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Obtaining Wave Equations from the Vlasov Equation for Plasmas in Inhomogeneous Magnetic Fields

by

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Submitted to the Faculty of Science in October 1988 for the Degree of Doctor of Philosophy

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Preface

In this thesis, techniques are developed for the self-consistent derivation of systems of coupled ordinary differential equations which describe the propagation of electromagnetic perturbations through inhomogeneously magnetized plasmas.

The Vlasov equation is used to model the reaction of the distribution of the plasma particles to a high frequency electromagnetic perturbation, its use being justified, for the timescales under consideration, from kinetic theory. Then the current carried in the plasma is obtained from the perturbed distribution function, and inserted in Maxwell's equations to give the wave equation.

The wave equation is first derived by Fourier transform techniques, for the case of a homogeneously magnetized plasma, and then derived as a set of differential equations for the case of an inhomogeneous magnetic field. The consistently derived differential equations are applied to a simple example of an inhomogeneously magnetised plasma, and then the equations and their solutions are compared with those obtained from the 'reverse Fourier transform' of the equations derived for the homogeneous case - a technique often used in the literature. While the comparison of results demonstrates the need for consistently derived equations, the derivation of the equations also reveals the limits of their validity. It is shown that while similar equations have been obtained before, they have been applied not only in their region of validity, but also well outside this region.

The technique used to obtain the consistent equations is successively generalised to describe short wavelength perturbations, anisotropic and inhomogeneous equilibrium particle distributions, and perturbations outside the plane formed by the magnetic field and its gradient.

Chapter 1

Plasma Theory

1.1. The Single Particle Distribution Function.

To describe in complete detail all the properties of a plasma would require a description of the motion of each individual particle in the plasma. Since there are typically $>10^{18}$ particles in plasma experiments, each one interacting with the others, this is not a practical approach to calculating plasma properties. Obviously some method for simplifying this set of equations is vital. One method that produces dramatic simplification is to ignore particle correlations and deal purely with the single particle distribution function f . This is the approach that leads to the Vlasov equation. Ignoring particle correlations is equivalent to saying that the probability of a particle being at position x at time t is unaffected by the presence of another particle in the immediate vicinity. In the statistical mechanics of neutral gases this approximation is often made for low density gases and is equivalent to ignoring the finite size of the gas molecules but, in a plasma, in addition to the finite size of the particles, there is also the long range of the electrostatic field interactions to be considered. However, the longer range interactions can be split into the 'self-consistent field' obtained from the single particle distribution function and a 'collision term' due to particle correlations. This gives the Boltzmann equation.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \underline{a} \cdot \frac{\partial f}{\partial \underline{v}} = \left(\frac{df}{dt}\right)_c \qquad (1.1.1)$$

To proceed any further a model of the collision term $\left(\frac{df}{dt}\right)_c$ is required. In general, the terms on the r.h.s. of equation (1.1.1) would contain, in addition to the effects of particle correlations, the effects of source and sink terms. Such source and sink terms themselves often arise from collisional effects, for example a fusion reaction would appear as a source term in the fusion product distributions, and as a sink term in the fuel distributions. However, before expending any effort on modelling the effects of fusion reactions on the distribution functions, it should be noted that the central motivation for this research is the use of R.F. waves as an auxiliary method of plasma heating in present day experiments in an attempt to achieve thermonuclear fusion parameters. In these experiments, the cross-section for nuclear fusion is far smaller than that for deflection. Therefore, in estimates of the size of the collision term in

(1.1.1), fusion reactions can be ignored.

The first step is to assess the effects of the approximation that the r.h.s. collisional terms are insignificant compared to the terms on the l.h.s. This is done by using a physical model of the collisions that take place in the plasma to estimate the collision term so that its size and dependence on the parameters of the plasma can be found. The dominant collisional effect on the plasma distribution is the Coulomb collision (fig. 1.1); the differential scattering cross-section for which is obtained in standard undergraduate textbooks (Goldstein,1980)

$$\sigma\left(|\underline{v}_1 - \underline{v}_2|, \theta\right) = \frac{b_0^2}{4} \operatorname{cosec}^4\left(\frac{\theta}{2}\right) \tag{1.1.2}$$

where

$$b_0 = \frac{e_1 e_2 (m_1 + m_2)}{4\pi \varepsilon_0 m_1 m_2 |\underline{v}_1 - \underline{v}_2|} \tag{1.1.3}$$

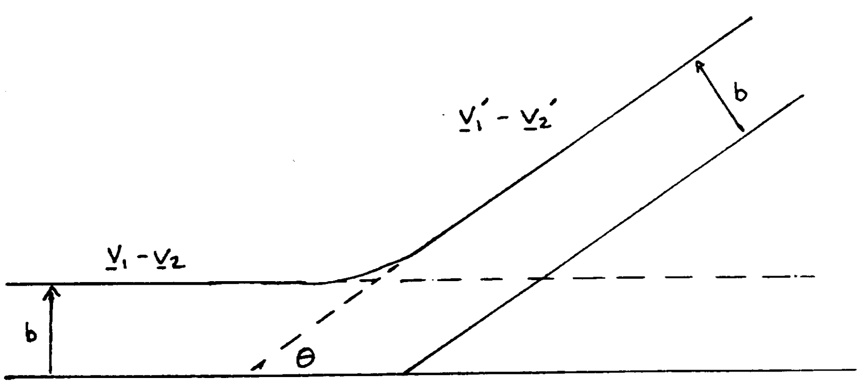


Fig. 1.1 Path of particle 1 in the centre of mass frame

This formula (1.1.2) however, suffers from the fact that it diverges in the $\theta \to 0$ limit. In atomic scattering the screening effects of the electrons around the nucleus give an upper bound on the impact parameter b, and therefore effectively a lower bound on θ. Fortunately a similar screening effect, due to the mobility of charged particles, exists in plasmas.

1.2. The Debye Length

The particles in moderate and low density plasmas in thermal equilibrium follow Boltzmann statistics

$$f_0(E) = \frac{n_0 m^{3/2}}{(2\pi kT)^{3/2}} e^{-E/kT} \tag{1.2.1}$$

therefore the equilibrium distributions would have a Maxwellian velocity dependence.

$$f_0\left(\tfrac{1}{2}mv^2\right) = \frac{n_0}{\pi^{3/2} v_T^3} e^{-v^2/v_T^2} \tag{1.2.2}$$

Given only an electrostatic potential field Φ, the spatial dependence of the equilibria would be solely due to the Φ dependence

$$\frac{\partial f_0}{\partial x} = \frac{\partial f_0}{\partial \Phi} \frac{d\Phi}{dx} \tag{1.2.3}$$

e.g. for a plasma consisting of ions and electrons

$$\nabla^2\Phi = \frac{-\rho}{\varepsilon_o} = \frac{n_o e}{\varepsilon_o}\left(\exp\left(\frac{e\Phi}{kT}\right) - \exp\left(\frac{-e\Phi}{kT}\right)\right) \tag{1.2.4}$$

If the potential Φ is due to the Coulomb field of a charged particle in the plasma, then the limit

$$\frac{e\Phi}{kT} << 1 \tag{1.2.5}$$

is appropriate, i.e. the plasma particles have greater kinetic than potential energy. (The opposite limit would be more descriptive of an ionic bonded crystal.) Given this approximation, the solution for Φ due to a charged particle in a plasma becomes

$$\Phi = \frac{e}{4\pi\varepsilon_o r}\exp\left(\frac{-r}{\lambda_D}\right) \tag{1.2.6}$$

where the Debye length

$$\lambda_D = \left(\frac{\varepsilon_o kT}{2ne^2}\right)^{1/2} \tag{1.2.7}$$

is the naturally arising screening length sought.

Using λ_D as a cutoff or upper bound on the impact parameter b, a value for θ_{min} is obtained.

$$\theta_{min} = \frac{2b_o}{\lambda_D} \tag{1.2.8}$$

Therefore the average plasma particle is simultaneously interacting with $n\lambda_D^3$ particles, and so if $n\lambda_D^3 >> 1$ it is reasonable to suspect that the dominant collisional effects could be due to multiple small angle scattering rather than individual large angle scattering.

The average time for a $\pi/2$ deflection due to an individual collision for a particle travelling with speed v is

$$t_1 = \frac{1}{n v \sigma_T(\pi/2)} \tag{1.2.9}$$

To compare this with the integrated effects of multiple small angle scattering the mean square deflection per second is estimated and then the time for a deflection of unity is calculated, a deflection of unity corresponding roughly to a total scattering of $\pi/2$.

1.3. Electron Scattering from Ions

The case of electron scattering from ions is particularly simple because of the great disparity of mass between ions and electrons. This difference in mass allows the ions to be regarded as stationary and in the centre of mass frame. The integrated sum of deflections through angles less than $\pi/2$ is

$$8\pi b_0^2 n v \int_{\theta_{min}}^{\pi/2} \frac{d\theta}{\theta} \tag{1.3.1}$$

and therefore the time for the electron to suffer a deflection of unity is

$$t_m = \frac{1}{8\pi b_0^2 n v \ln\left(\frac{\pi \lambda_0}{4 b_0}\right)} \tag{1.3.2}$$

As was suspected, the ratio of the 2 different collision times

$$\frac{t_1}{t_m} \approx \frac{8\pi b_o^2 n v \ \ln(\lambda_D/b_o)}{n v \pi b_o^2} = 8 \ln\left(\frac{\lambda_D}{b_o}\right) \qquad (1.3.3)$$

is large if the number of particles within the screening distance is large, since from (1.2.7) we have

$$n\lambda_D^3 = \left(\frac{\varepsilon_o kT}{2e^2}\right)\lambda_D \qquad (1.3.4)$$

and using (1.1.3) with $\frac{m}{2}|\underline{v}_1 - \underline{v}_2|^2$ replaced by the mean thermal value kT gives the result.

$$\frac{\lambda_D}{b_o} = \frac{8\pi\varepsilon_o kT}{e^2}\lambda_D = 8\pi n\lambda_D^3 \qquad (1.3.5)$$

If this limit is applicable then the collision term in equation (1.1.1) is best represented by a Fokker-Planck collision term.

1.4. Fokker-Planck

The Fokker-Planck collision term models the effects of large numbers of small collisions, by using the small size of velocity changes to justify a Taylor expansion of the collision operator. For this analysis it is assumed that there exists a function $\Psi(\underline{v},\Delta\underline{v})$ which is the probability of a collision causing a change in velocity of $\Delta\underline{v}$ to a particle that has a velocity $\underline{v}$. Then the distribution function at time t+Δt can be written in terms of the distribution at time t and the collision function.

$$f(\underline{r},\underline{v},t) = \int f(\underline{r},\underline{v}-\Delta\underline{v},t-\Delta t)\,\Psi(\underline{v},\Delta\underline{v})\; d\Delta\underline{v} \tag{1.4.1}$$

Performing the aforementioned Taylor expansion gives

$$f(\underline{r},\underline{v},t) = f(\underline{r},\underline{v},t-\Delta t) - \int \Delta\underline{v}\cdot\frac{\partial}{\partial\underline{v}}\left[f\Psi\right] + \tfrac{1}{2}\,\Delta\underline{v}\Delta\underline{v} : \frac{\partial^2}{\partial\underline{v}\partial\underline{v}}\left[f\Psi\right] \ldots \; d\Delta\underline{v} \tag{1.4.2}$$

usually written as

$$\left[\frac{\partial f}{\partial t}\right]_c = -\frac{\partial}{\partial\underline{v}}\cdot\left[f\langle\Delta\underline{v}\rangle\right] + \frac{1}{2}\frac{\partial^2}{\partial\underline{v}\partial\underline{v}}:\left[f\langle\Delta\underline{v}\Delta\underline{v}\rangle\right] \tag{1.4.3}$$

where

$$\langle Q\rangle = \frac{1}{\Delta t}\int Q\,\Psi(\underline{v},\Delta\underline{v})\; d\Delta\underline{v} \tag{1.4.4}$$

Until an expression is substituted for $\Psi(v,\Delta v)$ the r.h.s. of equation (1.4.3) is as formal as the l.h.s. If the assumption is made that 2 particle correlations are the dominant effect, which is consistent with the reasoning of section 1.1 then the probability of collisions causing a change Δv in the velocity of particle 1 is

$$|\underline{v}_1-\underline{v}_2|\,\sigma(|\underline{v}_1-\underline{v}_2|,\theta)\,f(\underline{v}_2)\,d\underline{v}_2$$

so

$$\begin{bmatrix}\langle\Delta\underline{v}_1\rangle \\ \langle\Delta\underline{v}_1\Delta\underline{v}_1\rangle\end{bmatrix} = \int f(\underline{v}_2)\,\sigma(|\underline{v}_1-\underline{v}_2|,\theta)\begin{bmatrix}\Delta\underline{v}_1 \\ \Delta\underline{v}_1\Delta\underline{v}_1\end{bmatrix}|\underline{v}_1-\underline{v}_2|\; d\underline{v}_2 \tag{1.4.5}$$

which leads to

$$\left[\frac{\partial f}{dt}\right]_c = \frac{e^4\ln(\lambda_0/b_0)}{2\pi\,\varepsilon_0^{\,2}\,m_1}\left[-\frac{\partial}{\partial\underline{v}_1}\left[f(\underline{v}_1)\frac{\partial H(\underline{v}_1)}{\partial\underline{v}_1}\right] + \frac{1}{2}\frac{d^2}{\partial\underline{v}_1\partial\underline{v}_1}:\left[f(\underline{v}_1)\frac{\partial^2}{\partial\underline{v}_1\partial\underline{v}_1}G(\underline{v}_1)\right]\right]$$

$$H(\underline{v}_1) = 2\int d\underline{v}_2 \, \frac{f(\underline{v}_2)}{|\underline{v}_1 - \underline{v}_2|} \qquad G(\underline{v}_1) = \int d\underline{v}_2 \, f(\underline{v}_2)\, |\underline{v}_1 - \underline{v}_2| \tag{1.4.6}$$

for a 2 species plasma. This equation can now be used to calculate collision times, as well as slowing times and energy exchange times, given the distribution functions $f(\underline{v}_1)$ and $f(\underline{v}_2)$. If the further assumption is made that the bulk of the plasma is in a Maxwellian distribution

$$f(\underline{v}_2) = \frac{n_0}{\pi^{3/2} v_T^3} e^{-v_2^2/v_T^2} \tag{1.4.7}$$

then $H(\underline{v}_1)$ and $G(\underline{v}_1)$ have the forms

$$H(\underline{v}_1) = \frac{m_1 + m_2}{m_2} \frac{n}{\underline{v}_1} \operatorname{erf}\left(\frac{\underline{v}_1}{\underline{v}_T}\right)$$

$$G(\underline{v}_1) = n\left(v_1 + \frac{v_T^2}{2v_1}\right)\operatorname{erf}\left(\frac{v_1}{v_T}\right) - \frac{v_T}{\sqrt{\pi}} e^{-v_1^2/v_T^2} \tag{1.4.8}$$

where erf(x) is the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \tag{1.4.9}$$

The distribution function for a test particle would be

$$f(\underline{v}_1) = \delta(\underline{v}_1 - \underline{u}(t))$$

$$x = u(t)/v_T \tag{1.4.9}$$

and so the collision time, which as before is the time for a deflection of unity, can be calculated from the $v_\perp^2$ moment of equation (1.4.6). The first Fokker-Planck term - usually referred to as the drag term - does not contribute to particle deflection, which arises purely from the second term - the coefficient of diffusion. The deflection rate is

$$\frac{\partial U_\perp^2}{\partial t} = \frac{n e^4 \ln(\lambda_0/b_0)}{2\pi \varepsilon_0^2 m_i^2 U_i}\left[\operatorname{erf}(x) + \frac{1}{2}\frac{d}{dx}\left(\frac{\operatorname{erf}(x)}{x}\right)\right] \tag{1.4.10}$$

and so the deflection time is

$$t^0 = \frac{U_\perp^2}{\left(\frac{\partial U_\perp^2}{\partial t}\right)} = \frac{U_\perp^3\, 2\pi \varepsilon_0^2 m_i^2}{n e^4 \ln\left(\frac{\lambda_0}{b_0}\right)\left[\operatorname{erf}(x) + \frac{1}{2}\frac{d}{dx}\left(\frac{\operatorname{erf}(x)}{x}\right)\right]} \tag{1.4.11}$$

By inserting the values of mass and charge corresponding to a test electron scattering from a thermal ion distribution, the estimate of the collision time made in equation (1.3.2) can be verified.

$$t_{ei}^0 = \frac{2\pi \varepsilon_0^2 m_e^{1/2} (2kT)^{3/2}}{n e^4 \ln(\lambda_0/b_0)} \tag{1.4.12}$$

More importantly the collision times for ion-ion,electron-electron and ion-electron scattering can be calculated (Sanderson,1981). They are

$$t_{ii}^0 = \left.\frac{2\pi \varepsilon_0^2 m_i^{1/2} (2kT)^{3/2}}{n e^4 \ln(\lambda_0/b_0)\left[\operatorname{erf}(x) + \frac{1}{2}\frac{d}{dx}\left(\frac{\operatorname{erf}(x)}{x}\right)\right]}\right|_{x=1} \tag{1.4.13}$$

$$t_{ee}^0 = \left(\frac{m_e}{m_i}\right)^{1/2} t_{ii}^0 \tag{1.4.14}$$

$$t_{ie}^0 = \frac{3\sqrt{\pi}}{4}\,\frac{m_i}{m_e}\, t_{ei} \tag{1.4.15}$$

Using these results it is clear that for the evolution of distribution functions that are close to collisional equilibrium, the collision terms are significant for timescales of order

$$t_{ei}^{D} = \frac{2\pi \varepsilon_0^2 m_e^{1/2} (2kT)^{3/2}}{ne^4 \ln(\lambda_D/b_o)} \tag{1.4.16}$$

or greater. For the evolution of such distribution functions on much shorter timescales the collision term can be dropped. In particular, for a R.F. perturbation with frequency considerably greater than the collision frequency, the r.h.s. of the Boltzmann equation can be dropped, giving the Vlasov equation.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \underline{a} \cdot \frac{\partial f}{\partial \underline{v}} = 0 \tag{1.4.17}$$

Thus for plasma parameters that satisfy the above criteria, the Vlasov equation is adequate to describe high frequency perturbations. The next question is whether or not the R.F. heating experiments referred to in section 1.1 will satisfy the restrictions to distribution functions that are near thermal equilibrium with large numbers of particles within a Debye sphere, and to perturbations on timescales much shorter than the collision time.

