Journal of Dynamics and Games
doi:10.3934/jdg.2017016

GAME THEORETICAL MODELLING OF A DYNAMICALLY EVOLVING NETWORK II: TARGET SEQUENCES OF SCORE 1

Chris Cannings

School of Mathematics and Statistics,
The University of Sheffield,
Hounsfield Road,
Sheffield, S3 7RH, UK.

Mark Broom*

Department of Mathematics,
City, University of London,
Northampton Square,
London EC1V 0HB, UK.

(Communicated by Yannick Viossat)

Abstract. In previous work we considered a model of a population where individuals have an optimum level of social interaction, governed by a graph representing social connections between the individuals, who formed or broke those links to achieve their target number of contacts. In the original work an improvement in the number of links was carried out by breaking or joining to a randomly selected individual. In the most recent work, however, these actions were often not random, but chosen strategically, and this led to significant complications. One of these was that in any state, multiple individuals might wish to change their number of links. In this paper we consider a systematic analysis of the structure of the simplest class of non-trivial cases, where in general only a single individual has reason to make a change, and prove some general results. We then consider in detail an example game, and introduce a method of analysis for our chosen class based upon cycles on a graph. We see that whilst we can gain significant insight into the general structure of the state space, the analysis for specific games remains difficult.

1. Introduction.

1.1. Modelling evolution in populations.

In this paper, following [3], we consider a population comprised of a network of individuals represented by a simple graph. The composition of the population does not change, but the connections between individuals do change following strategic decisions according to their preferences. In particular, each individual has a target number of neighbours that they prefer, and will try to achieve. There may be many types of individuals, where a type is this target number.

Such networks can occur throughout biology, economics and sociology, and this is the subject of a lot of recent research interest. Examples include companies

2010 *Mathematics Subject Classification.* Primary: 05C57, 91A43; Secondary: 05C81.
Key words and phrases. degree preferences; graphic sequences; Nash equilibrium; Markov process; stationary distribution.

* Corresponding author: Mark Broom.

which trade with each other (economics), individuals who are friends (sociology) or the owners of neighbouring territories or food webs ([10], biology). Social animals also have dominance and mutualist interactions and, for example, primate social structures can be complex and influence behaviours such as the level of cooperation [32, 33].

In particular, individuals may have different levels of desire to interact with others, known as "sociability". Sociability has been investigated in, for example, (non-human) primates [6], bottlenose dolphins [7, 34] and sheep [28]. In these cases temporary links occur between individuals. The likelihood of a given link occuring can depend upon many factors, such as gender, the relatedness of individuals, spatial factors or dominance relationships. Links can be reciprocal, or initiated or broken by the actions of a single individual. The presence or absence of a link may benefit one individual but not the other (e.g. a female and a poor quality male). This work is connected to an important related area of research on biological markets and partner choice [20, 21].

We note that, since the population composition is fixed, our process is not an evolutionary process. We discuss such models, and the relationship of our model to them, in more detail in [3]. It could be considered as a detailed examination of a snapshot in time of an evolutionary process; for instance a more complicated version of the type of scenario modelled in [22, 23], where the rates at which links are formed or broken depend upon the types of the individuals involved. It would be possible to embed our model into their, or similar, models. For reviews of evolutionary models involving structured populations see [1] and [25].

In the current paper we do not model complex behaviours, but simply the graph of interactions. Individuals are represented by vertices, and pairwise links by edges. Individuals are assumed to be identical, except in their target number of links. All will try to make changes that get their number of links closer to their target, but the actions of others can make an individual's situation better or worse.

1.2. **A dynamic network population model.** In [3] we introduced a population of individuals represented by the set $V = \{1, 2, \dots, n\}$ and the simple graph $G = (V, \mathbf{X})$ with $\mathbf{X} = (x_{ij})_{i \neq j=1,\dots n}$ representing the links between pairs of individuals, $x_{ij} = 1$ meaning there is a link, with $x_{ij} = 0$ otherwise. In particular we considered a random process in discrete time on the evolving edge set $\mathbf{X_t} = (x_{ij,t})_{i,j=1,\dots n}$.

At any time t individual i has $e_{i,t}$ edges, and the vector $\mathbf{e_t} = (e_{1,t}, e_{2,t}, \dots, e_{n,t})$ is referred to as the *sequence* $\mathbf{e_t}$. At each time point an individual is chosen and allowed to add or remove an edge. Each vertex has an acceptable range $[m_i, M_i]$ of edges to other vertices, where $0 \leq m_i \leq M_i \leq n-1$. In much of the work $m_i = M_i = t_i$, with t_i denoted as the unique target of i, giving a target sequence $\mathbf{t}$=$(t_i)_{i=1,\dots,n}$. We shall assume this from now onwards.

We shall always give the targets in decreasing order of size, i.e. $t_i \geq t_j$ for all $i < j$. If i was selected with $e_i < t_i$ (called a Joiner) then it formed a new edge, choosing one of the vertices it was not connected to at random. If $e_i > t_i$ (a Breaker) then it broke one of its edges at random. Otherwise, it neither created nor broke an edge (a Neutral vertex).

Definition 1.1. The *deviation* of individual/vertex i is given by $\epsilon_{i,t} = |t_i - e_{i,t}|$ and the *deviation* of the graph $\mathbf{X_t}$ is the sum of the vertex deviations, $\Upsilon_t = \sum_{i=1,n} \epsilon_{i,t}$.

Definition 1.2. The minimum value of the deviation over all possible graphs is termed the *score*.

If the score is 0 the sequence is called *graphic*. There has been a lot of work considering graphic sequences, for example [12], [13],[14],[19],[26].

In [4] the conjugate vector $\mathbf{v} = (v_i)$ of $\mathbf{t}$ was defined by $v_i = \#\{j : t_j \geq i\}$ (where # means "the number of").

There, and in the working below, we define f_k as

$$f_k = \sum_{i=1}^{k}(t_i + 1 - v_i), \tag{1}$$

with $f_0 = 0$. A sequence is graphic if and only if the sequence sum is even and $f_k \leq 0$ for $k = 1, \ldots, \lambda$, where $\lambda = \#\{i : t_i \geq i\}$ is the *Durfee number* [5].

Definition 1.3. The *deficit* of the sequence $\mathbf{t}$ is $\max_{0 \leq k \leq \lambda} f_k$.

We can explain this as follows. Suppose that we consider the first k vertices and try to connect them to others to achieve their targets. Given that we can only connect each of the k vertices to any other once, and assuming we are not allowed to take any of the other vertices over target, f_k is the number of such links that we fail to make (if f_k is negative, this represents the number of spare connections not used with other vertices after we achieve our k targets). The deficit corresponds to the largest value of f_k over all k. The Durfee number is the number of vertices which require at least some links with later vertices; we can thus stop the above process there, as the targets of subsequent vertices will be achieved by connecting with the previous vertices.

The *peak* μ denotes the largest value of $k \leq \lambda$ s.t. f_k achieves the deficit. In [5] we proved that the score is equal to the deficit or the deficit plus 1. Note that the score of a target with odd (even) sequence sum is also odd (even), so given the deficit of a sequence, the score immediately follows.

Examples A. Here we consider two example sequences to illustrate the definitions above, 43210 and 33311.

For the target sequence 43210, consider the graph where vertex 1 is connected to all other vertices, vertices 2 and 3 are connected and all other pairs of vertices are split. This graph then has sequence 42211. The deviations of the five vertices are thus 0, 1, 0, 0 and 1 repsectively, with graph deviation 2. This graph actually achieves the minimum possible such deviation, the score, so that the score for 43210 is 2. We note that in general the score of any sequence can be found using a modified Havel-Hakami algorithm [12],[14], following the method developed in [4].

To find the deficit, we need to find the v_is associated with 43210. Four of these five numbers are greater than or equal to 1, and hence $v_1 = 4$. Similarly we obtain $v_2 = 3, v_3 = 2, v_4 = 1$. Precisely the first two satisfy $v_i \geq i$, so that the Durfee number is 2. We have $f_0 = 0$ (as always), $f_1 = 4 + 1 - 4 = 1$ and $f_2 = (4 + 1 - 4) + (3 + 1 - 3) = 2$; the deficit is the maximum of these three numbers, and so is 2.

For 33311, consider the graph where vertices 1,2 and 3 are all connected, and the only other links are vertices 1 to 4 and 2 to 5. The graph has sequence 33211, deviations are 0, 0, 1, 0, 0 and so the graph deviation is 1, which is also the score. We have $v_1 = 5, v_2 = 3, v_3 = 3, v_4 = 0$, so the Durfee number is 3. We have $f_0 = 0, f_1 = 3 + 1 - 5 = -1, f_2 = (3 + 1 - 5) + (3 + 1 - 3) = 0, f_3 = (3 + 1 - 5) + (3 + 1 - 3) + (3 + 1 - 3) = 1$, so that the deficit is 1.

In general there is a set of sequences, with a corresponding set of graphs, which achieve the score; these were termed the minimal set(s) and labelled $J(min)$ and $K(min)$, respectively, in [4]. For the non-strategic case when random improving moves are selected, we proved that from any starting point there is a path of possible moves that reaches $K(min)$, and since the deviation of the graph can never increase, once $J(min)/K(min)$ is reached, it cannot be left. Further we showed that $K(min)$ was connected for non-graphic sequences, and the process always converges to a unique closed set of states. This is not the case for graphic sequences, where $J(min)$ of course has a unique element, but $K(min)$ may have a number, and in this case no further transitions can occur. We also showed in [4] a method to find all members of $K(min)$ (and hence $J(min)$) following Ruch-Gutman [26].

The Markov chain over $K(min)$ was considered in [3] (all states not in this set will be transient following the reasoning above). We showed that the process was reversible and so with a unique stationary distribution. We also showed how to find this stationary distribution. We finally considered the explicit form of the stationary distribution for a specific class of sequence.

In [5] we considered detailed aspects of the structure of $K(min)$. In [4] we proved certain restrictions to exist on its elements, e.g. that for any such graph we know that all Joiners (that is vertices that have degree less than their target) must be joined. In [5] we extended such analysis to consider the possible sequences of Joiners, Breakers (vertices with degrees greater than their target), and Neutrals (the remaining vertices) through time. Vertices fall into four classes; those which are always Neutral, those which are never Joiners, those never Breakers, and those which can be either Joiners or Breakers. We specified rules regarding the possible sequences of class memberships of the vertices (recall that we list all sequences in decreasing order of their targets).

We then considered a model, which in contrast to those of [3] and [4] considers the possibility that the individual at a vertex may choose between the available possibilities according to some aspect of the future costs at that vertex. This is complicated and hard to deal with in generality, and so we restricted ourselves to considering one example in detail, and demonstrating the important concepts to consider in any more extensive analysis. From that paper we saw this complexity, but also that strategic choices lead to clearly different results than the simply random process from [3].

In the current paper we shall focus on target sequences with score 1. These are the closest sequences to graphical sequences, and yield certain simplifications that will make them more amenabe to analysis. In particular, in the minimal set there will always be precisely one individual which is missing its target, so if all individuals play pure strategies, the subsequent evolution of the population state is completely determined. We make significant progress in understanding the structure of the state space, and find a method to analyse games for score 1 sequences by means of cycles of a graph. Analysis of specific examples remains challenging in general, however.

2. **A strategic model.** Our game involves n players trying to minimise the cost for missing their target number of links (equivalent to maximising their payoff), with the population following a random process where at each step an individual is selected uniformly at random to potentially form or break a link. In this section we define the other key elements of our game, namely the strategies available to the

players and their respective payoffs, as well as how the population can evolve in our system.

2.1. **Strategies and payoffs.** The population state is denoted by the edge set $\mathbf{X}$ and in each state any individual can be selected to potentially change one of their edges. When selected there are n available (pure) choices, namely to change their edge to any of the other $n-1$ individuals, or not to make any change. We use u_{ij} to represent the probability that individual i chooses to change edge x_{ij}, conditional on i being selected to make the change, with u_{ii} being the probability that no change is made. Thus $u_{ij}=1$ (with all other elements in the ith row 0) represents the pure strategy where individual i chooses to change edge x_{ij}; similarly i making no change is given by $u_{ii}=1$. The set of all selected changes can then be written as the strategy matrix $\mathbf{U}$.

