by gluing X and Y along this orbit is $\mathcal{MOD}$-formal (resp. has formal homotopy quotient).*

In the proof we make use of the following notation: if A and B are connected cdgas, then $A \oplus_{\mathbb{Q}} B$ is the sub-cdga of the product cdga $A \oplus B$ which in degree 0 is generated by $(1, 1)$ and agrees with $A \oplus B$ in positive degrees. We use the analogous notation for dg$R$ms whose degree 0 component is $\mathbb{Q}$.

Proof. The orbits are equivariant retracts of open G-invariant neighbourhoods in X and Y. Using these, we can cover $X \vee_G Y$ with open sets U, V which equivariantly retract onto X and Y and whose intersection retracts equivariantly onto the glued orbit. We consider the equivariant Mayer–Vietoris sequence of this cover. As the equivariant cohomology of an almost free orbit is just $\mathbb{Q}$, it follows that

$$H_G^*(X \vee_G Y) \cong H_G^*(X) \oplus_{\mathbb{Q}} H_G^*(Y).$$

We now turn our attention to models. Let $\mathcal{M}_G^{X\vee_G Y}$, $\mathcal{M}_G^X$, and $\mathcal{M}_G^Y$ denote the Sullivan models of the homotopy quotients of the respective G-spaces as in Remark 1.2.1 (*ii*). The previous discussion implies that the induced map $\mathcal{M}_G^{X\vee_G Y} \to \mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y$ is a quasi-isomorphism. In fact it can be chosen as a morphism of R-cdgas, where the R-module structure on the right is the diagonal one. If the actions have formal homotopy quotient, then we have a quasi-isomorphism

$$\mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y \to H_G^*(X) \oplus_{\mathbb{Q}} H_G^*(Y)$$

of cdgas which ends the proof. In case the actions are $\mathcal{MOD}$-formal, let $\mathfrak{M}_X$ and $\mathfrak{M}_Y$ denote the Hirsch–Brown models of the action and note that by the previous discussion, $\mathfrak{M}_X \oplus_{\mathbb{Q}} \mathfrak{M}_Y$ is a dg$R$m-model for $\mathcal{M}_G^{X\vee_G Y}$. We have a quasi-isomorphism

$$\mathfrak{M}_G^X \oplus_{\mathbb{Q}} \mathfrak{M}_G^Y \to H_G^*(X) \oplus_{\mathbb{Q}} H_G^*(Y)$$

of dg$R$ms, which finishes the proof. □

Proposition 2.2.3. *Let X and Y be formally based G-spaces. Then $X \vee_G Y$ obtained by gluing X and Y along an almost free orbit of the same orbit type is formally based. Moreover, if the actions have formal core and the Sullivan models for X_G and Y_G in a relative minimal model of the respective Borel fibrations are already minimal, then the action on $X \vee_G Y$ has formal core.*

If X_G and Y_G are nilpotent spaces, then the technical minimality condition on their models in the above proposition is equivalent to the surjectivity of

$$\pi_k(X_G) \otimes \mathbb{Q} \to \pi_k(BG) \otimes \mathbb{Q} \quad \text{and} \quad \pi_k(Y_G) \otimes \mathbb{Q} \to \pi_k(BG) \otimes \mathbb{Q}$$

for $k \geq 2$. In particular the condition is automatically fulfilled in case G is a torus and X and Y are simply-connected.

Proof. Suppose the G-spaces are formally based with respect to $A \subset H_G^*(X)$ and $A' \subset H_G^*(Y)$. Let $R \to \mathcal{M}_G^X$ and $R \to \mathcal{M}_G^Y$ be relative minimal models for the respective Borel fibrations. As in the proof of the previous proposition, we see that $\mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y$ and $\mathcal{M}_G^{X\vee_G Y}$ are quasi-isomorphic as R-cdgas, where the R-module structure of the sum is defined by the diagonal inclusion. Let $R \otimes C$ and $R \otimes C'$ be relative minimal models for $R \to A$ and $R \to A'$. Also let $R \to C''$ be a relative minimal model for $R \to A \oplus_{\mathbb{Q}} A'$. We may lift $C'' \to A \oplus_{\mathbb{Q}} A'$ to $C \oplus_{\mathbb{Q}} C'$ by applying the lifting lemma of Sullivan algebras to each summand separately. By assumption we have maps $C \to \mathcal{M}_G^X$ and $C' \to \mathcal{M}_G^Y$ inducing the inclusion on cohomology. Piecing everything together we obtain the composition

$$C'' \to C \oplus_{\mathbb{Q}} C' \to \mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y$$

which induces the inclusion $A \oplus_{\mathbb{Q}} A' \to H_G^*(X) \oplus_{\mathbb{Q}} H_G^*(Y) \cong H_G^*(X \vee_G Y)$ on cohomology. Lifting the morphism to $\mathcal{M}_G^{X\vee_G Y}$ relative to R shows that the action on $X \vee_G Y$ is formally based with respect to $A \oplus_{\mathbb{Q}} A'$.

Now assume that the actions on X and Y have formal core with respect to A and A'. For X (and analogously for Y) this means that we can assume the map $(C \otimes S, D) \to (\mathcal{M}_G^X \otimes S, D)$ between Hirsch extensions to be injective on cohomology, where D maps generators (s_i) of $S = \Lambda(s_i)$ to the generators X_i of $R = \Lambda(X_1, \ldots, X_r)$. In order to show that $X \vee_G Y$ has formal core, it suffices to show that

$$((C \oplus_{\mathbb{Q}} C') \otimes S, D) \to ((\mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y) \otimes S, D)$$

is injective on cohomology where $Ds_i = (X_i, X_i)$. Consider the commutative diagram

$$\begin{array}{ccc} (C \oplus_{\mathbb{Q}} C') \otimes S & \longrightarrow & (\mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y) \otimes S \\ \downarrow & & \downarrow \\ (C \oplus C') \otimes S & \longrightarrow & (\mathcal{M}_G^X \oplus \mathcal{M}_G^Y) \otimes S \end{array}$$

in which the bottom horizontal map is actually the direct sum of $C \otimes S \to \mathcal{M}_G^X \otimes S$ and $C' \otimes S \to \mathcal{M}_G^Y \otimes S$. By assumption this map is injective on cohomology, and it remains to prove that, on cohomology, the top horizontal map is injective on the kernel

$$K = \ker (H^*((C \oplus_{\mathbb{Q}} C') \otimes S) \longrightarrow H^*((C \oplus C') \otimes S))$$

of the morphism induced by the left vertical inclusion. Observe that the cokernel of this inclusion has a basis represented by a basis of $(1, 0) \otimes S$. The differential induced on the cokernel vanishes. The resulting short exact sequence of complexes

$$0 \longrightarrow ((C \oplus_{\mathbb{Q}} C) \otimes S, D) \longrightarrow ((C \oplus C) \otimes S, D) \longrightarrow ((1, 0) \otimes S, 0) \longrightarrow 0$$

induces a long exact sequence on homology from which we see that K is represented by $D((1, 0) \otimes S)$.

Thus it suffices to show that $D((1, 0) \otimes S)$ descends injectively to the cohomology of $(\mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y) \otimes S$. By assumption, $\mathcal{M}_G^X$ and $\mathcal{M}_G^Y$ are actually minimal as Sullivan models. Consequently, $D((\mathcal{M}_G^X \oplus \mathcal{M}_G^Y)^+ \otimes S) \subset ((\mathcal{M}_G^X)^+ \cdot (\mathcal{M}_G^X)^+ \oplus (\mathcal{M}_G^Y)^+ \cdot (\mathcal{M}_G^Y)^+) \otimes S)$. The only way to hit an element of $D((1, 0) \otimes S) \subset \langle (X_1, 0), \ldots, (X_r, 0) \rangle \otimes S$ with the differential of $(\mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y) \otimes S$ is as the image of some element in $(1, 1) \otimes S$. These, however, are never 0 in the second component which proves that K maps injectively to $H^*((\mathcal{M}_G^X \oplus_{\mathbb{Q}} \mathcal{M}_G^Y) \otimes S)$. □

Let M be a smooth G-manifold with (possibly empty) boundary. By the slice theorem, the orbit of any interior point $x \in M$ has a G-invariant tubular neighbourhood which is equivariantly diffeomorphic to

$$V_x = G \times_{G_x} T_x M / T_x(G \cdot x)$$

where G_x acts on the right hand side via the isotropy action and $G \cdot x$ corresponds to $[G, 0] \subset V_x$. Now if N is a G-manifold with boundary of the same dimension, with an interior point $y \in N$ such that $G_y = G_x$ and $T_x M / T_x(G \cdot x) \cong T_y N / T_y(G \cdot y)$ as representations, then we may form the equivariant connected sum as follows: the isomorphism defines an equivariant diffeomorphism $\varphi\colon V_x \cong V_y$ of tubular neighbourhoods. Choose some G_x-invariant inner product on $T_x M / T_x(G \cdot x)$, and let $S \subset T_x M / T_x(G \cdot x)$ be the associated unit sphere. The equivariant connected sum $M \#_G N$ is the quotient of $(M - \{G \cdot x\}) \sqcup (N - \{G \cdot y\})$ obtained by gluing the points $[g, ts]$ and $\varphi([g, (1 - t)s])$ for all $g \in G$, $t \in (0, 1)$, and $s \in S$. The result is a G-manifold with boundary whose homotopy type does however depend on the choice of φ.

Proposition 2.2.4. *Let M and N be almost free G-manifolds with boundary such that the equivariant connected sum $M\#_G N$ at some interior orbit is defined.*

(i) If M and N are $\mathcal{MOD}$-formal, then so is $M\#_G N$.

(ii) If M and N are simply-connected and M_G and N_G are formal, then also $(M\#_G N)_G$ is formal.

We want to point out that in the proof of *(ii)*, one of the cases is essentially the proof of the fact that the (nonequivariant) connected sum of compact simply-connected manifolds preserves formality (see e.g. [21, Theorem 3.13]).

Proof. In the notation of the construction of $M\#_G N$, let $D \subset T_xM/T_x(G\cdot x)$ be the unit disk. The collapsing of $D-\{0\}$ induces an equivariant map $G\times_{G_x}(D-\{0\})\to G\cdot x$ which induces an equivariant map

$$p\colon M\#_G N \longrightarrow M\vee_G N.$$

Both are almost free G-spaces so the map on equivariant cohomology can be determined from the orbit spaces (see Proposition 1.1.2). There it induces the collapse of the subspace

$$X := (G\times_{G_x}(D-\{0\}))/G \cong (D-\{0\})/G_x \simeq S/G_x.$$

Thus we may understand the map $p^*\colon H^*_G(M\vee_G N)\to H^*_G(M\#_G N)$ via the long exact homology sequence

$$\cdots\to H^k((M\#_G N)/G, X)\to H^k((M\#_G N)/G)\to H^k(X)\to\cdots$$

where we can identify $H^k((M\#_G N)/G,X)\cong H^k(M/G\vee N/G) = H^k_G(M\vee_G N)$ in positive degrees.

The algebra $H^*(X)=H^*(S)^{G_x}$ is isomorphic to either $\mathbb{Q}$ or $H^*(S)$. Let $n=\dim M-\dim G$, so $\dim S = n-1$. There are three possible scenarios. The map p^* is either surjective with 1-dimensional kernel in degree n, injective with 1-dimensional cokernel in degree $n-1$, or a quasi-isomorphism. In the last case we are done since $M\vee_G N$ is $\mathcal{MOD}$-formal by Proposition 2.2.2. Otherwise consider the map $\tilde p\colon \mathfrak{M}_\vee\to\mathfrak{M}_\#$ between minimal Hirsch–Brown models of $M\vee_G N$ and $M\#_G N$. As $M\vee_G N$ is $\mathcal{MOD}$-formal, the Hirsch–Brown model takes the form of a free resolution

$$\mathfrak{M}_\vee = \left(\bigoplus_{i\geq 0} F_i, d\right)$$

with $d\colon F_i\to F_{i-1}$ being exact at every $i\geq 1$. If p^* has nontrivial kernel, we add a generator α of degree $n-1$ to F_1 and define $d\alpha\in F_0$ to be a representative for the generator of $\ker p^*$. Also $\tilde p(d\alpha)$ is exact in $\mathfrak{M}_\#$ so we may extend $\tilde p$ to α. At this point, $\tilde p$ induces an isomorphism $F_0/d(F_1)\cong H^*_G(M\#_G N)$ but there may now be additional cohomology represented in F_1. As the newly introduced generator lives in degree $n-1$ and R is simply-connected, $\ker_d|_{F_1}$ remains unchanged up to degree $n+1$. But the cohomology of $M\#_G N$ vanishes in degrees above n so it follows that $\tilde p$ maps the newly introduced cohomology to exact elements. Hence if we introduce new generators in F_2 and use them to kill the cohomology generated in F_1, then $\tilde{p}$ extends to the new generators. We may repeat this process inductively and obtain a free resolution quasi-isomorphic to $\mathfrak{M}_\#$. Thus $M\#_G N$ is $\mathcal{MOD}$-formal.

If p^* is injective, then we start by adding a generator α to F_0 in degree $n-1$ and define $\tilde{p}(\alpha)$ to be a representative of the cokernel of p^*. Now add generators of degree $\geq n$ to F_1 and map them to a minimal generating set of $\ker(F_0 \to H_G^*(M\#_G N))$. Again $\tilde{p}$ extends to the new F_1. We are now in the same position as before and we analogously conclude that $M\#_G N$ is $\mathcal{MOD}$-formal. This proves (i).

The proof of (ii) works by applying the analogous argument to the cdga machinery. Consider the map $\varphi\colon \mathcal{M}_\vee \to \mathcal{M}_\#$ between the Sullivan minimal models of $(M \vee_G N)_G$ and $(M\#_G N)_G$. As $(M \vee_G N)_G$ is formal by Proposition 2.2.2, it has a bigraded minimal model with an additional lower grading

$$\mathcal{M}_\vee = \Lambda\left(\bigoplus_{i\geq 0} V_i\right)$$

and cohomology concentrated in $(\Lambda V)_0$ as in e.g. [21, Theorem 2.93]. We observe that $M \vee_G N$ is simply-connected if M and N are and therefore $\mathcal{M}_\vee^1 = 0$. The proof now proceeds as before with the role of F_i replaced by $(\Lambda V)_i$. If e.g. p^* is surjective with 1-dimensional kernel in degree n (and necessarily lower degree 0), then we add a generator to V_1 and use it to kill the existing kernel in cohomology. As $\mathcal{M}_\vee^1 = 0$, no new cohomology is generated up until cohomological degree $n+1$. Now for the same reasons as in the module case, we can extend φ to a quasi-isomorphism from a bigraded minimal model in the above sense, showing that $M\#_G N$ is formal. The other cases transfer analogously. □

Proposition 2.2.5. *Let M and N be almost free m-dimensional G-manifolds with boundary such that the equivariant connected sum along an interior orbit is defined. Assume that they have formal core with respect to $A \subset H_G^*(M)$, $A' \subset H_G^*(N)$, with $A^{m-\dim G} = A'^{m-\dim G} = 0$, and that the conditions of Proposition 2.2.3 are satisfied. Then $M\#_G N$ has formal core.*

We want to point out that the condition $A^{m-\dim G} = 0$ is automatically fulfilled if M has non-empty boundary, is non-compact, or is not orientable: in this case $H^m(M) = 0$ whence $H_G^{m-\dim G}(M) = 0$ by Proposition 1.1.2. In case M is closed and orientable, it just means that A is supposed to not contain the fundamental class of M/G.

Proof. Let $n = m - \dim G$ and recall the equivariant map $M\#_G N \to M \vee_G N$ from the proof of the previous proposition. As we showed there, on cohomology, it is either injective or has 1-dimensional kernel in degree n. Hence if A and A' are trivial in degree n, then

$$\psi\colon A \oplus_{\mathbb{Q}} A' \longrightarrow H_G^*(M \vee_G N) \longrightarrow H_G^*(M\#_G N)$$

is injective.

Let $\mathcal{M}_\vee$ and $\mathcal{M}_\#$ be models for $(M \vee N)_G$ and $(M\#_G N)$ arising from relative minimal models of the respective Borel fibrations and let $\varphi\colon \mathcal{M}_\vee \to \mathcal{M}_\#$ be a map of R-cdgas that is a Sullivan representative of the equivariant map $M\#_G N \to M \vee_G N$. By Proposition 2.2.3, $M \vee_G N$ has formal core with respect to $A \oplus_{\mathbb{Q}} A'$, which means we have a map $\phi\colon C'' \to \mathcal{M}_\vee$ of R-cdgas as in the proof of 2.2.3 (X and Y replaced by M and N).

Since ψ is injective, we see that $M\#_G N$ is formally based with respect to $\psi(A \oplus_{\mathbb{Q}} A')$ by considering the composition $\varphi \circ \phi$.

