Topics in the arithmetic of hypersurfaces and K3 surfaces

a thesis presented for the degree of
Doctor of Philosophy of Imperial College London
and the
Diploma of Imperial College
by

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September 2019

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

This thesis is a collection of various results related to the arithmetic of K3 surfaces and hypersurfaces which were obtained by the author during the course of his PhD studies.

The first part is related to Artin's conjecture on hypersurfaces over p-adic fields and solves the following question using tools from logarithmic geometry: Let $f : X \to Y$ be a proper, dominant morphism of smooth varieties over a number field k. When is it true that for almost all places v of k, the fibre X_P over any point $P \in Y(k_v)$ contains a zero-cycle of degree 1?

The second part proves new cases of Mazur's conjecture on the topology of rational points. Let E be an elliptic curve over $\mathbb{Q}$ with j-invariant 1728. For a class of elliptic pencils which are quadratic twists of E by quartic polynomials, the rational points on the projective line with positive rank fibres are dense in the real topology. This extends results obtained by Rohrlich and Kuwata-Wang for quadratic and cubic polynomials. We also give a proof of Mazur's conjecture for the Kummer surface associated to the product of two elliptic curves without any restrictions on the j-invariants.

The third and largest part presents a cohomological framework for determining the full Brauer group of a variety over a number field with torsion-free geometric Picard group. It investigates the middle cohomology of weighted diagonal hypersurfaces and implements the framework in the case of degree 2 K3 surfaces over $\mathbb{Q}$ which are double covers of the projective plane ramified in a diagonal sextic curve.

To my family.

ACKNOWLEDGMENTS

A PhD degree is not accurately described as the achievement of a single individual but of a whole community.

Foremost, I would like to thank my advisor Alexei Skorobogatov for accompanying me on this journey with expertise, enthusiasm, encouragement, commitment and the occasionally necessary care like a real “Doktorvater”.

Second, I would like to thank all my friends and colleagues at the London School of Geometry and Number Theory, Imperial, UCL and King’s for the comradeship, many questions answered, seminars, Friday pubs, parties, brunches, lunches and trips. Jonny Evans and Toby Gee provided crucial direction to me during the transition to a PhD student and I am grateful for this. Nicky Townsend has been an excellent manager and first point of contact for any administrative queries.

The results presented here have benefited from feedback by mathematicians apart from those already named, namely J.-L. Colliot-Thélène, M. Kuwata, Y. Liang, R. Newton, P. Satgé, M. Ulas, O. Wittenberg and Y. Zarhin.

This work was supported by the Engineering and Physical Sciences Research Council [EP/ L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London; I am indebted to their generous funding.

Finally, I would like to thank my partner and family who went with me through the highs and lows of a pure mathematics PhD. It wasn’t easy throughout but I hope that like me, you look back and say it was worth the struggle.

Declaration: The results in Parts I and II (apart from Appendix A and new proofs in Theorems 7.1 and 7.6) are a reproduction of previously appeared preprints by the author [Gvi19c, Gvi19a] and have been submitted for publication. Part II has been published in print [Gvi19d]. The results of Part III are new but Chapter 10 has overlaps with the joint publication [GS19].

CONTENTS

Part I

Arithmetic Surjectivity for Zero-Cycles

1

INTRODUCTION

In [LSS19], Loughran-Skorobogatov-Smeets develop, building upon work of Denef [Den16], a necessary and sufficient criterion to say when a morphism of varieties over a number field k is surjective on k_v-points for almost all finite places v. This property is called *arithmetic surjectivity* by Colliot-Thélène [CT11, §13]. More precisely, Loughran et. al. define a variety X to be *pseudo-split* if every Galois automorphism over the ground field fixes some geometric component of X of multiplicity 1. They then prove:

Theorem 1.1. *[LSS19, Theorem 1.4] Let $f : X \to Y$ be a dominant morphism between proper, smooth, geometrically integral varieties over a number field k with geometrically integral generic fibre.*

Then f is arithmetically surjective if and only if for each modification $f' : X' \to Y'$ of f and for each codimension 1 point ϑ' in Y', the fibre $f'^{-1}(\vartheta')$ is pseudo-split.

By a *modification* of f, we mean a morphism $f' : X' \to Y'$ of proper, smooth, geometrically integral varieties over k such that there exist proper, birational morphisms $\alpha_X : X' \to X$ and $\alpha_Y : Y' \to Y$ with $f' \circ \alpha_X = \alpha_Y \circ f$.

In this part of the thesis, we closely follow and extend the methods from

[LSS19] to deal with the analogous question for zero-cycles. We introduce the notion of *combinatorial cycle-splitness* and prove:

Theorem 1.2. *Let $f : X \to Y$ be a dominant morphism between proper, smooth, geometrically integral varieties over a number field k with geometrically integral generic fibre.*

The following statements are equivalent:

- (i) *For almost all places v, f has a v-adic zero-cycle of degree 1 in all fibres over k_v-points.*
- (ii) *For each modification $f' : X' \to Y'$ and for each codimension 1 point ϑ' in Y', the fibre $f'^{-1}(\vartheta')$ is combinatorially cycle-split.*

A situation where Theorem 1.2 applies but not Theorem 1.1 is given at the end in Example 4.14.

Note that we do not naively ask for surjectivity on zero-cycles but only for zero-cycles that are each entirely contained in a fibre. This has three reasons. First, if we allowed for zero-cycles whose summands lie in several distinct fibres, the question would not be fibre-wise anymore and our tools would not suffice to provide an answer for $\dim Y > 1$. Secondly, the naive version is not very well-behaved even in dimensions 0 and 1, which we can handle, where it already leads to rather complicated criteria.

Thirdly, it can be argued that the problem as posed above arises more naturally, for example when considering Artin's conjecture on p-adic forms in its variant for zero-cycles of degree 1.

Conjecture 1.3 (e.g. [KK86, Problem 3]). *If p is an arbitrary prime and if n and d are positive integers such that $n \geq d^2$, then a degree d hypersurface in $\mathbb{P}^n_{\mathbb{Q}_p}$ has a zero-cycle of degree 1.*

In other words, this open conjecture posits that the famous Ax-Kochen theorem, a special application of Theorem 1.1, holds without the need to exclude any primes when restated for zero-cycles of degree 1. In moduli terms, this asks for fibre-wise p-adic zero-cycles of degree 1 in the universal family of such hypersurfaces for every prime p.

1.1 NOTATION AND CONVENTIONS

By a variety, we mean a separated scheme of finite type over a field K. We denote by $\overline{X}$ the base change of a variety X along an algebraic closure $\overline{K}$ of K. For a field $K' \supset K$, we write $X_{K'}$ for $X \times_K K'$. If k is a number field and S a finite set of finite places in k, we write $\mathcal{O}_k$ for the ring of integers of k and $\mathcal{O}_{k,S}$ for the S-integers of k. Furthermore, for a finite place v of k, k_v shall denote the completion at v with ring of integers $\mathcal{O}_{k_v}$ and residue field $k(v)$ of size $N(v)$.

By a model of a variety X over k (respectively k_v), we mean a scheme $\mathcal{X}$ which is flat and of finite type over $\mathcal{O}_{k,S}$ for some finite set of places S (respectively $\mathcal{O}_{k_v}$) together with an isomorphism of its generic fibre to X. If X is proper, $\mathcal{X}$ a fixed model of X and $x \in X$ is a closed point, we write $\widetilde{x}$ for the closure of x in $\mathcal{X}$. By a model of a morphism of varieties $f : X \to Y$ over k (respectively k_v), we mean a morphism $f : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{O}_{k,S}$ (respectively $\mathcal{O}_{k_v}$) such that $\mathcal{X}$ and $\mathcal{Y}$ are models of X and Y compatible with f in the obvious way.

1.2 PRELIMINARY DEFINITIONS

To start, we introduce some terminology related to zero-cycles and our question.

Definition 1.4. A variety over a field K is *r-cycle-split*, if it contains a zero-cycle of degree r which is the sum of smooth points.

A variety over a number field k is *locally r-cycle-split* outside a finite set of places S, if for all finite places $v \notin S$ of k, the base change X_{k_v} is r-cycle-split.

Definition 1.5. A morphism of varieties over a field K is *r-cycle-surjective*, if the fibre over any rational point contains a zero-cycle of degree r.

A morphism of varieties over a number field k is *arithmetically r-cycle-surjective* outside a finite set of places S, if for all finite places $v \notin S$ of k, the base change $f \times_k k_v$ is r-cycle-surjective.

In the case $r = 1$, we propose the easier terminology *cycle-split*, *locally cycle-split*, *cycle-surjective* and *arithmetically cycle-surjective.* Although Theorem 1.2 is only concerned with arithmetic cycle-surjectivity, dealing with the case of general r does not add further complications. The terms are chosen in relation to [LSS19].

It turns out to be important to bound the degree of points appearing in zero-cycles.

Definition 1.6. For a zero-cycle $Z = \sum n_i x_i$ on a variety over a field K, define the *maximum degree* of Z

$$\operatorname{maxdeg} Z = \max[K(x_i) : K],$$

where $K(x_i)$ is the residue field of the point x_i.

We will make crucial use of a uniform version of the Lang-Weil estimates [LW54].

Lemma 1.7. *There exists a function $\overline{\Phi} : \mathbb{N}^3 \to \mathbb{N}$ with the following property. Let $U \subset \mathbb{P}^\nu$ be a geometrically irreducible, quasi-projective variety over a finite field, $\overline{U}$ its closure in $\mathbb{P}^\nu$ and $\partial U = \overline{U} \setminus U$.*

Then there exists a zero-cycle Z of degree 1 on U with

$$\operatorname{maxdeg} Z \leq \overline{\Phi}(N, \deg \overline{U}, \deg \partial U).$$

If X is proper and $\iota : X \dashrightarrow \mathbb{P}^\nu$ a rational embedding defined on an open $U \subset X$, then we will write $\overline{\Phi}(\iota)$ for $\overline{\Phi}(N, \deg \overline{\iota(U)}, \deg \partial(\iota(U)))$.

2

COMBINATORIAL CYCLE-SPLITNESS

We define the notion of combinatorial cycle-splitness, first for algebras and then for varieties.

2.1 IN DIMENSION 0

Let X be a finite étale scheme over a field K. It can be written as $X = \mathrm{Spec}(A)$ for some finite K-algebra $A = \oplus_{i=1}^{n} K_i$ (where K_i/K are finite field extensions but not necessarily normal). Let the Galois extension L/K be the compositum of the Galois closures of the K_i and denote $G := \mathrm{Gal}(L/K)$.

Let $H_i := \mathrm{Gal}(L/K_i)$, i.e. $L^{H_i} = K_i$. We note that X has a global zero-cycle of degree r, if and only if $\gcd_i(\#G/\#H_i)|r$. An element $g \in \mathrm{Gal}(L/K)$ acts on the set G/H_i of right cosets from the right and partitions it into r_i orbits of sizes which we denote by $m_{i1}^g, \dots, m_{ir_i}^g$.

Definition 2.1. Define the *combinatorial index of X at $g \in G$* as

$$I_X(g) := \gcd_{i,j}(m_{ij}^g).$$

We call X *combinatorially r-cycle-split* if and only if $I_X(g)|r$ for all $g \in G$. If $r = 1$, we say X is *combinatorially cycle-split.*

For the rest of this section, we take K to be a number field k. With notation as above, the extension L/k is unramified outside a finite set of places S. A finite place of k that is unramified in all K_i is also unramified in L. For a finite place $v \notin S$, let $\mathrm{Frob}_v \in G$ be the Frobenius automorphism at v.

Lemma 2.2. *Let $v \notin S$ be a finite place of k. Then*

$$A \otimes k_v = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{r_i} k_{ij}$$

where k_{ij}/k_v is a finite extension of degree $m_{ij}^{\mathrm{Frob}_v}$.

Proof. This is [Mar77, Theorem 33]. □

Note that the list of orbit sizes really only depends on the conjugacy class of g: the size of the orbit of $H_i t$ under g is the smallest integer j such that $tg^j \in H_i t$, or equivalently $g^j \in t^{-1} H_i t$.

Corollary 2.3. *Let X be a finite étale scheme over a number field k and S a finite set of places such that L/k is unramified outside S. For a finite place $v \notin S$, X_{k_v} is r-cycle-split, if and only if* $\gcd_{i,j}(m_{ij}^{\mathrm{Frob}_v})|r$.

Corollary 2.4. *Let X be a finite étale scheme over a number field k and S a finite set of places such that L/k is unramified outside S. Then X is locally r-cycle-split outside S, if and only if X is combinatorially r-cycle-split.*

Proof. One direction directly follows from the previous corollary. The other direction follows because by Cebotarev density, for every conjugacy class $C \subseteq G$, there exist infinitely many places v with $\mathrm{Frob}_v \in C$. Hence, if X is not combinatorially r-cycle split there exists a $v \notin S$ such that $\gcd_{i,j}(m_{ij}^{\mathrm{Frob}_v}) \nmid r$. □

Example 2.5. The preceding corollary gives a very explicit condition that can be explicitly checked for a finite group G. One example of an everywhere locally cycle-split scheme that is not cycle-split is

$$\operatorname{Spec}(k[t]/(t^2-a)(t^2-b)(t^6-ab))$$

with $a, b, a/b \notin k^2$. In the case where a or b is a square in k_v, the scheme has a rational point. If v does not lie over 2, $a, b \in \mathcal{O}_{k_v}^{\times}$ and neither a nor b are squares in k_v, then ab is a square in k_v and we get k_v-points of degree 2 and 3, hence a zero-cycle of degree 1.

In fact, this is a “modification” of an example by Colliot-Thélène for non-split pseudo-splitness where the exponent 6 is replaced by 2 [CT14, 4.1].

Example 2.6. Take

$$X = \operatorname{Spec}(\mathbb{Q}[t]/(t^2+1)(t^6-3t^2-1))$$

which has a local zero-cycle of degree 1 everywhere. The second factor $(t^6 - 3t^2 - 1)$ is an irreducible polynomial that is everywhere reducible. This is because its non-cyclic Galois group is A_4, of which a subgroup of order 2 leaves $\mathbb{Q}[t]/(t^6 - 3t^2 - 1)$ fixed. Moreover, due to the absence of subgroups of order 6 in A_4, locally there always is a factor of order dividing 3 which together with $(t^2 + 1)$ yields a zero-cycle of degree 1.

Moreover, X is not a finite cover of a non-split pseudo-split scheme X' over $\mathbb{Q}$ as in Example 2.5. This is because A_4 is the smallest counterexample to the converse Lagrange's theorem and thus one sees that any proper quotient of $\mathbb{Z}/2 \times A_4$ fails to satisfy even the group theoretic condition for combinatorial cycle-splitness.

It is a curious result that there is no connected example ($n = 1$) as the following theorem shows.

Theorem 2.7. *The Hasse principle for zero-cycles of degree* 1 *holds for connected, reduced zero-dimensional schemes over a number field k.*

Proof. As before, let L/k be a finite non-trivial Galois extension with Galois group G and $H \subsetneq G$ a proper subgroup. We want to show that $\operatorname{Spec} L^H$ is not locally cycle-split at infinitely many places. Equivalently, we want to find an element g such that

$$\gcd_{t\in G} \min\{k|g^k \in t^{-1}Ht\} > 1.$$

To do this we use the following fact proven "outrageous[ly]" in [FKS81, Theorem 1] via the classification of finite simple groups: for a finite group G, there exists a prime number p and an element $g \notin \bigcup_{t\in G} t^{-1}Ht$ of order a power of p. This is sufficient since then $p|\min\{k|g^k \in t^{-1}Ht\}$ for all $t \in G$. $\square$

In more down-to-earth language, there is no irreducible polynomial over k that factors into coprime degrees modulo almost all primes.

2.2 In higher dimensions

For the beginning of this section, let us again allow K to be any field. Let X be a proper variety over K. For X' a reduced, irreducible component of X, we define the *(apparent) multiplicity* of X' in X as the length of the local ring $\mathcal{O}_{X,\eta'}$ where η' is the generic point of X'. We define the *geometric multiplicity* of X' in X as the length of the local ring $\mathcal{O}_{\overline{X},\overline{\eta'}}$ where $\overline{\eta'}$ is a point of $\overline{X}$ lying over η'. If X' is geometrically reduced, for example when K is perfect, then multiplicity and geometric multiplicity coincide.

Let $X_1^m, \dots, X_n^m$ be the reduced, irreducible components of geometric multiplicity m in X. Let K_i be the separable closure of K in the function field of X_i^m.

Definition 2.8. Define the *algebra of irreducible components of geometric multiplicity* m as $Z_X^m := \operatorname{Spec}(\oplus_{i=1}^n K_i)$. (If there are no such components, then Z_X^m is empty.)

The reason for this definition is of course that the embedding of the ground field into the function field of a scheme controls, to some extent, its geometric properties and thus we can reduce to the previous section. A scheme T of finite type over K is geometrically irreducible if and only if T is irreducible and K is separably closed in the function field of T (see [Gro65, 4.5.9]).

However, using the functor of open irreducible components defined by Romagny we can obtain finer results. For a finite type morphism of schemes $T \to R$ with R integral, let $\mathrm{Irr}^m_{T/R}$ be the subfunctor of $\mathrm{Irr}_{T/R}$ defined in [Rom11, Def. 2.1.1] of open irreducible components of geometric multiplicity m. We recall that $\mathrm{Irr}_{T/R}$ parametrises open subschemes U of an R-scheme R' such that the geometric fibres of $U \times_R R' \to R'$ are interiors of irreducible components in the geometric fibres of $T \times_R R' \to R'$. Note that this is stable under base change and thus functorial because we use the geometric instead of the apparent multiplicity.

Lemma 2.9. *The functor* $\mathrm{Irr}^m_{T/R}$ *is representable over a dense open of* R *by a finite étale cover.*

Proof. Let η be the generic point of R and $T' \hookrightarrow T \to R$ be the reduced closure of the irreducible components of geometric multiplicity m in the fibre over η. Then after replacing R with a dense open subscheme, we have that $\mathrm{Irr}^m_{T/R} = \mathrm{Irr}_{T'/R}$ because the geometric multiplicity of the fibre over η spreads out to a dense open neighbourhood by [Gro66, Proposition 9.8.6].

After further replacement of R with a dense open subscheme, the functor $\mathrm{Irr}_{T'/R}$ is representable by a separated algebraic space which is finite étale over R by [Rom11, 2.1.2,2.1.3]. However, by Knutson's representability criterion, this algebraic space over R must in fact be a scheme (cf. [LS16, Proof of Proposition 3.7] for this last step). □

Lemma 2.10. *The functor* $\mathrm{Irr}^m_{X/K}$ *is represented by* Z^m_X.

Proof. This follows from [Rom11, 2.1.4]. □

Definition 2.11. Let G be the Galois group defined in Section 2.1 for the finite étale K-scheme Z^m_X. Define the *combinatorial index of* X *at* $g \in G$ as

$$I_X(g) := \gcd_m(m I_{Z^m_X}(g)).$$

We call X *combinatorially* r*-cycle-split* if and only if $I_X(g) | r$ for all $g \in G$. If $r = 1$, we say X is *combinatorially cycle-split*.

This is compatible with the previous definition of combinatorial index in dimension 0 and only depends on the conjugacy class of g in G.

Let us return to the case of k a number field and assume X is smooth and proper over k. Let v be a finite place of k. To tackle the question of zero-cycles on X_{k_v}, we need to relate closed points in the special and generic fibres of a model. This seems to be folkloric knowledge partly written down in [BLR90, §9, Cor. 9.1] but the author could not find a complete reference before [BL99] (see also [Wit15, 4.6] and [KN17, 2.4]).

Lemma 2.12. *Let $\mathcal{X}$ be a proper, flat model over $\mathcal{O}_{k_v}$ of X_{k_v}. Let $\overline{x} \in \mathcal{X}(k(v))$ be a point which is regular in $\mathcal{X}$ and regular in the reduction $\mathcal{X}_{k(v)}$ and lies on a geometrically irreducible component of $\mathcal{X}_{k(v)}$ of multiplicity m. Then there exists a closed point $x \in X_{k_v}$ of degree m with reduction $\overline{x}$.*

Proof. See [CTS96, 2.3]. ☐

Conversely, the following result applies.

Lemma 2.13. *Let $\mathcal{X}$ be a proper, regular, flat model over $\mathcal{O}_{k_v}$ of X_{k_v} and x a closed in point in X_{k_v} of degree d with reduction $\tilde{x}$.*

Let D_j, $j \in J$, be the irreducible components of $X_{k(v)}$ on which $\tilde{x}$ lies. Denote by m_j the multiplicity of D_j and by d_j the minimal degree of an extension of $k(v)$ over which D_j splits into geometrically irreducible components. Then $\gcd_{j\in J} m_j d_j$ divides d.

Proof. See [BL99, 1.6]. ☐

Lemma 2.14. *Let $\mathcal{X}$ be a proper, normal, flat model over $\mathcal{O}_{k_v}$ of X_{k_v}. Let Frob_v be the Frobenius element in the absolute Galois group of $k(v)$.*

If $I_{\mathcal{X}_{k(v)}}(\mathrm{Frob}_v)|r$, then X_{k_v} is r-cycle-split. If $\mathcal{X}$ is regular, then the converse holds.

There exists a function $\Phi : \mathbb{N}^3 \to \mathbb{N}$ not depending on X with the following property. Let $\iota : \mathcal{X}_{k(v)} \dashrightarrow \mathbb{P}^{\nu}_{k(v)}$ be a rational embedding. If

$$I_{\mathcal{X}_{k(v)}}(\mathrm{Frob}_v)|r,$$

then there exists a zero-cycle Z of degree r on X_{k_v} with $\operatorname{maxdeg} Z \leq \Phi(\iota)$ *(where $\Phi(\iota)$ is defined as after Lemma 1.7).*

Proof. Assume $I_{\mathcal{X}_{k(v)}}(\mathrm{Frob}_v)|r$. Then there exist geometrically irreducible components D_j, $j \in J$, of the special fibre of multiplicities m_j defined over extensions of $k(v)$ of degrees d_j s.t.

$$\gcd_{j \in J} d_j m_j = I_{\mathcal{X}_{k(v)}}(\mathrm{Frob}_v).$$

By the Lang-Weil estimates as formulated in Section 1.2, each D_j has a zero-cycle Z of degree d_j.

Let Z_j be the union of the non-regular locus of $\mathcal{X}$ and the non-regular locus of the reduction of D_j. Because $\mathcal{X}$ is assumed normal, hence regular in codimension 1, Z_j does not contain all of D_j. Then $\deg Z_j$ has an upper bound only depending on ν and the degree of the image of ι. By the Lang-Weil estimates as described in Section 1.2, one can arrange for the summands of Z to avoid all Z_j and satisfy $\operatorname{maxdeg} Z \leq \Phi(\iota)$ for a suitable function Φ.

Applying Lemma 2.12 to each of the summands, the existence of points of orders $m_j d_j$ and thus a zero-cycle of degree r in X_{k_v} follows.

The converse in the case of regular $\mathcal{X}$ follows from Lemma 2.13. □

Remark 2.15. We remark that to examine r-cycle-splitness of the special fibre itself, all components of multiplicity greater than 1 would have to be discarded. Thus, there are two notions, r-cycle-split and combinatorially r-cycle-split. This is a difference to the case of rational points with only one notion of pseudo-split.

Lemma 2.16. *Let X be a smooth, proper variety over a number field k. Let $\iota : X \dashrightarrow \mathbb{P}^\nu$ be a rational embedding of X. Then X is almost everywhere locally r-cycle-split if and only if X is combinatorially r-cycle-split. In this case, X_{k_v} has a zero-cycle Z of degree r with* $\operatorname{maxdeg} Z \leq \Phi(\iota)$ *for all $v \notin S$.*

Proof. Let $U \subseteq X$ be a dense open subvariety on which ι is defined. We can find a finite set S of places such that $U \hookrightarrow X$ and $\iota : U \hookrightarrow \mathbb{P}^\nu$ spread out to models $\mathcal{U} \hookrightarrow \mathcal{X}$ and $\iota_S : \mathcal{U} \hookrightarrow \mathbb{P}^{\nu}_{\mathcal{O}_{k,S}}$ over $\mathcal{O}_{k,S}$ where $\mathcal{U}$ and $\mathcal{X}$ are smooth over $\mathcal{O}_{k,S}$.

By Lemma 2.10 and Lemma 2.9, after possibly enlarging S, $\mathrm{Irr}^{m}_{\mathcal{X}/\mathcal{O}_{k,S}}$ is represented by $\mathrm{Spec}(\oplus_{i=1}^{n}\mathcal{O}_{K_i,S})$. The result now follows from Lemma 2.14. $\square$

3

s^0-INVARIANTS

In analogy to the s-invariants in [LSS19], we construct s^0-*invariants* that measure failure of combinatorial cycle-splitness in families. Let $f : X \to Y$ be a morphism of varieties over a number field k.

For any (possibly non-closed) point $y \in Y$, set $K := k(y)$. We get finite étale (possibly empty) K-schemes $Z^m_{f^{-1}(y)} = \mathrm{Irr}^m_{f^{-1}(y)/K}$ for all multiplicities m. We may pick L/K a minimal Galois extension which splits all $Z^m_{f^{-1}(y)}$ with Galois group G. Denote by k_K and k_L the algebraic closures of k in K and L. By replacing k_L with its Galois closure and extending L, we can assume that k_L/k is Galois. Let N be the subgroup of G acting trivially on k_L. Denote by Ω_{k_K} the set of finite places of k_K.

Definition 3.1. For a finite place v of k, define $s^{0,r}_{f,y}(v)$ in the following way:

(i) as 1, if v ramifies in k_L or there is no place in k_K of degree 1 over v

(ii) otherwise, as

$$\frac{\sum_{\substack{w \in \Omega_{k_K} \\ \mathrm{N}(w) = \mathrm{N}(v) \\ w | v}} \#\{g \in G : \mathrm{Frob}_w \equiv g \bmod N, I_{f^{-1}(y)}(g) | r\}}{\#N \#\{w \in \Omega_{k_K} \,|\, \mathrm{N}(w) = \mathrm{N}(v), w | v\}}.$$

One can see that $s^{0,r}_{f,y}(v)$ is constant on the conjugacy class of Frob_v, i.e. that this function is Frobenian in the sense of Serre [Ser12, §3.3.3.5] but this fact will not be directly needed.

Over finite fields, the s^0-invariants asymptotically quantify the failure of combinatorial r-cycle-splitness.

Proposition 3.2. *Assume Y is integral of dimension n with generic point η. Let $f : \mathcal{X} \to \mathcal{Y}$ be a model of f over $\mathcal{O}_k$. Then*

$$\begin{aligned} &\#\{y \in \mathcal{Y}(k(v)) | f^{-1}(y) \text{ is combinatorially } r\text{-cycle-split}\} \\ =& s^{0,r}_{f,\eta}(v)\#\mathcal{Y}(k(v)) + O(\mathrm{N}(v)^{n-1/2}) \end{aligned}$$

as $\mathrm{N}(v) \to \infty$, *where the asymptotic constant of the O-notation only depends on the chosen model.*

Proof. The main idea after [LSS19, Proposition 3.13] is to count and then compare both sides using Lang-Weil estimates and the Cebotarev density theorem for schemes. We divide the proof into several parts.

Set-up By the Lang-Weil estimates we can remove strict closed subsets of $\mathcal{Y}$ since for dimension reasons, their rational points only contribute to the error term. Hence, with the help of Lemma 2.9, we assume that $\mathrm{Irr}^m_{\mathcal{X}/\mathcal{Y}} \to \mathcal{Y}$ finite étale.

In the same way, we ensure that $\mathcal{Y}$ and its special fibres $\mathcal{Y}_{k(v)}$ are normal for all v not contained in some finite set S by removing the closed singular locus (including possibly finitely many special fibres). Set $y := \eta$ and from there on K, L, k_K and k_L, G and N as before. Enlarging S further, we may spread out and assume that L is the generic fibre of a Galois closure $\mathcal{L}$ of $\mathrm{Irr}^m_{\mathcal{X}/\mathcal{Y}} \to \mathcal{Y}$. From now on, let $v \notin S$

Counting points of $\mathcal{Y}_{k_v}$ with Lang-Weil The functor $\mathrm{Irr}_{\mathcal{Y}_{k(v)}/k(v)}$ is represented by

$$\operatorname{Spec} \mathcal{O}_{k_K} \otimes_{\mathcal{O}_k} k(v) = \bigoplus_{\substack{w \in \Omega_{k_K} \\ w | v}} k(w).$$

Therefore, geometrically irreducible components of $\mathcal{X}_{k(v)}$ correspond to places

w with $\mathrm{N}(w) = \mathrm{N}(v)$. We write $\mathcal{Y}_w$ for such a component. By the normality assumption, the irreducible components of $\mathcal{Y}_{k(v)}$ are all disjoint, so if there is none which is geometrically irreducible, $\mathcal{Y}_{k(v)}$ has no rational point. This is the trivial case of the proposition. In the non-trivial case, we can count points by Lang-Weil:

$$\begin{aligned}\#\mathcal{Y}(k(v)) &= \sum_{\substack{w\in\Omega_{k_K}\\ \mathrm{N}(w)=\mathrm{N}(v)\\ w|v}} \#\mathcal{Y}_w(k(w)) = \sum_{\substack{w\in\Omega_{k_K}\\ \mathrm{N}(w)=\mathrm{N}(v)\\ w|v}} \mathrm{N}(v)^n + O(\mathrm{N}(v)^{n-1/2}) \\ &= \#\{w \in \Omega_{k_K} \,|\, \mathrm{N}(w) = \mathrm{N}(v), w|v\}\, \mathrm{N}(v)^n + O(\mathrm{N}(v)^{n-1/2}).\end{aligned}$$

Counting combinatorially r-cycle-split fibres with Cebotarev For a rational point $y \in \mathcal{Y}(k(v))$, we can view the Frobenius Frob_y as an element of G up to conjugacy. The fibre $f^{-1}(y)$ is combinatorially r-cycle-split if and only if $I_{f^{-1}(\eta)}(\mathrm{Frob}_y) = I_{f^{-1}(y)}(\mathrm{Frob}_y)|r$. Let $\delta_f(g) \in \{0,1\}$ be the indicator function of the set of elements $g \in G$ for which $I_{f^{-1}(y)}(g)|r$. This function only depends on the conjugacy class of g. Applying the Cebotarev density theorem for étale morphisms as in [Ser12, 9.15] to δ_f one gets:

$$\begin{aligned}&\#\{y \in \mathcal{Y}(k(v)) | f^{-1}(y) \text{ is combinatorially } r\text{-cycle-split}\} \\ = &\frac{\mathrm{N}(v)^n}{\#N} \sum_{\substack{w\in\Omega_{k_K}\\ \mathrm{N}(w)=\mathrm{N}(v)\\ w|v}} \#\{g \in G : \mathrm{Frob}_w \equiv g \bmod N, I_{f^{-1}(\eta)}(g)|r\} + O(\mathrm{N}(v)^{n-1/2})\end{aligned}$$

Comparing both counts with the definition of $s^{0,r}_{f,\eta}(v)$, the result follows. □

The asymptotic formula gives a necessary condition for combinatorial cycle-splitness of all fibres.

Corollary 3.3. *With the same notation, if $s^{0,r}_{f,\eta}(v) < 1$ for some $v \notin S$, then there exists $y \in \mathcal{Y}(k(v))$ such that $f^{-1}(y)$ is not combinatorially r-cycle-split.*

Proof. For v large enough, there will be rational points on $\mathcal{Y}_{k(v)}$ but by Proposition 3.2, not all fibres over them can be combinatorially r-cycle-split. □

The asymptotics also give the other direction.

