arXiv:1906.01729v1 [quant-ph] 4 Jun 2019

Unruh Acceleration Radiation Revisited

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When ground-state atoms are accelerated and the field with which they interact is in its
normal vacuum state, the atoms detect Unruh radiation. We show that atoms falling
into a black hole emit acceleration radiation which, under appropriate initial conditions
(Boulware vacuum), has an energy spectrum which looks much like Hawking radiation.
This analysis also provides insight into the Einstein principle of equivalence between
acceleration and gravity. The Unruh temperature can also be obtained by using the
Kubo–Martin–Schwinger (KMS) periodicity of the two-point thermal correlation func-
tion, for a system undergoing uniform acceleration; as with much of the material in this
paper, this known result is obtained with a twist.

Ia. Introduction: Dedication

Julian Schwinger, that towering figure of 20th century physics, taught us how to
tame the infinities of quantum field theory and much more. For example, he and his
students taught us how to profitably apply the formalism of quantum field theory to
the problem of nonequilibrium quantum statistical mechanics;[1,2] yielding, among
other things, the famous KMS condition, which we use herein. Indeed, modern
quantum optics owes much to Schwinger's Green's function-correlation function
approach. In particular, we have found that the tools of quantum optics provide
another window into the problem of Unruh–Hawking radiation. It is therefore fitting
that we summarize and extend our work on acceleration radiation in this Schwinger
centennial collection.

Fig. 1.   Julian Schwinger rides a hay wagon at the New Mexico Scully ranch in 1987.

Ib. Introduction: Overview

The existence of black holes (BHs), regions of spacetime that nothing — not even light — can escape from, is one of predictions of Einstein's general relativity. Hawking's[3] demonstration that a non-rotating, uncharged BH of mass M emits thermal radiation at temperature[4]

$$T_{\mathrm{BH}} = \frac{\hbar c^3}{8\pi G M k_{\mathrm{B}}} \tag{1}$$

is mathematically based on quantum field theory in curved spacetime. This remarkable result is intriguing and beautiful but also a bit subtle and mysterious.

From a different point of view, our group of quantum optics and general relativity aficionados have teamed up to show[5,6] that atoms freely falling into a BH with the field in the Boulware vacuum (the state of the field in which no Hawking radiation is emitted by the black hole) emit radiation which has a thermal energy spectrum (but has phase correlations between the energy states making the emitted radiation a pure state rather than a thermal density matrix) which to a distant observer has aspects that look like (but also aspects that differ from) Hawking radiation. We call it Horizon Brightened Acceleration Radiation (HBAR).[5] It is produced solely by emission from the atom while outside the BH. This work was inspired by quantum optics in flat spacetime, which predicts that atoms moving with a uniform acceleration emit thermal radiation with Unruh[7] temperature. Although freely falling (having geodesic motion), the atoms seem to a distant observer to be accelerating in their fall into the black hole, and thus seem to that observer to be
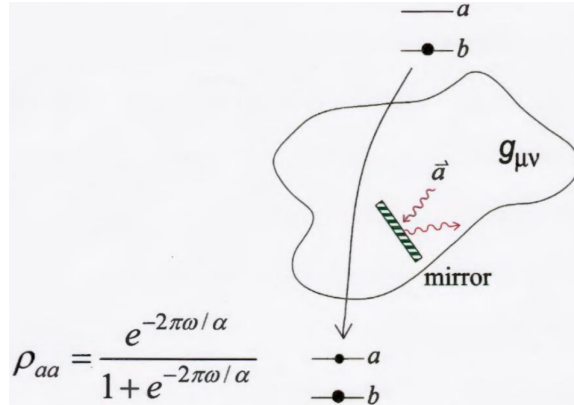
Fig. 2. A detector accelerating through a spacetime region in its field vacuum state detects Unruh radiation. This happens if in the frame relative-to-which the vacuum modes are defined, the atom is accelerating; whether or not the atom is actually accelerating. This could even happen if the atom is inertial, and the metric is flat, or if there are mirrors modifying the boundary conditions of the spacetime modes.

accelerated detectors in the Boulware vacuum (which for a distant observer is one with no particles).

However, rather than being excited as though in a thermal bath, they emit radiation whose energy spectrum as seen by the distant observer looks thermal with a temperature T_U proportional to their acceleration α,

$$T_U = \frac{\hbar \alpha}{2\pi c k_B} \ . \tag{2}$$

As is explained in the following section, this "acceleration radiation" arises from processes in which the atom jumps from the ground state to an excited state, together with the emission of a photon.[8,9] In quantum optics, such processes are usually discarded because they violate conservation of energy, and the virtual photons must be quickly reabsorbed in order to maintain the overall energy conservation. However, if the atom is accelerated away from the original point of virtual emission, there is a small probability that the virtual photon will "get away" before it is re-absorbed. Alternatively, the Doppler shift of the accelerated atom takes the otherwise re-absorbed photon out of the atom's bandwidth. Atom acceleration converts virtual photons into real ones at the expense of the energy supplied by the external force field driving the center-of-mass motion of the atom (in Unruh's original case, the acceleration results from an external force, while in our case, the seeming acceleration is due to gravity). In an alternate point of view, one can trace the excitation of the atom to a vacuum fluctuation, which in the usual case is canceled by a succeeding, correlated fluctuation. However, in the accelerated case, the velocity of the atom is different by the time that correlated fluctuation hits it, giving a Doppler shift which now means that the fluctuation has the wrong frequency for de-exciting the atom.

Near the event horizon, at radii close to $r_\mathrm{g} = 2MG/c^2$, the Schwarzschild metric is well-approximated by the constant-acceleration Rindler metric,[10] in which an atom would have a gravitational acceleration of $\alpha = c^2/2r_\mathrm{g}$ (even though to itself it has zero acceleration). The vacuum state through which it falls is one in which observers at rest in that frame see no particles. While in the usual Unruh effect, the atom is excited, in this case, the atom emits photons whose energy spectrum (as seen by distant stationary observers) appears to be thermal. As a result, the temperature, the HBAR temperature, can be obtained from the Unruh temperature by plugging $\alpha = c^2/2r_\mathrm{g}$ into Eq. (2) to find

$$T_\mathrm{HBAR} = \frac{\hbar}{2\pi c k_\mathrm{B}} \frac{c^2}{2r_\mathrm{g}} = T_\mathrm{BH} \,. \tag{3}$$

T_HBAR is equal to the temperature of Hawking radiation (1).

This radiation differs from Hawking radiation in that, although the probability of emission of the various possible energies is proportional to a thermal spectrum, the emission from any one atom is a pure state, with definite phase relations between the energies. Of course if one has many atoms with incoherent times of fall into the black hole, or if one took into account the recoil of the atom, some of that phase coherence could be destroyed, making the emission look closer to Hawking radiation.

However, the physics is very different from that of the Hawking effect. Here we have radiation coming from the atoms, whereas Hawking radiation requires no extra matter (e.g., atoms) and arises just from the BH geometry.

There are several features of this finding that some have found surprising. For example one objection could be that the atom is freely falling with proper acceleration of zero. Where then does the radiation come from? However this neglects that the state of the field is assumed to be the Boulware vacuum state in which the particle content near infinity is zero, but near the horizon is full of particles (the energy density actually diverges at the horizon). It is those particles that the atom is interacting with. And from far away, the atom looks as though it is accelerated as it falls into the black hole.

In the following section (Sec. II), we first follow a quantum optics path to Unruh radiation and compare it to the more usual treatment based on quantum fields in curved spacetime. In Sec. III, we use two scenarios where, surprisingly, acceleration radiation is emitted by inertial detectors, for discussing the equivalence principle of Einstein (in one case, we have a stationary atom interacting with a moving mirror, and in the other case, we have an atom freely-falling into a black hole). In Sec. IV, we discuss how Unruh radiation occurs because of the difference between mode definitions in different frames — a point of view in which it is not surprising that an inertial observer would detect acceleration radiation. In Sec. V, we present a KMS-inspired method for obtaining the Unruh temperature, an approach pioneered by Christensen and Duff.[11] There, we use the KMS periodicity approach to get the Unruh temperature from both a field and an atom perspective. We summarize in Sec. VI.
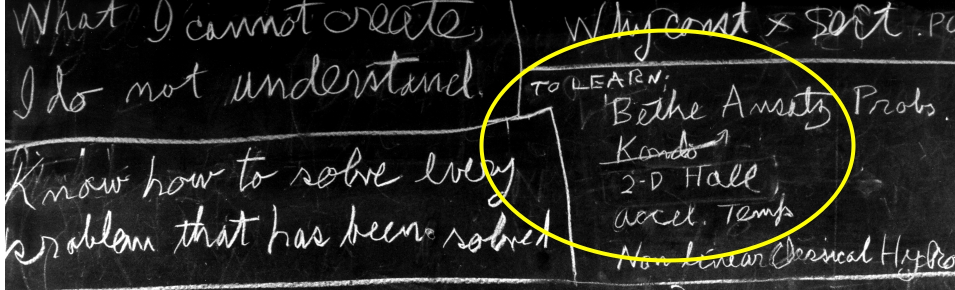
Fig. 3. Feynman's blackboard as he left it. On the bottom-right corner he inscribed "accel. Temp" under "TO LEARN:". This is a strong indication of the subtlety and interest in this problem (courtesy of the Archives, California Institute of Technology).

II. Quantum Optics Route to Obtaining Unruh Radiation in Minkowski Coordinates

In this section, we provide a simple first principles calculation of the radiation emitted by an accelerating atom. This calculation bears similarities to that of Unruh and Wald. [12] It answers, in part, the implied question of Feynman and Milonni, as in Fig. 3.