It is clear that $\mathbf{U}$ depends upon $\mathbf{X}$, and so the full set of strategies of the population, representing the choices of every individual in every conceivable situation, is denoted by $\mathbf{U_X}$, with elements $u_{ij(\mathbf{X})}$. The strategy of individual i is the set of the ith rows of the collection of matrices $\mathbf{U_X}$. For any $\mathbf{x}^*$ which differs from $\mathbf{x}$ in a single entry, where $x_{ij}=0, x^*_{ij}=1$ or $x_{ij}=1, x^*_{ij}=0$ for a given i,j,we have:

$$P(\mathbf{X_{t+1}}=\mathbf{x}^*|\mathbf{X_t}=\mathbf{x})=\frac{u_{ij(\mathbf{X})}+u_{ji(\mathbf{X})}}{n}. \tag{2}$$

In [5] we had a unique target $\mathbf{t}$, and changes were made that always reduced the deviation of the individual where possible, with no change made otherwise (as this would have made the immediate situation worse). This was called the *strict system*. In this paper we also limit ourselves to consideration of the strict system. We note, however, that in [5] we showed that assuming all other individuals play strictly, it can sometimes be optimal for an individual to choose a non-strict move even when the sequence is graphic (and indeed very simple in the example given).

Individuals want to minimise their deviation, and we shall simply consider their payoff as minus their expected long term deviation. If a process with strategies $\mathbf{U_X}$ has a unique stationary distribution $\pi(\mathbf{X})$ over $\mathbf{X}$, the payoff to i is

$$R_i(\mathbf{U_X})=-\sum_{\mathbf{X}}\epsilon_i(\mathbf{X})\pi(\mathbf{X}), \tag{3}$$

where $\epsilon_i(\mathbf{X})$ is the deviation of i in state $\mathbf{X}$.

2.2. **Stability and strategy switches.** Individuals can try to improve their payoffs by changing their strategy. In [5] two distinct types of allowable strategy changes were considered: *Local changes*, where i can change the ith row of $\mathbf{U_X}$ for a single state $\mathbf{X}$ only; *Global changes* - individual i changes the ith row of $\mathbf{U_X}$ for any number of states simultaneously. Making such global changes might be advantageous, since any individual change would potentially affect the probabilities of particular states being occupied and particular paths being followed, which in turn changes the optimal choices at other states. A significant calculating ability may be required to make good global choices, and we noted that an individual with limited abilities may be restricted to only using strict moves and local changes.

We only allow one individual to consider making a change at any given time. We assume that under all allowable changes $\mathbf{U_X}\rightarrow\mathbf{U^i_X}$ for individual i (including no change), it chooses a strategy that achieves $\max_i R_i(\mathbf{U^i_X})$, i.e. it chooses a best reply to the current strategies of others. A strategy set is a Nash equilibrium under local or global changes if, under all allowable changes by $i : \mathbf{U_X} \to \mathbf{U^i_X}$

$$R_i(\mathbf{U}) \geq R_i(\mathbf{U^i}) \qquad i = 1, \ldots, n. \tag{4}$$

3. **General results for score 1 sequences.** When a sequence is graphic, all individuals can eventually reach their target (though for some combinations of the strategic choices of all individuals they may not, see [5]) and on the minimum set there are no more changes in the population. Thus, assuming that $K(min)$ is reached, this situation is of minimal interest. The next simplest case is that of score 1 sequences. Here in the minimal set, there is exactly one individual not on target, so only one individual will want to change a link. If all individuals play pure strategies, this would lead to a unique choice being made at every member of the minimal set, and so the population will follow a unique path from the first time it enters the minimal set. We consider an example game for this case in Section 4 in detail.

In the current section, we mainly consider general results for score 1 sequences. However, we note that Theorem 3.2 applies to all sequences.

3.1. **The score and the deficit.** In Theorem 3.1 we show that a sequence is a score 1 sequence if and only if there is a graphic sequence to which it differs by 1 in a single position. We can find all graphic sequences either using the variant of the Havel-Hakimi procedure from [4], or the alternative criteria from [11]. Thus using Theorem 3.1 we can find all score 1 sequences. As well of being of interest in itself, this result is important in the categorisation that we use in Section 3.2 and in our main general result, Theorem 3.3.

Theorem 3.1. *For any sequence* $\mathbf{t}$*, there is a graphic sequence* $\mathbf{g}$ *such that for some* i*,* $|t_i - g_i| = 1$ *and* $t_j = g_j$ *for all* $j \neq i$*, if and only if* $\mathbf{t}$ *has score 1.*

Proof: Firstly note that from Theorem 8 of [4] adding or taking 1 from any element of a graphic sequence gives a score 1 sequence. It thus remains to show that every score 1 sequence is indeed reachable by this process from some graphic sequence.

Consider a score 1 sequence. It is either score 1 and deficit 0 or score 1 and deficit 1. In either case the sum of the targets $\sum t_i$ must be odd.

a) If the sequence has deficit 0, then add 1 to t_n. If after reordering this keeps t_n out of the first λ terms (or first $\lambda + 1$ terms if there is an increase in the Durfee number), then it leaves the deficit unchanged at 0. If it does not keep t_n out of the first λ (correspondingly $\lambda+1$) terms, then we must have $t_{\lambda+1} = \ldots = t_n = \lambda$ in the original sequence. There are thus two cases to consider:

i) the Durfee number remains at λ,

ii) the Durfee number becomes $\lambda + 1$.

Case i) occurs if $t_\lambda = \lambda$, and case ii) if $t_\lambda \geq \lambda + 1$. Note that in both cases we have $v_1 = \ldots = v_\lambda = n$.

i) Clearly $t_i + 1 - v_i$ is non-positive for the first λ terms so that the deficit is unchanged at 0.

ii) We now have the extra term $t_{\lambda+1} + 1 - v_{\lambda+1}$ to add. As the Durfee number has become $\lambda+1$, we know that this is $\lambda+1+1-(\lambda+1) = 1$. Thus the deficit is still 0, unless $t_i + 1 - v_i = 0$ for all $i \leq \lambda$. This only occurs if we have $t_1 = \ldots = t_\lambda = n-1$ in the original sequence. But then $\sum t_i = (n-1)\lambda + \lambda(n-\lambda) = \lambda(2n-1-\lambda)$ which is even, contradicting our assumption that the sequence has score 1.

Thus the deficit remains unchanged at 0. But $\sum t_i$ now changes from odd to even. A deficit of 0 and an even target sum implies that the score is 0.
b) If the sequence has deficit 1, then take 1 off t_1. If this keeps t_1 within the first μ terms (where μ is the peak), then the deficit is reduced by 1, since the deficit is $\sum_{i=1}^{\mu}(t_i+1-v_i)$. If it does not keep it within the first μ terms, this means that the first μ terms must all be equal. There are then four cases to consider. Recalling that the Durfee number is denoted by λ, we have:
i) $t_1 = \ldots = t_\mu > t_{(\mu+1)}$ where $\mu < \lambda$,
ii) $t_1 = \ldots = t_\mu = t_{(\mu+1)}$ where $\mu < \lambda$,
iii) $t_1 = \ldots = t_\mu > t_{(\mu+1)}$ where $\mu = \lambda$,
iv) $t_1 = \ldots = t_\mu = t_{(\mu+1)}$ where $\mu = \lambda$.

In cases i) and iii) t_1 is still the μth biggest, and clearly as the deficit is greater than 0, $t_\mu + 1 - v_\mu > 0$ (v_μ is the smallest of the first μ v's), so the deficit is reduced by 1 to 0. Thus $\sum t_i$ is even and the deficit is 0, which implies that the score is 0.
In case ii) prior to the change $t_\mu + 1 - v_\mu > 0$ so that $t_{(\mu+1)} + 1 - v_{(\mu+1)} > 0$, which would mean that the peak is not at μ but at $\mu + 1$ (or later). Thus this is a contradiction.
In case iv) $t_1 = \ldots = t_\lambda = t_{(\lambda+1)}$. The only way this can happen is if all of these take value λ (otherwise the Durfee number could not be λ). But then $v_1 = \ldots = v_\lambda \geq \lambda + 1$, so that the deficit is in fact 0. Thus this is also a contradiction.

Combining all of the above, the theorem is proved. $\square$

In [5] we considered the *dual sequence* of $\mathbf{t}$, denoted by $\mathbf{s}$, where $s_i = n-1-t_{n+1-i}$. $\mathbf{s}$ corresponds to the target number of breaks (as opposed to links) of the vertices in reverse order, i.e. in the order of increasing target of links and so decreasing target of breaks. We shall refer to the score of $\mathbf{s}$ as the reverse score of $\mathbf{t}$, and similarly the deficit of $\mathbf{s}$ as the reverse deficit of $\mathbf{t}$. We represent the conjugate sequence of $\mathbf{s}$ by $\mathbf{w}$. We denote λ^* as the Durfee number *from the back* of $\mathbf{t}$, which is $n+1$ minus the Durfee number of $\mathbf{s}$.

Examples A cont. For example 33311 we have $s_1 = 4 - t_5 = 3$, similarly $s_2 = 3, s_3 = 1, s_4 = 1, s_5 = 1$. Consider the graph where there are no links between vertices 3,4 and 5, vertices 1 to 4 and 2 to 5 are not linked, but all other pairs of vertices are. The graph has sequence 33211, deviations are 0, 0, 1, 0, 0 and so the graph deviation is 1, which is also the reverse score. All five s elements are at least as large as 1, so that $w_1 = 5$, and similarly $w_2 = 2, w_3 = 2, w_4 = 0$, so the Durfee number of $\mathbf{s}$ (i.e. from the back) is 2, and $\lambda^* = 6 - 2 = 4$. We have $f_0 = 0, f_1 = 3 + 1 - 5 = -1, f_2 = (3 + 1 - 5) + (3 + 1 - 2) = 1$, so that the reverse deficit is 1.

For example 43210 we have $s_i = 4 - t_i$ so that $\mathbf{s}$ is simply 43210, the same as $\mathbf{t}$. Consequently the reverse score and deficit of $\mathbf{t}$ are both 2, the conjugate sequence of $\mathbf{s}$ is 4321, the Durfee number of $\mathbf{s}$ is 2, so $\lambda^* = 6 - 2 = 4$.

In Theorem 3.2 we show two symmetry results relating to the two important sequence properties, the score and the deficit. This then means that results relating to the minimal set of a sequence can immediately be carried over to its dual without further analysis. This is intuitively obvious (but up to now not formally proved) for the score, but less so for the deficit.

Theorem 3.2. *The reverse score of a sequence* $\mathbf{t}$ *is equal to its score, and the reverse deficit of* $\mathbf{t}$ *is equal to its deficit.*

Proof: Firstly we prove the equality of the score and the reverse score. This is relatively straightforward, and in fact in [5] we stated that this result was easy to show and omitted it, but we include it for completeness here.

For $\mathbf{t}$ and any graph $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ the deviation is $D(\mathbf{t}, \mathbf{g}) = \sum_i |t_i - g_i|$. For $\mathbf{s}$ and $\mathbf{g}^* = (n-1-g_n, n-1-g_{n-1}, \ldots, n-1-g_1)$, the (reordered) complement of $\mathbf{g}$, the deviation is $D(\mathbf{s}, \mathbf{g}^*) = \sum_i |(n-1-t_{n+1-i}(n-1--g_{n+1-i}| = D(\mathbf{t}, \mathbf{g})$. It follows for target $\mathbf{t}$ that the set of deviations $D(\mathbf{t}, \mathbf{g})$ for $\mathbf{g} \in G_n$, where G_n is the set of all graphs with n vertices, is precisely the same as the set of deviations $D(\mathbf{s}, \mathbf{g}^*)$. Therefore the scores for $\mathbf{t}$ and $\mathbf{s}$ are equal, being the minima in the sets of deviations. Note also that the members of the minimal set for $\mathbf{t}$ are the complements of the members of the minimal set for $\mathbf{s}$.