1.5. Thermonuclear Fusion

In order to understand the need for auxiliary heating in these experiments, as well as the plasma parameters used, the ultimate objective - that of achieving first break-even and then ignition in a thermonuclear fusion reactor must be borne in mind.

For fusion reactions to occur, the particles to be fused must approach each other with enough kinetic energy to overcome the Coulomb repulsion of their like charged nuclei. For such reactions to occur with high frequency the average thermal energy of the particles in the plasma must be at least of comparable size to the required kinetic energy for fusion. This requires, for even the simplest fuel nuclei, those with only one proton in their nucleus, a temperature of around 100 million degrees centigrade. Furthermore, in order for ignition to occur the power production from nuclear reactions must balance the power losses from the plasma. These considerations lead to the Lawson criterion which can be expressed as

$$n\tau \geqslant 10^{20}\, m^{-3}\, s \tag{1.5.1}$$

where τ is the energy containment time. The extremely high temperature required for fusion rules out any material containment for the plasma and so research into controlled thermonuclear fusion has concentrated on inertial confinement and magnetic confinement.

1.6. Inertial Confinement

The philosophy behind inertial confinement is simple. Rather than try to contain the plasma, it is allowed to explode violently. The nuclear reaction only continues until the fireball has expanded sufficiently to cool below the required 10^8K.

This approach has the advantage of being well proven, as it is the method employed in the hydrogen bomb. However, for power production the violence of the explosion must be toned down. The problems of inertial confinement experiments stem from the necessarily very rapid reaction, which dictates that the size of the plasma must be very small. This restriction on size causes a tight restriction on τ which, if the Lawson criterion is to be satisfied, forces n to be of the order of 1000 times the solid density. The compression of the pellet of fuel to such high densities is usually performed by shining extremely powerful lasers on to the pellet surface to ablate an outer layer, the increased pressure in this region providing the large forces required to compress the rest of the pellet. Similar schemes using electron beams or ion beams in place of the laser do exist. For the plasma to reach such high densities without requiring huge amounts of power, it must be compressed adiabatically. Therefore, far from requiring additional heating of the plasma by R.F. waves, the problems of the core of the pellet being heated before full compression is achieved, are significant. Since there is no requirement for R.F. heating in these experiments and since the assumptions made in sections 1.2 and 1.4 are not all strictly valid for such high densities, no attempt will be made to model electromagnetic perturbations for such systems in this thesis. Instead, attention will be focused on lower density plasmas.

1.7. Magnetic Confinement

The fact that hot plasmas, by definition, contain large numbers of free charges makes them excellent conductors. Therefore, from Maxwell's equations, it can be seen that an element of plasma will maintain an almost constant magnetic flux. This property implies that plasmas can be contained by magnetic fields.

In this approach to obtaining a Lawson product of $10^{20}m^{-3}$ the energy confinement times are obviously far greater than those of inertial confinement experiments and so the required densities are considerably lower. This class of experiment is typically aimed at energy containment times of the order of 1s and therefore densities around $10^{20}m^{-3}$. In these machines several methods are used to heat the plasma, these methods include ohmic heating, adiabatic compression, neutral beam injection, and R.F. heating.

Ohmic heating - in this simple and very successful heating method, the plasma is treated like the heating element of an electric fire. A large current is run through the plasma and resistive losses cause the plasma to be heated. This method is particularly suitable for devices such as tokamaks and reversed field pinches (r.f.p's) which require large currents to flow in the plasma to produce part of the magnetic field used to contain the plasma. Unfortunately the conductivity of a plasma increases as $T^{3/2}$ which makes the use of ohmic heating at very high temperatures unattractive for tokamaks due to the very large current required. Since tokamaks require a toroidal magnetic field of greater magnitude than their poloidal field, a very large current flow in the plasma would require a very large current to flow in the toroidal field coils. This would cause considerable engineering problems due to the very great forces that would arise between coils and the large cross-section required for coils to carry such large currents without prohibitory power losses. This problem, which does not arise in the case of r.f.p's, has led to the use of auxiliary heating in most present day tokamak experiments and a revival of interest in r.f.p's.

Adiabatic Compression - in this method, the plasma is compressed rapidly enough to avoid major heat loss, i.e. on a timescale shorter than the energy confinement time τ, but slowly enough to remain in thermal equilibrium, i.e. on a timescale longer than the collision time t^{c}. Since the compression is adiabatic, this is a one shot heating method.

In a tokamak this is accomplished by moving the plasma inwards to a region of greater toroidal magnetic field. Since the plasma tries to conserve magnetic flux the minor radius of the plasma is reduced and this in combination with the reduction of the major radius leads to a greater reduction in plasma volume.

This process increases the plasma temperature as well as its density. However, unlike the case of inertial confinement the density cannot be increased dramatically as this would destabilise the plasma. Since the density increase is restricted, the temperature increase is also restricted and so this method, while useful, is of limited scope.

Neutral Beam Injection - the object behind this system is to fire large numbers of highly energetic particles into the plasma. These particles will then heat the plasma by collisions with the plasma particles. The difficulties with this method of heating arise from the fact that in order to penetrate the confining magnetic fields, the injected particles must be neutral. In order for the heat deposition to take place near the centre of the plasma, the particles must remain neutral as they pass through the outer edges of the plasma. To reduce the ionisation cross-section for these particles, the injected particles must be given very large energies by ion accelerators and then neutralised before entering the containment device. However, particles with too large an energy would pass through the plasma without interacting and strike the vacuum vessel wall. Present day devices use positive ion, e.g. H^+, beams which are then neutralised by collision with a gas target.

There are two major problems with this approach. First is the existence of other species in the ion source, e.g. H_2^+ and H_3^+ which on neutralisation, produce H neutrals with 1/2 and 1/3 the correct energy. Second, the cross-section for re-ionisation of the beam decreases more slowly than the cross-section for neutralisation, as the beam energy increases. This gives an effective upper limit on the neutral beam energy of around 80-100 keV per nucleon. Both of these problems can be avoided by the use of negative ion beams instead of positive ion beams. Only one negative ion species exists, H^-, and since the extra electron is weakly bound, neutralisation can be achieved by 'photo-detachment' with a laser without the production of positive ions.

Radio Frequency Heating - the subject of this thesis. In this method radio waves are beamed into the plasma to accelerate particles within it. These particles then heat the rest of the plasma by collision.

The parameters of magnetically confined plasmas satisfy all of the assumptions made in sections 1.2 and 1.4 : there are a large number of particles within a Debye length and the unperturbed distributions are near equilibrium. Therefore the properties of perturbations applied to the plasma with periods much less than the most rapid collision times are described adequately by the Vlasov equation. As will be shown in the next chapter, magnetically confined plasmas have many resonances where an electromagnetic perturbation can couple strongly to particle motion. Under such conditions, power can be absorbed from an electromagnetic wave and so heat the plasma.

Chapter 2

Wave Propagation

2.1 Introduction

To model the propagation of an electromagnetic wave in a medium the following Maxwell equations are employed.

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \qquad (2.1.1)$$

However, for a highly conducting medium like a plasma, the current density in the plasma must be obtained before equations (2.1.1) can be solved. The total electric current flowing in a plasma is simply the total of the currents carried by each species in the plasma. Since the current carried by a species is (charge of a particle) (number density) (average velocity) and the average velocity is the first velocity moment of the distribution function

$$\bar{\underline{V}}_\alpha = \int \underline{v}\, f_\alpha\, d^3\underline{v} \qquad (2.1.2)$$

all that are required now are the f as functions of the perturbing electric field.

$$\underline{J}(\underline{E}) = \sum_\alpha q_\alpha \bar{\underline{V}}_\alpha = \sum_\alpha q_\alpha \int \bar{\underline{v}}\, f_\alpha\, d^3\underline{v} \qquad (2.1.3)$$

2.2. Solving The Vlasov Equation.

In Chapter 1 it was shown that for timescales less than the collision time, the behaviour of the distribution functions of the particles making up the plasma was adequately described by the Vlasov equation. Recalling equation (1.4.17), it will be remembered that the acceleration term, a, contains, in addition to externally applied fields, the self consistent fields resulting from the distribution function f and so equation (1.4.17) is non-linear. Assuming that only electric and magnetic fields are significant, and restricting attention to velocites well below the speed of light,

$$\frac{\partial f}{\partial t} + \underline{v}.\frac{\partial f}{\partial \underline{x}} + \frac{q}{m}\left(\underline{E}(f) + \underline{v} \times B(f)\right).\frac{\partial f}{\partial \underline{v}} = 0 \qquad (2.2.1)$$

Although there do exist exact solutions for this nonlinear system - (Abraham-Shrauner,1984) (Lewis and Symon,1983) - many of the approaches used require the existence of an invariant Hamiltonian and therefore only produce undamped waves, which would not be suitable for heating a plasma (Leach,Lewis and Sarlet,1983) while others, based on the invariance of the one-dimensional Vlasov equation under infinitesimal Lie group transformations, can yield damped sinusoidal electric fields, but not with fixed frequency (Abraham-Shrauner,1984), which again limits their use in the modelling of R.F. heating.

If only small perturbations to an equilibrium plasma are considered, then equation (1.4.17) can be linearised, forming the system

$$\underline{v} \cdot \frac{\partial f_0}{\partial \underline{x}} + \frac{q}{m}\left(\underline{E}_0 + \underline{v} \times \underline{B}_0\right) \cdot \frac{\partial f_0}{\partial \underline{v}} = 0 \tag{2.2.2}$$

$$\frac{\partial f_1}{\partial t} + \underline{v} \cdot \frac{\partial f_1}{\partial \underline{x}} + \frac{q}{m}\left(\underline{E}_0 + \underline{v} \times \underline{B}_0\right) \cdot \frac{\partial f_1}{\partial \underline{v}} = -\frac{q}{m}\left(\underline{E}_1 + \underline{v} \times \underline{B}_1\right) \cdot \frac{\partial f_0}{\partial \underline{v}} \tag{2.2.3}$$

The most common approaches to calculating the propagation of waves in magnetic confinement devices are based on the solutions of the Vlasov equation in a homogeneous magnetic field. In order to understand the strengths and weaknesses of such methods, the standard derivation for such a field profile is performed below.

2.3. Constant B

Choosing a coordinate system with the z axis in the direction of the magnetic field and the wave-vector of the perturbation lying in the x-z plane, equations (2.2.2) and (2.2.3) become

$$\underline{v} \cdot \frac{\partial f_0}{\partial \underline{x}} + \frac{q}{m}\left(\underline{v} \times \underline{B}_0\right) \cdot \frac{\partial f_0}{\partial \underline{v}} = 0 \tag{2.3.1}$$

$$\frac{\partial f_1}{\partial t} + \underline{v} \cdot \frac{\partial f_1}{\partial \underline{x}} + \frac{q}{m}\left(\underline{v} \times \underline{B}_0\right) \cdot \frac{\partial f_1}{\partial \underline{v}} = -\frac{q}{m}\left(\underline{E}_1 + \underline{v} \times \underline{B}_1\right) \cdot \frac{\partial f_0}{\partial \underline{v}} \tag{2.3.2}$$

Noticing that equation (2.3.1) is satisfied by any function of $v_\perp^2$ and v_z, the Maxwellian distribution (1.2.2) is chosen for f_o. This particular choice of equilibrium has the advantage of eliminating the Lorentz force term.

$$(\underline{v} \times \underline{B}) \cdot \frac{\partial f_o(v^2)}{\partial \underline{v}} = 0 \qquad (2.3.3)$$

The next step is to Fourier transform all of the perturbed quantities.

$$i(\omega - \underline{k}.\underline{v}) f_1 + \omega_c \left(v_y \frac{\partial f_1}{\partial v_x} - v_x \frac{\partial f_1}{\partial v_y} \right) = \frac{2q\, \underline{E}.\underline{v}\, f_o}{m v_T^2} \qquad (2.3.4)$$

A further simplification is obtained by changing to cylindrical coordinates for the velocity space:

$$v_x = v_\perp \cos\phi \qquad v_y = v_\perp \sin\phi$$

$$v_y \frac{\partial f_1}{\partial v_x} - v_x \frac{\partial f_1}{\partial v_y} = -\frac{\partial f_1}{\partial \phi} \qquad (2.3.5)$$

Therefore (2.3.4) can be written as

$$\frac{\partial f_1}{\partial \phi} - \frac{i(\omega - k_z v_z - k_x v_\perp \cos\phi) f_1}{\omega_c}$$

$$= \frac{2q f_o}{m v_T^2 \omega_c} \left\{ \left[E_x \cos\phi + E_y \sin\phi \right] v_\perp + E_z v_z \right\} \qquad (2.3.6)$$

and so using the single valued nature of f_1 and the series expansion

$$e^{ib\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(b)\, e^{in\phi} \qquad (2.3.7)$$

with

$$b = \frac{k_x v_\perp}{\omega_c} \qquad (2.3.8)$$

f_1 can finally be expressed as

$$f_1 = \sum_n J_n(b)\, e^{i(\omega+n\omega_c-k_z v_z)\phi/\omega_c} \frac{q f_0}{\omega_c m v_T^2} \sum_j J_j(b)$$
$$\int^{\phi} \Big[v_\perp \Big[(E_x+iE_y)e^{-i\phi'} + (E_x-iE_y)e^{i\phi'} \Big] + 2 v_z E_z \Big]$$
$$e^{-i(\omega+j\omega_c-k_z v_z)\phi'/\omega_c} \quad d\phi' \tag{2.3.9}$$

This expression for f_1 can now be inserted in equation (2.1.3) to give the current density as a linear function of the perturbing electric field. This linear relationship is usually expressed by means of a conductivity tensor . Fortunately, only the +1,0 and -1 Fourier components of f_1 contribute to the conductivity tensor. This fact allows the double sum of equation (2.3.9) to be replaced by the single sum.

$$f_1 = \frac{q f_0}{m v_T^2} \sum_n \frac{1}{\omega+n\omega_c-k_z v_z} \Big(J_{n+1}(b)e^{i\phi} + J_n(b) + J_{n-1}(b)e^{-i\phi} \Big)$$
$$\Big(J_{n+1}(b)\, v_\perp (E_x+iE_y) + 2J_n(b)\, v_z E_z + J_{n-1}(b)\, v_\perp (E_x-iE_y) \Big)\Big)$$
$$+ \; O\,(e^{\pm 2i\phi} \ldots\;) \tag{2.3.10}$$

Using the Bessel function identities

$$J_{n-1}(b) + J_{n+1}(b) = \frac{2n}{b} J_n(b)$$
$$J_{n-1}(b) - J_{n+1}(b) = 2 J_n'(b) \tag{2.3.11}$$

The conductivity tensor $\underline{\underline{\sigma}}_\alpha$ can be written as

$$\underline{\underline{\sigma}}_\alpha = \frac{4\pi q_\alpha^2}{m v_{T\alpha}^2} \sum_n \int \frac{v_\perp dv_\perp dv_z f_{o\alpha}}{\omega + n\omega_{c\alpha} - k_z v_z} \underline{\underline{M}}_n$$

$$\underline{\underline{M}}_n = \begin{pmatrix} \frac{v_\perp^2 n^2 J_n^2(b)}{b^2} & \frac{-i v_\perp^2 n J_n'(b) J_n(b)}{b} & v_\perp v_z n J_n^2(b) \\ \frac{i v_\perp^2 n J_n'(b) J_n(b)}{b} & v_\perp^2 J_n'^2(b) & i v_\perp v_z J_n(b) J_n'(b) \\ \frac{v_z v_\perp n J_n^2(b)}{b} & i v_\perp v_z J_n'(b) J_n(b) & v_z^2 J_n^2(b) \end{pmatrix} \tag{2.3.12}$$

Performing the velocity integrals, and making use of relations between the Bessel function J(x) and the modified Bessel function I(x)=J(ix), can now be combined with Maxwell's equations to form the tensor equation.

$$\frac{c^2}{\omega^2} \underline{k} \times (\underline{k} \times \underline{E}) + \underline{E} + \sum_\alpha \frac{i \underline{\underline{\sigma}}_\alpha \cdot \underline{E}}{\varepsilon_0 \omega} = \underline{0} \tag{2.3.13}$$

$$\frac{i \underline{\underline{\sigma}}_\alpha}{\varepsilon_0 \omega} = \sum_n \frac{\omega_{p\alpha}^2 e^{-\lambda_\alpha}}{\omega k_z v_{T\alpha}} \underline{\underline{U}}_n$$

$$\underline{\underline{U}}_n = \begin{pmatrix} \frac{n^2 I_n Z}{\lambda_\alpha} & -in(I_n' - I_n) Z & \frac{n I_n Z'}{2\lambda_\alpha^{1/2}} \\ in(I_n' - I_n) Z & \left(\frac{n^2}{\lambda_\alpha} I_n + 2\lambda_\alpha (I_n' - I_n)\right) Z & \frac{\lambda_\alpha^{1/2} (I_n' - I_n) Z'}{2^{1/2}} \\ \frac{n I_n Z'}{2\lambda_\alpha^{1/2}} & \frac{\lambda_\alpha^{1/2} (I_n' - I_n) Z'}{2^{1/2}} & I_n Z \end{pmatrix} \tag{2.3.14}$$

Where λ_α, the argument of the modified Bessel functions, and ζ_n the argument of the plasma dispersion function are

$$\lambda_\alpha = \frac{k_x^2 v_{T\alpha}^2}{2\omega_{c\alpha}^2} \qquad \zeta_n = \frac{\omega + n\omega_{c\alpha}}{k_z v_{T\alpha}} \tag{2.3.15}$$

Non-trivial solutions of (2.3.12) exist $\Longleftrightarrow$ the determinant of the system equals zero. Setting the determinant equal to zero gives the dispersion relation.