Corollary 3.4. *With the same notation, assume $\mathcal{Y}$ is integral normal and Irr_f^m finite étale over $\mathcal{Y}$ for all m. Then there exists a finite set of places S such that for all $v \notin S$, $s_{f,\eta}^{0,r}(v) = 1$ if and only if the fibre of f over every $y \in \mathcal{Y}(k(v))$ is combinatorially r-cycle-split.*

Proof. One direction has just been proven. For the other direction, we use the same notation as in Proposition 3.2.

A point $y \in \mathcal{Y}(k(v))$ must lie on a geometrically irreducible component corresponding to the degree 1 place w of k_K. Let $l \in \mathcal{L}$ be a closed point over y and u be the corresponding place of its irreducible component. Then

$$k(y) = k(w) \subset k(u) \subset k(l)$$

and there exist natural embeddings

$$\mathrm{Gal}(k(l)/k(y)) \hookrightarrow G$$

and

$$\mathrm{Gal}(k(u)/k(y)) \hookrightarrow G/N.$$

By functoriality of Frobenius, we have

$$\mathrm{Frob}_{l/y} \bmod N = \mathrm{Frob}_{u/w}\,.$$

Because of the assumption that $s_{f,\eta}^{0,r}(v) = 1$, we deduce that $\mathrm{Frob}_{l/y}$ acts on $\mathrm{Irr}_{f^{-1}(y)/y}^m$ such that $I_{f^{-1}(y)}(\mathrm{Frob}_{l/y})|r$. Hence $f^{-1}(y)$ is combinatorially r-cycle-split. □

Corollary 3.5. *The fibre $f^{-1}(y)$ is combinatorially r-cycle-split if and only if $s_{f,y}^{0,r}(v) = 1$ for almost all v.*

Proof. This is Corollary 3.4 in the case of a zero-dimensional base. □

4

ARITHMETIC CYCLE-SURJECTIVITY

Let $f : X \to Y$ be a dominant morphism between proper, smooth, geometrically integral varieties with geometrically integral generic fibre over a number field k.

4.1 BIRATIONAL INVARIANCE

We want to prove that arithmetic r-cycle-surjectivity is a property invariant under modifications. The argument here is more subtle than in the case of rational points.

Definition 4.1. Let v be a place of k. If a fibre over a k_v-point y of Y contains a zero-cycle of degree r we call this cycle a *witness for r-cycle-surjectivity over y at v.*

Lemma 4.2. *Let v be a place of k. Let V be a dense open subset of Y. Assume that there exists $B \in \mathbb{N}$ such that $f^{-1}(V) \to V$ is r-cycle-surjective at v and there exist witnesses Z_v for r-cycle-surjectivity over y at v for all $y \in V(k_v)$ with* $\operatorname{maxdeg} Z_v \leq B$. *Then f is r-cycle-surjective at v.*

Proof. Assume cycle-surjectivity on an open V with a uniform bound B as described above. Let $k_v(i)$ denote the compositum of all degree i extensions of k_v. Then $X(k_v(i)) \subseteq X(\overline{k_v})$ is the set of $\overline{k_v}$-points fixed by all elements in $\mathrm{Gal}(\overline{k_v}/k_v(i))$ and this a closed subset. Hence

$$Y_B(f) := \bigcup_{\substack{I \subset \{1,\dots,B\} \\ \gcd(I) | r}} \bigcap_{i \in I} f(X(k_v(i)))$$

is a finite union of closed subsets of $Y(\overline{k_v})$.

Let y be a k_v-rational point in V for which the fibre $f^{-1}(y)$ contains a zero-cycle Z of degree r with $\operatorname{maxdeg} Z \leq B$. There exists $I \subset \{1,\dots,B\}$ such that

$$y \in \bigcap_{i \in I} f(X(k_v(i))) \subset Y_B(f).$$

On the other hand, a point $y \in Y_B(f)$ lies in $\bigcap_{i \in I} f(X(k_v(i)))$ for some $I \subset \{1,\dots,B\}$ with $\gcd(I)|r$, so its fibre has a closed $k_v(i)$-point for all $i \in I$. Let j_i be the degree of this point. It follows that the prime factors of j_i are contained in the prime factors of i. In particular, the fibre has a zero-cycle of degree $\gcd_{i \in I} j_i = \gcd I | r$.

Now $Y_B(f)$ is closed and contains $V(k_v)$ which is dense and open in $Y(k_v)$, hence $Y(k_v) \subseteq Y_B(f)$. □

Remark 4.3. The above proof generalises to k_v any Henselian (non-trivially) valued field.

Lemma 4.4. *To show arithmetic r-cycle-surjectivity of f, it is enough to show arithmetic r-cycle-surjectivity of $f^{-1}(V) \to V$ for a dense open V in Y.*

Proof. By Lemma 4.2 all we have to show is that for v large enough, if f is arithmetically r-cycle-surjective over V, there is a uniform bound on the maximum degree of witnesses. By generic smoothness [Har77, Corollary 10.7], after shrinking V, we may assume that all fibres over V are smooth.

Let $\iota : X \dashrightarrow \mathbb{P}^{\nu}_{Y}$ be a rational embedding. But now by Lemma 2.16 (which has a smoothness assumption), a fibre over a point in V is almost everywhere locally r-cycle-split if and only if it is almost everywhere locally r-cycle-split with zero-cycles as witnesses that have maximum degree less than $\Phi(\iota)$. $\square$

4.2 NECESSARY CONDITION

From the results over finite fields, we can deduce a necessary condition for arithmetic r-cycle-surjectivity.

Proposition 4.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper model of f over $\mathcal{O}_{k,S}$ for a finite set of places S of k with regular source and target. Let $\mathcal{T} \subset \mathcal{Y}$ be a reduced divisor such that f is smooth away from $\mathcal{T}$. Then after possibly enlarging S, we can find a subset $\mathcal{R} \subset \mathcal{T}_{\mathcal{O}_{k,S}}$ of codimension at least 2 in $\mathcal{Y}_{\mathcal{O}_{k,S}}$ such that for all $v \notin S$ the following holds.*

Choose $\widetilde{y} \in \mathcal{Y}(\mathcal{O}_{k_v})$; denote its generic point by $y \in Y(k_v)$. If $\tilde{y}$ intersects $\mathcal{T}_{\mathcal{O}_{k,S}}$ transversally outside $\mathcal{R}_{\mathcal{O}_{k,S}}$ and the fibre at $(\tilde{y} \bmod \pi_v)$ is not combinatorially r-cycle-split, then $f^{-1}(y)$ is not r-cycle-split.

Proof. This is a variant of [LS16, Theorem 2.8]: After possibly enlarging S, $\mathcal{R}$ can be chosen of codimension 2 in a way such that

- (i) by generic flatness for regular schemes, f is flat on the complement $\mathcal{Y} \setminus \mathcal{R}$, and
- (ii) by generic submersivity [LS16, Theorem 2.4] in characteristic 0, f is submersive (i.e. surjective on tangent spaces) over $\mathcal{T} \setminus \mathcal{R}$.

Then $\mathcal{X} \times_{\mathcal{Y}} \widetilde{y}$ is regular and its special fibre is not combinatorially r-cycle-split. The rest follows by Lemma 2.14. $\square$

Proposition 4.6. *Let $\vartheta \in Y^{(1)}$ be a codimension 1 point of Y. There exists a finite set of places S such that for all $v \notin S$ the following holds: if $s^{0,r}_{f,\vartheta}(v) < 1$, then f is not arithmetically r-cycle-surjective.*

Proof. If $s^{0,r}_{f,\vartheta}(v) < 1$, let $\mathcal{E}$ be the closure of ϑ in $\mathcal{Y}$. By Corollary 3.3, for suitable S we can find a point y in the special fibre of $\mathcal{E}$ above which the fibre is not combinatorially r-cycle-split. By Proposition 4.5, it therefore suffices to lift y to an integral point intersecting $\mathcal{E}$ transversally. The argument for this is well-known and literally the same as in [LSS19, Theorem 4.2] via blowing-up $\mathcal{Y}$ in y and choosing a point on the exceptional divisor. □

4.3 SUFFICIENT CONDITION AND PROOF OF MAIN THEOREM

Finally, using tools from logarithmic geometry, we can give a necessary and sufficient criterion for arithmetic r-cycle-surjectivity. A brief overview of the foundations of logarithmic geometry is provided in Appendix A. The experienced reader may ignore it and read this section on its own. All log schemes in this section will be fs Zariski log schemes.

For this section assume that we have a log smooth, proper model

$$f : (\mathcal{X}, \mathcal{D}) \to (\mathcal{Y}, \mathcal{E})$$

of f where $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{Y}, \mathcal{E})$ are Zariski log regular schemes (with divisorial log structure induced by $\mathcal{D}$ and $\mathcal{E}$) that are log smooth and proper over $\mathcal{O}_{k,S}$ equipped with the trivial log structure for some finite set of places S. This can be achieved after a modification of f by using Abramovich-Denef-Karu's Toroidalisation Theorem (cf. Theorem A.17) and spreading out. Denote by D and E the generic fibres of $\mathcal{D}$ and $\mathcal{E}$. Set $\mathcal{U} := \mathcal{X} \setminus \mathcal{D}$, $U := X \setminus D$, $\mathcal{V} := \mathcal{Y} \setminus \mathcal{E}$, and $V := Y \setminus E$. On these open sets, the log structures are trivial.

By possibly enlarging S in the spreading-out procedure above, we may assume that all irreducible components $\mathcal{E}'$ of $\mathcal{E}$ intersect the generic fibre nontrivially, i.e. their generic points lie in Y. This property of our chosen model is absolutely crucial for the method presented here. Namely, one can control the splitting behaviour of the fibre of f over a point in the interior of $\mathcal{E}'$ by the behaviour of the fibre of f over the generic (characteristic 0) point ϑ' of $\mathcal{E}'$ (see Lemma 4.7).

Let v be a finite place of k. Let k'/k be a finite extension and w an extension of v to k'. By the valuative criterion of properness, any closed point

$$y : \operatorname{Spec} k'_w \to Y$$

extends to a morphism

$$\widetilde{y} : (\operatorname{Spec} \mathcal{O}_{k'_w})^\dagger \to (\mathcal{Y}, \mathcal{E}),$$

where $(\operatorname{Spec} \mathcal{O}_{k'_w})^\dagger$ is the log scheme equipped with the standard divisorial log structure defined by a uniformiser π_w (i.e. with monoid given by $\mathcal{O}_{k'_w} \setminus 0$).

A morphism g of log regular schemes induces a morphism $F(g)$ of Kato fans. Because $F_{(\operatorname{Spec} \mathcal{O}_{k'_w})^\dagger} \cong \operatorname{Spec} \mathbb{N}$, this defines a logarithmic height $h(y)$ for any $y \in Y(k'_w)$. Morally, the height of y quantifies how often $\widetilde{y}$ intersects the special fibre.

Lemma 4.7. *For any $t \in F_{\mathcal{Y}}$ and $m \in \mathbb{N}$, the functor $\operatorname{Irr}^m_{f^{-1}(U(t))/U(t)}$ is representable by a finite étale scheme over $U(t)$.*

Proof. It is shown in [LSS19, Proposition 5.18] that $\operatorname{Irr}_{f^{-1}(U(t))/U(t)}$ is representable by a finite étale scheme over $U(t)$. By [LSS19, Proposition 5.16], apparent multiplicity is constant along logarithmic strata for proper, log smooth morphisms of log regular schemes, and because log smoothness is stable under base change, the same is true for geometric multiplicity. Thus the subfunctor $\operatorname{Irr}^m_{f^{-1}(U(t))/U(t)}$ is represented by the closure of $\operatorname{Irr}^m_{f^{-1}(t))/t}$ in $\operatorname{Irr}_{f^{-1}(U(t))/U(t)}$. □

The following two propositions bound the intersection behaviour of points in Y the fibres above which we have to consider.

Proposition 4.8. *There is a positive integer N with the following property. Let $B \in \mathbb{N}$ be arbitrary and $v \notin S$ a place of k. If the fibre over each point $y \in V(k_v)$ with $h_{\mathcal{Y}}(y) \leq N$ has a zero-cycle Z of degree r with* $\operatorname{maxdeg} Z \leq B$, *then $f \times_k k_v$ is r-cycle-surjective.*

Proof. The proof is very similar to the one in [LSS19, Proposition 6.1], which itself is an adaptation of [Den16, 4.2], and we only sketch the steps and highlight the necessary changes.

Let $F(f)_* : F_X(\operatorname{Spec}\mathbb{N}) \to F_Y(\operatorname{Spec}\mathbb{N})$ be the morphism induced by f. Then define for all $s \in F_X$ and $t = F(f)_*(s) \in F_Y$:

$$N_t = \min\{h_Y(t') | t' \in F_Y^t(\operatorname{Spec}\mathbb{N}), t' \notin F(f)_*(F_X(\operatorname{Spec}\mathbb{N}))\},$$

$$N_{s,t} = \min\{h_Y(t') | t' \in F(f)_*(F_X^s(\operatorname{Spec}\mathbb{N})) \subset F_Y^t(\operatorname{Spec}\mathbb{N})\}$$

and $N = \max\{N_t, N_{s,t}\}$.

We have thus a finite partition

$$\begin{aligned} F_Y(\operatorname{Spec}\mathbb{N}) = & \bigsqcup_{t \in F_Y} F_Y^t(\operatorname{Spec}\mathbb{N}) \setminus F(f)_*(F_X(\operatorname{Spec}\mathbb{N})) \\ & \sqcup \bigsqcup_{s \in F_X} F(f)_*(F_X^s(\operatorname{Spec}\mathbb{N})), \end{aligned}$$

where each partition subset contains at least one element with height less than N.

Given some arbitrary $y \in V(k_v)$, we have to show that its fibre is r-cycle-split with a uniform bound B on the maximum degree of witnesses so that we can conclude by Lemma 4.2. The proof works by twice applying the logarithmic analogue of Hensel's lemma for log smooth morphisms (cf. Lemma A.16).

By the above, we may find $b \in F_Y(\operatorname{Spec}\mathbb{N})$ in the same partition subset as $F(\widetilde{y})$ with $h_Y(b) \leq N$. Write $b = \sum_{i \in I} b_i v_i$, where $(v_i)_{i \in I}$ are the cones in F_Y corresponding to the irreducible components $(\mathcal{E}_i)_{i \in I}$ of $\mathcal{E}$.

Let $(\pi_i)_{i \in I}$ be local equations for $(\mathcal{E}_i)_{i \in I}$ in an affine neighbourhood $\operatorname{Spec} A$ of $(\tilde{y} \bmod \pi_v)$ in $\mathcal{Y}$.

Let $\overline{\varphi}$ be the canonical morphism

$$\overline{\varphi} : \mathcal{O}_{k_v} \setminus 0 \to (\mathcal{O}_{k_v} \setminus 0)/(1 + \pi_v \mathcal{O}_{k_v}) \cong k(v)^* \oplus \mathbb{N} \to k(v)^*.$$

The first application of logarithmic Hensel's lemma is to the diagram

$$\begin{tikzcd}
\operatorname{Spec}(k(v))^{\dagger} \arrow[r] \arrow[d] & (\mathcal{Y}, \mathcal{E}) \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_{k_v})^{\dagger} \arrow[r] & \operatorname{Spec}(\mathcal{O}_{k_v})^{\mathrm{tr}}
\end{tikzcd}.$$

Here, $\operatorname{Spec}(\mathcal{O}_{k_v})^{\mathrm{tr}}$ denotes the trivial log structure with monoid $\mathcal{O}_{k_v}^*$ and $\operatorname{Spec}(k(v))^{\dagger}$ denotes the standard log point with log structure $k(v)^* \oplus \mathbb{N}$, the restriction of $\operatorname{Spec}(\mathcal{O}_{k_v})^{\dagger}$.

On the level of monoids, the upper horizontal arrow is defined by

$$\begin{aligned}
A^* \times \mathbb{N}^I &\to k(v)^* \oplus \mathbb{N}, \\
\alpha \in A^* &\mapsto (\alpha(\tilde{y} \bmod \pi_v), 0), \\
1_i \in \mathbb{N}^I &\mapsto (\overline{\varphi}(\pi_i(\widetilde{y})), b_i),
\end{aligned}$$

where 1_i is the generator of the i-th factor. All other morphisms are the obvious ones.

The point $y' \in Y(k_v)$ yielded by logarithmic Hensel's lemma has the same reduction as y but satisfies

$$F(\widetilde{y'}) = b$$

and

$$\overline{\varphi}(\pi_i(\widetilde{y})) = \overline{\varphi}(\pi_i(\widetilde{y'})).$$

This is the first half of the proof and works verbatim as in [LSS19, Proposition 6.1].

For the second half, the assumption of our proposition now states that $f^{-1}(y')$ contains a zero-cycle of degree r which we write as $\sum_h n_h x'_h$. Here, x'_h is a closed point defined over a finite extension l_{w_h}/k_v with $[l_{w_h} : k_v] \leq B$. We are done with the proof, if we can lift each $(\widetilde{x'_h} \bmod \pi_{w_h})$ to an l_{w_h}-point $x_h \in f^{-1}(y)$.

To do so, we only have to slightly alter diagram (6.3) from the original proof in [LSS19] and apply (for the second time) logarithmic Hensel, namely to

$$\begin{tikzcd}
\operatorname{Spec}(k(w_h))^\dagger \arrow[r] \arrow[d] & (\mathcal{X}, \mathcal{D}) \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_{l_{w_h}})^\dagger \arrow[r] & (\mathcal{Y}, \mathcal{E})
\end{tikzcd}.$$

On schemes, the upper horizontal morphism is given by $(\widetilde{x'_h} \bmod \pi_{w_h})$ and the lower horizontal morphism is defined by $\widetilde{y}$ composed with $\operatorname{Spec}(\mathcal{O}_{l_{w_h}})^\dagger \to \operatorname{Spec}(\mathcal{O}_{k_v})^\dagger$.

Let e_h be the ramification index of l_{w_h}/k_v. Then on fans

$$\operatorname{Spec}(\mathcal{O}_{l_{w_h}})^\dagger \to \operatorname{Spec}(\mathcal{O}_{k_v})^\dagger$$

is just $\operatorname{Spec}\mathbb{N} \to \operatorname{Spec}\mathbb{N}$ induced by multiplication with e_h and hence

$$F(\operatorname{Spec}(\mathcal{O}_{l_{w_h}})^\dagger \to (\mathcal{Y}, \mathcal{E})) = e_h F(\widetilde{y}).$$

In an affine neighbourhood $\operatorname{Spec}(B)$ of $(\widetilde{x'_h} \bmod \pi_{w_h})$ in $\mathcal{X}$, $(\mathcal{X}, \mathcal{D})$ has a chart $\mathbb{N}^J \to B$ given by sending the generator 1_j to a local equation ω_j of the irreducible component $\mathcal{D}_j$. Let u_j be the Kato subcone corresponding to $\mathcal{D}_j$. Since $F(\widetilde{y})$ and b were chosen in the same partition subset and

$$F(f)_*(\widetilde{x}'_h) = F(\widetilde{y'}) = b,$$

there exists $a = \sum_j a_j u_j \in F^s_X(\operatorname{Spec}\mathbb{N})$ such that $F(\widetilde{x}'_h) \in F^s_X(\operatorname{Spec}\mathbb{N})$ and $F(f)_*(a) = F(\widetilde{y})$, so

$$F(f)_*(e_h a) = F(\operatorname{Spec}(\mathcal{O}_{l_{w_h}})^\dagger \to (\mathcal{Y}, \mathcal{E})).$$

Then the log structure of $\operatorname{Spec}(k(w_h))^\dagger \to (\mathcal{X}, \mathcal{D})$ should be defined by the morphism of monoids

$$\mathbb{N}^J \to k(w_h)^* \oplus \mathbb{N}, 1_j \mapsto (\overline{\varphi}(\omega_j(\widetilde{x}'_h)), e_h a_j).$$

The proof that this defines a commuting diagram of log schemes works as in [LSS19, Proposition 6.1]. □

The next proposition [LSS19, Proposition 5.10 and Proposition 6.2] gives us a modification of f (which was obtained obtained by pulling back along $N-1$ barycentric log blow-ups of $\mathcal{Y}$) which will turn out to be optimal in the sense that it is all we need to check arithmetic r-cycle-surjectivity.

Proposition 4.9. *Let N be a positive integer. There is a log smooth modification $f' : (\mathcal{X}', \mathcal{D}') \to (\mathcal{Y}', \mathcal{E}')$ of f with $\mathcal{X}'$ and $\mathcal{Y}'$ smooth, proper over $\mathcal{O}_{k,S}$ and geometrically integral with the following property:*

Let Y' be the generic fibre of $\mathcal{Y}'$ and E' be the generic fibre of $\mathcal{E}'$. For any $v \notin S$ and each point $y \in (Y \setminus E)(k_v) = (Y' \setminus E')(k_v)$ with $1 \leq h_{\mathcal{Y}}(y) \leq N$, $h_{\mathcal{Y}'}(y) = 1$ and its reduction in $\mathcal{Y}'$ is a smooth point of the reduction of $\mathcal{E}'$.

Now we can prove a sufficient criterion:

Proposition 4.10. *Let $v \notin S$ and $f' : (\mathcal{X}', \mathcal{D}') \to (\mathcal{Y}', \mathcal{E}')$ a log smooth modification of f as in Proposition 4.9. If $s^{0,r}_{f,\vartheta'}(v) = 1$ for each generic point ϑ' of D' (the generic fibre of $\mathcal{D}'$), then $f \times_k k_v$ is r-cycle-surjective.*

Proof. Pick a rational embedding $\iota : \mathcal{X}'_{k(v)} \dashrightarrow \mathbb{P}^{\nu}_{\mathcal{Y}'_{k(v)}}$ and let $B = \Phi(\iota)$. Let $\mathcal{V}' := \mathcal{X}' \setminus \mathcal{E}'$. It is enough to prove that the fibre over a point $y \in V'(k_v) = V(k_v)$ has a zero-cycle Z of degree r with $\operatorname{maxdeg} Z \leq B$. If the reduction of y in $\mathcal{Y}$ is in $\mathcal{V}$, we know that $f'^{-1}(\tilde{y} \bmod \pi_v) \cap \mathcal{U}$ is non-empty smooth and geometrically integral (by assumption on the generic fibre), so $f^{-1}(y)$ has a zero-cycle of degree 1 with maximum degree less than B by the Lang-Weil estimates.

Otherwise, assume that $\widetilde{y}$ intersects $\mathcal{E}'$. By Proposition 4.8, we can restrict ourselves to y with $h(y) \leq N$.

Because of Proposition 4.9, $\widetilde{y}$ intersects transversally a codimension 1 logarithmic stratum $\mathcal{Z}$ of $(\mathcal{Y}', \mathcal{E}')$. By Lemma 4.7 Irr^m_f is representable by a finite étale cover over logarithmic strata. Hence by assumption of $s^{0,r}_{f,\eta_{\mathcal{Z}}}(v) = 1$ and Corollary 3.4, the fibre $f'^{-1}(\tilde{y} \bmod \pi_v)$ is combinatorially r-cycle-split.

The closure $\widetilde{y}$ of y in $\mathcal{Y}'$ lies outside the Zariski closure of E'_{sing} (the singular locus of E'). Therefore f' is integral outside the closure of E'_{sing} by [Kat89, Cor. 4.4(ii)]. Hence, the fibre product $\mathcal{X}'_y := (\mathcal{X}', \mathcal{D}') \times_{f', (\mathcal{Y}', \mathcal{E}'), \widetilde{y}} (\operatorname{Spec} \mathcal{O}_{k_v})^\dagger$, taken in the category of Zariski log schemes, is fine. Its underlying scheme agrees with the fibre product in schemes [Kat89, (1.6)]. Since $\widetilde{y}$ intersects $\mathcal{E}'$ transversally, it follows that $\widetilde{y} : (\operatorname{Spec} \mathcal{O}_{k_v})^\dagger \to (\mathcal{Y}', \mathcal{E}')$ is a saturated morphism as in [Tsu19]. Hence by [Tsu19, I.3.14], $\mathcal{X}'_y \to (\mathcal{X}', \mathcal{D}')$ is saturated and so is $\mathcal{X}'_y$ [Tsu19, II.2.12]. Thus $\mathcal{X}'_y$ coincides with the fibre product taken in the category of fs log schemes.

Log smoothness is stable under fs base change [GR18, Proposition 12.3.24], so $\mathcal{X}'_y$ is log regular, being log smooth over the log regular base $(\operatorname{Spec} \mathcal{O}_{k_v})^\dagger$ [Kat94, Theorem 8.2]. It follows that $\mathcal{X}'_y$ is Cohen-Macaulay and in particular normal [Kat94, Theorem 4.1].

That $f'^{-1}(y) = f^{-1}(y)$ is r-cycle-split with a witness Z of $\operatorname{maxdeg} Z \leq B$ now follows from its reduction being combinatorially r-cycle-split and Lemma 2.14. □

The main result Theorem 1.2 reformulated for any $r \in \mathbb{N}$ is now an easy corollary of Proposition 4.6 and Proposition 4.10.

Theorem 4.11. *Let $f : X \to Y$ be a dominant morphism between proper, smooth, geometrically integral varieties over a number field k with geometrically integral generic fibre.*

Then f is arithmetically r-cycle-surjective outside a finite set S, if and only if for each modification $f' : X' \to Y'$ and for each codimension 1 point ϑ' in Y', the fibre $f'^{-1}(\vartheta')$ is combinatorially r-cycle-split.

Remark 4.12. The above result cannot be applied directly to Conjecture 1.3, which requires to prove that the exceptional set S in Theorem 1.2 is empty. We can nevertheless say the following.

In contrast to the case of Theorem 1.1, the set S for which we prove Theorem 1.2 does not depend on Lang-Weil estimates but only on the existence of a sufficiently nice log smooth model of f as stated in Section 4.3. However, the existence of such models remains open. As far as zero-cycles are concerned, one may try to construct log smooth models by allowing alterations of f instead of modifications and [Tem17] contains strong results in this direction. Unfortunately, even those models do not suffice since the creation of codimension 1 logarithmic strata is not controlled.

Remark 4.13. Because the criterion of the preceding main theorem is stable under extensions of the ground field k, we could have also defined r-cycle-surjective to mean the existence of a zero-cycle of degree r on each fibre over *closed* points of Y_{k_v} (instead of fibres over k_v*-rational* points as in Definition 1.5). The criterion of Theorem 4.11 then shows that either definition leads to equivalent notions of arithmetic r-cycle-surjectivity.

While using closed points is arguably the more natural definition, we prefer to keep Definition 1.5 in analogy with [LSS19].

Example 4.14. We give an example of a morphism for which one can show that it is arithmetically cycle-surjective but not arithmetically surjective.

Let $A = \oplus_{i=1}^{n} k_i$ be a finite étale algebra over a number field k. Assume that A is almost everywhere locally cycle-split but not pseudo-split (e.g. one of the algebras in Examples 2.5 and 2.6). Then one can define the multinorm torus $\mathrm{R}^1_{A/k}\mathbb{G}_m$ through

$$0 \to \mathrm{R}^1_{A/k}\mathbb{G}_m \to \mathrm{R}_{A/k}\mathbb{G}_m \xrightarrow{N_{A/k}} \mathbb{G}_m \to 0$$

where the middle term maps to $\mathbb{G}_m$ via the norm maps.

The 1-parameter family of torsors for $\mathrm{R}^1_{A/k}\mathbb{G}_m$ given by

$$\mathrm{N}_{A/k}(x) = t \neq 0$$

can be compactified to a proper, smooth, geometrically integral variety X with a morphism f to $\mathbb{P}^1_k$.

It is easy to see that for all $v \notin S$, all smooth fibres over k_v-points have a zero-cycle of degree 1. Hence, f is arithmetically cycle-surjective. On the other hand, since $A \otimes_k k_v$ is non-split for infinitely many v, it follows from

[LS16, Lemma 5.4], that f is not arithmetically surjective.

A

Foundations of logarithmic geometry

Logarithmic geometry (or short: log geometry) has been dubbed by K. Kato, who together with P. Deligne, G. Faltings, J.-M. Fontaine and L. Illusie is one of the founding fathers of the subject, the “magic powder” of algebraic geometry. Its basic structure, the log scheme, is an enrichment of the structure of classical schemes that, in vague terms, remembers information on degeneration and by doing so allows us to treat non-smooth situations almost as if they were smooth.

The aim of this appendix is to explain this statement and enable the reader to follow the arguments in the main body of Part I. The experienced reader may ignore the appendix and read the main body on its own. We will give a brief overview of the foundations of log geometry without any proofs. The selection will be idiosyncratic because we restrict ourselves to the tools needed in the present work. More complete treatments can be found in [Ogu18, ACG$^+$13, GR18], as well as in the original sources [Kat89, Kat94, Niz06].

A.1 MONOIDS AND LOG SCHEMES

In a similar way in which commutative rings are the local objects of scheme theory, monoids underlie the local theory of log schemes.

Definition A.1.

- (*i*) A *monoid* is a set P with a commutative, associative binary operation $\cdot$ and a neutral element $1 \in P$. A *morphism of monoids* is a map that preserves the binary operations and neutral elements.
- (*ii*) The *group envelope* P^{gp} of a monoid P is the group

$$P^{\mathrm{gp}} = \{(x, y) \in P \times P | (x, y) \sim (w, z) \text{ if } \exists s \in P : sxz = syw\}.$$

The functor which associates to P its group envelope is the left adjoint functor to the inclusion functor of groups into monoids. There is a natural map of monoids $P \to P^{\mathrm{gp}}$.

The fundamental example of a monoid is $P = \mathbb{N}$ with addition as composition. Its group envelope is $\mathbb{Z}$.

As for rings, one can develop a geometry of monoids.

Definition A.2. Let P be a monoid.

- (*i*) An *ideal* of P is a subset $I \subseteq P$ closed under the composition law.
- (*ii*) A *prime ideal* of P is an ideal $I \subsetneq P$ such that for all $x, y \in P$ with $xy \in I$, we have $x \in I$ or $y \in I$.
- (*iii*) The *spectrum* $\operatorname{Spec} P$ is the set of prime ideals of P. It is equipped with a topology for which the closed sets are of the form

$$V(I) = \{J \subsetneq P \text{ prime} | I \subseteq J\}$$

for some ideal $I \subseteq P$.

The spectrum of $\mathbb{N}$ consists of a generic point $\emptyset$ and a closed point $\mathbb{N}_{>0}$.

We define some basic properties of monoids.

Definition A.3. A monoid P is

(*i*) *integral*, if $P \to P^{\mathrm{gp}}$ is injective. The functor which associates to P its image P^{int} in P^{gp} is the left adjoint functor to the inclusion functor of integral monoids in monoids.

(*ii*) *saturated*, if it is integral and for any $p \in P^{\mathrm{gp}}$ and positive integer n, $p^n \in P$ implies $p \in P$. The functor which associates to an integral module P the monoid $P^{\mathrm{sat}} = \{p \in P^{\mathrm{gp}} : p^n \in P\}$ is the left adjoint functor to the inclusion functor of saturated monoids in integral monoids.

(*iii*) *fine*, if it is integral and finitely generated (as a monoid).

(*iv*) *fs*, if it is fine and saturated.

(*v*) *free*, if it is isomorphic to a power of $\mathbb{N}$.

Let $q_1 : P \to Q_1$ and $q_2 : P \to Q_2$ be morphisms of monoids. The *pushout* $Q_1 \oplus_{q_1,P,q_2} Q_2$ exists in the category of monoids and is given by the quotient of $Q_1 \times Q_2$ by the smallest equivalence relation which is preserved by $\cdot$ and contains $(1, q_2(x)) \sim (q_1(x), 1)$ for all $x \in P$. In general, the pushout is not easy to compute and does not preserve the full subcategories of integral, fine and saturated monoids. We will return to this latter problem in the treatment of fibre products of log schemes.