Milonni wrote:

> [A] uniformly accelerated detector [i.e., atom] in the vacuum responds as it would if it were at rest in a thermal bath at temperature $T = \hbar a/2\pi c k_{\mathrm{B}}$. *It is hardly obvious why this should be* [emphasis added] — it took half a century after the birth of the quantum theory of radiation for the thermal effect of uniform acceleration to be discovered.

IIa. Accelerating atom in a vacuum

We consider a two-level atom (a is the excited level and b is the ground state) with transition frequency ω moving along the z-axis in a $1 + 1$-dimensional spacetime with a uniform acceleration α. The atom trajectory is given by

$$ct(\bar{t}) = \ell \sinh\left(\frac{c\bar{t}}{\ell}\right), \quad z(\bar{t}) = \ell \cosh\left(\frac{c\bar{t}}{\ell}\right), \tag{4}$$

where t is the lab time and $\bar{t}$ is the proper time for the accelerated atom, [13] and where

$$\ell = c^2/\alpha \tag{5}$$

is the length-scale in the problem. The interaction Hamiltonian between the atom and an outward-propagating photon with wave number k reads

$$\hat{V}(\bar{t}) = \hbar g \left[\hat{a}_k e^{-i\nu t(\tau) + ikz(\tau)} + \text{H.a.}\right]\left(\hat{\sigma} e^{-i\omega\bar{t}} + \text{H.a.}\right), \tag{6}$$

where operator $\hat{a}_k$ is the photon annihilation operator, $\hat{\sigma}$ is the atomic lowering operator, and g is the atom-field coupling constant which depends on the atomic dipole moment and on the electric field in the frame of the atom.

Initially the atom is in the ground state and there are no photons. If the interaction is weak enough, the state vector of the atom-field system at the atomic proper time $\bar{t}$ can be found using first-order time-dependent perturbation theory,

$$|\psi(\bar{t})\rangle = |\psi(\tau_0)\rangle - \frac{i}{\hbar} \int_{\bar{t}_0}^{\tau} d\bar{t}' \, \hat{V}(\bar{t}') \, |\psi(\tau_0)\rangle . \tag{7}$$

The probability of excitation of the atom (frequency ω) with simultaneous emission of a photon with frequency ν is due to a counter-rotating term $\hat{a}_k^+ \hat{\sigma}^+$ in the interaction Hamiltonian.

The probability of this event is

$$P = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} d\bar{t}' \, \langle 1_k, a | \hat{V}(\bar{t}') | 0, b \rangle \right|^2 = g^2 \left| \int_{-\infty}^{\infty} d\bar{t}' \, e^{i\nu t(\bar{t}') - i k z(\tau')} e^{i\omega\tau'} \right|^2 , \tag{8}$$

where $|b\rangle$ and $|a\rangle$ are the ground and excited state of the atom respectively, and $t(\bar{t}')$ and $z(\tau')$ are obtained from Eqs. (4), and using that $k = \nu/c$ and changing the variable of integration to $x = \frac{\nu\ell}{c} e^{-c\bar{t}'/\ell}$, and taking into account that

$$\int_0^{\infty} dx \, e^{-ix} x^{-i\frac{\omega\ell}{c} - 1} = e^{-\frac{1}{2}\frac{\pi\omega\ell}{c}} \Gamma\left(-\frac{i\omega\ell}{c}\right) ,$$

where $\Gamma(x)$ is the gamma function, and the property $|\Gamma(-ix)|^2 = \pi/[x \sinh(\pi x)]$, we finally obtain that the probability is

$$P = \frac{2\pi c g^2}{\alpha\omega} \frac{1}{\exp\left(2\pi\frac{\omega\ell}{c}\right) - 1} . \tag{9}$$

We find that P is proportional to the Planck factor $1/\left[\exp\left(\frac{2\pi\omega c}{\alpha}\right) - 1\right]$ which is the probability that the atom is excited and a photon is emitted. The Planck factor corresponds to excitation probability with a temperature that is proportional to the acceleration α,

$$T_{\text{U}} = \frac{\hbar\alpha}{2\pi c k_{\text{B}}} .$$

This can be understood as was discussed in the previous section, as generating a photon by breaking adiabaticity due to the acceleration of the atom. Another physical picture involved the promotion of vacuum fluctuations. In any case, the operator product $\hat{\sigma}^{\dagger}(\bar{t})\hat{a}_k^{\dagger}(t, z)$ tells us that the (Minkowski) photon is emitted and the atom is excited.
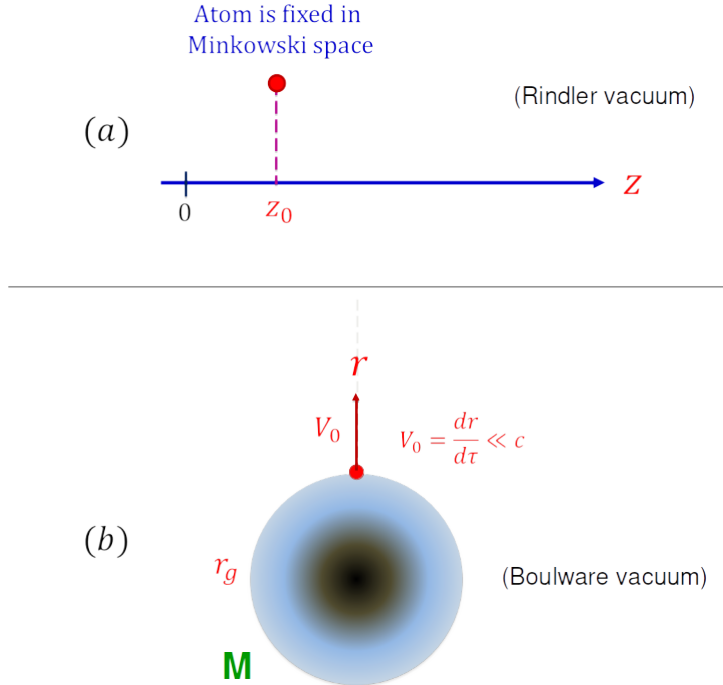
Fig. 4. (a) An atom is fixed in Minkowski spacetime at coordinate z_0 and the field is in the Rindler vacuum (created by an accelerated mirror). (b) An atom moves in the vicinity of the BH event horizon in the Boulware vacuum, emitting acceleration radiation. These two cases are equivalent to each other, given that the acceleration of the mirror is related to the BH mass by Eq. (60).

IIb. Excitation of a Static Atom by the Rindler Vacuum

Having seen that an atom accelerating through the Minkowski vacuum emits (Minkowski) photons, we consider the "inverse" problem of a stationary atom in an accelerating Rindler vacuum. To put this in perspective, Sec. IIa represents the Cavity QED problem of an atom passing through a stationary cavity. In this section (IIb), we are essentially dealing with an accelerating mirror [14] (with the state of the field being a Rindler-like vacuum) and stationary atom, as in Fig. 4b. This is the physics behind the present Rindler coordinate analysis.

We proceed by assuming that an atom is fixed in the inertial reference frame (t, z) at position $z = z_0$ (see Fig. 4a). We make a coordinate transformation into a uniformly accelerating reference frame,

$$ct = \ell e^{\bar{z}/\ell} \sinh\left(\frac{c\bar{t}}{\ell}\right) , \quad z = \ell e^{\bar{z}/\ell} \cosh\left(\frac{c\bar{t}}{\ell}\right) , \tag{10}$$

where ℓ is defined in the same way as in Eq. (4), which gives that the proper acceleration at $\bar{z} = 0$ is α. See Fig. 5.
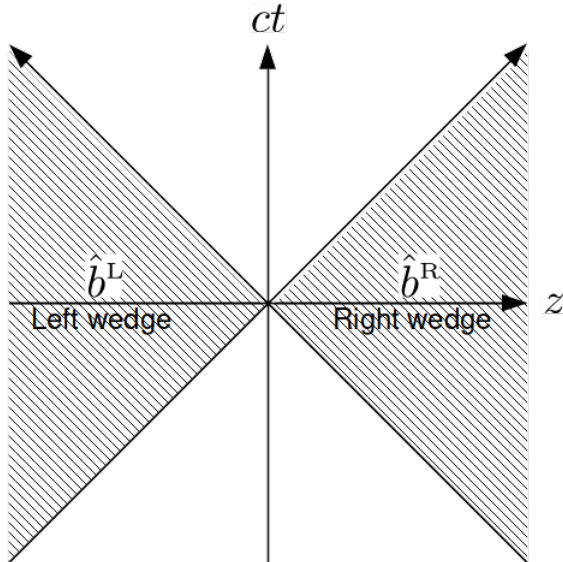
Fig. 5.   Minkowski space divided into four wedges. Of particular relevance are the right and left wedges, which are called "the right Rindler wedge" and "the left Rindler wedge," respectively.

The coordinate transformation (10) covers only the part of the Minkowski spacetime with $z > c|t|$ (right Rindler wedge). It converts the Minkowski spacetime line element $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}z^2$ to the Rindler line element,[10,15]

$$\mathrm{d}s^2 = e^{2a\bar{z}/c^2}\left(c^2\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{z}^2\right) . \tag{11}$$

An observer moving along the trajectory $\bar{z} = 0$ in the Rindler space is uniformly-accelerating in the Minkowski space along the trajectory (4), which is a special case ($\bar{z} = 0$) of Eq. (10). Normal modes of scalar photons in the conformal metric (11) take the same form as the usual positive frequency normal modes in the Minkowski metric, e.g., one can take them as traveling waves,

$$\phi_\nu(\bar{t}, \bar{z}) = \frac{1}{\sqrt{\nu}}e^{-i\nu\bar{t}+ik\bar{z}} , \tag{12}$$

where ν is the photon angular frequency in the reference frame of the Rindler space and $k = \pm\nu/c$. However, the modes (12) are a mixture of positive and negative frequency modes with respect to the physical Minkowski spacetime. Therefore, the vacuum state of these modes is not the Minkowski vacuum but rather the Rindler vacuum, which is what we assume for those modes.