Since solving a transcendental equation like the dispersion relation requires numerical techniques, most of the research done in this field concentrates on long wavelength solutions. If the wavenumber k_x is small enough for the argument of the Bessel functions to be considerably smaller than 1, then in the dielectric tensor they can be approximated with only the first few terms of their series expansions.

Therefore, for perturbations that vary slowly, i.e. changing significantly only over several Larmor orbits, the dispersion relation becomes a polynomial in k_x^2. The lowest order polynomial obtained by this method can be written as

$$\left(a_1\left(a_1 - \frac{c^2k_x^2}{\omega^2}\right) - a_2^2\right)\left(a_3 - \frac{c^2k_x^2}{\omega^2}\right) - \frac{c^4}{\omega^4}k_x^2k_z^2\left(a_1 - \frac{c^2k_x^2}{\omega^2}\right) = 0 \tag{2.3.16}$$

$$a_1 = 1 - \frac{c^2k_z^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{2\omega k_z V_{T\alpha}}\left(Z(\zeta_{-1}) + Z(\zeta_1)\right)$$

$$a_2 = \sum_\alpha \frac{\omega_{p\alpha}^2}{2\omega k_z V_{T\alpha}}\left(Z(\zeta_{-1}) - Z(\zeta_1)\right)$$

$$a_3 = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega k_z V_{T\alpha}}\frac{2\omega}{k_z V_{T\alpha}}\left(\zeta_0 Z(\zeta_0) - 1\right) \tag{2.3.17}$$

for large arguments the plasma dispersion functions can be approximated.

$$Z(\zeta_{-1}) = \frac{k_z v_T}{\omega - \omega_c} \qquad Z(\zeta_1) = \frac{k_z v_T}{\omega + \omega_c}$$

$$\frac{2\omega}{k_z v_T}\left(\zeta_0 Z(\zeta_0) - 1\right) = \frac{k_z v_T}{\omega} \tag{2.3.18}$$

(2.3.17) becoming

$$a_1 = 1 - \frac{c^2 k_z^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2}$$

$$a_2 = \sum_\alpha \frac{\omega_{p\alpha}^2\ \omega_{c\alpha}}{\omega(\omega^2 - \omega_{c\alpha}^2)}$$

$$a_3 = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \tag{2.3.19}$$

Since there are now no thermal effects in the equation, i.e. no dependence on v_T, this limit is called cold plasma theory.

Another simplification can be achieved by noting that the plasma conductivity along the magnetic field is far greater than the conductivity perpendicular to the magnetic field and so the electric field in the z direction tends to be suppressed. For a large number of cases this implies that only the electric fields perpendicular to the magnetic field need be considered. Another way of considering this is that the determinant of the dielectric tensor is dominated by the product of the determinant of the 3,3 minor with the 3,3 element, for a large range of frequencies and wave-vectors.

So w and k that lie within this range and simultaneously give zero for the determinant of the 3,3 minor are good approximations to solutions of the full dispersion relation. Taking the elements of the 3,3 minor to first order in λ, the following dispersion relation is obtained.

$$(a_1 + B_1)\left(a_1 + B_2 - \frac{c^2 k_x^2}{\omega^2}\right) - (a_2 + B_3)^2 = 0 \tag{2.3.20}$$

$$B_1 = \sum_\alpha \frac{\lambda_\alpha \omega_{p\alpha}^2}{2\omega k_z V_{T\alpha}} \left(Z(J_{-2}) + Z(J_2) - Z(J_{-1}) - Z(J_1) \right)$$

$$B_2 = \sum_\alpha \frac{\lambda_\alpha \omega_{p\alpha}^2}{2\omega k_z V_{T\alpha}} \left(Z(J_{-2}) + Z(J_2) - 3\left[Z(J_{-1}) + Z(J_1) \right] + 4 Z(J_0) \right)$$

$$B_3 = \sum_\alpha \frac{\lambda_\alpha \omega_{p\alpha}^2}{2\omega k_z V_{T\alpha}} \left(Z(J_{-2}) - Z(J_2) + 2\left[Z(J_1) - Z(J_{-1}) \right] \right) \tag{2.3.21}$$

2.4. The Cyclotron Resonances

One interesting feature of the v_z integration is the existence of simple poles, which occur when the Doppler shifted wave frequency $\omega - k_z v_z$ is an integer multiple of the cyclotron frequency of the species. These poles give rise to the imaginary part of the plasma dispersion function

$$Z(J_n) = 2 e^{-J_n^2} \int_0^{J_n} e^{t^2} dt + i\sqrt{\pi}\, e^{-J_n^2} \tag{2.4.1}$$

and can also cause significant cyclotron damping of waves if large numbers of particles satisfy the resonance condition.

$$\operatorname{im} Z(\zeta_n) \sim f_o(v_r) \qquad \underline{v_r = \omega - n\omega_c} \atop k_z \qquad (2.4.2)$$

Cyclotron damping has been successfully employed as an auxiliary form of plasma heating in several experiments e.g. JET,TFR,PLT. Part of the attraction of this particular type of R.F. heating is that in an equilibrium magnetic field which is a function of position, cyclotron damping will be significant only where $\omega - n\omega_c \leq k_z v_T$. Therefore, by tuning the frequency of the wave launched into the plasma, the heating effect can be localised.

2.5. Inhomogeneous B_o

In order to calculate the heating effects in a magnetically confined plasma, it is necessary to model the behaviour of the perturbation in an inhomogeneous magnetic field. This in turn requires that a differential equation be obtained. One method that has been widely used is to take a dispersion relation from the homogeneous case, replace the k_x^2 with $\frac{-d^2}{dx^2}$ and then allow the coefficients of this differential equation to become functions of position. It is the last stage which invalidates the method, for while the differential equation obtained is perfectly valid for the homogeneous case, there is no reason to expect it even to be of similar form for the case of inhomogeneous fields. In fact, on the contrary, in the similar case of warm magnetohydrodynamic equations, it has been shown that the differential equations obtained consistently have singularities that do not appear in equations based on the homogeneous dispersion relation (Diver,1986).

Despite the flawed derivation of these equations, they are still widely used in the literature, the main interest of the authors being the method of solution of the differential equations thus formed.

Other writers, dissatisfied with these methods, particularly because of inconsistencies in the expressions derived for power flows in the plasma, have attempted to obtain the differential equations more consistently; by use of variational techniques (Colestock and Kashuba,1983) or by perturbation techniques (Swanson,1981). The first technique suffers from its complexity and the resultant fact that some assumptions are made implicitly and their consequences ignored. The approach used is to take a variational integral

$$\int d\underline{x}\, \underline{E}^{+}(\underline{x})\left[-\nabla\times\nabla\times\underline{E}(\underline{x}) + \frac{\omega^2}{c^2}\underline{E}(\underline{x}) + i\omega\mu_0\int d\tilde{\underline{x}}\, \underline{\underline{\sigma}}(\underline{x},\tilde{\underline{x}})\underline{E}(\tilde{\underline{x}})\right] \qquad (2.5.1)$$

from which the wave equation

$$\nabla \times \nabla \times \underline{E} - \frac{\omega^2}{c^2}\underline{E} - i\omega\mu_0 \underline{\underline{\sigma}} . \underline{E} = \underline{0} \qquad (2.5.2)$$

can be derived, Fourier transform the fields and so recast the variational integral in k-space.

$$\int \frac{dk}{(2\pi)^3} \left[\underline{E}^+(-\underline{k}) . \left(\underline{k} \times \underline{k} \times \underline{E}(\underline{k}) + \frac{\omega^2}{c^2} \underline{E}(\underline{k}) + i\omega\mu_0 \int \frac{d\tilde{k}}{(2\pi)^3} \underline{\underline{\sigma}}(\underline{k},\tilde{\underline{k}}) \underline{E}(\tilde{\underline{k}}) \right) \right] \qquad (2.5.3)$$

So far, the manipulations have been purely formal. The problem of the form of $\underline{\underline{\sigma}}(\underline{x},\tilde{\underline{x}})$ and $\underline{\underline{\sigma}}(\underline{k},\tilde{\underline{k}})$ has not been addressed. It is at this point that approximations are made. The conductivity tensor, $\underline{\underline{\sigma}}$, is obtained by using the method of characteristics to solve the perturbed Vlasov equation. The path of the characteristic is approximated by the unperturbed orbit of a single particle in a constant magnetic field. This is analogous to the choice of a drift free equilibrium made in Chapter 3 of this thesis. The critical assumption comes next: it is assumed that when the conductivity tensor elements are expanded as series in the Larmor radius, only zeroeth order terms need be included except for the harmonic resonance terms, which are taken to first order. This finite Larmor radius expansion is only valid if the elements of $\underline{\underline{\sigma}}$ are themselves slowly varying. If more terms had been included, it would have been noticed that higher derivatives of the resonances themselves appear in the coefficients of even the lowest order derivatives of the electric field. In Chapter 3, the restrictions imposed by the small Larmor orbit expansion are obtained explicitly. In Colestock and Kashuba 1983, the failure to identify these restrictions leads to the use of the equations obtained for k_z values for which they would not appear to be valid.

The second approach (Swanson 1981) starts from the perturbed Vlasov equation

$$\frac{\partial f_1}{\partial \phi} + \frac{i(\omega - k_z v_z) f_1}{\omega_c} - \frac{v_\perp \cos\phi}{\omega_c} \frac{\partial f_1}{\partial x} = \frac{-2 v_\perp}{V_T} \left[\frac{\cos\phi \, E_x + \sin\phi \, E_y}{B_o} \right] f_o \tag{2.5.4}$$

which is integrated at once.

$$f_1 = \exp\left(\frac{i(k_z v_z - \omega)\phi}{\omega_c} \right) \int^{\phi} \exp\left(\frac{i(\omega - k_z v_z)\tilde{\phi}}{\omega_c} \right) \left[\frac{v_\perp \cos\tilde{\phi}}{\omega_c} \frac{\partial f_1}{\partial x} - \frac{2 v_\perp f_o}{V_T^2} \left(\frac{\cos\tilde{\phi} \, E_x + \sin\tilde{\phi} \, E_y}{B_o} \right) \right] d\tilde{\phi} \tag{2.5.5}$$

This integral equation is then solved using a perturbation expansion

$$f_1 = f_1^{(0)} + \left(\frac{v_\perp}{\omega_c} \right) f_1^{(1)} + \left(\frac{v_\perp}{\omega_c} \right)^2 f_1^{(2)} \tag{2.5.6}$$

Again this is a small Larmor orbit expansion, and again the expansion is only carried to second order for the harmonic resonant term, and to zeroeth order in the rest. Because only the resonant term is expanded to second order, the equation produced is only valid locally. Finally, as in the variational technique, the restrictions on the gradients of the elements of $\underline{\underline{\sigma}}$, and therefore on the value of k_z are not identified.

To deal with these shortcomings, a different approach is used in Chapter 3 and extended in later chapters to produce differential equations consistently from the Vlasov equation. This simple approach has the advantage that assumptions are made explicitly and so the resulting limits on validity of the approach are also clear. The new equations and their solutions are compared with those obtained by other methods for a simple case, and some more general conclusions made.

However, before discussing improved derivation techniques for the equations, a further major difference between the behaviour of the constant B_0 case and that of a spatially dependent B_0 should be noted.

2.6. Mode Conversion

Possibly the most significant difference between the behaviour of o.d.e's with constant coefficients and o.d.e's with coefficients that are functions of the independent variable, is that the latter may exhibit mode conversion. Swanson, in his review article (Swanson,1985), established that in the vicinity of a plasma resonance, further terms in the series expansions of the dielectric tensor would become significant. Moreover, since the order of the dispersion relation would be increased, he deduced that the additional solutions, or modes, introduced were evidence that mode conversion was involved in resolving the resonance.

A slightly more satisfactory, and certainly more useful approach to establish the existence of mode conversion in inhomogeneous plasmas, is to change variables from the electric field to the eigenfunctions or modes of the o.d.e's. This transformation (Heading,1961) diagonalises the matrix of coefficients of the o.d.e's and generates an additional matrix of coefficients as is shown below for the case of a second order o.d.e.

$$u'' - a(x)\,u' - b(x)\,u = 0 \tag{2.6.1}$$

Written as a system of first order o.d.e's.

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ b(x) & a(x) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} \tag{2.6.2}$$

and then transformed

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha_1(x) & \alpha_2(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.6.3}$$

gives

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(\begin{bmatrix} \alpha_1(x) & 0 \\ 0 & \alpha_2(x) \end{bmatrix} + \frac{1}{\alpha_1 - \alpha_2} \begin{bmatrix} -\alpha_1' & -\alpha_2' \\ \alpha_1' & \alpha_2' \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{2.6.4}$$

where α_1 and α_2 are the eigenvalues of the system.

From this it can be seen that the coupling between the eigenfunctions or modes is due to the spatial variation of the eigenvalues and is inversely proportional to their difference. Thus it is observed that mode coupling is a consequence of the variation of the eigenvalues and is therefore non-existent in systems of o.d.e's with constant coefficients. The connection with the 'mode conversion theorem' of Swanson is clearer when the dependence on the differences between eigenvalues is considered, as normally some of these differences are considerably reduced in the presence of resonances.

It can therefore be seen that in order to satisfactorily describe the effects of cyclotron heating on a plasma, not only must a theory include the properties of the incident wave and the impact of cyclotron harmonic damping, but also the phenomenon of mode conversion and the properties of the mode converted wave. Some of the difficulties that this poses are addressed in the next section.

Once a differential equation has been obtained, there are several methods of solution that have been proposed in the literature.

2.7. The Laplace Transform Technique

For this approach to be useful analytically, the coefficients of the differential equation must be constant or linear. Then the Laplace transform of the differential equation is itself a first order o.d.e. which can formally be solved (Ngan and Swanson,1977), (Gambier and Schmitt,1983). Taking their example fourth order equation

$$y^{IV} + \lambda^2 z y'' + (\lambda^2 z + \gamma) y = 0 \tag{2.7.1}$$

which transforms to the first order equation

$$(p^4 - 2\lambda^2 p + \gamma) Y = \lambda^2 (p+1) \frac{dY}{dp} \tag{2.7.2}$$

with an almost trivial solution.

$$Y = \frac{\exp}{1+p^2} \left(\frac{p^3}{3\lambda^2} - \frac{p}{\lambda^2} + \frac{(1+\gamma)}{\lambda^2} \tan^{-1}(p) \right] \tag{2.7.3}$$

Thus the problem has now been reduced to finding the inverse transform of (2.7.3)

One of the main weaknesses of the Laplace transform method is that the differential equations describing the perturbation in the actual plasma cannot, as a rule, be manipulated into the required form. However, the use of numerical methods raises the possibility of generalising the technique to equations with coefficients that are asymptotically linear in the independent variable (Swanson,1978).

If the equation is now written in the form

$$y^{iv} + \lambda^2 z y'' + (\lambda^2 z + \gamma) y = g(z, y, y', y'', y''') \tag{2.7.4}$$

then an iteration technique can be employed treating the r.h.s. as a driving term and using a Green function formed from the solutions of the adjoint to the homogeneous equation, i.e. the adjoint of the equation formed by removing the r.h.s. of equation (2.7.4). The iteration process will converge provided that the kernel of the integral equation is bounded. This method has been employed quite extensively (Stix and Swanson,1983), (Swanson,1985) despite the increase in complexity over the original equation.

To understand the advantages of this method it is necessary to examine the nature of the solutions to the equations that are being considered. In the vicinity of a cyclotron harmonic resonance there are several solutions of the dispersion relation that satisfy $k_x^2 v_T^2 << 2\omega_c^2$. In addition to the solutions of the 'cold plasma' dispersion relation, (2.3.17) there are the two almost purely electrostatic solutions that arise from the thermal corrections. These solutions, the Bernstein modes (Bernstein,1958), are wavelike on the high magnetic field side of the resonant region but become strongly evanescent on the low field side. The problem with direct numerical integration of these equations from one side of the resonant region to the other is the exponential growth of one of the Bernstein solutions. Any numerical error will tend to excite this mode, which will then rapidly grow to dominate the solution. In the integral equation approach each growing exponential is multiplied by a decaying one in such a way as to avoid any numerical difficulties. This approach requires that the homogeneous equations are already solved - not merely asymptotically as before, but for the entire region. In principle this is no easier than solving the original set of equations numerically. However there is a major advantage claimed for this method, in that consistency relations can be used to detect errors and a contour integration starter can be used to restart the differential equation solver, utilising the solutions of the Laplace transformed adjoint equation when the errors grow above a preset level.