Definition A.4. A Zariski (resp. étale) *pre-log scheme* is a scheme X together with a *log structure*, i.e. a Zariski (resp. étale) sheaf of monoids $\mathcal{M}_X$ on X and a structure morphism $\alpha : \mathcal{M}_X \to \mathcal{O}_X$. It is called a *log scheme* if α restricts to an isomorphism on $\alpha^{-1}(\mathcal{O}_X^*)$. (Thus, one can view $\mathcal{O}_X^*$ as a subsheaf of $\mathcal{M}_X$).

A *morphism between pre-log structures* $\mathcal{M} \to \mathcal{O}_X$ and $\mathcal{M}' \to \mathcal{O}_X$ is a morphism $\mathcal{M} \to \mathcal{M}'$ compatible with the structure morphisms.

A *morphism of pre-log schemes* $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a morphism of the underlying schemes together with a morphism $f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that the following diagram commutes

$$\begin{tikzcd}
f^{-1}\mathcal{M}_Y \arrow[r] \arrow[d] & \mathcal{M}_X \arrow[d] \\
f^{-1}\mathcal{O}_Y \arrow[r, "f^{\#}"] & \mathcal{O}_X.
\end{tikzcd}$$

A *morphism of log structures* (resp. *log schemes*) is defined as a morphism of pre-log structures (resp. pre-log schemes).

We will work with Zariski log schemes. The pushout can easily be defined for sheaves of monoids by taking the pushout in presheaves and sheafifying. Using this construction, one can associate a log structure to a pre-log structure and this procedure yields a left adjoint functor $(\_)^{\log}$ to the inclusion of log structures in pre-log structures.

Definition A.5. Let $f : X \to Y$ be a morphism of schemes and $(Y, \mathcal{M}_Y)$ a log scheme. We can define an *inverse image functor* by

$$f^*(\mathcal{M}_Y) = (f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X)^{\log}.$$

A direct image can be defined similarly. With the above definition in mind, a morphism of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is nothing else but a morphism of the underlying schemes together with a morphism $f^b : f^*\mathcal{O}_Y \to \mathcal{O}_X$. We call f *strict* if f^b is an isomorphism.

Intuitively, the sheaf of monoids $\mathcal{M}_X$ contains the elements of which we can take logarithmic derivatives (and there is a way to make this precise). The following examples are fundamental to the theory of logarithmic geometry.

Example A.6. (*i*) Every scheme X can be equipped with the *trivial log structure* $\mathcal{M}_X = \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$, an inital object among all log structures on X. (There also is the terminal log structure $\mathcal{M}_X = \mathcal{O}_X \xrightarrow{\text{id}} \mathcal{O}_X$, which in general however is not fine in the sense below and hence less useful.)

(*ii*) For a monoid P and a commutative ring R, let $R[P]$ be the monoid algebra. One can define a *canonical log structure* on $X = \operatorname{Spec} R[P]$

given by the map $P_X \to \mathcal{O}_X$ induced from $P \hookrightarrow R[P]$ where P_X is the constant sheaf on X with values in P.

(*iii*) For a locally Noetherian scheme X and a divisor $D \hookrightarrow X$ with complement $U = X \setminus D \xrightarrow{j} X$, the log scheme (X, D) is given by the *divisorial log structure* $\alpha : j_*\mathcal{O}_U^* \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$. This is one of the prime sources for logarithmic schemes that the reader should keep in mind. We will return to this example throughout the appendix. A special case is the divisorial log structure $(\operatorname{Spec} R)^\dagger$ on the spectrum of a local ring R induced by its maximal ideal.

(*iv*) Let $r \in \mathbb{N}$ and let k be a field. A *log point* is the log scheme $\operatorname{Spec} k$ with log structure

$$k^* \oplus \mathbb{N}^r \to k, \quad (a, n_1, \ldots, n_r) \mapsto a \cdot 0^{n_1 + \ldots + n_r}$$

where by convention $0^0 = 1$. If $r = 1$, we write $(\operatorname{Spec} k)^\dagger$ for the log scheme and call it the *standard log point*.

To study the local constituents of log schemes, one uses so-called charts.

Definition A.7. Let P be a monoid and let $(X, \mathcal{M}_X)$ be a log scheme. To a morphism $P \to \Gamma(X, \mathcal{M}_X)$ we can associate a pre-log structure $P_X \to \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_X$. The associated log structure $P^{\log}$ has a natural morphism to $\mathcal{M}_X$. We say that $P \to \Gamma(X, \mathcal{M}_X)$ is a *chart subordinate to* P, if $P^{\log} \to \mathcal{M}_X$ is an isomorphism.

Equivalently, a *chart* is given by a strict morphism $(X, \mathcal{M}_X) \to \operatorname{Spec} \mathbb{Z}[P]$.

Using charts we can now define properties of log schemes.

Definition A.8. A (Zariski or étale) log scheme is *coherent*, resp. *fine*, resp. *fs* if (Zariski or étale) locally, there exists a chart subordinate to a *finitely generated*, resp. *fine*, resp. *fs* monoid.

There are left adjoint functors $(\_)^{\text{fine}}$ and $(\_)^{\text{fs}}$ to the natural categorical inclusions of fine and fs log schemes in coherent log schemes.

An important subtlety of log geometry which can cause great pain but is not to be ignored is that the fibre product depends on whether we work in the category of coherent, fine or fs log schemes. The fibre product of coherent log schemes is given by the fibre product of the underlying schemes together with the pushout log structure. However, this does not preserve the fine and fs properties, and so the fibre product of fine/fs schemes is given by applying $(\_)^{\mathrm{fine}}/(\_)^{\mathrm{fs}}$ to the fibre product in coherent log schemes. This can change the underlying scheme!

A.2 LOG REGULARITY AND LOG SMOOTHNESS

After having set up the foundations of log schemes, we are in a position to fulfil the promise of "magic powder" by defining log versions of regularity and smoothness.

Definition A.9. An fs log scheme $(X, \mathcal{M}_X)$ is *log regular at* $x \in X$, if with $I(x, \mathcal{M}) \subset \mathcal{O}_{X,x}$ denoting the ideal generated by the image of $\mathcal{M}_{X,x} \setminus \mathcal{O}^*_{X,x}$,

(*i*) the ring $\mathcal{O}_{X,x}/I(x, \mathcal{M}_X)$ is regular and

(*ii*) the equality

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}/I(x, \mathcal{M}_X) + \operatorname{rk}(\mathcal{M}^{\mathrm{gp}}_{X,x}/\mathcal{O}^*_{X,x})$$

holds.

We call $(X, \mathcal{M}_X)$ *log regular* if it is log regular at each point $x \in X$.

Over a field k, there is an equivalent characterisation which requires that the completion of the local ring $\mathcal{O}_{X,x}$ be isomorphic to a formal power series ring $k(x)[[P]][[T_1, \ldots, T_r]]$ (defined by $k[[P]] = \varprojlim_n R[P]/(P \backslash P^\times)^n R[P]$) for some chart subordinate to a torsion-free fs monoid P with trivial $P^\times$. The reader may compare this to analogous results for regular schemes. Intuitively, log regularity is thus "regularity up to the local monoid". A scheme equipped with the trivial log structure is log regular if and only if it is regular. One also sees that if P is free, then the underlying scheme X is regular at x.

A closely related, older notion is that of toroidal embeddings as introduced by KEMPF, KNUDSON, MUMFORD and SAINT-DONAT [KKMSD73].

Definition A.10. A *toroidal embedding* over a perfect field k is a variety X over k together with a divisor $D \subset X$ such that étale locally around every point, there is an isomorphism of X to a toric variety such that D is identified with the toric boundary.

There is a one-to-one correspondence between toroidal embeddingsand étale log regular varieties over k. If we restrict to toroidal embeddings (X, D) such that D is strict, i.e. each of its irreducible components is normal, we get an equivalence to the category of Zariski log regular varieties over k. One can always recover $X \setminus D$ as the maximal open subscheme of X on which the log structure is trivial.

However, log regularity is more general in that it applies to schemes and not just varieties. For example, a regular scheme equipped with the divisorial log structure of a strict normal crossing divisor is log regular.

In analogy to toric varieties, one can attach a special notion of fan to log regular schemes.

Definition A.11. (*i*) A *Kato fan* is a locally monoidal topological space which has a cover by spectra of fs monoids. These spectra are also called *Kato cones*. A *morphism of Kato fans* is a morphism of locally monoidal spaces.

(*ii*) A Kato fan is *smooth* if it can be covered by spectra of free monoids.

(*iii*) To a log regular fs scheme $(X, \mathcal{M}_X)$, we associate a Kato fan as follows. The underlying set of points is

$$F_X = \{x \in X | I(x, \mathcal{M}) \subset \mathcal{O}_{X,x} \text{ is maximal}\}.$$

We equip it with the inverse image of the topology on X and the inverse image of the sheaf $\mathcal{M}_X / \mathcal{O}_X^*$.

(*iv*) If f is a morphism of log regular schemes, we write $F(f)$ for the natural morphism induced on the associated Kato fans.

In the language of a toroidal embedding or strict normal crossing pair (X, D), the points of the associated Kato fan are exactly the generic points of repeated intersections of the components of D.

If $(X, \mathcal{M}_X)$ is a log regular scheme, there is a map $c_X : (X, \mathcal{M}_X/\mathcal{O}_X^*) \to F_X$ of locally monoidal spaces which gives rise to a stratification.

Definition A.12. Let $(X, \mathcal{M}_X)$ be a log regular scheme. Then for $x \in F_X$, the preimage $U(x) = c_X^{-1}(x)$ is a locally closed subset of X called the *logarithmic stratum* attached to x.

The logarithmic strata of a pair (X, D) as above are described easily. If $(D_i)_{i\in I}$ are the irreducible components of D, the logarithmic strata are given by the connected components of

$$\left(\bigcap_{j\in J} D_j\right) \setminus \left(\bigcup_{j\in I\setminus J} D_j\right)$$

where J runs over all subsets of I. The points of the Kato fan are exactly the generic points of the strata.

Given a morphism of smooth Kato fans $x : \operatorname{Spec}\mathbb{N} \to F$, the closed point $\mathbb{N}_{>0}$ is sent to the closed point of a unique Kato subcone $\operatorname{Spec}\mathbb{N}^r$ of F for some suitable integer r.

Definition A.13. The *height* $h(x)$ of the $\mathbb{N}$-valued point $x \in F(\mathbb{N})$ is defined as the sum of the images of the generators of $\mathbb{N}^r$ under the map $\mathbb{N}^r \to \mathbb{N}$ induced by x.

A particular source of $\mathbb{N}$-valued points is as follows. If X is a flat, proper, regular scheme over a discrete valuation ring R such that the special fibre is a strict normal crossing divisor, the log scheme with the induced divisorial log structure is log regular with a smooth Kato fan. Now a section $\tilde{x} \in X(R)$ induces a point $x \in F_X(\mathbb{N})$ and the height $h(x)$ is the intersection number of $\tilde{x}$ with the special fibre.

By the preceding characterisation of log regularity, a log regular scheme with a smooth Kato fan is regular in the classical sense. Thus the resolution of singularities for log regular schemes reduces to finding modifications such that the Kato fan becomes smooth. This is achieved by subdivisions of the Kato fan, in a way similar to resolution of singularities for toric varieties.

Definition A.14. A *subdivision* of a Kato fan is a morphism $F' \to F$ which is surjective on the group enevelopes of stalks and injective on $\mathbb{N}$-valued points. A subdivision is called *proper*, if it has finite fibres and $F'(\mathbb{N}) \to F(\mathbb{N})$ is a bijection.

One can show that a Kato fan always has a subdivision which is smooth.

The relationship of subdivisions to log schemes is as follows. If $(X, \mathcal{M}_X)$ is a log regular scheme and $f : F' \to F_X$ a subdivision, then there exists a morphism of log regular schemes $(X', \mathcal{M}'_X) \to (X, \mathcal{M}_X)$ which induces the morphism f on the associated fans. We call this the *log blow-up of* $(X, \mathcal{M}_X)$ *along* f. It is a birational morphism of schemes and proper if f is proper.

A particular type of subdivision on a smooth fan, which we only mention en passant, is the *barycentric subdivision* [ACM⁺16, Example 4.10.(ii)]. On a pair (X, D) it corresponds to iterated classical blow-ups of X in the proper transforms of $\bigcap_{j \in J} D_j$ running over all $J \subset I$ ordered by increasing dimension.

Arguably, the most surprising property of log blow-ups is that they are log étale morphisms in the sense of the next definition.

Definition A.15. Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of (Zariski or étale) fs log schemes. Consider commutative diagrams of fine log schemes of the form

$$\begin{tikzcd}
(T_0, \mathcal{M}_0) \arrow[r] \arrow[d] & (X, \mathcal{M}_X) \arrow[d] \\
(T_1, \mathcal{M}_1) \arrow[r] & (Y, \mathcal{M}_Y)
\end{tikzcd}$$

where $T_0 \to T_1$ is a closed immersion defined by an ideal I with $I^2 = 0$ and $(T_0, \mathcal{M}_0) \to (T_1, \mathcal{M}_1)$ is strict. Then f is called *log smooth* (resp. *log étale*) if (*i*) it is locally of finite presentation on the underlying schemes and

(*ii*) for each diagram as above, the following infinitesimal lifting property holds: (Zariski or étale) locally on T_1 there exists at least one (resp. exactly one) morphism $(T_1, \mathcal{M}_1) \to (X, \mathcal{M}_X)$ making the diagram commute.

There exists an analogue of a relative sheaf of logarithmic differentials Ω^1_f with a universal logarithmic derivation, which behaves similarly to relative sheaves of differentials in the smooth case, but we will not need it. In the language of toroidal embeddings, log smooth morphisms correspond to toroidal morphisms.

From the lifting criterion of log smoothness, one deduces a logarithmic version of Hensel's lemma (see [Cao16] or [LSS19, §5.2]).

Lemma A.16. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a log smooth morphism of fs log schemes. Let R be a complete discrete valuation ring with residue field k. Assume that we have a commutative diagram*

$$\begin{tikzcd}
(\operatorname{Spec} k)^\dagger \arrow[r] \arrow[d, hook] & (X, \mathcal{M}_X) \arrow[d] \\
(\operatorname{Spec} R)^\dagger \arrow[r] & (Y, \mathcal{M}_Y)
\end{tikzcd}$$

where the left vertical arrow is the natural inclusion map. Then there exists a lift $(\operatorname{Spec} R)^\dagger \to (X, \mathcal{M}_X)$ making the diagram commute.

The relation between log smoothness and log regularity is as expected: An fs log scheme with a log smooth map to a log regular scheme is itself log regular. In particular, a log smooth scheme over a field k with the trivial log structure is log regular, and the converse is true if k is perfect.

We end the appendix with a powerful, modern development outside the classical foundations of log geometry. The notion of log smoothness would be of limited (but of course still valuable) use, were it not for the next theorem by D. Abramovich and K. Karu [AK00], who show that after a modification, we can always get ourselves in a log smooth situation. Of course, the same statement would fail terribly for classical smoothness.

Theorem A.17. *Let $f : X \to Y$ be a dominant morphism of integral varieties over an algebraically closed field of characteristic 0. Let $Z \subset X$ be a proper closed subset. Then there exist a modification $f' : X' \to Y'$ of f and strict normal crossing divisors $D' \subset X'$ and $E' \subset Y'$ such that*

(*i*) *the preimage of Z in X' is strict normal crossing and contained in D',*

(*ii*) *$f'^{-1}(Y' \setminus E') = X' \setminus D'$ and*

(*iii*) *f induces a log smooth morphism of log regular varieties $(X', D') \to (Y', E')$.*

A later version of Abramovich-Karu-Denef [ADK13] shows that the assumption that k be algebraically closed can be dropped. On the other hand, the characteristic of the field is used in the construction in an essential way. However, work by Illusie–Temkin [IT14] and Temkin [Tem17] has proved a generalisation for schemes in mixed characteristic: Let S be the set of all primes appearing as the characteristic of residue fields of points in Y. Then one can still achieve log smoothness if instead of a modification, one allows an alteration of degree d where d is only divided by primes in S.

Part II

Mazur's Conjecture and an Unexpected Rational Curve on Kummer Surfaces and Their Superelliptic Generalisations

5

INTRODUCTION

In the study of the distribution of rational points on varieties, two methods are frequently used to generate new points from existing ones: One can apply automorphisms defined over the ground field, e.g. arising from a group law on an elliptic curve. Or one can look for rational subvarieties that will be guaranteed to have many rational points. Often, a combination of both is needed. The prevalence of these methods is paramount to the whole subject.

A famous, successful example is ELKIES' solution to Euler's conjecture on $A^4 + B^4 + C^4 = D^4$ [Elk88]. In this part of the thesis, we consider another example given by KUWATA and WANG in [KW93]. Let A be an abelian variety which is the product of two elliptic curves E_1 and E_2 over $\mathbb{Q}$. Assume that E_1 and E_2 do not both have equal j-invariants 0 or 1728. Let

$$\begin{aligned} E_1 &: y_1^2 = x^3 + ax + b =: g(x) \\ E_2 &: y_2^2 = t^3 + ct + d =: f(t) \end{aligned}$$

be affine equations for the elliptic curves in Weierstrass form (in particular $a = b = 0$ and $c = d = 0$ are excluded). The assumption on the j-invariant excludes exactly the cases $a = c = 0$ and $b = d = 0$. An affine model of the

Kummer surface K associated to A is given after setting $y = y_1/y_2$:

$$K : (t^3 + ct + d)y^2 = x^3 + ax + b.$$

On this surface, [KW93, §1] constructs a parametric curve C as the scheme-theoretic image of the morphism

$$\sigma : \mathbb{P}^1 \to K, u \mapsto (x, y, t)(u) := \left(\frac{du^6 - b}{a - cu^4}, u^3, \frac{du^6 - b}{u^2(a - cu^4)} \right).$$

Using this curve, one can prove the following theorem:

Theorem 5.1. *[KW93, Theorem 3] The set of rational points on K is dense in the Zariski and real topologies.*

This verifies, for the surface K, Mazur's conjecture on the topology of rational points: For any smooth variety V over $\mathbb{Q}$, if the rational points are Zariski dense in V, then their topological closure in the real locus $V(\mathbb{R})$ of V is a union of real connected components of $V(\mathbb{R})$ [Maz92, Maz95]. It has been shown by a concrete counterexample [CTSSD97] that Mazur's conjecture does not hold without further assumptions on the variety, although refined versions have been proposed that so far have resisted attempts at disproving them.

The same curve C or rather its preimage C' on A was also independently found by Mestre in [Mes92] and used to prove that there are infinitely many elliptic curves over $\mathbb{Q}$ of rank at least 2 with a fixed j-invariant.

The appearance of C is somewhat surprising and mysterious, given that the construction of K starts with two generic elliptic curves and a priori there is not much reason to expect a rational curve over $\mathbb{Q}$ on it apart from the obvious ones.

The discovery that prompted the present results is that the curve C found by Mestre and Kuwata-Wang arises from a rather simple equation, which generalises to a wider class of surfaces. The precise statements and applications are contained in Sections 6-7, containing to the author's knowledge the first known case of Mazur's conjecture dealing with a class of quadratic twists of an elliptic curve by a quartic polynomial in Theorem 7.5.

The last section does not utilise the curve C and exhibits a proof of Mazur's conjecture for the Kummer surface K without any assumptions on the j-invariants.

For the questions discussed in Part II, it is not necessary to have projective models. We will thus mostly work with affine models that yield a dense open subvariety of the respective surface or curve. In our terminology, an affine, not necessarily geometrically irreducible curve has genus 0 if it has a birational map to a projective curve whose desingularisation has genus 0. A rational curve will always be an integral genus 0 curve with a smooth rational point over the ground field.

After the publication of Part II, the author was kindly informed by M. Ulas that the curve considered in Theorem 6.1 had previously been discovered by him [Ula07, Lem. 2.1].

6

A Rational Curve on K and Superelliptic Generalisations

Theorem 6.1. *Let* D_1 *and* D_2 *be two superelliptic curves over a field with arbitrary characteristic of the form*

$$D_1 : y_1^k = x^n + ax + b$$

$$D_2 : y_2^k = t^n + ct + d,$$

with a *or* b *nonzero and* c *or* d *nonzero. The group* μ_k *of* k*-th roots of unity acts diagonally on* $D_1 \times D_2$*. Let*

$$X = (D_1 \times D_2)/\mu_k.$$

An affine equation of X *is given by*

$$(t^n + ct + d)y^k = x^n + ax + b.$$

Then there exists a genus 0 *curve* C *on* X *which is the closure of the subvariety of* X *cut out by the affine equation*

$$(ct+d)y^k = ax+b.$$

Moreover, if a *and* c *are not both equal to* 0*,* C *has a rational component. If* k *and* n *are coprime,* $b \neq 0$ *and* $a^n d^{n-1} - b^{n-1} c^n \neq 0$*, then* C *is geometrically irreducible.*

The condition $a^n d^{n-1} - b^{n-1} c^n \neq 0$ excludes the cases when there is an isomorphism between D_1 and D_2 that is compatible with the μ_k-action.

For $k = 2$ and $n = 3$, this recovers MESTRE'S and KUWATA-WANG'S curve on K. In this special case, these equations already appear in [Sat01] (cf. Section 6.1.1 below).

Proof. We derive an alternative affine model of C after which a brief analysis of the geometrically irreducible components yields the desired results. A transformation of the equations for C gives

$$\begin{aligned} \frac{ax+b}{ct+d} &= \left(\frac{x}{t}\right)^n, \\ y^k &= \left(\frac{x}{t}\right)^n. \end{aligned}$$

Setting $r := x/t$, the first equation is equivalent to

$$tr(cr^{n-1} - a) = b - dr^n$$

which defines a plane curve $\tilde{C}$ in the variables (r, t). Note that this equation is linear in t. We distinguish three different cases:

(i) There exists no point $(r_0, t_0) \in \tilde{C}$ with $r_0(cr_0^{n-1} - a) = 0$: In this case

$$\pi : \mathbb{P}^1 \to \tilde{C} : r \mapsto (r, (b - dr^n)/(r(cr^{n-1} - a))$$

defines a birational map, hence $\tilde{C}$ is a rational curve parametrised by r.

(ii) There exist points $(r_0, t_0) \in \tilde{C}$ with $r_0(cr_0^{n-1} - a) = 0$, and neither $a = c = 0$ nor $b = d = 0$: If $r_0 = 0$, we must have $b = 0$. If $cr_0^{n-1} - a = 0$, we must have $r_0^n d = b$ and thus $a^n d^{n-1} - b^{n-1}c^n = 0$. The map π is from above is non-constant and yields a component of $\tilde{C}$ parametrised by r. However, additional components with $r = r_0$ appear, onto which π does not map dominantly.

(iii) $a = c = 0$ or $b = d = 0$: If $a = c = 0$, then $y^k = r^n = b/d$ and thus C decomposes into components with constant y and r. If $b = d = 0$, then C has three components cut out by $r^{n-1} = a/c$, $x = y = 0$ and $x = t = 0$ respectively.

From now on, assume that we are in one of the first two cases and let $\tilde{C}_1$ be the closure of the image of π in $\tilde{C}$. Since C is obtained from $\tilde{C}$ by the affine equation $y^k = r^n$ and r is locally constant outside $\tilde{C}_1$, we only have to consider $\tilde{C}_1$.

Let p be the characteristic exponent of the ground field. Let s be the p-primary part of the greatest common divisor of k and n, so that $k = sp^i k'$ and $n = sp^i n'$ where $(k', n') = 1$. Then geometrically, C decomposes into components

$$C_\zeta : (y^{k'} - \zeta r^{n'})^{p^i} = 0$$

where ζ runs over all s-th roots of unity. For $\zeta = 1$, we get a reduced, geometrically irreducible component

$$C_1 : y^{k'} = r^{n'}$$

defined over the ground field since it is fixed by the Galois action. The curve C_1 is well-known to be rational and a parametrisation is given by $\theta \mapsto (r, y)(\theta) := (\theta^{k'}, \theta^{n'})$. The other C_ζ are Galois twists of C_1 and so have genus 0 too.

If k and n are coprime, i. e. $sp^i = 1$, then C coincides with the geometrically irreducible component C_1. □

A direct computation gives:

Theorem 6.2. *A parametrisation of C_1 is given by:*

$$\sigma : \mathbb{P}^1 \to C_1, u \mapsto (x, y, t)(u) = \left(\frac{du^{kn} - b}{a - cu^{kn-k}}, u^n, \frac{du^{kn} - b}{u^k(a - cu^{kn-k})}\right)$$

6.1 Further Remarks

6.1.1 Geometric Considerations involving $\mathbb{P}^1 \times \mathbb{P}^1$. The original example by Mestre has been studied by P. Satgé in [Sat01]. There, he utilises the natural map from K to $\mathbb{P}^1 \times \mathbb{P}^1$ together with the Riemann–Hurwitz theorem to develop a combinatorical criterion for when the preimage of a rational curve on the latter surface yields a rational curve on the former. Amongst the low-degree examples he retrieves with the help of this criterion is the Mestre curve.

6.1.2 Geometric Considerations involving A. A new different approach that we mention for geometric insight is to first understand the preimage C' on $A = E_1 \times E_2$. In what follows, we show how to derive that C has genus 0 by such arguments in the case of two *generic* elliptic curves, i. e. with distinct j-invariants and without complex multiplication.

Let O_1 and O_2 be the points at infinity of E_1 and E_2. The Néron-Severi group of $E_1 \times E_2$ is given by $\mathbb{Z}h \oplus \mathbb{Z}v$ where

$$h := E_1 \times \{O_2\},\ v := \{O_1\} \times E_2.$$

By Bézout's theorem, the intersection numbers of C' (cut out of A by a quadric in each of the factors of an embedding $E_1 \times E_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ by Weierstrass equations) with h and v are both 6, therefore the class of C' in the Néron–Severi group is $6h + 6v$. Hence by the adjunction formula we deduce $p_a(C') = 37$. We can compute the singularities of C': (O_1, O_2) is a singularity with multiplicity 4,

$$\{(p_1, p_2) \in E_1[2] \setminus O_1 \times E_2[2] \setminus O_2\}$$

is a set of 9 singularities with multiplicity 2 and $V(x=t=0)$ is a set of 4 singularities with multiplicity 3. All singularities are ordinary and C' does not pass through other torsion points than the ones mentioned. Hence C' has geometric genus 10.

We now use the Riemann-Hurwitz theorem. Before applying it to the double cover $C' \to C$, we first have to blow up the torsion points which are singular to get non-singular ramification points. If such a point P has multiplicity m_P, then in the resolution we will have m_P points of ramification index 2. Indeed, after doing this, in the case $j(E_1) \neq j(E_2)$, Riemann-Hurwitz substitutes to

$$18 = 2g(C')-2 = 2(g(C)-2)+\sum_{P\in\widetilde{C'}}(e_P-1) = 2(g(C)-2)+(1\cdot 4+9\cdot 2)(2-1)$$

and thus $g(C)=0$.

6.1.3 Degenerate cases.

In the cases of geometrically isomorphic D_1 and D_2 (i.e. $a^n d^{n-1} - b^{n-1}c^n = 0$ and in particular $a=c=0$ or $b=d=0$), C acquires geometric components which are the graphs of isomorphisms between D_1 and D_2.

6.2 TWISTS OF SUPERELLIPTIC CURVES

As a corollary of Theorem 6.1, we obtain similarly to [KW93, Thm. 3]:

Corollary 6.3. *Let D_1 and D_2 be superelliptic curves over a number field L of the same form as in Theorem 6.1. Assume that we are not in one of the cases $a=c=0$ or $b=d=0$. Then there exist infinitely many $[l] \in L^*/(L^*)^k$ such that the twists of D_1 and D_2 by $[l]$ both have an L-rational point.*

By *twist by* $[l]$, we mean in the case of D_1 the curve given by $ly_1^k = x^n + ax + b$ for a representative l in the class $[l]$, and analogously for D_2. Up to L-isomorphism, it does not depend on the chosen representative. In the special case of $k=2, n=4$ and $k=n=3$, the theorem is a statement about genus 1 models.

Proof. This proof follows the same idea as KUWATA-WANG but uses the newly found curve C on X. Let $x(u), y(u), t(u)$ be as in Theorem 6.2. For a superelliptic curve D, denote by D^l the twist by $l(L^*)^k$. Using C gives us infinitely many points (x, y, t) such that $D_1^{t^n+ct+d}$ and $D_2^{x^n+ax+b}$ have a rational point. Because $(t^n + ct + d)y^k = x^n + ax + b$, these are isomorphic to twists by the same class. We thus have a map

$$[\phi] : L^* \to L^*/(L^*)^k, u \mapsto (t(u)^n + ct(u) + d)(L^*)^k = (x(u)^n + ax(u) + b)(L^*)^k$$

such that $D_1^{[\phi](u)}$ and $D_2^{[\phi](u)}$ have a rational point.

Let $\phi(u) := t(u)^n + ct(u) + d$. It remains to show that $[\phi]$ does not have finite image. Suppose the image of $[\phi]$ is finite. Then there exists a finite set S of places of L such that $k \mid v(\phi(u))$ for all $u \in L, v \notin S$. This means by continuity that $v(\phi(u)) \equiv 0 \bmod k$ for all $u \in L_v$. However, since $\phi \notin (L(u)^*)^k$ as a rational function (just by computing its numerator and denominator), there exists a point $P \in \mathbb{P}^1_L$ such that ϕ has multiplicity m prime to k at P. Let $L(P)/L$ be the residue field extension of P. There are infinitely many places of L that split completely in $L(P)$, so pick one $v \notin S$ amongst them and denote by w an extension of v to $L(P)$. Now ϕ has a zero or pole P of multiplicity m in $\mathbb{P}^1_{L(P)_w} = \mathbb{P}^1_{L_v}$ and in a neighbourhood of P, $v(\phi(u))$ cannot be divisible by k, yielding a contradiction. $\square$

7

Further Generalisations

The equation for C in Theorem 6.1 gives rise to rational curves on an even wider class of surfaces where the exponents of x and t are chosen differently. Some of these curves have genus 0 but do not contain a rational point. We give a few interesting examples and applications.

7.1 Elliptic Curves with j-invariant 1728

Let E be an elliptic curve with j-invariant 1728 over a field F of characteristic $\neq 2, 3$. Let

$$E : y^2 = x^3 + ax$$

be an affine model of E in Weierstrass form, in particular $a \neq 0$, and $f(t) := t^4 + ct + d$ a polynomial with rational coefficients. Assume $c, d \neq 0$. Quadratic twisting by $f(t)$ yields an elliptic pencil $E^{f(t)}$. The situation at the degenerate fibres is irrelevant for our purposes.

Theorem 7.1. *The surface over F which is the total space of the pencil $E^{f(t)}$ contains a curve C given by $(ct + d)y^2 = ax$ with an irreducible component C_1 given by $x = y = 0$ and another rational irreducible component C_2.*

Proof. The proof method is similar to that of Theorem 6.1 and we only sketch the steps. In an affine model, a transformation of the equations for C gives

$$\begin{aligned} \frac{ax}{ct+d} &= \frac{x^3}{t^4}, \\ y^2 &= \frac{x^3}{t^4}. \end{aligned}$$

Setting $r = x/t^2$ and $y_2 = y/t$, this becomes

$$\begin{aligned} \frac{a}{ct+d} &= r^2, \\ y_2^2 &= r^3. \end{aligned}$$

Because of our assumption that $c \neq 0$, the first line is equivalent to

$$t = \frac{a}{cr^2} - \frac{d}{c}$$

and defines a plane rational curve $\tilde{C}$ in the variables (r, t) parametrised by r. The component C_2 then is a cover of $\tilde{C}$ given by $y_2^2 = r^3$, so is itself rational. □

A direct computation gives:

Theorem 7.2. *A parametrisation of C_2 is given by:*

$$\begin{aligned} \sigma :&\mathbb{P}^1 \to C, \\ &u \mapsto (x, y, t)(u) \\ &= \left(\frac{(d^2/c^4)u^8 - 2dau^4 + c^4a^2}{u^6}, \frac{(-d/c^4)u^4 + a}{u}, \frac{(-d/c)u^4 + c^3a}{u^4} \right) \end{aligned}$$

In what follows we fix the parametrisation σ above.