From Eq. (10) we obtain $\bar{t}$ and $\bar{z}$ in terms of t and z,

$$c\bar{t}(t, z) = \frac{\ell}{2}\ln\left(\frac{z+ct}{z-ct}\right) , \quad \bar{z}(t, z) = \frac{\ell}{2}\ln\left[\frac{z^2-c^2t^2}{\ell^2}\right] . \tag{13}$$

The atomic trajectory is obtained from Eq. (13) by setting the Minkowski space

position to $z = z_0$. In the Rindler space, the atomic velocity is

$$\bar{V} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}} = -\frac{c^2 t}{z_0} . \tag{14}$$

From the perspective of the atom, it passes through the right Rindler wedge within the proper time interval

$$-\frac{z_0}{c} < t < \frac{z_0}{c}$$

for which the atom velocity in the Rindler space $\bar{V}$ changes from c to $-c$. During this time the atom interacts with the mode (12). The probability P that the static atom gets excited and a photon in the mode (12) is generated is given by the integral

$$P = g^2 \left| \int_{-\frac{z_0}{c}}^{\frac{z_0}{c}} \mathrm{d}t \, \phi_\nu^*(t, z_0) e^{i\omega t} \right|^2 , \tag{15}$$

where t is the proper time for the atom, and z is taken at the atomic position z_0. Using Eqs. (12) and (13), we obtain (assuming $k = \nu/c$)

$$P = g^2 \left| \int_{-\frac{z_0}{c}}^{\frac{z_0}{c}} \mathrm{d}t \, e^{-i\frac{\nu\ell}{c} \ln[(z_0 - ct)/\ell] + i\omega t} \right|^2 . \tag{16}$$

Changing the integration variable to $x = \omega(z_0/c - t)$, we have

$$P = \frac{g^2}{\omega^2} \left| \int_0^{\frac{2\omega z_0}{c}} \mathrm{d}x \, e^{ix} x^{i\frac{\nu\ell}{c}} \right|^2 . \tag{17}$$

Using that

$$\int_0^{\frac{2\omega z_0}{c}} \mathrm{d}x \, e^{ix} x^{i\frac{ic\ell}{c}} = e^{-\frac{\pi}{2}\frac{\nu\ell}{c}} \gamma \left(1 + i\frac{\nu\ell}{c}, -i\frac{2\omega z_0}{c} \right) ,$$

where $\gamma(s, x)$ is the incomplete lower gamma function which has the asymptotic behavior $\gamma(s, -ix) \to i\Gamma(s)$, as $x \to \infty$, we find that the probability in Eq. (17) is

$$P = \frac{g^2}{\omega^2} e^{-\pi\frac{\nu\ell}{c}} \left| \gamma \left(1 + i\frac{\nu\ell}{c}, -i\frac{2\omega z_0}{c} \right) \right|^2 . \tag{18}$$

In the limit $z_0 \gg c/\omega$ we have

$$\left| \gamma \left(1 + i\frac{\nu\ell}{c}, -i\frac{2\omega z_0}{c} \right) \right|^2 \approx \left| \Gamma \left(1 + i\frac{\nu\ell}{c} \right) \right|^2 = \left(\frac{\nu\ell}{c} \right)^2 \left| \Gamma \left(i\frac{\nu\ell}{c} \right) \right|^2 = \frac{\nu\ell}{c} \frac{\pi}{\sinh \left(\pi\frac{\nu\ell}{c} \right)} \tag{19}$$

which yields that the probability for exciting the atom along with emission of a ν-photon is

$$P \approx \frac{2\pi\nu\ell g^2}{c\omega^2} \frac{1}{\exp\left[2\pi\frac{\nu\ell}{c}\right] - 1} . \tag{20}$$

Notice that in our present case of a stationary atom in the Rindler vacuum the Planck factor is ν-dependent, whereas in the case of the accelerating atom, the Planck factor in the analogous Eq. (9) in Sec. II is ω-dependent. It is the emitted radiation by the stationary atom which is thermal, not the excitation of the atom.

The probability of photon absorption is obtained by changing $\nu \to -\nu$. Equation (20) yields

$$P_{\mathrm{abs}} = \exp\left[2\pi\frac{\nu\ell}{c}\right] P_{\mathrm{exc}} . \tag{21}$$

However, if we use the more accurate Eq. (18), we obtain

$$P_{\mathrm{abs}} = \exp\left[2\pi\frac{\nu\ell}{c}\right] \frac{\left|\gamma\left(1 + i\frac{\nu\ell}{c}, i\frac{2\omega z_0}{c}\right)\right|^2}{\left|\gamma\left(1 + i\frac{\nu\ell}{c}, -i\frac{2\omega z_0}{c}\right)\right|^2} P_{\mathrm{exc}} , \tag{22}$$

which is thermal only in the limit $z_0 \gg c/\omega$.

III. Acceleration radiation and the equivalence principle using Unruh–Minkowski modes

Let us approach the question of the relation between accelerated motion of either the mirror or the atom in an accelerated vacuum in a different way.

Consider the function

$$f(t) = \lim_{\lambda \to 0^+} \left(\frac{t \pm i\lambda}{\ell/c}\right)^{i\Omega} \tag{23}$$

where $\Omega = \nu'\ell/c$ is some dimensionless frequency, and $\lambda \to 0_+$. This $i\lambda$ prescription is to indicate the sector of the complex t plane in which we place the branch-cut of the function. I.e., in both cases, we take $\lambda \to 0^+$, but $-i\lambda$ indicates that the branch cut is in the upper-half complex t-plane, while $+i\lambda$ would indicate that is in the lower-half complex t-plane. See Fig. 6.

To determine the frequency content of $f(t)$ in Eq. (23), we consider the integral

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} \left(\frac{t - i\lambda}{\ell/c}\right)^{i\Omega} . \tag{24}$$

If $\omega < 0$, the integral can be completed in the lower-half complex t-plane, giving $F(\omega) = 0$ for all values of Ω. Thus, the Fourier transform of $f(t)$ is non-zero only for positive ω, i.e., it is a purely positive-frequency function.
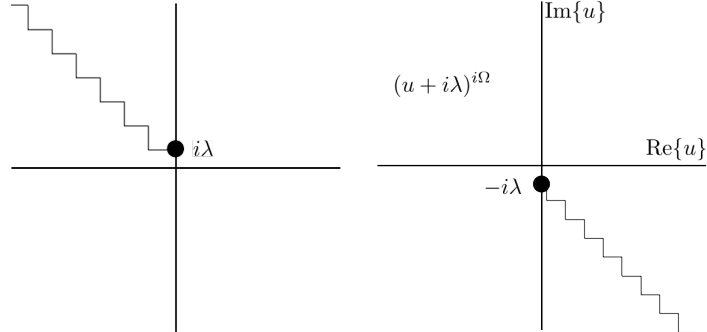
Fig. 6.   The definition of the branch cut in Eq. (23) for positive λ.

Similarly, $[c(t + i\lambda)/\ell]^{i\Omega}$ is a purely negative-frequency function for all values of Ω. These functions thus form a complete set of functions over t under the Klein–Gordon inner-product

$$\langle f, g \rangle = -\frac{i}{2} \int \mathrm{d}t \left(f \frac{\partial}{\partial t} g^* - g^* \frac{\partial}{\partial t} f \right). \tag{25}$$

We will use a complete set of modes similar to Eq. (23) to examine two different situations: the motion of a mirror by a stationary atom, and the motion of a two-level atom (or detector) in the presence of a mirror, both interacting with a massless scalar field $\hat{\Phi}$. We will work in 1+1 dimensional spacetime. These will be special cases of systems which some of us have examined previously.[5,6,16]

In the first case, we will have a mirror at rest in the ordinary Minkowski vacuum state, i.e., the state in which one would ordinarily say there are no particle excitations of the scalar field. The atom, however, is accelerated with constant acceleration α, following the trajectory in Eq. (4). We specialize to the case where the atom's closest approach to the mirror is given by the distance ℓ, defined in Eq. (5). See Fig. 7. In the second case, we swap the behavior of the atom and the mirror, and choose a different initial state for the field. The mirror will follow a trajectory of constant acceleration, Eq. (4), while the atom will be at rest. Again, the distance of closest approach of the mirror to the atom would be ℓ. In this case, we will take the state of the quantum field to be the so-called Rindler vacuum. This is the state in which the accelerated mirror sees the quantum field as containing no particles. See Fig. 8. It is in some sense an approximate weak equivalence principle[*] analog

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[*]This is of course only a crude approximation for the weak equivalence principle, since when one is in a bumper car that decelerates rapidly when it hits another one against a rail that prevents it from accelerating, one will feel different from when one is in a bumper car against a rail that does not accelerate when another hits it, even though the relative acceleration is the same in the two cases. See Fig. 9, where it is seen that the full equivalence principle is between an accelerating mirror (B) and an atom freely-falling into a black hole (C). There, we find that the spectra are equivalent. However, while the spectra of the accelerating atom (A) and the accelerating mirror (B) are strikingly similar they are different, and therefore, the two cases are not equivalent.

of the first case.[17] In both cases, the mirror sees no photons, and the mirror and atom have the same relative accelerations with respect to each other.

In each case, we look at the interaction between the atom and the field only to lowest order in the coupling constant.