To prove that the action has formal core it remains to see that this map is still cohomologically injective when extending it to the Hirsch extensions

$$C'' \otimes S \longrightarrow \mathcal{M}_\vee \otimes S \longrightarrow \mathcal{M}_\# \otimes S$$

in which the differential maps generators of $S = \Lambda(s_i)$ bijectively to generators of R. Cohomological injectivity of $\phi \otimes \mathbf{1}_S$ is part of Proposition 2.2.3, so we only need to prove injectivity of $(\varphi \otimes \mathbf{1}_S)^*$ on the cohomological image of $\phi \otimes \mathbf{1}_S$. To see this we consider the maps between the Serre spectral sequences arising by filtering the Hirsch extensions above in the degrees of the left hand cdgas. The second pages are isomorphic to the tensor products of the (non-twisted) cohomologies of the respective factors so by naturality we obtain the maps

$$(A \oplus_{\mathbb{Q}} A') \otimes S \longrightarrow H^*_G(M \vee_G N) \otimes S \longrightarrow H^*_G(M\#_G N) \otimes S.$$

Again, we distinguish the three possible cases for the map φ^*. If it is an isomorphism, then so is $\varphi^* \otimes \mathbf{1}_S$ and we are done. If φ^* is injective and its cokernel is generated by some $\alpha \in H^{n-1}(M\#_G N)$, then the map $\varphi^* \otimes \mathbf{1}_S$ on the second pages is an isomorphism up to the column $\alpha \otimes S \subset E_2^{n-1,*}$. As the differentials in the spectral sequence vanish on this column for degree reasons, we deduce that $\varphi \otimes \mathbf{1}_S$ induces an injective map between the E_∞-pages which implies injectivity on cohomology.

Finally, consider the case where φ^* is an isomorphism up to 1-dimensional kernel contained in $H^n_G(M \vee_G N)$. We claim that the kernel of the map induced by $\varphi \otimes \mathbf{1}_S$ on the E_∞-pages is contained in $E_\infty^{n,*}$. Since the image of $\phi^* \otimes \mathbf{1}_S$ on the E_2 pages is contained in $E_2^{<n,*}$, and the same degree restrictions carry over to the E_∞ pages, this will imply injectivity of $(\varphi \circ \phi) \otimes \mathbf{1}_S$ on the E_∞-pages and thus finish the proof of the proposition. The claim can be verified via induction: assume the map between the rth pages is injective on $E_r^{<n,*}$ and an isomorphism on $E_r^{\leq n-r,*}$. Then it is a straightforward diagram chase to show that on the $(r+1)$th pages the induced map is injective on $E_{r+1}^{<n,*}$ and an isomorphism on $E_{r+1}^{\leq n-r-1,*}$.

□

2.2.3 Subgroups

We investigate how the previously defined notions behave under restriction of the action to subgroups. As it turns out, problems arise when restricting to subgroups of smaller rank. The only one of the discussed concepts which behaves well under restriction to arbitrary subgroups is the classical equivariant formality. Example 2.4.5 is an action with formal homotopy quotient such that the restriction to a certain subgroup is neither formally based nor spherical. However, we have the following

Proposition 2.2.6. *A G-action fulfils one of the conditions of being spherical, almost $\mathcal{MOD}$-formal, or $\mathcal{MOD}$-formal if and only if the respective condition is fulfilled by the action of a maximal subtorus.*

Proof. Let G act on X and let T be a maximal torus of G. The central observation needed for the proof is the fact that the Borel fibration of the T-action is the pullback of the Borel fibration of the G-action along the map $BT \to BG$. We denote the minimal models of BT and BG by R and S. Now if

$$(S,0) \to (S \otimes \Lambda V, D) \to (\Lambda V, d)$$

is a minimal model for the fibration of the G-action, a minimal model for the pullback fibration is given by

$$(R,0) \to (R \otimes_S (S \otimes \Lambda V), \mathbf{1}_R \otimes_S D) \to (\Lambda V, d)$$

(see [20, Prop. 15.8]). Since T is maximal, the map $S \to R$ turns R into a finitely generated, free S-module. In particular, $R \otimes_S (S \otimes \Lambda V) \cong R \otimes \Lambda V$ splits as a sum of multiple degree shifted copies of $S \otimes \Lambda V$. This decomposition is respected by the differential so it induces an analogous splitting of the cohomology. We obtain $H_T^*(X) = R \otimes_S H_G^*(X)$ (actually as algebras although the splitting is one of S-modules).

As tensoring with R over S is exact, we deduce that the minimal free resolution of $H_T^*(X)$ is obtained from the one of $H_G^*(X)$ by tensoring with R. This implies that the G-action is almost $\mathcal{MOD}$-formal if and only if this holds for the T-action.

In view of the statement for spherical actions, this also shows that a minimal generating set of $H_G^*(X)$, i.e. one which descends to a basis of $H_G^*(X)/S^+ \cdot H_G^*(X)$, is also a minimal generating set of $H_T^*(X)$. By Lemma 2.1.15, the condition of being spherical is equivalent to the restriction to $H^*(X)$ being injective on the span of such a generating set. The claim now follows from the observation that the inclusion

$$H_G^*(X) = S \otimes_S H_G^*(X) \subset R \otimes_S H_G^*(X) = H_T^*(X)$$

commutes with the restriction to $H^*(X)$.

We turn our attention to $\mathcal{MOD}$-formal actions. Let $(\mathfrak{M}, \tilde{D}) \to (S \otimes \Lambda V, D)$ be the minimal Hirsch–Brown model of the G-action, with $\mathfrak{M} = S \otimes H^*(X)$. By the previous discussion it follows that

$$(R \otimes H^*(X), \tilde{D}) \cong (R \otimes_S \mathfrak{M}, \mathbf{1}_R \otimes_S \tilde{D}) \to (R \otimes_S (S \otimes \Lambda V), D)$$

induces a quasi-isomorphism. Note that the induced differential on $R \otimes_S \mathfrak{M}$ satisfies the minimality condition so this is indeed the minimal Hirsch–Brown model of the T-action.

A quasi-isomorphism $(\mathfrak{M}, \tilde{D}) \to (H_G^*(X), 0)$ induces a quasi-isomorphism

$$(R \otimes_S \mathfrak{M}, \tilde{D}) \to (R \otimes_S H_G^*(X), 0).$$

Thus $\mathcal{MOD}$-formality of the G-action implies $\mathcal{MOD}$-formality of the T-action. For the converse implication we use criterion (iii) of Lemma 2.1.8 by which we obtain a vector space splitting $R \otimes_S \mathfrak{M} = \ker \tilde{D} \oplus C$ such that $C \oplus \operatorname{im} \tilde{D}$ is an R-submodule. As argued above, $R \otimes_S \mathfrak{M}$ splits as the sum of multiple degree shifted copies of $\mathfrak{M}$ when regarded as a differential graded S-module. We identify $\mathfrak{M}$ with the summand $S \otimes_S \mathfrak{M} \subset R \otimes_S \mathfrak{M}$. Denote by $\pi\colon R \otimes_S \mathfrak{M} \to \mathfrak{M}$ the projection onto this summand. The differential restricts to an isomorphism $\tilde{D}\colon C \to \operatorname{im}\tilde{D}$ and we denote its inverse by $\tilde{D}^{-1}$. Set

$$C' = \pi \circ \tilde{D}^{-1}(\mathfrak{M} \cap \operatorname{im}\tilde{D}).$$

We claim that C' fulfils the requirements of Lemma 2.1.8 (iii) with respect to the G-action. We observe that in the commutative diagram

$$\begin{array}{ccc} \tilde{D}^{-1}(\mathfrak{M}\cap \mathrm{im}\tilde{D}) & & \\ \downarrow \pi & \searrow^{\tilde{D}} & \\ C' & \xrightarrow{\tilde{D}} & \mathfrak{M}\cap \mathrm{im}\tilde{D} \end{array}$$

all maps are isomorphisms. It follows that C' is a complement of $\ker \tilde{D}|_{\mathfrak{M}}$ in $\mathfrak{M}$ and it remains to show that a closed S-linear combination $\sum s_i c_i'$ of elements in C' is already exact in $\mathfrak{M}$. As $\tilde{D}|_C$ is an isomorphism onto $\mathrm{im}\tilde{D}$, there are unique elements $c_i \in C$ with $\tilde{D}c_i = \tilde{D}c_i'$. They fulfil $\pi(c_i) = c_i'$ and have closed elements in all the other components with respect to the decomposition of $R\otimes_S \mathfrak{M}$ (because the differential respects the decomposition). Consequently, the element $\sum s_i c_i$ is closed. By the choice of C it follows that it is already exact which is equivalent to exactness in every component. In particular, $\sum s_i c_i'$ is exact in $\mathfrak{M}$, which proves the claim. □

Proposition 2.2.7. *If the G-action is formally based or has formal core, then the same holds for the action of its maximal torus.*

Proof. Let S, R as above and assume the G-action is formally based with respect to some $A \subset H_G^*(X)$. Let $(C, d) \simeq (A, 0)$ be a relative minimal model of the canonical morphism $(S, 0) \to (A, 0)$. The map $R \to H_T^*(X)$ corresponds to $R \otimes_S S \to R \otimes_S H_G^*(X)$ which has image $R\otimes_S A$, and the induced map $(R\otimes_S C, d) \simeq (R\otimes_S A, 0)$ is a relative minimal model for $(R \otimes_S S, 0) \to (R \otimes_S A, 0)$. Clearly, a morphism $(C, d) \to (S \otimes \Lambda V, D)$ of S-cdgas induces a morphism $(R\otimes_S C, d) \to (R\otimes_S (S\otimes \Lambda V), D)$ of R-cdgas. Thus the T-action is formally based if the G-action is.

If the G-action has formal core with respect to A, then we may take $C/S^+ \to \Lambda V$ to be cohomologically injective. It factors through the morphism $C/S^+ \to (R \otimes_S C)/R^+$, which is an isomorphism. This implies that also $(R \otimes_S C)/R^+ \to \Lambda V$ is cohomologically injective. Consequently, the T-action has formal core as well. □

2.3 Higher operations on the cohomology

As established earlier (see Prop. 1.2.4 and before), fixing a base space Y, there is a correspondence between free torus actions with orbit space Y and degree 2 cohomology classes of Y. This correspondence is one-to-one in a suitable rational sense so it is a natural question how the formality properties of those actions are encoded in the corresponding cohomology classes. The usual algebra structure on the cohomology is not sufficient for answering this kind of question. Instead, in this section we attack the problem via certain higher operations on the cohomology.

We consider, more generally, any G action on X. As before, let $R \otimes \Lambda V$ be a Sullivan model for X_G. Then we can consider its minimal C_∞-model $(H_G^*(X); 0, m_2, m_3, \ldots)$ which is unique up to isomorphism of C_∞-algebras (see Section 1.4.4). It is known that X_G is formal if and only if it admits a C_∞-model of the form $(H_G^*(X), 0, m_2, 0, \ldots)$ where all higher operations vanish (see Theorem 1.4.22). Thus there is a characterization of actions with formal homotopy quotient in terms of the higher operation on the equivariant cohomology. Our goal is to find something similar for $\mathcal{MOD}$-formal actions.

Theorem 2.3.1. *Let $A \subset H_G^*(X)$ be the image of $H^*(BG) \to H_G^*(X)$. If the unital minimal C_∞-model $(H_G^*(X); m_i)$ can be chosen in a way that m_i vanishes on the subspace $H_G^*(X) \otimes A^{\otimes i-1}$ for $i \geq 3$, then the action is $\mathcal{MOD}$-formal.*

Remark 2.3.2. The above theorem is particularly useful for constructing free $\mathcal{MOD}$-formal torus actions (see Example 2.4.9): isomorphism classes of simply-connected, minimal, unital C_∞-models are in one-to-one correspondence with simply-connected rational homotopy types. Hence, starting with any finite-dimensional, simply-connected, minimal, unital C_∞-algebra (H, m_i), we find a finite CW-complex Y with minimal C_∞-model $(H; m_i)$. Now any choice of r elements in $H^2(Y) = H^2$ defines (the rational homotopy type of) a free T^r-space with orbit space M (see Prop. 1.2.4). If we choose the degree 2 classes in a way that their spanned subalgebra $A \subset H$ fulfils $m_i(x, a_1, \ldots, a_{i-1}) = 0$ for $a_i \in A$, $i \geq 3$, then the corresponding action will be $\mathcal{MOD}$-formal.

Proof. Let
$$(R, 0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$$
be a Sullivan minimal model of the Borel fibration of the action. Suppose there is a minimal unital C_∞-model $\varphi\colon (H_G^*(X); m_i) \to (R \otimes \Lambda V, D)$ satisfying the properties from the theorem. Then the canonical map $f_1\colon R \to H_G^*(X)$ can be extended to a C_∞-morphism $f\colon (R, 0) \to (H_G^*(X); m_i)$ by setting the higher components to be trivial. This yields a diagram
$$\begin{array}{ccc} (R,0) & \longrightarrow & (R \otimes \Lambda V, D) \\ & \overset{f}{\searrow} & \uparrow \varphi \\ & & (H_G^*(X); m_i) \end{array}$$
of augmented C_∞-algebras (see Remark 1.4.10) which commutes on the level of cohomology. We claim that it commutes up to homotopy of A_∞-algebras.

The inclusion functor $\text{cdga}^+ \to \mathcal{C}_\infty\text{-alg}^+$ between the augmented cdgas and augmented C_∞-algebras induces an equivalence between the homotopy categories $\text{Ho}(\text{cdga}^+)$ and $\text{Ho}(\mathcal{C}_\infty\text{-alg}^+)$, which are the localizations of the respective categories at the quasi-isomorphisms (see Theorem 1.4.19). Also, since $(R, 0)$ and $(R \otimes \Lambda V, D)$ are both free cdgas, they are both fibrant and cofibrant with respect to a model category structure whose weak equivalences are the quasi-isomorphisms (see e.g. [47, Section B.6.11]). Through the equivalence
$$\text{cdga}^+_{cf}/ \sim \to \text{Ho}(\text{cdga}^+)$$
($\sim$ being the homotopy relation), we deduce that the equivalence class of the morphism $\varphi \circ f$ in $\text{Ho}(\mathcal{C}_\infty\text{-alg}^+)$ contains a (unital) cdga-morphism $\psi\colon (R, 0) \to (R \otimes \Lambda V, D)$. This is not necessarily the standard inclusion, which we denote by i, but it induces the same map in cohomology. By Lemma 1.2.2, this already implies that i and ψ are homotopic. It follows that i and $\varphi \circ f$ give rise to the same morphism in $\text{Ho}(C_\infty\text{-alg})$, which implies they are in particular homotopic when considered as A_∞-morphisms through the forgetful functor (see [45, Cor. 1.3.1.3]).

Now by Lemma 1.4.17 the two A_∞-R-module structures on $R \otimes \Lambda V$ defined by the morphisms i and $\varphi \circ f$ are quasi-isomorphic as A_∞-R-modules and by Lemma 1.4.16 they are also quasi-isomorphic to the A_∞-R-module structure on $H_G^*(X)$ defined by f. As the higher operations of $H_G^*(X)$ vanish on the image of f by assumption, all but the binary operation of the A_∞-R-module $H^*_G(X)$ vanish. Thus the latter is just the differential graded R-module $(H^*_G(X), 0)$.

We have shown that the differential graded R-modules $(H^*_G(X), 0)$ and $(R \otimes \Lambda V, D)$ are quasi-isomorphic as A_∞-R-modules. But then Theorem 1.4.19 implies that they are also quasi-isomorphic as ordinary differential graded R-modules. □

By the formal cohomogeneity of a G-action on X we mean the difference $\mathrm{fd}(X) - \dim G$ of the formal dimensions of X and G, where formal dimension is the highest degree in which nontrivial cohomology exists.

Corollary 2.3.3. *Let G act almost freely on X. Let c be the formal cohomogeneity of the action and assume that one of the following holds:*

(i) $c \leq 3$.

(ii) G is semisimple, X is k-connected for $0 \leq k \leq 3$ and $c \leq 7 + k$.

Then the action is $\mathcal{MOD}$-formal.

Proof. It follows from Theorem 1.1.2 that $H^*_G(X)$ vanishes in degrees above the codimension c. We argue that in the situation of (i) and (ii), the conditions of Theorem 2.3.1 are fulfilled for degree reasons. Choose a unital minimal C_∞-model structure on $H^*_G(X)$, which means that the higher operations m_i, $i \geq 3$ vanish if the argument has a tensor component of degree 0. Thus we only need to check the vanishing of the m_i on $H^+_G(X) \otimes A^+ \otimes \ldots \otimes A^+$ where A is the image of $H^*(BG) \to H^*_G(X)$. In the situation of (i), the minimal nonzero and nontrivial degree of A^+ is at least 2. Hence m_i, which is of degree $2 - i$, takes values in degrees $\geq i + 1$ when restricted to this subspace. This proves (i).

If G is semisimple, then the first nontrivial degree of A^+ is 4. If X is k-connected, $0 \leq k \leq 3$, so is X_G and it follows that m_i takes values in degrees $\geq k + 3i - 1$ when restricted to $H^+_G(X) \otimes A^+ \otimes \ldots \otimes A^+$. This implies (ii). □

Remark 2.3.4. Instead of arguing via minimal C_∞-models, the corollary above could also be deduced from analogous degree considerations in the minimal A_∞-R-module model without the detour through algebras made in Theorem 2.3.1.

The precise nature of the connection between Massey products of algebras and $\mathcal{MOD}$-formality is hard to grasp and the sufficient condition of Theorem 2.3.1 is not necessary as shown by example 2.4.11: in the example, the action is $\mathcal{MOD}$-formal despite the existence of nontrivial quadruple Massey products which cause m_4 to be nontrivial on $A^{\otimes 4}$ for any C_∞-model structure on the equivariant cohomology. We want to add that contrary to this observation, the nontriviality of certain quadruple Massey products in A^4 can be an obstruction to $\mathcal{MOD}$-formality in the right situation.

Other than $\mathcal{MOD}$-formality, the notion of being formally based has a precise description via higher C_∞-operations and is equivalent to a weakened form of the requirement of Theorem 2.3.1.