2.8. Reduced Order Equations.

A different approach relies on reducing the order of the o.d.e. to second order (Cairns and Lashmore-Davies,1983). This is valid locally if only two modes are coupled in the region. Reducing the order to second order produces considerable benefits not only for numerical solutions but also for analytic solutions. The analytic method is quoted below.

First the dispersion relation is reduced to a second order equation by factoring out the solutions that are not expected to be directly coupled in the region to be modelled.

$$\left(\omega - \omega_1(k,x)\right)\left(\omega - \omega_2(k,x)\right) = \eta(\omega,k,x) \qquad (2.8.1)$$

Then the crossing point of the asymptotic forms of ω_1 and ω_2 is identified $(k_o, x=0)$, and ω_1 and ω_2 are linearly expanded about this point.

$$\omega_1 = \omega + a\Delta k + b\Delta x$$

$$\omega_2 = \omega + f\Delta k + g\Delta x \qquad (2.8.2)$$

Now substituting (2.8.2) in (2.8.1) and then splitting (2.8.1) into two parts determines the 'wavenumbers' k as functions of position.

$$\left[(k_o - k) - \frac{b}{a}x\right]\left[(k_o - k) - \frac{g}{f}x\right] = \frac{\eta}{af} \qquad (2.8.3)$$

$$k - \left(k_o - \frac{b}{a}x\right) = \lambda_1 \qquad (2.8.4)$$

$$k - \left(k_o - \frac{g}{f}x\right) = \frac{\eta_o}{af\lambda_1} \qquad (2.8.5)$$

Replacing k with $-i\frac{d}{dx}$ gives two coupled first order equations each describing a wave (Ψ_1 and Ψ_2).

$$\frac{d\Psi_1}{dx} - i\left(k_0 - \frac{b}{a}x\right)\Psi_1 = i\lambda_1\Psi_2 \tag{2.8.6}$$

$$\frac{d\Psi_2}{dx} - i\left(k_0 - \frac{g}{f}x\right)\Psi_2 = i\lambda_2\Psi_1 \tag{2.8.7}$$

The 'wavenumbers' are not eigenvalues or solutions of the dispersion relation, and the discrepancy between the asymptotic form of the solutions and the local form of the equation gives rise to η_o which later forms the basis of the coupling.

Obviously equation (2.8.1) can be split in many ways. The choice of $\lambda_1 = \lambda_2 = \left(\frac{\eta_o}{af}\right)^{1/2}$ made by Cairns and Lashmore-Davies has the advantage that when k is replaced by $-i\frac{d}{dx}$ the sum $|\Psi_1|^2 + |\Psi_2|^2$ is a constant. It should be mentioned here that Ψ_1 and Ψ_2 are not modes of the system in the sense of section 2.6 . In a homogeneous plasma, equations (2.8.6) and (2.8.7) would become identical, would not individually represent homogeneous solutions and would still be coupled. It should perhaps also be noted that Ψ_1 and Ψ_2 do not obey the same second order o.d.e. unless $\frac{b}{a} = \frac{g}{f}$, in which case only one mode is being discussed.

Then, returning to a second order system, but this time a differential equation, by eliminating from equations (2.8.6) and (2.8.7), the transformation

$$\Psi_1 = \exp\left(i k_0 x - \frac{ib}{4a} x^2 - \frac{ig}{4f} x^2 \right) \Psi \qquad (2.8.8)$$

combined with the scaling

$$\xi = \left(\frac{ag - bf}{af} \right)^{1/2} \exp\left(\frac{i3\pi}{4} \right) x \qquad (2.8.9)$$

gives the Weber equation.

$$\frac{d^2 \Psi}{d\xi^2} + \left(\frac{i\eta_0}{ag-bf} + \frac{1}{2} - \frac{\xi^2}{4} \right) \Psi = 0 \qquad (2.8.10)$$

The parabolic cylinder function $D_n(\xi)$ is a solution of the Weber equation and, by comparing the asymptotic form of it with the asymptotic forms of Ψ_1 and Ψ_2, an expression for the transmission factor of Ψ_1 is obtained.

$$T = \exp\left(\frac{-2\pi \eta_0}{|ag - bf|} \right) \qquad (2.8.11)$$

Since T is defined from (2.8.10), and $|\Psi_1|^2 + |\Psi_2|^2$ is a constant, the mode conversion factor is 1 - T. Having obtained this expression for T, any equation that can be manipulated into the form of equation (2.8.1) is of course also solved. Further, as the writers point out, many more complicated interactions can be broken down into individual coupling events. The drawbacks of this analytic method are twofold; first, it cannot, as it stands, be applied to equations which involve damping processes, and second, the identification of Ψ_1 and Ψ_2 as modes is not standard.

A further development of this method

(Lashmore-Davies et al,1987) treats only the fast modes (those that would be obtained from the cold plasma equations) as of interest and ignores the other modes. The thermal terms from the dispersion relation are given k values calulated from the cold plasma equations and then treated as a local perturbation to the fast wave. The differential equation is obtained by replacing the $\frac{c^2 k_x^2}{\omega^2}$ in the dispersion relation with $-\frac{c^2 d^2}{\omega^2 dx^2}$. For example, (2.3.19) would give rise to the differential equation

$$\frac{d^2\Psi}{dx^2} = \frac{\omega^2}{c^2}\left(\frac{\alpha_2+\beta_3}{\alpha_1+\beta_1} - \alpha_1 - \beta_2\right)\Psi \tag{2.8.12}$$

This method is probably inspired by (Cairns and Lashmore-Davies,1986) which attempted to identify Ψ_1 and Ψ_2 with the modes of section 2.6. In this paper, equation (2.8.1) was obtained by splitting the conductivity tensor into the parts that would give rise to the fast mode and a part that had a pole for $x = x_0$.

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_f + \frac{\lambda_f}{x-x_0}\underline{\underline{\sigma}}_r \tag{2.8.13}$$

The second part was then treated as a resonant response in the plasma, approximating k_x by its value from the asymptotic or cold plasma expression for the fast wave at the resonant layer. This process identified Ψ_1 as the fast mode but did not identify Ψ_2 as a propagating mode.

Since this derivation applied only to the case where the conductivity tensor had a pole, the desire to extend the theory to cases that did not have a pole in the conductivity tensor could lead to splitting the dispersion relation into fast wave and other terms rather than splitting the conductivity tensor.

This method has the obvious disadvantage, in addition to the questionable method employed to obtain a differential equation, that only the behaviour of the fast mode is modelled. Therefore, while transmission and reflection of the fast mode can be calculated, there is no information about how much of the power lost from the fast wave is mode converted to the slow wave and how much is simply lost to cyclotron damping. The one exception is in the case of zero damping where all the power lost from the fast wave is assumed to be mode converted.

The stability of the transmission factor T to such different treatments of the mode conversion phenomenon is quite remarkable. Furthermore, it will be shown in Chapter 3 that while the amount of mode conversion that occurs is very sensitive to the form of the o.d.e., the transmission factor for the fast wave is the same within numerical error for equations obtained rigorously and those obtained from the dispersion relation.

Chapter 3

The Differential Equation

3.1. Introduction

Most of the theoretical work on R.F. heating has been based on the 'inverse Fourier transform' of the homogeneous dispersion relation. The coefficients of the differential equation thus obtained are then allowed to become functions of position (Cairns and Lashmore-Davies, 1983, Lashmore-Davies, Fuchs, Gauthier, Ram and Bers, 1987). Unfortunately, since homogeneity is assumed before the dispersion relation is obtained by Fourier transform techniques, this method does not reproduce any of the terms arising from parameter gradients.

In this chapter the wave differential operator is obtained directly from the perturbed Vlasov equation in a systematic manner, and so includes self consistently the effects of parameter gradients as well as those of strong wave damping and linear mode conversion. The advantages of such a systematic approach are the ease with which it can be extended not only to the cases of anisotropic (Chapter 5) and inhomogeneous (Chapter 6) equilibrium distributions, but also to the case of finite k_y (Chapter 7).

Perhaps more importantly, this method allows the explicit determination of conditions on the parallel wavenumber and the magnetic field gradient for which such methods are valid.

From these equations it is shown that inclusion of the parameter gradient terms is important for accurate calculation of mode conversion from fast wave to ion Bernstein wave when propagating nearly perpendicular to the unperturbed magnetic field, although the dispersion relation based operator can be sufficient to describe transmission and reflection of the fast wave.

3.2. Method

The starting point for this method is the Vlasov equation. As in the case of wave propagation in a hot homogeneous plasma, the field quantities f,$\underline{B}$ and $\underline{E}$ are linearised and then the perturbed equation is Fourier transformed in z and t. But, unlike the homogeneous case, it is not Fourier transformed in x (which is chosen to be the direction of the inhomogeneity).

For clarity and to facilitate comparison with other methods, the plasma equilibrium chosen here has no associated electric field, and the equilibrium distribution function is taken to be Maxwellian with no particle drift. From equation (2.3.1) it can be seen that such a distribution must be spatially homogeneous. It should be emphasised that in this chapter the only equilibrium quantity that is a function of position is the magnetic field; temperatures and densities are constant.

$$i(k_z v_z - \omega) f_1 + v_x \frac{\partial f_1}{\partial x} + \omega_c \left(v_y \frac{\partial f_1}{\partial v_x} - v_x \frac{\partial f_1}{\partial v_y} \right) = \frac{q}{m} \underline{E} \cdot \frac{\partial f_0}{\partial \underline{v}} \quad (3.2.1)$$

Again, as in the case of a constant background magnetic field, the number of dependent variables can be further reduced by a change of velocity coordinate system to cylindrical coordinates.

$$\omega_c \frac{\partial f_1}{\partial \phi} + i(\omega - k_z v_z) f_1 - \frac{v_\perp}{2} \left(e^{i\phi} + e^{-i\phi} \right) \frac{\partial f_1}{\partial x} =$$

$$-\frac{2q}{m} \frac{f_0}{v_T^2} \left[(E_- e^{i\phi} + E_+ e^{-i\phi}) v_\perp + E_z v_z \right] \quad (3.2.2)$$

$$E_- = \frac{E_x - iE_y}{2} \qquad E_+ = \frac{E_x + iE_y}{2} \quad (3.2.3)$$

So far the manipulations have been closely modelled on the standard procedures. However, the standard method for dealing with the x derivative is not useful: instead we first utilise the single valued nature of f_1, which implies that f_1 is periodic in ϕ, to write f_1 as a Fourier series in ϕ.

$$f_1(\phi) = \sum_{n=-\infty}^{\infty} f_{1n} e^{in\phi} \quad (3.2.4)$$

Then considering the coefficient of $e^{in\phi}$ in (3.2.2)

$$f_{1n} = A_n \frac{d}{dx}\left[f_{1n+1} + f_{1n-1}\right] + \frac{2A_n q}{mV_\perp}\left[P(E_- \delta_{n,1} + E_+ \delta_{n,-1}) + RE_z \delta_{n,0}\right] \tag{3.2.5}$$

$$A_n = \frac{V_\perp}{2ik_z V_T (J_n - \bar{V}_z)} \qquad J_n = \frac{\omega + n\omega_c}{k_z V_T} \tag{3.2.6}$$

$$P = \frac{\partial f_0}{\partial V_\perp} \qquad R = \frac{\partial f_0}{\partial V_z} \qquad \bar{V}_z = \frac{V_z}{V_T} \tag{3.2.7}$$

These equations have the following important properties.

Just as in the homogeneous case (section 2.3), only the -1 and +1 Fourier components contribute directly to J_x and J_y (and only the 0 component contributes directly to J_z), since only these components give rise to a non-zero value for the integral of the velocity moments of f_1. The higher harmonics are of interest because of their coupling to the -1,0 and +1 components.

Since the equilibrium distribution is isotropic in velocity space, it is ϕ independent and so

$$-\frac{2qf_0}{V_T^2}\left[(E_- e^{i\phi} + E_+ e^{-i\phi})V_\perp + E_z V_z\right] \tag{3.2.8}$$

has only $e^{i\phi}$, e^0 and $e^{-i\phi}$ terms. Therefore only the -1,0 and +1 components are directly driven by the perturbing electric field, hence the Kronecker delta terms in (3.2.5). The other components are excited via the coupling to these fundamental components.

At each level the nth Fourier component is coupled only to the gradient of the (n+1)th and (n-1)th components.

3.3. The Tree Diagram

If all relevant quantities are slowly varying on the scale of a Larmor radius, i.e.

$$\left| \frac{V_T}{2\omega_c} \frac{dQ}{dx} \right| \ll |Q| \tag{3.3.1}$$

- which should be compared with the condition for the validity of a polynomial form of the homogeneous dispersion relation, which is

$$\left| \frac{K_x V_T}{2\omega_c} \right| \ll 1 \tag{3.3.2}$$

- then the tree diagram (figure 3.1) can be used to give a perturbation expansion for the relevant components of f_1 in terms of the electric field and its derivatives and the equilibrium plasma parameters and their derivatives. The expressions obtained can be put into a more convenient form by using

$$A_n A_m = \frac{iV_\perp}{2\omega_c} \left(\frac{A_n - A_m}{n-m} \right) \qquad n \neq m \tag{3.3.3}$$

to replace the products of A_n's with differences. Then, taking the first velocity moment of f_1, $\underline{J}$ can be obtained. Since the relationship between $\underline{E}$ and $\underline{J}$ is linear, it can conveniently be expressed in the

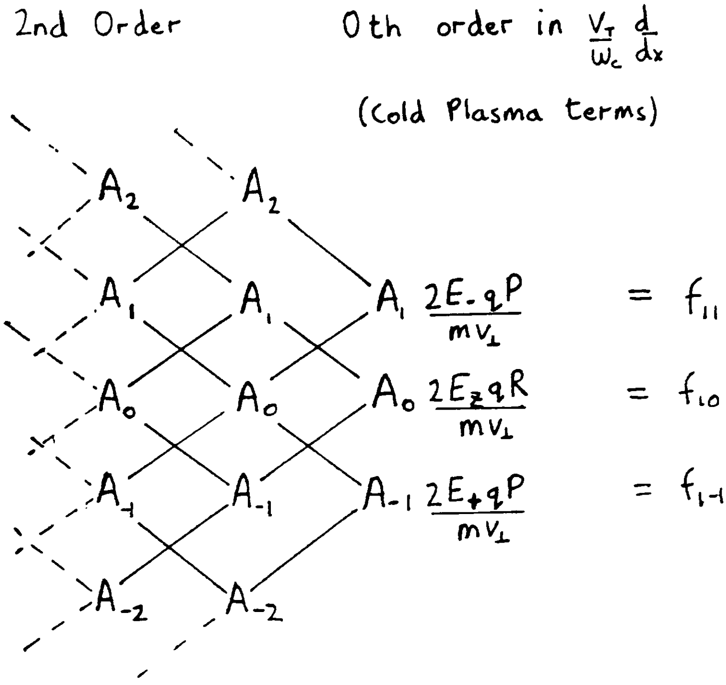


Figure 3.1 Tree Diagram showing the coupling between the Fourier components of f_1 with straight lines representing the operator $\frac{d}{dx}$.

form of a conductivity tensor $\underline{\underline{\sigma}}$.

$$\underline{J} = \underline{\underline{\sigma}} \cdot \underline{E} \tag{3.3.4}$$

Finally the expressions for $\underline{J}$ can be incorporated with Maxwell's equations in the usual manner to give a system of coupled o.d.e's which govern the spatial evolution of the perturbing electric field.

$$\frac{c^2}{\omega^2} \nabla \times \nabla \times \underline{E} - \underline{E} - \frac{i\underline{\underline{\sigma}} \cdot \underline{E}}{\varepsilon_0 \omega} = \underline{0} \tag{3.3.5}$$

3.4. Restrictions

To truncate the order of the differential equation describing it, $\underline{E}$ is required to be slowly varying - just as, in the homogeneous case in Chapter 2, the same condition was used to truncate the series expansions in k_x.

In addition, since the differential operator also acts on the A_n, these too must be slowly varying. This is necessary in order to be able to truncate the expressions for the coefficients of the differential equation describing $\underline{E}$.

Performing the velocity integrations first, the restriction can be written

$$\left| \frac{V_T}{2\omega_c} \frac{d}{dx} Z(\zeta_n) \right| \ll |Z(\zeta_n)| \tag{3.4.1}$$

where

$$Z(\zeta_n) = \frac{1}{n_o} \int \frac{f_o}{\zeta_n - \bar{v}_z} d^3\underline{v} \tag{3.4.2}$$

For the Maxwellian equilibrium distribution already chosen, Z is the plasma dispersion function (2.4.1). In which case (3.4.1) becomes

$$\left| \frac{V_T}{\omega_c} \zeta_n' \left(1 - \zeta_n Z(\zeta_n)\right) \right| \ll |Z(\zeta_n)| \tag{3.4.3}$$

For non-resonant terms this simply implies that B_o must be slowly varying;

$$\left|\frac{V_T\, J_n'}{2\omega_c\, J_n^2}\right| \ll \left|\frac{1}{J_n}\right|$$

$$\Rightarrow \quad \left|\frac{V_T}{2\omega_c} B_0'\right| \ll |B_0| \tag{3.4.4}$$

More importantly, the resonant coefficients must also be slowly varying. Thus, if the wave enters a region where the effects of the $\omega \simeq n\omega_c$ harmonic resonance become significant, then, in order to truncate the parameter gradient terms, it is required that

$$\left|\frac{V_T\, J_n'}{\omega_c}\right| \ll 1 \tag{3.4.5}$$

If the magnetic scale length is L

$$\left|\frac{V_T}{\omega_c}\ \frac{n\omega_c}{L k_z V_T}\right| \ll 1 \tag{3.4.6}$$

The condition $k_z L \gg n$ can perhaps be more clearly understood from the following argument.