Let us look at the flat spacetime examples first and take the coordinates (t, z) to be the usual Minkowski coordinates such that the metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}z^2 \,. \tag{26}$$

Let us define dimensionless null coordinates u and v,

$$u = (ct - z)/\ell \,, \quad v = (ct + z)/\ell \,. \tag{27}$$

The equation of motion for a massless scalar field is (the massless Klein–Gordon equation)

$$\frac{1}{c^2} \partial_t^2 \phi - \partial_z^2 \phi = 0 \,, \tag{28}$$

which, in terms of the null coordinates u and v in Eq. (27) is

$$\partial_u \partial_v \phi = 0 \,, \tag{29}$$

where we use the notation $\partial_\xi \equiv \frac{\partial}{\partial \xi}$.

The plane-wave modes of the field, which are commonly used for expanding solutions of Eqs. (28) or (29), are

$$\phi_{\omega+} = \frac{1}{\sqrt{4\pi |\omega|}} e^{-i\omega(ct-z)/c} = \frac{1}{\sqrt{4\pi |\omega|}} e^{i\omega\ell/c \; u} \,, \tag{30}$$

$$\phi_{\omega-} = \frac{1}{\sqrt{4\pi |\omega|}} e^{-i\omega(ct+z)/c} = \frac{1}{\sqrt{4\pi |\omega|}} e^{i\omega\ell/c \; v} \,, \tag{31}$$

where $\omega\pm$ correspond to right- and left-moving solutions, respectively.

In terms of the Klein–Gordon norm for the fields, Eq. (25), the modes with $\omega > 0$ have a positive value for the norm, while those for $\omega < 0$ have a negative norm. We however, use a different complete set of modes, Eq. (32) below, which are similar to Eq. (23), for expanding solutions of Eq. (29).

Instead of the solutions (30) and (31), we elect to use a complete set of modes for the field by

$$\hat{\phi}_{\Omega+} = \frac{e^{-\pi\Omega/2}}{\sqrt{8\pi\Omega \sinh(\pi\Omega)}} \lim_{\lambda \to 0^+} (u - i\lambda)^{i\Omega} \,, \tag{32}$$

where we normalized Eq. (23) and use a different variable, u. These are a complete set of positive norm (often called the positive frequency Unruh–Minkowski modes,[7,12]) even though Ω takes all values positive and negative. The negative-norm modes are just the complex-conjugate of these (due to the sign of $i\lambda$, or ultimately, the definition of the branch-cut).

IIIa. Accelerating atom

We are now going to place a mirror at position $z = 0$. We will take the boundary conditions on the solutions ϕ that they be zero at the mirror. The solutions of Eq. (29) then are of the form

$$\phi(u, v) = g(u) - g(v),\tag{33}$$

for some function g. Since at $z = 0$, the null coordinates are both $u = v = ct/\ell$, then we see that $\phi(z = 0, t) = g(t) - g(t) = 0$, whish satisfies the boundary conditions. Using Eq. (33) and the modes (32), we have that the modes satisfying the boundary conditions are

$$\phi_\Omega(u, v) = \frac{e^{-\pi\Omega/2}}{\sqrt{4\Omega\sinh(\pi\Omega)}} \lim_{\lambda \to 0^+} \left[(u - i\lambda)^{i\Omega} - (v - i\lambda)^{i\Omega} \right].\tag{34}$$

For the two-level atom, let us define the two states $|b\rangle$ as the ground state of the atom and $|a\rangle$ as the excited state, with proper energy ω, and the atomic raising operator $\hat{\sigma}^\dagger$, which takes $\sigma^\dagger |b\rangle = |a\rangle$, having time dependence $e^{i\omega\bar{t}}$ in the interaction picture, where $\bar{t}$ is the proper time of the atom.

We can write the quantum field $\hat{\Phi}$ in terms of the null coordinates u and v

$$\hat{\Phi}(u, v) = \int d\Omega \left(\hat{a}_\Omega \phi_\Omega(u, v) + \hat{a}_\Omega^\dagger \phi_\Omega^*(u, v) \right).\tag{35}$$

In terms of the null coordinates (27), the path of the particle (4) is

$$u(\bar{t}) = -e^{-c\bar{t}/\ell}, \quad v(\bar{t}) = e^{c\bar{t}/\ell}.\tag{36}$$

The interaction between the atom and the field will be taken to be

$$\hat{H}_{\mathrm{I}} = g \left(\hat{\sigma} e^{-i\omega\bar{t}} + \hat{\sigma}^\dagger e^{i\omega\bar{t}} \right) w^\mu \partial_\mu \hat{\Phi},\tag{37}$$

where w^μ is the four velocity of the atom, and $\bar{t}$ is the proper time along the path of the detector. In the frame of the atom, it is stationary, thus we have

$$w^\mu \partial_\mu \hat{\Phi} = \partial_{\bar{t}} \hat{\Phi},\tag{38}$$

where the derivative is evaluated along the path of the the atom. This interaction is chosen because it makes the field $\hat{\Phi}$ an ohmic-coupled bath for the detector, in the nomenclature of Caldera and Leggett.[18] See Fig. 7.

Since the atom begins in its ground state, and the quantum field in the Minkowski vacuum state, in the atom-field interaction, the only term that contributes to the probability amplitude that the atoms becomes excited is the "counter-rotating" term, in the language of quantum optics. I.e., we need terms that look like $\hat{\sigma}^\dagger \hat{a}^\dagger$. If the atom is not accelerated, such counter-rotating terms will give zero when integrated over time. However, using the above definition of the field, and the fact that the time-dependence of the atomic raising operator $\hat{\sigma}^\dagger$ is
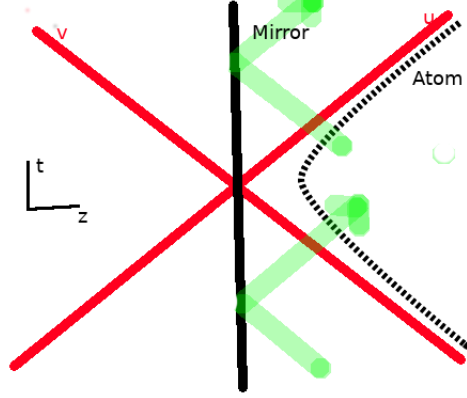
Fig. 7. An atom accelerates in the presence of a stationary mirror. Initially, the atom is in its ground state $|b\rangle$, and the field is in the Minkowski vacuum $|0_\mathrm{M}\rangle$. There is some amplitude $|\mathcal{A}_\mathrm{ex}\rangle$ for the atom to become excited. We note that since the mirror destroys the Lorentz invariance (in contrast to the true, Lorentz-invariant vacuum case in Sec. V), the state seen by the accelerated atom is not a static thermal bath, but this aspect does not matter for our conclusions.

$e^{i\omega\bar{t}}$, we get an excitation amplitude of

$$|\mathcal{A}_\mathrm{ex}\rangle = \langle a|\, g \int \mathrm{d}\bar{t} \left[\hat{\sigma}^\dagger e^{i\hat{\omega}\bar{t}} + \hat{\sigma} e^{-i\hat{\omega}\bar{t}}\right] \partial_{\bar{t}} \hat{\Phi}\left(-e^{-c\bar{t}/\ell}, e^{c\bar{t}/\ell}\right) |b, 0_\mathrm{M}\rangle \tag{39}$$

$$= g \int_{-T}^{T} \mathrm{d}\bar{t}\, e^{i\omega\bar{t}}$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\Omega\, \frac{i\Omega c}{\ell} \left[(-e^{-c\bar{t}/\ell})^{-i\Omega} + (e^{c\bar{t}/\ell})^{-i\Omega}\right] \frac{e^{-\pi\Omega/2}}{\sqrt{8\pi\Omega\sinh(\pi\Omega)}} \hat{a}_\Omega^\dagger |0_\mathrm{R}\rangle\,, \tag{40}$$

since $\partial_{\bar{t}}\phi_\Omega^*(u, v)$ is

$$\partial_{\bar{t}}\phi_\Omega^*(u, v) = \frac{i\Omega c}{\ell} \frac{e^{-\pi\Omega/2}}{\sqrt{8\pi\Omega\sinh(\pi\Omega)}} \lim_{\lambda\to 0^+} \left[(u + i\lambda)^{-i\Omega} + (v + i\lambda)^{-i\Omega}\right]\,, \tag{41}$$

where we used Eq. (34). I.e., the first-order excitation is due to the $\hat{\sigma}^\dagger \hat{a}^\dagger$ term, a product of the counter-rotating terms in the quantum optics nomenclature. If $\Omega > 0$, then the second term in the square brackets will be zero after integration over $\bar{t}$, while if $\Omega < 0$, it is the first term that will be zero. Now $(-x+i\lambda)^{i\Omega} = x^{i\Omega} e^{-\pi\Omega}$ for positive x since one must take the contour around the upper $-x$ complex values so that $(-1)^{i\Omega} = (e^{+i\pi})^{i\Omega} = e^{-\pi\Omega}$. See Fig. 6. The integral in Eq. (40) thus becomes (in the limit that $T \to \infty$)

$$|\mathcal{A}_\mathrm{ex}\rangle \approx 2T \frac{e^{-\pi\omega/2a}}{\sqrt{8\pi\omega\sinh(\pi\omega/a)/a}} \left(\hat{a}_{\omega/a}^\dagger - \hat{a}_{-\omega/a}^\dagger\right) |0_\mathrm{M}\rangle\,, \tag{42}$$

where $\hat{a}^\dagger_\omega$ and $\hat{a}^\dagger_{-\omega}$ are the creation operators for the right- and left-moving modes, Eqs. (30) and (31), respectively. The probability of atomic excitation is

$$P_{\text{ex}} = \langle \mathcal{A}_{\text{ex}} | \mathcal{A}_{\text{ex}} \rangle \,, \tag{43}$$

which is proportional to the thermal factor $1/(e^{2\pi\omega/a} - 1)$.