Now consider the deficit and the reverse deficit. Firstly note that we know that for any sequence, $\sum_{i=1}^{n} t_i = \sum_{i=1}^{n-1} v_i$.

a) Suppose first that a sequence has deficit 0. This then means that

$$\sum_{i=1}^{\lambda}(t_i + 1 - v_i) \leq 0 \Rightarrow \sum_{i=1}^{\lambda} t_i \leq \sum_{i=1}^{\lambda} v_i - \lambda. \tag{5}$$

We know from Theorem 4 of [5] that for all $r = 1, \ldots, n$,
$s_r + 1 - w_r = v_{n-r} - t_{n+1-r}$.

We know from Theorem 5 of [5] that the reverse deficit is either
(i) $\sum_{r=1}^{n-\lambda-1}(s_r + 1 - w_r)$ if there is a "gap" ($\lambda^* = \lambda + 2$) or
(ii) $\sum_{r=1}^{n-\lambda}(s_r + 1 - w_r)$ otherwise ($\lambda^* = \lambda + 1$).

In case (i) we have the reverse deficit given by the maximum of 0 and $\sum_{r=1}^{n-\lambda-1}(s_r + 1 - w_r) = \sum_{r=1}^{n-\lambda-1}(v_{n-r} - t_{n+1-r}) = \sum_{i=\lambda+1}^{n-1} v_i - \sum_{i=\lambda+2}^{n} t_i \leq -\lambda + t_{\lambda+1}$, from inequality (5).

The right hand side of the above inequality is clearly less than or equal to 0 otherwise the Durfee number is bigger than λ, and so the reverse deficit is 0.

In case (ii) we add the extra term $s_{n-\lambda} + 1 - w_{n-\lambda} = v_\lambda - t_{\lambda+1}$, so we have $\sum_{r=1}^{n-\lambda}(s_r + 1 - w_r) \leq -\lambda + t_{\lambda+1} + v_\lambda - t_{\lambda+1} = v_\lambda - \lambda = 0$ (since in the "no gap" case $t_{\lambda+1} < \lambda$).

Thus in either case, the reverse deficit is 0.

b) Following essentially identical working to the proof of Theorem 3.1 part b) above, whenever the deficit is non-zero, taking 1 from t_1 always reduces the deficit by 1.

Similarly, whenever the deficit is greater than 0, adding 1 to t_n always reduces the deficit by 1. We can see this since $v_1 = \ldots = v_{t_n} = n$ and so if $\mu \leq t_n$ the deficit must be zero. Thus $\mu > t_n$. Increasing t_n by 1 increases v_{t_n+1} by 1, and so reduces the deficit $\sum_{i=1}^{\mu}(t_i + 1 - v_i)$ by 1.

Suppose that the deficit is x and the reverse deficit is $x + 1$. From the above, taking 1 from t_1 gives a new sequence with deficit $x - 1$, but the new reverse sequence is the old reverse sequence with s_n increased by 1, so it has reverse deficit x. Carrying on this process, eventually the deficit of the sequence will be 0, but its reverse deficit will be 1. From a) we know that this cannot be true, so we have a contradiction.

Thus the deficit and the reverse deficit are always the same. $\square$

3.2. **Vertex categorisation.** In [5], we saw that for any given sequence we can categorise the vertices into four classes, associated with the corresponding minimal set K(min). There were:

a) Vertices which are sometimes a Joiner and sometimes a Breaker (and also sometimes Neutral) for some elements of $K(min)$, denoted by set S_A;
b) vertices which are Joiners (or Neutrals) for some elements but never Breakers, denoted by set S_J;
c) vertices which are Breakers (or Neutrals) for some elements but never Joiners, denoted by set S_B;
d) vertices which are always Neutral, denoted by set S_N.

Below we shall investigate the categorisation of vertices for score 1 sequences. We do this by considering the score of sequences close to any given score 1 sequence. In particular, we know from Theorem 3.1 that any sequence of score 1 can be obtained by starting from some graphic sequence, and adding 1 to, or taking 1 from, some element of the sequence. Similarly any addition or subtraction of 1 from a graphical sequence will yield a score 1 sequence. Here we consider what additional changes of 1 in a single element of a score 1 sequence can recover a graphic sequence. For the Durfee number λ, we consider four different possibilities for the first change:
A - an addition of 1 within the vertices with index $1, \dots, \lambda$,
B - a subtraction of 1 within the vertices with index $1, \dots, \lambda$,
C - an addition of 1 within the vertices with index $\lambda+1, \dots, n$,
D - a subtraction of 1 within the vertices with index $\lambda+1, \dots, n$.
There are the same four possibilities for the second change, which we shall equivalently denote by a, b, c and d respectively.

We call a change an *outer move* if it increases the target for $i \leq \lambda$ or decreases the target for $i > \lambda$ (i.e. is an A or D move), and an *inner move* otherwise. Further we define:
$g(i) = i, h(m) = m$ if $i \leq \lambda$ and $m \leq \lambda$,
$g(i) = i, h(m) = t_m$ if $i \leq \lambda$ and $m > \lambda$ (if $t_m = \lambda$ and the Durfee number increases then $h(m) = \lambda + 1$),
$g(i) = t_i, h(m) = m$ if $i > \lambda$ and $m \leq \lambda$, (if $t_i = \lambda$ and the Durfee number increases then $g(i) = \lambda + 1$),
$g(i) = t_i, h(m) = t_m$ if $i > \lambda$ and $m > \lambda$ (if $t_i = \lambda$ or $t_m = \lambda$ and the Durfee number increases then $g(i) = \lambda + 1, h(m) = \lambda + 1$ respectively).
We note that given the potential change in Durfee number, for a given pair i, m, the value of $g(i)$ $(h(m))$ may depend upon whether the move at vertex i (m) was an outer or an inner move, and so we denote these below as $g_o(i)$ $(h_o(m))$ and $g_i(i)$ $(h_i(m))$, respectively. In the majority of cases, we will have $g_o(i) = g_i(i)$ and $h_o(m) = h_i(m)$.

Using the above, we can now prove Theorem 3.3, our main theoretical result. This result enables us to find the set classifications, i.e. the membership of S_A, S_J, S_B and S_N, for an arbitrary sequence of score 1 using straightforward calculations, without having to investigate the membership of the minimal set, which is a much more complicated task.

We shall consider a score 1 sequence obtained by a change at vertex i from a graphic sequence with associated values $f_1, \dots, f_\lambda$ where K is the largest index such that $f_k \geq -1$ for $k = 1, \dots, \lambda$, and K' is the largest index such that $f_k = 0$ for $k = 1, \dots, \lambda$. If there is no such f_k in either case, then $K = 0, K' = 0$ respectively. For succintness of exposition, in Theorem 3.3 we shall consider the following:
$C_o =$ "the change was an outer move"
$C_i (= C_o^c) =$ "the change was an inner move"
$A_o =$ "there is no $f_k = 0$ s.t. $g_o(i) \leq k < h_i(m)$"

$A_i =$ "there is no $f_k = 0$ s.t. $h_i(m) \leq k < g_i(i)$"
$B =$ "neither of $K \geq \max(g_o(i), h_o(m))$, $K' \geq \min(g_o(i), h_o(m))$ hold"
$D_l =$ "$m \leq \lambda$"
$D_g (= D_l^c) =$ "$m > \lambda$"

Theorem 3.3. *For an arbitrary score 1 sequence we have the following complete set classifications for its elements:*

Element m is in S_A (except if the target after the change is 0 or $n-1$, when m is in S_B and S_J respectively) if $(C_o \cap A_o \cap B) \cup (C_i \cap A_i)$.

Element m is in S_N if $C_o \cap A_o^c \cap B^c$.

Element m is in S_J if $(C_o \cap ((D_l \cap A_o \cap B^c) \cup (D_g \cap A_o^c \cap B))) \cup (C_i \cap D_l \cap A_i^c)$.

Element m is in S_B if $(C_o \cap ((D_l \cap A_o^c \cap B) \cup (D_g \cap A_o \cap B^c))) \cup C_i \cap D_g \cap A_i^c)$.

Theorem 3.3 is proved in Appendix A.

Examples B. Below we consider example sequences 766543221 and 766444221, shown in Tables 1 and 2. In each case we identify the appropriate f_i values as well as the values of λ, K and K', and list the changes, the score 1 sequence derived from them (and which element is changed to reach it) and then the categories of all elements within that sequence A, B, J or N (in the corresponding position to the target sequence). For each sequence we also identify any changes which lead to a new Durfee number, as this can change the values of $g(i)$ or $h(m)$.

Note that the two sequences labelled ^{a} and ^{b} in both tables are actually the same sequence in each table, and so of course yield the same result. In general, most score 1 sequences can be generated from a number of graphic sequences. For the sequences from Table 1 in particular, extra care is needed when considering the fifth element, as here often the Durfee number changes. For example, for the sequence 766553221 from Table 1 we have that $g_o(5) = 4$ and $h_i(5) = 5$, which makes the fifth element a member of S_J and not S_N (this had to be true since here the fourth and fifth elements have the same target, and so must be in the same set).

We note that whenever we have a vertex in S_N, it is easy to see that we cannot also have another vertex in the sequence that is in S_A, as we already knew from [5].

3.3. **Pure and Mixed Equibria; the 111 case.** Suppose that we consider the case with $n = 3$ and target 111, when the minimum score is 1. It is easy to specify the transition graph, which has 6 vertices and is shown in Figure 1. Now suppose the process which we consider allows when in state i for the deficit vertex individual to choose either to move clockwise with probability p_i and anticlockwise with probability $1-p_i$. Here there are (at least) three Nash equilibria. If $p_i = 1$ for all i each choice moves the system clockwise and if $p_i = 0$ for all i, anticlockwise. Each state occurs with frequency 1/6 so the cost to each individual is 1/3. Should any individual in any state opt to play differently then the system will immediately return to that state and the system will oscillate giving a cost of 1/2 to the individual who switches play. Thus these two sets of choice are pure Nash equilibria.

We also have a mixed Nash equilibrium when $p_i = 1/2$ for all i. In general, for any $p_i = p$ for all i we will have a uniform stationary distribution over the states and so each vertex has cost 1/3. Suppose now we consider the case where the individual off target in state 1 (and so also in state 4) considers switching to r and $1-r$ when

766543221	$f_1 = -1, f_2 = -2, f_3 = -1, f_4 = 0,$ $\lambda = 4, K = 4, K' = 4$	
866543221	$i = 1, \lambda* = 5$ for $m = 5$ up	J J J JNBBBB
666543221	$i = 1, \lambda* = 5$ for $m = 5$ up	AAAAAAAAA
776543221	$i = 2, 3, \lambda* = 5$ for $m = 5$ up	J J J JNBBBB
765543221	$i = 2, 3, \lambda* = 5$ for $m = 5$ up	AAAAAAAAA
766643221	$i = 4, \lambda* = 5$ for $m = 5$ up	J J J JNBBBB
766443221[a]	$i = 4, \lambda* = 5$ never	AAAAAAAAA
766553221	$i = 5, \lambda* = 5$ for all but $m = 4, 5$ down	J J J J JBBBB
766533221	$i = 5, \lambda* = 5$ never	J J J JBBBBB
766544221[b]	$i = 6, \lambda* = 5$ for $m = 5$ up	AAAAAAAAA
766542221	$i = 6, \lambda* = 5$ for $m = 5$ up	J J J JNBBBB
766543321	$i = 7, 8, \lambda* = 5$ for $m = 5$ up	AAAAAAAAA
766543211	$i = 7, 8, \lambda* = 5$ for $m = 5$ up	J J J JNBBBB
766543222	$i = 9, \lambda* = 5$ for $m = 5$ up	AAAAAAAAA
766543220	$i = 9, \lambda* = 5$ for $m = 5$ up	J J J JNBBBB

TABLE 1. Score 1 sequences generated from the graphic sequence 766543221. In each case we identify the set, one of S_A, S_B, S_J or S_N that each element is in, in the corresponding position to the target sequence.