Lemma 7.3. *Assume $F = \mathbb{Q}$. The set of $u \in \mathbb{Q}$ such that $\sigma(u)$ has infinite order in its fibre $E^{f(t(u))}$ is dense in $\mathbb{R}$.*

Proof. Define $E'_u : f(t(u))y^2 = g(x)$, a family of elliptic curves parametrised by u. It has a section $\sigma'(u) := (x(u), y(u))$. After a finite base change

$k(\sqrt{f(t(u))})/k(u)$, this family becomes trivial and σ' is pulled back to the section $\sigma'' : u \mapsto (x(u), y(u)\sqrt{f(t(u))})$. We infer that σ'' is not a torsion section since it intersects the identity section for $u = 0$ but distinct torsion sections on elliptic surfaces have to be disjoint at smooth fibres ([Huy16, Rem. 11.3.8] – compare to the similar argument in [Mir89, VII.3.2] for singular fibres). Hence, σ' is not torsion either. Now the specialisation theorem ([Sil94, III.11.4]) says that for almost all u, $\sigma(u)$ is not torsion in its fibre. ☐

From this, one immediately deduces Zariski density of rational points:

Corollary 7.4. *Assume $F = \mathbb{Q}$. Infinitely many fibres of $E^{f(t)}$ have positive rank. More precisely, there is a set of $W \subset \mathbb{Q}$, which is dense in the half-interval $(-d/c, \infty)$ if $ac > 0$, respectively dense in $(-\infty, -d/c)$ if $ac < 0$, such that the $E^{f(t)}$ has positive rank for all $t \in W$.*

Proof. The respective half-intervals given above are the images of $u \mapsto t(u)$. Now use Lemma 7.3. ☐

Note that density of the positive rank fibres in $E^{f(t)}$ over a non-empty open interval should be true if E is any elliptic curve over $\mathbb{Q}$ and f is any polynomial with a real zero of odd order by a result of Rohrlich [Roh93, Thm. 2], conditional on the parity conjecture.

We deduce a new special case of Mazur's conjecture applied to elliptic pencils [Maz92, Conj. 4].

Theorem 7.5. *Let E be an elliptic curve over $\mathbb{Q}$ with j-invariant 1728 and let $f(t) = t^4 + ct + d$ be a quartic polynomial over $\mathbb{Q}$. Assume that $c, d \neq 0$ and $f(t)$ is non-negative for all $t \in \mathbb{R}$. Then the set of $t \in \mathbb{Q}$ with $\operatorname{rk} E^{f(t)} > 0$ is dense in $\mathbb{R}$.*

A result by ROHRLICH [Roh93, Thm. 3] settled the case of f being a quadratic polynomial using similar ideas as [KW93] for cubic polynomials. Theorem 7.5 complements KUWATA and WANG'S quartic example $(t^4 + 1)y^2 = x^3 - 4x$ [KW93, p. 121] which they derived from the work by ELKIES mentioned in the introduction. A recent preprint by HUANG [Hua18] deals with $d(t^4+1)y^2 = x^3-x$ for some d. By entirely different methods and under some additional assumption, [HS16, Prop. 1.1] proves Mazur's conjecture for the Kummer quotient associated to the product of non-trivial 2-coverings of elliptic curves.

Proof. View $E^{f(t)}$ as a genus 1 pencil E_x with respect to projection to x. A priori, the fibres do not have rational points but there are infinitely many which do. Namely, $O_u := (x(u), y(u), t(u))$ and $O_{-u} := (x(u), y(-u), t(u))$ are two (generically distinct) rational points in their fibre $E_{x(u)}$.

Now by the same argument as in Lemma 7.3, for some choice of $u_0 \in \mathbb{Q}$ the point $(x_0, y_0, t_0) := O_{-u_0}$ has infinite order in E_{x_0} with respect to the identity chosen as $O_{u_0} \in E_{x_0}$, as well as infinite order in $E^{f(t_0)}$. Using the group law on E_{x_0}, we spread O_{-u_0} to get a dense set T in a connected component of $E_{x_0}(\mathbb{R})$. By Mazur's torsion bound [Maz78] the rational points (x, y, t) that are torsion in their fibre $E^{f(t)}$ lie in a proper Zariski-closed subset S of the total space. The intersection $E^{f(t_0)} \cap S$ is finite because otherwise, one would have $E^{f(t_0)} \subset S$ but $O_{-u_0} \in E^{f(t_0)} \setminus S$. It follows that $T' := T \setminus S$ is dense in a connected component of $E_{x_0}(\mathbb{R})$. But by assumption on f, connected components of $E_{x_0}(\mathbb{R})$ project surjectively to t so that the image of T' projects densely to t. $\square$

7.2 Elliptic Curves with j-invariant 0

Let E be an elliptic curve with j-invariant 0 over a field F of characteristic $\neq 2, 3$. Let

$$E : y^2 = x^3 + b$$

be an affine model of E in Weierstrass form and $f(t) := t^6+ct+d$ a polynomial with rational coefficients. Assume $b, c \neq 0$. Quadratic twisting by $f(t)$ yields an elliptic pencil $E^{f(t)}$. Once again, the situation at the degenerate fibres is irrelevant for our purposes.

Theorem 7.6. *The surface which is the total space of the pencil $E^{f(t)}$ contains a curve given by $(ct+d)y^2 = b$ with a rational irreducible component.*

Proof. The proof method is similar to that of Theorem 6.1 and we only sketch the steps. In an affine model, a transformation of the equations for C gives

$$\begin{aligned}\frac{b}{ct+d} &= \frac{x^3}{t^6},\\ y^2 &= \frac{x^3}{t^6}.\end{aligned}$$

Setting $r = x/t^2$, this becomes

$$\begin{aligned}\frac{b}{ct+d} &= r^3,\\ y^2 &= r^3.\end{aligned}$$

Because of our assumption that $c \neq 0$, the first line is equivalent to

$$t = \frac{b - dr^3}{cr^3}$$

and defines a plane rational curve $\tilde{C}$ in the variables (r, t) parametrised by r. The rational component postulated in the theorem then is a cover of $\tilde{C}$ given by $y^2 = r^3$. $\square$

A direct computation gives:

Theorem 7.7. *A parametrisation is given by:*

$$\begin{aligned}\sigma :& \mathbb{P}^1 \to C,\\ &u \mapsto (x, y, t)(u)\\ &= \left(\left(\frac{d^2}{b^2c^2}u^{12} - \frac{2db^5}{c^2}u^6 + \frac{b^{12}}{c^2}\right)/u^{10}, u^3/b^3, \left(\frac{b^7}{c} - \frac{d}{c}u^6\right)/u^6\right).\end{aligned}$$

In what follows we fix the parametrisation σ above. We can then prove an analogue to Corollary 7.4.

Lemma 7.8. *Assume $d \neq 0$ and $F = \mathbb{Q}$. Then infinitely many fibres of $E^{f(t)}$ have positive rank. More precisely, there is a set $W \subset \mathbb{Q}$ which is dense in the half-interval $(-d/c, \infty)$ if $ac > 0$, respectively dense in $(-\infty, -d/c)$ if $ac < 0$, such that $E^{f(t)}$ has positive rank for all $t \in W$.*

Proof. By clearing denominators, the coefficients a, b and d can be assumed integral. We want to show that $\sigma(u)$ is non-torsion for a dense set of u. Set $u := k/l$ with coprime $k, l \in \mathbb{Z}$ and $s := ck^6$. Then an integral model of $E^{f(t(u))}$ is given by:

$$y'^2 = x'^3 + s^{24} f(t(u))d$$

where $y' = s^{12} f(t(u))^2 y$ and $x = s^8 f(t(u))x$. For l large enough $y'(u) = s^{12} f(t(u))^2 y(u)$ is not integral and thus by the Lutz-Nagell criterion [Sil09, VIII.7.2], $\sigma(u)$ cannot be a torsion point.

The respective half-intervals given above are the images of $u \mapsto t(u)$. $\square$

8

PROOF OF MAZUR'S CONJECTURE FOR THE KUMMER SURFACE OF A PRODUCT ABELIAN SURFACE

In [KW93, Thm. 3'], a sketch was given that extends Theorem 5.1 to a proof of Mazur's conjecture for all j-invariants. It has to proceed along lines different from Theorem 5.1 because the parametric curve is not available in the cases of equal j-invariants 0 or 1728. The strategy was to rely on the two elliptic pencils given by projections to x and t to spread rational points using the group laws. As communicated between the author and M. KUWATA, it is not clear whether this method is sufficient to get density in the real locus. We thus give a first proof.

Theorem 8.1. *Let K be the Kummer surface associated to the product of two arbitrary elliptic curves E_1 and E_2 over $\mathbb{Q}$. Assume the rational points are Zariski dense in K. Then they are dense in the real topology of K.*

Proof. Recall that an affine equation of K was given in Chapter 5 by

$$f(t)y^2 = g(x).$$

Let K_t and K_y be the fibrations given by projections to the respective coordinates. Note that only the first comes equipped with a section and thus a natural group law. The fibres of K_y are cubic curves and may not have a rational point.

Let $t_1 \in \mathbb{R}$ be arbitrary. If we show that for any $\epsilon > 0$, there exists an approximating $t' \in \mathbb{Q}$ with $|t' - t_1| < \epsilon$ such that the topological closure $\overline{K_{t'}(\mathbb{Q})}$ is $K_{t'}(\mathbb{R})$, then we are done.

Let S be the Zariski closure of the set of rational points on K that are torsion in their fibre K_t or torsion in their fibre K_y with respect to any of the inflection points chosen as identity. (The latter does not depend on the chosen inflection point since $3[I_1] = 3[I_2]$ in $\mathrm{Pic}(K_y)$ for any inflection points $I_1, I_2 \in K_y(\mathbb{C})$ where $[\cdot]$ denotes the class of a divisor modulo linear equivalence.)

Claim: $S \neq K$. Assume $Q = (x, y, t) \in S(\mathbb{Q})$ is torsion in its fibre K_t. Then by Mazur's torsion bound, Q lies in a proper closed subset S_1 of K.

Now assume $Q = (x, y, t) \in S(\mathbb{Q})$ is torsion in its fibre K_y with respect to some inflection point $I \in K_y(\mathbb{C})$. Then by Merel's torsion bound [Mer96] for the number field $k(I)$, there is a bound N (only depending on the uniformly bounded degree of the residue field $k(I)/\mathbb{Q}$) such that $n_Q Q = I$ for some positive $n_Q < N$. This can again be expressed by some necessary algebraic relations so that Q lies in a proper closed subset S_2 of K. This proves the claim since $S \subset S_1 \cup S_2$.

By assumption of Zariski density, there exists a point $P = (x_0, y_0, t_0) \in K(\mathbb{Q})$ outside of S. Because S is algebraic, we know that $K_{y_0} \cap S$ is finite. Otherwise, one would have $K_{y_0} \subset S$ which is impossible since $P \in K_{y_0} \setminus S$. In the same way, we conclude that $K_{t_0} \cap S$ is finite.

Multiples of P with respect to the group law on K_{t_0} are dense in the identity component of $K_{t_0}(\mathbb{R})$, which maps surjectively to the y-coordinate. Therefore we can replace P without loss of generality by one such multiple (x_0, y_0, t_0) which is not in S with arbitrarily small $|y_0|$. Using this we may make two assumptions about P:

- (i) We can assume $|y_0|$ is sufficiently small such that $K_{y_0}(\mathbb{R})$ is connected. To see this, after setting $u := \sqrt[3]{y_0} \in \mathbb{R}$ and $\tau := tu^2$, we can write $K_{y_0}(\mathbb{R})$ as:
$$\tau^3 + c\tau u^4 + du^6 = g(x).$$
This is a family of curves parameterised by u which is smooth in a neighbourhood of $u = 0$. By Ehresmann's lemma, for small $|u|$ (and hence small $|y_0|$) its fibre is homeomorphic to the real curve $\tau^3 = g(x)$, which in turn is homeomorphic to the connected real curve $v = g(x)$, where we set $v := \tau^3$.

- (ii) Moreover, if $g(x)$ has three real roots, we define $m < 0$ and $M > 0$ as local minimum and maximum of $g(x)$ and assume that $|y_0|$ is sufficiently small such that
$$m < f(t_1)y_0^2 < M,$$
where $t_1 \in \mathbb{R}$ is as in the beginning of the proof.

Choose some inflection point $I_0 \in K_{y_0}(\mathbb{C})$ as identity for the group law on K_{y_0}. Then by Lemma 8.2 below,

$$T := \{(3n+1)P | n \in \mathbb{Z}\} \subset K_{y_0}(\mathbb{Q}).$$

By Assumption ((i)), $K_{y_0}(\mathbb{R})$ is isomorphic to the real Lie group $\mathbb{R}/\mathbb{Z}$ and T is dense in it since P is not torsion in K_{y_0}. Let $T' := T \setminus S$. Because $(T \cap S) \subset (K_{y_0} \cap S)$ is finite, the set of rational points T' is also dense in $K_{y_0}(\mathbb{R})$.

We distinguish two cases to finish the proof of the theorem:

$g(x)$ **has only one real root:** Then $K_t(\mathbb{R})$ is connected for all $t \in \mathbb{R}$. We have to find a non-torsion point in $K_{t'}(\mathbb{Q})$ for some $t' \in \mathbb{Q}$ with $|t' - t_1| < \epsilon$.

The set T' is dense in $K_{y_0}(\mathbb{R})$ and the projection from $K_{y_0}(\mathbb{R})$ to the t-coordinate is surjective. Hence the image of T' under this projection is dense in $\mathbb{R}$ and we can find $(x', y_0, t') \in T'$ with $|t' - t_1| < \epsilon$.

$g(x)$ **has three real roots:** Then $K_t(\mathbb{R})$ has two connected components for all $t \in \mathbb{R}$ and we denote its non-identity component by $N_t(\mathbb{R})$. It remains to show the existence of a rational point $P' \in N_{t'}(\mathbb{R})$ of infinite order in $K_{t'}$ for some $t' \in \mathbb{Q}$ with $|t' - t_1| < \epsilon$.

Observe that $K_{y_0}(\mathbb{R}) \cap K_{t_1}(\mathbb{R})$ is the intersection of the elliptic curve $K_{t_1}(\mathbb{R})$ with the line $\{y = y_0\}$. By Assumption ((ii)), this intersection consists of three points, of which exactly two lie in the *oval* component $N_{t_1}(\mathbb{R})$. As K_{y_0} is connected and T' dense in $K_{y_0}(\mathbb{R})$, for any of these two intersection points $(x, y_0, t_1) \in N_{t_1}(\mathbb{R})$ we can find $P' = (x', y_0, t') \in T'$ such that $|t' - t_1| < \epsilon$ and $P' \in N_{t'}(\mathbb{R})$. ☐

In classical geometric terms, the following lemma spreads rational points using secants and tangents without the need of a group law defined over the ground field.

Lemma 8.2. *Let E be a plane cubic curve over a field F and let $P \in E(F)$. Let F'/F be a finite field extension and let $I \in E(F')$ be an inflection point. Equip $E_{F'}$ with the group structure with I as neutral point. Then for all $n \in \mathbb{Z}$, the multiple $(3n+1)P$ is F-rational.*

Proof. Denoting by $H \in \operatorname{Pic}(E)$ the class of a hyperplane section and by $[\cdot]$ the class of a divisor modulo linear equivalence, we have that

$$D := (3n+1)[P] - nH$$

has degree 1, so there exists a point $Q \in E(F)$ with $[Q] = D$. Then:

$$(3n+1)([I] - [P]) = [I] + nH - (3n+1)[P] = [I] - [Q].$$

☐

Remark 8.3. Relating the proof in the last section to the rest of Part II, it should be mentioned that there is no possibility of applying the method of using several elliptic fibrations to cases beyond K3. Only K3 and abelian surfaces can contain distinct elliptic fibrations with sections [SS10, Lem. 12.18]. In particular, the case of quintic f is out of reach.

Part III

On the Transcendental Brauer Group

9

INTRODUCTION

Let X be a smooth, projective, geometrically integral variety over a number field K. The *(cohomological) Brauer group*

$$\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X})$$

has been a fundamental object of study in the area of rational points since Y. MANIN [Man71] realised that its elements can often obstruct the Hasse principle or weak approximation on X. More recently, conjectures by A. VÁRILLY-ALVARADO [VA17] have focused on analogies between the Brauer group of K3 surfaces and the torsion group of elliptic curves over number fields.

It is thus an interesting question to be able to determine the Brauer group. To develop and apply methods that achieve this is the aim of this part of the thesis.

Let us briefly summarise the state of the art in computing the Brauer group. A first step in understanding its structure is the natural filtration

$$\mathrm{Br}_0(X) \subseteq \mathrm{Br}_1(X) \subseteq \mathrm{Br}(X)$$

whose terms are defined as follows. The subgroup of *constant* Brauer classes $\operatorname{Br}_0(X)$ is the image of the canonical map

$$\operatorname{Br}(k) \to \operatorname{Br}(X).$$

Constant classes never provide an obstruction and so one is usually interested in the quotient $\operatorname{Br}(X)/\operatorname{Br}_0(X)$. If X is isomorphic to a projective space, a smooth quadric hypersurface or smooth complete intersection of dimension ≥ 3, then $\operatorname{Br}_0(X) = \operatorname{Br}(X)$.

The *algebraic* Brauer group $\operatorname{Br}_1(X)$ is the kernel of the canonical map

$$\operatorname{Br}(X) \to \operatorname{Br}(\overline{X}),$$

where $\overline{X} = X \times_K \overline{K}$ is the base change of X to an algebraic closure $\overline{K}/K$. Since all elements of $\operatorname{Br}(X)$ are defined over the ground field K, the image of $\operatorname{Br}(X)$ in fact lies in the Galois invariant part $\operatorname{Br}(\overline{X})^\Gamma$ where $\Gamma = \operatorname{Gal}(\overline{K}/K)$. Examples with $\operatorname{Br}(X) = \operatorname{Br}_1(X)$ are curves, cubic surfaces or more generally geometrically rational varieties.

From the Hochschild-Serre spectral sequence

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_{m,\overline{X}})) \implies \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_{m,X})$$

we derive the exact sequence

$$\operatorname{Br} k \to \operatorname{Br}_1(X) \to \mathrm{H}^1(\Gamma, \operatorname{Pic}(\overline{X})) \to \mathrm{H}^3(\Gamma, \overline{k}^\times).$$

Since the last term vanishes for number fields, we get an isomorphism

$$\operatorname{Br}_1(X)/\operatorname{Br}_0(X) \cong \mathrm{H}^1(\Gamma, \operatorname{Pic}(\overline{X})).$$

In the case that $\operatorname{Pic}(\overline{X})$ is explicitly known as a finitely generated Galois module split over a finite extension of K, this isomorphism can be effectively computed. Nowadays there are various examples for this in the literature, most notably M. Bright's classification of the algebraic Brauer group of diagonal quartic surfaces [Bri02] (but see also [KT04, KT08, LM19, CN18]).

Hence, the main open problem is to get hold of the transcendental part. We solve this task for X with torsion-free $\operatorname{Pic}(\overline{X})$ by dividing it into three challenges:

(a) Determine the action of Γ on the geometric Brauer group $\operatorname{Br}(\overline{X})$ and its invariants $\operatorname{Br}(\overline{X})^\Gamma$. In certain cases $\operatorname{Br}(\overline{X})^\Gamma$ is finite and indeed, we will see that this is the case for the varieties we treat.

(b) Determine the image $\operatorname{im}(\operatorname{Br}(X) \to \operatorname{Br}(\overline{X})^\Gamma)$. In general, there is no reason why this map should be surjective. Indeed, an argument in homological algebra will characterise those Galois invariant geometric Brauer classes that come from $\operatorname{Br}(X)$.

(c) If the output of (b) is such that $\operatorname{Br}_1(X) \subsetneq \operatorname{Br}(X)$, the last challenge is to determine the extension

$$0 \to \operatorname{Br}_1(X)/\operatorname{Br}_0(X) \to \operatorname{Br}(X)/\operatorname{Br}_0(X) \to \operatorname{Br}(X)/\operatorname{Br}_1(X) \to 0.$$

This will involve a hypercohomology argument in derived categories.

Recall that the determination of the algebraic Brauer group as sketched above required knowledge of $\operatorname{Pic}(\overline{X})$. In order to make our methods work, we require additional information. Namely, to be able to implement (a) we need an understanding of the Galois action on the transcendental cycles. For (b) and (c), the structure of the geometric Picard group *and* the transcendental cycles as Galois modules as well as their relation to each other via the discriminant group has to be understood.

Under the assumption that this understanding is available, Chapter 10 develops a framework for determining $\operatorname{Br}(X)/\operatorname{Br}_0(X)$. We refrain from calling it "algorithmic" since it is not presented in a formalised way but it is reasonable to expect that our methods be applicable to a wide class of varieties. The first successfully implemented case is that of diagonal quartic surfaces in joint work by the author [GS19] where the following main result was proved. (In the original statement of (i), it was assumed that all coefficients are in $\mathbb{Q}$ but an argument similar to Lemma 12.18 shows that this assumption can be dropped.)

Theorem 9.1. *(i) Let X be a diagonal quartic surface over $\mathbb{Q}(i)$. Then*

$$\mathrm{Br}(X)[2^\infty] \subset \mathrm{Br}_1(X),$$

unless X is isomorphic to the surface given by

$$x^4 + y^4 + 2z^4 + 8w^4 = 0,$$

in which case the extension in (c) becomes

$$0 \to \mathbb{Z}/2 \times \mathbb{Z}/4 \to \mathbb{Z}/4 \times \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

(ii) Let X be a diagonal quartic surface over $\mathbb{Q}$. Then

$$\mathrm{Br}(X)[2^\infty] \subset \mathrm{Br}_1(X)$$

unless X is isomorphic to one of the surfaces given by

$$x^4 + y^4 + 2z^4 - 2w^4 = 0$$

or

$$x^4 + y^4 + 8z^4 - 8w^4 = 0,$$

in which case the extension in (c) becomes

$$0 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \to \mathbb{Z}/2 \to 0.$$

The necessary information about the Galois action on transcendental cycles can sometimes be obtained in the presence of *complex multiplication* when Γ acts through Größencharacters. This is the case in Chapter 11 where we give a complete description of the integral cohomology of weighted diagonal hypersurfaces.

The final Chapter 12 combines these results to classify the Brauer group of degree 2 K3 surfaces

$$X_{A,B,C} : y^2 = Ax^6 + By^6 + Cz^6 \subset \mathbb{P}^3_K(3,1,1,1)$$

over $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-3})$ with $A, B, C \in K^\times$. These are double covers of the projective plane ramified in a diagonal sextic. These surfaces have previously been studied by Bouyer-Costa-Festi [BCF+19] and Corn-Nakahara [CN18]. In conjunction with [CN18], we are able to show in Corollary 12.11 that degrees do not capture the Brauer-Manin obstruction for K3 surfaces answering a question in an article by B. Creutz and B. Viray [CV18].

The main results of this part, which rely on Magma computations [BCP97, Gvi19b], are as follows. We call $X_{A,B,C}$ and $X_{A',B',C'}$ *equivalent*, if we can obtain one from the other by permuting the variables x, y, z and multiplying each coefficient with sixth powers in $\mathbb{Q}^\times$ and -27 (which is a sixth power in $\mathbb{Q}(\sqrt{-3})$). This is an a priori stronger notion than isomorphy over $\mathbb{Q}(\sqrt{-3})$.

Theorem A *Let* $K = \mathbb{Q}(\sqrt{-3})$ *and*

$$X = X_{A,B,C} : y^2 = Ax^6 + By^6 + Cz^6 \subset \mathbb{P}^3_K(3, 1, 1, 1)$$

with $A, B, C \in K^\times$.

(*i*) *We have* $\mathrm{Br}(X)[2^\infty] = \mathrm{Br}_1(X)[2^\infty]$ *unless* X *is equivalent to*

$$(I) : X_{-2c_1^4c_2^4, -8c_1^2, -8c_2^2} \text{ or } (II) : X_{-2c_1^4, 8c_1^2c_2^5, -c_2}$$

for some $c_1, c_2 \in K^\times$. *In these two cases*

$$(\mathrm{Br}(X)/\mathrm{Br}_1(X))[2^\infty] = (\mathbb{Z}/2)^2.$$

(*ii*) *We have* $\mathrm{Br}(X)[3^\infty] = \mathrm{Br}_1(X)[3^\infty]$ *unless* X *is equivalent to*

$$(III) : X_{-9c_1^3c_2, 3c_1^3c_2^4, -c_2}$$

for some $c_1, c_2 \in K^\times$. *In this case*

$$(\mathrm{Br}(X)/\mathrm{Br}_1(X))[3^\infty] = (\mathbb{Z}/3)^2.$$

Theorem B *Consider the surface*

$$X = X_{A,B,C} : y^2 = Ax^6 + By^6 + Cz^6 \subset \mathbb{P}^3_{\mathbb{Q}}(3,1,1,1)$$

with $A, B, C \in \mathbb{Q}^\times$*.*

(*i*) *We have* $\operatorname{Br}(X)[2^\infty] = \operatorname{Br}_1(X)[2^\infty]$ *unless* $X \times_{\mathbb{Q}} \mathbb{Q}(\sqrt{-3})$ *is equivalent to a surface of type* (I) *or* (II)*. In this case,*

$$(\operatorname{Br}(X)/\operatorname{Br}_1(X))[2^\infty] = \mathbb{Z}/2.$$

(*ii*) *We have* $\operatorname{Br}(X)[3^\infty] = \operatorname{Br}_1(X)[3^\infty]$ *unless* $X \times_{\mathbb{Q}} \mathbb{Q}(\sqrt{-3})$ *is equivalent to a surface of type* (III)*. In this case,*

$$(\operatorname{Br}(X)/\operatorname{Br}_1(X))[3^\infty] = \mathbb{Z}/3.$$

For step (c), instead of listing all the exceptional cases, which differ in their algebraic Brauer groups, we give a general structure theorem for the extension

$$0 \to \operatorname{Br}_1(X)/\operatorname{Br}_0(X) \to \operatorname{Br}(X)/\operatorname{Br}_0(X) \to \operatorname{Br}(X)/\operatorname{Br}_1(X) \to 0.$$

Supplement to Theorem A *Let* $K = \mathbb{Q}(\sqrt{-3})$ *and* $\ell = 2$*, respectively* 3*. Let* X *be one of the exceptional surfaces in Theorem A.*(*i*)*, respectively* (*ii*)*. Then* $\operatorname{Br}_1(X)/\operatorname{Br}_0(X)$ *is an* ℓ*-group and writing*

$$\operatorname{Br}_1(X)/\operatorname{Br}_0(X) = \bigoplus_{i=1}^{k} \mathbb{Z}/\ell^{n_i}$$

with $n_i \leq n_j$ *for all* $1 \leq i < j \leq k$*, we have*

$$\operatorname{Br}(X)/\operatorname{Br}_0(X) = \bigoplus_{i=1}^{k-2} \mathbb{Z}/\ell^{n_i} \oplus \mathbb{Z}/\ell^{n_{k-1}+1} \oplus \mathbb{Z}/\ell^{n_k+1}.$$

Supplement to Theorem B *Let* $K = \mathbb{Q}$ *and* $\ell = 2$*, respectively* 3*. Let* X *be one of the exceptional surfaces in Theorem B.*(*i*)*, respectively* (*ii*)*. Then*

$\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ *is an ℓ-group and writing*

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) = \bigoplus_{i=1}^{k} \mathbb{Z}/\ell^{n_i}$$

with $n_i \leq n_j$ for all $1 \leq i < j \leq k$, we have

$$\mathrm{Br}(X)/\mathrm{Br}_0(X) = \bigoplus_{i=1}^{k-1} \mathbb{Z}/\ell^{n_i} \oplus \mathbb{Z}/\ell^{n_k+1}.$$

Comparing with Theorem 9.1, we notice some similarities and differences:

- Like in Theorem 9.1, the results in step (c) point to a difficulty in lifting elements of the transcendental part to the full Brauer group. A 2- or 3-torsion element of the transcendental part will generally only lift to a 4-, 8- or 9-torsion Brauer class.
- As seen in Theorem 9.1.(*iii*), it is not the case that all diagonal quartic surfaces over $\mathbb{Q}$ which are descended from the exceptional case over $\mathbb{Q}(i)$ have a nontrivial transcendental Brauer group. In fact, only those with coefficients $1, 1, 2, -2$ and $1, 1, 8, -8$ have a nontrivial transcendental 2-torsion Brauer class. This result stands in contrast to Theorem B.
- Unlike Theorem 9.1, there isn't a finite list of surfaces with nontrivial transcendental 2- or 3-torsion. Instead, the exceptional cases appear in families.

Upcoming work by the present author will also study other members of Festi's family using the main theorem of complex multiplication for K3 surfaces due to J. Rizov [Riz05] and D. Valloni [Val18].

We end this introduction by returning to the original motivation of computing the Brauer-Manin obstruction. A remaining problem in relating our results to rational points is that our cohomological framework only outputs the Brauer group as an abstract group. In order to compute the obstruction presented by Brauer classes, one would like to find a representation as Azumaya algebras.

At the moment, we are not able to do this. On the other hand recent work by A. Várilly-Alvarado and J. Berg [BVA18] has demonstrated that the obstruction can sometimes be computed by a geometric argument.

Regardless of this caveat, experience shows that the transcendental part is usually much smaller than $\mathrm{Br}(\overline{X})^\Gamma$ and very often trivial, in which case there obviously is no Brauer-Manin obstruction by transcendental elements. Upcoming joint work by the present author uses this fact to give asymptotic results on the Brauer-Manin obstruction for the family of K3 surfaces considered in Chapter 12.

Conventions. All tensor products are over $\mathbb{Z}$ unless stated otherwise. The leftmost term of complexes as written down is in degree 0.

10

A framework for computing the Brauer group

This chapter explains a cohomological framework to determine the full Brauer group of varieties over number fields with torsion-free Picard group (up to constant Brauer classes). In the next chapters we will apply the method successfully to weighted diagonal surfaces. The framework upgrades an argument in spectral sequences from [CTS13] to an argument in derived categories as was done for surfaces in [GS19], following the mantra of derived categories to take cohomology *as late as possible*. This upgrade is essential as it allows us to complete step (c) using hypercohomology whereas the spectral sequence argument only returns the graded pieces of the filtration $\mathrm{Br}_0(X) \subseteq \mathrm{Br}_1(X) \subseteq \mathrm{Br}(X)$.

10.1 Cohomological tools

10.1.1 The following lemma in homological algebra is proved in [CTS13].

Lemma 10.1. *Let $G : \mathcal{A} \to \mathcal{B}$ be an additive, left exact functor of abelian* *categories. Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in $\mathcal{A}$. Let $B^\bullet$ be an injective resolution of B.

Then there exists a complex $E^\bullet$ concentrated in degrees 0 and 1 and morphisms

$$(\tau^{[1,2]}G(B^\bullet))[1] \xleftarrow{e_1} E^\bullet \xrightarrow{e_2} [R^1G(C) \to R^2G(A)]$$

where $\tau^{[1,2]}$ denotes natural truncation and $[1]$ shift to the left, with the following properties:

(*i*) *On cohomology, e_1 induces an isomorphism in degree 0 and the natural map*

$$\operatorname{coker}(R^1G(C) \to R^2G(A)) \to R^2G(B)$$

in degree 1.

(*ii*) *On cohomology, e_2 induces the natural map*

$$R^1G(B) \to \ker(R^1G(C) \to R^2G(A))$$

in degree 0 and an isomorphism in degree 1.