Theorem 2.3.5. *Let $A \subset H^*_G(X)$ be an R-subalgebra. The action is formally based with respect to A if and only if the unital minimal C_∞-model $(H^*_G(X); m_i)$ can be chosen in a way that m_i vanishes on $A^{\otimes i}$ for $i \geq 3$.*

Proof. Suppose that the action is formally based which means we have a morphism

$$\varphi\colon (C,d) \to (R \otimes \Lambda V, D)$$

of R-cdgas, where (C,d) is a relative minimal model for $R \to A$. Then by part (*iii*) of Lemma 1.4.23, we can construct the unital C_∞-model of X_G in the desired way.

Conversely suppose we have a unital minimal model $(H_G^*(X); m_i^{X_G})$ where the $m_i^{X_G}$ vanish on A. The cdga (C,d) is formal with cohomology equal to A so by part (*ii*) of Lemma 1.4.23 we can choose a unital minimal C_∞-model of the form $(A; m_i^A)$ with $m_i^A = 0$ for $i \neq 2$ and m_2^A the ordinary multiplication. The inclusion $(A; m_i^A) \to (H_G^*(X); m_i^{X_G})$ defines a unital morphism of C_∞-algebras with trivial higher components. Thus we have a morphism $(C,d) \to (R \otimes \Lambda V, D)$ in $\mathrm{Ho}(C_\infty\text{-alg}^+)$ defined by

$$(C,d) \leftarrow (A; m_i^A) \to (H_G^*(X); m_i^{X_G}) \to (R \otimes \Lambda V, D).$$

We observe that (C,d) and $(R\otimes\Lambda V, D)$ are Sullivan cdgas and conclude as in the proof of Theorem 2.3.1 that the morphism in $\mathrm{Ho}(C_\infty\text{-alg}^+)$ is represented by a morphism $(C,d) \to (R \otimes \Lambda V, D)$ of cdgas. Then it is also represented by a morphism of R-cdgas by Lemma 1.2.2.

□

2.4 Examples

2.4.1 Symplectic actions and Hard Lefschetz spaces

In recent decades, the topology of torus actions on symplectic and Kähler manifolds has been a very successful field of study. In particular it was proved in [3] that Hard Lefschetz manifolds satisfy the TRC (going back to the study of derivations on the cohomology algebra by [8]) and the result has been generalized to free actions on cohomologically symplectic spaces of Lefschetz type in [2] and [48] (we come back to this in Section 3.2.2). It is therefore no surprise that among those spaces, we find classes of examples satisfying our formality properties. However, it will turn out that in the generalized setting of Lefschetz type spaces, the topology alone will not yield the formality properties we seek and we will need to resort to the geometric condition of the action being symplectic. Let us recall some notions

Definition 2.4.1. A $2n$-dimensional compact symplectic manifold (M,ω) is said to be of Lefschetz type if multiplication with ω^{n-1} defines an isomorphism $H^1(M;\mathbb{R}) \to H^{2n-1}(M;\mathbb{R})$.

Theorem 2.4.2. *Let T be a torus acting on X such that one of the following holds:*

(i) Any derivation of negative odd degree on $H^*(X)$ vanishes if it vanishes on $H^1(X)$.

(ii) X is a compact symplectic manifold of Lefschetz type and the T-action is smooth and symplectic.

Then the action is $\mathcal{MOD}$-formal and has formal core.

The property in (i) is fulfilled in particular for compact Kähler or more generally Hard Lefschetz manifolds (see [8, Théorème II.1.2]). It is however a little more general as it holds e.g. for any space with cohomology concentrated in even degrees and is stable under products ([7, Prop. 3.5]). The proof of the theorem relies on the following property that unifies both types of actions.

Lemma 2.4.3. *Suppose the T-action on X satisfies either condition (i) or (ii) above and that the map $H^2(BT) \to H^2_T(X)$ is injective. Then the action is equivariantly formal.*

Proof. Observe that the injectivity assumption is equivalent to the vanishing of the transgression on the second page of the Serre spectral sequence of the Borel fibration. Under condition (i) the lemma follows from [21, Proposition 4.40] (note that differentials on odd pages vanish automatically as R is concentrated in even degrees). In case (ii), note first that it suffices to consider real coefficients. We may thus work with the spectral sequence of the Borel fibration which is obtained from the Cartan model by filtering in polynomial degree. The symplectic form induces an element $[\omega] \in E_2^{0,2}$ and we claim that $d_2[\omega] = 0$. If $d_2[\omega] = v \in E_2^{2,1} = H^2(BT) \otimes H^1(X)$ was nontrivial, Poincaré duality of X would imply the existence of some element $x \in E_2^{0,2n-1} = H^{2n-1}(X)$ such that $xv \neq 0$. As X is of Lefschetz type, we may write $x = [\omega]^{n-1}u$ for some $u \in E_2^{0,1}$. By assumption we have $d_2 u = 0$ and $[\omega]^n u = 0$ for degree reasons. This implies $0 = d_2[\omega]^n u = n[\omega]^{n-1}uv$ which is a contradiction.

By the definition of the Cartan differential, this means precisely that contractions of ω with the fundamental vector fields are exact. Consequently the action is Hamiltonian and thus equivariantly formal ([43, Prop. 5.8]). □

Remark 2.4.4. Interestingly, although the proof of the TRC generalizes to cohomologically symplectic spaces of Lefschetz type in the purely topological setting (see Section 3.2.2), the statement of (ii) in the above lemma is false without the geometric assumptions: [1, Example 1] is an S^1-action with a fixed point on a simply-connected cohomologically symplectic space that is however not equivariantly formal.

Proof of Theorem 2.4.2. Set $V = \ker(R^2 \to H^*_T(X))$. There is a subtorus $T' \subset T$ such that the kernel of $H^2(BT) \to H^2(BT')$ is exactly V. Let $X_i \in V$ be a basis and $Y_i \in R^2$ be a basis of a complement of V. Then we have the following commutative diagram

$$\begin{array}{ccccc} (\Lambda(X_i, Y_j), 0) & \longrightarrow & (\Lambda(X_i, Y_i) \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, d) \\ \downarrow & & \downarrow & & \downarrow \mathbf{1}_{\Lambda V} \\ (\Lambda(Y_j), 0) & \longrightarrow & (\Lambda(Y_j) \otimes \Lambda V, \overline{D}) & \longrightarrow & (\Lambda V, d) \end{array}$$

where the top row is a model for the Borel fibration of the T action on X, the bottom row is a model for the Borel fibration of the restricted T'-action (note that the T'-Borel fibration is up to homotopy the pullback of the T-Borel fibration along $BT' \to BT$), and the vertical maps are defined by sending the X_i to 0. By construction, the Y_j map injectively into the cohomology of $(\Lambda(Y_j) \otimes \Lambda V, \overline{D})$ and thus the bottom row fibration is TNHZ by Lemma 2.4.3.

By assumption there are $s_i \in V^1$ with $D(s_i) = X_i$. In particular $\Lambda V = \Lambda(s_i) \otimes \Lambda W$ where $d(s_i) = 0$ and $\overline{D}(s_i) = 0$. Thus in the bottom row it makes sense to quotient out the ideal generated by the s_i, which yields an extension sequence

$$(\Lambda(Y_j), 0) \to (\Lambda(Y_j) \otimes \Lambda W, \tilde{D}) \to (\Lambda W, \overline{d}). \qquad (*)$$

We argue that this is TNHZ as well. By naturality of spectral sequences it suffices to argue that the projection $(\Lambda V, d) \to (\Lambda W, \overline{d})$ is surjective on cohomology. To see this, consider the composition

$$(\Lambda(X_i, s_i) \otimes \Lambda W, \overline{D'}) \to (\Lambda V, d) \to (\Lambda W, \overline{d})$$

where $\overline{D'}$ is obtained from D by dividing out the Y_j. It is defined by sending the contractible algebra $(\Lambda(X_i, s_i), D)$ to 0 and is thus a quasi-isomorphism. In particular this shows cohomological surjectivity of the second morphism and thus degeneracy of the Serre spectral sequence of $(*)$ at E_2.

As in the proof of Lemma 2.1.3, we obtain a quasi-isomorphism between the free differential graded $\Lambda(Y_j)$-modules $(\Lambda(Y_j) \otimes \Lambda W, \tilde{D})$ and $(H^*(\Lambda(Y_j) \otimes \Lambda W), 0)$, which we consider as $\Lambda(X_i, Y_j)$-modules via the projection. As before, sending $\Lambda(X_i, s_i)$ to 0 yields a quasi-isomorphism

$$(\Lambda(X_i, Y_i) \otimes \Lambda V, D) \to (\Lambda(Y_j) \otimes \Lambda W, \tilde{D})$$

of $\Lambda(X_i, Y_j)$ modules. We have shown that the action is $\mathcal{MOD}$-formal.

To see that it has formal core, set $A = \mathrm{im}(R \to H^*_G(X))$ and note that in the language of Definition 2.1.16, ΛC is given by $(\Lambda(X_i, Y_j, s_i), D)$. Thus the morphism $\overline{C} \to \Lambda V$ is just given by the inclusion of $(\Lambda(s_i), 0)$. By Lemma 3.2.2 the s_i span an exterior algebra in $H^*(\Lambda V, d)$. This shows the cohomological injectivity of $\overline{C} \to \Lambda V$.

□

2.4.2 Counterexamples

Example 2.4.5. We show that the notions we defined in Section 2.1 are **not preserved by restriction to subgroups of smaller rank**. Consider the threefold Hopf action of T^3 on $M = (S^3)^3$. The space M_{T^3} is formal so it is in particular $\mathcal{MOD}$-formal as well as formally based. However, if we restrict the action to T^2 along the homomorphism $(s, t) \mapsto (s, st, t)$, the model of the Borel fibration becomes

$$(\Lambda(X, Y), 0) \to (\Lambda(X, Y, a, b, c), D) \to (\Lambda(a, b, c), 0),$$

with $|X| = |Y| = 2$, $|a| = |b| = |c| = 3$, $D(a) = X^2$, $D(b) = X^2 + 2XY + Y^2$, and $D(c) = Y^2$. Now $2Ya - X(b - a - c)$ defines a nonzero cohomology class in degree 5 which maps to 0 in $H^5(M) = 0$. Also $H^3_{T^2}(M) = 0$ so said class does not lie in $\mathfrak{m}H^*_{T^2}(M)$, where $\mathfrak{m} = (X, Y)$. It follows that the action is not spherical. It is also not formally based because of the nontrivial Massey product $\langle X, X, Y\rangle$: the image of $H^*(BT^2) \to H^*_{T^2}(M)$ is isomorphic to $\Lambda(X, Y)/(X^2, XY, Y^2)$. Let (C, d) be as in Definition 2.1.16 and $\alpha, \beta, \gamma \in C$ with $d\alpha = X^2$, $d\beta = X^2 + 2XY + Y^2$, and $d\gamma = Y^2$. Any morphism $(C, d) \to (\Lambda(X, Y, a, b, c), D)$ that is the identity on R also has to map $\alpha \mapsto a$, $\beta \mapsto b$, $\gamma \mapsto c$, which is not possible since there is nontrivial cohomology represented in the $\Lambda(X, Y)$-span of a, b, and c but not in that of α, β, and γ.

Example 2.4.6. We construct an action which is **spherical but not almost $\mathcal{MOD}$-formal**. Consider $\Lambda(X_1, X_2, X_3)$ with X_i in degree 2 and set the differential D to be trivial on the X_i. Add generators a_{ij} for $i \leq j \in \{1,2,3\}$ and set $D(a_{ij}) = X_iX_j$. One checks that a basis of the kernel of D in degree 5 is given by

- $m_{iij} = X_ja_{ii} - X_ia_{ij}$, for $(i,j) \in \{1,2,3\}^2$ with $i \neq j$, where $a_{ij} := a_{ji}$ in case $i > j$.
- $m_{312} = X_3a_{12} - X_2a_{13}$, $m_{123} = X_1a_{23} - X_2a_{13}$.

The easiest way to verify this is by observing that the listed elements are linearly independent and that the differential maps the 18-dimensional degree 5 component surjectively onto the 10-dimensional $\Lambda^3(X_1, X_2, X_3)$.

Now for each of the m_{ijk} except for m_{112}, we introduce a generator c_{ijk} in degree 4 with $D(c_{ijk}) = m_{ijk}$. Furthermore, we add a generator b in degree 3 with $Db = 0$ and glue it to the remaining Massey product by introducing the generator c_{112} in degree 4 and setting $D(c_{112}) = m_{112} - X_1b$. We extend the cdga

$$(A, D) = (\Lambda(X_i, a_{ij}, b, c_{ijk}), D)$$

to a minimal Sullivan algebra (C, D) by adding generators in degrees ≥ 5 to inductively kill all cohomology in degrees ≥ 6.

Setting $R = \Lambda(X_1, X_2, X_3)$, we have $H^*(C) = R/R^{\geq 3} \otimes \Lambda(b)$. By Proposition 1.2.4 we find a compact space X with a T^3 action such that the Borel fibration has minimal model

$$R \to (C, D) \to (\overline{C}, \overline{D})$$

with the latter cdga being the quotient by the ideal generated by R^+. As an R-module, $H^*_{T^3}(X) = H^*(C)$ is generated by the classes of 1 and b which map injectively to $H^*(X) = H^*(\overline{C})$ so the action is spherical by Lemma 2.1.15.

Let us now argue that it is not almost $\mathcal{MOD}$-formal. By Proposition 2.1.11, it suffices to find a nontrivial differential on the E_3-page of the spectral sequence of the fibration $T^3 \to X \to X/T^3$. It is obtained by filtering $(C \otimes \Lambda S, D)$ in the degree of C where $S = \Lambda(s_1, s_2, s_3)$ with $Ds_i = X_i$. We find a nontrivial differential by defining a suitable zigzag: consider the element $\alpha_2 = X_1s_{123}$ in filtration degree 2, where the multi index stands for the respective product of the s_i. We have $D(\alpha_2) = X_1^2s_{23} - X_1X_2s_{13} + X_1X_3s_{12}$. Thus for $\alpha_3 = -a_{11}s_{23} + a_{12}s_{13} - a_{13}s_{12}$, we obtain

$$D(\alpha_2 + \alpha_3) = (X_2a_{13} - X_3a_{12})s_1 + (X_3a_{11} - X_1a_{13})s_2 + (X_1a_{12} - X_2a_{11})s_3$$

in filtration degree 5. We claim that this defines a nontrivial element in $E_3^{5,1}$, which implies that we have a nontrivial differential starting from $E_3^{2,3}$.

First note that $D(\alpha_2+\alpha_3) \equiv D(\alpha_2+\alpha_3)+D(c_{312}s_1-c_{113}s_2) \equiv -m_{112}s_3+X_1c_{213}-X_2c_{113}$ in $E_3^{5,1}$. For this to be trivial in $E_3^{5,1}$ we need to have

$$m_{112}s_3 - D(e_{12}s_{12} + e_{13}s_{13} + e_{23}s_{23} + e_1s_1 + e_2s_2 + e_3s_3) \in C^6$$

for some $e_{ij} \in A^3$ and $e_i \in A^4$. We see that the e_{ij} are necessarily closed and hence multiples of b. If the e_{ij} were nontrivial, then their image under D would produce error terms of the form $X_ib_js_k$ which cannot cancel with the $D(e_is_i)$ because no element in $R^2 \cdot \langle b \rangle$ is exact. Thus we have $e_{ij} = 0$. But then it follows that the projection of the whole right hand expression $D(\cdots)$ to the $A^5 \otimes \langle s_3 \rangle$ component is given by $D(e_3)s_3$, which can never be equal to $m_{112}s_3$ because m_{112} is not exact in A.

Example 2.4.7. We construct an **almost $\mathcal{MOD}$-formal but not $\mathcal{MOD}$-formal** free action. Consider the cdga (C, D) from Example 2.4.6. We set $R = \Lambda(X_1, X_2)$ and obtain a compact T^2-space Y such that the Borel fibration is given by $R \to C$. The R-module $H^*_{T^2}(Y)$ is generated by the classes of 1, X_3, b, and $X_3 b$ which map injectively to $H^*(Y)$. Thus the action is spherical by Lemma 2.1.15. A nontrivial differential past the second page of the spectral sequence from Proposition 2.1.11 would need to map from $E_3^{*,2}$ to $E_3^{*,0}$ because we only have a T^2-action. But those are also trivial according to Proposition 2.1.13. Thus the action is almost $\mathcal{MOD}$-formal.

Finally, we argue that the action is not $\mathcal{MOD}$-formal. To see this, we construct the lower part of the Hirsch–Brown model $\varphi\colon R \otimes V \to C$ (see 1.3.3). In degree 0 we introduce the generator $\overline{1}$ and set $\varphi(\overline{1}) = 1$. In degree 1 there is nothing to do and in degree 2 we introduce generators $\overline{X_3}$ and define $\varphi(\overline{X_3}) = X_3$. To achieve surjectivity in degree 3, we introduce $\overline{b}$, with $\varphi(\overline{b}) = b$. Injectivity of φ in degree 4 is achieved by introducing $\overline{a_{11}}, \overline{a_{12}}, \overline{a_{22}}$ with $D(\overline{a_{ij}}) = X_i X_j \cdot \overline{1}$ as well as $\overline{a_{13}}$ and $\overline{a_{23}}$ with $D(\overline{a_{i3}}) = X_i \cdot \overline{X_3}$ and setting $\varphi(\overline{a_{ij}}) = a_{ij}$. The procedure is continued by adding generators of degree ≥ 4 to form the Hirsch–Brown model.