In order for the resonance to be slowly varying, its width must be greater than the Larmor orbit of the species concerned. The frequency range of the resonance is finite due to Doppler broadening $k_z v_T$, and the physical width that this corresponds to is given by the equation

$$W = \frac{k_z V_T}{n\left|\frac{\partial \omega_c}{\partial x}\right|} \tag{3.4.7}$$

If the scale length for the variation of B_0 is L then

$$W = \frac{k_z V_T L}{n \omega_c} \tag{3.4.8}$$

Therefore requiring that

$$|W| \gg \left|\frac{V_T}{\omega_c}\right| \tag{3.4.9}$$

implies that $|k_z L| \gg |n|$.

To demonstrate the use of this formalism and the effect of including parameter gradient terms, a simple case is examined below.

3.5. Example

Mode conversion and damping at the first harmonic resonance with the ion cyclotron frequency.

The equilibrium chosen has two plasma species (ions and electrons), both described by homogeneous Maxwellian distribution functions, in a linearly increasing magnetic field.

The perturbation applied is a fast mode wave incident from the high field side of the mode conversion region. The fast mode corresponds to a solution of the cold plasma equations, whereas the mode converted wave (the ion Bernstein mode) arises from thermal effects.

With the assumptions made earlier in this chapter, only terms up to second order in the differential operator are needed to include the dominant effects of the first ion cyclotron harmonic resonance. To facilitate comparison both with methods based on the 3,3 minor of the homogeneous dielectric tensor and those based on the determinant of the same 3,3 minor or dispersion relation, only E_x and E_y equations are considered.

The system of equations obtained using the methods described in sections 3.2 and 3.3 is

$$\underline{\underline{U}}.\underline{E}'' + \underline{\underline{V}}.\underline{E}' + \underline{\underline{W}}.\underline{E} = \underline{0} \tag{3.5.1}$$

where

$$U_{11} = \sum_{\text{Species}} \frac{\omega_p^2}{\omega k_z V_T} \left(\frac{V_T}{2\omega_c}\right)^2 \left[Z(\zeta_{-2}) + Z(\zeta_2) - Z(\zeta_{-1}) - Z(\zeta_1) \right]$$

$$U_{12} = -U_{21} = \sum_{\text{Species}} \frac{\omega_p^2}{\omega k_z V_T} \left(\frac{V_T}{2\omega_c}\right)^2 \left[Z(\zeta_{-2}) - Z(\zeta_2) - 2\left[Z(\zeta_{-1}) - Z(\zeta_1) \right] \right]$$

$$U_{22} = -\frac{c^2}{\omega^2} + \sum_{\text{Species}} \frac{\omega_p^2}{\omega k_z V_T} \left(\frac{V_T}{2\omega_c}\right)^2 \left[Z(\zeta_{-2}) + Z(\zeta_2) - 3\left[Z(\zeta_{-1}) + Z(\zeta_1) \right] + 4Z(\zeta_0) \right] \tag{3.5.2}$$

$$V_{11} = U_{11}' \quad V_{12} = U_{12}' + \text{corr} \quad V_{21} = U_{21}' + \text{corr} \quad V_{22} = U_{22}'$$

$$\text{corr} = i \sum_{\text{Species}} \frac{\omega_p^2}{\omega k_z V_T} \frac{V_T}{2\omega_c} \left(\frac{V_T}{2\omega_c} \left[Z(\zeta_{-1}) - Z(\zeta_1) \right] \right)' \tag{3.5.3}$$

$$W_{11} = W_{22} = \frac{c^2 k_z^2}{\omega^2} - 1 - \frac{1}{2} \sum_{\text{Species}} \frac{\omega_p^2}{\omega k_z V_T} \left[Z(\zeta_{-1}) + Z(\zeta_1) \right]$$

$$W_{12} = -W_{21} = \frac{i}{2} \sum_{\text{Species}} \frac{\omega_p^2}{\omega k_z V_T} \left[Z(\zeta_1) - Z(\zeta_{-1}) \right] \tag{3.5.4}$$

3.6. Comparison of Equations

Note that if the approximation

$$A_{-1} = \frac{V_\perp}{2i(\omega-\omega_c)} \qquad (3.6.1)$$

is made, and only the $(A_{-1}-A_{-2}-A_{-1})$ contribution to the coefficient of the second derivative of $\underline{E}$ is considered, then the resulting expression is simply that obtained by Swanson (1981). Using our more systematic approach it can be seen that the second order pole from $(A_{-1}-A_{-2}-A_{-1})$ cancels exactly with that from $(A_{-1}-A_{0}-A_{-1})$ and so the same set of equations can be used with impunity in the vicinity of the fundamental resonance.

Similar equations have been obtained by Romero and Scharer (1987). However, the restriction on $k_z L$ is not identified in their paper and as a result the equations are used not only for k_z values for which they are valid but also for k_z values where they would not appear to be justified.

The dominant terms in equation (3.5.1) have also been obtained from a variational technique (Colestock and Kashuba,1983), but again the restriction on $k_z L$ is not identified. In this case the critical point can be seen, in retrospect, to be their assumption that the elements of $\underline{\sigma}(\underline{k},\underline{\tilde{k}})$ are only first order in $\tilde{k}$. This is equivalent to the assumption that all quantities are slowly varying and therefore requires, as above, that $|k_z L| >> |n|$. Again, as in the paper by Romero and Scharer, this failure leads to the use of the equations for $|k_z L| < 1$ where their use is not justified.

The coefficients of the even order derivatives are simply the homogeneous terms, c.f. Stix (1962).

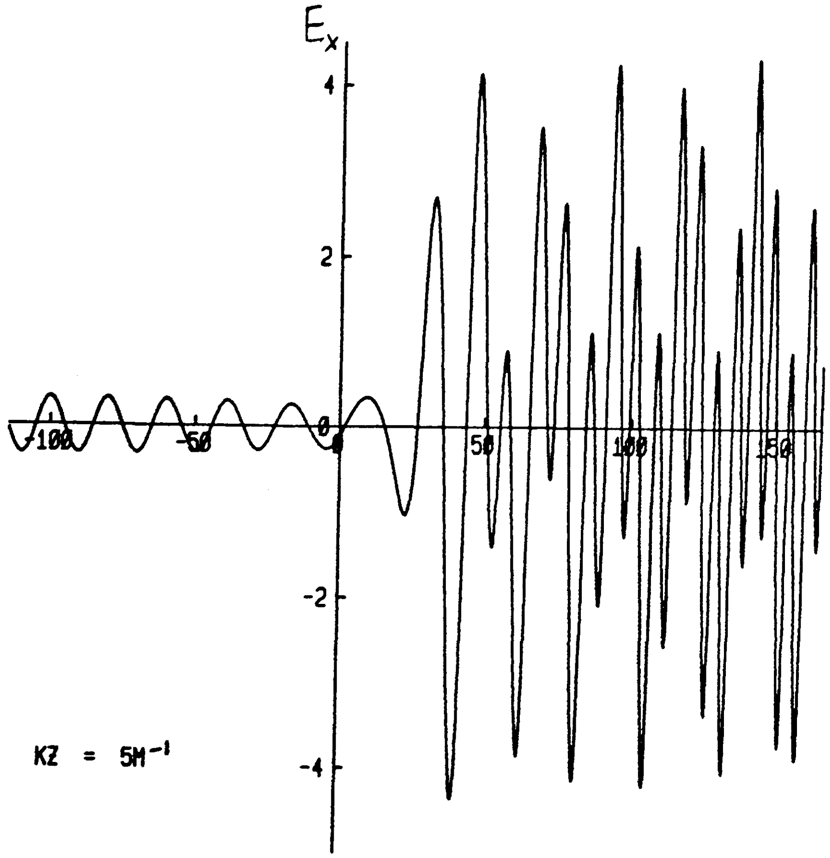
3.7. Comparison of Results

These equations can be readily solved numerically as a complete set of linearly independent boundary value problems. On this occasion a finite diference scheme (Nag library routine D02GBF) based on PASVA3 (Pereyra, 1979) was used. The equations were integrated over a region extending from well below the resonance where the eigenvalues of the modes were well separated, to far enough above the resonance for the eigenvalues to be well separated once again. Over this region the fast mode's eigenvalue changes only a little, so the integration is performed from where the ion Bernstein mode has a large (and real) eigenvalue to where it has a large (and imaginary) eigenvalue. (Where large in this context means relative to the fast mode's eigenvalue.) The integration is stopped before the

Bernstein eigenvalue becomes large relative to ω_c/v_T where the perturbed electric field would no longer be slowly varying. Since the Bernstein mode has a much shorter wavelength everywhere except in the mode conversion region, it has a much slower phase velocity, and so is often referred to as the slow mode. Then a linear combination of the solutions is formed, corresponding to a pure fast wave leaving on the low field side, yielding the results shown in figures 3.2 - 3.4. The graphs are of the electric fields and have their horizontal scale normalized to the ion Larmor radius corresponding to the magnetic field at the origin, the origin being the position of the harmonic resonance. The vertical normalisation is to the amplitude of the incident fast wave. These graphs should be compared with those obtained by setting the explicit parameter gradient terms to zero. (figures 3.5 - 3.7).

3.8. Conclusions

The difference in the amount of mode conversion arising from the two different sets of equations is far more obvious in the graphs of E_x than in those of E_y because the mode converted wave is almost purely electrostatic. As can be seen from these graphs and



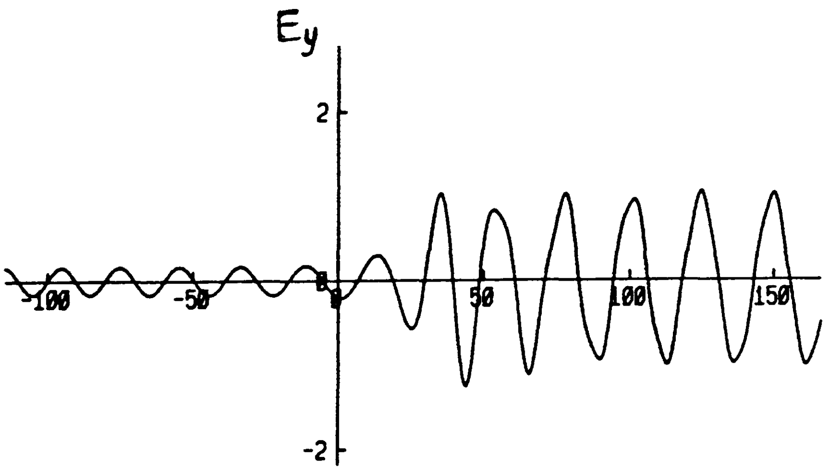


Figure 3.2 E_x and E_y plotted from 0.55m below the resonance to 0.8m above it. $B_o(0)=3T$, $L=4m$, $T_e=T_i=5kev$ and $n_e=n_i=10^{20}\,m^{-3}$. Explicit parameter gradient terms included.

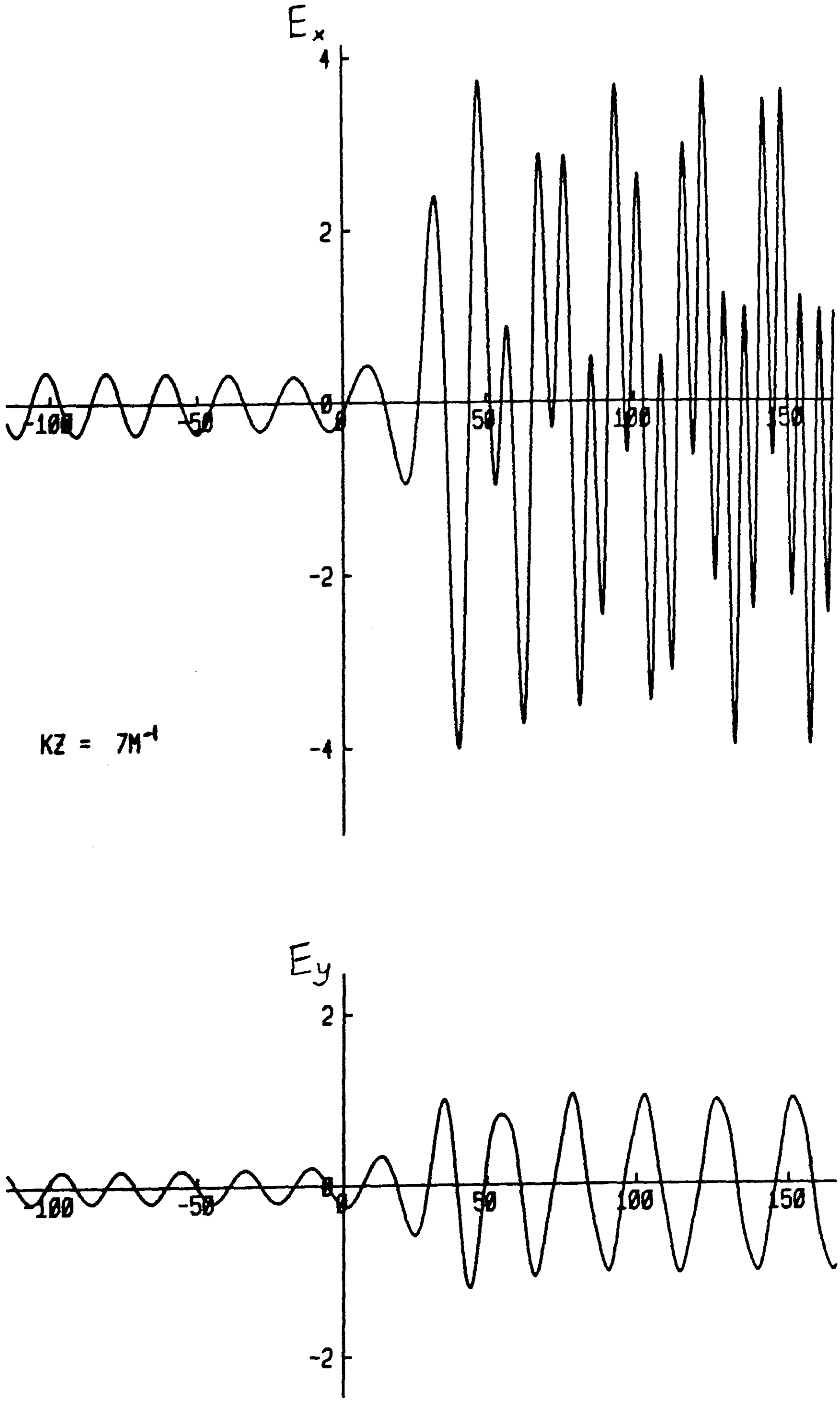


Figure 3.3 E_x and E_y plotted from 0.55m below the resonance to 0.8m above it. $B_o(0)=3T$, $L=4m$, $T_e=T_i=5kev$ and $n_e=n_i=10^{20}\ m^{-3}$. Explicit parameter gradient terms included.

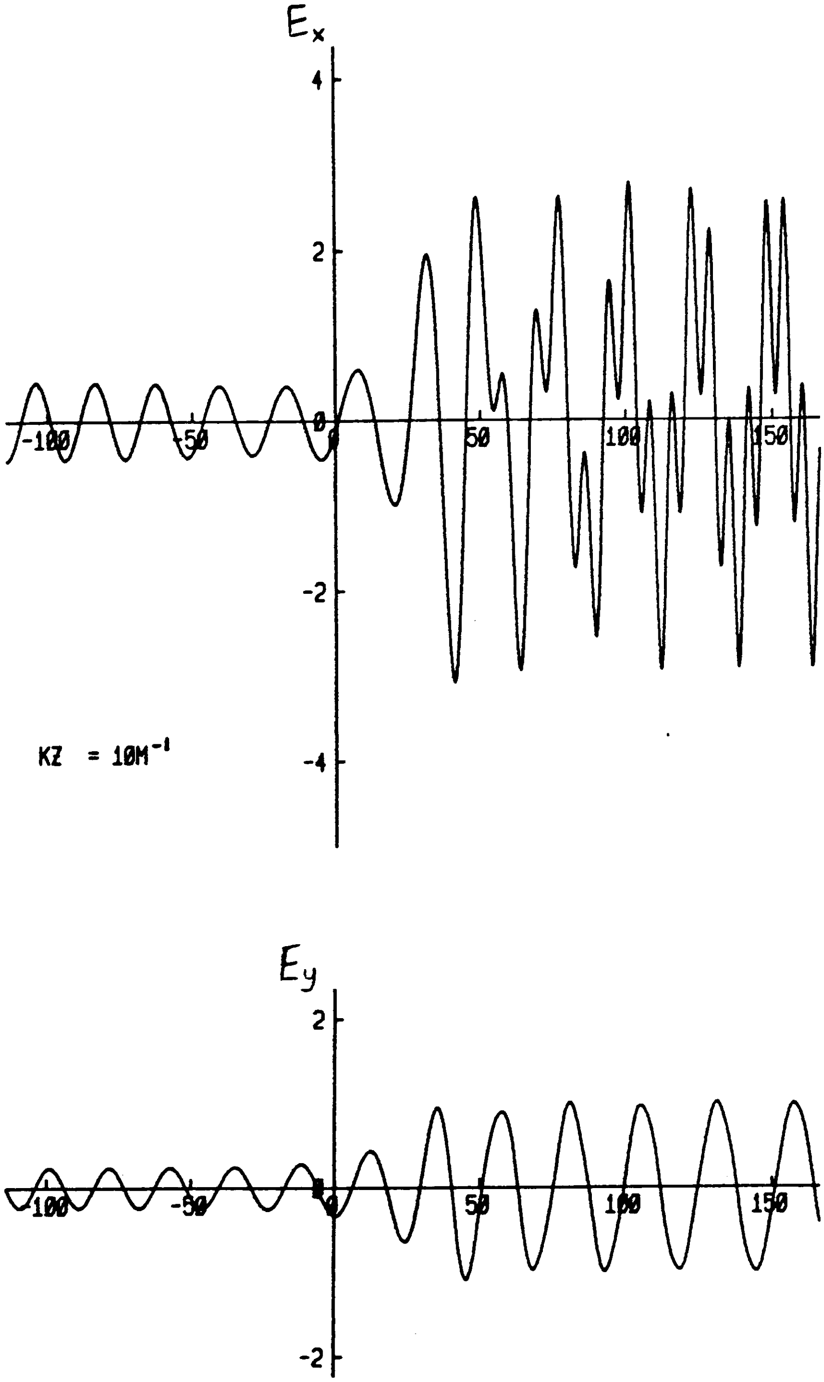


Figure 3.4 E_x and E_y plotted from 0.55m below the resonance to 0.8m above it. $B_o(0)=3T$, L=4m, $T_e=T_i=5$kev and $n_e=n_i=10^{20}$ m^{-3}.
Explicit parameter gradient terms included.