Proof. The complex $E^\bullet$ with these properties is constructed in the proof of [CTS13, Lemme 3.2]. □

10.1.2 We now want to apply Lemma 10.1 to our problem of computing the Brauer group. Let $p : X \to \operatorname{Spec} K$ be an n-dimensional smooth, projective, geometrically integral variety over a number field K. Fix an embedding $K \subset \mathbb{C}$. Let $\Gamma = \Gamma_K = \operatorname{Gal}(\overline{K}/K)$ be the absolute Galois group of K and $\overline{X} = X \times_K \overline{K}$. Assume that $\operatorname{Pic}(\overline{X})$ is torsion-free.

The goal is to arrive at a complex representing the derived object

$$(\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1]$$

(or a complex closely related to it) where [1] denotes shift to the left. This will be enough, as the following lemma shows.

Lemma 10.2. *There is a natural isomorphism from* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *to the hypercohomology* $\mathbb{H}^1(\Gamma, \tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X})[1])$.

Proof. The derived functor of $\mathrm{H}^0(\Gamma, \cdot)$ applied to the distinguished triangle

$$\tau^{[0]}\mathrm{R}p_*(\mathbb{G}_{m,X}) \to \tau^{[0,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}) \to \tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X})$$

yields the short exact sequence

$$\mathrm{Br}(K) \to \mathrm{Br}(X) \to \mathbb{H}^1(\Gamma, (\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1]) \to \mathrm{H}^3(\Gamma, \overline{K}^\times)$$

and the last term vanishes for any number field K [NSW08, 8.3.11(iv)]. ☐

10.1.3 We set $\mathcal{A} = \mathcal{D}(X)$, the bounded below derived category of étale abelian sheaves on X, and $\mathcal{B} = \mathcal{D}(K)$, the bounded below derived category of étale abelian sheaves on $\operatorname{Spec} K$, or equivalently of Γ-modules. Then the functor $G = p_*$ is an additive, left exact functor between $\mathcal{A}$ and $\mathcal{B}$.

Looking, for any positive integer n, at the Kummer exact sequence

$$0 \to \mu_n \to \mathbb{G}_{m,X} \xrightarrow{()^n} \mathbb{G}_{m,X} \to 0$$

of étale sheaves on X, we get from Lemma 10.1 a diagram of complexes

$$(\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1] \xleftarrow{e_1} E^\bullet \xrightarrow{e_2} [\mathrm{Pic}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n)].$$

Induced from the Kummer exact sequence is the exact sequence

$$\overline{K}^\times \to \overline{K}^\times \to \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n) \to \mathrm{Pic}(\overline{X}) \xrightarrow{\cdot n} \mathrm{Pic}(\overline{X}).$$

Because we assumed $\mathrm{Pic}(\overline{X})_{\text{tors}} = 0$, it follows that $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n) = 0$ for any positive integer n. Therefore, e_2 is a quasi-isomorphism. In $\mathcal{D}(K)$ we obtain a morphism

$$[\mathrm{Pic}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n)] \to (\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1].$$

On cohomologies, it induces the isomorphism $n\mathrm{Pic}(\overline{X}) \xrightarrow{\simeq} \mathrm{Pic}(\overline{X})$ in degree 0 and the natural inclusion $\mathrm{Br}(\overline{X})[n] \to \mathrm{Br}(\overline{X})$ in degree 1.

Taking the inverse limit over all powers of ℓ of

$$[\mathrm{Pic}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{l^n})] \to (\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1]$$

and then taking the sum of these limits in $\mathcal{D}(K)$ over all primes ℓ yields the morphism

$$\beta : [\mathrm{Pic}(\overline{X}) \otimes \mathbb{Q} \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(1))] \to (\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1]$$

where

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(1)) = \bigoplus_{\ell} \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) = \bigoplus_{\ell} \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

and (1) denotes the Tate twist.

The morphism β induces an isomorphism in degree 0 and induces the natural inclusion $\mathrm{Br}^0(\overline{X}) \to \mathrm{Br}(\overline{X})$ in degree 1. Here $\mathrm{Br}^0(\overline{X})$ is the maximal divisible subgroup of $\mathrm{Br}(\overline{X})$.

Proposition 10.3. *Let $p : X \to \operatorname{Spec} K$ be a smooth, projective, geometrically integral variety over a number field K. Assume that $\mathrm{Pic}(\overline{X})$ is torsion-free.*

Then

$$\beta : [\mathrm{Pic}(\overline{X}) \otimes \mathbb{Q} \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(1))] \to (\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1]$$

induces the natural inclusion $\mathrm{Br}^0(X)/\mathrm{Br}_0(X) \to \mathrm{Br}(X)/\mathrm{Br}_0(X)$ on first hypercohomologies where $\mathrm{Br}^0(X)$ is the preimage of $\mathrm{Br}^0(\overline{X})$ under $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$.

Proof. By [Gro68][(8.9)], there is a short exact sequence

$$0 \to \mathrm{Br}^0(\overline{X}) \to \mathrm{Br}(\overline{X}) \to \bigoplus_{\ell} \mathrm{H}^3(\overline{X}, \mathbb{Z}_\ell(1))_{\text{tors}} \to 0,$$

yielding a distinguished triangle

$$\begin{aligned}[\mathrm{Pic}(\overline{X}) \otimes \mathbb{Q} \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(1))] &\to (\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1] \\ &\to [0 \to \bigoplus_\ell \mathrm{H}^3(X, \mathbb{Z}_\ell(1))_{\mathrm{tors}}].\end{aligned}$$

Applying hypercohomology and Lemma 10.2 we get an injection

$$\mathbb{H}^1(\Gamma, [\mathrm{Pic}(\overline{X}) \otimes \mathbb{Q} \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(1))]) \hookrightarrow \mathrm{Br}(X)/\mathrm{Br}_0(X)$$

since $\mathbb{H}^0(\Gamma, [0 \to \bigoplus_\ell \mathrm{H}^3(X, \mathbb{Z}_\ell(1))_{\mathrm{tors}}]) = 0$. Looking at the Cartan-Eilenberg resolution computing the hypercohomology, we see that an element in

$$\mathrm{Br}(X)/\mathrm{Br}_0(X)$$

lies in the image of this injection if and only if its image in

$$\mathrm{H}^1((\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1]) = \mathrm{Br}(\overline{X})$$

lies in the subgroup $\mathrm{H}^1([\mathrm{Pic}(\overline{X}) \otimes \mathbb{Q} \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(1))]) = \mathrm{Br}^0(\overline{X})$. □

10.1.4 There is an isomorphism between the groups $\mathrm{Pic}(\overline{X})$ and $\mathrm{Pic}(X_{\mathbb{C}})$ and between their maximal divisible subgroups $\mathrm{Pic}^0(\overline{X}) \cong \mathrm{Pic}^0(X_{\mathbb{C}})$. Let

$$\mathrm{NS}(\overline{X}) = \mathrm{Pic}(\overline{X})/\mathrm{Pic}^0(\overline{X})$$

be the Néron-Severi group, a saturated subgroup of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z})$.

When $n = 2$, $\mathrm{Pic}(\overline{X})_{\mathrm{tors}} = 0$ implies

$$0 = \mathrm{NS}(\overline{X})_{\mathrm{tors}} = \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))_{\mathrm{tors}} = \mathrm{H}^3(X_{\mathbb{C}}, \mathbb{Z})_{\mathrm{tors}}$$

where the last equality follows from Poincaré duality. Hence for surfaces, our assumptions imply that $\mathrm{Br}^0(X) = \mathrm{Br}(X)$ and β is a quasi-isomorphism.

10.2 TRANSCENDENTAL CYCLES

10.2.1 Following [CTS13, §4.1], we can improve the complex of Proposition 10.3 further using transcendental cycles.

Let $N(X_{\mathbb{C}})$ be the subgroup of algebraic cycles in $\mathrm{H}^{2n-2}(X_{\mathbb{C}}, \mathbb{Z}(n-1))$. It naturally carries an action of Γ. To avoid issues of torsion in the construction, we assume that $\mathrm{H}^{2n-2}(X_{\mathbb{C}}, \mathbb{Z}(n-1))$ is torsion-free and $N(X_{\mathbb{C}})$ is primitive in $\mathrm{H}^{2n-2}(X_{\mathbb{C}}, \mathbb{Z}(n-1))$. (Otherwise, one has to replace $N(X_{\mathbb{C}})$ with a certain "saturation" as done in [CTS13, ibid.] but after doing so, the results of this section still apply.)

The cup product defines a non-degenerate bilinear pairing

$$\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \times \mathrm{H}^{2n-2}(X_{\mathbb{C}}, \mathbb{Z}(n-1)) \to \mathrm{H}^{2n}(X_{\mathbb{C}}, \mathbb{Z}(n)) = \mathbb{Z}$$

inducing an isomorphism

$$\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \cong \mathrm{Hom}(\mathrm{H}^{2n-2}(X_{\mathbb{C}}, \mathbb{Z}(n-1)), \mathbb{Z}).$$

This defines a short exact sequence

$$0 \to \mathrm{Pic}(X_{\mathbb{C}}) \to \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}) \to \Delta \to 0$$

and we call the hereby defined quotient Δ the *discriminant group* of X.

Definition 10.4. The group of *transcendental cycles* $T(X_{\mathbb{C}}) \subset \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ (resp. $S(X_{\mathbb{C}}) \subset \mathrm{H}^{2n-2}(X_{\mathbb{C}}, \mathbb{Z}(n-1))$) is the orthogonal complement to $N(X_{\mathbb{C}})$ (resp. $\mathrm{Pic}(X_{\mathbb{C}})$) under the cup product.

From primitivity, we deduce short exact sequences

$$0 \to \mathrm{Pic}(X_{\mathbb{C}}) \to \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \to \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z}) \to 0$$

and

$$0 \to S(X_{\mathbb{C}}) \to \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \to \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}) \to 0.$$

Since $\mathrm{Pic}(X_{\mathbb{C}}) \to \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z})$ is injective, it follows that $\mathrm{Pic}(X_{\mathbb{C}}) \cap$

$S(X_{\mathbb{C}}) = 0$.

10.2.2 Applying the snake lemma to the commutative diagrams with exact columns

$$\begin{tikzcd}
0 & 0 \\
\mathrm{Pic}(X_{\mathbb{C}}) \arrow[u] \arrow[r] & \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}) \arrow[u] \\
\mathrm{Pic}(X_{\mathbb{C}}) \oplus S(X_{\mathbb{C}}) \arrow[u] \arrow[r] & \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \arrow[u] \\
S(X_{\mathbb{C}}) \arrow[u] \arrow[r, equal] & S(X_{\mathbb{C}}) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 & 0 \\
S(X_{\mathbb{C}}) \arrow[u] \arrow[r] & \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z}) \arrow[u] \\
\mathrm{Pic}(X_{\mathbb{C}}) \oplus S(X_{\mathbb{C}}) \arrow[u] \arrow[r] & \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \arrow[u] \\
\mathrm{Pic}(X_{\mathbb{C}}) \arrow[u] \arrow[r, equal] & \mathrm{Pic}(X_{\mathbb{C}}) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

we get that

$$\begin{aligned}
\Delta &\cong \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z})/\mathrm{Pic}(X_{\mathbb{C}}) \cong \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))/(\mathrm{Pic}(X_{\mathbb{C}}) \oplus S(X_{\mathbb{C}})) \\
&\cong \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z})/S(X_{\mathbb{C}}).
\end{aligned}$$

10.2.3 For any prime ℓ, there are comparison isomorphisms

$$\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Z}_\ell \cong \mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1))$$

between singular and ℓ-adic cohomology, compatible with the cup product and cycle class maps. Thus we can define $T(\overline{X})_\ell$ and $S(\overline{X})_\ell$ analogously to their complex counterparts such that

$$T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell \cong T(\overline{X})_\ell, \quad S(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell \cong S(\overline{X})_\ell$$

and obtain a short exact sequence of Γ-modules

$$0 \to \operatorname{Pic}(\overline{X}) \otimes \mathbb{Z}_\ell \oplus S(\overline{X})_\ell \to \mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1)) \to \Delta[\ell^\infty] \to 0.$$

After tensoring this with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, we arrive at

$$0 \to \Delta[\ell^\infty] \to \operatorname{Pic}(\overline{X}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \oplus S(\overline{X})_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \to 0.$$

This gives rise to a commutative diagram of Γ-modules with exact rows

$$\begin{tikzcd}
\Delta[\ell^\infty] \arrow[r, hook] \arrow[d] & S(\overline{X})_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r, two heads] \arrow[d, "\cdot(-1)"] & \operatorname{Hom}(T(\overline{X})_\ell, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[d, "\cong"] \\
\operatorname{Pic}(\overline{X}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r, hook] & \mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \arrow[r, two heads] & \operatorname{Hom}(T(\overline{X})_\ell, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell
\end{tikzcd}$$

which after summing over all primes ℓ becomes

$$\begin{tikzcd}
\Delta \arrow[r, hook] \arrow[d] & S(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r, two heads] \arrow[d, "\cdot(-1)"] & \operatorname{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}) \arrow[d, "\cong"] \\
\operatorname{Pic}(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r, hook] & \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z}(1)) \arrow[r, two heads] & \operatorname{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}).
\end{tikzcd}$$

10.2.4 At this point, the reader should keep in mind that while $S(X_{\mathbb{C}})$ and $T(X_{\mathbb{C}})$ are not equipped with Galois actions,

$$S(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \text{ and } \operatorname{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z})$$

are indeed Galois modules owing to the comparison isomorphisms.

10.2.5 Another commutative diagram of Γ-modules with exact rows is given by

$$\begin{tikzcd}
\mathrm{Pic}(X_{\mathbb{C}}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}(N(X_{\mathbb{C}}),\mathbb{Z}) \arrow[r, two heads] \arrow[d] & \Delta \arrow[d] \\
\mathrm{Pic}(X_{\mathbb{C}}) \arrow[r, hook] & \mathrm{Pic}(X_{\mathbb{C}})\otimes\mathbb{Q} \arrow[r, two heads] & \mathrm{Pic}(X_{\mathbb{C}})\otimes\mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

In conclusion, we realise that there is an isomorphism of complexes in $\mathcal{D}(K)$

$$[\mathrm{Hom}(N(X_{\mathbb{C}}),\mathbb{Z}) \to S(X_{\mathbb{C}})\otimes\mathbb{Q}/\mathbb{Z}] \cong [\mathrm{Pic}(\overline{X})\otimes\mathbb{Q} \to \mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Q}/\mathbb{Z}(1))].$$

In particular, it becomes clear that $\mathrm{Br}^0(\overline{X}) \cong \mathrm{Hom}(T(X_{\mathbb{C}}),\mathbb{Q}/\mathbb{Z})$ as Γ-modules and both are isomorphic to $(\mathbb{Q}/\mathbb{Z})^{b_2(X_{\mathbb{C}})-\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}})} = (\mathbb{Q}/\mathbb{Z})^{\mathrm{rk}\,T(X_{\mathbb{C}})}$ as abstract groups, where $b_2(X_{\mathbb{C}})$ is the second Betti number of $X_{\mathbb{C}}$.

Moreover, we can reformulate Proposition 10.3 as follows.

Proposition 10.5. *Let X be a smooth, projective, geometrically integral variety over a number field K. Assume that $\mathrm{Pic}(\overline{X})$ is torsion-free.*

Then

$$\mathbb{H}^1(\Gamma,[\mathrm{Hom}(N(X_{\mathbb{C}}),\mathbb{Z}) \to S(X_{\mathbb{C}})\otimes\mathbb{Q}/\mathbb{Z}]) \cong \mathrm{Br}^0(X)/\,\mathrm{Br}_0(X).$$

10.2.6 When $n = 2$, we have that $\mathrm{Pic}(X_{\mathbb{C}}) = N(X_{\mathbb{C}})$ and $T(X_{\mathbb{C}}) = S(X_{\mathbb{C}})$ and all our assumptions are satisfied. Therefore, we have a quasi-isomorphism

$$[\mathrm{Hom}(\mathrm{Pic}(X_{\mathbb{C}}),\mathbb{Z}) \to T(X_{\mathbb{C}})\otimes\mathbb{Q}/\mathbb{Z}] \cong (\tau^{[1,2]}\mathrm{R}p_*(\mathbb{G}_{m,X}))[1]$$

recovering [GS19, Proposition 1.2].

We recover a theorem of [CTS13].

Corollary 10.6. *Let X be a smooth, projective, geometrically integral variety over a number field K. Assume that $\mathrm{Pic}(\overline{X})$ is torsion-free.*

Then $\mathrm{im}(\mathrm{Br}^0(X) \to \mathrm{Br}^0(\overline{X}))$ *is equal to the kernel of the map*

$$\delta : \mathrm{H}^0(\Gamma,\mathrm{Br}^0(\overline{X})) \to \mathrm{H}^2(\Gamma,\mathrm{Pic}(\overline{X}))$$

which is obtained from the 2*-extension of* Γ*-modules*

$$0 \to \operatorname{Pic}(\overline{X}) \to \operatorname{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}) \to S(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \to \operatorname{Br}^0(\overline{X}) \to 0.$$

10.2.7 Since the morphism δ in Corollary 10.6 is the composition of natural maps

$$\mathrm{H}^0(\Gamma, \operatorname{Br}^0(\overline{X})) \to \mathrm{H}^1(\Gamma, \Delta) \to \mathrm{H}^2(\Gamma, \operatorname{Pic}(\overline{X})),$$

it follows that for any $\ell \nmid \#\Delta$, we have

$$\operatorname{Br}^0(\overline{X})[\ell^\infty]^\Gamma = (\operatorname{Br}^0(X)/\operatorname{Br}_1(X))[\ell^\infty].$$

In particular, step (b) and (c) in our framework are only relevant for $\ell | \#\Delta$.

10.2.8 We derive the following corollary.

Corollary 10.7. *Let* X *and* Y *be two smooth, projective, geometrically integral surfaces over a number field* K *with torsion-free geometric Picard groups. Assume that there is an isomorphism*

$$\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \cong \mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Z})$$

which respects the cup products and induces an isomorphism of Hodge structures as well as an isomorphism of Galois modules on ℓ*-adic cohomology (by the comparison theorem of singular and* ℓ*-adic cohomology) for all primes* ℓ*.*

Then

$$\operatorname{Br}(X)/\operatorname{Br}_0(X) \cong \operatorname{Br}(X)/\operatorname{Br}_0(X).$$

Proof. The left hand side is computed by the hypercohomology of the complex

$$[\operatorname{Hom}(\operatorname{Pic}(X_{\mathbb{C}}), \mathbb{Z}) \to T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z}]$$

and the same is true for the right hand side after replacing X by Y. Because of our assumption,

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \cong \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)).$$

Since $\operatorname{Pic}(X_{\mathbb{C}}) \cong \operatorname{Pic}(Y_{\mathbb{C}})$ by the Lefschetz $(1,1)$-theorem and $\operatorname{Pic}(\overline{X}) \otimes \mathbb{Z}_\ell \cong \operatorname{Pic}(\overline{Y}) \otimes \mathbb{Z}_\ell$, we have $\operatorname{Pic}(\overline{X}) \cong \operatorname{Pic}(\overline{Y})$. The same holds for the duals and ℓ-adic orthogonal complements. Thus, the two complexes of which we take the hypercohomology are isomorphic. □

10.3 Reduction to a finite computation

10.3.1 In order to make the hypercohomology with respect to the infinite profinite group Γ amenable to calculations, we would like to replace Γ by a finite group.

However, while the Galois action on $\operatorname{Pic}(X_{\mathbb{C}})$ and N does factor through a finite quotient, the action on $S(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z}$ and $\operatorname{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z})$ does not do so in general, as predicted by the Tate conjecture. Another perspective to think about this issue is that as one takes larger algebraic extensions L of K, the group $\operatorname{Br}^0(\overline{X})^{\operatorname{Gal}(\overline{L}/L)}$ will continue to grow, until it eventually encompasses the whole $\operatorname{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z})$ when $L = \overline{K}$.

10.3.2 Hence, we will adapt our method to compute

$$\operatorname{Br}_B(X) \subset \operatorname{Br}(X)/\operatorname{Br}_0(X),$$

the preimage of a finite Γ-submodule $B \subset \operatorname{Br}^0(\overline{X})$.

In all our applications, we will have $\operatorname{Br}(\overline{X}) = \operatorname{Br}^0(\overline{X})$, and once we fix the ground field K, the Galois invariant part $\operatorname{Br}(\overline{X})^\Gamma$ will be finite. In such a situation, if we take B to be a finite submodule containing $\operatorname{Br}(\overline{X})^\Gamma$, our cohomological machinery indeed outputs the full group

$$\operatorname{Br}(X)/\operatorname{Br}_0(X) = \operatorname{Br}_B(X).$$

10.3.3 Of course, substituting Γ by a finite quotient changes some of the group cohomologies. However, the resulting hypercohomology is unaffected as the following proposition shows.

Proposition 10.8. *Let X be a smooth, projective, geometrically integral variety over a number field K. Let B be a finite Γ-submodule of*

$$\mathrm{Br}^0(\overline{X}) \cong \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}).$$

Let S_B be the preimage of B under $S(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z})$. Let $\Gamma_{K_B} \subset \Gamma$ be the finite index subgroup which is the kernel of the action morphism

$$\Gamma \to \mathrm{Aut}(N(X_{\mathbb{C}})) \times \mathrm{Aut}(S_B).$$

Define the finite extension $K_B = (\overline{K})^{\Gamma_{K_B}/K}$ and set $G_B = \mathrm{Gal}(K_B/K)$.

The following statements hold.

(*i*) *The image of* $\mathrm{Br}_B(X)$ *under the natural map* $\mathrm{Br}(X)/\mathrm{Br}_0(X) \to \mathrm{Br}(\overline{X})$ *is equal to the kernel of the map*

$$\delta_B : \mathrm{H}^0(G_B, B) \to \mathrm{H}^2(G_B, \mathrm{Pic}(\overline{X}))$$

obtained from the 2-extension of G_B*-modules*

$$0 \to \mathrm{Pic}(\overline{X}) \to \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}) \to S_B \to B \to 0.$$

(*ii*) *The group* $\mathrm{Br}_B(X)$ *is isomorphic to*

$$\mathbb{H}^1(G_B, [\mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}) \to S_B]).$$

Proof. (*i*) The 2-extension is indeed well-defined since the action of Γ on all terms factors through G_B. We get a commutative diagram

$$\begin{tikzcd}
B^{\Gamma} \arrow[r, "\delta"] \arrow[d, equal] & \mathrm{H}^2(\Gamma, \mathrm{Pic}(\overline{X})) \\
B^{G_B} \arrow[r, "\delta_B"] & \mathrm{H}^2(G_B, \mathrm{Pic}(\overline{X})). \arrow[u, "\mathrm{inf}"']
\end{tikzcd}$$

By Corollary 10.6, the image of $\mathrm{Br}_B(X)$ in B^{Γ} is given by $\ker(\delta)$. However, the inflation-restriction exact sequence implies that

$$\ker(\inf) = \mathrm{H}^1(\Gamma_{K_B}, \mathrm{Pic}(\overline{X})) = \mathrm{Hom}(\Gamma_{K_B}, \mathrm{Pic}(\overline{X})) = 0$$

since the profinite Galois group Γ_{K_B} acts trivially on $\mathrm{Pic}(\overline{X})$. Hence

$$\ker(\delta) = \ker(\delta_B).$$

(*ii*) This is a combination of the first item and Proposition 10.5.

□

If $B = \mathrm{Br}^0(\overline{X})[n]$ for some integer n, we will also write $S_B = S_n$, $K_B = K_n$ and $G_B = G_n$.

10.3.4 In practice, instead of considering the connecting map

$$\mathrm{H}^0(G_B, B) \to \mathrm{H}^1(G_B, \Delta) \to \mathrm{H}^2(G_B, \mathrm{Pic}(\overline{X})),$$

which requires the computation of second cohomology, it is often easier to use the exact sequence

$$\mathrm{H}^1(G_B, \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z})) \to \mathrm{H}^1(G_B, \Delta) \to \mathrm{H}^2(G_B, \mathrm{Pic}(\overline{X})).$$

Then $\ker(\delta_B)$ is the preimage of

$$\mathrm{im}(\mathrm{H}^1(G_B, \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z})) \to \mathrm{H}^1(G_B, \Delta))$$

under the connecting map $\mathrm{H}^0(G_B, B) \to \mathrm{H}^1(G_B, \Delta)$.

10.4 Finiteness of the Brauer group

This section collects a few results on the finiteness of the Brauer group. In particular, as remarked before, if $\mathrm{Br}(X)^\Gamma \subset \mathrm{Br}^0(\overline{X})[n]$ for some integer n, we can apply Lemma 10.8 to determine $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

10.4.1

Definition 10.9. A *K3 surface* is a smooth, projective, geometrically integral variety X with trivial canonical bundle, satisfying $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Together with abelian varieties, K3 surfaces provide a higher-dimensional analogue of elliptic curves. A standard reference is [Huy16].

The Hodge diamond of a K3 surface X over $\mathbb{C}$ is known to be

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

so that $\operatorname{rk}\operatorname{Pic}(X)$ can take values between 1 and 20. Furthermore, $\operatorname{Pic}(X)$ is torsion-free. The Tate conjecture is known for K3 surfaces in all characteristics [Mad15, KM16].

Proposition 10.10. *Let X be a K3 surface over a number field K. Then the groups $\operatorname{Br}(\overline{X})^\Gamma$ and $\operatorname{Br}(X)/\operatorname{Br}_0(X)$ are finite.*

Proof. See [SZ08]. ☐

10.4.2

Definition 10.11. A *variety dominated by a product of curves (DPC variety)* is a smooth, projective, geometrically integral variety X such that there exists a dominant rational map from a product of smooth, projective, geometrically integral curves to X.

In fact, the dominant rational map in the above definition can be chosen to be generically finite [Sch96, Lemma 6.1]. Many properties of DPC varieties can be proved inductively starting with the curve case. For example, the Tate conjecture is known for DPC varieties [Tat94, Section 5].

Proposition 10.12. *Let X be a DPC variety over field K which is finitely generated over $\mathbb{Q}$. Then the groups $\operatorname{Br}(\overline{X})^\Gamma$ and $\operatorname{Br}(X)/\operatorname{Br}_1(X)$ are finite.*

Proof. Since $\operatorname{Br}(X)/\operatorname{Br}_1(X) \subset \operatorname{Br}(\overline{X})^\Gamma$, it suffices to show that $\operatorname{Br}(\overline{X})^\Gamma$ is finite.

Let $Y = \prod_{i=1}^n Y_i$ be a product of smooth, projective, geometrically integral curves over K and let $Y \dashrightarrow X$ be a dominant, generically finite, rational map. Due to the general behaviour of the Brauer group under products of varieties (see [SZ14, Theorem A]), the cokernel of

$$\bigoplus_{i=1}^{n} \operatorname{Br}(\overline{Y}_i)^\Gamma \to \operatorname{Br}(\overline{Y})^\Gamma$$

is finite. Because the Brauer group of a smooth, projective, geometrically integral curve over an algebraically closed field is trivial by Tsen's theorem, this implies that $\operatorname{Br}(\overline{Y})^\Gamma$ is finite.

We now find a resolution of the indeterminacy locus of $Y \dashrightarrow X$, i.e. a smooth, projective geometrically integral variety Y' over K with a birational morphism $Y' \to Y$ and a dominant generically finite morphism $Y' \to X$. The Brauer group is a birational invariant, hence $\operatorname{Br}(\overline{Y}')^\Gamma = \operatorname{Br}(\overline{Y})^\Gamma$.

Let $\overline{K}(X)$ be the function field of $\overline{X}$ and analogously for $\overline{K}(Y')$. We have restriction and corestriction maps

$$\begin{tikzcd} \operatorname{Br}(\overline{K}(X)) \arrow[r, shift left, "\text{res}"] & \operatorname{Br}(\overline{K}(Y')) \arrow[l, shift left, "\text{cores}"] \end{tikzcd}$$

and $\text{cores} \circ \text{res}$ equals multiplication by $[\overline{K}(Y') : \overline{K}(X)]$. Because $\operatorname{Br}(\overline{X}) \hookrightarrow \operatorname{Br}(\overline{K}(X))$ is injective [Gro68, Corollaire 1.10], it follows that the kernel of $\operatorname{Br}(\overline{X}) \to \operatorname{Br}(\overline{Y})$ is annihilated by $[\overline{K}(Y') : \overline{K}(X)]$, hence finite. The same holds for $\operatorname{Br}(\overline{X})^\Gamma \to \operatorname{Br}(\overline{Y}')^\Gamma$ and the result follows. □

Corollary 10.13. *Let X be a variety over a number field K such that $\operatorname{Pic}(\overline{X})$ is finitely generated. Then $\operatorname{Br}_1(X)/\operatorname{Br}_0(X)$ is finite.*

Proof. We have $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) = \mathrm{H}^1(\Gamma, \mathrm{Pic}(\overline{X}))$ and the latter is finite because $\mathrm{Pic}(\overline{X})$ is finitely generated. $\square$

Combining the finiteness of the geometric and algebraic Brauer group, we obtain:

Corollary 10.14. *Let X be a DPC variety over a number field K such that $\mathrm{Pic}(\overline{X})$ is finitely generated. Then $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite.*

11

Cohomology of weighted diagonal surfaces

The aim of this chapter is to develop a full description of the middle cohomology of smooth weighted diagonal hypersurfaces. By “full”, we mean an explicit understanding of the integral singular cohomology including the cup product, the Hodge cohomology, the Galois action on ℓ-adic cohomology and the comparison isomorphisms between those. We will build on previous work by Pham, Looijenga, Weil, Shioda, Ulmer and Gvirtz-Skorobogatov.

One beauty of the subject that the reader will surely be able to appreciate is how rather different branches of mathematics come together. While Pham’s work is very topological in nature and works with explicit singular chains, the famous work by Weil on counting points of diagonal hypersurfaces over finite fields is purely arithmetic. Nevertheless they lead to the same combinatorial structures. It is unknown to the author whether Pham and Weil, who published their results during the same time period, were aware of each other’s work.

11.1 SETUP

11.1.1 Fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$ and set $\zeta_m = e^{2\pi i/m}$.

Let $\boldsymbol{q} = (q_0, \dots, q_n) \in \mathbb{N}^{n+1}$. We define the group

$$\mu_{\boldsymbol{q}} = \mu_{q_0} \times \cdots \times \mu_{q_{n+1}}$$

where $\mu_{q_i} = \langle t_i \rangle$ is the group of q_i-th roots of unity with generator t_i.

Definition 11.1. The *weighted projective space* $\mathbb{P}^{n+1}_{\mathbb{Q}}(\boldsymbol{q})$ is the $(n+1)$-dimensional projective scheme

$$\operatorname{Proj} \mathbb{Q}[x_0, \dots, x_{n+1}]$$

where the grading of the polynomial ring is given by $\deg(x_i) = q_i$.

Alternatively, $\mathbb{P}^{n+1}_{\mathbb{Q}}(\boldsymbol{q})$ can be defined by the quotient

$$\pi_{\boldsymbol{q}} : \mathbb{P}^{n+1}_{\mathbb{Q}} \to \mathbb{P}^{n+1}_{\mathbb{Q}}(\boldsymbol{q})$$

of n-dimensional projective space by the $\mu_{\boldsymbol{q}}$-action for which t_i multiplies the i-th coordinate of $\mathbb{P}^{n+1}_{\mathbb{Q}}$ with ζ_{q_i}. It is easy to see that every weighted projective space is isomorphic to one satisfying $\gcd(\boldsymbol{q}) = 1$. We assume this holds and write shorthand $\mathbb{P} = \mathbb{P}^{n+1}_{\mathbb{Q}}(\boldsymbol{q})$. (Indeed, every weighted projective space is isomorphic to a *normalised* one satisfying $\gcd(q_0, \dots, q_{i-1}, q_{i+1}, \dots, q_{n+1}) = 1$ for all $i = 0, \dots, n+1$, but not so linearly [Dol82, 1.3.1].)

11.1.2 Let d be a positive integer such that $q_i \mid d$ for all $0 \leq i \leq n+1$. To ease notation, we write $\varepsilon = \zeta_d$. We set $d_i = d/q_i$.