If the action were $\mathcal{MOD}$-formal, the criterion in Lemma 2.1.8 would tell us that we can choose preimages $\alpha, \beta, \gamma \in V^3$ of the exact cocycles $X_1^2 \cdot \overline{1}$, $X_1 X_2 \cdot \overline{1}$, and $X_2^2 \cdot \overline{1}$ such that every closed element in $R \otimes \langle \alpha, \beta, \gamma \rangle_{\mathbb{Q}}$ is exact. Every such α, β, and γ are of the form $\alpha = \overline{a_{11}} + t_\alpha \overline{b}$, $\beta = \overline{a_{12}} + t_\beta \overline{b}$, $\gamma = \overline{a_{22}} + t_\gamma \overline{b}$ for $t_\alpha, t_\beta, t_\gamma \in \mathbb{Q}$. But no choice of scalars fulfils the condition that

$$\begin{aligned} &\varphi(X_2\alpha - X_1\beta) = m_{112} + (t_\alpha X_2 - t_\beta X_1) b \\ \text{and} \quad &\varphi(X_2\beta - X_1\gamma) = -m_{221} + (t_\beta X_2 - t_\gamma X_1) b \end{aligned}$$

are simultaneously exact: exactness of the first element would require $(t_\alpha, t_\beta) = (0, 1)$, whereas for the second one we need $(t_\beta, t_\gamma) = (0, 0)$.

Remark 2.4.8. We point out that the counterexamples 2.4.6 and 2.4.7 can be modified to produce **simply-connected compact manifolds**. By Proposition 1.2.4 it suffices to extend the cdga (C, D) to a cdga whose cohomology satisfies Poincaré duality and check that the arguments carry over.

To do this, choose any big enough degree N in which the fundamental class shall live. We assume $N \geq 11$ for convenience in the arguments below. Now introduce generators e_1, e_2, and e_3 in degree $N - 5$ and map them to 0 under the differential. In degree $N - 3$, cohomology is now generated by the $X_i e_j$ for $1 \leq i, j \leq 3$. We introduce new generators in degree $N - 4$ and map them (under the differential) to $X_i e_j$ for $i \neq j$ as well as to $X_1 e_1 - X_2 e_2$ and $X_2 e_2 - X_3 e_3$. Let V^{N-4} denote the space spanned by the newly introduced generators. Now the cohomology in degree $N - 3$ is 1-dimensional and represented by any of the $X_i e_i$. Also, degree $N - 2$ cohomology is represented by $b e_1$, $b e_2$, and $b e_3$ and cocycles in $V^{N-4} \cdot \langle X_1, X_2, X_3 \rangle$. Introduce new generators V^{N-3} to kill all cohomology of the latter kind in degree $N - 2$. Since the differential is injective on $C^3 \cdot V^{N-4}$, cohomology in degree $N - 1$ is entirely represented in $V^{N-3} \cdot \langle X_1, X_2, X_3 \rangle$. We kill this cohomology by introducing generators in V^{N-2}.

We claim that in degree N, the elements of the form $X_i b e_i$ are not exact (although they are cohomologous to one another). Indeed, since for $i = N-3, N-2$ the differential maps V^i into the ideal generated by V^{i-1}, it suffices to check whether e.g. $X_1 b e_1$ is in the image of $C^3 \cdot V^{N-4}$, which is clearly not the case. We choose a complement of $\langle [X_1 b e_1] \rangle$ in degree N cohomology and introduce generators V^{N-1} which map bijectively to representatives of a basis of the complement. Now inductively kill all cohomology in degrees $> N$. Representatives for a basis of the cohomology are given by

degree	0	2	3	5	$N-5$	$N-3$	$N-2$	N
		X_1		X_1b	e_1		be_1	
generators	1	X_2	b	X_2b	e_2	X_ie_i	be_2	X_ibe_i
		X_3		X_3b	e_3		be_3	

and we observe that Poincaré duality holds.

To check that the arguments in Examples 2.4.6 and 2.4.7 carry over, note first that both the T^3-action defined by X_1, X_2, X_3 as well as the T^2-action defined by X_1, X_2 are still spherical: for $R = \Lambda(X_1, X_2, X_3)$, a generating set of the cohomology as an R-module is given by 1, b, e_i, and e_ib for $i = 1, 2, 3$. For $R = \Lambda(X_1, X_2)$ we need to also add X_3 and X_3b to the list. In any case, none of those generators become exact when dividing by the ideal of R^+ so both actions are spherical by Lemma 2.1.15. The arguments showing that the actions are not (almost) $\mathcal{MOD}$-formal took place only in the lower half of the cdgas, which we did not modify.

Example 2.4.9. We construct a **$\mathcal{MOD}$-formal action** that is **neither equivariantly formal nor has a formal homotopy quotient**. We expand on a discussion from [38]. Consider the graded vector space H with Betti numbers $1, 0, 3, 0, 0, 1$. For degree reasons, all operations of a unital A_∞-algebra structure on H vanish except for $m_3 \colon H^2 \otimes H^2 \otimes H^2 \to H^5$. In turn one checks that any specification of m_3 does indeed yield an A_∞-structure, where the formal one is just given by the cohomology of $S^2 \vee S^2 \vee S^2 \vee S^5$. For an A_∞-structure to be C_∞ it is required that it vanishes on all shuffles. In our case this is equivalent to m_3 vanishing on all elements of the form $a \otimes b \otimes c - a \otimes c \otimes b + c \otimes a \otimes b$ for $a, b, c \in H^2$. Let $\alpha \in H^5$ be a generator and let $X, Y, Z \in H^2$ be a basis. Then by setting $m_3(Z \otimes Z \otimes X) = \alpha$, $m_3(X \otimes Z \otimes Z) = -\alpha$, and $m_3(a \otimes b \otimes c) = 0$ for all other tensors with $a, b, c \in \{X, Y, Z\}$, we obtain a C_∞-structure on H. We observe that m_3 vanishes on $H^2 \otimes A \otimes A$, where $A = \langle X, Y \rangle$ is the sub-algebra of H (with trivial multiplication) generated by X and Y. Consequently, the classes X, Y satisfy the requirements of Remark 2.3.2 and we obtain a free $\mathcal{MOD}$-formal T^2-action on a compact space such that the orbit space has the rational homotopy type of (H, m_3).

To see that this is indeed not the formal rational homotopy type, note that the only nontrivial component of a C_∞-morphism $f \colon (H, m_3) \to (H, m_3')$ is f_1, again for degree reasons. Thus m_3 and m_3' yield isomorphic C_∞-structures if and only if $f_1 \circ m_3 = m_3' \circ (f_1 \otimes f_1 \otimes f_1)$ for an automorphism f_1 of the graded vector space H. In particular (H, m_3) and $(H, 0)$ are not isomorphic.

Example 2.4.10. We construct a **hyperformal action** which is **not spherical**. Consider the nilmanifold with model

$$(N, d) = \Lambda(y_1, \ldots, y_5, z_1, \ldots, z_4, d)$$

with $dy_i = 0$, $dz_i = y_i y_{i+1}$. The extension

$$(\Lambda(X_1, X_2), 0) \to (\Lambda(X_1, X_2) \otimes N, D) \to (N, d)$$

with X_i of degree 2, $Dz_1 = X_1 + y_1y_2$, $Dz_4 = X_2 + y_4y_5$, and $D = d$ on the other generators is the model of the Borel fibration of a free T^2-action. By dividing out a contractible ideal, we obtain a commutative diagram

$$\begin{array}{ccccc} (\Lambda(X_1, X_2), 0) & \longrightarrow & (\Lambda(X_1, X_2) \otimes N, D) & \longrightarrow & (N, d) \\ & \searrow & \downarrow & \nearrow & \\ & & (\Lambda(y_1, \dots, y_5, z_2, z_3), d) & & \end{array}$$

where the vertical map is a quasi-isomorphism with $X_1 \mapsto -y_1y_2$, $X_2 \mapsto -y_4y_5$, $z_1, z_4 \mapsto 0$ and which is the identity on the remaining generators. Since $y_1y_2y_4y_5$ defines a nonzero cohomology class in the lower cdga while $(y_1y_2)^2 = (y_4y_5)^2 = 0$, the kernel of $\Lambda(X_1, X_2) \to H^*(\Lambda(X_1, X_2) \otimes N, D)$ equals (X_1^2, X_2^2). Hence the action is hyperformal.

Now consider the cocycle $\alpha = -y_1y_4y_5z_2 + y_1y_2y_5z_3$ in the bottom cdga of the above diagram. One checks that the cohomology class $[\alpha]$ is not in the algebra span of the degree 2 classes which are represented by y_iy_j, y_2z_2, y_3z_2, y_3z_3, and y_4z_3. In particular $[\alpha]$ does not lie in $\Lambda^+(X_1, X_2) \cdot H^*(\Lambda(y_i, z_2, z_3), d)$ with respect to the module structure defined by the diagram. However, it becomes exact in (N, d) where

$$\alpha = d(y_3z_1z_4 - y_1z_2z_4 - y_5z_1z_3).$$

This shows that the action is not spherical.

Example 2.4.11. We show the existence of a $\mathcal{MOD}$**-formal** action that is **not formally based** by constructing the cdga $(\Lambda W, D)$ in the following way: introduce generators X, Y in degree 2 and a, b, c in degree 3 with $D(a) = X^2$, $D(b) = XY$, and $D(c) = Y^2$. Then the kernel in degree 5 is generated by $\alpha_1 = Ya - Xb$ and $\alpha_2 = Yb - Xc$. Now introduce generators d and e in degree 4 and set $D(d) = \alpha_1$, $D(e) = \alpha_2$. At this point, $(\Lambda W^{\leq 4}, D)$ has no cohomology in degrees 3, 4, and 5 while in degree 6 representatives for a basis are given by $\alpha_3 = Yd + ac + Xe$, $\alpha_4 = Xd + ab$, and $\alpha_5 = bc + Ye$. Now complete ΛW to a minimal cdga by inductively killing all cohomology starting from degree 7. By Proposition 1.2.4, there is a free T^2-action on a finite CW-complex M whose homotopy quotient M_{T^2} has $(\Lambda W, D)$ as minimal model. One quickly sees that $(\Lambda W, D)$ is not formal because α_1, α_2, and α_3 represent nontrivial quadruple Massey products. However, we argue that it is formal as a $\Lambda(X, Y)$-module by constructing a quasi-isomorphism $\Lambda W \to H^*(M_{T^2})$. We set this map to be the canonical projection in degrees 0 and 2 and trivial in all other degrees except degree 6. In degree 6, we choose the canonical basis given by the products of the generators and map $\Lambda^3 W^2$ and $W^2 \cdot W^4$ to 0 while sending ac, ab, and bc to $[\alpha_3]$, $[\alpha_4]$, and $[\alpha_5]$. It is easy to check that this map does have the desired properties.

To see that the action is not formally based, we observe that for degree reasons the only possible nontrivial operation of a unital C_∞-structure on $H = H^*(M_{T^2})$ is

$$m_4 \colon H^2 \otimes H^2 \otimes H^2 \otimes H^2 \longrightarrow H^6.$$

This map has to be nontrivial for every C_∞-model structure because M_{T^2} is not formal. But $H^*(BT^2) = \Lambda(X, Y) \to H^*(\Lambda W, D) = H^*(M_{T^2})$ is surjective in degree 2 so the requirements of Theorem 2.3.5 are not met.

Example 2.4.12. We give an example of an **action on a homogeneous space** which has a **formal homotopy quotient** but **does not have formal core with respect to a smaller subalgebra** of $H_G^*(X)$. Consider the action of the maximal diagonal torus T^4 on $\mathrm{U}(4)$ by multiplication from the left. If $X_1, \ldots, X_4 \in H^2(BT^4)$ is the basis dual to the standard basis of the Lie-Algebra of T^4 and $R = \Lambda(X_1, \ldots, X_4)$, then the model of the borel fibration is given as

$$(R, 0) \to (R \otimes \Lambda Z, D) \to (\Lambda Z, 0)$$

where Z is 4-dimensional and D maps a basis to the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_4$ in the variables X_i. Now we make a change of basis by pulling back the action along the automorphism ϕ of T^4 which, in the standard basis of the Lie algebra, is given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This induces an automorphism ϕ^* of R which, in the basis $X_1, \ldots, X_4$, is represented by A^t. The model of the Borel fibration of the new action is the same except D is replaced by the differential $\tilde{D}$ which maps a basis of Z to the polynomials $\phi^*(\sigma_i)$. We consider the splitting $T^4 = T \times T'$ with T consisting of the two circle factors on the left and T' of those on the right. We claim that the T-action on $\mathrm{U}(4)/T'$ has a formal homotopy quotient but does not have formal core with respect to $\mathrm{im}(H^*(BT) \to H_T(\mathrm{U}(4)/T')$.

The model of the Borel fibration of the T-action is

$$(\Lambda(X_1, X_2), 0) \to (R \otimes \Lambda Z, \tilde{D}) \to (\Lambda(X_3, X_4) \otimes \Lambda Z, \overline{D}).$$

The middle cdga is formal so the action does indeed have a formal homotopy quotient. Let $S = \Lambda(X_1, X_2)$ and $i\colon S \to R$ be the inclusion. Then $J := \ker(S \to H_T^*(\mathrm{U}(4)/T')) = i^{-1}(\phi^*(I))$, where $I \subset R$ is the ideal generated by the σ_i. One computes that $(\phi^*)^{-1} \circ i(X_1) = X_1$ as well as $(\phi^*)^{-1} \circ i(X_2) = X_2 + 2X_3$ and, using appropriate tools, we obtain

$$J = (X_1^4, 28X_1^3X_2^2 + 12X_1^2X_2^3 + 3X_1X_2^4, X_2^6).$$

We recommend the freely available software Macaulay2 for such computations and the ones that follow below. Let (C, d) be a relative minimal model of $(S, 0) \to (S/J, 0)$. It is our goal to show that a morphism

$$(C \otimes \Lambda Z', d) \to (R \otimes \Lambda Z \otimes \Lambda Z', \tilde{D})$$

can not be cohomologically injective, where the differentials map the generators $Z' = \langle z_1, z_2 \rangle$ to X_1 and X_2. This follows for dimensional reasons: formality provides us with quasi-isomorphisms

$$(C \otimes \Lambda Z', d) \to (S/J \otimes \Lambda Z', d) \quad \text{and} \quad (R \otimes \Lambda Z \otimes \Lambda Z', \tilde{D}) \to (R/\phi^*(I) \otimes \Lambda Z', d).$$

The cohomologies are thus given by

$$\mathrm{Tor}_*^S(S/J, S/(X_1, X_2)) \quad \text{and} \quad \mathrm{Tor}_*^S(R/\phi^*(I), S/(X_1, X_2))$$

which can again be computed with standard software. Doing so, one finds the first one to be nontrivial in degree 11 (in the cdga grading) while the second one is 0 in this degree. It follows that the action has the desired properties. As a side note we want to add that without the change of basis via ϕ, the analogous construction does yield an action that has formal core with respect to $\mathrm{im}(S \to H_T(\mathrm{U}(4)/T')$.

Chapter 3

Bounds on the toral rank

3.1 Toral rank and formality

3.1.1 The total rank of free resolutions

Formality can help link the Buchsbaum–Eisenbud–Horrocks Conjecture (see below) and the Toral Rank Conjecture. For example it was observed in [60] (see also [52]) that such a link is given by the fact that the Serre spectral sequence of the homotopy fibration

$$T \to X \to X_T$$

collapses at E_3 if X_T is a formal space. In our language this comes down to the fact that those actions are in particular almost $\mathcal{MOD}$-formal (see Prop. 2.1.11). Rather recently, in [61], the following breakthrough theorem was proved, solving a weak form of the Buchsbaum–Eisenbud–Horrocks conjecture.

Theorem 3.1.1. *Let R be a commutative Noetherian ring that is locally a complete intersection such that spec(R) is connected. Further, let M be a nonzero finitely generated R-module of finite projective dimension such that M is 2-torsion free and*

$$0 \leftarrow M \leftarrow P_0 \leftarrow \ldots \leftarrow P_d \leftarrow 0$$

a projective resolution. Then

$$\sum_{i=0}^{d} \mathrm{rk}_R(P_i) \geq 2^c,$$

where c is the codimension of M.

In its strong incarnation one conjectures the more specific bounds

$$\mathrm{rk}_R(P_i) \geq \binom{c}{i}$$

to hold. The link to the TRC is provided by the equivariant cohomology: for a torus T acting on a space X, take $R = H^*(BT)$ and $M = H_T^*(X)$. Then R is a polynomial ring and thus regular which implies all the conditions of the above theorem. Also $H_T^*(X)$ is finitely generated (see [4, Prop. 3.10.1]) and of finite projective dimension. The codimension c

of $H_T^*(X)$ has a nice geometrical interpretation: if X is compact, then c is the minimal dimension among the orbits (see Lemma 1.1.6).

Thus the only missing piece is linking the projective resolution of $H_T^*(X)$ to $H^*(X)$. The number $\sum \mathrm{rk}_R(P_i)$ from the above theorem gives an upper bound for $\dim H^*(X)$, which is not sharp in general as it is the dimension of the E_2 page in Remark 2.1.10. Equality holds if and only if said spectral sequences collapse at E_2.

Theorem 3.1.2. *Suppose the T-action on the compact space X is (almost) $\mathcal{MOD}$-formal or has formal core. Then*

$$\dim H^*(X) \geq 2^c,$$

where c is the minimal dimension among the orbits. In particular the TRC holds for those kinds of actions.

Proof. The rank of the minimal Hirsch–Brown model is precisely $\dim H^*(X)$ so the statement for (almost) $\mathcal{MOD}$-formal actions follows directly from Theorems 2.1.5 and 3.1.1.