We note that this is interesting in that there is really no horizon hiding the partner particles from the quantum field from the detector. There is entanglement between the incoming field in the right Rindler wedge and that behind the incoming horizon. But the latter gets reflected out by the mirror. Thus the entanglement in the Minkowski vacuum occurs between the ingoing modes in the right Rindler wedge and the outgoing modes in that same wedge, instead of being hidden behind the horizon.

We can ask whether or not the system is truly thermal by comparing the probability of emission of radiation by an excited accelerated atom with the absorption of the counter-rotating term by the unexcited atom.

IIIb. Accelerating mirror

In the second case, we consider an accelerated mirror, with a stationary detector whose surface is at $uv = -1$, and the field initially in the Rindler vacuum (as defined by Fulling[19]). With the mirror accelerated, the field is expanded in terms of the positive-norm Rindler modes,[10]

$$\bar{\phi}_{\Omega++} = \frac{1}{\sqrt{4\pi\Omega}} \begin{cases} u^{-i\Omega}, & u > 0 \\ 0, & u < 0 \end{cases} \qquad \bar{\phi}_{\Omega+-} = \frac{1}{\sqrt{4\pi\Omega}} \begin{cases} 0, & u > 0 \\ (-u)^{i\Omega}, & u < 0 \end{cases}$$

$$\bar{\phi}_{\Omega-+} = \frac{1}{\sqrt{4\pi\Omega}} \begin{cases} v^{-i\Omega}, & v > 0 \\ 0, & u < 0 \end{cases} \qquad \bar{\phi}_{\Omega--} = \frac{1}{\sqrt{4\pi\Omega}} \begin{cases} 0, & u > 0 \\ (-v)^{i\Omega}, & v < 0 \end{cases} \tag{44}$$

with positive Ω.

Because of the mirror, this spacetime features the following modes, which are superposition of the basic positive-frequency modes, (44). See Fig. 8. We have the positive-norm "1-modes," which are left-moving modes in the negative v region (and are zero elsewhere),

$$\bar{\phi}_{\Omega 1} = \frac{1}{\sqrt{4\pi\Omega}} (-v)^{i\Omega} \,; \qquad v < 0 \,, \tag{45}$$

and we have the positive-norm "3-modes," which are right-moving modes in the positive u region (and zero elsewhere),

$$\bar{\phi}_{\Omega 3} = \frac{1}{\sqrt{4\pi\Omega}} (u)^{-i\Omega} \,; \qquad u > 0 \,. \tag{46}$$

In these regions, $v < 0$ and $u > 0$, there is no mirror. We have the positive-norm "2-modes," which interact with the mirror. The region of the spacetime with negative u and positive v contains the mirror, which lies on the surface $uv = -1$.
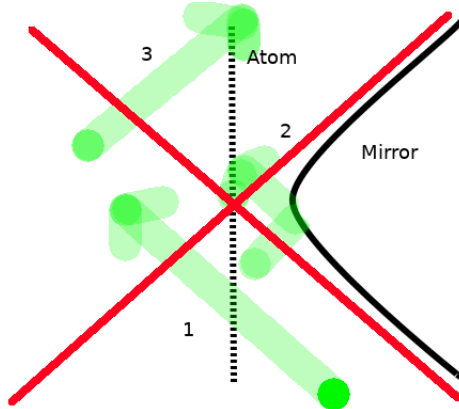
Fig. 8. A stationary atom with a moving mirror. While usually the spatial left- and right-moving modes are independent, in this scenario, we have three families of modes which interact with the atom. The first (labeled '2'), consists of right- and left-moving components, with relative phase (-1) between them, so that they vanish at the mirror. Also in this spacetime are left-moving modes and right-moving modes (labeled '1' and '3', respectively) which do not interact with the mirror. Those have a random phase relationship to one another. The mirror follows a trajectory of constant acceleration, and the atom is at rest. The three cases depicted are (1) the positive-norm "1-modes" that originate before the past right null asymptote of the mirror trajectory and travel to the left; (2) the positive-norm "2-modes" that originate from the left before the past extension of the future null asymptote for the mirror, bounce off the mirror, and continue traveling to the left after the past null asymptote of the mirror; (3) the positive-norm "3-modes" that originate from the left after the extension of the future null asymptote of the mirror and travel to the right. The field is in the Rindler vacuum state, which means that each of the three types of modes above (and the "4-modes" (not depicted) that are to the right of the mirror and do not interact with the atom) are independent of (unentangled and uncorrelated with) any of the other modes and have no particles detectable by accelerated observers in either Rindler wedge. The atom is at rest (moving along an inertial static world line) at distance ℓ from the closest approach of the mirror.

These modes are a superposition of the positive-norm left- and right-moving Rindler modes, Eq. (44), which vanish at the mirror. They are

$$\bar{\phi}_{\Omega 2}(u,v) = \frac{1}{\sqrt{4\pi\Omega}} \left[(u)^{i\Omega} - (-v)^{-i\Omega} \right] , \tag{47}$$

and zero elsewhere. These are a bit subtly-defined, because the right-moving piece is defined for $u < 0$, but the left-moving part is in $v > 0$, see Fig. 8. We also have the "4-modes" (not shown in the figure). These are confined to the region $u < 0$ and $v > 0$ (the right wedge) and vanish at the mirror, but do not interact with the atom, so we ignore them.

In terms of the positive-norm mode families which interact with the atom, Eqs. (45), (46), and (47), the field $\hat{\Phi}$ is

$$\hat{\Phi}(u,v) = \sum_{i=1}^{3} \left(\int_0^\infty d\Omega \, \hat{b}_{\Omega i} \phi_{\Omega i} + \text{H.a.} \right) , \tag{48}$$

where the summation over i is to include all three mode types. The atom travels

along the path $u = v = ct/\ell$. The state of the field, the Rindler vacuum, $|0_R\rangle$, is defined by $\hat{b}_{\Omega i} |0_R\rangle = 0$ for all values of Ω, and the only terms in the amplitude which survive if the detector is initially in its ground state $|b\rangle$ are

$$|A_{\text{ex}}\rangle = \int_{-T}^{T} \mathrm{d}t \int_0^\infty \mathrm{d}\Omega \; e^{i\omega t} \partial_t \phi_{\Omega i}^*(t,t) \hat{b}_{\Omega i}^\dagger |0_R\rangle \;, \tag{49}$$

where $2T$ is the interaction time, and we used Eq. (48) for the field $\hat{\Phi}$ and Eq. (37) for the interaction Hamiltonian.

To calculate (49) for infinite interaction time T, we first compute $W_{\Omega\pm}$, where

$$W_{\Omega\pm} = \pm \int_0^{\pm\infty} \mathrm{d}t \; e^{i\omega t} \partial_t \left(\pm \frac{ct}{\ell} \right)^{\pm i\Omega} . \tag{50}$$

To compute $W_{\Omega+}$, we rotate the contour of integration from the real t-axis to the imaginary t-axis, with $t_I = \mathrm{Im}\{t\}$ and where the branch-cut is not in the first quadrant of the complex t-plane.

$$W_{\Omega+} = \int_0^\infty \mathrm{d}t_I \; e^{-\omega t_I} \partial_{t_I} \left(\frac{ct_I}{\ell} \right)^{i\Omega} e^{-\pi\Omega/2} . \tag{51}$$

Changing integration variables from t_I to $x = \omega t_I$ we get

$$W_\Omega = \left(\frac{c}{\omega\ell} \right)^{i\Omega} e^{-\pi\Omega/2} \int_0^\infty \mathrm{d}x \, e^{-x} \partial_x x^{i\Omega} = (i\Omega) \left(\frac{c}{\omega\ell} \right)^{i\Omega} e^{-\pi\Omega/2} \, \Gamma(i\Omega)$$
$$= i \frac{\sqrt{\pi\Omega} \, e^{-\pi\Omega/2}}{\sqrt{\sinh(\pi\Omega)}} e^{i\varphi(\Omega)} \left(\frac{c}{\omega\ell} \right)^{i\Omega} , \tag{52}$$

where $\varphi(\Omega)$ is the slowly-varying phase of the complex argument gamma function $\Gamma(i\Omega)$, which starts at $-\pi/2$ for $\Omega = 0$ and reaches 0 only once $\Omega \approx 3$, by which time $e^{-\pi\Omega} \sqrt{\frac{\Omega}{2\sinh(\pi\Omega)}}$ will have dropped by a factor of about 10^3. I.e., the phase of $\Gamma(i\Omega)$ is essentially constant over the range in which the $\Gamma(i\Omega)$ is non-zero.

Similarly, one can rotate the contour in Eq. (50) the other way and evaluate

$$-W_{\Omega-} = \int_\infty^0 \mathrm{d}t \, e^{i\omega t} \partial_t \left(i\frac{ct}{\ell} \right)^{i\Omega} = \left(\frac{c}{\omega\ell} \right)^{i\Omega} \int_\infty^0 \mathrm{d}x \, e^{-\pi\Omega/2} e^{-x} \partial_x x^{i\Omega} . \tag{53}$$

We thus find that $W_\Omega = W_{\Omega+} = (W_{\Omega-})^*$, and therefore, the excitation amplitude per Ω is

$$|\mathcal{A}_\Omega\rangle = \frac{g}{\sqrt{4\pi\Omega}} \left[\hat{b}_{\Omega 1}^\dagger W_\Omega^* + \hat{b}_{\Omega 3}^\dagger W_\Omega + \hat{b}_{\Omega 2}^\dagger \left(W_\Omega - W_\Omega^* \right) \right] , \tag{54}$$

where, using (49), the full amplitude is

$$|A_{\text{ex}}\rangle = \int \mathrm{d}\Omega \, |A_\Omega\rangle . \tag{55}$$

Integrating the amplitude $|A_\Omega\rangle$ in Eq. (54) over Ω gives some constant which is independent of the frequency of the atom, and certainly not thermal. However, the

probability of emitting a mode with frequency Ω is proportional to a thermal factor

$$P_\Omega = |\langle 0_\mathrm{R}| \, \hat{b}_{\Omega i} \, |\mathcal{A}_\Omega\rangle|^2 \propto \frac{1}{1 - \exp[2\pi\Omega]} \tag{56}$$

which was also found in Ref. 6.