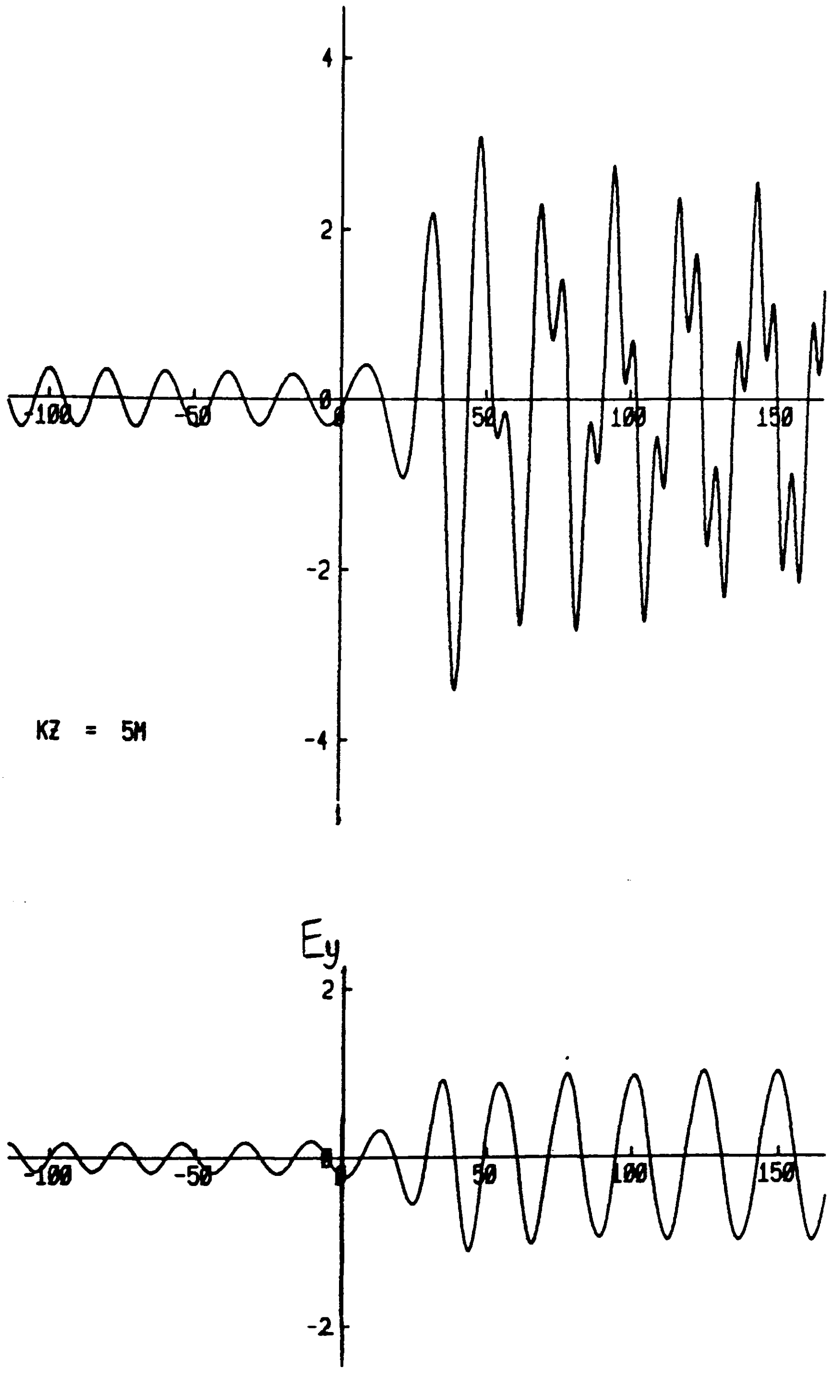


Figure 3.5 E_x and E_y plotted from 0.55m below the resonance to 0.8m above it. $B_o(0)=3T$, $L=4m$, $T_e=T_i=5kev$ and $n_e=n_i=10^{20}\ m^{-3}$. Explicit parameter gradient terms excluded.

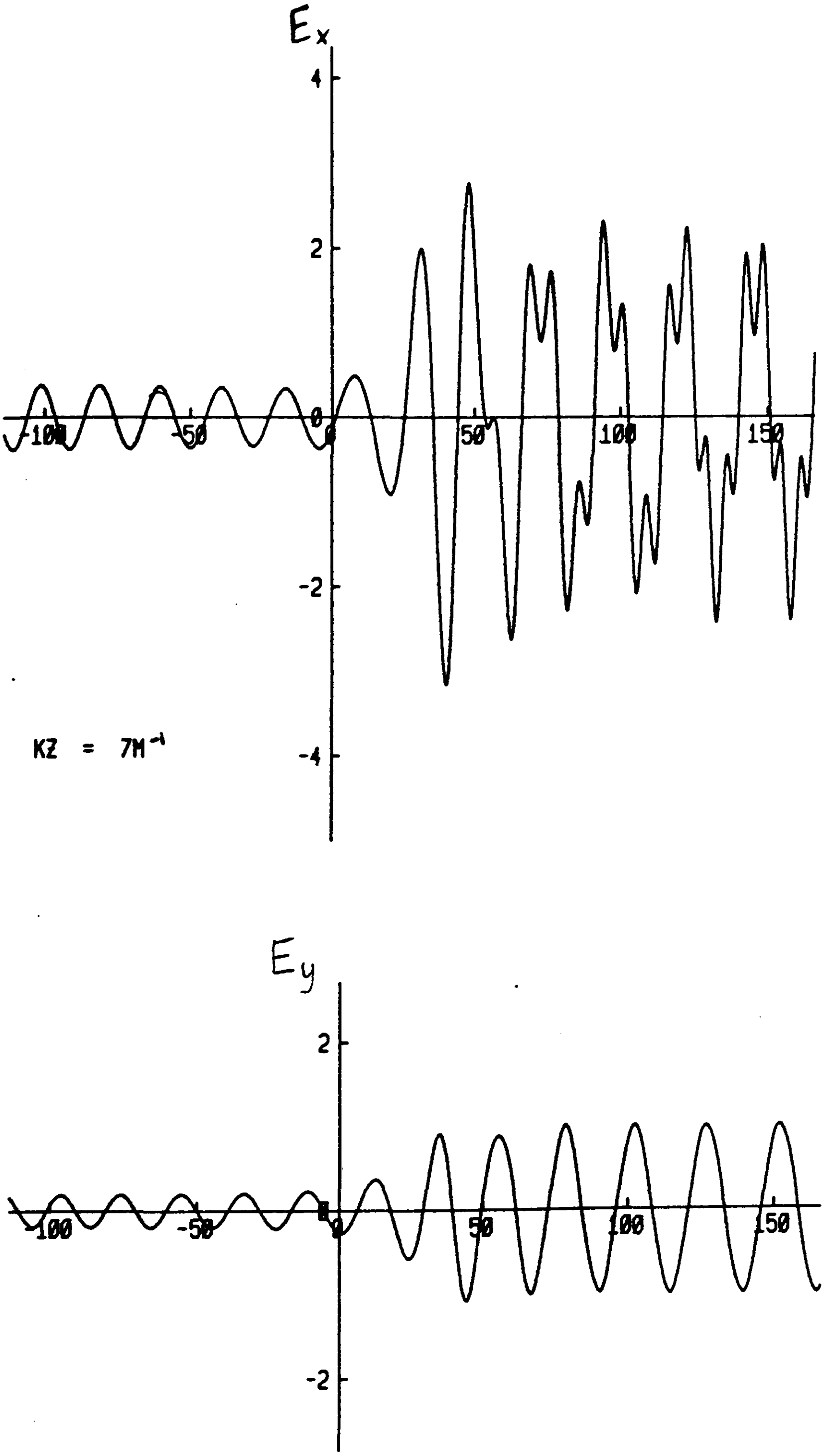


Figure 3.6 E_x and E_y plotted from 0.55m below the resonance to 0.8m above it. $B_o(0)=3T$, $L=4m$, $T_e=T_i=5kev$ and $n_e=n_i=10^{20}\ m^{-3}$.
Explicit parameter gradient terms excluded.

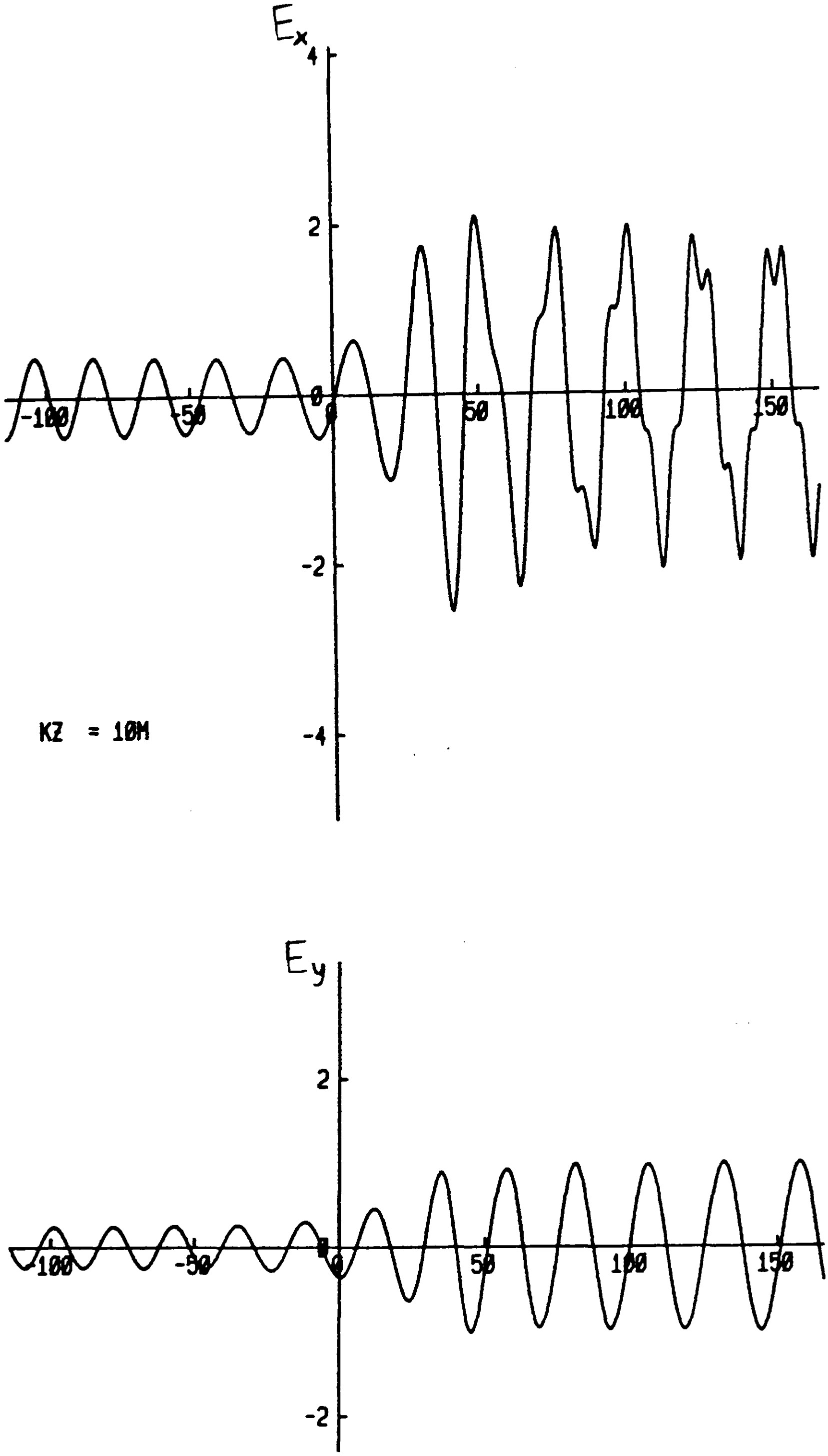


Figure 3.7 E_x and E_y plotted from 0.55m below the resonance to 0.8m above it. $B_o(0)=3T$, L=4m, $T_e=T_i=5$kev and $n_e=n_i=10^{20}m^{-3}$.

Explicit parameter gradient terms excluded.

Table 1 there is considerably more mode conversion when explicit parameter gradient terms are included.

Table 1.

	E_y amplitudes						
	without explicit B_0' terms				with explicit B_0' terms		
K	%	$\lvert T\rvert$	$\lvert R\rvert$	$\lvert Mc\rvert$	$\lvert T\rvert$	$\lvert R\rvert$	$\lvert Mc\rvert$
10		24·2	0·2	0·6	23·5	0·1	2
7		19	0·2	1·3	18·4	0·1	4·1
5		16·7	0·1	1·6	16·2	0·2	4·9

It may seem rather strange that even when the parameter gradient terms are relatively small they can still make a considerable difference to the amount of mode conversion that occurs; however this should not be too surprising since the same phenomenon is already implied by the different properties of the two modes.

Note that in the homogeneous case the spatial evolution of E_x and E_y is described by the same dispersion relation (2.3.20) whether written as a polynomial in k_x^2 or replacing each k_x^2 with $-\frac{d^2}{dx^2}$.

$$\left(a_1 - \frac{\beta_1}{k_x^2}\frac{d^2}{dx^2}\right)\left(a_1 - \left(\frac{\beta_2}{k_x^2} - \frac{c^2}{\omega^2}\right)\frac{d^2}{dx^2}\right)\Psi - \left(a_2 - \frac{\beta_3}{k_x^2}\frac{d^2}{dx^2}\right)^2\Psi = 0 \qquad (3.8.1)$$

In the case of the fast mode which is a mixed electromagnetic and electrostatic mode, $|E_x|$ and $|E_y|$ are comparable. Since the slow mode is almost purely electrostatic, it has $|E_x|$ much greater than $|E_y|$.

Therefore proceeding from a region where only the fast mode is excited to a region where both modes are excited, E_x and E_y must evolve differently, since E_x must excite far more of the short wavelength solution than E_y does. Thus to be consistent with the known properties of the two different modes, the differential equation describing E_x must give rise to far more 'mode conversion' than the equation describing E_y.

The difference between the differential equations describing E_x and E_y is purely due to the parameter gradient terms; in the absence of such terms both E_x and E_y are described by exactly the same dispersion relation-based equation (3.8.1). It should also be pointed out that these parameter gradient terms can be made as small as desired (in the case of fixed non-zero k_z) simply by increasing the scale lengths over which the parameters vary. Yet no matter how great the scale length, the ratio of mode conversion experienced by E_x to that experienced by E_y must remain constant and large. It is also noteworthy that transmission and reflection coefficients must be very similar for E_x and E_y, since the fast wave does not radically change its mix of transverse and longitudinal electric fields.

The ratio of E_x/E_y can be shown to depend on the value of 'k_x'. Using the notation of (2.3.20), and ignoring the non-resonant thermal corrections for clarity, the constant B_0 equations are

$$\begin{pmatrix} a_1 + \beta k_x^2 & i(a_2 + \beta k_x^2) \\ -i(a_2 + \beta k_x^2) & a_1 + \left(\beta - \frac{c^2}{\omega^2}\right) k_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \underline{0} \tag{3.8.2}$$

where

$$\beta = \frac{\omega_{p_i}^2}{\omega k_z V_{T_i}} \left(\frac{V_{T_i}}{2\omega_{c_i}}\right)^2 Z(\zeta_{-2}) \tag{3.8.3}$$

From equation (3.8.2), the ratio of E_x/E_y is

$$\frac{E_x}{E_y} = i\left[\frac{c^2 k_x^2}{\omega^2(a_1 - a_2)} - 1\right] \tag{3.8.4}$$

The fast wave's nearly constant polarisation can therefore be traced back to its quasi-constant wavenumber; whereas the increasingly electrostatic nature of the Bernstein mode is due to its steadily increasing wavenumber.