Regarding actions with formal core with respect to some $A \subset H_G^*(X)$, note first that $\dim H^*(X) \geq \dim H^*(\overline{C}, \overline{d})$ where we use the notation surrounding Definitions 2.1.16 and 2.1.20. The map $(R, 0) \to (C, d)$ turns (C, d) into a formal dg$R$m so by the same arguments as in the $\mathcal{MOD}$-formal case we obtain $\dim H^*(\overline{C}, \overline{d}) \geq 2^{c'}$, where c' is the height of the annihilator of $H^*(C)$ as an R-module. But the annihilators of $H_T^*(X)$ and $H^*(C, d)$ are just given by the kernel of the map $R \to H^*(C, d) \subset H_T^*(X)$. In particular, $c = c'$ and

$$\dim H^*(X) \geq \dim H^*(\overline{C}, \overline{d}) \geq 2^c.$$

When T acts almost freely, we have $c = \dim T$ which yields the TRC. □

Also, there is the following addendum to Theorem 3.1.1 from [61].

Theorem 3.1.3. *Suppose R is a local (Noetherian, commutative) ring of Krull dimension d which is the quotient of a regular local ring by a regular sequence. Assume further that 2 is invertible in R and let M be a finitely generated R-module of finite projective dimension and finite length. If the sum of the Betti numbers of M is 2^d then M is the quotient of R by a regular sequence of d elements.*

For torus actions, we deduce the following

Proposition 3.1.4. *Suppose a T-action on a compact space X is almost $\mathcal{MOD}$-formal or has formal core and fulfils*

$$\dim H^*(X) = 2^c,$$

where c is the minimal dimension among the orbits. Then X is rationally equivalent to a product of c odd-dimensional spheres.

Proof. If the action is formally based with respect to some $A \subset H_G^*(X)$, observe that, in the notation surrounding Definitions 2.1.16 and 2.1.20, $\dim H^*(\overline{C}, \overline{d}) \geq 2^c$. As it cohomologically injects into $H^*(X)$, we deduce that $(\overline{C}, \overline{d})$ is a model for X. Consequently, (C, d) is a model for X_T and the action is $\mathcal{MOD}$-formal.

For an almost $\mathcal{MOD}$-formal action, the total rank of the minimal graded free resolution of $H_T^*(X)$ as an R-module is $\dim H^*(X) = 2^c$. Let $\mathfrak{p}$ be a minimal prime containing

$\mathrm{Ann}(H_T^*(X))$. Then $H_T^*(X)_\mathfrak{p}$ has finite length. Since localization is exact, we obtain a free resolution of $H_T^*(X)_\mathfrak{p}$ by localizing the minimal graded free resolution of $H_T^*(X)$. The codimension of $H_T^*(X)_\mathfrak{p}$ is also c and thus this resolution has to be minimal by Theorem 3.1.1. This means that the sum of the Betti numbers of $H_T^*(X)_\mathfrak{p}$ is equal to 2^c.

We may now apply Theorem 3.1.3 and conclude that $H_T^*(X)_\mathfrak{p}$ is a quotient of $R_\mathfrak{p}$ by a regular sequence of c elements. Since the minimal free resolution of $H_T^*(X)_\mathfrak{p}$ was constructed from the one of $H_T^*(X)$, we conclude that also $H_T^*(X)$ is a quotient of R by c elements. Those elements span $\mathrm{Ann}(H_T^*(X))$ which is of height c, so it follows that they also form a regular sequence in R. To see this formally, note that a sequence of homogeneous elements of positive degree is regular in R if and only if it is regular in $R_\mathfrak{m}$, where $\mathfrak{m} = R^+$. After localizing at R^+ we may use [14, Corollary 17.7] combined with the fact that R is a Cohen–Macaulay ring.

We observe that, in this special case, the R-module structure determines also the algebra structure on $H_T^*(X)$. A quotient of a polynomial ring by a regular sequence is intrinsically formal (see [18, Remark 3.1]) so a Sullivan model for X_T is given by the Koszul complex $(R \otimes \Lambda Z, D)$ where D maps a basis of Z to the regular sequence. It follows that $(\Lambda Z, 0)$ is a model for X.

□

3.1.2 Small dimensions

As we have shown in Corollary 2.3.3, actions of small enough codimension are $\mathcal{MOD}$-formal and hence fulfil the TRC by Theorem 3.1.2. We can achieve stronger results by placing additional topological restrictions on X. Recall that by the formal cohomogeneity of a G-action on X we mean the number $\mathrm{fd}(X) - \dim G$.

Lemma 3.1.5. *Let G act almost freely on X with formal cohomogeneity c and assume one of the following holds*

- *(i) X is simply-connected and $c \leq 4$.*
- *(ii) X is simply-connected, satisfies Poincaré duality, and $c \leq 2k$, where $k \geq 3$ is the minimal odd degree such that $\pi_k(X) \otimes \mathbb{Q} \neq 0$.*

Then X_G is formal.

Proof. By Proposition 1.1.2 we have $\mathrm{fd}(X_G) = c$. If X is simply-connected, so is X_G. Any simply-connected space of formal dimension ≤ 4 is formal as higher operations in the minimal unital C_∞-model vanish for degree reasons. This proves the lemma under condition (i).

In the situation of (ii), X_G is also a Poincaré duality space by Proposition 1.1.2. The Sullivan minimal model of X does not have an odd degree generator up until degree k. As the rational homotopy of BG is concentrated in even degrees, we deduce that the minimal model of X_G also does not have odd generators of degree $< k$. This implies that the differential in the minimal model of X_G vanishes in degrees from 1 up to $k-1$. Thus X_G is $(k-1)$-formal and by [22, Theorem 3.1], X_G is formal because $\mathrm{fd}(X_G) \leq 2k$. □

Not only do we know that the TRC holds for small cohomogeneities but it is also a classical result that it holds for actions of T^r if $r \leq 3$ ([4, Theorem 4.4.3]). Together with Corollary 2.3.3, Lemma 3.1.5, and Theorem 3.1.2 this yields

Theorem 3.1.6. *The toral rank conjecture holds for*

(i) *spaces of formal dimension* ≤ 7.

(ii) *simply-connected spaces of formal dimension* ≤ 8.

(iii) *simply-connected Poincaré duality spaces of formal dimension* $\leq 2k+4$, *where* $k \geq 3$ *is the minimal odd degree such that* $\pi_k(X) \otimes \mathbb{Q} \neq 0$.

Case (*iii*) does in particular imply the TRC for simply-connected orientable manifolds of dimension ≤ 10. This was proved earlier in [33, Théorème A], also using formality of the homotopy quotient but concluding differently.

3.1.3 Boij–Söderberg theory and consequences

We give a brief introduction to the theory surrounding the solved conjectures by Boij and Söderberg (see [24] for a detailed survey) and illustrate their applications to $\mathcal{MOD}$-formal torus actions. The purpose here is twofold: while we obtain obstructions to the existence of $\mathcal{MOD}$-formal torus actions that are much more refined than the TRC, the theory below turns out to be a powerful tool, even in the absence of formality, which will be further explored in Section 3.2.1.

In this section, R denotes a polynomial ring in r variables over an arbitrary field. The sources surrounding Boij–Söderberg theory usually work with a grading in which R has variables of degree 1 and differentials are usually of degree 0. We refer to this as the free resolution grading (FRG). This is of course different from the topological grading (TG) where variables of $R = H^*(BT)$ are of degree 2 and differentials are of degree 1. When discussing the algebraic theory, we stick to the FRG conventions and later translate to the TG conventions for applications.

Let us briefly recall the basic facts about graded free resolutions (see e.g. the first chapter of [15]) and introduce notation. All modules will be assumed to be finitely generated. For any graded R-module $M = \bigoplus_k M^k$ and integer n, we denote by $M(n)$ the graded module with $M(n)^k = M^{n+k}$. A (graded) free resolution of M is an exact complex

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \dots$$

of free R-modules $F_i = \bigoplus R(-j)^{\beta_{i,j}}$ and degree 0 maps. Thus the numbers $\beta_{i,j}$ correspond to the degrees of the generators of the free R-module F_i. The module M has a so called minimal free resolution F_* with the property that for any other free resolution F'_* of M we have $F'_* \cong F_* \oplus C_*$ as complexes of graded R-modules for some complex C_* of free R-modules. In particular, the minimal resolution requires the minimal amount of generators. It is unique up to isomorphism of complexes and is characterized by the fact that at each stage, the image of the map $F_i \leftarrow F_{i+1}$ is contained in $\mathfrak{m}F_i$, where $\mathfrak{m}$ is the maximal homogeneous ideal in R. The (uniquely determined) integers $\beta_{i,j}$ in the minimal free resolution are called the graded Betti numbers of M.

By Hilberts syzygy theorem, the length of the minimal free resolution of M is at most r, which means that $F_i = 0$ for $i > r$. Only finitely many of the Betti numbers are nonzero. Hence, we can see the collection of the $\beta_{i,j}$ as an element of $\bigoplus_{j\in\mathbb{Z}} \mathbb{Z}^{r+1} \subset \mathbb{D} := \bigoplus_{j\in\mathbb{Z}} \mathbb{Q}^{r+1}$. This element is called the Betti diagram of M.

Example 3.1.7. For $R = \mathbb{Q}[x,y]$, consider the module $M = R/(x,y^2)$. A free resolution is given by

$$0 \leftarrow M \leftarrow R \xleftarrow{\begin{pmatrix} x & y^2 \end{pmatrix}} R(-1) \oplus R(-2) \xleftarrow{\begin{pmatrix} y^2 \\ -x \end{pmatrix}} R(-3) \leftarrow 0.$$

As the maps between the free modules in the resolution have image in the multiples of $\mathfrak{m}$, this is the minimal free resolution of M. If we display the corresponding Betti diagram $(\beta_{i,j}) \in \mathbb{D}$ as an array, showing only the window of $\mathbb{D}$ where $(\beta_{i,j})$ is nontrivial, we obtain

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array},$$

where $\beta_{i,j}$ is located in the ith column and jth row.

A very important class of modules is given by those whose depth is equal to their dimension, which by the Auslander-Buchsbaum formula is equivalent to the fact that their codimension coincides with the length of their minimal free resolution. Those modules are called Cohen–Macaulay.
The main result of Boij–Söderberg theory is to classify all possible Betti diagrams of Cohen–Macaulay R-modules up to multiplication by a rational number. The result then extends to arbitrary R-modules, even without the Cohen–Macaulay assumption. Let us make this precise.

Definition 3.1.8. (i) An element $d = (d_0, \ldots, d_c) \in \mathbb{Z}^{c+1}$ such that $d_0 < \ldots < d_c$ is called a degree sequence of length c.

(ii) Let d be a degree sequence of length c. We say that a finitely generated graded R-module has a pure diagram of type d if its Betti numbers satisfy $\beta_{i,j} = 0$ whenever $i > c$ or $j \neq d_i$.

A digram is pure if and only if it has only one nontrivial entry in each column when displayed as above. Given a degree sequence d of length c, we define the associated pure diagram $\pi(d) \in \mathbb{D}$ by

$$\pi(d)_{i,j} = \begin{cases} \prod_{k \neq i} \frac{1}{|d_k - d_i|} & \text{if } 0 \leq i \leq c \text{ and } j = d_i \\ 0 & \text{else} \end{cases}.$$

One can prove that any Betti diagram of a codimension c Cohen–Macaulay R-module with pure diagram of type d is a rational multiple of $\pi(d)$. This is a consequence of certain restrictions on Betti diagrams known as the *Herzog-Kühl equations*, which simplify to the above formulas in the pure case. Thus pure diagrams are understood up to multiplication by a rational number. The Boij–Söderberg conjectures enable us to extend this understanding to the Betti diagrams of arbitrary (Cohen–Macaulay) modules. We summarize some of the main results in the following

Theorem 3.1.9. *(i) The Betti diagram of an R-module of codimension c is a positive rational linear combination of the Betti diagrams of Cohen–Macaulay R-modules of codimension $\geq c$.*

(ii) The Betti diagram of a Cohen–Macaulay module of codimension c is a positive rational linear combination of Betti diagrams of codimension c Cohen–Macaulay R-modules with pure diagrams.

(iii) For any degree sequence d, there is a Cohen–Macaulay module with pure diagram of type d.

In the theorem above, a positive rational linear combination really means that all scalars are positive. Part (*ii*) and (*iii*) were proved in [17]. Note that the characteristic zero case of (*iii*) was proved earlier in [16]. Part (*i*) was then proved last in [10], extending the previous results to the non-Cohen–Macaulay case.

Remark 3.1.10. Let M and N be two R-modules with Betti-diagrams $\pi_M, \pi_N \in \mathbb{D}$. Then any positive rational linear combination $\frac{p_1}{q_1}\pi_M + \frac{p_2}{q_2}\pi_N$ is the Betti diagram of the sum of p_1q_2 copies of M and p_2q_1 copies of N, up to multiplication with $(q_1q_2)^{-1}$, where we assume $p_i, q_i \geq 0$. So the theorem above indeed classifies all Betti diagrams up to rational scalars: Betti diagrams of codimension c Cohen–Macaulay R-modules are, up to scalars, precisely the positive linear combinations of the $\pi(d)$, where d is a degree sequence of length c. Without the Cohen–Macaulay condition we allow degree sequences of length $\geq c$.

Let us translate this to the realm of torus actions. Let $T = T^r$ act on a space X. Then (in TG) we have $H_T^*(X) = H_0 \oplus H_1$ as dg$R$ms, where $H_0 = H_T^{even}(X)$ and $H_1 = H_T^{odd}(X)$. We consider now the graded vector spaces $\overline{H_0}$ and $\overline{H_1}$ with $\left(\overline{H_0}\right)^n = H_0^{2n}$ and $\left(\overline{H_1}\right)^n = H_1^{2n+1}$. The graded R-module structure on H_0 and H_1 in TG translates to graded R-module structures on $\overline{H_0}$ and $\overline{H_1}$ with respect to FRG. Let

$$\overline{H_0} \leftarrow F_* \quad \text{and} \quad \overline{H_1} \leftarrow F'_*$$

be free resolutions in FRG. Then we set $\tilde{F}_i$ to be the graded vector space with $(\tilde{F}_i)^n = (F_i)^{\frac{n+i}{2}}$ if $n+i$ is even and $(\tilde{F}_i)^n = 0$ if $n+i$ is odd. We also define $\tilde{F}'_i$ via $(\tilde{F}'_i)^n = {F'_i}^{\frac{n+i-1}{2}}$ if $n+i$ is odd and $(\tilde{F}'_i)^n = 0$ if $n+i$ is even. Then on $\tilde{F}_i$ and $\tilde{F}'_i$ we have canonically induced graded R-module structures in TG and the degree 0 differentials of the free resolutions induce degree 1 differentials

$$\tilde{F}_{i-1}^{n+1} \leftarrow \tilde{F}_i^n \quad \text{and} \quad \tilde{F'}_{i-1}^{n+1} \leftarrow \tilde{F'}_i^n.$$

We observe that the canonical degree 0 maps $\tilde{F}_0 \to H_0$ and $\tilde{F}'_0 \to H_1$ make $\bigoplus_i(\tilde{F}_i \oplus \tilde{F}'_i)$ the minimal dg$R$m-model of $(H_T^*(X), 0)$ in TG.

If the action is (almost) $\mathcal{MOD}$-formal, we have $R \otimes H^*(X) \cong \bigoplus_i(\tilde{F}_i \oplus \tilde{F}'_i)$ as graded R-modules, so the Betti numbers of the (FRG) graded modules $\overline{H_0}$ and $\overline{H_1}$ translate to the (TG) Betti numbers of X in the above fashion. In particular they are given by positive rational linear combinations of pure diagrams by Theorem 3.1.9 (with a degree shift). In case X is compact, the codimension c from Theorem 3.1.9 is given by the minimal orbit dimension so in this case we only need to consider degree sequences of length $\geq c$. Also if $\mathrm{fd}(X) = n$ is the highest degree with $H^n(X) = 0$, then this gives a natural bound on the degrees that can occur in the degree sequences: if d is a degree sequence of length $l \geq c$ contributing to the Betti diagram of $\overline{H_0}$, then degree restrictions imply $d_l \leq (n+l)/2$. As the sequence is strictly increasing we obtain $i \leq d_i \leq (n+l)/2 - (l-i) = (n-l)/2 + i$. Similar restrictions apply to the sequences contributing to the Betti diagram of $\overline{H_1}$. We sum up the discussion in the corollary below. For a degree sequence d of length l, we define the associated vector $\tilde{\pi}(d) \in \mathbb{Q}^{\mathbb{Z}}$ of Betti numbers via

$$\tilde{\pi}(d)_j = \begin{cases} \prod_{k \neq i} \frac{1}{|d_k - d_i|} & \text{if } j = 2d_i - i, \text{ for } 0 \leq i \leq l \\ 0 & \text{else} \end{cases}.$$

and also set $\tilde{\pi}'(d)_j = \tilde{\pi}(d)_{j-1}$. The degree sequences d^k below correspond to the free resolution of $\overline{H_0}$, while the δ^l correspond to $\overline{H_1}$.