Thus, an accelerated atom above a stationary mirror with the field in the Minkowski vacuum (no particles detected by the mirror as striking the stationary mirror) is excited with a probability proportional to the thermal factor, while an accelerated mirror above a stationary atom, with the field in the Rindler vacuum (i.e., no particles detected by the mirror as striking the mirror) emits Rindler modes with a probability proportional to the thermal factor. We must distinguish this statement from stating that the atom emits particles into a thermal state. The atom emits modes with correlations between the modes, given by the phase factor $i(a/\omega)^{i\Omega}e^{\varphi(\Omega)}$, as in Eq. (52). I.e., what an unaccelerated atom below the accelerated mirror emits is a pure state, not a thermal state (a mixed state); albeit, the probability distribution over Rindler energies is proportional to a thermal factor. There is thus some crude approximate form of the equivalence principle in play here.

Hawking showed that a black hole emits thermal radiation. While an observer at infinity sees the black hole as in some sense stationary, a static observer or atom near the horizon is accelerated with constant acceleration. The Hartle–Hawking state of the field near the black hole looks like a thermal state to such a static observer, but looks much more like a vacuum state to a freely-falling observer. We can again look at two cases, the one analyzed by Hawking, in which the atom is accelerated and near the horizon, while the state is the vacuum state as far as the horizon is concerned (although it is a state in thermal equilibrium with a temperature inversely-proportional to the mass for an observer far away). The second case is where the atom is in free fall into the horizon, while the state of the field is the so-called Boulware vacuum (the analog of the Rindler vacuum in the curved spacetime of the Schwarzschild metric of a non-rotating black hole), where a distant observer sees nothing coming out of the black hole.

IV. Acceleration Radiation and the Equivalence Principle

In this section, we discuss acceleration radiation from atoms which do not accelerate, and show the approximate equivalence between atoms freely-falling into a Schwarzschild black hole and stationary atoms (in Minkowski space) in the presence of an accelerating mirror. See Fig. 9.

When Einstein first formulated the equivalence principle he was mainly concerned with the laws of classical physics. Ginzburg and Frolov in their review paper [20] mentioned that: "The question of whether or not the equivalence principle holds for the description of phenomena for which their quantum nature is important is by no means trivial."
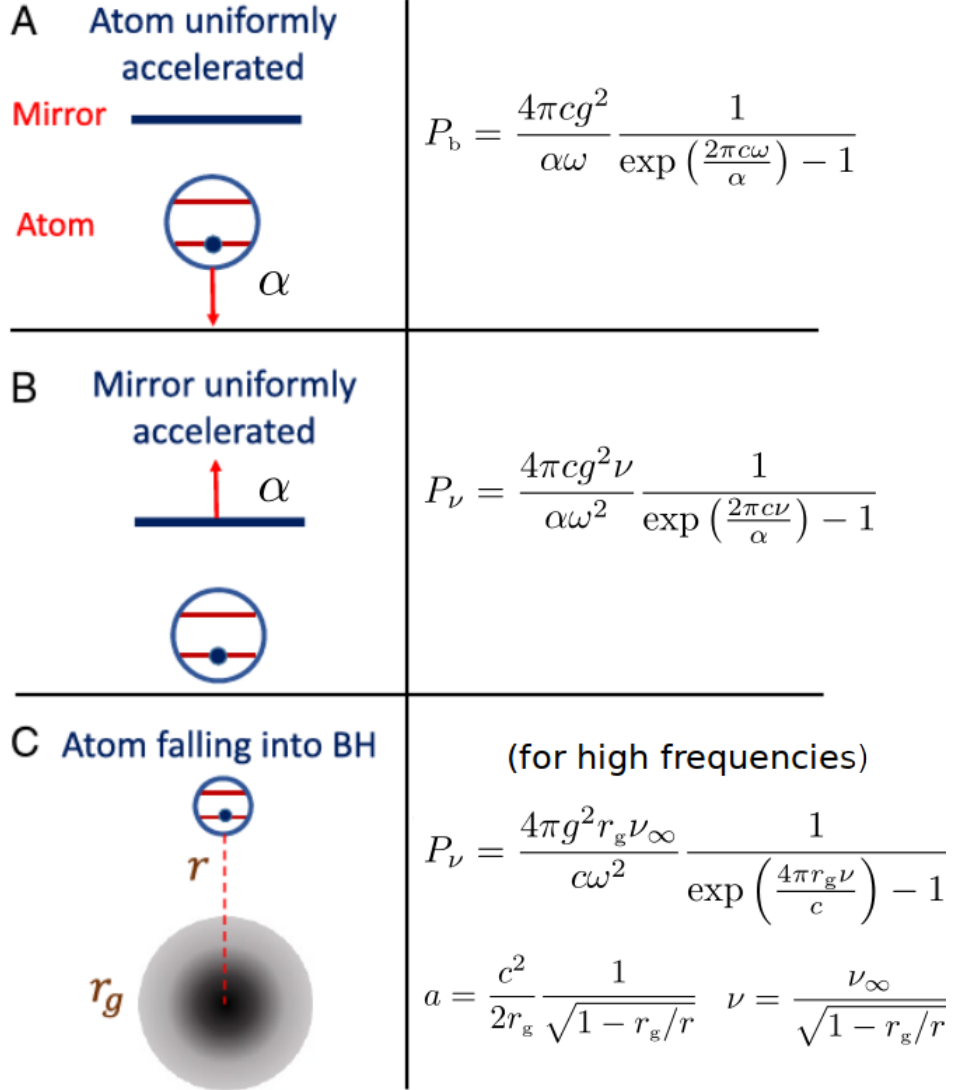
Fig. 9. The three physical cases which we consider: (A) An atom uniformally accelerated in Minkowski space in the presence of a stationary mirror with the Minkowski vacuum. (B) A stationary atom in Minkowski space in the presence of an accelerating mirror and the Rindler vacuum. (C) An atom in free-fall in the Schwarzschild metric in the Boulware vacuum. In all three sub-figures, we indicate the probability of atomic excitation (atomic frequency ω) in the first case or with an excitation probability at high frequency for the electromagnetic field mode with frequency ν. Cases (B) and (C) are similar because in both cases, the atom is freely-falling, but still emits radiation. Case A is a physically-different case, because the *atom* has non-zero proper acceleration, and it is the atom that is thermally-excited, giving physically-different results.
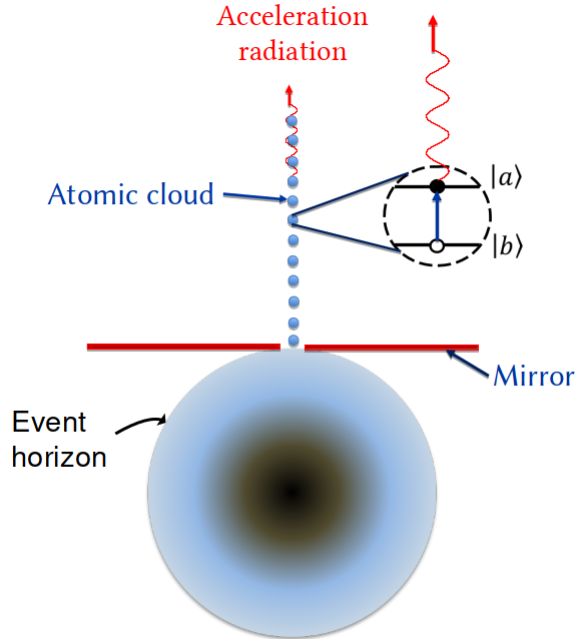
Fig. 10. Atoms in the ground state $|b\rangle$ are freely-falling in a Schwarzschild black hole metric, where the state of the field is the Boulware vacuum. The atoms, though inertial and moving along geodesics, emit acceleration radiation. When the atoms are released at random times from infinity, the outgoing field is thermal.

Here we discuss acceleration radiation of an atom freely-falling in the gravitational field of a static BH. The equivalence principle tells us that the atom essentially falls "force-free" into the BH, that is, the atom's acceleration is equal to zero. How then could it emit something which looks like acceleration radiation? To answer this question we consider modes of the field in the reference frame of the black hole. In the Schwarzschild metric the field modes are stationary, even though they are modified by the gravitational field of the BH. However, in the reference frame of the freely falling atom the field modes are changing with time.