If at this stage k_x^2 is replaced by $-\frac{d^2}{dx^2}$, the equations formed will retain implicit parameter gradient terms, although explicit ones will not be included. These equations must reproduce the electrostatic nature of the Bernstein mode, although the actual amount of mode conversion will not be accurate. Applying this technique to equation (3.8.2) gives the following set of coupled o.d.e's.

$$\left(a_1 E_x - \beta E_x''\right) + i\left(a_2 E_y - \beta E_y''\right) = 0$$

$$-i\left(a_2 E_x - \beta E_x''\right) + \left(a_1 E_y - \left(\beta - \frac{c^2}{\omega^2}\right) E_y''\right) = 0$$

(3.8.5)

If E_x is eliminated from (3.8.5) then the differential equation for E_y is obtained.

$$\frac{c^2}{\omega^2}\beta(a_1-a_2)\frac{d^2}{dx^2}\left(\frac{E_y''}{a_1-a_2}\right) + \left[2\beta(a_1-a_2) - a_1\frac{c^2}{\omega^2}\right]E_y''$$

$$-(a_1^2-a_2^2)E_y = 0$$

(3.8.6)

If instead E_y is eliminated, the equation for E_x is obtained.

$$\frac{c^2}{\omega^2}\left[\beta\left(\beta-\frac{c^2}{\omega^2}\right) - a_1\beta\right]\frac{d^2}{dx^2}\left[\frac{\beta E_x''}{a_2\left(\beta-\frac{c^2}{\omega^2}\right) - a_1\beta}\right] +$$

$$\left[a_2\left(\beta-\frac{c^2}{\omega^2}\right) - a_1\beta\right]\frac{d^2}{dx^2}\left[\frac{\left(a_1\left(\beta-\frac{c^2}{\omega^2}\right) - a_2\beta\right)E_x}{a_2\left(\beta-\frac{c^2}{\omega^2}\right) - a_1\beta}\right] +$$

$$(a_1-a_2)\beta E_x'' - (a_1^2-a_2^2)E_x = 0$$

(3.8.7)

The huge difference in the behaviour of the E_x and E solutions can therefore be seen to be due to implicit gradient terms proportional to

$$\frac{d}{dx}\beta$$

(3.8.8)

which is the only significant difference between the two equations in the vicinity of the resonance. Since it is precisely terms of the form of (3.8.8) that are ignored by reverse Fourier transform techniques, these techniques cannot accurately describe mode conversion.

An intuitively reasonable interpretation of these results is that for a mode which undergoes a significant change in wavenumber during its propagation, the explicit parameter gradient terms are required for accurate calculation of wave amplitudes. Whereas, for modes that only undergo small changes in 'k_x' the parameter gradient terms are unimportant. It is worth emphasising that it is the cumulative change in wavenumber that (if this interpretation is valid) indicates whether or not the cumulative effect of parameter gradient terms are significant.

What is certain is that it is perfectly possible for parameter gradient terms to cause major differences in mode conversion factors without dramatically affecting transmission and reflection coefficients.

Chapter 4

Large Larmor Orbit Effects.

4.1. Introduction

In Chapter 3 it was established that it was possible to obtain, self consistently, a system of coupled o.d.e's which govern the spatial evolution of an electromagnetic wave propagating through a plasma in an inhomogeneous magnetic field. However, in order to obtain the equations several rather stringent restrictions were made. These restrictions concerned the form of the unperturbed plasma distribution function, the magnetic field profile, and the wavelengths and direction of propagation of the perturbation applied.

Many of the restrictions imposed had their roots in the general application of the small Larmor orbit expansion. In order to allow the wider use of the techniques of Chapter 3, it is desirable to relax these constraints wherever possible. In this chapter the constraint on the range of wavenumbers in the direction of the magnetic field gradient will be relaxed, allowing shorter wavelength modes to be modelled and also permitting the effects of additional more energetic species, with their correspondingly greater Larmor radii, to be incorporated. The latter consideration being particularly relevant to the modelling of R.F. heating effects in fusion plasmas, where significant numbers of highly energetic alpha particles will be created.

Returning to the analogy of the homogeneous case (Chapter 2) it will be remembered that the restriction to a spatially slowly varying electric field was only necessary to force rapid convergence of the series expansions of the products of Bessel functions of $k_x v_T / 2\omega_c$ and that the same series expansions will eventually converge, no matter how rapidly varying the electric field is.

It would seem reasonable to carry this analogy further and investigate the possibility of a similar convergence of terms in the differential equation obtained in the case of an inhomogeneous equilibrium magnetic field.

4.2. The Infinite Tree

Retaining all of the constraints of Chapter 3 except that of requiring $\underline{E}$ to be slowly varying, it can be seen that still only the -1,0 and 1 Fourier components contribute to the electric current, and only the -1,0 and 1 components are directly driven by the perturbing electric field. Therefore, the tree diagram approach is still valid in principle. Referring back to the tree diagram (figure 3.1), and bearing in mind the objective of investigating the possible convergence of the coefficients of the higher order derivatives of the electric field, the requirement for an expression for the contribution to f_l from a general parallelogram region of the tree diagram becomes very obvious.

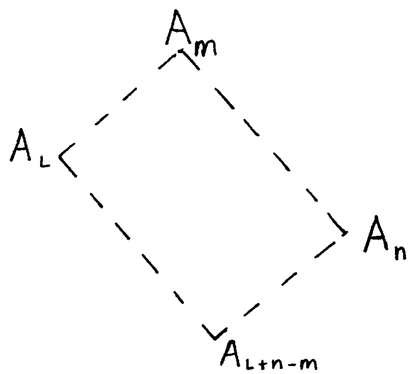


Fig. 4.1

Fortunately such an expression can be obtained. The parameter gradient free contribution from a general parallelogram element of the mesh (figure 4.1) can be shown (appendix A) to be

$$S^m_{ln}\left(\frac{d}{dx}\right)^{2m-l-n} \tag{4.2.1}$$

where

$$S^m_{ln} = \sum_{j=n+l-m}^{m} \frac{P^m_{ln}(j)\, A_j}{(m-l)!\,(m-n)!}\left(\frac{l V_\perp}{2\omega_c}\right)^{2m-l-n} \tag{4.2.2}$$

$$P_{Ln}^{m}(j) = \binom{2m-L-n}{m-j} (-1)^{m-j} \tag{4.2.3}$$

The first order parameter gradient terms from figure 4.1, i.e. the contribution to f from figure 4.1 of terms that are first order in the equilibrium magnetic field gradient, can be shown, using the same induction technique (appendix B), to be

$$T_{Ln}^{m} \left(\frac{d}{dx}\right)^{2m-L-n-1} \tag{4.2.4}$$

where

$$T_{Ln}^{m} = \sum_{j=n+L-m}^{m} \Bigg\{ \left[\frac{2m-L-n}{2} P_{Ln}^{m}(j) + \frac{n-L}{2} H_{Ln}^{m}(j) \right] A_j' \left(\frac{iV_\perp}{2\omega_c}\right) +$$

$$\left[(n(n-L) + 2(m-L)(m-n)) P_{Ln}^{m}(j) + \frac{(n-L)(2m-L-n-1)}{2} H_{Ln}^{m}(j) \right] A_j \left(\frac{iV_\perp}{2\omega_c}\right)' \Bigg\}$$

$$\frac{1}{(m-L)!\,(m-n)!} \left(\frac{iV_\perp}{2\omega_c}\right)^{2m-L-n-1} \tag{4.2.5}$$

$$H_{Ln}^{m}(j) = \frac{2j-L-n}{2m-L-n} P_{Ln}^{m}(j) \tag{4.2.6}$$

Using the above expressions the contribution to f_{11}, f_{10} and f_{1-1} of the derivatives of the 'driving' terms can be calculated.

$$f_{11} = \sum_{m=1}^{\infty} G_{11}^{m} D_1 + G_{10}^{m} D_0 + G_{1-1}^{m} D_{-1}$$

$$f_{10} = \sum_{m=0}^{\infty} G_{01}^{m+1} D_1 + G_{00}^{m} D_0 + G_{0-1}^{m} D_{-1}$$

$$f_{1-1} = \sum_{m=0}^{\infty} G_{-11}^{m+1} D_1 + G_{-10}^{m} D_0 + G_{-1-1}^{m-1} D_{-1} \tag{4.2.7}$$

where

$$D_1 = \frac{2q}{mv_\perp} P E_- \qquad D_0 = \frac{2q}{mv_\perp} R E_z$$

$$D_{-1} = \frac{2q}{mv_\perp} P E_+ \qquad (4.2.8)$$

and G^m_{ln}, the parallelogram operator (fig 4.1) is (to first order in $\frac{dB_0}{dx}$)

$$G^m_{ln} = S^m_{ln}\left(\frac{d}{dx}\right)^{2m-l-n} + T^m_{ln}\left(\frac{d}{dx}\right)^{2m-l-n-1} \qquad (4.2.9)$$

4.3. The Conductivity Tensor

Taking the first moments of the perturbed distribution function, the general expression for the conductivity tensor in a spatially dependent equilibrium magnetic field is found to be (to first order in magnetic field gradient)

$$\underline{\underline{\sigma}} = \sum_{Species} \frac{4\pi q^2}{m} \int_0^\infty dv_\perp \int_{-\infty}^{\infty} dv_z\ \underline{\underline{V}} \cdot \sum_{m=0}^{\infty} \left(\underline{\underline{c}}(m) + \underline{\underline{d}}(m)\right) . \underline{\underline{F}}$$

(4.3.1)

$$\underline{\underline{c}}(m) = \begin{pmatrix} c_{11}(m)\left(\frac{d}{dx}\right)^{2m} & c_{12}(m)\left(\frac{d}{dx}\right)^{2m} & c_{13}(m)\left(\frac{d}{dx}\right)^{2m+1} \\ c_{21}(m)\left(\frac{d}{dx}\right)^{2m} & c_{22}(m)\left(\frac{d}{dx}\right)^{2m} & c_{23}(m)\left(\frac{d}{dx}\right)^{2m+1} \\ c_{31}(m)\left(\frac{d}{dx}\right)^{2m+1} & c_{32}(m)\left(\frac{d}{dx}\right)^{2m+1} & c_{33}(m)\left(\frac{d}{dx}\right)^{2m} \end{pmatrix} \qquad (4.3.2)$$

$$\underline{\underline{d}}(m) = \begin{pmatrix} d_{11}(m)\left(\frac{d}{dx}\right)^{2m-1} & d_{12}(m)\left(\frac{d}{dx}\right)^{2m-1} & d_{13}(m)\left(\frac{d}{dx}\right)^{2m} \\ d_{21}(m)\left(\frac{d}{dx}\right)^{2m-1} & d_{22}(m)\left(\frac{d}{dx}\right)^{2m-1} & d_{23}(m)\left(\frac{d}{dx}\right)^{2m} \\ d_{31}(m)\left(\frac{d}{dx}\right)^{2m} & d_{32}(m)\left(\frac{d}{dx}\right)^{2m} & d_{33}(m)\left(\frac{d}{dx}\right)^{2m-1} \end{pmatrix} \tag{4.3.3}$$

$$F = \mathrm{diag}(P,P,R) \qquad V = \mathrm{diag}(v_\perp, v_\perp, v_z) \tag{4.3.4}$$

where

$$c_{11}(m) = \left(S_{11}^{m+1} + S_{1-1}^{m} + S_{-11}^{m} + S_{-1-1}^{m-1}\right)/4$$

$$c_{12}(m) = -c_{21}(m) = i\left(S_{-1-1}^{m-1} - S_{11}^{m+1}\right)/4$$

$$c_{13}(m) = c_{31}(m) = \left(S_{10}^{m+1} + S_{-10}^{m}\right)/2$$

$$c_{22}(m) = \left(S_{11}^{m+1} - S_{1-1}^{m} - S_{-11}^{m} + S_{-1-1}^{m-1}\right)/4$$

$$c_{23}(m) = -c_{32}(m) = i\left(S_{10}^{m+1} - S_{-10}^{m}\right)/2 \qquad c_{33}(m) = S_{00}^{m} \tag{4.3.5}$$

$$d_{11}(m) = m\,c_{11}'(m) \qquad d_{12}(m) = m\,c_{12}'(m) + ib_m$$

$$d_{13}(m) = m\,c_{13}'(m) \qquad d_{21}(m) = m\,c_{21}'(m) + ib_m$$

$$d_{22}(m) = m\,c_{22}'(m) \qquad d_{23}(m) = m\,c_{23}'(m) + iA_0 b_m$$

$$d_{31}(m) = (m+1)\,c_{31}'(m) \qquad d_{32}(m) = (m+1)\,c_{32}'(m) + iA_0 b_m$$

$$d_{33}(m) = m\,c_{33}'(m) \tag{4.3.6}$$

$$S_{LL}^{m+L} = \frac{1}{(m!)^2}\left(\frac{v_\perp}{2w_c}\right)^{2m} \sum_{n=-m}^{m} \binom{2m}{m+n} (-1)^n A_{n+L} \quad \text{(figure 4.2)}$$

$$S^{m}_{1\,-1} = S^{m}_{-1\,1} = \frac{m}{m+1}\, S^{m}_{00} \qquad \text{(figure 4.3)}$$

$$b_m = \frac{-2}{m!\,(m+1)!}\,\frac{V_\perp}{2\omega_c} \sum_{n=-m}^{m} \binom{2m}{m+n} (-1)^n\, n \left[\left(\frac{V_\perp}{2\omega_c} \right)^{2m-1} A_n \right]' \qquad \text{(figure 4.4)}$$

In order to clarify the methods used to obtain the elements of the conductivity tensor, the derivation of one particular element will be examined in detail.

The 3,3 element of the conductivity tensor represents the dependence of J_z on E_z and its derivatives. By considering the ϕ integral, it can be seen that J_z depends solely on f_{10}. Examining the f_{10} of the tree diagram (figure 3.1), it is clear that to lowest order in $\frac{d}{dx}$, the coefficient of E_z is $\frac{qA_0R}{m}$. Performing the velocity integrals gives

$$\sigma_{33} = i\varepsilon_0 \frac{\omega_p^2}{k_z V_T}\, Y(\zeta_0) \qquad (4.3.7)$$

which can be compared with (2.3.17)

$$Y(\zeta_n) = 2\zeta_n\left(\zeta_n Z(\zeta_n) - 1\right) \qquad (4.3.8)$$

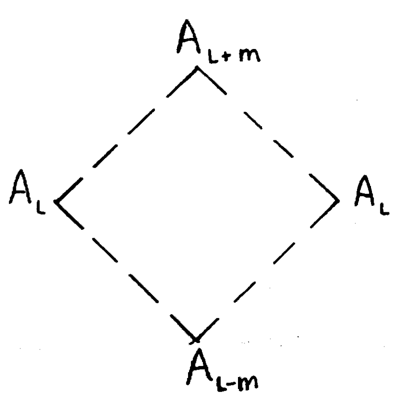


Fig. 4.2

The calculation of the derivative terms is particularly simple in this case, as all of them come from the diamond shaped elements - the G_{00}^{m}, figure 4.2, - of the tree diagram. For example the first diamond element G_{00}^{1} gives rise to the first thermal corrections to J_z

$$i\varepsilon_0 \frac{\omega_p^2}{k_z v_T} \frac{v_T^2}{4\omega_c^2} \left[Y(\zeta_1) - 2Y(\zeta_0) + Y(\zeta_{-1}) \right] \frac{d^2}{dx^2} \tag{4.3.9}$$

which can be compared with any standard textbook, and to the parameter gradient term

$$i\varepsilon_0 \frac{\omega_p^2}{k_z v_T} \left\{ \frac{v_T^2}{4\omega_c^2} \left(Y(\zeta_1) - 2Y(\zeta_0) + Y(\zeta_{-1}) \right) \right\}' \frac{d}{dx} \tag{4.3.10}$$

The first order corrections due to the equilibrium field gradient are easily calculated for diamond shaped elements of the tree diagram by symmetry arguments.

Consider a path through diamond from left to right (and if the route is not symmetric about the vertical diagonal, its image on reflection in the vertical diagonal). If m is the length of the sides of the diamond, then for any element A_n on this path 1 steps from the vertical diagonal, there are (m-1) differential operators acting on it, and (m+1) differential operators acting on its mirror image. Therefore, the $2A_n$ of the zeroth order expression, each multiplied by the same path elements, give rise to $2mA_n'$ in the first order correction. This process can be repeated for every element in the diamond, with the result that the first order correction to the diamond is simply m times the derivative of the zeroth term.

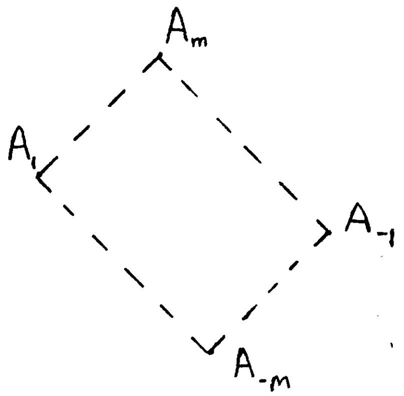


Fig. 4.3

A similar simplification occurs for the sum of G^m_{1-1} and G^m_{-11}. Since the sum of these two elements is symmetric about the vertical bisector, the first order correction is again m times the derivative of the zeroth order term.