766444221	$f_1 = -1, f_2 = -2, f_3 = -1, f_4 = -2,$ $\lambda = 4, K = 3, K' = 0$	
866444221	$i = 1, \lambda* = 5$ never	J J J AAABBB
666444221	$i = 1, \lambda* = 5$ never	AAAAAAAAA
776444221	$i = 2, 3, \lambda* = 5$ never	J J J AAABBB
765444221	$i = 2, 3, \lambda* = 5$ never	AAAAAAAAA
766544221[b]	$i = 4, 5, 6, \lambda* = 5$ for $m = 5$ up	AAAAAAAAA
766443221[a]	$i = 4, 5, 6, \lambda* = 5$ never	AAAAAAAAA
766444321	$i = 7, 8, \lambda* = 5$ never	AAAAAAAAA
766444211	$i = 7, 8, \lambda* = 5$ never	J J J AAABBB
766444222	$i = 9, \lambda* = 5$ never	AAAAAAAAA
766444220	$i = 9, \lambda* = 5$ never	J J J AAABBB

TABLE 2. Score 1 sequences generated from the graphic sequence 766444221. In each case we indentify the set, one of S_A, S_B, S_J or S_N that each element is in, in the corresponding position to the target sequence.

in state 1. We have, where $x[i]$ is the stationary distribution frequency for state i,

$$x[1] = x[6] * p + x[2] * (1 - p);$$
$$x[2] = x[1] * r + x[3] * (1 - p);$$
$$x[3] = x[2] * p + x[4] * (1 - p);$$
$$x[4] = x[3] * p + x[5] * (1 - p);$$
$$x[5] = x[4] * p + x[6] * (1 - p);$$
$$x[6] = x[5] * p + x[1] * (1 - r).$$

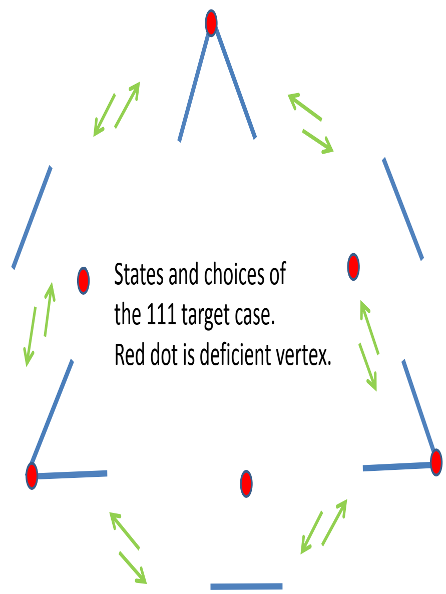

Figure 1. The transition graph for the target 111 with six states. The vertex in deficit in each state is highlighted by a dot, and corresponding possible transitions are shown.

Now the payoff for vertex 1 is $x[1]+x[4]=p(x[3]+x[6])+(1-p)(x[2]+x[5])$, and when $p=1/2$ this implies that for any r the payoff for vertex 1 is still 1/3 so $p=1/2$ is a mixed Nash equilibrium. We observe, from computer simulations, that when $p<1/2$ then the cost for vertex 1 is less when $r<p$, and more when $r>p$, which suggests that this solution is unstable, and any slight deviation from the equilibrium moves the population to one of the two pure Nash equilibria.

4. **The example sequence 11111.** We shall now consider a specific example involving strategic decisions. This example allows us to demonstrate a methodology for analysing score 1 games using cycles on a graph (the *choice graph*, defined below), but at the same time demonstrates the general complexity of investigating games for specific targets, even relatively simple-looking ones like this.

Suppose we have target 11111 which has score 1. There are two possible configurations, which we refer to as states, for a graph in the minimal set. These are: $(\bar{i})(j,k)(l,m)$, the graph with edges (j,k) and (l,m), and $(i,\bar{j},k)(l,m)$, the graph with edges $(i,j),(j,k)$ and (l,m); the overline indicating the deficit vertex and i,j,k,l,m taking distinct values 1,2,3,4,5. There are 15 and 30 of these configurations respectively. Examples (1)(2,5)(3,4) and (2,1,5)(3,4) are shown in Figure 2.

In our model, in any state a vertex is selected at random; if it is a deficit vertex it makes a move to improve that deficit. In the target 11111 case the deficit is always 1, so that there is precisely one vertex that can make a change at any time. The possible transitions are:

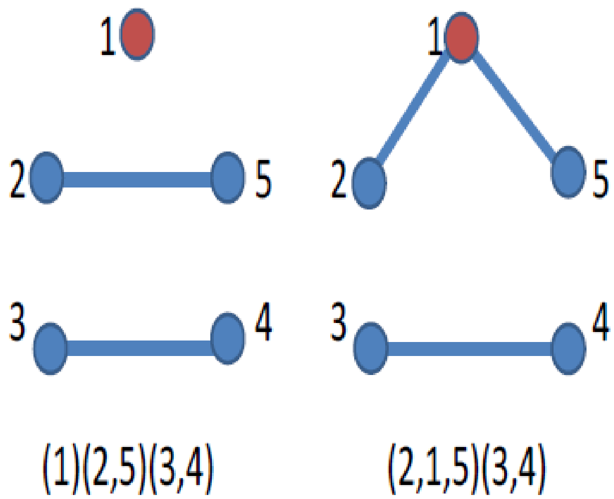

FIGURE 2. The two possible configurations for members of the minimal set with target 11111. The uppermost vertex is the one not attaining its target. There are 15 different cases of the configuration on the left, and 30 of the configuration on the right.

$(\bar{i})(j,k)(l,m) \to (i,\bar{j},k)(l,m); (i,\bar{k},j)(l,m); (i,\bar{l},m)(j,k)$ or $(i,\bar{m},l)(j,k)$ and $(i,\bar{j},k)(l,m) \to (\bar{i})(j,k)(l,m)$ or $(\bar{k})(i,j)(l,m)$.

At any state an individual will decide which of these options to take, which we shall refer to as its *choice*. There will thus be such a choice at every state, and we refer to this as the *choice graph*. Recalling that only strict moves are allowed, an individual's strategy is thus the set of all of its choices at the states for which it is in deficit. The choice graph then simply represents the collection of strategies selected by the players. We are thus interested in investigating the possible choice graphs that can occur in our game, the long-term behaviours that result, and whether these are stable (restricting ourselves to local changes, as defined in Section 2.2).

Consider for the moment the set of six states where there is always an edge (3,4); specifically $(\bar{1})$(2,5)(3,4); (1,$\bar{2}$,5)(3,4); (1,2)($\bar{5}$)(3,4), (2,$\bar{1}$,5)(3,4); (1,5)($\bar{2}$)(3,4) and (1,$\bar{5}$,2)(3,4). The transitions between these states are just those of the set of transitions for a target 111 on the remaining vertices 1,2,5, and the transition graph for these vertices is thus just equivalent to the one shown in Figure 1, where the numbering of the vertices shown is as in Figure 2. In the context of target 11111 we observe that the three states in Figure 1 with a single edge are linked only to other vertices in Figure 1, and not to any others in the transition graph of 11111. The vertices with two edges, on the other hand, are each members of other restricted sets. For example, if we consider, as above, the case with fixed edge (3,4) then the configuration with edges (3,4) and (1,2) will also belong to the transition graph of the case where only (1,2) is fixed. Thus the ($\bar{5}$)(1,2)(3,4) vertex will belong to two 111-transition graphs, as will the vertices ($\bar{1}$)(2,5)(3,4) and ($\bar{2}$)(1,5)(3,4). For ease of reference we will refer to the transition graphs for the various 111 sub-cases as (111)-triangles, where the vertices are the states with two edges, and the states with one edge are placed at the centre of the line joining adjacent vertices. We have

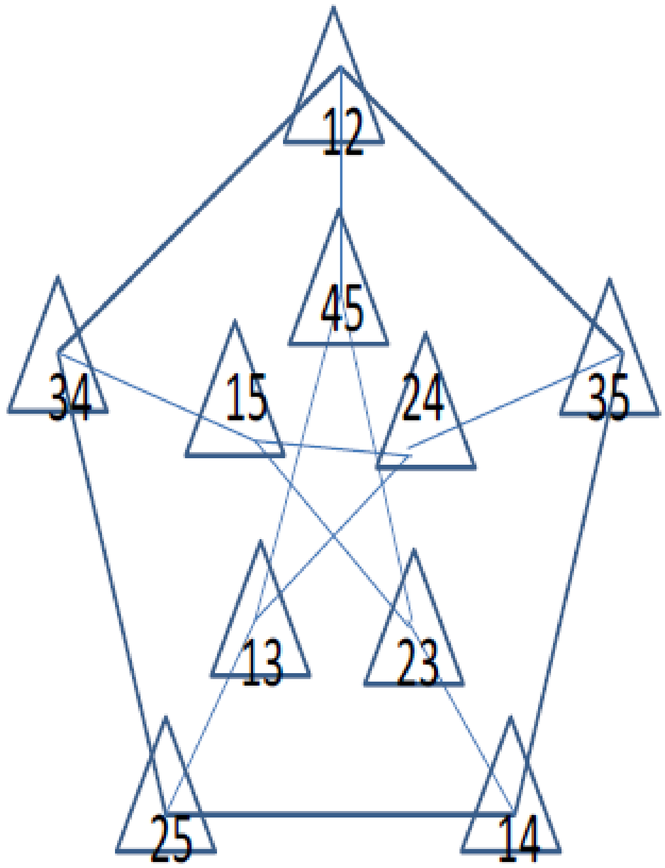

FIGURE 3. The Petersen triangle with associated sets. Vertices are joined if the set-intersection is empty.

in the 11111 transition graph ten such triangles corresponding to the ten possible pairs from {1,2,3,4,5}.

4.1. **The associated extended Petersen Graph.** Figure 3 shows the famous Petersen graph, a non-planar, non-Hamiltonian cubic graph. There are ten vertices each of which can be associated with a set of two elements drawn from the set {1,2,3,4,5}. Two vertices are joined if, and only if, the associated sets are disjoint. A possible labelling of the vertices is shown in Figure 3. The transition graph for 11111 can be constructed by actions on a Petersen graph.

We begin by constructing the triangle-replaced graph from the Petersen graph. A triangle-replaced graph is derived from a cubic graph by replacing the subgraph $(a,b)(a,c)(a,d)$ by $(e,b)(f,c)(g,d)(e,f)(e,g)(f,g)$ where f,g and h are new vertices, for all vertices in the initial graph. Essentially this is a process in which at each vertex a small "tetrahedron" is snipped off. Now we apply a further step by merging the two new vertices which have been created on every original edge. In this way we have created a 4-regular graph. This produces precisely the graph in Figure 4, which shows the set of possible transitions between the 45 states for the minimal set for target 11111. Note that there are 15 *corner states* which are vertices in the Petersen graph, and 30 *edge states* which appear midway along an edge of the Petersen graph. Both corner states and edge states are vertices in the transition graph. Each vertex of the Petersen graph gives rise to a triangle which is labelled with a pair of numbers from {1,2,3,4,5}, matching those of the Petersen graph.

Now in our model each vertex, when it is the deficit vertex, can choose which state to move to, and this is indicated by a directed edge. We see a graph where the current choices, the choice graph, are shown in Figure 5. There are 60 edges in the transition graph and 45 in the choice graph. We note in passing that the Petersen graph is a hemi-dodecahedron; a hemi-graph is that obtained from a graph by merging the opposite vertices (assuming these are well defined). Now an icosidodecahedron, a polyhedron with 20 triangular faces and 12 pentagonal ones, can be obtained from a dodecahedron by removing a tetrahedron at each vertex so that the two tetrahedra on each edge share a common vertex. Our extended Petersen graph is just a hemi-icosidodecahedron and the process used to obtain it is analogous to that in obtaining an icosidodecahedron from a dodecahedron.