Definition 11.2. We define the *weighted diagonal hypersurface of multidegree* $(d_0, \dots, d_{n+1})$ to be

$$F = F_{(d_0,\dots,d_{n+1})} \subset \mathbb{P} : \quad x_0^{d_0} + \cdots + x_{n+1}^{d_{n+1}} = 0.$$

By the coprimality assumption on $\boldsymbol{q}$, we know that $d = \operatorname{lcm}(d_0, \dots, d_{n+1})$.

There is a natural quotient map

$$\pi_{\boldsymbol{q}} : F_d = F_{(d,\dots,d)} \to F$$

from the Fermat hypersurface of degree d and dimension n to the weighted quotient.

11.1.3 The singularities of F have been analysed by Y. Goto in [Got96, §2]. He proves that F is smooth if and only if $\gcd(q_i, q_j) = 1$ for all $i \neq j$ between 0 and $n+1$.

Otherwise, under the assumption that $\mathbb{P}$ is normalised, the singularities of F coincide with those of $\mathbb{P}$ and are cyclic, hence their resolution is explicitly described by a Hirzebruch continuous fraction. We will however assume smoothness for the rest of this chapter.

11.1.4 Another computation in [Dol82, Theorem 3.2.4, Theorem 3.3.4] shows that $\mathrm{H}^i(F, \mathcal{O}_F) = 0$ for $1 < i < n$ and that the dualising sheaf of F is $\omega_F = \mathcal{O}_F(d - q_0 - q_1 - \cdots - q_{n+1})$. In the case of $n = 2$, this implies a finite list of weighted diagonal surfaces whose minimal resolution is K3, of which two cases, with degrees $(4,4,4,4)$ and $(2,6,6,6)$, are smooth.

Moreover, the analogue of the Lefschetz hyperplane theorem holds [Dol82, Corollary 4.2.2]:

$$\mathrm{H}^i(F_{\mathbb{C}}, \mathbb{C}) \cong \mathrm{H}^{i+1}(\mathbb{P}_{\mathbb{C}}, \mathbb{C}), \quad i \neq n$$

For this reason, our interest lies in the middle cohomology, and as far as the transcendental Brauer group is concerned, in surfaces.

11.1.5 It is known since Shioda and Katsura [SK79, Theorem I] that Fermat hypersurfaces are dominated by a product of Fermat curves, thus the same is true for weighted diagonal hypersurfaces. In particular, Proposition 10.14 is applicable. Many properties of Fermat hypersurfaces can be shown inductively using the DPC structure. For example they have complex multiplication, i.e. the Mumford-Tate group is abelian [Dol14, Example

14.12]. We will however not use this fact.

11.1.6 The group $\mu_{d_0}\times\cdots\times\mu_{d_{n+1}}$ acts on $F_{\mathbb{C}}$. Namely, if we write $\mu_{d_i}=\langle u_i\rangle$, then u_i multiplies x_i with ζ_{d_i}. This action restricts trivially to μ_d where μ_d acts via

$$\varepsilon\mapsto(\varepsilon^{q_0},\ldots,\varepsilon^{q_{n+1}}).$$

Let $G=(\mu_{d_1}\times\cdots\times\mu_{d_{n+1}})/\langle(u_1\ldots u_{n+1})^{d_0}\rangle$. There is an isomorphism

$$G\cong(\mu_{d_0}\times\cdots\times\mu_{d_{n+1}})/\mu_d$$

that identifies u_0 with $(u_1\ldots u_{n+1})^{-1}$. Thus, the action of G on $F_{\mathbb{C}}$ can be described in a coordinate-symmetric or asymmetric way, depending on which is more convenient.

11.1.7 We define polynomials

$$\phi_i(x):=1+x+x^2+\cdots+x^{d_i-1}$$

for later use.

11.1.8 Set $n'=\lfloor n/2\rfloor$. Poincaré duality induces a unimodular bilinear form on the singular cohomology

$$H=\mathrm{H}^n(F_{\mathbb{C}},\mathbb{Z}(n')).$$

It is symmetric for even n and antisymmetric for odd n. Our first goal is to describe H together with its cup product.

One feature of the weighted ambient space is that coordinate hyperplane section classes differ depending on the chosen coordinate. The following lemma clarifies the situation.

Lemma 11.3. *Assume n is even. Let $l\in\mathrm{H}^2(F_{\mathbb{C}},\mathbb{Z})$ be the saturation of any hyperplane section class. Then $L=l^{n/2}\in\mathrm{H}^n(F_{\mathbb{C}},\mathbb{Z})$ has self-intersection $d_{\boldsymbol{q}}:=d/\prod_{j=0}^{n+1}q_j$.*

Proof. Let l_i be the class given by the hyperplane $x_i = 0$. Then the intersection product of $l_i^{n/2}$ and $l_j^{n/2}$ is $q_i q_j d / \prod_{j=0}^{n+1} q_j$ by the weighted version of Bézout's theorem Lemma 11.4 below for all $0 \leq i \leq n+1$. By the coprimality assumption on $\boldsymbol{q}$, there exists a linear combination l such that $L = l^{n/2}$ has self-intersection $d / \prod_{j=0}^{n+1} q_j$. It follows that the image of L under the pullback map

$$\pi_{\boldsymbol{q}}^* : \mathrm{H}^n(F_{\mathbb{C}}, \mathbb{Z}) \to \mathrm{H}^n((F_d)_{\mathbb{C}}, \mathbb{Z}),$$

is the hyperplane section class of $(F_d)_{\mathbb{C}}$, which is saturated. Hence L is saturated. □

Lemma 11.4 (Weighted Bézout's Theorem)**.** *Let $F_1, \ldots, F_{n+1}$ be hypersurfaces in $\mathbb{P}$. Then*

$$\deg(F_1 \ldots F_{n+1}) = \frac{\deg F_1 \ldots \deg F_{n+1}}{q_0 \cdots q_{n+1}}$$

where deg *is the weighted degree function.*

Proof. [EH99, Theorem 3.6]. □

11.1.9 It turns out that it is easier to first determine the primitive cohomology of F which is defined as follows.

Definition 11.5. The *primitive cohomology* $P = \mathrm{P}^n(F_{\mathbb{C}}, \mathbb{Z}(n')) \subseteq H$ is the kernel of the intersection pairing with a hyperplane section class.

One finds that P is the orthogonal complement to L if n is even, and equal to H if n is odd.

If M is a $\mathbb{Z}$-lattice, we write $M^* = \mathrm{Hom}(M, \mathbb{Z})$ for the dual. We have that $P^*/P \cong \mathbb{Z}/d_{\boldsymbol{q}}$ for even n by Lemma 11.3. Since G fixes L, we know that P is a $\mathbb{Z}[G]$-module.

11.2 Homology of affine diagonal hypersurfaces

11.2.1 Let $Z \subset F$ be the hyperplane section $x_0 = 0$. Its complement $U = F \setminus Z$ is the affine diagonal hypersurface in $\mathbb{A}^{n+1}_{\mathbb{Q}}$ given by

$$x_1^{d_1} + \cdots + x_{n+1}^{d_{n+1}} = -1.$$

We recall a topological description of the singular middle homology of $U_{\mathbb{C}}$ due to V. Pham [Pha65].

11.2.2 Define a simplex $\boldsymbol{e}$ as follows. Let

$$\Delta^n = \{z \in \mathbb{R}^{n+1} : z_1 + \cdots + z_{n+1} = 1, z_i \geq 0, \forall i = 1, \ldots, n+1\}$$

be the standard n-simplex. Then set

$$\begin{array}{rcl} \boldsymbol{e} : \Delta^n & \to & F(\mathbb{C}) \\ (z_1, \ldots, z_{n+1}) & \mapsto & (\zeta_{2d_1} z_1^{1/d_1}, \ldots, \zeta_{2d_{n+1}} z_1^{1/d_{n+1}}) \end{array}$$

where the roots of the z_i are chosen to be positive real numbers.

Note that this definition differs from Pham's because we introduced a minus sign on the right hand side of the affine equation of U so that the real structure of $F_{\mathbb{C}}$ is preserved.

Then

$$e = (1 - u_1^{-1}) \ldots (1 - u_{n+1}^{-1}) \boldsymbol{e}$$

is a cycle and generates $\mathrm{H}_n(U, \mathbb{Z})$ as a $\mathbb{Z}[G]$-module. Again, our definition of e differs from Pham's cycle, but only by the element $(-1)^{n+1} u_0 = \prod_{i=1}^{n+1} (-u_i)^{-1}$ which is invertible in $\mathbb{Z}[G]$.

Pham now shows that the $\mathbb{Z}[G]$-module morphism

$$\mathbb{Z}[G] \to \mathrm{H}_n(U_{\mathbb{C}}, \mathbb{Z}), \quad x \mapsto xe$$

is surjective and its kernel is the ideal

$$I = (\phi_i(u_i) : i = 1, \ldots, n+1) \subset \mathbb{Z}[G].$$

This identifies the middle homology of $U_{\mathbb{C}}$ with the group algebra quotient

$$R = \mathbb{Z}[G]/I.$$

11.2.3

Definition 11.6. For an abelian group M with a bilinear form Q and an action of G on M preserving Q, i.e. $Q(x, y) = Q(gx, gy)$ for all $x, y \in M, g \in G$, define the *sesquilinear extension*

$$\begin{array}{rcl} M \times M & \to & \mathbb{Z}[G] \\ x, y & \mapsto & x * y := \displaystyle\sum_{g \in G} Q(x, gy)g \in \mathbb{Z}[G]. \end{array}$$

Here, sesquilinearity means that $g(x * y) = gx * y = x * g^{-1}y$.

Note that $Q(x, y)$ can be recovered from $x * y$ by looking at the constant coefficient. The sesquilinear extension of the intersection product $(\cdot, \cdot)$ on $\mathrm{H}_n(U_{\mathbb{C}}, \mathbb{Z})$, which is invariant under G, is then characterised in [Pha65] as follows:

$$e * e = (-1)^{n(n+1)/2}(1 - u_0)(1 - u_1) \ldots (1 - u_{n+1}).$$

This value determines $*$ completely by sesquilinearity.

11.2.4 Complex conjugation induces an involution τ on the singular (co)homologies of $U_{\mathbb{C}}$ and $F_{\mathbb{C}}$, which anti-commutes with the action of G. One checks that τ sends $\boldsymbol{e}$ to $(u_1 \ldots u_{n+1})^{-1}\boldsymbol{e}$ and thus

$$\tau(e) = \frac{(1 - u_1) \ldots (1 - u_{n+1})}{(1 - u_1^{-1}) \ldots (1 - u_{n+1}^{-1})}(u_1 \ldots u_{n+1})^{-1} e = (-1)^{n+1} e.$$

11.3 PRIMITIVE COHOMOLOGY OF WEIGHTED PROJECTIVE DIAGONAL HYPERSURFACES

11.3.1 The Gysin sequence in homology of the smooth pair (F, Z) yields an exact sequence:

$$\begin{aligned} 0 \;\to\;& \mathrm{H}_{n+1}(F_{\mathbb{C}}, \mathbb{Z}) \to \mathrm{H}_{n-1}(Z_{\mathbb{C}}, \mathbb{Z}) \to \mathrm{H}_n(U_{\mathbb{C}}, \mathbb{Z}) \\ \to\;& \mathrm{H}_n(F_{\mathbb{C}}, \mathbb{Z}) \to \mathrm{H}_{n-2}(Z_{\mathbb{C}}, \mathbb{Z}) \to 0. \end{aligned}$$

As in [Loo10, §2], one obtains the following short exact sequence

$$0 \to \mathrm{P}_{n-1}(Z_{\mathbb{C}}, \mathbb{Z}) \to \mathrm{H}_n(U_{\mathbb{C}}, \mathbb{Z}) \to \mathrm{P}_n(F_{\mathbb{C}}, \mathbb{Z}) \to 0,$$

where the outer terms denote primitive homology, i.e. the kernel of the intersection pairing with a hyperplane class. This realises $\mathrm{P}_n(F_{\mathbb{C}}, \mathbb{Z})$ as the maximal non-degenerate quotient of $\mathrm{H}_n(U_{\mathbb{C}}, \mathbb{Z})$. We write e' for the image of e in $\mathrm{P}_n(F_{\mathbb{C}}, \mathbb{Z})$.

Lemma 11.7. *The maximal non-degenerate quotient of R is*

$$R' := R/(\phi_0(u_0)).$$

Proof. Since $(1 - u_i)$ is invertible in R for $i = 1, \ldots, n+1$ (cf. the later Lemma 11.13), we find

$$\mathrm{Ann}_R((1-u_0)(1-u_1)(1-u_2)\ldots(1-u_{n+1})) = \mathrm{Ann}_R(1-u_0) = \phi_0(u_0).$$

This shows that the kernel of the intersection product is generated by $\phi_0(u_0)$. □

11.3.2 The cap product with the fundamental class $[F] \in \mathrm{H}_{2n}(F_{\mathbb{C}}, \mathbb{Z})$ gives a Poincaré duality isomorphism

$$\mathrm{H}_n(F_{\mathbb{C}}, \mathbb{Z}) \cong \mathrm{H}^n(F_{\mathbb{C}}, \mathbb{Z}),$$

which identifies the intersection product on homology and the cup product $\langle\cdot,\cdot\rangle$ on cohomology. The isomorphism sends a multiple of a hyperplane class to a multiple of a hyperplane class and thus restricts to an isomorphism of primitive homology with primitive cohomology. Furthermore, since the action of G preserves $[F]$ and τ sends $[F]$ to $(-1)^n[F]$, this is an isomorphism of $\mathbb{Z}[G]$-modules which (anti-)commutes with τ.

11.3.3 Finally, to take the Tate twist by $\mathbb{Z}(n')$ into account, note that there is an isomorphism

$$\begin{array}{rcl} \mathrm{H}^n(F_{\mathbb{C}},\mathbb{Z}) & \to & H \\ x & \mapsto & x(n') := (2\pi i)^{n'}x \end{array}$$

The twisted action $\tau(n')$ of complex conjugation on H is thus given by the product of τ and complex conjugation acting on the coefficients $\mathbb{C}$:

$$\begin{array}{rcl} \tau(n')(e'(n')) = \tau((2\pi i)^{n'})\tau(e') & = & (-1)^{n'+n+1+n}e'(n') \\ & = & -(-1)^{n'}e'(n'). \end{array}$$

This is a good point to clarify the relation between three different "complex conjugations" on $\mathrm{H}^n(F_{\mathbb{C}},\mathbb{C})$ as described in [Del79]. The "complex conjugation" τ (or F_∞ in the notation of [Del79, 0.2.5]) is induced by the involution on the points $F(\mathbb{C})$. The second "complex conjugation" is induced by the action of complex conjugation on the coefficients. Each of these actions swaps the Hodge spaces $\mathrm{H}^{p,q}(F_{\mathbb{C}})$ and $\mathrm{H}^{q,p}(F_{\mathbb{C}})$ (see [Sil89, I.2.4] for the latter) and their composition is the "complex conjugation" induced by the comparison isomorphism $\mathrm{H}^n(F_{\mathbb{C}},\mathbb{C}) \cong \mathrm{H}^n_{\mathrm{dR}}(F_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C}$, which hence preserves the Hodge spaces $\mathrm{H}^{p,q}(F_{\mathbb{C}})$ [Del79, Proposition 1.4, Corollaire 1.6].

11.3.4 The above discussion shows:

Proposition 11.8. *There is an isomorphism of $\mathbb{Z}[G]$-modules*

$$P \cong R' = \mathbb{Z}[G]/\left(\phi_i(u_i) : i = 0,\dots,n+1\right)$$

which sends $e'(n')$ *to* 1*. Under this isomorphism,*

$$\tau(n')(g) = -(-1)^{n'} g^{-1}$$

for all $g \in G$*.*

The sesquilinear extension $*$ *of the cup product is induced by*

$$e'(n') * e'(n') = (-1)^{n(n+1)/2}(1-u_0)(1-u_1)\dots(1-u_{n+1}).$$

11.4 STRUCTURE AS A $\mathbb{Z}[G]$-MODULE

The presence of the $\mathbb{Z}[G]$-module structure on P is a crucial tool to relate the singular, Hodge and ℓ-adic cohomologies of F to each other. We thus have to understand it first.

11.4.1 Let $E = \mathbb{Q}(\varepsilon)$, the d-th cyclotomic field. We write

$$\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^\times) = \left\{ a \in (q_1\mathbb{Z}/d \times \cdots \times q_{n+1}\mathbb{Z}/d) : d_0 \mid \sum_{i=1}^{n+1} a_i \right\}$$

for the group of complex characters on G. In the symmetric notation,

$$\widehat{G} \cong \left\{ a \in (q_0\mathbb{Z}/d \times \cdots \times q_{n+1}\mathbb{Z}/d) : \sum_{i=0}^{n+1} a_i = 0 \in \mathbb{Z}/d \right\}.$$

A tuple $(a_1, \dots, a_{n+1}) \in q_1\mathbb{Z}/d \times \cdots \times q_{n+1}\mathbb{Z}/d$ corresponds to the character

$$\chi(u_1^{l_1} \dots u_{n+1}^{l_{n+1}}) = \varepsilon^{a_1 l_1 + \cdots + a_{n+1} l_{n+1}}.$$

Attached to χ is an element

$$\alpha_\chi = \alpha_{a_1}(u_1) \dots \alpha_{a_{n+1}}(u_{n+1}) \in E[G]$$

where

$$\alpha_i(u) = \frac{1}{d}\sum_{j=0}^{d-1} \varepsilon^{-ij} u^j.$$

The family $(\alpha_\chi)_{\chi\in\widehat{G}}$ is an orthogonal basis of idempotent eigenvectors: it satisfies $\alpha_\chi\alpha_\rho = \delta_{\chi,\rho}$ where δ is the Kronecker delta. One easily checks that $g\alpha_\chi = \chi(g)\alpha_\chi$ for all $g \in G, \chi \in \widehat{G}$.

The classical representation theory of finite groups now gives that after extending the base to E, the $\mathbb{Z}[G]$-module $\mathbb{Z}[G]$ decomposes into a sum of 1-dimensional eigenspaces $V_\chi = \langle\alpha_\chi\rangle$:

$$E[G] = \bigoplus_{\chi\in\widehat{G}} V_\chi.$$

By Proposition 11.8, $P \otimes E$ is the quotient of $E[G]$ by the ideal

$$I' = (\phi_i(u_i) : i = 0, \ldots, n+1).$$

We find that

$$\phi_i(u_i)E[G] = \bigoplus_{\chi\in S_i} V_\chi$$

for $i = 0, \ldots, n+1$, where S_i is the set of all characters $\chi \in \widehat{G}$ restricting trivially to the factor μ_{d_i}. Thus

$$P \otimes E = \bigoplus_{\chi\in S} V_\chi$$

where $S = \widehat{G} \setminus \bigcup_{i=0}^{n+1} S_i$. In other words S comprises the characters corresponding to $(a_0, \ldots, a_{n+1}) \in \{1, \ldots, d-1\}^{n+2}$ such that $q_i \mid a_i$ for all $i = 0, \ldots, n+1$ and $d \mid \sum_{i=0}^{n+1} a_i$.

11.4.2 The following lemma shows that our idempotent basis is orthogonal with respect to the cup product.

Lemma 11.9. *Set* $\Xi = \mathrm{Re}$ *(the real part) for even* n *and* $\Xi = \mathrm{Im}$ *(the* *imaginary part) for odd n. For all* $\chi, \rho \in S$,

$$\langle \alpha_\rho, \alpha_\chi \rangle = (-1)^{n(n+1)/2} \prod_{i=0}^{n+1} q_i \frac{2}{d^{n+1}} \Xi((1-\varepsilon^{a_1}) \dots (1-\varepsilon^{a_{n+1}}))\delta_{\rho^{-1},\chi}$$

where χ *corresponds to* $(a_1, \dots, a_{n+1})$.

Proof. Using the bilinearity of the cup product, we find that

$$\langle \alpha_\rho, \alpha_\chi \rangle = \langle 1, \alpha_{\rho^{-1}} \alpha_\chi \rangle.$$

From the idempotency property, it follows that $\langle \alpha_\rho, \alpha_\chi \rangle = 0$ if $\chi \neq \rho^{-1}$. If $\chi = \rho^{-1}$, then

$$\langle \alpha_{\chi^{-1}}, \alpha_\chi \rangle = \langle 1, \alpha_\chi^2 \rangle = \langle 1, \alpha_\chi \rangle$$

is the coefficient of 1 in the expression

$$(-1)^{n(n+1)/2} \alpha_{\chi^{-1}} (1-u_0) \dots (1-u_{n+1}) \in E[G],$$

which evaluates to

$$\prod_{i=0}^{n+1} q_i \frac{(-1)^{n(n+1)/2}}{d^{n+1}} \left(\prod_{i=1}^{n+1} (1-\varepsilon^{a_i}) + (-1)^n \prod_{i=1}^{n+1} (1-\overline{\varepsilon}^{a_i}) \right).$$

□

11.5 HODGE STRUCTURE

11.5.1 Due to our chosen twist, P carries with it a pure Hodge structure of weight 0 for even n, respectively weight 1 for odd n. It is preserved by the action of G. For a character χ given by $(a_1, \dots, a_{n+1})$, define

$$q(\chi) = \lfloor \frac{\sum_{i=1}^{n+1} a_i}{d} \rfloor - n' = \frac{\sum_{i=0}^{n+1} a_i}{d} - 1 - n'$$

In [Gri69], Griffiths describes the Hodge structure of a smooth projective hypersurface (see also [Voi03, Theorem 6.10]). This is generalised by Dolgachev in [Dol82, §4.2] to the weighted projective case.

Theorem 11.10. *Let $q \in \mathbb{Z}$. The graded piece of the Hodge filtration*

$$F^{(n \bmod 2)-q}P/F^{(n \bmod 2)-q-1}P$$

has a basis given by the differential forms

$$\mathrm{Res}_F \, \mathrm{Res}_{\mathbb{P}} \left(\prod_{i=0}^{n+1} x_i^{a_i} \right) \frac{dx_0 \wedge \cdots \wedge dx_{n+1}}{(x_0^{d_0} + \cdots + x_{n+1}^{d_{n+1}})^{q+1}}$$

where $(a_0, \ldots, a_{n+1})$ runs over all tuples in $\{1, \ldots, d-1\}$ such that $q_i \mid a_i$ and $q + 1 + n' = \frac{1}{d}\sum_{i=0}^{n+1} a_i$.

Proposition 11.11. *The Hodge summand $P^{p,q}$ is the direct sum of V_χ such that $q(\chi) = q$.*

Proof. From Theorem 11.10, we deduce that G acts on $P^{p,q}$ via those $\chi \in S$ with $q(\chi) = q$. ☐

The Hodge structure on P can also be recovered from the one on $\mathrm{P}^n((F_d)_{\mathbb{C}}, \mathbb{Z})$ via the quotient map $\pi_{\boldsymbol{q}}$ but then the determination of the Hodge structure of F_d would use the classical Griffiths theorem.

11.5.2 Let us briefly assume that $n = 2$. The transcendental lattice $T(F_{\mathbb{C}})$ is the smallest saturated sublattice of P such that $P^{-1,1} \subset T(F_{\mathbb{C}}) \otimes E$. In particular $V_\chi \subset T(F_{\mathbb{C}}) \otimes E$ for all χ with $q(\chi) = 1$. The group $\mathrm{Gal}(E/\mathbb{Q}) = (\mathbb{Z}/d\mathbb{Z})^\times$ acts on $P \otimes E$ via the second factor so that an element $t \in (\mathbb{Z}/d\mathbb{Z})^\times$ sends α_χ to α_{χ^t}. Hence $V_\chi \subset T(F_{\mathbb{C}}) \otimes E$ for all $\chi \in S$ whose $\mathrm{Gal}(E/\mathbb{Q})$-orbit contains χ' with $q(\chi') = 1$. Denote this subset of S by S_T.

It follows that:

Lemma 11.12.

$$T(F_{\mathbb{C}}) = P \cap \bigoplus_{\chi \in S_T} V_\chi.$$

11.5.3 We conjecture that Δ, the discriminant of the transcendental or algebraic cycles on F, always divides a power of d. In particular, this would imply that

$$\mathrm{Br}(F)[\ell^\infty] \to \mathrm{Br}(\overline{F})[\ell^\infty]^{\mathrm{Gal}(\overline{K}/K)}$$

is always surjective for $\ell \nmid d$ by Section 10.2.7.

The conjecture is known for the Fermat surface $F_{(d,d,d,d)}$ where $d \leq 4$ or $\gcd(d,6) = 1$ because in this case, lines generate $\mathrm{Pic}(F_{\mathbb{C}})$ [Deg15] and the discriminant of the lattice they span is known to divide $d^{9(d-1)(d-2)+4-3(d \bmod 2)}$ [SSvL10, Corollary 3.2].

11.6 Recovering full cohomology

11.6.1 When n is even, the task remains to recover the full cohomology lattice. We have shown in Lemma 11.3 that the saturation L of the $(n/2)$-fold power of any hyperplane section class has self-intersection $d_{\boldsymbol{q}}$. Since H is unimodular and $\mathbb{Z}L$ and P are orthogonal, saturated sublattices of H, we get

$$P^*/P \cong H/(P \oplus \mathbb{Z}L) \cong (\mathbb{Z}L)^*/\mathbb{Z}L \cong \mathbb{Z}/d_{\boldsymbol{q}}.$$

The group $(\mathbb{Z}L)^*/\mathbb{Z}L$ is generated by the class of the linear map $\langle \frac{1}{d_{\boldsymbol{q}}} L, \cdot \rangle$. We deduce that $H/(P \oplus \mathbb{Z}L)$ is generated by $\frac{1}{d_{\boldsymbol{q}}}(L + \xi)$ for some $\xi \in P$. The integrality of the cup product requires that $\langle \xi, P \rangle \subset d_{\boldsymbol{q}}\mathbb{Z}$. Note that ξ is only uniquely determined in $P/d_{\boldsymbol{q}}P$. Our aim is to determine one of the many possible lifts to P.

11.6.2 We need the following easy identities involving the polynomial functions

$$\phi(x) = \sum_{i=0}^{d-1} x^i, \quad \rho(x,y) = \sum_{0 \leq l \leq m \leq d-2} y^l x^m.$$

Lemma 11.13. *(i)* $(1-y)\phi(xy) = (1-x)(1-y)\rho(x,y)$ *inside the ring* $\mathbb{Z}[x,y]/(x^d - 1, y^d - 1)$.

(ii) $(1-x)\rho(1,x) = d$ *inside the ring* $\mathbb{Z}[x]/(\phi(x))$*, in particular* $(1-x)$ *is invertible in* $\mathbb{Q}[x]/(\phi(x))$*.*

(iii) $\phi(xy) = (1-x)\rho(x,y)$ *inside the ring* $\mathbb{Z}[x,y]/(\phi(x),\phi(y))$*.*

11.6.3 Recall that there is a quotient map

$$\pi_{\boldsymbol{q}} : F_d \to F_{(d_0,\ldots,d_{n+1})}.$$

Let $\Lambda \in \mathrm{H}_n((F_d)_{\mathbb{C}},\mathbb{Z})$ be the homology class of the linear subspace given by

$$x_0 = \zeta_{2d}x_1, x_2 = \zeta_{2d}x_3, \ldots, x_n = \zeta_{2d}x_{n+1}.$$

Because the intersection number of Λ with a hyperplane section L_d of F_d is 1, it follows that Λ generates $\mathrm{H}_n((F_d)_{\mathbb{C}},\mathbb{Z})$ modulo primitive homology.

Proposition 11.14. *Let*

$$\begin{aligned} c &= (1-u_0)^{-1}\phi(u_0u_1)(1-u_2)^{-1}\phi(u_2u_3)\ldots(1-u_n)^{-1}\phi(u_nu_{n+1}) \\ &= \rho(u_0,u_1)\rho(u_2,u_3)\ldots\rho(u_n,u_{n+1}) \in \mathrm{P}_n((F_d)_{\mathbb{C}},\mathbb{Z}). \end{aligned}$$

Then $\frac{1}{d}(L_d + (-1)^{n(n+1)/2}c) = \Lambda$*.*

Proof. The intersection product $(\cdot,\cdot)$ on $\mathrm{H}_n((F_d)_{\mathbb{C}},\mathbb{Z})$ is non-degenerate, hence we only need to show that the images of $\frac{1}{d}(L_d + (-1)^{n(n+1)/2}c)$ and Λ in $\mathrm{H}_n((F_d)_{\mathbb{C}},\mathbb{Z})^*$ are equal. It is clear that

$$(\Lambda, L_d) = 1 = \frac{1}{d}(L_d, L_d) = (\frac{1}{d}(L_d + (-1)^{n(n+1)/2}c), L_d)$$

and $(\frac{1}{d}(L_d + (-1)^{n(n+1)/2}c), x) = \frac{1}{d}((-1)^{n(n+1)/2}c, x)$ for all $x \in \mathrm{P}_n((F_d)_{\mathbb{C}},\mathbb{Z})$.

Degtyarev and Shimada have computed in [DS16, p. 12, Proof of Part (a) of Theorem 1.1] that the image of Λ under the map

$$\mathrm{ev} : \mathrm{H}_n((F_d)_{\mathbb{C}},\mathbb{Z}) \to \mathbb{Z}[(\mu_d)^{n+1}], x \to \sum_{g\in(\mu_d)^{n+1}} (x,g)g$$

is given by $\psi := (1-u_1)(1-u_3)\dots(1-u_{n+1})\phi(u_2u_3)\dots\phi(u_nu_{n+1})$. So it remains to show that $\mathrm{ev}(c) = (-1)^{n(n+1)/2}d\psi$.

Using the $(\mu_d)^{n+1}$-invariance of the intersection product on F_d, we get

$$\mathrm{ev}(h) = \sum_{g\in(\mu_d)^{n+1}} (h,g)g = \sum_{g\in(\mu_d)^{n+1}} (1,gh^{-1})g = \sum_{g\in(\mu_d)^{n+1}} (1,g)gh = \mathrm{ev}(1)h$$

for all $h \in (\mu_d)^{n+1}$ and by bilinearity of the intersection product, the same equation holds for $h \in \mathrm{P}_n((F_d)_{\mathbb{C}}, \mathbb{Z})$.

Recall that by Proposition 11.8 $\mathrm{ev}(1) = (-1)^{n(n+1)/2}(1-u_0)\dots(1-u_{n+1})$. Thus $(-1)^{n(n+1)/2}\mathrm{ev}(c)$ equals

$$\begin{aligned}
& (1-u_0)(1-u_1)\dots(1-u_{n+1})\rho(u_0,u_1)\rho(u_2,u_3)\dots\rho(u_n,u_{n+1}) \\
= \; & (1-u_1)(1-u_3)\dots(1-u_{n+1})\phi(u_0u_1)\phi(u_2u_3)\dots\phi(u_nu_{n+1}) \\
= \; & (1-u_1)(1-u_3)\dots(1-u_{n+1})\phi(u_2\dots u_{n+1})\phi(u_2u_3)\dots\phi(u_nu_{n+1}) \\
= \; & (1-u_1)(1-u_3)\dots(1-u_{n+1})d\phi(u_2u_3)\dots\phi(u_nu_{n+1}) = d\psi.
\end{aligned}$$

□

11.6.4 The image of $\frac{1}{d}(L_d + (-1)^{n(n+1)/2}c)$ under the pushforward map

$$(\pi_{\boldsymbol{q}})_* : \mathrm{H}_n((F_d)_{\mathbb{C}}, \mathbb{Z}) \to \mathrm{H}_n(F_{\mathbb{C}}, \mathbb{Z})$$

is

$$\frac{1}{d_{\boldsymbol{q}}}L + \frac{1}{d}(-1)^{n(n+1)/2}c.$$

Here we use that $(\pi_{\boldsymbol{q}})_*(\pi_{\boldsymbol{q}})^*$ equals $\deg \pi_{\boldsymbol{q}} = \prod_{j=0}^{n+1} q_j$. As a consequence, we infer that $\xi = \frac{1}{\prod_{i=0}^{n+1} q_i}(-1)^{n(n+1)/2}c \in P$ is a possible choice such that $H/(P \oplus \mathbb{Z}L)$ is generated by $\frac{1}{d_{\boldsymbol{q}}}(L+\xi)$.