Corollary 3.1.11. *Let X be a compact space of formal dimension n with a $\mathcal{MOD}$-formal T-action whose minimal orbits are of dimension c. Then there are degree sequences $d^1, \ldots, d^s$ and $\delta^1, \ldots, \delta^t$ with $c \leq \mathrm{length}(d^k), \mathrm{length}(\delta^l) \leq r$, satisfying $i \leq d^k_i \leq (n - \mathrm{length}(d^k))/2 + i$ and $i \leq \delta^l_i \leq (n - \mathrm{length}(\delta^l) - 1)/2 + i$ for $1 \leq k \leq s$, $1 \leq l \leq t$, such that the vector of Betti numbers $(b_0, \ldots, b_n) \subset \mathbb{Q}^{n+1} \subset \mathbb{Q}^{\mathbb{Z}}$ of X is a positive rational linear combination*

$$\sum_{k=1}^{s} a_k \tilde{\pi}(d^k) + \sum_{l=1}^{t} \alpha_l \tilde{\pi}'(\delta^l).$$

Example 3.1.12. As an application, let us examine possible Betti numbers of an almost free compact T^2-space X with $\mathrm{fd}(X) \leq 5$. By Corollary 2.3.3 the action is automatically $\mathcal{MOD}$-formal. As the minimal orbit dimension is 2, we only need to consider degree sequences of length 2. In the Betti diagram of $\overline{H_0}$ and $\overline{H_1}$, the degree restrictions allow only the degree sequences $(0,1,2)$, $(0,1,3)$, $(0,2,3)$, and $(1,2,3)$. We deduce that the vector $(b_0, \ldots, b_5)$ of Betti numbers of X lies in the rational cone which is generated by the vectors

$$(1,2,1,0,0,0),\ (2,3,0,0,1,0),\ (1,0,0,3,2,0),\ \text{and } (0,0,1,2,1,0),$$

coming from $\overline{H_0}$ and the same set of vectors, with nontrivial entries shifted one position to the right, from $\overline{H_1}$. If we number the above degree sequences by $d^1, \ldots, d^4$ from left to right, then e.g. for the Betti numbers of T^5 we have

$$(1,5,10,10,5,1) = \frac{1}{2}\pi(d^1) + \frac{1}{2}\pi(d^3) + \frac{5}{2}\pi(d^4) + \frac{7}{2}\pi'(d^1) + \frac{1}{2}\pi'(d^3).$$

The same considerations carry over more generally to the case of an almost free T^r-action when $\mathrm{fd}(X) \leq r+3$. In this case we obtain $r+2$ admissible degree sequences for each of the $\overline{H_i}$, giving rise to a cone spanned by $2r+4$ vectors.

As Boij–Söderberg theory is a complete classification, up to scalars, of possible Betti diagrams, it is natural to ask for a classification of possible vectors of Betti numbers of $\mathcal{MOD}$-formal T-spaces, in particular for compact spaces. For (almost) free actions there seems to be hope since it is possible to pass from algebra to geometry within the compact realm (see Proposition 1.2.4). Still the task seems out of reach as algebra structures are far more complex than R-modules. Also note that the idea of a stable approach, i.e. classifying Betti vectors up to multiplication by scalars, does not seem suited since on each path component we normalize to $b_0 = 1$.

The last condition does however only concern $H_T^{even}(X)$ and we will see below that there are no intrinsic restrictions on which R-modules can occur as $H_T^{odd}(X)$. This provides a partial converse to the previous corollary within the realm of almost free actions.

Corollary 3.1.13. *Let $v = \sum_k \alpha_k \tilde{\pi}'(\delta^k)$ be a positive rational linear combination where the δ^k are degree sequences of length r in degrees $\delta_i^k \geq 0$ for all k and $0 \leq i \leq r$. Let N be the maximum among the δ_r^k and set $d = (0, N-r+1, N-r+2, \ldots, N)$. Then there are positive integers a, α such that the vector $a\tilde{\pi}(d) + \alpha v$ consists of the Betti numbers of a compact free $\mathcal{MOD}$-formal T-space X.*

Proof. By Theorem 3.1.9 there is a finitely generated graded (in FRG) R-module $\overline{H_1}$ of dimension 0 whose Betti diagram is $\alpha\overline{v}$, where α is some positive integer and $\overline{v} = \sum_k \alpha_k \pi(\delta^k)$. We construct the dg$R$m $(H_1, 0)$ (in TG) as above. It is concentrated in odd, positive degrees and the number of generators of its minimal Hirsch–Brown model, which we build from the minimal free resolution of $\overline{H_1}$, is described by the vector αv in each degree. Observe that $\overline{H_1}$ has finite length and that the maximal nontrivial degree is $N-r$ (see [15, Corollary 4.4]). This translates to degree $2(N-r)+1$ as the top degree of H_1.

Set $H_0 = R/R^{2(N-r)+1}$ and consider the R-module $H = H_0 \oplus H_1$. We claim that there is a graded commutative algebra structure on H such that the previous R-module structure is induced by the canonical map $R \to H_0 \subset H$. We need to define a multiplication map $\mu\colon H \otimes H \to H$. On $H_0 \otimes H_0$ we define μ to be the canonical multiplication $H_0 \otimes H_0 \to H_0$, while on $H_1 \otimes H_1$ we set $\mu = 0$. Finally the multiplication on $H_0 \otimes H_1$ is the one induced from the module structure $R \otimes H_1 \to H_1$. This is well defined since $R^{2(N-r)+1}$ annihilates H_1 for degree reasons. We do the same for $H_1 \otimes H_0$ after applying the switching map $H_1 \otimes H_0 \cong H_0 \otimes H_1$. With this definition, μ is easily checked to be commutative and associative.

Consider a relative minimal Sullivan model for $(R, 0) \to (H, 0)$. Then by Proposition 1.2.4, there is a free T-action on a compact space X such that the relative model is a model for the Borel fibration of X. Since $(H, 0)$ is a formal cdga, the action is in particular $\mathcal{MOD}$-formal. The Betti numbers of X agree, in each degree, with the number of generators of the Hirsch–Brown model of $(H, 0)$. The latter decomposes as the model of $(H_0, 0)$ and the model of $(H_1, 0)$ which are just the respective minimal free resolutions. We conclude the proof by the observation that $R/R^{(2(N-r)+1}$ corresponds to R/R^{N-r+1} in FRG, which has a pure Betti diagram of type $d = (0, N-r+1, N-r+2, \ldots, N)$. □

In the corollary above, the construction of course works more generally if we replace R/R^{N-r+1} by R/J for some ideal J which is generated in degrees $\geq N-r+1$ (in FRG). In this case, $\tilde{\pi}(d)$ in the statement of the corollary is replaced by a suitable degree shifted version of the Betti diagram of R/J.

3.1.4 Formal elliptic spaces

Assuming formality of the space X itself does not seem to yield immediate results when attacking the TRC in full generality. In this section however, we provide a proof of the

TRC in case X is formal and elliptic.

Recall that an elliptic space of positive Euler characteristic, i.e. a *positively elliptic space*, has a minimal model given by a pure algebra $(\Lambda V, d)$ such that d maps a basis of V^{odd} to a maximal regular sequence of V^{even}. Recall that we say a fibration is totally non-homologous to zero (TNHZ) if its Serre spectral sequence collapses at E_2.

Proposition 3.1.14. *Let X be a formal elliptic space. Then rationally it is the total space of a TNHZ fibration with model*

$$(\Lambda B, 0) \to (\Lambda B \otimes \Lambda V, D) \to (\Lambda V, d),$$

where $B = B^{odd}$ and $(\Lambda V, d)$ is positively elliptic.

Proof. By [18], X has a two-stage model of the form $(\Lambda Z, D)$, where $Z = Z_0 \oplus Z_1$, such that $Z_1 = Z_1^{odd}$ and D maps a basis of Z_1 to a regular sequence $a_1, \ldots, a_k$ in ΛZ_0. Now set $V = Z_0^{even} \oplus Z_1$ and $B = Z_0^{odd}$, which produces the desired extension sequence. We now show that $(\Lambda V, d)$ is positively elliptic, where d is the differential obtained by projecting B^+ to 0.

Observe that $a_1, \ldots, a_k$ is in particular a regular sequence in the strictly commutative ring

$$(\Lambda Z_0)^{even} = \Lambda Z_0^{even} \otimes (\Lambda Z_0^{odd})^{even}.$$

The right hand tensor factor has Krull dimension 0 whence the Krull dimension of $(\Lambda Z_0)^{even}$ is equal to $r := \dim Z_0^{even}$. In particular, we have $k \leq r$. Denote by $J \subset \Lambda Z_0^{even}$ the ideal generated by the canonical projections $\overline{a_1}, \ldots, \overline{a_k}$ of the a_i to ΛZ_0^{even}. It follows from the odd spectral sequence of the elliptic algebra $(\Lambda Z, D)$ that the quotient $\Lambda Z_0^{even}/J$ is finite-dimensional. In particular, J has height r. Since J is generated by $k \leq r$ elements, it follows that $r = k$ and that the $\overline{a_i}$ form a regular sequence in ΛZ_0^{even}.

It remains to prove that the Serre spectral sequence collapses at E_2. Observe that $E_2^{*,*} = H^*(\Lambda B, d) \otimes H^*(\Lambda V, d)$. In particular we have $E_2^{0,*} = H^*(\Lambda V, d)$. This is completely represented by polynomials in Z_0^{even} due to the fact that $(\Lambda V, d)$ is positively elliptic. As D vanishes on Z_0^{even}, it follows that also the differentials of the spectral sequence vanish on $E_r^{0,*}$, $r \geq 2$, which causes the spectral sequence to collapse. □

Corollary 3.1.15 ([44])**.** *The TRC holds for formal elliptic spaces.*

Proof. Let X be formal and elliptic and display its model as in the previous proposition. For the homotopy Euler characteristic of X, we obtain $\chi_\pi(X) = -\dim B$ which implies $\mathrm{rk}_0(X) \leq \dim B$. On the other hand, due to the collapse of the Serre spectral sequence, we have

$$\dim H^*(X) = \dim H^*(\Lambda V, d) \cdot 2^{\dim B}.$$

□

3.2 Bounds in the general setting

3.2.1 Under assumptions on certain Betti numbers

One of the classical results regarding the toral rank is the inequality

$$\dim H^*(X) \geq 2\mathrm{rk}_0(X)$$

by Allday and Puppe. The idea of the proof is that since $R \otimes H^0(X)$ gets mapped to 0 by the differential in the minimal Hirsch–Brown model of a T^r-action, almost all of $R \otimes H^0(X)$ has to get killed in cohomology in order for the cohomology of X_{T^r} to be finite-dimensional. By projecting to this submodule, the differential induces a map

$$R \otimes H^{odd} \to R \otimes H^0(X)$$

with image in $\mathfrak{m} \otimes H^0(X)$, whose cokernel is finite-dimensional over $\mathbb{Q}$. One can show, e.g. by means of the Krull height theorem, that such a map $R^l \to R$ requires l to be at least r since its image is an ideal of maximal height. From this, the above inequality can be deduced using the fact that the Euler characteristic of X is 0.

Our goal is to investigate how the above result can be improved if there are multiple copies of R getting mapped to 0 by the differential of the Hirsch–Brown model. In other words: if we have a map $R^l \to R^k$ with image in $\mathfrak{m}R^k$ and finite-dimensional cokernel, what can be said about the relation of k and l? The R-linear maps defined by the matrices

$$\begin{pmatrix} 0 & X & Y \\ X & Y & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & W & X & Y & Z \\ W & X & Y & Z & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & X & Y & Z \\ 0 & X & Y & Z & 0 \\ X & Y & Z & 0 & 0 \end{pmatrix}$$

fulfil $l = k+r-1$ (if R is considered to be the polynomial ring over the variables occurring in the respective matrix). The author suspects the pattern seen in the matrices can be used to construct examples where $l = k + r - 1$ for all values of k and r.

This rather unsatisfactory lower bound may however be improved under certain additional assumptions: in the examples above, the cokernel, whilst being finite-dimensional, has rather large formal dimension. It is our goal to obtain a better estimate by taking this factor into consideration. For the following proposition we want to apply Boij–Söderberg theory and thus momentarily use the free resolution grading (FRG, see Section 3.1.3).

Proposition 3.2.1. *Let $k, l \geq 1$ and $f\colon R^l \to R^k$ be a graded R-linear map with respect to some grading on R^l and the grading with generators concentrated in degree 0 on R^k. Assume further that* $\operatorname{im}(f) \subset \mathfrak{m}R^k$ *and that* $\operatorname{coker}(f)$ *has finite length. Let N be the maximal degree in which* $\operatorname{coker}(f)$ *is nontrivial. Then*

$$l \geq \frac{N+r}{N+1}k.$$

Proof. We may change the grading on R^l such that f becomes a degree 0 map of graded R-modules. Consider the minimal free resolution $\operatorname{im}(f) \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots$ of $\operatorname{im}(f)$. As minimality is equivalent to the image of the map $F_i \leftarrow F_{i+1}$ being contained in $\mathfrak{m}F_i$ at each stage, we deduce that

$$\operatorname{coker}(f) \leftarrow R^k \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots$$

is a minimal free resolution of $\operatorname{coker}(f)$. For the graded Betti numbers $\beta_{i,j}$ of $\operatorname{coker}(f)$, we set $\beta_i = \sum_j \beta_{i,j}$. Note that the original sequence

$$\operatorname{coker}(f) \leftarrow R^k \leftarrow R^l$$

can also be completed to a (not necessarily minimal) free resolution of $\operatorname{coker}(f)$ by just taking any free resolution of $\ker(f)$. In particular, minimality of the first resolution implies that R^l contains F_0 as a direct summand so $l \geq \beta_1$. We also have $k = \beta_0$ because R^k is the first step of the minimal free resolution of $\operatorname{coker}(f)$.

We have a closer look at the Betti diagram of $\operatorname{coker}(f)$. The Castelnuovo-Mumford regularity of $\operatorname{coker}(f)$ is equal to N (c.f. [15, Cor. 4.4]). This means that $\beta_{i,j} = 0$ for $j > N + i$. Furthermore, note that the Krull dimension and the depth of $\operatorname{coker}(f)$ are equal to 0 because $\operatorname{coker}(f)$ has finite length. So $\operatorname{coker}(f)$ is a Cohen–Macaulay module of codimension r. By Theorem 3.1.9, the Betti diagram of $\operatorname{coker}(f)$ can be written as a positive linear combination of pure diagrams of length r. This implies that a lower bound on the ratio of the first two Betti numbers of such a pure diagram gives a lower bound for β_1/β_0 and thus for l/k.
Let $b_0, \ldots, b_r$ be the Betti numbers of a codimension r Cohen–Macaulay module C with pure Betti diagram of type d, where d is a degree sequence of length r. We can assume that $d_0 = 0$ and $d_i \leq N+i$, for otherwise the Betti diagram of C can not occur nontrivially in a positive linear combination forming the Betti diagram of $\operatorname{coker}(f)$ due to the degree restrictions above. The Herzog-Kühl equations yield

$$\frac{b_1}{b_0} = \prod_{i=2}^{r} \frac{d_i}{d_i - d_1}.$$

This ratio is minimal when the d_i take their maximum values $d_i = N + i$ and d_1 takes its minimum value 1. In total, we get

$$\frac{b_1}{b_0} \geq \prod_{i=2}^{r} \frac{N+i}{N+i-1} = \frac{N+r}{N+1},$$

which proves the claim. □

Note that this bound is sharp: By part (*iii*) of Theorem 3.1.9, for any $N \geq 0$, there is a codimension r Cohen–Macaulay module with pure diagram of type d, where $d_0 = 0$, $d_1 = 1$, and $d_i = N + i$ for $2 \leq i \leq r$. This module is of finite length and its maximal nontrivial degree coincides with the Castelnuovo-Mumford regularity N. Consequently, the first map in the free resolution has the desired properties.
The question that ensues is when we actually have generators in the minimal Hirsch–Brown model that map to 0 under the differential. The following trick, used in the proof of the TRC for Hard Lefschetz manifolds in [3], will be fundamental to assume the existence of such generators.

Lemma 3.2.2. *Let X be a finite CW-complex with an almost free T^r-action and*

$$(R, 0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$$

a Sullivan model for the associated Borel fibration. If $Z \subset V^1$ is a subspace such that $d(Z) = 0$ and $D|_{1\otimes Z}$ is injective, then Z generates an exterior algebra in $H^*(X)$. In particular, if $\operatorname{fd}(X) - 1 \leq r \leq \operatorname{fd}(X)$, then $H^*(X)$ contains an exterior algebra on r-generators of degree 1.

Proof. We show the first part via induction. Assume that $\mathrm{im}(d) \cap \Lambda^l Z = 0$ and $d(x) \in \Lambda^{l+1} Z$ for some $x \in \Lambda V$. We can write $D(x) = d(x) + y$ for some $y \in R^{\geq 2} \otimes \Lambda V$. By assumption, $D(Z)$ is contained in $R^2 \otimes 1$ so D maps $d(x)$ into $R^2 \otimes \Lambda^l Z$. As $D^2(x) = 0$, it follows that $D(y) = -D(d(x)) \in R^2 \otimes \Lambda^l Z$. But as the $R^2 \otimes \Lambda V$ component of $D(y)$ is, for degree reasons, equal to $(1 \otimes d)(y)$, this implies $D(y) = 0$ by the induction hypothesis. Hence, $D(d(x)) = 0$ and also $d(x) = 0$ because D is injective on ΛZ. By induction, we obtain that $\mathrm{im}(d) \cap \Lambda Z$ is trivial so ΛZ projects injectively into cohomology.
Now, if $\mathrm{fd}(X) - 1 \leq r \leq \mathrm{fd}(X)$, then $\mathrm{fd}(X_{T^r}) \leq 1$ so D must map surjectively onto $R^2 \otimes 1$, which is r-dimensional. Preimages must, for degree reasons, lie in $V^1 \cap \ker(d)$ so the second part follows. □

Together, the previous considerations enable the main result of this section.