4.2. **Cycles.** For any given set of transitions, representing choices by the individuals, for any initial state, changes will continue until a state is revisited. Subsequent changes will then follow those from before, so that we will continuously follow a cycle of states. We begin by identifying what directed cycles can occur in the choice graph. The Petersen graph has cycles of length 5, 6, 8 and 9. With the exception of cycles of length two (e.g. $16 \rightarrow 17 \rightarrow 16$ in Figure 4) and a 6 cycle round a single triangle, a cycle in the choice graph (as distinct from a cycle in the Petersen graph) will necessarily contain an edge, or two edges of a triangle, and then an edge or two of a neighbouring triangle. We will call two triangles *neighbours* in a sequence, if the sequence leads directly from one triangle to the other. Any cycle in the Petersen graph corresponds to a sequence of neighbouring triangles and to multiple potential cycles in the choice graph. For a Petersen cycle of length 5, e.g. (24)(13)(45)(23)(15) we have cycles in the choice graph of length 2(2)20 (i.e. all values from 2 to 20 inclusive in steps of 2). We need to specify the number of edges each triangle contributes to the cycle. The lists of the number of triangle edges in successive triangles are 11111, 11112, 11122, 11212, 11222, 12122, 12222 and 22222. Those with 7 and 8 triangle edges have 2 distinct patterns. For a Petersen cycle of length 6, e.g. (24)(13)(25)(14)(23)(15) there are cycles of length 12(2)24; those with length 12, 14, 22, 24 have one pattern each while those with lengths 16, 18 and 20 have two patterns.

The situation for the other cycles of the Petersen graph is somewhat different. In the cases with 5 and 6 triangles in the Petersen graph, triangles involved are linked only to their immediate neighbours in that cycle; e.g. in the 6 cycle above the triangle 24 is disjoint from 25, 14 and 23. This does not happen when we consider 8 and 9 triangles. For example the 8 cycle in the Petersen graph (35)(14)(25)(13)(45)(23)(15)(24)(35) has two pairs of triangles which are joined (i.e. share a common vertex) although not neighbours, (14) and (23), and (13) and (24). It follows that the shared vertex in each case can only be used in one triangle in a cycle in the choice graph. We have therefore two triangles in which we can only count one vertex in the cycle so that we have a maximum cycle length of 28. For the cycle of length 9 we will have 3 intersecting triangles which are not neighbours in the cycle so that the maximum cycle length will be 30. For example for the 9-cycle, (12)(45)(23)(14)(25)(13)(24)(15)(34)(12), we

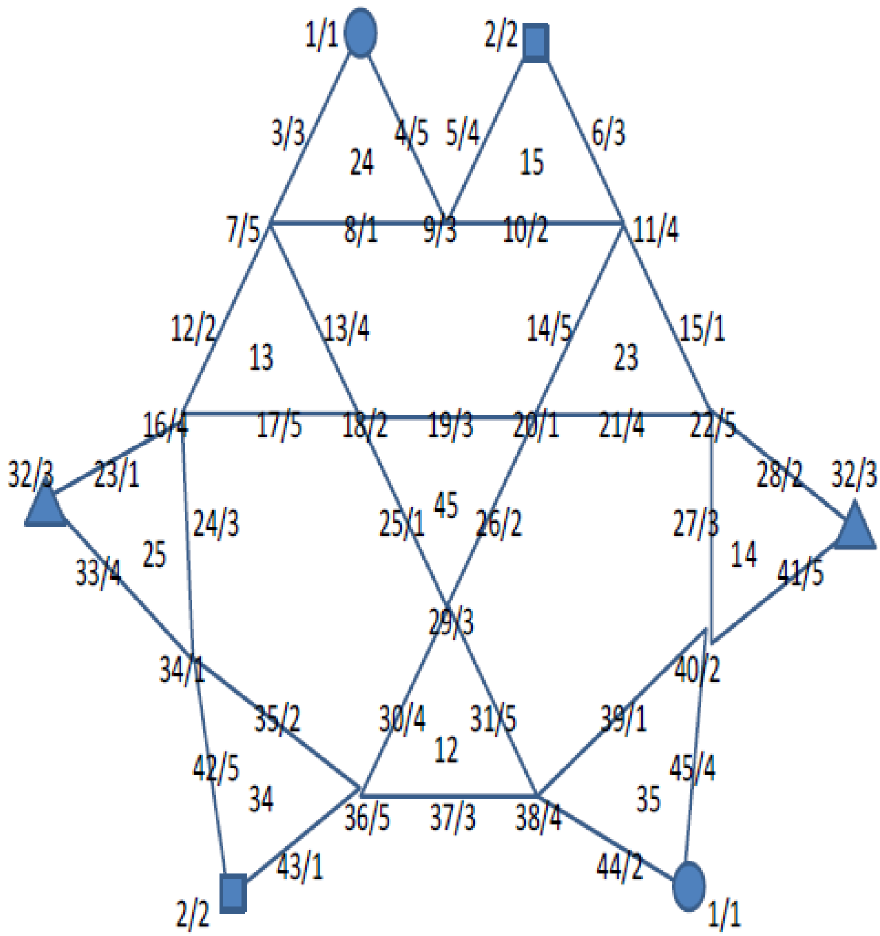

FIGURE 4. The transition graph for the graphs within the minimal set for target 11111. Each triangle is labelled internally with the edge fixed in its configurations. Note that each edge has a vertex at its mid-point. Each vertex is labelled a/b with a unique state number a, and the vertex b which is deficient in state a. Three vertices 1, 2 and 32 are duplicated (shown as different shapes) to allow a planar representation.

have three pairs of triangles joined other than within the cycle, (13) and (45), (15) and (23), and (34) and (25). In these pairs the first has one edge within the cycle, the second has two edges in the cycle. The triangle (35) has zero edges in the cycle being missing from the nine cycle on the Petersen graph. All other triangles have two edges in the cycle. The cycle is (in terms of states) (36,37,38,31,29,25,18,19,20,14,11,15,22,27,40,41,32,33,34,24,16,12,7,3,1,4,9,5,2,43,36) and is shown in Figure 5.

We are interested in elucidating when there can be two directed cycles which do not intersect, leading to two distinct long-term behaviours, depending upon the

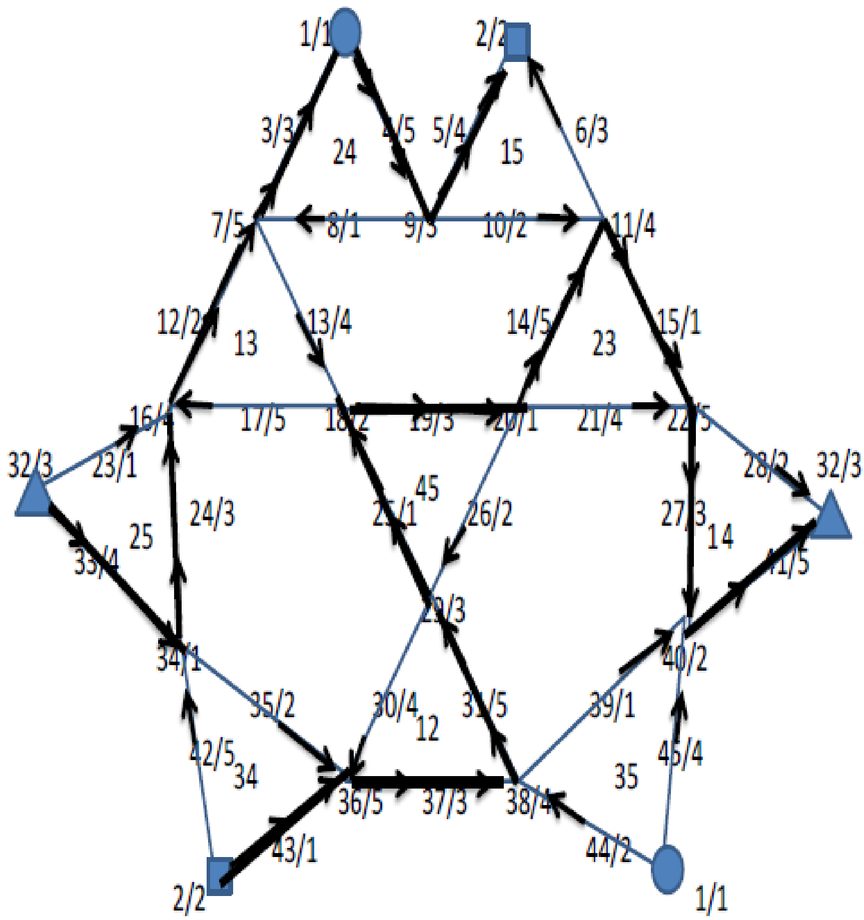

Figure 5. The choice graph for the graphs within the minimal set for target 11111. Each vertex has an arrow which is the choice that the deficient vertex would make in that state. There is a stable cycle of length 30, specifically (36; 37; 38; 31; 29; 25; 18; 19; 20; 14; 11; 15; 22; 27; 40; 41; 32; 33; 34; 24; 16; 12; 7; 3; 1; 4; 9; 5; 2; 43; 36), which is shown in bold.

initial state. Clearly this can only happen in the Petersen graph when we consider 5 cycles (if we ignore for the moment the 2 and 6 cycles in the choice graph). Examples are (12)(34)(25)(14)(35) and (15)(23)(45)(13)(24), and in fact for any five cycle the complementary vertices form another five cycle. As discussed above, when we have a 5-cycle in the Petersen graph this can lead to a 10-cycle in the choice graph. Associated with this cycle will be 5 vertices (representing corner states), those not used in the 5-cycle but in the set of the 5 triangles. Now this same set of 5 corner vertices will also occur as the unused vertices of the 5-cycle for the complementary vertices. In forming two non-overlapping cycles we can take the two 10-cycles, and for each of the 5 extra vertices add it to neither cycle, to the first or to the second cycle, but not to both. The addition of each vertex increases the cycle length by 2 (the corresponding corner state and a neighbouring edge state). Thus we have possible non-overlapping cycle lengths $2i$ and $2j$ where $i \in [5, 10]$ and $j \in [5, 15-i]$].

4.3. **Dynamics of the system.** At any point in time there will be a single vertex which is in deficit and there will be a choice graph. The deficit vertex will consider the possible choices it may make (either 2 or 4). Figure 5 shows arrows, one at each vertex, which are the choices which deficit vertices would make if the system were in that state. For each possible move the path from the deficit vertex along the various existing choices will be followed until a cycle is reached. The long-term cost for a possible move will be the cost for the deficit vertex around this cycle. The choice is then made between possible moves so as to minimise the cost, and possibly a switch made in the choice graph. We consider in the main the case where a switch is only made if a definite improvement in costs is achieved.

Suppose there is a cycle through states $\{v_1, v_2, \ldots, v_k\}$, with $k > 2$, then if we begin the process with choices $v_i \to v_{i+1}$ (all mod(k)) and for any other vertex w such that w is a neighbour of some v_i the individual in deficit at w chooses that v_i, then either that cycle will be stable, or a sub-cycle comprising a subset of the same elements, but in the same order, will be stable. Necessarily the cost to each individual in the cycle will be less than 1/2, and if it considers a possible swap then either that state is pointing straight back so there is a cost of 1/2 (as per state 27 considering choice 22 in Figure 6) or the neighbouring state will point to another member of the cycle and either the new cost will be at least equal to the current cost, so that no change is made, or the states that lie between the two in the sequence will be missed out, leading to the corresponding sub-cycle.

Figures 4 and 5 show the numbering of the vertices and the deficit vertex for that state. The cost incurred by each of the five vertices if the system were to move around a directed cycle is just computed by counting the occurrence of each vertex in deficit. For example for a three cycle in the Petersen graph (a 6-cycle in the choice graph), necessarily a cycle around a specific triangle, the three vertices have costs 1/3, e.g. triangle 24 has costs 1/3 for each of vertices 1,3,5 and costs 0 for vertices 2,4. We refer to the lists of costs as the cost vector, so for the 6 cycle just referred to the cost vector is (2,0,2,0,2)/6 while for the 30 cycle in Figure 5 it is (6,4,8,5,7)/30. These costs naturally play a key part in the dynamics which we apply.