The equivalence principle is manifested as a symmetry between emission by a static atom in Minkowski spacetime in the Rindler vacuum (discussed in the previous section), and an atom freely falling in a gravitational field of a BH in the Boulware vacuum. Moreover, there is an analogy between the Rindler horizon and the BH event horizon. Indeed, the time-radius part of Schwarzschild metric interval,

$$\mathrm{d}s^2 = \frac{r - r_\mathrm{g}}{r}c^2\mathrm{d}t^2 - \frac{r}{r - r_\mathrm{g}}\mathrm{d}r^2 \,, \tag{57}$$

which could be approximated near the event-horizon $r = r_\mathrm{g}$ by

$$\mathrm{d}s^2 \simeq \frac{r - r_\mathrm{g}}{r_\mathrm{g}}c^2\mathrm{d}t^2 - \frac{r_\mathrm{g}}{r - r_\mathrm{g}}\mathrm{d}\bar{r}^2 \,, \tag{58}$$

and using the coordinate $\bar{r}$ such that $r - r_g = e^{\bar{r}/r_g}$ to describe space-time events outside the event horizon, the time-radius Schwarzschild interval becomes

$$\mathrm{d}s^2 \simeq e^{\bar{r}/r_g} \left(c^2 \mathrm{d}t^2 - \mathrm{d}\bar{r}^2 \right) , \tag{59}$$

which is the interval of the Rindler space metric, Eq. (11). Comparing with the interval of Rindler space, Eq. (11), we obtain an effective acceleration corresponding to a free fall near the event horizon

$$\frac{\mathrm{d}^2 \bar{r}}{\mathrm{d}s^2} = \alpha = \frac{1}{2r_g} . \tag{60}$$

Next we consider an atom launched radially from the event horizon with an initial radial velocity $V_0 = c\,\mathrm{d}r/\mathrm{d}s$ (see Fig. 10b). Using the Schwarzschild metric in Eq. (57), the equations of atomic radial motion are

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s} \right)^2 = \frac{V_0^2}{c^2} + \frac{r_g}{r} - 1 ,$$
$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{V_0}{c^2 \left(1 - \frac{r_g}{r} \right)} . \tag{61}$$

For $V_0 \ll c$ we find the following solution

$$\frac{r}{r_g} = 1 + \frac{V_0^2}{c^2} - \frac{s^2}{4r_g^2} ,$$
$$t = \frac{r_g}{c} \ln \left(\frac{2r_g V_0 + cs}{2r_g V_0 - cs} \right) . \tag{62}$$

In terms of the coordinate $\bar{r}$, the atomic trajectory is

$$\bar{r} = r_g \ln \left[\frac{1}{4r_g^2} \left(\frac{4r_g^2 V_0^2}{c^2} - s^2 \right) \right] . \tag{63}$$

The trajectory of the atom near the BH event horizon, given by Eqs. (62) and (63), has the same form as the trajectory of the atom fixed in Minkowski spacetime at

$$z_0 = 2\frac{V_0}{c} r_g \tag{64}$$

viewed in the Rindler coordinates (10) when relating the acceleration in the Rindler case to the effective acceleration near the BH, Eq. (60). Since near the event horizon the Schwarzschild metric (57) can be approximated as the Rindler metric (59), the probability of atomic excitation and photon emission for an atom falling into a Schwarzschild black hole is given by the same expressions, (18) and (20), only where α and z_0 are replaced with the corresponding values, (60) and (64), respectively.

V. The "Bogoliubov" Path to Unruh Radiation

In this section, we present yet another interpretation of Unruh radiation. It could be understood as a difference of perspective between two observers. For simplicity, we consider a real scalar field $\hat{\Phi}(z,t)$ to represent the photons. This field is an operator, which can be expanded in different basis sets. Let us consider two observers — a stationary and an accelerating observer (in Minkowski space) — which naturally have two basis sets to describe the modes of the field. The stationary observer has the line element $\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - \mathrm{d}z^2$, while the accelerating observer's line element is $\mathrm{d}s^2 = e^{-2\bar{z}/\ell}(c^2\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{z}^2)$, which is obtained from the stationary observer's line element by transforming to "accelerating" coordinates, Eq. (10). The normal modes ϕ_ω in each coordinate systems are different, satisfying the wave equation

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g}g^{\mu\nu}\partial_\nu \phi_\omega = 0\,, \tag{65}$$

where $g^{\mu\nu}$ is the metric, which could be read-off from the expression for the line element, and g is its determinant. Using Eq. (65), with the metrics corresponding to Minkowski (stationary) and Rindler (11) (accelerating) observers, the normal modes for the stationary and accelerated observer both satisfy $[(\partial_0)^2 - (\partial_1)^2]\phi = 0$, albeit in different coordinate systems. So in both cases the normal modes are complex exponentials, but in terms of different coordinates. The stationary observer's modes ϕ_ν, evaluated at some spacetime event, $(\bar{z}, \bar{t})$ specified in Rindler coordinates, are

$$\phi_\nu(\bar{z},\bar{t}) = \frac{1}{\sqrt{\nu}}e^{-i\nu/c\left(z(\bar{z},\bar{t})-ct(\bar{z},\bar{t})\right)} = \frac{1}{\sqrt{\nu}}e^{-i\nu\ell/c\,\exp[(c\bar{t}-\bar{z})/\ell]} \tag{66}$$

and for the accelerating observer

$$\overline{\phi}_\nu(\bar{z},\bar{t}) = \frac{1}{\sqrt{\nu}}e^{-i\nu/c(\bar{z}-c\bar{t})}\,. \tag{67}$$

So in the right Rindler wedge, the two observers describe the field as

$$\hat{\Phi}(\bar{z},\bar{t}) = \sum_\nu \left(\phi_\nu(\bar{z},\bar{t})\hat{a}_\nu + \phi_\nu^*(\bar{z},\bar{t})\hat{a}_\nu^\dagger\right)$$
$$= \sum_{\bar{\nu}} \left(\bar{\phi}_{\bar{\nu}}(\bar{z},\bar{t})\hat{b}_{\bar{\nu}} + \bar{\phi}_{\bar{\nu}}^*(\bar{z},\bar{t})\hat{b}_{\bar{\nu}}^\dagger\right)\,. \tag{68}$$

Using the orthogonality of the modes, $\langle\phi_\nu(z,t),\phi_{\bar{\nu}}(z,t)\rangle = \delta_{\nu,\bar{\nu}}$, where the inner-product is given by Eq. (25), we see that we could obtain $\hat{a}_\nu$'s in terms of the $\hat{b}_{\bar{\nu}}$'s,

$$\hat{a}_\nu = \left\langle\phi_\nu(z,t),\hat{\Phi}(z,t)\right\rangle = \sum_{\bar{\nu}}\left(\alpha_{\nu\bar{\nu}}\hat{b}_{\bar{\nu}} + \beta_{\nu\bar{\nu}}\hat{b}_{\bar{\nu}}^\dagger\right)\,, \tag{69}$$

where $\alpha_{\nu\bar{\nu}} = \langle\phi_\nu, \bar{\phi}_{\bar{\nu}}\rangle$, and $\beta_{\nu\bar{\nu}} = \langle\phi_\nu, \bar{\phi}_{\bar{\nu}}^*\rangle$. Alternatively, one can obtain the $\hat{b}_{\bar{\nu}}$'s in terms of the $\hat{a}_\nu$'s,

$$\hat{b}_{\bar{\nu}} = \left\langle\bar{\phi}_{\bar{\nu}},\hat{\Phi}\right\rangle = \sum_n \left(\alpha_{\nu\bar{\nu}}^*\hat{a}_\nu - \beta_{\nu\bar{\nu}}^*\hat{a}_\nu^\dagger\right)\,, \tag{70}$$

where we have used the properties of the inner-product (25),

$$\langle f, g \rangle = \langle g, f \rangle^* = -\langle g^*, f^* \rangle = -\langle f^*, g^* \rangle^* . \tag{71}$$

Particles in the vacuum

We can use Eq. (69) to make calculations, for instance, the number of S particles in the $\bar{S}$ vacuum is

$$\langle \hat{n} \rangle = \langle 0_{\bar{S}} | \hat{a}_\nu^\dagger \hat{a}_\nu | 0_{\bar{S}} \rangle = \sum_{\bar{\nu}} |\beta_{\nu\bar{\nu}}|^2 \tag{72}$$

and using Eq. (70), we find that the number of $\bar{S}$ particles in the S vacuum is

$$\langle \hat{\bar{n}} \rangle = \left\langle 0_S \left| \hat{b}_{\bar{\nu}}^\dagger \hat{b}_{\bar{\nu}} \right| 0_S \right\rangle = \sum_\nu |\beta_{\nu\bar{\nu}}|^2 . \tag{73}$$

An interesting symmetry is that in both cases, the number of particles in the other frame's vacuum is given by a summation of $|\beta_{\nu\bar{\nu}}|^2$; albeit, the two quantities involve summations over different indices. If we use the Unruh–Minkowski modes for the modes ϕ_ν,

$$\phi_\nu(u) = \frac{e^{-\pi\nu\ell/2c}}{\sqrt{\sinh(\pi\nu\ell/c)}} \lim_{\lambda \to 0^+} \left(\frac{u - i\lambda}{\ell} \right)^{i\nu\ell/c} , \tag{74}$$

whose annihilation operator corresponds to a superposition of plane wave $e^{i\nu'(ct-z)}/\sqrt{\nu'}$ annihilation operators $\hat{\alpha}_{\nu'}$,

$$\hat{a}_\nu = \Gamma\left(1 + \frac{\nu\ell}{c}\right) \int \frac{\mathrm{d}\nu'}{i\nu'} \left[\frac{(i\nu')^{i\nu\ell/c}}{e^{\pi\nu\ell/c}} - (-i\nu')^{i\nu\ell/c} \right] \hat{\alpha}_{\nu'} . \tag{75}$$

$\alpha_{\nu\bar{\nu}}$ and $\beta_{\nu\bar{\nu}}$ are

$$\alpha_{\nu\bar{\nu}} = \frac{e^{-\frac{\pi}{2}\nu\ell/c}}{\sqrt{2\sinh\left(\pi\nu\ell/c\right)}} \delta_{\nu\bar{\nu}} , \quad \beta_{\nu\bar{\nu}} = \frac{e^{-\frac{\pi}{2}\nu\ell/c}}{\sqrt{2\sinh\left(\pi\nu\ell/c\right)}} \delta_{\nu\bar{\nu}} , \tag{76}$$

we find that the number of Rindler photons in the Minkowski vacuum state, and the number of Unruh–Minkowski photons in the Rindler vacuum state are both

$$\langle \hat{n} \rangle = \langle \hat{\bar{n}} \rangle = \frac{1}{2} \frac{1}{\exp\left(2\pi\nu\ell/c\right) - 1} , \tag{77}$$

which is the Planck factor corresponding to the temperature of $T_{\mathrm{U}} = \hbar a / 2\pi c k_{\mathrm{B}}$.