Theorem 3.2.3. *Let X be a compact space of formal dimension n with an almost free T^r-action.*

(i) Let b be the first Betti number of X and $m = \max(b - r, 0)$. Then

$$\dim H^*(X) \geq \min_{k=m,\dots,b} \frac{n+r-1}{n-r+1} 2k + 2^{b-k}.$$

(ii) Let k be the degree corresponding to the first nontrivial odd Betti number. Then

$$\dim H^*(X) \geq \frac{n+r-1}{n-r+1} 2 \dim H^{<k}(X).$$

Proof. For the proof of (i), consider the relative minimal model

$$(R, 0) \to (R \otimes \Lambda V, D) \to (\Lambda V, d)$$

for the Borel fibration of the action. By assumption, $\ker(d|_{V^1})$ is b-dimensional. Now decompose $\ker(d|_{V^1}) = Z \oplus Z'$, where $Z' = \ker(D|_{V^1})$ and let $k \in \{\max(b - r, 0), \dots, b\}$ be the dimension of Z'. By Lemma 3.2.2, Z generates an exterior algebra in cohomology. If $k = 0$, Theorem 3.2.3 holds so in what follows we will assume $k \geq 1$. If $r = n$, then $H^1(X_{T^r}) = 0$, which implies $k = 0$ so we will assume $r < n$ as well.
Let us now construct the minimal Hirsch–Brown model of the action. We decompose $\Lambda V = A \oplus B \oplus C$ as vector spaces as in the construction presented in Section 1.3.2. Note that since $\Lambda Z \oplus Z'$ projects injectively into cohomology, the decomposition above can be chosen in a way that $\Lambda Z \oplus Z'$ is contained in A. Let $(R \otimes H^*(X), \delta)$ be the resulting Hirsch–Brown model and recall that on $R \otimes H^{\leq l}(X)$, δ is explicitly given by

$$\delta = \tilde{g} \Sigma_l \tilde{f}$$

with notation as in Section 1.3.2. Note that since t maps $R \otimes (\Lambda Z \oplus Z')$ to itself and thus into $R \otimes A$, we have $\tilde{\phi} t|_{\Lambda Z \oplus Z'} = 0$ and in particular $\delta|_{\overline{\Lambda Z} \oplus \overline{Z'}} = \tilde{g} t \tilde{f} = \tilde{g} D \tilde{f}$, where $\overline{\Lambda Z}, \overline{Z'} \subset H^*(X)$ denote the corresponding subspaces of cohomology. This implies

$$\delta(R \otimes \overline{\Lambda Z}) \subset R \otimes \overline{\Lambda Z} \quad \text{and} \quad \delta(R \otimes \overline{Z'}) = 0.$$

Hence, when composed with a suitable projection onto $R\otimes\overline{Z'}$, δ induces a map

$$R\otimes (H^*(X)/\overline{\Lambda Z})^{even}\to R\otimes\overline{Z'}$$

with image in $\mathfrak{m}\otimes\overline{Z'}$, whose cokernel is finite-dimensional because the cohomology of the Hirsch–Brown model is.
This cokernel vanishes in degrees above the formal dimension of X_{T^r} which is $n-r$ in TG. This implies that when we give $R\otimes\overline{Z'}$ the grading from Proposition 3.2.1, where generators are in degree 0 and the variables of R have degree 1, the maximal degree N in which the cokernel is nontrivial fulfils $2N+1\leq n-r$, so $N\leq\frac{n-r-1}{2}$. Proposition 3.2.1 yields

$$\dim(H^*(X)/\overline{\Lambda Z})^{even}\geq\frac{n+r-1}{n-r+1}k.$$

As $(\overline{\Lambda Z})^{even}$ has dimension 2^{b-k-1}, we obtain

$$\dim H^*(X)^{even}\geq\frac{n+r-1}{n-r+1}k+2^{b-k-1}.$$

Using that the Euler characteristic of X is 0 completes the proof of (i).

In the situation of (ii), note that in a minimal Hirsch–Brown model of an almost free T^r-action, $R\otimes H^{<k}(X)$ gets mapped to 0 for degree reasons. Projecting onto this submodule, the differential induces a map

$$R\otimes H^{odd}(X)\to R\otimes H^{<k}(X)$$

which has image in $\mathfrak{m}\otimes H^{<k}(X)$ and finite-dimensional cokernel as the cohomology of X_{T^r} is finite-dimensional. Applying Proposition 3.2.1 as in (i) together with the fact that the Euler characteristic is 0 yields the desired lower bound. □

Let us take some time to put Theorem 3.2.3 into perspective and compare it to the known results. The quotient $(n+r-1)/(n-r+1)$ approaches $2r-1$ as r approaches n. In this sense, for a fixed space X, we can say that our lower bound in (ii) approaches a linear bound of slope $4\dim H^{<k}(X)$ for smaller cohomogeneities.

For the bound in (i), calculations show that for positive $a,b\in\mathbb{R}$, the real valued function $f(x)=ax+2^{b-x}$ has global minimum $a(b-\log_2(a)+(1+\log\log(2))/\log(2))$. Thus, in the same vague sense as before and neglecting logarithmic terms, the bound approximates to a linear bound with slope close to $4b$ when approaching the extreme case $r=n$. However, note that in both cases, the approximation occurs rather late and that the result is not interesting for very small cohomogeneities, such as $n-1\leq r\leq n$, where the TRC is proved easily as seen in Lemma 3.2.2 above. Still, the theorem gives an improved lower bound in many cases.
The tables below are meant to give a feeling for the behaviour of our lower bounds and when Theorem 3.2.3 is an improvement of the established linear bound of slope 8/3 from [5]. In both tables, we have set $n=10$. Let us begin with estimate (i):

r	1	2	3	4	5	6	7	8	9	10
$b=4$	8	9	10	12	13	14	16	16	16	16
$b=6$	12	14	16	19	22	26	32	39	50	64
$b=10$	20	24	28	34	41	50	64	84	122	216

Now let us have have a look at scenario (ii) of Theorem 3.2.3 where we set $l = \dim H^{<k}(X)$.

r	1	2	3	4	5	6	7	8	9	10
$l = 4$	8	10	12	15	19	24	32	46	72	152
$l = 6$	12	15	18	23	28	36	48	68	108	228
$l = 10$	20	25	30	38	47	60	80	114	180	380

3.2.2 Application to c-symplectic spaces

As an application of the previous section, we investigate the cohomological consequences of almost free torus actions on the following class of spaces.

Definition 3.2.4. A space M is called c-symplectic if $H^*(M)$ satisfies Poincaré duality with fundamental class in degree $2n$ and there is an element $\omega \in H^2(M)$ such that ω^n is nontrivial.

The most important examples of c-symplectic spaces are of course compact symplectic manifolds. However, the notion of a c-symplectic space is a little more general, even within the realm of compact manifolds. For instance certain connected sums of copies of $\mathbb{C}\mathrm{P}^n$ are known to not carry almost complex structures and thus, in particular, can not be symplectic (see [26]).

Torus actions on c-symplectic spaces are quite well understood as a lot of ideas from equivariant symplectic geometry carry over to the purely topological setting. In the context of this thesis, the most interesting result is the following: the orbit map $T^r \to M$ of an almost free T^r-action on a c-symplectic space M of Lefschetz type (which means that multiplication with ω^{n-1} induces an isomorphism $H^1(M) \cong H^{2n-1}(M)$) induces a surjection on cohomology. This was proved in [48] and [2], where the first reference has a homotopy theoretical approach while the second reference operates purely on the level of equivariant cohomology. As an immediate consequence we see that $H^*(M)$ contains an exterior algebra on r generators of degree 1 and in particular the TRC holds. Without the Lefschetz type assumption, a weaker form of this statement is still true in the form of the theorem below which was proved in [48]. We give a simplified proof through equivariant cohomology. Note that the core idea of our proof (the fact that a c-Hamiltonian S^1-action has a fixed point) was also shown in [2] using similar machinery.

Theorem 3.2.5. *Let M be a c-symplectic compact space with an almost free T^r-action. Then*

$$r \leq b_1,$$

where b_1 is the first Betti number of M.

Proof. Assume that $r > b_1$. Let $a_1, \ldots, a_{b_1}$ be a basis of $H^1(M)$, denote by d_2 the differential on the second page of the Serre spectral sequence of the Borel fibration associated to the action, and by ω the symplectic class in $H^2(M)$. We have

$$d_2(\omega) = \sum_{i=1}^{b_1} a_i \otimes p_i$$

for certain $p_1, \ldots, p_{b_1} \in H^2(BT^r)$. Since $r > b_1$ there exists a sub-circle $S^1 \subset T^r$ such that the p_i lie in the kernel of the map $H^*(BT^r) \to H^*(BS^1)$. The morphism

$$\begin{array}{ccccc} M & \longrightarrow & M_{T^r} & \longrightarrow & BT^r \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & M_{S^1} & \longrightarrow & BS^1 \end{array}$$

of the Borel fibrations of the T^r- and the restricted S^1-actions induces a morphism of the associated spectral sequences. Therefore, by construction, the second differential of the spectral sequence associated to the lower row vanishes on ω. Observe that the same holds for all subsequent differentials due to degree reasons. This implies that ω^n lives to infinity, where $2n$ is the formal dimension of M. But this contradicts the fact that the formal dimension of M_{S^1} is less than $2n$. □

Remark 3.2.6. To put the above proof into a geometric context, note that we essentially show that, in case of an almost free action, the element $d_2(\omega) \in H^2(BT^r) \otimes H^1(M) \cong \mathfrak{t}^* \otimes H^1(M) \cong \hom(\mathfrak{t}, H^1(M))$ has to be injective, where $\mathfrak{t}$ denotes the Lie-algebra of T^r. Compare this to the situation of a symplectic action on a smooth symplectic manifold (M, ω): any element in the kernel of the homomorphism $\mathfrak{t} \to H^1(M)$ sending X to the contraction of ω along the fundamental vector field of X generates a subgroup of T^r that acts in a Hamiltonian fashion and thus has fixed points, preventing the action to be almost free. The connection between the two homomorphisms can be made explicit using the Cartan model.

Corollary 3.2.7. *Let M be a c-symplectic compact space with* $\mathrm{fd}(M) = 2n \geq 4$ *and an almost free T^r-action. Then*

$$\dim H^*(M) \geq 4r.$$

Proof. By Theorem 3.2.5, we have $\dim H^1(M) \geq \mathrm{rk}_0(M)$. As $H^*(M)$ fulfils Poincaré duality, the same is true for $\dim H^{2n-1}(M)$. If $r \geq 1$, then the Euler characteristic of M is equal to 0. Both 1 and $2n-1$ are odd so the corollary follows. □

With regards to the toral rank conjecture one might hope that the r linearly independent elements of $H^1(M)$ whose existence is assured by Theorem 3.2.5 pull back nontrivially to $H^1(M)$ via the orbit map. In this case they would span an exterior algebra and the TRC would follow. To see that this fails without the Lefschetz type assumption, one needs to look no further than the standard T^2-action on the Kodaira-Thurston manifold: here the orbit map has 1-dimensional kernel in cohomology. The example below makes it clear that we can not hope to directly obtain an improved lower bound on $\dim H^*(M)$ by using the algebra structure of $H^*(M)$ and the elements in $H^1(M)$ given by Theorem 3.2.5.

Example 3.2.8. Consider the cdga $(\Lambda(a_1, a_2, a_3, b_1, b_2, b_3), d)$ where all generators are of degree 1 and $d(a_1) = d(a_2) = d(a_3) = 0$, $d(b_1) = a_2a_3$, $d(b_2) = a_3a_1$ and $d(b_3) = a_1a_2$. This is the Sullivan model of a compact nilmanifold M which is a T^3-bundle over T^3. The element $\omega = a_1b_2 + a_2b_3 + a_3b_1$ fulfils $d(\omega) = 0$ and $\omega^3 = -6a_1a_2a_3b_1b_2b_3$ so M is c-symplectic. Notice that $H^1(M)$ is spanned by the classes of the a_i and that the cohomology class of any element of the form a_ia_j is trivial in $H^2(M)$. Thus every two elements of $H^1(M)$ have trivial product.

The results of the previous section enable us to make use of Theorem 3.2.5 without using the multiplicative structure. Combining it with Theorem 3.2.3 immediately yields

$$\dim H^*(M) \geq \min_{k=0,\ldots,r} \frac{2n+r-1}{2n-r+1} 2k + 2^{r-k} \tag{$*$}$$

for a c-symplectic compact space M of formal dimension $2n$ with an almost free T^r-action. To see this, one quickly checks that the bound (i) from Theorem 3.2.3 grows monotonously in b because this is true for a fixed k and the minimum is never realized by $k = b$. Thus by Theorem 3.2.5, we can replace b by r. The bound $(*)$, in contrast to Theorem 3.2.3, requires no assumptions on Betti numbers and, in the same vague spirit as the discussion after Theorem 3.2.5, approaches a quadratic bound for small cohomogeneities. Note that the proof of $(*)$ does not yet make any use of Poincaré duality, which holds by definition for c-symplectic spaces. The remainder of this section is dedicated to the improvements that can be made by refining the previous arguments with regards to using Poincaré duality.

Theorem 3.2.9. *Let M be a c-symplectic compact space of formal dimension $2n$ with an almost free T^r-action. Then*

(i)

$$\dim H^*(M) \geq \min_{k=0,\ldots,r} \frac{2n+r-1}{2n-r+1} 4k + 4 \sum_{i=0}^{\frac{n-1}{2}} \binom{r-k}{2i}$$

if n is odd and $r \geq n+1$.

(ii)

$$\dim H^*(M) \geq \min_{k=0,\ldots,r} \frac{2n+r-1}{2n-r+1} 4k + 4 \sum_{i=0}^{\frac{n-2}{2}} \binom{r-k}{2i} + 2\binom{r-k}{n}$$

if n is even and $r \geq n$.

In particular, the TRC holds for M if $n \leq 4$.

Remark 3.2.10. Calculations show that, for $n \geq 7$, the minimum of the bounds in Theorem 3.2.9 is realized for $k > r - n$. In particular, in both the even and the odd bound, we can replace the sum of the binomial coefficients by 2^{r-k+1}. So both bounds coincide and are exactly double the bound $(*)$. However, for $n \leq 6$ the distinction is necessary.

Proof. The proof is, for the most part, identical to the proof of Theorem 3.2.3 but we pay more attention to the degrees of elements to be able to use Poincaré duality. Let k, Z, Z' be as in the proof of Theorem 3.2.3 and construct the map

$$p\colon R \otimes (H^*(M)/\overline{\Lambda Z})^{even} \to R \otimes \overline{Z'}$$

as before. Again, we treat the case $k = 0$ (and thus also $r = 2n$) separately: $H^*(M)$ contains an exterior algebra on r generators of degree 1 and fulfils Poincaré duality so the theorem holds. Thus in what follows we will assume $k > 0$ and $r < 2n$.

Now instead of just applying Proposition 3.2.1, let us refine the argument a little. As in the proof of Proposition 3.2.1, we can consider p as the first step in a free resolution of $\operatorname{coker}(p)$. Now decompose $R\otimes(H^*(M)/\overline{\Lambda Z})^{even} = F\oplus F'$, where F, F' are free R modules and $p|_F\colon F\to R\otimes\overline{Z'}$ is the first stage of a minimal free resolution of $\operatorname{coker}(p)$. The space of generators $F\otimes_R\mathbb{Q}\cong F/\mathfrak{m}F$ of F injects into $(H^*(M)/\overline{\Lambda Z})^{even}$. A generator of F that is of degree l with respect to the free resolution grading as in Proposition 3.2.1 (meaning that variables of R have degree 1, $\overline{Z'}$ is concentrated in degree 0 and p has degree 0) injects into the degree $2l$ component of $(H^*(M)/\overline{\Lambda Z})^{even}$ (in TG; see the discussion on the degree shifts in Section 3.1.3).

Now assume first that n is odd and $r\geq n+1$. Since $H^*_{T^r}(M)$ vanishes in degrees above $2n-r$ (in TG), the Castelnuovo-Mumford regularity N of $\operatorname{coker}(p)$ satisfies $N\leq(2n-r-1)/2\leq(n-2)/2$. In particular, the generators of F lie in degrees $\leq n/2$ and thus contribute to $(H^*(M)/\overline{\Lambda Z})^{<n,\ even}$. Analogous to the proof of Theorem 3.2.3, adding the part of $\overline{\Lambda Z}$ that lies in even degrees below n, we obtain

$$\dim H^{<n,\ even}(M)\geq\frac{2n+r-1}{2n-r+1}k+\sum_{i=0}^{\frac{n-1}{2}}\binom{r-k}{2i}.$$

Now the lower bound for $\dim H^*(M)$ follows by using Poincaré duality and the fact that the Euler characteristic vanishes to multiply this bound by 4.