We now consider the possibility of there being multiple cycles arising from an initial set of choices. Suppose that there is a stable cycle of length 6 (involving three vertices). Then for each deficit vertex within that cycle we have cost 1/3. At any of those vertices there are four possible choices, one forward around the cycle (cost 1/3), one backwards around the cycle (cost 1/2), and two away from the cycle. For a stable cycle, if either of these leads to a vertex which chooses something outside the three cycle it must give a higher cost. It is clear that no two stable triangles can share a vertex. Accordingly there can be only as many such stable triangles as the independence number of the Petersen graph, 4. We could have any set of four triangles which share a number being simultaneously stable; e.g. (12)(13)(14)(15).

We observe that the system may move between cycles which intersect. For example see Figure 7. Here there are two 14 cycles which share the 9 vertices in the middle line, but differ in the 5 vertices in the two other rows. All missing edges point into the cycles so have no influence. If the system at some stage reaches the vertex 40, then the target vertex is 2 and since there is one target 2 in each of the

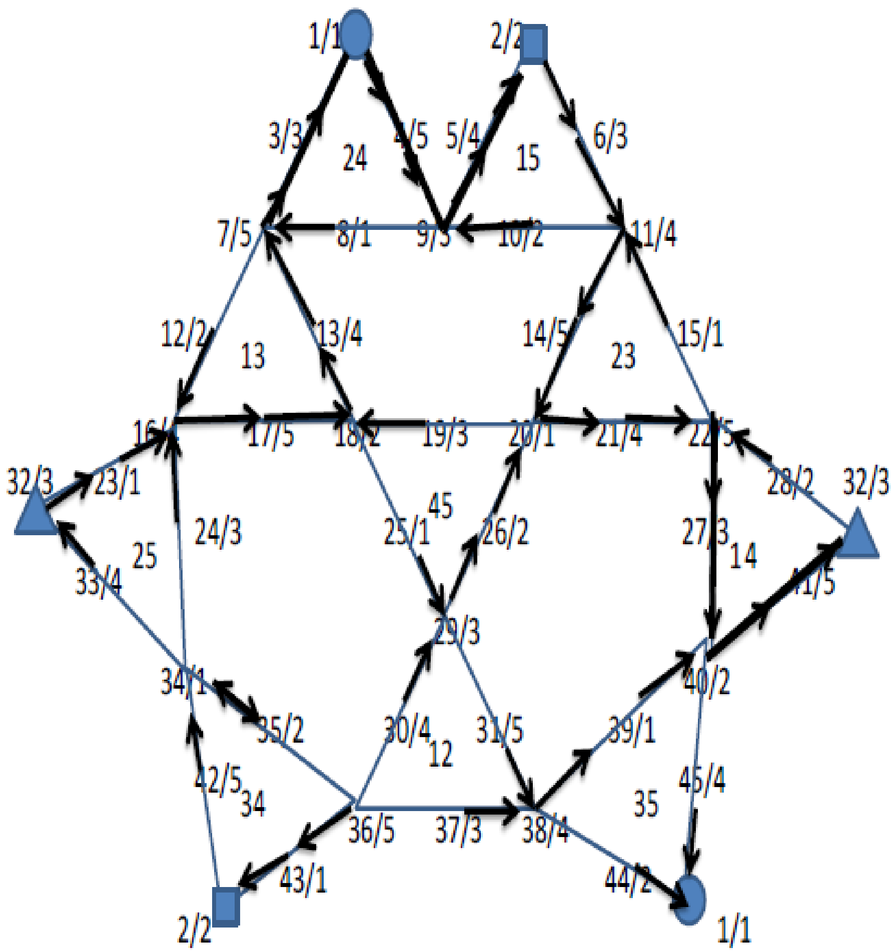

Figure 6. The choice graph for the minimal set for target 11111. Here we have a stable 24-cycle. (1; 4; 9; 5; 2; 6; 11; 14; 20; 21; 22; 27; 40; 41; 32; 23; 16; 17; 18; 13; 7; 3; 1).

sets of 5 the payoff for target 2 for the vertex 40 is the same for choices 27 and 45 in the model where a switch to a cycle with equal payoff to the current one is allowed. If a switch is only allowed where an improvement is made then whichever of the two 14 cycles is current will persist. Note that if we consider the reverses of these “cycles” then when the vertex “29” is reached, the choice “31” is preferred to “26” and so the system will only spend time on the lower cycle.

Finally we consider the case where a switch is made to the direction which leads to the minimal cost for the current deficit vertex. If there is more than one such minimal cost then we pick one of the corresponding directions at random with the same probabilities. This implies that a switch may be made to a direction which gives the same cost as the current direction. We suppose that the system becomes fixed when a cycle occurs from which there is no switch. Thus if there is a pair of cycles like those in Figure 7 one of the two would become fixed by chance; whichever

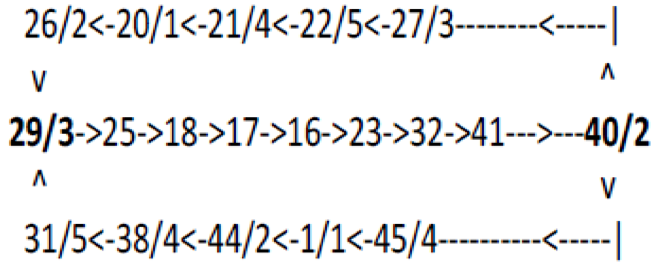

FIGURE 7. Two intersecting 14-cycles with 9 common vertices.

Cycle-length	6	10	12	14	16	18	20	22	24	26	28	30
Frequency	358	436	735	319	96	29	17	7	3	0	0	0

TABLE 3. Frequencies of stable cycles resulting from 2,000 simulations starting from randomly generated initial choice graphs.

was completed first. Now we exclude cycles of length 2 and 4. We exclude the first since it would give a cost of 1/2 and this is the maximum possible, so here it cannot be worth stopping the process. We exclude 4 since such a cycle always contains a 2-cycle. Thus we are left with possible cycles of length (recall we cannot have a cycle of length 8) 6 and 10(2)30. Table 3 records the frequecies of stable cycles arising from 2,000 random initial choice graphs.

5. **Discussion.** In this paper we have considered a special case of the dynamically evolving model of [3], [4], [5], where the score of the sequence is 1. This special case is one where at any point within the minimal set, only one individual has an incentive to change their number of links and so potentially simplifies analysis. It also means that we consider sequences as close as possible to classical graphical sequences [11],[12],[13], [14], [19],[26]. We analysed the set of score 1 sequences, and found a general method of classifying the vertices of the sequences into one of four types, which in turn provides information about the graphs that comprise the minimal set of the sequence $K(min)$. In particular using our Theorem 3.3 we can specify the set membership of all vertices for any score 1 sequence.

We then considered a special case of the model. We were able to identify stable sets of strategies which led to cycles around the choice graph. For this example the minimal set contained 45 graphs. There were stable cycles of varying lengths, from 6 elements (as in Figure 1) to 30 (see Figure 5); in Figure 6 we saw a cycle of length 24. In Figure 7 we saw a case where potential cycles had elements in common, and that one or other cycle would be fixed by chance as the process evolves. Our simulations showed that the shorter to medium length cycles were more likely to evolve in practice, with a third of simulations leading to a cycle of length 12.

For even such a relatively simple example, we thus saw that there were many stable solutions, and which was reached was due to chance and initial conditions. It would seem likely that this feature is present in real situations too; for example with a given group of people, many friendship structures are possible, depending upon the order in which they met. Of course for real situations, there are heterogeneities between individuals, and the simple preference for a given number of links in our model would be replaced by more detailed preferences.

Whilst significant progress can be made on the general classification, as described above, the analysis of particular cases, even with small sequences, is still very hard, as this example and Figures 5 and 6 illustrate. The most important determinants of the complexity of the game appear to be the size of the minimal set and the score. As shown in [5] the game leads to a Markov process on the minimal set, so that the transition matrix is $|K(min)| \times |K(min)|$. A larger score, as well as often leading to a larger minimal set, also makes more transitions have non-zero probability from any given state, as many individuals may wish to make a move from any given position. From [5] we saw that any number of elements of the set of individuals that are always neutral S_N can effectively be removed from a sequence without changing the minimal set. Thus the number of elements that are not of this type is also an important indicator of the potential complexity of analysing the sequence.

How can progress be made? There are a number of ways that we can adapt our game to potentially make the analysis more amenable (although in some cases this might make analysis even more complex). We will consider finding the right "simple" classes of strategy that keep the important features of the model but simplify analysis; for example, maybe individuals cannot evaluate the stationary distribution, and have a fixed preference order (e.g. always connecting to individual 2 first if possible). Alternative models, for example where both individuals must agree to form and/or break a link, are likely to be easier to solve, but the long-term dynamic nature of the process might be lost. We will also consider cases where some links are banned or enforced. The case where links are banned corresponds to our process occurring on a fixed underlying graph of potential links, with our current work taking place over the complete graph (for any enforced link, analysis will be identical to a banned link with appropriate adjustment of the targets of the vertices involved). This can potentially make the process simpler or more complex, depending upon the nature of the restrictions, but we will of course start to consider the former cases, for example with underlying graphs such as the circle or star.

The models that we considered here are of a specific type, but they relate to a number of models by other authors. They can be thought of as a special class of the economic models of Jackson and colleagues [15, 16, 17, 18] where individuals form connections to others and their payoff is determined by the network (see in particular [2, 9] where links are formed unilaterally, as in our model). The work of Southwell and Cannings [29, 30, 31] considers a population where individuals are born and (in some cases) die, but where the strategic aspect of link formation is absent. Thus the relationship between our model and these two classes of model could be more fully explored. More generally, our models are examples of stochastic games [27], albeit quite complex ones, and there is thus the potential of reformulating our models correspondingly to see what existing stochastic game theory can tell us.

Finally, there are a variety of potential applications, as discussed in Section 1. For example there is the area of biological markets developed by Noe and colleagues [20, 21]. Alternatively, models of animal group formation and maintenance, such as in chimpanzees [8, 24], can be considered. In such models heterogeneity plays an important role, with some links being more valuable than others, and this would thus be an important feature to include in the model.

Acknowledgments. This work was supported by funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 690817.

REFERENCES

[1] B. Allen and M. A. Nowak, Games on graphs, *EMS Surveys in Mathematical Sciences,* **1** (2014), 113–151.

[2] V. Bala and S. Goyal, A model of non-cooperative network formation, *Econometrica,* **68** (2000), 1181-1230.

[3] M. Broom and C. Cannings, A dynamic network population model with strategic link formation governed by individual preferences, *J. Theor. Biol.,* **335** (2013), 160–168.

[4] M. Broom and C. Cannings, Graphic Deviation, *Discrete Mathematics,* **338** (2015), 701–711.

[5] M. Broom and C. Cannings, Game theoretical modelling of a dynamically evolving network I: General target sequences, *Journal of Dynamic Games* **4** (2017), 285-318.

[6] C. Capitanio, Sociability and response to video playback in adult male rhesus monkeys (macac mulatta), *Primates*, **43** (2002), 169–177.

[7] R. C. Connor, M. R. Helthaus and L. M. Barre, Superalliances of bottlenose dolphins, *Nature*, **397** (1999), 571–572.

[8] P. I. M. Dunbar, Neocortex size as a constraint on group size in primates, *J.Human Evoluion,* **22** (1992), 468–493.

[9] B. Dutta and M. O. Jackson, The Stability and Effeciency of Directed Communication Networks, *Rev. Econ. Design,* **5** (2015), 251-272.

[10] C. S. Elton, *Animal Ecology,* Sidgwick & Jackson, London, 1927.

[11] P. Erdos and P.; T. Gallai, Grafok eloirt fokszamu pontokkal, *Matematikai Lapo,* **11** (1960), 264274

[12] S. L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph, *Siam J.Appl.Math.,* **10** (1960), 496–506.