11.7 TWISTING AND GALOIS REPRESENTATION

11.7.1 Let k be a number field containing E with integer ring $\mathcal{O}_k$ and let ℓ be a prime number. We write $\overline{k}$ for an algebraic closure of k, and for a variety X over k, we write $\overline{X} = X \times_k \overline{k}$. We fix an embedding $k \subset \mathbb{C}$.

For an $(n+1)$-tuple $(c_0, \ldots, c_{n+1})$ with values in $k^\times$, we consider the hypersurface in $\mathbb{P}_k$ given by

$$c_0 x_0^{d_0} + \cdots + c_{n+1} x_{n+1}^{d_{n+1}} = 0.$$

Without loss of generality we assume that $c_0 = 1$ and denote this hypersurface by $X_{\boldsymbol{c}}$ where $\boldsymbol{c} = (c_1, \ldots, c_{n+1})$. The "untwisted" hypersurface F is given as $X_{(1,\ldots,1)}$ and $X_{\boldsymbol{c}}$ is obtained from F by twisting with the 1-cocycle which is the image of $\boldsymbol{c}$ under the composition of natural maps

$$(k^\times)^{n+1} \to \prod_{j=1}^{n+1} (k^\times / k^{\times d_j}) = \mathrm{H}^1(\Gamma_k, G) \to \mathrm{H}^1(\Gamma_k, \mathrm{Aut}_{\overline{k}}(\overline{F})).$$

In particular, $F_{\mathbb{C}} \cong (X_{\boldsymbol{c}})_{\mathbb{C}}$ and the previous discussion of the Betti and de Rham cohomologies also applies to $X_{\boldsymbol{c}\mathbb{C}}$, except that the action of τ has to be twisted by the above 1-cocycle.

11.7.2 The absolute Galois group $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ acts on $\mathrm{H}^n_{\text{ét}}(\overline{X}_{\boldsymbol{c}}, \mathbb{Z}_\ell(n'))$. The comparison isomorphism between ℓ-adic and singular cohomology identifies $\mathrm{H}^n_{\text{ét}}(\overline{X}_{\boldsymbol{c}}, \mathbb{Z}_\ell(n'))$ and $\mathrm{H}^n((X_{\boldsymbol{c}})_{\mathbb{C}}, \mathbb{Z}(n')) \otimes \mathbb{Z}_\ell$. Because the action of Γ_k preserves hyperplane classes, Γ_k also acts on the primitive ℓ-adic cohomology

$$P_\ell = P \otimes \mathbb{Z}_\ell \cong \mathrm{P}^n_{\text{ét}}(\overline{X}_{\boldsymbol{c}}, \mathbb{Z}_\ell(n')).$$

We write $P_{\ell,F}$ in the case $X_{\boldsymbol{c}} = F$. Furthermore, if $\mathbb{Q}(c) \subset \mathbb{R}$, $X_{\boldsymbol{c}}$ can be naturally defined over $\mathbb{R}$ and the comparison isomorphism identifies the induced action of complex conjugation on $\overline{X}_{\boldsymbol{c}}$ with the action of $\tau(n')$ [Del79, 0.2.5].

Let $\mathcal{O}$ be the ring of integers of E. Let λ be a prime of $\mathcal{O}$ lying above ℓ. We have that

$$P_\ell \otimes_{\mathbb{Z}_\ell} E_\lambda = \bigoplus_{\chi \in S} V_\chi \otimes_E E_\lambda$$

and because the action of G commutes with the action of Γ_k, this decomposition is preserved by Γ_k. By the Chebotarev density theorem, to determine the action of Γ_k on P, it suffices to determine the action of $\mathrm{Frob}_\mathfrak{p} \in \Gamma_k$ for all prime ideals $\mathfrak{p} \subset \mathcal{O}_k$ such that $\mathfrak{p} \nmid d\ell$.

If $\mathfrak{p}$ is such a prime, we let $\mathbb{F}_\mathfrak{p}$ be the residue field, of characteristic p with $\mathrm{N}(\mathfrak{p})$ elements. We define the multiplicative character $\psi : \mathbb{F}_\mathfrak{p}^\times \to \mu_d$ by the condition that

$$\psi(x) \bmod \mathfrak{p} = x^{(\mathrm{N}(\mathfrak{p})-1)/d}.$$

From this discussion, the relation between the Galois representations on the ℓ-adic cohomologies of F and $X_{\boldsymbol{c}}$ is as follows.

Lemma 11.15. *Let* $\chi = (a_1, \ldots, a_{n+1}) \in S$. *Let* $h(\chi)$ *be the eigenvalue by which* $\mathrm{Frob}_\mathfrak{p}$ *acts on*

$$V_\chi \otimes_E E_\lambda \subset P_{\ell,F}.$$

Then the eigenvalue of $\mathrm{Frob}_\mathfrak{p}$ *on*

$$V_\chi \otimes_E E_\lambda \subset P_\ell$$

is given by

$$\frac{h(\chi)}{\prod_{j=1}^{n+1} \psi(c_j)^{a_j}}.$$

11.7.3 Fix a p-th root of unity ζ.

Definition 11.16. Let $r \in \mathbb{Z}/d$. The *Gauss sum* $g(r) \in \mathbb{Q}(\varepsilon, \zeta)$ is the element

$$g(r) = \sum_{x \in \mathbb{F}_\mathfrak{p}^\times} \psi(x)^r \zeta^{\mathrm{Tr}_{\mathbb{F}_\mathfrak{p}/\mathbb{F}_p}(x)}.$$

Let $\chi \in \widehat{G}$ correspond to $(a_0, \ldots, a_{n+1})$. Define the *Jacobi sum* $J_\mathfrak{p}(\chi) \in \mathcal{O}_E$

by

$$\begin{aligned} J_{\mathfrak{p}}(\chi) &= \sum_{x_1+\dots+x_{n+1}=1} \psi(x_1)^{a_1}\dots\psi(x_{n+1})^{a_{n+1}} \\ &= \frac{g(a_1)\dots g(a_{n+1})}{g(a_1+\dots+a_{n+1})} = \mathrm{N}(\mathfrak{p})^{-1}\psi(-1)g(a_0)\dots g(a_n). \end{aligned}$$

The equalities in the above definition of $J_{\mathfrak{p}}(\chi)$ follow from [IR82, Chapter 8, Theorem 3 and Corollary 1].

11.7.4 In [Wei49], A. Weil has shown that the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $P_{\ell,F}$ are exactly $(\psi(-1)\,\mathrm{N}(\mathfrak{p})^{-n'}J_{\mathfrak{p}}(\chi))_{\chi\in S}$. It remains to match these to the known eigenspace decomposition under the action of G. In the classical projective Fermat case, this was done by D. Ulmer [Ulm02, 7.6] but the statement goes back to Shioda.

It is however possible to give a short and simple proof using the Fourier transform on G. The inspiration comes from the equivariant Lefschetz trace formula by Deligne and Lusztig [DL76, p. 119].

Proposition 11.17. *Let λ be a prime of $E=\mathbb{Q}(\mu_d)$ above ℓ. Let $\mathfrak{p}$ be a prime not dividing $d\ell$. Then for all $\chi\in S$, the action of* $\mathrm{Frob}_{\mathfrak{p}}$ *on $V_\chi\otimes E_\lambda\subset P_{\ell,F}$ is multiplication by*

$$\psi(-1)\,\mathrm{N}(\mathfrak{p})^{-n'}J_{\mathfrak{p}}(\chi).$$

Proof. We define two functions $h_1,h_2:\widehat{G}\to E_\lambda$ and show that their Fourier transform agrees.

Let $h_1(\chi)=\psi(-1)\,\mathrm{N}(\mathfrak{p})^{-n'}J_{\mathfrak{p}}(\chi)$.

Let $h_2(\chi)$ be the eigenvalue by which $\mathrm{Frob}_{\mathfrak{p}}$ acts on $V_\chi\otimes E_\lambda$ for $\chi=(a_1,\dots,a_{n+1})\in S$ and let $h_2(\chi)=0$ for $\chi\in\widehat{G}\backslash S$. For arbitrary $(c_1,\dots,c_{n+1})\in G$, we choose preimages $\tilde{c}_i\in\mathbb{F}_{\mathfrak{p}}$ under the multiplicative character ψ.

By Lemma 11.15, the hypersurface $X_{\tilde{\boldsymbol{c}}}$ has eigenvalues $h_2(\chi)/(\psi(\tilde{c}_1)^{a_1}\dots\psi(\tilde{c}_{n+1})^{a_{n+1}})$. The Lefschetz trace formula thus gives

$$\#X_{\tilde{\boldsymbol{c}}}(\mathbb{F}_{\mathfrak{p}})=\#\mathbb{P}(\mathbb{F}_{\mathfrak{p}})+(-1)^n f(\boldsymbol{c})\,\mathrm{N}(\mathfrak{p})^{n'}$$

where

$$f(\boldsymbol{c}) = \sum_{\chi \in \widehat{G}} h_2(\chi)/(\psi(\tilde{c}_1)^{a_1} \ldots \psi(\tilde{c}_{n+1})^{a_{n+1}}) = \sum_{\chi \in \widehat{G}} \chi(-\boldsymbol{c}) h_2(\chi).$$

According to [Wei49], the same holds true with h_2 replaced by h_1 in the formula for f. Hence, the inverse Fourier transform gives

$$h_1(\chi) = \frac{1}{\#G} \sum_{c \in G} \chi(\boldsymbol{c}) f(\boldsymbol{c}) = h_2(\chi)$$

$\square$

11.7.5 Up to a unit, Jacobi sums can be computed using Stickelberger elements.

Definition 11.18. Let σ_t be the image of $t \in (\mathbb{Z}/d\mathbb{Z})^\times$ under the isomorphism $(\mathbb{Z}/d\mathbb{Z})^\times \xrightarrow{\simeq} \mathrm{Gal}(E/\mathbb{Q})$. For $x \in \mathbb{Q}$, let $\langle x \rangle = x - \lfloor x \rfloor$ be the fractional part of x.

For an integer a, define the *Stickelberger element*

$$\theta(a) = \sum_{\bar{t} \in (\mathbb{Z}/d\mathbb{Z})^\times} \left\langle \frac{ta}{d} \right\rangle \sigma_{-\bar{t}}^{-1} \in \mathbb{Q}[\mathrm{Gal}(E/\mathbb{Q})],$$

where t is a lift of $\bar{t}$ to $\mathbb{Z}$.

For a character $\chi = (a_0, \ldots, a_{n+1}) \in S$, define

$$\omega(\chi) = \sum_{i=0}^{n+1} \theta(a) - \sum_{\bar{t} \in (\mathbb{Z}/d\mathbb{Z})^\times} \sigma_{\bar{t}} = \sum_{\bar{t} \in (\mathbb{Z}/d\mathbb{Z})^\times} \left\lfloor \sum_{i=1}^{n+1} \langle \frac{ta_i}{d} \rangle \right\rfloor \sigma_{-\bar{t}}^{-1} \in \mathbb{Z}[\mathrm{Gal}(E/\mathbb{Q})].$$

Weil shows in [Wei52]:

Proposition 11.19. *Let $\chi \in S$. Then the following equality of ideals holds:*

$$(J_{\mathfrak{p}}(\chi)) = \omega(\chi)(\mathfrak{p}).$$

11.7.6 The explicit determination of Gauss and Jacobi sums including their sign is in general a difficult subject. The case $d = 4$ was treated by Swinnerton-Dyer in [PSD91] and Chapter 12 will treat the case $d = 6$. The following property of Gauss sums will be helpful.

Lemma 11.20. *We have* $g(r)g(-r) = \psi(-1)^r \operatorname{N}(\mathfrak{p})$.

Proof. See for example [IR82, Exercise 10.22(d)]. □

In the case of regular primes p however, we can say more. First, we define a notion of primary elements in cyclotomic rings as found in the statement of Eisenstein reciprocity.

Definition 11.21. We call $x \in \mathcal{O}$ *primary*, if it is congruent to a rational integer (i.e. an element in $\mathbb{Z}$) modulo $(1-\varepsilon)^2$.

For every element $x \in \mathcal{O}$, there exists a unit $u \in \mathcal{O}^\times$ such that ux is primary.

Proposition 11.22. *Assume that d is prime and does not divide the class number $h = h(\mathbb{Q}(\varepsilon))$. Let $\mathfrak{p}$ be a prime not dividing $d\ell$. Let x be a primary generator of the principal ideal $\mathfrak{p}^h$. Then $J_{\mathfrak{p}}(\chi)$ equals up to sign an h-th root of $\pm\omega(\chi)(x)$. If n is even and h is odd, the sign is positive.*

Proof. From Proposition 11.19, we infer that $(J_{\mathfrak{p}}(\chi))^h = \varepsilon(\mathfrak{p})(\omega(\chi)(x))$ for some unit $\varepsilon(\mathfrak{p}) \in E^\times$. However, the usual argument gives $|J_{\mathfrak{p}}(\chi)^h|^2 = \operatorname{N}(\mathfrak{p})^{nh} = \operatorname{N}(\omega(\chi)(x))$. This is true for all Galois conjugates of $\mathfrak{p}$ and so by a theorem of Kronecker $\varepsilon(\mathfrak{p}) = J_{\mathfrak{p}}(\chi)^h/\omega(\chi)(x)$ is a root of unity ε^i for some $0 \le i \le d-1$. We want to show that $\varepsilon(\mathfrak{p}) = \pm 1$.

To do so, notice that $J_{\mathfrak{p}}(\chi)^h \equiv 1 \bmod (1-\varepsilon)^2$ [Lem00, Lemma 11.6]. Furthermore, $\omega(\chi)(x)$ is primary since Galois conjugates and products of primary elements are primary. Therefore, $\varepsilon(\mathfrak{p})$ is a primary root of unity but the only primary roots of unity are ± 1.

More precisely, if $x \equiv z \bmod (1-\varepsilon)^2$ for some $z \in \mathbb{Z}$, then

$$\omega(\chi)(x) \equiv \omega(\chi)(z) \equiv z^{n(d-1)/2} \bmod (1-\varepsilon)^2.$$

If n is even, it follows that $\omega(\chi)(x) \equiv 1 \bmod (1-\varepsilon)^2$ and so $\varepsilon(\mathfrak{p}) = 1$. □

11.7.7 Unfortunately, we are not able to descend the Galois action to $K = \mathbb{Q}$ unless E is a quadratic number field. To do so would require determining the Galois action of lifts of automorphisms $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ to $\Gamma_{\mathbb{Q}}$. One can formally imitate our approach to the action of complex conjugation by substituting ζ_{2d_i} with $\sigma(\zeta_{2d_i})$ in the definition of $\boldsymbol{e}$. This yields a potential candidate for the missing action.

12

DIAGONAL SURFACES OF DEGREE $(2, 6, 6, 6)$

In the notation of Chapter 11 we now restrict to $n = 2$, $d = 6$ and $\boldsymbol{q} = (3, 1, 1, 1)$. We work over the Eisenstein numbers

$$k = E = \mathbb{Q}(\zeta_6) = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$$

and write $\mathcal{O} = \mathcal{O}_E$ for their ring of integers. Note that $\mathcal{O}$ is a principal ideal domain. Explicitly, we consider the surface

$$F : x_0^2 + x_1^6 + x_2^6 + x_3^6 = 0$$

in $\mathbb{P}^{n+1}_{\mathbb{Q}}(3, 1, 1, 1)$.

The set S then equals the 21-element set

$$\begin{aligned} \{(1,1,1),(5,5,5),\\ (1,3,5),(1,5,3),(3,1,5),(3,5,1),(5,1,3),(5,3,1),\\ (2,2,5),(2,5,2),(5,2,2),\\ (4,4,1),(4,1,4),(1,4,4),\\ (2,3,4),(2,4,3),(3,2,4),(3,4,2),(4,2,3),(4,3,2),\\ (3,3,3)\}. \end{aligned}$$

12.1 EXPLICIT TRANSCENDENTAL LATTICE

12.1.1 By Lemma 11.12, we have that

$$T(F_{\mathbb{C}}) = P \cap (V_{(1,1,1)} \oplus V_{(5,5,5)})$$

Lemma 12.1. *We have $T(F_{\mathbb{C}}) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where*

$$w_1 = 24\sqrt{-3}(\varepsilon\alpha_{(1,1,1)} + \varepsilon^2\alpha_{(5,5,5)})$$

$$w_2 = 24\sqrt{-3}(\varepsilon^2\alpha_{(1,1,1)} + \varepsilon\alpha_{(5,5,5)})$$

Furthermore,

$$\langle w_1, w_1\rangle = \langle w_2, w_2\rangle = 24$$

and

$$\langle w_1, w_2\rangle = \langle w_2, w_1\rangle = 12.$$

Proof. Clearly, $T(F_{\mathbb{C}}) \otimes \mathbb{Q} = \mathbb{Q}w_1 \oplus \mathbb{Q}w_2$.

We calculate from Lemma 11.9

$$\begin{aligned} \langle \alpha_{(1,1,1)}, u_1^{\alpha}u_2^{\beta}u_3^{\gamma}\rangle &= \langle u_1^{-\alpha}u_2^{-\beta}u_3^{-\gamma}\alpha_{(1,1,1)}, 1\rangle \\ &= \langle \varepsilon^{-(\alpha+\beta+\gamma)}\alpha_{(1,1,1)}, 1\rangle = \frac{1}{36}\varepsilon^{-(\alpha+\beta+\gamma)} \end{aligned}$$

and similarly

$$\langle \alpha_{(5,5,5)}, u_1^\alpha u_2^\beta u_3^\gamma \rangle = \frac{1}{36}\varepsilon^{\alpha+\beta+\gamma}.$$

Thus,

$$\begin{aligned} \langle w_1, u_1^\alpha u_2^\beta u_3^\gamma \rangle &= \frac{24\sqrt{-3}}{36}(\varepsilon^{-(\alpha+\beta+\gamma-1)} + \varepsilon^{\alpha+\beta+\gamma+2}) \\ &= \frac{2}{\sqrt{3}}\mathrm{Im}(\varepsilon^{-(\alpha+\beta+\gamma-1)}) \end{aligned}$$

and similarly

$$\langle w_1, u_1^\alpha u_2^\beta u_3^\gamma \rangle = \frac{2}{\sqrt{3}}\mathrm{Im}(\varepsilon^{-(\alpha+\beta+\gamma-2)}).$$

Therefore, if $sw_1 + tw_2 \in H$ for some $s, t \in \mathbb{Q}$, then $s, t \in \mathbb{Z}$. However, H was the direct sum of P and an algebraic class $\pi_{q_*}\Lambda$. Because $\langle w_i, P\rangle = \mathbb{Z}$ and $\langle w_i, \pi_{q_*}\Lambda\rangle = 0$ for $i = 1, 2$, we get that $\langle w_i, \cdot\rangle \in H^*$. By unimodularity of H, this means $w_i \in H$, hence

$$\mathbb{Z}w_1 \oplus \mathbb{Z}w_2 = H \cap (T(X_\mathbb{C}) \otimes \mathbb{Q}) = T(X_\mathbb{C}).$$

The formula for the cup product follows directly from Lemma 11.9. □

12.1.2 The group μ_6 acts on $T(F_\mathbb{C})$ and $T(F_\mathbb{C}) \otimes E$ via u_1, u_2, or u_3 and the preceding proposition shows that it does not matter which of the three variables we pick. Denote the action of $x \in \mu_6$ by $[x]$. We have that $[x]\alpha_{(1,1,1)} = x\alpha_{(1,1,1)}$ and $[x]\alpha_{(5,5,5)} = x^{-1}\alpha_{(5,5,5)}$, hence

$$[\varepsilon]w_1 = w_2, \quad [\varepsilon]w_2 = w_2 - w_1.$$

The free $\mathcal{O}$-module $T(F_\mathbb{C})$ is thus (non-uniquely) isomorphic to $\mathcal{O}$ itself sending w_1 to 1 and w_2 to ε. The intersection product under this identification is given by $\langle x, y\rangle = 12\,\mathrm{Tr}_{E/\mathbb{Q}}(x\overline{y})$. Because $12\,\mathrm{Tr}_{E/\mathbb{Q}}(1/(12\sqrt{-3})) = 0$ and $12\,\mathrm{Tr}_{E/\mathbb{Q}}(\varepsilon/(12\sqrt{-3})) = -1$, it follows that the dual lattice of $\mathcal{O}$ is $\frac{1}{12\sqrt{-3}}\mathcal{O}$.

The exact sequence

$$0 \to T(F_{\mathbb{C}}) \to T(F_{\mathbb{C}})^* \to \Delta \to 0$$

then becomes

$$0 \to \mathcal{O} \xrightarrow{12\sqrt{-3}} \mathcal{O} \to \mathcal{O}/12\sqrt{-3} \to 0$$

12.1.3 We recover the fact that $\Delta = \mathcal{O}/12\sqrt{-3} = \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$, as shown in [CN18, 3.1] with explicit divisors.

12.2 EXPLICIT GALOIS REPRESENTATION IN THE UNTWISTED CASE

12.2.1 Let ℓ be a prime number and let λ be a prime in k above ℓ. Let $\pi \in \mathcal{O}$ be a prime element not dividing $d\ell$. The multiplicative character ψ becomes the sextic residue character $(\cdot/\pi)_6$.

12.2.2 We will require a very particular notion of primary generators of prime ideals due to Eisenstein [Lem00, 7.3]. This notion is a strengthening of the cubic notion of primary primes so that we can apply sextic reciprocity.

Definition 12.2. We call $x = a + b\varepsilon \in \mathcal{O}$ *primary*, if $3|b$ and

$$\begin{cases} a + b \equiv 1 \bmod 4, & \text{if } 2|b \\ b \equiv 1 \bmod 4, & \text{if } 2|a \\ a \equiv 3 \bmod 4, & \text{if } 2 \nmid ab. \end{cases}$$

Every prime ideal in $\mathcal{O}$ has a primary generator.

Theorem 12.3. *Let $x, y \in \mathcal{O}$ be primary and relatively prime. Then*

$$\left(\frac{x}{y}\right)_6 = (-1)^{\frac{\mathrm{N}(x)-1}{2}\frac{\mathrm{N}(y)-1}{2}} \left(\frac{y}{x}\right)_6.$$

Proof. This is a combination of cubic reciprocity and a quadratic reciprocity law for $\mathcal{O}$, see [Lem00, Theorem 7.10]. $\square$

12.2.3

Proposition 12.4. *Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal generated by a primary element π. Let $\lambda \subset \mathcal{O}$ be a prime ideal over the rational prime ℓ. Assume $\mathfrak{p}$ does not divide $d\ell$.*

Set $\zeta_\pi = (-4/\pi)_6$. Then for $\chi \in S$, the eigenvalue μ of $\mathrm{Frob}_\mathfrak{p} \in \Gamma_k$ on $V_\chi \otimes_k k_\lambda$ is as follows:
If χ is $(3,3,3)$ or a permutation of $(1,3,5)$, then $\mu = 1$.
If χ is a permutation of $(2,3,4)$, then $\mu = \zeta_\pi^3$.
If χ is a permutation of $(2,2,5)$, then $\mu = \zeta_\pi$.
If χ is a permutation of $(4,4,1)$, then $\mu = \overline{\zeta_\pi}$.
If $\chi = (1,1,1)$, then $\mu = \pi/\overline{\pi}$.
If $\chi = (5,5,5)$, then $\mu = \overline{\pi}/\pi$.

Proof. We treat each item individually via Proposition 11.17 which states that μ is given by $\mathrm{N}(\mathfrak{p})^{-1}\left(\frac{-1}{\pi}\right)_6 J_\mathfrak{p}(\chi)$.

If χ is $(3,3,3)$ or a permutation of $(1,3,5)$, we find by Lemma 11.20 that

$$g(3)^2 = \left(\frac{-1}{\pi}\right)_6^3 \mathrm{N}(\mathfrak{p}) = \left(\frac{-1}{\pi}\right)_6 \mathrm{N}(\mathfrak{p}) = g(1)g(5)$$

from which it follows that $g(3)^4 = g(3)^2 g(1) g(5) = \mathrm{N}(\mathfrak{p})^2$.

If χ is a permutation of $(2,3,4)$, we find by Lemma 11.20 that

$$g(2)g(4) = (-1/\pi)_6^2\, \mathrm{N}(\mathfrak{p}) = \mathrm{N}(\mathfrak{p})$$

from which it follows that

$$\mathrm{N}(\mathfrak{p})^{-2} g(3)^2 g(2) g(4) = \left(\frac{-1}{\pi}\right)_6^3 = \left(\frac{(-4)^3}{\pi}\right)_6 = \left(\frac{-4}{\pi}\right)_6^3.$$

If χ is a permutation of $(2,2,5)$, we find by [BEW98, Theorem 3.1.1] that

$$\frac{g(1)g(2)}{g(3)} = \left(\frac{4^2}{\pi}\right)_6 \frac{g(2)^2}{g(4)}.$$

Now $\mathrm{N}(\mathfrak{p})^2 \left(\frac{-1}{\pi}\right)_6 (g(4)g(5))^{-1} = g(1)g(2)$, hence

$$\mathrm{N}(\mathfrak{p})^2 \left(\frac{-4}{\pi}\right)_6 = g(3)g(4)g(5)\frac{g(2)^2}{g(4)} = g(2)^2 g(5)g(3).$$

If χ is a permutation of $(4,4,1)$, this is the conjugate case to $(2,2,5)$.

If $\chi = (1,1,1)$, we find by [BEW98, Theorem 3.1.1] that

$$\frac{g(1)g(2)}{g(3)} = \left(\frac{-4}{\pi}\right)_6 \frac{g(1)^2}{g(2)}.$$

Hence

$$J(\chi) = \left(\frac{-4}{\pi}\right)_6^{-1} \left(\frac{g(1)g(2)}{g(3)}\right)^2.$$

But now applying [BEW98, Theorem 3.1.1] in combination with [BEW98, (3.1.6)], yields

$$\frac{g(1)g(2)}{g(3)} = \pm \left(\frac{4}{\pi}\right)_6^{-1} \pi.$$

The result follows as $(4/\pi)_6^3 = 1$.

If $\chi = (5,5,5)$, this is the conjugate case to $(1,1,1)$. □

12.2.4

Corollary 12.5. *The element* $\mathrm{Frob}_{\mathfrak{p}} \in \Gamma_k$ *acts on* $T(F_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ *as multiplication by* $\pi/\overline{\pi}$. *Complex conjugation (the generator of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}} \cap \mathbb{R})$*) acts on* $T(F_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ *as the usual complex conjugation under the identification* $T(F_{\mathbb{C}}) \cong \mathcal{O}$.

Proof. In Lemma 12.1, $T(F_{\mathbb{C}}) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ was identified with $\mathcal{O}$ as an $\mathcal{O}$-module such that $[\varepsilon]w_1 = w_2$. By Proposition 12.4, Frob_π acts with eigenvalue $\pi/\overline{\pi}$ on $\alpha_{(1,1,1)}\mathcal{O}_\lambda$ and with eigenvalue $\overline{\pi}/\pi$ on $\alpha_{(5,5,5)}\mathcal{O}_\lambda$. Therefore, the matrix representing the action of Frob_π in the basis (w_1, w_2) is given by multiplication with $\pi/\overline{\pi}$. $\square$

12.3 GALOIS INVARIANT PART OF THE BRAUER GROUP

12.3.1 We work over the ground field $K = k = \mathbb{Q}(\zeta_6)$ or $K = \mathbb{Q}$ and write $\Gamma = \Gamma_K$. Let $X_{A,B,C}$ be the Galois twist of F given in $\mathbb{P}^3_K(3,1,1,1)$ by

$$w^2 = Ax^6 + By^6 + Cz^6$$

where $A, B, C \in K^\times$. Note that this differs from the convention used in Chapter 11 by multiplying A, B and C with -1.

12.3.2 Using Corollary 12.5, we can bound and compute the Galois invariant part of the geometric Brauer group of $X_{A,B,C}$.

Proposition 12.6. *The exponent of* $\mathrm{Br}(\overline{X}_{A,B,C})[\ell^\infty]^\Gamma$ *is at most* 4 *if* $\ell = 2$, *at most* ℓ *if* $\ell \in \{3, 5, 7\}$, *and* 1 *if* $\ell \geq 11$.

Proof. Since Frob_π acts as multiplication by $x\pi/\overline{\pi}$ where $x \in \mu_6$, for the group $\mathrm{Br}(\overline{X}_{A,B,C})[\ell^m]$ with $\pi \nmid \ell$ to be Γ_k-invariant, we require that

$$x\pi \equiv \overline{\pi} \bmod \ell^m$$

or equivalently $\ell^m | (x\pi - \overline{\pi})$. Set $\pi = 3\varepsilon - 1$ so that $\mathrm{N}(\pi) = 7$. Then a calculation shows that the set of maximal rational prime powers that divide $(x\pi - \overline{\pi})$ is $\{4, 3, 5\}$. Doing the same for $\pi = 3\varepsilon - 4$, so that $\mathrm{N}(\pi) = 13$, yields $\{4, 3, 5, 7\}$. $\square$

One may compare Proposition 12.6 with the bounds obtained in [Val18, Example 11.2].

12.3.3

Proposition 12.7. *Let $k = \mathbb{Q}(\zeta_6)$. The group $\mathrm{Br}(\overline{X}_{A,B,C})[\ell^\infty]^{\Gamma_k}$ equals*

- $\ell = 2: \begin{cases} \mathcal{O}/4, & \textit{if } ABC/16 \in k^{\times 6}, \\ \mathcal{O}/2, & \textit{if } ABC/16 \in k^{\times 3} \setminus k^{\times 6}, \\ 0, & \textit{otherwise.} \end{cases}$
- $\ell = 3: \begin{cases} \mathcal{O}/3, & \textit{if } -ABC \in k^{\times 6}, \\ (1+\varepsilon)\mathbb{Z}/3\mathbb{Z}, & \textit{if } -ABC \in k^{\times 2} \setminus k^{\times 6}, \\ 0, & \textit{otherwise.} \end{cases}$
- $\ell = 5: \begin{cases} \mathcal{O}/5, & \textit{if } -5ABC \in k^{\times 6}, \\ 0, & \textit{otherwise.} \end{cases}$
- $\ell = 7: \begin{cases} \mathcal{O}/7, & \textit{if } ABC/7 \in k^{\times 6}, \\ 0, & \textit{otherwise.} \end{cases}$
- $\ell > 7: \quad 0.$

Proof. We use the Sextic Reciprocity Theorem 12.3 throughout in order to express $\pi/\overline{\pi}$ as Dirichlet characters with respect to the chosen modulus.

- $\ell = 2$: We have

$$\begin{aligned} \left(\frac{-16}{\pi}\right)_6 &= \left(\frac{-1}{\pi}\right)_2 \left(\frac{16}{\pi}\right)_2 \left(\frac{16}{\pi}\right)_3^{-1} = (-1)^{(\mathrm{N}(\pi)-1)/2} \left(\frac{2}{\pi}\right)_3^2 \\ &\equiv (-1)^{(\mathrm{N}(\pi)-1)/2} \pi^2 \equiv \mathrm{N}(\pi)^{-1}\pi^2 \equiv \pi/\overline{\pi} \bmod 4. \end{aligned}$$

So for primary π, Frob_π acts via $\left(\frac{16/ABC}{\pi}\right)_6$. Thus if $ABC/16$ is a sixth power, then $\left(\frac{16/ABC}{\pi}\right)_6 \equiv 1 \bmod 4$, so all of $\mathcal{O}/4$ is invariant. If $ABC/16$ is a third but not a sixth power, then the sextic residue symbol assumes the value the -1 for some π by Chebotarev density, so the invariants are $\mathcal{O}/2$. In all other cases, the sextic residue symbol assumes the value ζ_3 for some π. The invariants of $\mathcal{O}/4$ under multiplication by ζ_3 are trivial.