In case n is even, we have to pay extra attention to degree n because it can not be doubled using Poincaré duality. First, we deal with the case when $n\leq r\leq n+1$. The case $r\geq n+2$ will be exposed as an easy special case. For some $d_1<\frac{2n-r+3}{2}$, define

$$S(d_1):=\prod_{i=2}^{r}\frac{2n-r-1+2i}{2n-r-1+2i-2d_1}.$$

Recall that for a codimension r Cohen–Macaulay module of Castelnuovo-Mumford regularity $\leq\frac{2n-r-1}{2}$ with pure Betti diagram of type d, we can bound the ratio of the first two Betti numbers by

$$\frac{\beta_1}{\beta_0}\geq S(d_1)$$

as in the proof of Proposition 3.2.1. The Betti diagram of $\operatorname{coker}(p)$ decomposes as a positive linear combination $a_1\pi(d^1)+\ldots+a_l\pi(d^l)$ of pure diagrams. The sum of all $a_i\pi(d^i)_{0,0}$ equals k, the zeroth Betti number of $\operatorname{coker}(p)$. Define α to be the sum of those $a_i\pi(d^i)_{0,0}$ for which $d^i_1=n/2$. Note that $d_i>n/2$ is not possible due to the degree restrictions of the regularity. At the first stage of the minimal free resolution of $\operatorname{coker}(p)$ we obtain

$$\dim(F\otimes_R\mathbb{Q})^{\frac{n}{2}}\geq S\left(\frac{n}{2}\right)\alpha\quad\text{and}\quad\dim(F\otimes_R\mathbb{Q})^{<\frac{n}{2}}\geq S(1)(k-\alpha)$$

because $S(1)\leq S(d^i_1)$ for any of the d^i. As above, adding $\overline{\Lambda Z}$, we obtain

$$\dim H^n(M)\geq S\left(\frac{n}{2}\right)\alpha+\binom{r-k}{n}$$

and

$$\dim H^{<n,\ even}(M) \geq S(1)(k-\alpha) + \sum_{i=0}^{\frac{n-2}{2}} \binom{r-k}{2i}.$$

When bounding all of $H^*(M)$, we can count the $< n$ part twice, due to Poincaré duality and then double everything using the Euler characteristic. In total, we obtain

$$\dim H^*(M) \geq S(1)4(k-\alpha) + S\left(\frac{n}{2}\right)2\alpha + 4\sum_{i=0}^{\frac{n-2}{2}} \binom{r-k}{2i} + 2\binom{r-k}{n}.$$

Note that, for $\alpha = 0$, this is the bound claimed in the theorem. For $n \leq r \leq n+1$ and $n \geq 4$, Lemma 3.2.11 below shows that $S\left(\frac{n}{2}\right) \geq 2S(1)$ so the above bound is minimal for $\alpha = 0$ and the theorem holds. When $n = 2$, the desired bound is dominated by Corollary 3.2.7. If $r \geq n+2$, $\alpha = 0$ must hold from the start because of the degree restrictions of the regularity. To verify that this (in combination with Corollary 3.2.7) proves the TRC for $n \leq 4$, see Remark 3.2.13 below. □

For the remaining values of r, the bound $(*)$ can not be doubled with the help of Poincaré duality. However, we still get a rather complicated, intermediate result that lies between $(*)$ and the bound from Theorem 3.2.9. Let us do some calculations first, where $S(\cdot)$ is defined as in the proof of Theorem 3.2.9.

Lemma 3.2.11. *Let $n \geq 4$ even and $3 \leq r \leq n+1$. Then $S\left(\frac{n}{2}\right) \geq 2S(1)$.*

Proof. Observe first that $S(1) = \frac{2n+r-1}{2n-r+1} \leq \frac{3n}{n} = 3$ and that

$$S\left(\frac{n}{2}\right) = \prod_{i=2}^{r} \frac{2n-r-1+2i}{n-r-1+2i} \geq \prod_{i=2}^{r} \frac{2n+r-1}{n+r-1} \geq \left(\frac{3}{2}\right)^{r-1}.$$

This implies that, independent of n, the claim is true for $r \geq 6$. For the remaining cases, we use induction over n. Assume that the claim is true for some n, r which is equivalent to $P(n) \geq 2$ where

$$P(n) := \frac{S\left(\frac{n}{2}\right)}{S(1)} = \prod_{i=2}^{r} \frac{2n-r-3+2i}{n-r-1+2i}.$$

Leaving r fixed, we obtain

$$P(n+2) = \prod_{i=2}^{r} \frac{2n-r-3+2(i+2)}{n-r-1+2(i+1)} = P(n)\frac{f}{g}$$

where $f = (2n+r+1)(2n+r-1)(n-r+3)$ and $g = (2n-r+1)(2n-r+3)(n+r+1)$. So $P(n+2) \geq P(n)$ is equivalent to $f - g = 12n(r-1) - 2r^3 + 6r^2 + 2r - 6 \geq 0$. As $n \geq r-1$, this expression is bounded from below by $-2r^3+18r^2-22r+6$, which is positive for $3 \leq r \leq 5$. This implies that if the claim is true for (n, r) with $3 \leq r \leq 5$, it is also true for $(n+2, r)$. The lemma now follows by checking that it holds for $(n, r) = (4, 3), (4, 4)$, and $(4, 5)$. □

Theorem 3.2.12. *Let M be a c-symplectic compact space of formal dimension $2n$ with an almost free T^r-action where $r \leq n$ if n is odd and $r \leq n-1$ if n is even. Then*

$$\dim H^*(M) \geq \min_{\substack{k=0,\ldots,r\\ \gamma\in[0,k]}} \max\left(B_1(k,\gamma), B_2(k,\gamma)\right)$$

where

$$B_1(k,\gamma) = 2S(1)(k-\gamma) + 2S\left(\left\lfloor \frac{n}{2}+1\right\rfloor\right)\gamma + 2^{r-k}$$

and

$$B_2(k,\gamma) = 4S(1)(k-\gamma) + 2^{r-k+1}.$$

Proof. Define k, Z, Z', p, F as in the proof of Theorem 3.2.9. Again, the theorem holds for $k=0$ so assume $k>0$. Let us treat the case when n is even. We decompose the Betti diagram of coker(p) as a positive linear combination $a_1\pi(d^1)+\ldots+a_l\pi(d^l)$ of pure diagrams. Now define α and γ as the sum of those $a_i\pi(d^i)_{0,0}$ for which $d_1^i = \frac{n}{2}$ and $d_1^i \geq \frac{n}{2}+1$. As before, we obtain bounds on $\dim F\otimes\mathbb{Q}$ in certain degrees which translate into the bounds

$$\begin{aligned}\dim H^{<n,\ even}(M) &\geq S(1)(k-\alpha-\gamma) + 2^{r-k-1}\\ \dim H^n(M) &\geq S\left(\frac{n}{2}\right)\alpha\\ \dim H^{>n,\ even}(M) &\geq S\left(\frac{n}{2}+1\right)\gamma,\end{aligned}$$

where we have used that the even part of $\overline{\Lambda Z}$ is contained in $H^{<n,\ even}(M)$. With the help of the Euler characteristic, we obtain

$$\dim H^*(M) \geq C_1(k,\alpha,\gamma) := 2S(1)(k-\alpha-\gamma) + 2S\left(\frac{n}{2}\right)\alpha + 2S\left(\frac{n}{2}+1\right)\gamma + 2^{r-k}.$$

On the other hand, if we forget about cohomology in degree $>n$ and use Poincaré duality first, we get

$$\dim H^*(M) \geq C_2(k,\alpha,\gamma) := 4S(1)(k-\alpha-\gamma) + 2S\left(\frac{n}{2}\right)\alpha + 2^{r-k+1}.$$

This implies $\dim H^*(M) \geq \max\left(C_1(k,\alpha,\gamma), C_2(k,\alpha,\gamma)\right)$. If we can prove that this expression takes its minimum for $\alpha = 0$, this proves the claim. For $r \leq 2$, the bounds get dominated by Corollary 3.2.7 so there is nothing to prove. For $n \geq 4$ and $r \geq 3$, this is a consequence of Lemma 3.2.11. Hence, the theorem is proved in case n is even. If n is odd, the proof is completely analogous except we can assume $\alpha = 0$ from the start because F has no elements in degree $\frac{n}{2}$. □

Let us have a look at some values of the lower bounds given by Theorems 3.2.9 and 3.2.12.

r	1	2	3	4	5	6	7	8	9	10
$n=2$	3	6	10	16						
$n=3$	3	6	12	28	44	64				
$n=4$	3	7	13	25	40	65	110	214		
$n=5$	3	6	11	20	33	52	80	123	208	428

Remark 3.2.13. Recall that the TRC is fulfilled for cohomogeneity 3 or less by Corollary 2.3.3. Combining this with the values of the above table, it follows that the TRC holds for c-symplectic spaces of formal dimension 8 or less. In dimension 10, the only missing case is $r = 6$ where we have a lower bound of 52 instead of the necessary 64. It is also interesting to note that, for some small r and n, our lower bounds are actually stronger than the TRC which is sharp for compact manifolds in general.
This is of course due to the fact that we consider special spaces in a fixed formal dimension. However, dropping either of both assumptions results in the TRC becoming sharp again: the torus acting on itself shows that the TRC is sharp for c-symplectic spaces. Also, if we only fix a formal dimension n and omit the assumption of being c-symplectic, the TRC can be seen to be sharp by considering e.g. a suitable product of r odd spheres with total dimension n, at least in the case $r \equiv n \mod 2$. Improvements on the TRC may be possible if r and n are of different parity: for $\mathcal{MOD}$-formal actions this is a consequence of Proposition 3.1.4.

Bibliography

[1] C. Allday. Examples of circle actions on symplectic spaces. In *Homotopy and geometry (Warsaw, 1997)*, volume 45 of *Banach Center Publ.*, pages 87–90. Polish Acad. Sci. Inst. Math., Warsaw, 1998.

[2] C. Allday. *Transformation Groups: Symplectic Torus Actions and Toric Manifolds.* Hindustan Book Agency, India, 2005.

[3] C. Allday and V. Puppe. Bounds on the torus rank. In *Transformation groups, Poznań 1985*, volume 1217 of *Lecture Notes in Math.*, pages 1–10. Springer, Berlin, 1986.

[4] C. Allday and V. Puppe. *Cohomological methods in transformation groups*, volume 32 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1993.

[5] M. Amann. Cohomological consequences of (almost) free torus actions. arXiv:1204.6276, 2012.

[6] L. Avramov and S. Halperin. Through the looking glass: a dictionary between rational homotopy theory and local algebra. In *Algebra, algebraic topology and their interactions (Stockholm, 1983)*, volume 1183 of *Lecture Notes in Math.*, pages 1–27. Springer, Berlin, 1986.

[7] G. Bazzoni, G. Lupton, and J. Oprea. Hereditary properties of co-Kähler manifolds. *Differential Geom. Appl.*, 50:126–139, 2017.

[8] A. Blanchard. Sur les variétés analytiques complexes. *Ann. Sci. Ecole Norm. Sup. (3)*, 73:157–202, 1956.

[9] R. Body and R. Douglas. Rational homotopy and unique factorization. *Pacific J. Math.*, 75(2):331–338, 1978.

[10] M. Boij and J. Söderberg. Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen-Macaulay case. *Algebra Number Theory*, 6(3):437–454, 2012.

[11] G. E. Bredon. *Introduction to compact transformation groups.* Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.

[12] J. D. Carlson and C.-K. Fok. Equivariant formality of isotropy actions. *J. Lond. Math. Soc. (2)*, 97(3):470–494, 2018.

[13] C. Deninger and W. Singhof. On the cohomology of nilpotent Lie algebras. *Bull. Soc. Math. France*, 116(1):3–14, 1988.

[14] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[15] D. Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[16] D. Eisenbud, G. Fløystad, and J. Weyman. The existence of equivariant pure free resolutions. *Ann. Inst. Fourier (Grenoble)*, 61(3):905–926, 2011.

[17] D. Eisenbud and F.-O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. *J. Amer. Math. Soc.*, 22(3):859–888, 2009.

[18] Y. Félix and S. Halperin. Formal spaces with finite-dimensional rational homotopy. *Trans. Amer. Math. Soc.*, 270(2):575–588, 1982.

[19] Y. Félix, S. Halperin, and J.-C. Thomas. Gorenstein spaces. *Adv. in Math.*, 71(1):92–112, 1988.

[20] Y. Félix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[21] Y. Félix, J. Oprea, and D. Tanré. *Algebraic models in geometry*, volume 17 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2008.

[22] M. Fernández and V. Muñoz. Formality of Donaldson submanifolds. *Math. Z.*, 250(1):149–175, 2005.

[23] B. L. Fine and G. Triantafillou. On the equivariant formality of Kähler manifolds with finite group action. *Canad. J. Math.*, 45(6):1200–1210, 1993.

[24] G. Fløystad. Boij-Söderberg theory: introduction and survey. In *Progress in commutative algebra 1*, pages 1–54. de Gruyter, Berlin, 2012.

[25] M. Franz and V. Puppe. Exact sequences for equivariantly formal spaces. *C. R. Math. Acad. Sci. Soc. R. Can.*, 33(1):1–10, 2011.

[26] O. Goertsches and P. Konstantis. Almost complex structures on connected sums of complex projective spaces. *Ann. K-Theory*, 4(1):139–149, 2019.

[27] O. Goertsches and S. Rollenske. Torsion in equivariant cohomology and Cohen-Macaulay G-actions. *Transform. Groups*, 16(4):1063–1080, 2011.

[28] O. Goertsches and D. Töben. Torus actions whose equivariant cohomology is Cohen-Macaulay. *J. Topol.*, 3(4):819–846, 2010.

[29] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.

[30] V. K. A. M. Gugenheim. On the chain-complex of a fibration. *Illinois J. Math.*, 16:398–414, 1972.

[31] S. Halperin. Rational homotopy and torus actions. In *Aspects of topology*, volume 93 of *London Math. Soc. Lecture Note Ser.*, pages 293–306. Cambridge Univ. Press, Cambridge, 1985.

[32] S. Halperin and J. Stasheff. Obstructions to homotopy equivalences. *Adv. in Math.*, 32(3):233–279, 1979.

[33] M. R. Hilali. *Action du tore T^n sur les espaces simplement connexes.* PhD thesis, Université catholique de Louvain, 1980.

[34] V. Hinich. Homological algebra of homotopy algebras. *Comm. Algebra*, 25(10):3291–3323, 1997.

[35] W.-Y. Hsiang. *Cohomology theory of topological transformation groups.* Springer-Verlag, New York-Heidelberg, 1975. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85.

[36] S. B. Iyengar and M. E. Walker. Examples of finite free complexes of small rank and small homology. *Acta Math.*, 221(1):143–158, 2018.

[37] B. Jessup and G. Lupton. Free torus actions and two-stage spaces. *Math. Proc. Cambridge Philos. Soc.*, 137(1):191–207, 2004.

[38] T. Kadeishvili. Cohomology C_∞-algebra and rational homotopy type. In *Algebraic topology—old and new*, volume 85 of *Banach Center Publ.*, pages 225–240. Polish Acad. Sci. Inst. Math., Warsaw, 2009.

[39] T. V. Kadeishvili. The algebraic structure in the homology of an $A(\infty)$-algebra. *Soobshch. Akad. Nauk Gruzin. SSR*, 108(2):249–252 (1983), 1982.

[40] T. V. Kadeishvili. *Struktura $A(\infty)$-algebry v kogomologiyakh i ratsional′nyĭ gomotopicheskiĭ tip*, volume 107 of *Tbilisis A. Razmadzis Saxelobis Matematikis Institutis Shromebi [Tbilisi A. Razmadze Mathematics Institute Works].* "Metsniereba", Tbilisi, 1993.

[41] T. V. Kadeisvili. On the theory of homology of fiber spaces. *Uspekhi Mat. Nauk*, 35(3(213)):183–188, 1980. International Topology Conference (Moscow State Univ., Moscow, 1979).

[42] B. Keller. Introduction to A-infinity algebras and modules. *Homology Homotopy Appl.*, 3(1):1–35, 2001.

[43] F. C. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*, volume 31 of *Mathematical Notes.* Princeton University Press, Princeton, NJ, 1984.

[44] Y. Kotani and T. Yamaguchi. A lower bound for the LS category of a formal elliptic space. *Math. J. Okayama Univ.*, 47:141–145, 2005.

[45] K. Lefèvre-Hasegawa. *Sur les A_∞-catégories*. PhD thesis, Université Paris 7, 2003.

[46] S. Lillywhite. Formality in an equivariant setting. *Trans. Amer. Math. Soc.*, 355(7):2771–2793, 2003.

[47] J.-L. Loday and B. Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.

[48] G. Lupton and J. Oprea. Cohomologically symplectic spaces: toral actions and the Gottlieb group. *Trans. Amer. Math. Soc.*, 347(1):261–288, 1995.

[49] M. Markl. A cohomology theory for $A(m)$-algebras and applications. *J. Pure Appl. Algebra*, 83(2):141–175, 1992.

[50] J. McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[51] S. A. Merkulov. Strong homotopy algebras of a Kähler manifold. *Internat. Math. Res. Notices*, (3):153–164, 1999.

[52] V. Muñoz. Torus rational fibrations. *J. Pure Appl. Algebra*, 140(3):251–259, 1999.

[53] V. V. Prasolov. *Elements of homology theory*, volume 81 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007.

[54] A. Prouté. A_∞-structures. Modèles minimaux de Baues-Lemaire et Kadeishvili et homologie des fibrations. *Repr. Theory Appl. Categ.*, (21):1–99, 2011. Reprint of the 1986 original, With a preface to the reprint by Jean-Louis Loday.

[55] A. Roig. Minimal resolutions and other minimal models. *Publ. Mat.*, 37(2):285–303, 1993.

[56] A. Roig. Formalizability of dg modules and morphisms of cdg algebras. *Illinois J. Math.*, 38(3):434–451, 1994.

[57] M. Schlessinger and J. Stasheff. The Lie algebra structure of tangent cohomology and deformation theory. *J. Pure Appl. Algebra*, 38(2-3):313–322, 1985.

[58] L. Scull. Equivariant formality for actions of torus groups. *Canad. J. Math.*, 56(6):1290–1307, 2004.

[59] L. Scull. Formality and S^1-equivariant algebraic models. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 463–471. Amer. Math. Soc., Providence, RI, 2004.

[60] Y. M. Ustinovskiĭ. On almost free torus actions and the Horrocks conjecture. *Dal′nevost. Mat. Zh.*, 12(1):98–107, 2012.

[61] M. E. Walker. Total Betti numbers of modules of finite projective dimension. *Ann. of Math. (2)*, 186(2):641–646, 2017.

[62] L. Zoller. New bounds on the toral rank with application to cohomologically symplectic spaces. *Transform. Groups*, to appear. https://doi.org/10.1007/s00031-019-09525-8.