[13] W. Hässelbarth, Die Verzweightheit von Graphen, *Comm. in Math. and Computer Chem. (MATCH)*, **16** (1984), 3–17.

[14] V. Havel, A remark on the existence of finite graphs, (Czech)*Časopis Pěst. Mat.,* **80** (1955), 477–480.

[15] M. O. Jackson, The stability and efficiency of economic and social networks, *In: Networks and Groups,* (Ed. B.Dutta, M.O.Jackson) Springer, 2003.

[16] M. O. Jackson, Social and Economic Networks, *Princeton Univ. Press,* Princeton NJ.

[17] M. O. Jackson, An overview of Social Networks and Economic Applications, *In: Handbook of Social Economics* (Ed. J.Benhabib, A.Bisin, M.O.Jackson,) 512-585 Elsevier, 2011.

[18] M. O. Jackson and A. Wolinsky, A strategic model of social and economic networks, *J.Econ. Theory,* **71(1)** (1996), 44-74.

[19] R. Merris and T. Roby, The lattice of threshold graphs, *J. Inequal. Pure and Appl. Math.*, 6 (2005), Article 2, 21pp.

[20] R. Noë, Biological markets: Partner choice as the driving force behind the evolution of cooperation. In: *Economics in Nature. Social Dilemmas, Mate Choice and Biological Markets,* (Ed. by Noë, R., van Hooff, J. A. R. A. M. and Hammerstein, P.), (2001), 93–118. Cambridge: Cambridge University Press.

[21] R. Noë and P. Hammerstein, Biological markets: Supply and demand determine the effect of partner choice in cooperation, Mutualism and Mating *Behav.Ecol.Sociobio,* **35** (1994), 1–11.

[22] J. M. Pacheco, A. Traulsen and M. A. Nowak, Active linking in evolutionary games, *J.Theor. Biol.,* **243** (2006), 437–443.

[23] J. M. Pacheco, A. Traulsen and M. A. Nowak, Coevolution of strategy and structure in complex networks with dynamical linking, *Phys. Rev. Lett.,* **97** (2006), 258103.

[24] J. Pepper, J. Mitani and D. Watts, General gregariousness and specific social preferences among wild chimpanzees, *Int. J. Primatol.,* **20** (1999), 613–632.

[25] M. Perc and A. Szolnoki, Coevolutionarygames - a mini review, *BioSystems,* **99** (2010), 109–125.

[26] E. Ruch and I. Gutman, The branching extent of graphs, *J. Combin. Inform. Systems Sci.,* **4** (1979), 285–295.

[27] L.S. Shapley, Stochastic Games, *PNAS,* **39** (1979), 1095–1100.

[28] A. M. Sibbald and R. J. Hooper, Sociability and willingness of individual sheep to move away from their companions in order to graze, *Applied Animal Behaviour*, **86** (2004), 51–62.

[29] R. Southwell and C. Cannings, Some models of reproducing graphs: I pure reproduction, *Applied Mathematics,* **1** (2010), 137–145.

[30] R. Southwell and C. Cannings, Some models of reproducing graphs: II age capped reproduction, *Applied Mathematics,* **1** (2010), 251–259.

[31] R. Southwell and C. Cannings, Some models of reproducing graphs: III game based reproduction, *Applied Mathematics,* **1** (2010), 335–343.

[32] B. Voelkl and C. Kasper, Social structure of primate interaction networks facilitates the emergence of cooperation, *Biology Letters,* **5** (2009), 462–464.

[33] B. Voelkl and R. Noë, The influence of social structure on the propagation of social information in artificial primate groups: A graph-based simulation approach, *J. Theor. Biol.,* **252** (2008), 77–86.

[34] J. Wiszniewski, C. Brown and L. M. Moller, Complex patterns of male alliance formation in dolphin social networks, *Journal of Mammalogy*, **93** (2012), 239–250.

Appendix A: Proof of Theorem 3.3. 1) We start by considering the different pairs of changes labelled A-D, a-d as discussed prior to the statement of Theorem 3.3. It is possible that the Durfee number of the new sequence, following the two changes, will have changed, either increasing by 1 or decreasing by 1 (it is easy to see that it cannot change by more than 1). In every case only one of the changes is possible, if any, and it is clear which one, as we see below; we shall refer to no change being case (i) and a change being case (ii). Thus potentially we have all combinations of A-D, a-d and (i), (ii) giving 32 in total.

We note that for any subsequences containing identical values, we shall assume additions are at the top of the sequence (i.e. the element with the lowest index) and subtractions from the bottom, to avoid unnecessary reordering of the elements. It is possible that the two changes are of the same element in the same direction, which might then lead to a potential reordering. We note that if this is the case, then the higher element must only be one more than the lower one (in the case of a double addition to the original lower one or a double subtraction from the original higher one), but that this is equivalent to just adding 1 to both elements.

It is easy to see that Aa(ii), Dd(ii), Ad(ii) and aD(ii) are not possible. In addition some combinations are equivalent, for example Ab(i) leads to the same possibilities as aB(i), the only difference being the order of the changes. We refer to them differently, since we will need to make comparisions between sequences after the first change and after the second, making the order important. For now though, we will treat them as equivalent. This then leaves us 17 distinct cases.

2) If we take 1 from two places within the first set of vertices $1, \ldots, \lambda$, add 1 to two places within the second set of vertices $\lambda+1, \ldots, n$ or take 1 from one of the first set and add 1 to one of the second set, then none of the terms f_k can be increased and so the score cannot be increased if the Durfee number is unchanged or reduced. Suppose that such a change leads to an increase in the Durfee number; in this case, for the original sequence we would have to have one of the changes being an addition in the second set with $t_{\lambda+1} = \lambda$ and $t_\lambda > \lambda$. This means that the new $(t_{\lambda+1}+1-v_{\lambda+1})$ can add at most 1 to f_λ, which was previously less than or equal to 0. The second change, however, must take 1 from one of the t_is or add 1 to one of the v_is in the summation for f_λ, which means that the new $f_{\lambda+1}$ is no greater than the old f_λ. Thus the score cannot be increased. The same is true for any combination of one increase and one decrease where the decrease happens at the lower index (by the convention above, if both targets are the same, we denote the addition as occurring

at the lower index). Thus $Bb(i), Bb(ii), Bc(i) = bC(i), Bc(ii) = bC(ii), Cc(i)$ and $Cc(ii)$ all yield graphic sequences.

3) This leaves 11 remaining cases, which we shall consider in turn. In each one we shall consider vertices $j < l$ as those where the changes happen, with $+$ indicating an addition of 1 and $-$ indicating a subtraction of 1. Following the above arugment, if there is an addition and a subtraction, we only need to consider the addition occurring at the lower index.

Aa(i): assuming +s occur in positions j and $l \geq j$, f_k increases by 1 for $j \leq k < l$ and by 2 for $l \leq k \leq \lambda$. Thus the sequence is graphic if $f_k < 0 \quad k : j \leq k < l$ and $f_k < -1 \quad k : l \leq k \leq \lambda$.

Ab(i)=aB(i): assume $+$ occurs in position j and $-$ in position $l > j$. S_k increases by 1 for $j \leq k < l$; v_λ will decrease by 1 if $t_l = \lambda$ (we would also have $f_k < -1$ if $k = t_l = \lambda$, but this cannot occur, as this addition and subtraction leads to a reduction of the Durfee number, and so case (ii)). Thus the sequence is graphic if $f_k < 0 \quad k : j \leq k < l$.

Ab(ii)=aB(ii): for the change of Durfee number to occur, we must have $+$ in a position with a lower index than $-$. A change implies that $t_l = l = \lambda$, so that the new Durfee number is $\lambda - 1$. We then have f_k increasing by 1 from j to $\lambda - 1$. Thus the sequence is graphic if $f_k < 0 \quad k : j \leq k < l = \lambda$.

Ac(i)=aC(i): f_k increases by 1 from j and decreases by 1 from t_l. Thus the sequence is graphic if $t_l \leq j$ and $f_k < 0 \quad k : j \leq k < t_l$.

Ac(ii)=aC(ii): A change of Durfee number implies that $t_l = \lambda$ and $l = \lambda + 1$, so that the new Durfee number is $\lambda + 1$. f_k increases by 1 from j to $t_l - 1 = \lambda - 1$ and f_λ is unchanged. We then add $t_{\lambda+1} + 1 - v_{\lambda+1} = \lambda + 1 + 1 - (\lambda + 1) = 1$ to f_λ. This means that we need $f_\lambda < 0$ for the new sequence to be graphic. Thus the sequence is graphic if $f_k < 0 \quad k : j \leq k \leq t_l = \lambda$.

Ad(i)=aD(i): f_k increases by 1 for $\min(j, t_l) \leq k < \max(j, t_l)$, and by 2 for $\max(j, t_l) \leq k \leq \lambda$. Thus the sequence is graphic if $f_k < 0 \quad k : \min(j, t_l) \leq k < \max(j, t_l)$ and $f_k < -1 \quad \max(j, t_l) \leq k \leq \lambda$.

Bd(i)=bD(i): f_k decreases by 1 from j and increases by 1 from t_l. Thus the sequence is graphic if $j \leq t_l$ and $f_k < 0 \quad k : t_l \leq k < j$.

Bd(ii)=bD(ii): A change of Durfee number implies that $t_j = j = \lambda$, so that the new Durfee number is $\lambda - 1$. f_k increases by 1 from t_l to $j - 1$. Thus the sequence is graphic if $s_k < 0 \quad k : t_l \leq k < j = \lambda$.

Cd(i)=cD(i): similarly to the above, we assume that $+$ occurs in position j and $-$ in position $l > j$. f_k increases by 1 for $t_l \leq k < t_j$ (we would also have $f_k < -1$ if $k = t_j = \lambda$, but this would lead to a change in Durfee number, and so case (ii)). Thus the sequence is graphic if $f_k < 0 \quad k : t_l \leq k < t_j$.

Cd(ii)=cD(ii): for the change of Durfee number to occur, we must have $+$ in a position with a lower index than $-$. A change implies that $t_j = \lambda$ and $j = \lambda + 1$, so that the new Durfee number is $\lambda + 1$. f_k increases by 1 from t_l to $t_j - 1 = \lambda - 1$ and f_λ is unchanged. We then add $t_{\lambda+1} + 1 - v_{\lambda+1} = \lambda + 1 + 1 - (\lambda + 1) = 1$ to f_λ. This means that we need $f_\lambda < 0$ for the new sequence to be graphic. Thus the sequence is graphic if $f_k < 0 \quad k : t_l \leq k \leq t_j = \lambda$.

Dd(i): f_k increases by 1 from t_l to $t_j - 1$, and by two from t_j to λ. Thus the sequence is graphic if $f_k < 0 \quad k : t_l \leq k < t_j$ and $f_k < -1 \quad k : t_j \leq k \leq \lambda$.

4) Now we shall consider the possible values of f_k for $k = 1, \ldots, \lambda$ (see equation 1) for our starting sequence, which is graphic. In the working that follows the ordering of the changes matters, e.g. Ad(i) is different from aD(i), since we will

consider all subsequent changes from the score 1 sequence obtained from the first change. Clearly for a graphic sequence $f_k \leq 0$ for $k = 1, \ldots, \lambda$.

a) Suppose that $f_k < -1$ for $k = 1, \ldots, \lambda$. Following the working above, it is clear that any two changes in the sequence lead to a graphic sequence. Thus for a score 1 sequence that is obtained from a graphic sequence of this type by any single change, any subsequent change (an increase or decrease at any vertex) leads to a graphic sequence. Thus such a score 1 sequence has all elements in S_A (except for any with targets 0 or $n-1$).

b) Now suppose that $f_k \leq -1$ for $k = 1, \ldots, \lambda$, with at least one satisfying this with equality. Let K be the largest such index. We note that this means that K is the largest index for which $f_k \geq -1$ since there is no f_k which takes value greater than -1 (this is important for consistency with c) below).
Following the above working, there are only four cases where some pair of changes lead to a sequence that is non-graphic. These are Aa(i), Ad(i), aD(i) and Dd(i).