- $\ell = 3$: We have
$$\pi/\overline{\pi} \equiv 1 \bmod 3.$$
So for primary π, Frob_π acts via $\left(\frac{-1/ABC}{\pi}\right)_6$. Thus if $-ABC$ is a sixth power, then $\left(\frac{-1/ABC}{\pi}\right)_6 \equiv 1 \bmod 3$, so all of $\mathcal{O}/3$ is invariant. If $-ABC$ is a square but not a sixth power, then the sextic residue symbol assumes all values in μ_3 infinitely often by Chebotarev density, so the invariants are $\mathbb{Z}/3(1+\varepsilon)$. In all other cases, the sextic residue symbol assumes the value -1 for some π. The invariants of $\mathcal{O}/3$ under multiplication by -1 are trivial.

- $\ell = 5$: We have
$$\left(\frac{5}{\pi}\right)_6^{-1} = \left(\frac{\pi}{5}\right)_6^{-1} \equiv \pi^{-4} \equiv \pi/\overline{\pi} \bmod 5.$$
So for primary π, Frob_π acts via $\left(\frac{-1/(5ABC)}{\pi}\right)_6$. Thus if $-5ABC$ is a sixth power, then $\left(\frac{-5ABC}{\pi}\right)_6 \equiv 1 \bmod 5$, so all of $\mathcal{O}/5$ is invariant. In all other cases, the sextic residue symbol assumes a nontrivial value in μ_6 for some π. The invariants of $\mathcal{O}/5$ under multiplication by nontrivial $x \in \mu_6$ are trivial.

- $\ell = 7$: We have $7 = \theta\overline{\theta}$ where $\theta = -1 + 2\varepsilon$. Furthermore -7 is primary and $\mathrm{N}(-7) - 1 = 48$, hence
$$\begin{aligned}\left(\frac{-7}{\pi}\right)_6 &= \left(\frac{\pi}{-7}\right)_6 = \left(\frac{\pi}{7}\right)_6 \\ &= \left(\frac{\pi}{\theta}\right)_6 \left(\frac{\pi}{\overline{\theta}}\right)_6 = \left(\frac{\pi}{\theta}\right)_6 \left(\frac{\overline{\pi}}{\theta}\right)_6^{-1} \equiv \pi/\overline{\pi} \bmod 7.\end{aligned}$$
So for primary π, Frob_π acts via $\left(\frac{7/ABC}{\pi}\right)_6$. Thus if $ABC/7$ is a sixth power, then $\left(\frac{ABC/7}{\pi}\right)_6 \equiv 1 \bmod 7$, so all of $\mathcal{O}/7$ is invariant. In all other cases, the sextic residue symbol assumes a nontrivial value in μ_6

for some π. The invariants of $\mathcal{O}/7$ under multiplication by nontrivial $x \in \mu_6$ are trivial.

$\square$

We can express the conditions of Proposition 12.7 in terms of $\mathbb{Q}$ by the following easy lemma.

Lemma 12.8. *Let $k = \mathbb{Q}(\zeta_6)$ and $x \in \mathbb{Q}$. Then $x \in k^{\times 6}$ if and only if either $x \in \mathbb{Q}^{\times 6}$ or $x \in (-27)\mathbb{Q}^{\times 6}$; $x \in (k^\times)^3$ if and only if $x \in \mathbb{Q}^{\times 3}$; $x \in k^{\times 2}$ if and only if either $x \in \mathbb{Q}^{\times 2}$ or $x \in (-3)\mathbb{Q}^{\times 2}$.*

Proof. One direction is clear, since -3 is a square in k. For $m \in \{2, 3, 6\}$, the inflation-restriction sequence yields

$$\mathrm{H}^1(\mathbb{Z}/2, \mu_m) \cong \ker(\mathbb{Q}^\times/\mathbb{Q}^{\times m} \to k^\times/k^{\times m}).$$

Since the Herbrand quotient of finite coefficient modules is 1, we deduce that $\#\mathrm{H}^1(\mathbb{Z}/2, \mu_m) = \#\mathrm{H}^0(\mathbb{Z}/2, \mu_m)$ equals $\#\mu_2 = 2$ if $m = 2, 6$ and $\#\mu_1 = 1$ if $m = 3$. $\square$

12.3.4 Incorporating the action of complex conjugation into the picture, we get the following.

Proposition 12.9. *The group $\mathrm{Br}(\overline{X}_{A,B,C})[\ell^\infty]^{\Gamma_\mathbb{Q}}$ equals*

- $\ell = 2:$ $$\begin{cases} \frac{1}{4}\mathbb{Z}/\mathcal{O}, & \textit{if } ABC/16 \in (-27)\mathbb{Q}^{\times 6}, \\ (1+\varepsilon)\mathbb{Z}/4\mathbb{Z}, & \textit{if } ABC/16 \in \mathbb{Q}^{\times 6}, \\ \mathbb{Z}/2, & \textit{if } ABC/16 \in \mathbb{Q}^{\times 3} \setminus \mathbb{Q}^{\times 6}, \\ 0, & \textit{otherwise.} \end{cases}$$

- $\ell = 3:$ $$\begin{cases} \frac{1}{3}\mathbb{Z}/\mathcal{O}, & \textit{if } -ABC \in \mathbb{Q}^{\times 6}, \\ (1+\varepsilon)\mathbb{Z}/3\mathbb{Z}, & \textit{if } -ABC \in (-3)\mathbb{Q}^{\times 2}, \\ 0, & \textit{otherwise.} \end{cases}$$

- $\ell = 5: \begin{cases} \frac{1}{5}\mathbb{Z}/\mathcal{O}, & \textit{if } -5ABC \in \mathbb{Q}^{\times 6}, \\ (1+\varepsilon)\mathbb{Z}/5\mathbb{Z}, & \textit{if } -5ABC \in (-27)\mathbb{Q}^{\times 6}, \\ 0, & \textit{otherwise.} \end{cases}$
- $\ell = 7: \begin{cases} (1+\varepsilon)\mathbb{Z}/7\mathbb{Z}, & \textit{if } ABC/7 \in \mathbb{Q}^{\times 6}, \\ \frac{1}{7}\mathbb{Z}/\mathcal{O}, & \textit{if } ABC/7 \in (-27)\mathbb{Q}^{\times 6}, \\ 0, & \textit{otherwise.} \end{cases}$
- $\ell > 7: \quad 0.$

Proof. Complex conjugation acts as the usual complex conjugation on $\mathcal{O}$ followed by multiplication with the sextic character $\sqrt[6]{-ABC}/\tau(\sqrt[6]{-ABC})$. The cases follow from a simple computation of the invariants of

$$\mathrm{Br}(\overline{X}_{A,B,C})[\ell^\infty]^{\Gamma_k}$$

under this additional automorphism. □

This completes step (a) of our framework.

12.3.5 In [CV18], Creutz and Viray ask which Brauer classes can obstruct the Hasse principle on K3 surfaces. More precisely, they ask whether degrees capture the Brauer-Manin obstruction in the following sense.

Definition 12.10. Let X be a smooth, projective, geometrically integral variety over a number field. We say that *degrees capture the Brauer-Manin obstruction on* X if for every globally generated ample line bundle of degree δ on X, the following implication holds.

$$X(\mathbb{A})^{\mathrm{Br}(X)} = \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[\delta^\infty]} := \bigcap_{\beta \in \mathrm{Br}(X)[\delta^\infty]} X(\mathbb{A})^{\{\beta\}} = \emptyset.$$

Here $X(\mathbb{A})$ are the adelic points of X and $X(\mathbb{A})^B$ is the Brauer-Manin set relative to B for any subset $B \subset \mathrm{Br}(X)$, i.e. the set of adelic points which pair trivially with any element of B.

We can answer their question negatively for K3 surfaces by giving a counterexample.

Corollary 12.11. *Degrees do not capture the Brauer-Manin obstruction for the surface* $X = X_{-3,97,97\cdot 28\cdot 8}$.

Proof. The surface X is a degree 2 K3 surface. It was shown in [CN18] that $\mathrm{Br}_1(X) = \mathbb{Z}/3$ and the generator of this group obstructs the Hasse principle. Now it follows from Proposition 12.7 that $(\mathrm{Br}(X)/\mathrm{Br}_0(X))[2^\infty] = 0$, so the 2-primary Brauer classes cannot obstruct. □

12.4 DETERMINING THE TRANSCENDENTAL BRAUER GROUP

12.4.1 We can now apply Proposition 10.8 in order to compute the transcendental Brauer group of $X_{A,B,C}$ over $K = k$ or $K = \mathbb{Q}$. It suffices to treat the preimages of $\mathrm{Br}(\overline{X})[\ell^\infty]$ separately for each $\ell \in \{2, 3, 5, 7\}$. By Section 10.2.7,

$$(\mathrm{Br}(X)/\mathrm{Br}_1(X))[\ell^\infty] = \mathrm{Br}(\overline{X})[\ell^\infty]$$

for $\ell = 5, 7$. Thus, only the cases $\ell = 2, 3$ remain to be treated.

Lemma 12.12. *Let* $L = k(\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C})$.

(i) *The action of* Γ_L *on* $\mathrm{Pic}(\overline{X})$ *factors through* $\mathrm{Gal}(L(\varepsilon, \sqrt{-1}, \sqrt[3]{2})/L)$.

(ii) *One has* $S_2 = S_{\mathrm{Br}(\overline{X})[2]} = \mathcal{O}/24\sqrt{-3}$ *and the action of* Γ_L *on* $\mathcal{O}/24\sqrt{-3}$ *factors through* $\mathrm{Gal}(L(\varepsilon, \sqrt{-1}, \sqrt[6]{2})/L)$.

(iii) *One has* $S_3 = S_{\mathrm{Br}(\overline{X})[3]} = \mathcal{O}/36\sqrt{-3}$ *and the action of* Γ_L *on* $\mathcal{O}/24\sqrt{-3}$ *factors through* $\mathrm{Gal}(L(\varepsilon, \sqrt{-1}, \sqrt[3]{2}, \sqrt[3]{3})/L)$.

(*i*) This is immediately clear from Proposition 12.4 since -4 is a sixth power in this finite extension.

(*ii*) The first claim is clear. We have that $\pi/\overline{\pi}$ acts trivially on $\mathcal{O}/24\sqrt{-3}$ if and only if $\pi \equiv \overline{\pi} \bmod 24\sqrt{-3}$ if and only if $\pi \in \mathbb{Z} + 12\sqrt{-3}\mathbb{Z}$. Thus the Galois group $\Gamma_{k(\mathbb{Z}+12\sqrt{-3}\mathbb{Z})}$ corresponding to the ring class field $k(\mathbb{Z}+12\sqrt{-3}\mathbb{Z})$ of the non-maximal order $\mathbb{Z}+12\sqrt{-3}\mathbb{Z} \subset \mathcal{O}$ acts trivially on $\mathcal{O}/24\sqrt{-3}$. It can be checked that

$$k(\sqrt{-1}, \sqrt[6]{2}) = k(\mathbb{Z}+12\sqrt{-3}\mathbb{Z}).$$

(*iii*) Analogous to the previous case.

Note that (*i*) recovers the splitting field of $\operatorname{Pic}(\overline{F})$ found in [CN18]. Coincidentally, $k(\sqrt{-1}, \sqrt[3]{2})$ is the ring class field of the order $\mathbb{Z} + 6\sqrt{-3}\mathbb{Z} \subset \mathcal{O}$.

Proposition 12.13. *Let* $B_2 = \operatorname{Br}(\overline{X}_{A,B,C})[4]$ *and* $B_3 = \operatorname{Br}(\overline{X}_{A,B,C})[3]$. *Then* $(\operatorname{Br}(X_{A,B,C})/\operatorname{Br}_1(X_{A,B,C}))[\ell^\infty]$ *is isomorphic to*

$$\begin{aligned} &\operatorname{im}[\mathrm{H}^1(G_\ell, \operatorname{Hom}(\operatorname{Pic}(\overline{X}_{A,B,C})^*, \mathbb{Z})) \to \mathrm{H}^1(G_\ell, \Delta)] \\ \cap\ &\operatorname{im}[\mathrm{H}^0(G_\ell, B_\ell) \to \mathrm{H}^1(G_\ell, \Delta)]. \end{aligned}$$

Proof. Since $\operatorname{Br}(\overline{X}_{A,B,C})[\ell^\infty]^\Gamma \subseteq B_\ell$, it follows that

$$\operatorname{Br}(X_{A,B,C})/\operatorname{Br}_1(X_{A,B,C})[\ell^\infty] = \operatorname{Br}_{B_\ell}(X_{A,B,C})/\operatorname{Br}_1(X_{A,B,C}).$$

The right hand side now equals the preimage of

$$\begin{aligned} &\operatorname{im}[\mathrm{H}^1(G_\ell, \operatorname{Hom}(\operatorname{Pic}(\overline{X}_{A,B,C})^*, \mathbb{Z})) \to \mathrm{H}^1(G_\ell, \Delta)] \\ \cap\ &\operatorname{im}[\mathrm{H}^0(G_\ell, B_\ell) \to \mathrm{H}^1(G_\ell, \Delta)] \end{aligned}$$

under the connecting map $\mathrm{H}^0(G_\ell, B_\ell) \to \mathrm{H}^1(G_\ell, \Delta)$ induced by

$$0 \to \Delta \to S_\ell \xrightarrow{\cdot 12\sqrt{-3}} B_\ell \to 0.$$

Since $12\sqrt{-3}$ annihilates S_ℓ, $\mathrm{H}^0(G_\ell, B_\ell) \to \mathrm{H}^1(G_\ell, \Delta)$ is an injection and the proposition follows. □

12.4.2 We can now complete step (b) using Magma [BCP97]. In order to list all possible cases how Γ can act on

$$[\mathrm{Pic}(\overline{X}_{A,B,C})^* \to S_{\mathrm{Br}(\overline{\mathcal{V}_s})[\ell]}],$$

we employ the moduli viewpoint taken in [Bri02, 3.3] for the classification of the algebraic Brauer groups of diagonal quartic surfaces.

The fine moduli space of diagonal hypersurfaces in $\mathbb{P}$ of degree $(2,6,6,6)$ is isomorphic to the space

$$\mathcal{M} : \mathbb{A}^3_{A,B,C,\mathbb{Q}} \setminus \{ABC = 0\}.$$

It carries a universal family

$$\mathcal{V} : y^2 = Ax^6 + By^6 + Cz^6 \subset \mathbb{P} \times_{\mathbb{Q}} \mathcal{M}.$$

Let $\eta = \operatorname{Spec} \mathbb{Q}(A,B,C)$ be the generic point of $\mathcal{M}$. Define $\mathbb{Q}(A,B,C)$-algebras

$$A_{\mathrm{Pic}} = \mathbb{Q}[\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C}, i, \sqrt{-3}, \sqrt[3]{2}],$$
$$A_2 = \mathbb{Q}[\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C}, i, \sqrt{-3}, \sqrt[6]{2}],$$
$$A_3 = \mathbb{Q}[\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C}, i, \sqrt[6]{-3}, \sqrt[3]{2}]$$

and consider the schemes

$$\begin{aligned} \mathcal{M}_{\mathrm{Pic}} &: \operatorname{Spec}(A_{\mathrm{Pic}}) \setminus \{ABC = 0\}, \\ \mathcal{M}_2 &: \operatorname{Spec}(A_2) \setminus \{ABC = 0\}, \\ \mathcal{M}_3 &: \operatorname{Spec}(A_3) \setminus \{ABC = 0\}, \end{aligned}$$

which are finite Galois covers of $\mathcal{M}$. We can pull back the universal family $\mathcal{V}$ along these covers and get families $\mathcal{V}_{\mathrm{Pic}} = \mathcal{V} \times_{\mathcal{M}} \mathcal{M}_{\mathrm{Pic}}$, $\mathcal{V}_2 = \mathcal{V} \times_{\mathcal{M}} \mathcal{M}_2$ and $\mathcal{V}_3 = \mathcal{V} \times_{\mathcal{M}} \mathcal{M}_3$.

From Lemma 12.12, it is clear that these covers trivialise the Galois action on the fibres of $\mathcal{V}$ in the following sense. Let $s \in \mathcal{M}_{\mathrm{Pic}}$ be a point with residue field $k(s)$ and let $(\mathcal{V}_{\mathrm{Pic}})_s$ be the fibre of $\mathcal{V}_{\mathrm{Pic}} \to \mathcal{M}_{\mathrm{Pic}}$ over s. Then the Galois group $\Gamma_{k(s)}$ acts trivially on $\mathrm{Pic}(\overline{(\mathcal{V}_{\mathrm{Pic}})_s})$. Furthermore, for $\ell = 2, 3$ and $s \in \mathcal{M}_\ell$, $\Gamma_{k(s)}$ acts trivially on $\mathrm{Pic}(\overline{(\mathcal{V}_\ell)_s})$ and $S_{\mathrm{Br}(\overline{(\mathcal{V}_\ell)_s})[\ell]}$.

12.4.3 We will now use the Galois theory of schemes as presented in [Gro63, V.1-2]. The reader may compare this with the theory of decomposition groups in classical Galois theory over $\mathbb{Q}$.

For $\bullet = \mathrm{Pic}, 2, 3$, we have associated *generic Galois groups* of the covers

$$G_\bullet = \mathrm{Gal}(k(A_\bullet)/k(\eta))$$

where $k(A_\bullet)$ denotes the field of fractions of $A_\bullet$.

These groups are easily described as semidirect products of the abelian subgroups

$$\begin{aligned} G^{\mathrm{ab}}_{\mathrm{Pic}} = & \ \mathrm{Gal}(k(\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C}, \sqrt{3}, \sqrt[3]{2})/k(A,B,C)) \cong & (\mathbb{Z}/6)^3 \times \mathbb{Z}/2 \times \mathbb{Z}/3, \\ G^{\mathrm{ab}}_{2} = & \ \mathrm{Gal}(k(\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C}, \sqrt{3}, \sqrt[6]{2})/k(A,B,C)) \cong & (\mathbb{Z}/6)^3 \times \mathbb{Z}/2 \times \mathbb{Z}/6, \\ G^{\mathrm{ab}}_{3} = & \ \mathrm{Gal}(k(\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C}, \sqrt[6]{3}, \sqrt[3]{2})/k(A,B,C)) \cong & (\mathbb{Z}/6)^3 \times \mathbb{Z}/6 \times \mathbb{Z}/3 \end{aligned}$$

with the subgroup $\mathbb{Z}/2 = \langle \tau \rangle$ where the generator τ leaves $(\sqrt[6]{A}, \sqrt[6]{B}, \sqrt[6]{C})$ fixed and acts as complex conjugation otherwise.

12.4.4 The actions of $\Gamma_{k(\eta)}$ on $\mathrm{Pic}(\overline{\mathcal{V}_\eta})$, $S_{\mathrm{Br}(\overline{\mathcal{V}_\eta})[2]}$ and $S_{\mathrm{Br}(\overline{\mathcal{V}_\eta})[3]}$ factor through the respective generic Galois groups.

For any other point $s \in \mathcal{M}$, the action of $\Gamma_{k(s)}$ on $\mathrm{Pic}(\overline{\mathcal{V}_s})$, $S_{\mathrm{Br}(\overline{\mathcal{V}_s})[2]}$ and $S_{\mathrm{Br}(\overline{\mathcal{V}_s})[3]}$ will factor through the *decomposition group*

$$H_s = \mathrm{Gal}(k(s'), k(s)) \subset G_\bullet$$

where $s' \in \mathcal{M}_\bullet$ is a point over s and $\bullet = \mathrm{Pic}, 2, 3$ as appropriate.

We can thus list all possible cases of Galois actions on the complex

$$[\mathrm{Pic}(\overline{X}_{A,B,C})^* \to S_{\mathrm{Br}(\overline{\mathcal{V}_s})[\ell]}]$$

by iterating over the subgroups of G_ℓ. Each action then arises by restriction of the G_ℓ-action to a subgroup. We call two subgroups equivalent if they coincide up to conjugation and the $\mathfrak{S}_3$-action permuting A, B and C. It suffices to list the subgroups up to equivalence. This yields a complete list of cases for $\mathrm{Br}_{\mathrm{Br}(\overline{X})[\ell]}(X)$.

12.4.5 Concretely, we represent subgroups $H \subseteq G_\ell$ as follows. Define $H^{\mathrm{ab}} = H \cap G_\ell^{\mathrm{ab}}$, the abelian part of H. One can represent H^{ab} by a matrix with each row a generator of H^{ab}. If $H = H^{\mathrm{ab}}$, we are done. Otherwise, $H/H^{\mathrm{ab}} = \mathbb{Z}/2$ and a lift to H of the generator of this quotient will have the form τh for some $h \in G_\ell^{\mathrm{ab}}$.

12.4.6 The equivalence classes of subgroups induce an "arithmetic stratification" on the points of $\mathcal{M}$. We would like to associate conditions on the coefficients A, B and C to the decomposition groups.

Let $s \in \mathcal{M}$ be a closed point of $\mathcal{M}$ and $H_s \subset G_\ell$ its decomposition group. Then for $\ell = 2$,

$$k(s) = \mathbb{Q}(i, \sqrt{-3}, \sqrt[6]{2}) \cap k(A_2)^{H_s}$$

while for $\ell = 3$,

$$k(s) = \mathbb{Q}(i, \sqrt[6]{-3}, \sqrt[3]{2}) \cap k(A_3)^{H_s}.$$

Moreover, the following holds.

Lemma 12.14. *Let $H \subset G_\ell$ be a subgroup and $s \in \mathcal{M}$. Then $H \supseteq H_s$ if and only if the fibre over s of*

$$(\mathrm{Spec}(A_\ell^H) \setminus \{ABC = 0\}) = \mathcal{M}_\ell/H \to \mathcal{M}_\ell/G_\ell = \mathcal{M}$$

contains a $k(s)$-point.

12.4.7 Hence, in order to compute conditions on the coefficients of $X_{A,B,C}$ so that the corresponding point $s \in \mathcal{M}$ satisfies $H \supseteq H_s$ for a given subgroup $H \subseteq G_\ell$, we need to characterise points

$$\operatorname{Spec} k(s) \to \operatorname{Spec}(A_\ell^H).$$

The invariants A_ℓ^H can be computed by the following three easy lemmas.

Lemma 12.15. *(i) The action of G_2 on A_2 decomposes into a sum of G_2-modules*

$$A_2 = \bigoplus \mathbb{Q}(\varepsilon)\sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt{3}^{r_4}\sqrt[6]{2}^{r_5}$$

where r_1, r_2, r_3, r_5 run from 0 to 5 and r_4 runs from 0 to 1.

(ii) The action of G_3 on A_3 decomposes into a sum of G_3-modules

$$A_3 = \bigoplus \mathbb{Q}(\varepsilon)\sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt[6]{3}^{r_4}\sqrt[3]{2}^{r_5}$$

where r_1, r_2, r_3, r_4 run from 0 to 5 and r_5 runs from 0 to 2.

Proof. The action of G_2 on any element of the form

$$\sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt[6]{3}^{r_4}\sqrt[3]{2}^{r_5}$$

as above multiplies it with a power of ε. The same works for G_3. □

Lemma 12.16. *Let x be an element of the form*

$$\sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt{3}^{r_4}\sqrt[6]{2}^{r_5} \text{ resp. } \sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt[6]{3}^{r_4}\sqrt[3]{2}^{r_5}$$

as in Lemma 12.15. Then $(\mathbb{Q}(\varepsilon)x)^{H^{\mathrm{ab}}}$ equals $\mathbb{Q}(\varepsilon)x$ if H_{ab} fixes x and 0 otherwise.

Proof. The action of H^{ab} commutes with multiplication by ε, so $\mathbb{Q}(\varepsilon)x$ is a free $\mathbb{Q}(\varepsilon)$-module with a compatible H^{ab}-action. It is fixed under H_{ab} if its generator is fixed, otherwise $(\mathbb{Q}(\varepsilon)x)^{H^{\mathrm{ab}}} = 0$ □

Lemma 12.17. *Let x be an element of the form*

$$\sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt{3}^{r_4}\sqrt[6]{2}^{r_5}\ \textit{resp.}\ \sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt[6]{3}^{r_4}\sqrt[3]{2}^{r_5}$$

as in Lemma 12.15. Assume $H/H^{\mathrm{ab}} = \mathbb{Z}/2$ and choose $h \in G_\ell^{\mathrm{ab}}$ such that $\tau h \in H \setminus H_{\mathrm{ab}}$.

If $hx = \varepsilon^j x$, then $(\mathbb{Q}(\varepsilon)x)^{\tau h} = \mathbb{Q}(1+\varepsilon)^{-j}x$.

Proof. We have

$$\tau h(1+\varepsilon)^{-j}x = \tau(1+\varepsilon)^{-j}\varepsilon^j x = \tau(\frac{1+\varepsilon}{\varepsilon})^{-j}x = (1+\varepsilon)^{-j}x.$$

Since τh anti-commutes with ε, the invariant space over $\mathbb{Q}$ is one-dimensional. □

It is now easy to compute $(A_2)^H$, which will be spanned as a $\mathbb{Q}$-algebra by generators of the form

$$\sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt{3}^{r_4}\sqrt[6]{2}^{r_5}(1+\varepsilon)^{r_6}$$

where r_1, r_2, r_3, r_5, r_6 run from 0 to 5 and r_4 from 0 to 1, and similarly for $(A_3)^H$. A $k(s)$-point is given by a morphism

$$(A_2)^H \to k(s)$$

which is determined by the images of the generators in $k(s)$. We thus get conditions of the form

$$\sqrt[6]{A}^{r_1}\sqrt[6]{B}^{r_2}\sqrt[6]{C}^{r_3}\sqrt{3}^{r_4}\sqrt[6]{2}^{r_5}(1+\varepsilon)^{r_6} \in k(s)$$

or

$$A^{r_1}B^{r_2}C^{r_3}27^{r_4}2^{r_5}(-27)^{r_6} \in k(s)^{\times 6}.$$

It follows that every case can be realised with rational coefficients, or more precisely:

Lemma 12.18. *Let $\ell = 2$ or 3 and let l be a number field. Let*

$$X_{A,B,C} : y^2 = Ax^6 + By^6 + Cz^6 \subset \mathbb{P}^3_l(3,1,1,1)$$

with $A, B, C \in l^\times$. Then one can find $A', B', C' \in \mathbb{Q}^\times$ such that there is an isomorphism of complexes of Γ_l-modules

$$[\mathrm{Pic}(\overline{X}_{A,B,C}) \to S_{\mathrm{Br}(\overline{X}_{A,B,C})[\ell]}] \cong [\mathrm{Pic}(\overline{X}_{A',B',C'}) \to S_{\mathrm{Br}(\overline{X}_{A',B',C'})[\ell]}].$$

12.4.8 We are now ready to complete the classification of the Brauer group over $K = \mathbb{Q}$ and $K = \mathbb{Q}(\varepsilon)$. The cases $\ell \neq 2, 3$ we have previously completed in Propositions 12.7 and 12.9 by Section 10.2.7.

A Magma computation shows that there are:

- 7486 equivalence classes of groups $H_s \subset G_2$ with $k(s) = k$. Of these, 60 cases have nontrivial $(\mathrm{Br}(X)/\,\mathrm{Br}_1(X))[2]$ and they are precisely the subcases (i.e. subgroups $H_{s'} \subset H_s$ with $k(s') = k$) of 2 exceptional cases, which we call (I) and (II).

- There are 34966 equivalence classes of groups $H_s \subset G_2$ with $k(s) = \mathbb{Q}$. Of these, 386 have nontrivial $(\mathrm{Br}(X)/\,\mathrm{Br}_1(X))[2]$ and they are precisely the subcases (i.e. subgroups $H_{s'} \subset H_s$ with $k(s') = \mathbb{Q}$) of 8 exceptional cases.

  There are exactly 4 equivalence classes of groups $H_s \subset G_2$ with $k(s) = \mathbb{Q}$ whose abelian part is equivalent to (I) and the same is true for (II). Taken together, they are precisely the 8 exceptional cases mentioned in the preceding statement.

- 18448 equivalence classes of groups $H_s \subset G_3$ with $k(s) = k$. Of these, 38 cases have nontrivial $(\mathrm{Br}(X)/\,\mathrm{Br}_1)[3]$ and they are precisely the subcases of one exceptional case, which we call (III).

- 71264 equivalence classes of groups $H_s \subset G_3$ with $k(s) = \mathbb{Q}$. Of these, 196 cases have nontrivial $(\mathrm{Br}(X)/\,\mathrm{Br}_1)[3]$ and they are precisely the subcases of 2 exceptional cases. There are exactly 2 equivalence classes of groups $H_s \subset G_2$ with $k(s) = \mathbb{Q}$ whose abelian part is equivalent to (III) and they are the 2 exceptional cases mentioned in the preceding statement.

Furthermore, one can see that the 60 cases in the first item all have cohomology $\mathrm{H}^1(\Gamma, \operatorname{Pic}(\overline{X})^*)$ of exponent 2. Thus, it follows from Proposition 12.13 that there is no transcendental 4-torsion and

$$(\operatorname{Br}(X)/\operatorname{Br}_1(X))[2] = (\operatorname{Br}(X)/\operatorname{Br}_1(X))[2^\infty].$$

Since $\operatorname{Br}(\overline{X})[3^\infty]^\Gamma = \operatorname{Br}(\overline{X})[3]$, we also have

$$(\operatorname{Br}(X)/\operatorname{Br}_1(X))[3] = (\operatorname{Br}(X)/\operatorname{Br}_1(X))[3^\infty].$$

Taken together, our computations yield Theorems A and B.

12.5 Determining the full Brauer group

In order to complete step (c) for the exceptional cases of Theorems A and B, we use a partial Cartan-Eilenberg resolution of

$$[\operatorname{Pic}(\overline{X})^* \xrightarrow{\phi} S_{\operatorname{Br}(\overline{X})[\ell]}]$$

and compute the first cohomology of the total complex.

More precisely, let $\mathrm{Z}^1(M)$ denote the space of 1-cocycles for a group module M. Then we construct the complex

$$\begin{tikzcd}
0 & & \\
\mathrm{Z}^1(\operatorname{Pic}(\overline{X})^*) \arrow[u] \arrow[r, "h_{01}"] & \mathrm{Z}^1(\operatorname{Br}(\overline{X})[\ell]) & \\
\operatorname{Pic}(\overline{X})^* \arrow[u, "v_{00}"] \arrow[r, "\phi"] & \operatorname{Br}(\overline{X})[\ell] \arrow[u, "v_{10}"] \arrow[r] & 0
\end{tikzcd}$$

where h_{01} is the map induced by ϕ and the vertical maps v_{00} and v_{10} are the differentials of the standard resolution. Then

$$\begin{aligned} & \mathbb{H}^1(G_\ell, [\mathrm{Pic}(\overline{X})^* \xrightarrow{\phi} S_{\mathrm{Br}(\overline{X})[\ell]}]) \\ \cong \; & \frac{\ker((h_{01}, -v_{10}) : \mathrm{Z}^1(\mathrm{Pic}(\overline{X})^*) \oplus \mathrm{Br}(\overline{X})[\ell] \to \mathrm{Z}^1(\mathrm{Br}(\overline{X})[\ell]))}{\mathrm{im}((v_{00}, \phi) : \mathrm{Pic}(\overline{X})^* \to \mathrm{Z}^1(\mathrm{Pic}(\overline{X})^*) \oplus \mathrm{Br}(\overline{X})[\ell])}. \end{aligned}$$

The rest is linear algebra.

After implementing this functionality in Magma, we verify the Supplements to Theorems A and B.

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