An accelerating observer in Minkowski vacuum

Notice that the Minkowski-space mode ϕ in Eq. (66) is only defined in the right Rindler wedge, see Fig. 5. However, the extension to the rest of Minkowski space (into the left Rindler wedge) is unique if we demand that it correspond to an annihilation operator, and that it not have any creation operator "components" (for all

values of the frequency parameter ν). To correspond to an annihilation operator, it must have positive-norm, and demanding that its norm be positive for all ν, we find that it is

$$\phi_\nu^{\mathrm{R}} = \frac{1}{\sqrt{2\sinh(\pi\nu\ell/c)}} \begin{cases} e^{-\pi\nu\ell/2c}\bar{\phi}_\nu^{*\mathrm{L}}\,, & \text{left wedge,} \\ e^{\pi\nu\ell/2c}\bar{\phi}_\nu^{\mathrm{R}}\,, & \text{right wedge.} \end{cases} \tag{78}$$

There is another family of Minkowski modes, ϕ_ν^{L}, which is concentrated mostly in the left Rindler wedge,

$$\phi_\nu^{\mathrm{L}} = \frac{1}{\sqrt{2\sinh(\pi\nu\ell/c)}} \begin{cases} e^{\pi\nu\ell/2c}\bar{\phi}_\nu^{\mathrm{L}}\,, & \text{left wedge,} \\ e^{-\pi\nu\ell/2c}\bar{\phi}_\nu^{*\mathrm{R}}\,, & \text{right wedge.} \end{cases} \tag{79}$$

Consider a two-level atom with constant acceleration in the right Rindler wedge, with trajectory given by Eq. (4). In its frame, the atom interacts with the mode $\bar{\phi}_\nu^{\mathrm{R}}$ in the right Rindler wedge, Eq. (10), which corresponds to the annihilation operator $\hat{b}_\nu^{\mathrm{R}}$. Thus, the time evolution of the state of the field–atom system is given by the time-evolution operator $\hat{U}$ (first-order time-dependent perturbation theory)

$$\hat{U} \simeq \hat{1} + \frac{1}{i\hbar}\int_0^{\tau'} \mathrm{d}\tau\, \hat{\sigma}^\dagger e^{i\omega\tau}\hat{b}_\nu^{\mathrm{R}} e^{-i\nu\tau}\,, \tag{80}$$

which means that the atomic excitation process is accompanied by the annihilation of a a right Rindler wedge photon.

For the Minkowski observer, however, the mode which the atom interacts with is zero in the left wedge, and he describes the annihilation operator $\hat{b}_\nu^{\mathrm{R}}$ using Eqs. (78) as

$$\hat{b}_\nu^{\mathrm{R}} = \frac{1}{\sqrt{2\sinh(\pi\nu\ell/c)}} \left(e^{\pi\nu\ell/2c}\hat{a}_\nu^{\mathrm{R}} + e^{-\pi\nu\ell/2c}\hat{a}_\nu^{\dagger\mathrm{L}} \right)\,. \tag{81}$$

Thus, since $\hat{a}_\nu^{\mathrm{R}}\,|0_{\mathrm{M}}\rangle = 0$, the time-evolution operator, operating on the initial Minkowski vacuum state, is

$$\hat{U} \simeq \hat{1} + \frac{1}{i\hbar}\frac{1}{\sqrt{2\sinh(\pi\nu\ell/c)}}e^{-\pi\nu\ell/2c}\int_0^{\tau'} \mathrm{d}\tau\, \sigma^\dagger e^{i\omega\tau}\hat{a}_\nu^{\dagger\mathrm{L}} e^{-i\nu\tau}\,. \tag{82}$$

VI. Periodicity Trick for Unruh Temperature

Now we will give a "trick" for deriving the Unruh temperature. The trick is to argue that, in the Rindler metric, the time coordinate must be periodic in the imaginary direction and this imaginary periodicity implies that Rindler spacetime has a temperature. The original derivation of the Unruh temperature using periodicity in imaginary time may be found in a paper by one of us,[11,21,22] following a similar derivation of the Hawking temperature.[23]

Quantum field theory at finite temperature is periodic in imaginary time, with periodicity

$$t \to t + i\hbar\beta\,, \tag{83}$$

where $\beta = 1/k_{\mathrm{B}}T$. One way to see this is by looking at the thermal average, which possesses the property

$$\left\langle \hat{Q}(t)\hat{Q}(t') \right\rangle = \left\langle \hat{Q}(t')\hat{Q}(t + i\hbar\beta) \right\rangle . \tag{84}$$

Indeed, using the equation for the time evolution of the $\hat{Q}$ operator and the invariance of the trace under cyclic permutation, we obtain

$$\begin{aligned}
\left\langle \hat{Q}(t)\hat{Q}(t') \right\rangle &= \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}} e^{i\hat{H}t/\hbar}\hat{Q}(0)e^{-i\hat{H}t/\hbar}\hat{Q}(t') \right) \\
&= \frac{1}{Z}\mathrm{Tr}\left(e^{i\frac{\hat{H}}{\hbar}(t+i\hbar\beta)}\hat{Q}(0)e^{-i\frac{\hat{H}}{\hbar}(t+i\hbar\beta)}e^{-\beta\hat{H}}\hat{Q}(t') \right) \\
&= \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{Q}(t')\hat{Q}(t + i\hbar\beta) \right) .
\end{aligned} \tag{85}$$

Equation (84) is commonly referred to as the Kubo–Martin–Schwinger (KMS) condition. Since the ordering of the field operators on the two sides are interchanged, the corresponding periodicity along the imaginary time direction is referred to as "periodicity with a twist."

Now let us assume that state of the field is the Minkowski vacuum $|0_{\mathrm{M}}\rangle$. That is, in the inertial reference frame the temperature is equal to zero. Then the zero temperature average over this state can be written as

$$G(t, z; t', z') = \langle 0_{\mathrm{M}}| \hat{\Phi}(t, z)\hat{\Phi}(t', z') |0_{\mathrm{M}}\rangle , \tag{86}$$

where $\hat{\Phi}(t, z)$ is the field operator at the spacetime event (z, t).

Since the vacuum is Lorentz-invariant, the two-point function (86) must depend only on the Lorentz-invariant spacetime interval $c^2(t - t')^2 - (z - z')^2$. If we make a coordinate transformation into the Rindler spacetime using Eq. (10) to express the interval in terms of the Rindler coordinates, the average (86) depends on

$$\begin{aligned}
c^2(t - t')^2 - (z - z')^2 = \ell^2 &\left[\left(e^{\bar{z}/\ell}\sinh\left(\frac{c\bar{t}}{\ell}\right) - e^{\bar{z}'/\ell}\sinh\left(\frac{c\bar{t}'}{\ell}\right) \right)^2 \right. \\
&\left. - \left(e^{\bar{z}/\ell}\cosh\left(\frac{c\bar{t}}{\ell}\right) - e^{\bar{z}'/\ell}\cosh\left(\frac{c\bar{t}'}{\ell}\right) \right)^2 \right] .
\end{aligned} \tag{87}$$

Hence, because of the periodicity of hyperbolic sine and cosine functions under the addition of the imaginary increment $2\pi i$ to their argument, we have

$$\sinh\left(\frac{c}{\ell}\bar{t}\right) = \sinh\left[\frac{1}{\ell}\left(\bar{t} + \frac{2\pi\ell}{c}i \right) \right] = \sinh(c\bar{t}/\ell)\cos(2\pi) ,$$

$$\cosh\left(\frac{c}{\ell}\bar{t}\right) = \cosh\left[\frac{1}{\ell}\left(\bar{t} + \frac{2\pi\ell}{c}i \right) \right] = \cosh(c\bar{t}/\ell)\cos(2\pi) , \tag{88}$$

and we conclude that in the Rindler spacetime the two-point function $G(\bar{t}, \bar{z}; \bar{t}', \bar{z}')$ obeys the KMS condition, namely

$$G(\bar{t}, \bar{z}; \bar{t}', \bar{z}') = G\left(\bar{t}', \bar{z}'; \bar{t} + \frac{2\pi\ell}{c}i, \bar{z} \right) . \tag{89}$$

Comparing this with Eq. (84), we see that $2\pi\ell/c = 2\pi c/\alpha = \hbar/k_{\rm B}T_{\rm U}$, which yields the Unruh temperature

$$T_{\rm U} = \frac{\hbar\alpha}{2\pi c k_{\rm B}} \; . \tag{90}$$

In other words, when viewed from a uniformly accelerating frame (i.e., the Rindler frame), the two-point function computed in the Minkowski vacuum appears to satisfy the KMS condition (84). Therefore, one may conclude that with respect to the Rindler observer, the Minkowski vacuum looks like a thermal reservoir of temperature $T_{\rm U}$.

VII. Conclusions

We revisit Unruh Radiation and arrive at the effect by different means. Using a quantum-optics route, we treat both the accelerating atom and accelerating mirror cases, which we also treat using the Unruh–Minkowski modes. The case of an atom freely-falling into a black hole is also discussed, and we discuss its relation to Einstein's Equivalence Principle. Then, we show how the effects could be obtained from Bogoliubov transformations, and finally, we show the relation to the KMS condition, of which Schwinger is among the namesakes.

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