For the four cases above, we have the following situations when the new sequence is not graphic:
Aa(i): assuming that the original change (increase) is at vertex $i \leq \lambda$, if $K \geq i$ then the new sequence is not graphic if and only if $m \leq K$, where m is the index of the second change (increase).
Ad(i): assuming that the original change (increase) is at vertex $i \leq \lambda$, if $K \geq i$ then the new sequence is not graphic if and only if $t_m \leq K$, where m is the index of the second change (increase).
aD(i): assuming that the original change (decrease) is at vertex $i > \lambda$, if $K \geq t_i$ then the new sequence is not graphic if and only if $m \leq K$, where m is the index of the second change (increase).
Dd(i): assuming that the original change (decrease) is at vertex $i > \lambda$, if $K \geq t_i$ then the new sequence is not graphic if and only if $t_m \leq K$, where m is the index of the second change (increase).

Thus we have:
I. A score 1 sequence that is obtained from such a graphic sequence by an A move, has all elements in S_A (except for any with targets 0 or $n-1$) if $K < i$. Otherwise, it has all elements with index $1, \ldots, K$ in S_J, all elements with target less than or equal to K in S_B, and the remaining elements in S_A (except for any with targets 0 or $n-1$).
II. A score 1 sequence that is obtained from such a graphic sequence by a D move, has all elements in S_A (except for any with targets 0 or $n-1$) if $K < t_i$. Otherwise, it has all elements with index $1, \ldots, K$ in S_J, all elements with target less than or equal to K in S_B, and the remaining elements in S_A (except for any with targets 0 or $n-1$).
III. A score 1 sequence that is obtained from such a graphic sequence by a B or C move, has all elements in S_A (except for any with targets 0 or $n-1$).

c) Now suppose that $f_k = 0$ for at least some values of k.

The only cases that lead to graphic sequences for all pairs of changes are $Bb(i)$, $Bb(ii), Bc(i) = bC(i), Bc(ii) = bC(ii), Cc(i)$ and $Cc(ii)$. The four cases $Aa(i), Ad(i)$, $aD(i)$ and $Dd(i)$ are a little different, and we deal with these at the end. We consider each of the other cases below: we have the following situations when the new sequence is not graphic:
Ab(i): assuming that the original change (increase) is at vertex $i < \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $i < m$ and there

is a k s.t. $f_k = 0$ and $i \le k < m$.
aB(i): assuming that the original change (decrease) is at vertex $i < \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $m < i$ and there is a k s.t. $f_k = 0$ and $m \le k < i$.
Ab(ii): assuming that the original change (increase) is at vertex $i < \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $m = \lambda$ and there is a k s.t. $f_k = 0$ and $i \le k < m$.
aB(ii): assuming that the original change (decrease) is at vertex $i = \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $m < i$ and there is a k s.t. $f_k = 0$ and $m \le k < i$.
Ac(i): assuming that the original change (increase) is at vertex $i < \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $i < t_m$ and there is a k s.t. $f_k = 0$ and $i \le k < t_m$.
aC(i): assuming that the original change (increase) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $m < t_i$ and there is a k s.t. $f_k = 0$ and $m \le k < t_i$.
Ac(ii): assuming that the original change (increase) is at vertex $i \le \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $i \le t_m = \lambda$ and there is a k s.t. $f_k = 0$ and $i \le k \le t_m$.
aC(ii): assuming that the original change (increase) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $m \le t_i = \lambda$ and there is a k s.t. $f_k = 0$ and $m \le k \le t_i$.
Bd(i): assuming that the original change (decrease) is at vertex $i < \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $t_m < i$ and there is a k s.t. $f_k = 0$ and $t_m \le k < i$.
bD(i): assuming that the original change (decrease) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $t_i < m$ and there is a k s.t. $f_k = 0$ and $t_i \le k < m$.
Bd(ii): assuming that the original change (decrease) is at vertex $i = \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $t_m < i$ and there is a k s.t. $f_k = 0$ and $t_m \le k < i$.
bD(ii): assuming that the original change (decrease) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $m = \lambda$ and there is a k s.t. $f_k = 0$ and $t_i \le k < m$.
Cd(i): assuming that the original change (increase) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $t_m < t_i$ and there is a k s.t. $f_k = 0$ and $t_m \le k < t_i$.
cD(i): assuming that the original change (decrease) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $t_i < t_m$ and there is a k s.t. $f_k = 0$ and $t_i \le k < t_m$.
Cd(ii): assuming that the original change (increase) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $t_m \le t_i$ and there is a k s.t. $f_k = 0$ and $t_m \le k \le t_i = \lambda$.
cD(ii): assuming that the original change (decrease) is at vertex $i > \lambda$, then a second change at vertex m leads to a non-graphic sequence if and only if $t_i \le t_m$ and there is a k s.t. $f_k = 0$ and $t_i \le k \le t_m = \lambda$.

We note in the above that the conditions for the cases Ac(ii)/aC(ii)/Cd(ii)/cD(ii) all involve $k \le \lambda$ rather than $k < \lambda$ for Ab(ii)/aB(ii)/(Bd(ii)/bD(ii) and k less than some term in all other cases. In these four cases the change increases the value of the Durfee number from λ to $\lambda+1$, with an increased element being equal to the old Durfee number before the change and the new Durfee number after the change.

Now suppose that K is the largest index such that $f_k \geq -1$ for $k = 1, \dots, \lambda$, and K' is the largest index such that $f_k = 0$ for $k = 1, \dots, \lambda$.

Aa(i): assuming that the original change (increase) is at vertex $i \leq \lambda$, if $K \geq \max(i, m)$ or $K' \geq \min(i, m)$, where m is the index of the second change (increase), then the new sequence is not graphic.

Ad(i): assuming that the original change (increase) is at vertex $i \leq \lambda$, if $K \geq \max(i, t_m)$ or $K' \geq \min(i, t_m)$, where m is the index of the second change (increase), then the new sequence is not graphic.

aD(i): assuming that the original change (increase) is at vertex $i > \lambda$, if $K \geq \max(t_i, m)$ or $K' \geq \min(t_i, m)$, where m is the index of the second change (increase), then the new sequence is not graphic.

Dd(i): assuming that the original change (increase) is at vertex $i > \lambda$, if $K \geq \max(t_i, t_m)$ or $K' \geq \min(t_i, t_m)$, where m is the index of the second change (increase), then the new sequence is not graphic.

Thus we have:

I. A score 1 sequence that is obtained from such a graphic sequence by an A move, has the following.

Element m is in S_A (except for any with targets 0 or $n-1$) if:

$m \leq \lambda$, there is no $f_k = 0$ s.t. $i \leq k < m$, and neither of $K \geq \max(i, m)$, $K' \geq \min(i, m)$ hold.

$m > \lambda$, there is no $f_k = 0$ s.t. $i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and not both of $K \geq \max(i, t_m)$, $K' \geq \min(i, t_m)$ hold.

Element m is in S_N if:

$m \leq \lambda$, there is an $f_k = 0$ s.t. $i \leq k < m$, and at least one of $K \geq \max(i, m)$, $K' \geq \min(i, m)$ hold.

$m > \lambda$, there is an $f_k = 0$ s.t. $i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and both of $K \geq \max(i, t_m)$, $K' \geq \min(i, t_m)$ hold.

Element m is in S_J if:

$m \leq \lambda$, there is no $f_k = 0$ s.t. $i \leq k < m$, and at least one of $K \geq \max(i, m)$, $K' \geq \min(i, m)$ hold.

$m > \lambda$, there is an $f_k = 0$ s.t. $i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and not both of $K \geq \max(i, t_m)$, $K' \geq \min(i, t_m)$ hold.

Element m is in S_B if:

$m \leq \lambda$, there is an $f_k = 0$ s.t. $i \leq k < m$, and neither of $K \geq \max(i, m)$, $K' \geq \min(i, m)$ hold.

$m > \lambda$, there is no $f_k = 0$ s.t. $i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and both of $K \geq \max(i, t_m)$, $K' \geq \min(i, t_m)$ hold.

II. A score 1 sequence that is obtained from such a graphic sequence by a B move, has the following.

Element m is in S_A (except for any with targets 0 or $n-1$) if:

$m \leq \lambda$ and there is no $f_k = 0$ s.t. $m \leq k < i$.

$m > \lambda$ and there is no $f_k = 0$ s.t. $t_m \leq k < i$.

Element m is never in S_N.

Element m is in S_J if:

$m \leq \lambda$ and there is an $f_k = 0$ s.t. $m \leq k < i$.

For $m > \lambda$, m cannot be in S_J.

Element m is in S_B if:

$m > \lambda$ and there is an $f_k = 0$ s.t. $t_m \leq k < i$.
For $m \leq \lambda$, m cannot be in S_B.
III. A score 1 sequence that is obtained from such a graphic sequence by a C move, has the following.
Element m is in S_A (except for any with targets 0 or $n-1$) if:
$m \leq \lambda$ and there is no $f_k = 0$ s.t. $m \leq k < t_i$ ($\leq t_i$ if $t_i = \lambda$ and there is a Durfee change).
$m > \lambda$ and there is no $f_k = 0$ s.t. $t_m \leq k < t_i$ ($\leq t_i$ if $t_i = \lambda$ and there is a Durfee change).
Element m is never in S_N.
Element m is in S_J if:
$m \leq \lambda$ and there is an $f_k = 0$ s.t. $m \leq k < t_i$ ($\leq t_i$ if $t_i = \lambda$ and there is a Durfee change).
For $m > \lambda$, m cannot be in S_J.
Element m is in S_B if:
$m > \lambda$ and there is an $f_k = 0$ s.t. $t_m \leq k < t_i$ ($\leq t_i$ if $t_i = \lambda$ and there is a Durfee change).
For $m \leq \lambda$, m cannot be in S_B.
IV. A score 1 sequence that is obtained from such a graphic sequence by a D move, has the following.
Element m is in S_A (except for any with targets 0 or $n-1$) if:
$m \leq \lambda$, there is no $f_k = 0$ s.t. $t_i \leq k < m$, and not both of $K \geq \max(t_i, m)$, $K' \geq \min(t_i, m)$ hold.
$m > \lambda$, there is no $f_k = 0$ s.t. $t_i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and not both of $K \geq \max(t_i, t_m)$, $K' \geq \min(t_i, t_m)$ hold.
Element m is in S_N if:
$m \leq \lambda$, there is an $f_k = 0$ s.t. $t_i \leq k < m$, and both of $K \geq \max(t_i, m)$, $K' \geq \min(t_i, m)$ hold.
$m > \lambda$, there is an $f_k = 0$ s.t. $t_i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and both of $K \geq \max(t_i, t_m)$, $K' \geq \min(t_i, t_m)$ hold.
Element m is in S_J if:
$m \leq \lambda$, there is no $f_k = 0$ s.t. $t_i \leq k < m$, and both of $K \geq \max(t_i, m)$, $K' \geq \min(t_i, m)$ hold.
$m > \lambda$, there is an $f_k = 0$ s.t. $t_i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and not both of $K \geq \max(t_i, t_m)$, $K' \geq \min(t_i, t_m)$ hold.
Element m is in S_B if:
$m \leq \lambda$, there is an $f_k = 0$ s.t. $t_i \leq k < m$, and not both of $K \geq \max(t_i, m)$, $K' \geq \min(t_i, m)$ hold.
$m > \lambda$, there is no $f_k = 0$ s.t. $t_i \leq k < t_m$ ($\leq t_m$ if $t_m = \lambda$ and there is a Durfee change), and both of $K \geq \max(t_i, t_m)$, $K' \geq \min(t_i, t_m)$ hold.

Combining 4) parts a), b) I-III and c) I-IV leads to Theorem 3.3 as stated. □

E-mail address: Mark.Broom@city.ac.uk
E-mail address: c.cannings@sheffield.ac.uk