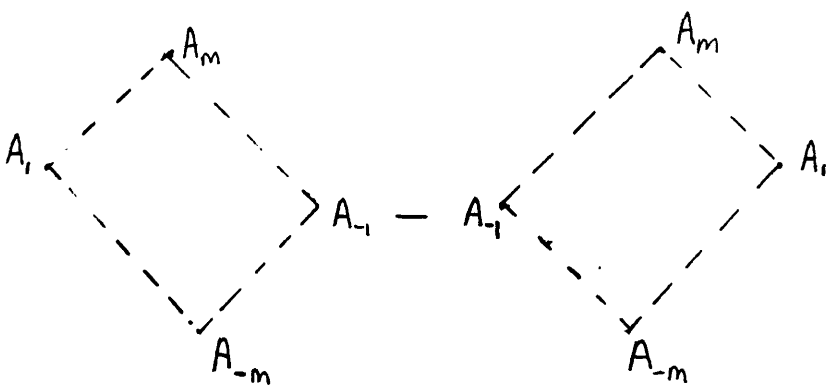


Fig. 4.4

Clearly, such a symmetry does not exist in the case of the difference between G^m_{1-1} and G^m_{-11}. This asymmetry gives rise to the b_m term, whereas to zeroth order in parameter gradients, the difference between G^m_{1-1} and G^m_{-11} is zero. In the case of homogeneous B_o, this cancellation causes $\sigma_{12} = -\sigma_{21}$. In the inhomogeneous case, the b_m terms break this symmetry.

Note, if the A_n were not all slowly varying, i.e. if in the plasma being considered there existed an n for which

$$\left| \frac{v_T}{2\omega_c} \frac{d}{dx} Z(\zeta_n) \right| << \left| Z(\zeta_n) \right| \tag{4.3.11}$$

was not satisfied, then restricting attention to only 1st order terms in B_o' would no longer be justified. Including higher order derivatives of the A_n would mean that, in addition to producing additional derivative terms, the parameter gradients would alter existing coefficients, dramatically complicating the algebra.

The importance of the restriction to slowly varying A_n, which in turn requires the restrictions (3.4.4) and (3.4.6) can now be appreciated, although the requirement $|k_z L| >> |n|$ has not previously been realised in the literature.

Returning to the specific case of J_z, it can be seen that S^m_{oo}, the coefficient of the 2m th derivative of E_z, is calculated from the diamond shaped element of side m, G^m_{oo} (figure 4.2), of the tree diagram while the coefficient of the (2m-1)th derivative of E_z, calculated from the 1st order parameter gradient terms of the same element G^m_{oo}, is simply m times the derivative of S^m_{oo}.

4.4. Comparison With Homogeneous Case

Comparison of $\underline{\underline{\sigma}}$ with textbook treatments of the homogeneous case is facilitated by first using the following identities to tidy up the gradient independent terms (the c_{ij}).

$$S^{m+1}_{11} + S^{m+1}_{-1-1} + S^m_{1-1} + S^m_{-11} = \frac{-1}{[(m+1)!]^2}\left(\frac{V_\perp}{2\omega_c}\right)^{2m} \sum_{n=-m-1}^{m+1} \binom{2(m+1)}{m+1+n} (-1)^n n^2 A_n \tag{4.4.1}$$

$$i\left(S^{m-1}_{-1-1} - S^{m+1}_{11}\right) = \frac{i}{m!\,(m+1)!}\left(\frac{V_\perp}{2\omega_c}\right)^{2m} \sum_{n=-m-1}^{m+1} \binom{2(m+1)}{m+1+n} (-1)^n n A_n \tag{4.4.2}$$

$$S^{m+1}_{10} + S^m_{-10} = \frac{i}{[(m+1)!]^2}\left(\frac{V_\perp}{2\omega_c}\right)^{2m+1} \sum_{n=-m-1}^{m+1} \binom{2(m+1)}{m+1+n} (-1)^n n A_n \tag{4.4.3}$$

$$i\left(S^{m+1}_{10} - S^m_{-10}\right) = \frac{1}{m!\,(m+1)!}\left(\frac{V_\perp}{2\omega_c}\right)^{2m+1} \sum_{n=-m-1}^{m+1} \binom{2(m+1)}{m+1+n} (-1)^n A_n \tag{4.4.4}$$

The second step is to compare these expressions with the series expansions of the products of Bessel functions in the corresponding terms for the homogeneous case obtained in Chapter 2. Before this comparison of the coefficients of k_x with those of $-i\frac{d}{dx}$ can usefully be made the following identity

$$\sum_{s=0}^{m-n} \frac{(-1)^{m-n}}{s!\,(m-n-s)!\,(r-s)!\,(m+n+s-r)!} = \frac{(-1)^{m-n}}{(m-n)!\,(m+n)!} \binom{2m}{r} \tag{4.4.5}$$

(appendix D) is used to order the homogeneous series in powers of k_x^2.

Perhaps the simplest approach is to first consider the homogeneous expression for .

$$J_n^2(\lambda) = \left(\frac{\lambda}{2}\right)^{2n} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{s+t}}{s!\,(n+s)!\,t!\,(n+t)!} \left(\frac{\lambda}{2}\right)^{2(s+t)}$$

$$\text{coef } \left(\frac{\lambda}{2}\right)^{2m} = \sum_{s=0}^{m-n} \frac{(-1)^{m-n}}{s!\,(n+s)!\,(m-n-s)!\,(m-s)!} \tag{4.4.6}$$

Which, with

$$\lambda = \frac{k_x V_T}{\omega_c} \tag{4.4.7}$$

and using (4.4.5) with r=m, gives the coefficient of $A_n k_x^{2m}$ as

$$\frac{(-1)^{m-n}\,(2m)!}{(m-n)!\,(m+n)!\,[m!]^2} \left(\frac{V_T}{2\omega_c}\right)^{2m} \tag{4.4.8}$$

Then the homogeneous results of (2.3.11) corresponding to (4.4.1) - (4.4.4) can be obtained easily.

$$\sum_n \frac{n^2 J_n^2(\lambda)\, A_n}{\lambda^2}$$

(4.4.9)

$$i\sum_n \frac{nJ_n(\lambda)}{\lambda}\frac{dJ_n(\lambda)}{d\lambda}A_n$$

$$=\sum_m\sum_n \frac{(-1)^{m-n}}{m!(m+1)!}\binom{2(m+1)}{m+1+n}\left(\frac{V_T}{2\omega_c}\right)^{2m} nA_n \quad K_x^{2m} \tag{4.4.10}$$

$$\sum_n \frac{nJ_n^2(\lambda)A_n}{\lambda}$$

$$=\sum_m\sum_n \frac{(-1)^{m-n}}{[(m+1)!]^2}\binom{2(m+1)}{m+1+n}\left(\frac{V_T}{2\omega_c}\right)^{2m+1} nA_n \quad K_x^{2m+1} \tag{4.4.11}$$

$$i\sum_n J_n(\lambda)\frac{d}{d\lambda}J_n(\lambda)$$

$$=i\sum_m\sum_n \frac{(-1)^{m-n}}{m!(m+1)!}\binom{2(m+1)}{m+1+n}\left(\frac{V_T}{2\omega_c}\right)^{2m+1} A_n \quad K_x^{2m+1} \tag{4.4.12}$$

It can then be seen that the c terms are exactly those that would be obtained by taking the inverse Fourier transform of the series solutions for the homogeneous case.

It should be noted that the factorials in the denominators of the coefficients of the derivatives guarantee convergence of the series for all but pathological cases.

The expression for can be incorporated with Maxwell's equations in the usual manner to give a system of coupled o.d.e's which govern the spatial evolution of the perturbing electric field.

$$\frac{c^2}{\omega^2}\nabla\times\nabla\times \underline{E} - \underline{E} - \frac{i\underline{\sigma}.\underline{E}}{\varepsilon_0\omega} = \underline{0} \tag{4.4.13}$$

4.5. Summary

This extension to the analysis of Chapter 3 has produced several benefits. First, all the results of the homogeneous case are recovered from this analysis. This is, in the opinion of the author, quite compelling evidence for the validity of the formalism first developed in Chapter 3. Second, in addition to showing that the gradient free coefficients of the differential equation converge like 1/n!, which is a consequence of the first benefit, it is also established that the gradient dependent terms similarly converge as 1/n!. This implies that for any physically reasonable electromagnetic perturbation of the plasma, provided that the restrictions on $k_z L$ and $L\omega_c/v_T$ are not violated, a description based on a set of 3 finite order differential equations can be formed without inconsistencies.

There is, however, one major problem raised by this extension to rapidly varying electromagnetic perturbations or equivalently to include species with much larger Larmor orbits. If the value of the differential operator, $\frac{v_T}{2\omega_c}\frac{d}{dx}$, when applied to the perturbing electric field is around 1 or 2, the order of the differential equations is only increased to 4 or 6, but larger values of the operator rapidly lead to much higher order differential equations. The practical considerations of numerically solving such large systems of equations make this direct differential equation approach very unattractive. In Chapter 8 (future work) a possible approach to circumventing this problem is discussed.

Chapter 5

Anisotropic Velocity Distributions

5.1. Introduction

So far, in this thesis, only the simplest possible equilibrium distribution function has been considered. The restriction to a completely isotropic f_o in Chapter 3 had the consequence of nullifying the effects of the perturbed magnetic field in the Vlasov equation, while the restriction to spatially invariant f_o allowed f_o to commute with the differential operator, with a corresponding simplification of the coefficients of the derivatives of the electric field.

A more detailed examination of the terms involved shows that these restrictions are far from vital, and that generalising the theory to include spatially dependent plasmas with anisotropic velocity distributions can be done independently of the generalisations performed earlier in Chapter 4. In this chapter the analysis required in order to apply the formalism of Chapter 3 and Chapter 4 to equilibria that have only cylindrical symmetry in velocity space is performed, and a particular example of such an equilibrium is investigated.

5.2. Cylindrically Symmetric F

The motivation for extending the theory to include plasmas that have only cylindrical symmetry is due to the fact that most magnetic confinement devices have equilibria with their velocity distributions parallel to the magnetic field considerably different to their velocity distribution perpendicular to the magnetic field. Such distributions are often set up deliberately due to the need (for confinement devices like the tokamak) for the plasma itself to carry a current along the externally applied field. Another source of anisotropic velocity distributions is the use of neutral beam injection as a plasma heating mechanism in tokamaks such as J.E.T. The fast ion velocity distribution produced by neutral beam injection is biased by the original injection velocity.

The effect on the Vlasov equation of changing to a f_o that only has cylindrical symmetry is that now

$$\underline{v} \times \underline{B}_1 \cdot \frac{\partial f_o}{\partial \underline{v}} \neq 0 \qquad (5.2.1)$$

Therefore the first step in modifying the theory of the preceeding chapters to include this class of equilibria is to express the perturbing magnetic field ($\underline{B}_1$) as a function of the perturbing electric field and its derivatives. Fortunately this is very simple using Faraday's law..

$$\frac{\partial \underline{B}_1}{\partial t} = -\nabla \times \underline{E}_1 \tag{5.2.2}$$

Using the same Fourier transforms for z and t as were used in Chapter 3

$$\underline{B}_1 = \frac{1}{\omega}\left(-k_z E_y \,,\, k_z E_x + iE_z' \,,\, -iE_y'\right) \tag{5.2.3}$$

therefore the Lorentz force term (5.2.1) becomes

$$\frac{1}{\omega}\Big(\left(-iv_y E_y' - v_z k_z E_x - iv_z E_z'\right), \left(-v_z k_z E_y + iv_x E_y'\right),$$

$$\left(v_x k_z E_x + iv_x E_z' + v_y k_z E_y\right)\Big) \cdot \frac{\partial f_0}{\partial \underline{v}} \tag{5.2.4}$$

Noting that once again only f_{11}, f_{10} and f_{1-1} are directly driven by the perturbing electric field and its derivative, and that, as always, only these three components of f_1 contribute to the flow of electric current, the only alteration to the tree diagram (figure 3.1) is in the form of the driving terms.

$$D_1 = \frac{2q}{mv_\perp}\left[\left(P + \frac{k_z}{\omega}Q\right)E_- + \frac{i}{\omega}Q\frac{dE_z}{dx}\right]$$

$$D_0 = \frac{2q}{mv_\perp} R E_z$$

$$D_{-1} = \frac{2q}{mv_\perp}\left[\left(P + \frac{k_z}{\omega}Q\right)E_+ + \frac{i}{\omega}Q\frac{dE_z}{dx}\right] \tag{5.2.5}$$

where

$$Q = v_\perp R - v_z P \tag{5.2.6}$$

Replacing the old driving terms

$$D_1 = \frac{2q}{mv_\perp} P E_-$$

$$D_0 = \frac{2q}{mv_\perp} R E_z$$

$$D_{-1} = \frac{2q}{mv_\perp} P E_+ \tag{5.2.7}$$

with the new ones (5.2.5) can be seen to be equivalent to the mapping

$$P E_x \longrightarrow \left(P + \frac{k_z}{\omega} Q\right) E_x + \frac{i}{\omega} Q \frac{dE_z}{dx}$$

$$P E_y \longrightarrow \left(P + \frac{k_z}{\omega} Q\right) E_y$$

$$R E_z \longrightarrow R E_z$$

Thus, the alteration to equation (4.2.7) is conveniently confined to a modification of the matrix $\underline{\underline{F}}$, which now has the form

$$\begin{pmatrix} P + \frac{k_z}{\omega} Q & 0 & \frac{i}{\omega} Q \frac{d}{dx} \\ 0 & P + \frac{k_z}{\omega} Q & 0 \\ 0 & 0 & R \end{pmatrix} \tag{5.2.8}$$

which of course reduces to the original $\underline{\underline{F}}$ for fully isotropic f_o since in that case Q is zero.

As was mentioned in the introduction, one of the reasons for being interested in f (v ,v) is the need for tokamak plasmas to carry electric currents along the toroidal magnetic field. However, such currents cause a twist in the magnetic field, which now has a dependence on 2 spatial variables, both of these effects causing considerable complications in the theory, beside which the effects of cylindrical symmetry pale by comparison.

In an attempt to be more self consistent, a current free example, the equilibrium induced by co and counter neutral beam injection, is considered, as this does not of itself imply a current in the plasma.

5.3. Fast Ions

In order to model the effects of Neutral Beam Injection on the plasma's conductivity and so its wave propagation properties, the first step must be to obtain the unperturbed distribution function for the fast ions created from the injected neutral atoms by electron impact,

$$H_f + e \longrightarrow H_f^+ + 2e$$

ion impact

$$H_f + H^+ \longrightarrow H_f^+ + H^+ + e$$

and charge exchange.

$$H_f + H^+ \longrightarrow H_f^+ + H$$

The fast ions can then be treated as an additional species in the plasma, and their contribution to the conductivity tensor calculated accordingly.

Fast-ion transport in a tokamak, in the presence of collisions can be described by the steady-state, drift kinetic equation (Cordey, 1976). Using $v_\perp/(2\omega_c L) \ll 1$ (which is required for the techniques of Chapters 3 and 4) and averaging over a magnetic flux surface, the equation can be written (Mudford, 1985)

$$0 = \frac{1}{v^2\tau_s}\frac{\partial}{\partial v}\left[\left[v^3+v_c^3\right]f\right] - \frac{f}{\tau_{cx}} + \frac{v_c^3}{v^3\tau_s}\frac{d}{d\gamma}\left[(1-\gamma^2)\frac{\partial f}{\partial \gamma}\right]$$

$$+ S_0\,\delta(v_{inj}-v)\,K(\gamma) \qquad (5.3.1)$$

Since a lot of new notation has been introduced, including changes in the velocity coordinate system, and bearing in mind the fact that interparticle collisions have been ignored since Chapter 1, it is perhaps helpful to consider the constituents of the drift kinetic equation individually, in terms of their physical effects.

The first term on the right-hand side of equation (5.3.1) is the velocity drag term which gives rise to the loss of energy of the fast ions.

The coefficient τ_s is simply the Spitzer slowing-down time,

$$\tau_s = -U \Big/ \left(\frac{\partial U}{\partial t}\right) \tag{5.3.2}$$

which can be calculated in the same way as the deflection times were calculated in section 1.4.

Using a test particle distribution (1.4.9) $\frac{\partial U}{\partial t}$ can be calculated from the first velocity moment of equation (1.4.6). Only the first term in the Fokker Planck equation - the drag term - contributes. The drag on the test particle due to thermal electrons gives

$$\tau_s = \frac{3\pi^{3/2}\varepsilon_0^2 m_i m_e V_e^3}{ne^4 \ln(\lambda_0/b_0)} \tag{5.3.3}$$

which is independent of U. Performing the calculation for the drag due to thermal ions gives

$$\tau_{s\,(\mathrm{ions})} = \frac{2\pi\varepsilon_0^2 m_i^2 U^3}{ne^4 \ln(\lambda_0/b_0)} \tag{5.3.4}$$

Equating the ion and electron drag terms gives the critical velocity

$$U_c = \left(\frac{3}{2}\pi^{1/2}\,\frac{m_e}{m_i}\right)^{1/3} V_e \tag{5.3.5}$$

at which energy is transferred equally to ions and electrons. It can be seen that both sources of drag are included in the first term on the right hand side of (5.3.1).

The second term represents the loss of fast ions by charge-exchange with neutral particles diffusing in from the edge regions of the plasma.

The penultimate term on the right-hand side of equation (5.3.1) models the pitch-angle scattering of the fast ions as they thermalise. Where γ is the cosine of the pitch angle.

$$\gamma = \frac{U_z}{U} \tag{5.3.6}$$

In section 1.4 it was shown that the scattering of ions from electrons, equation (1.4.15), was a much slower process than the scattering of ions from other ions, equation (1.4.13). Therefore only ion-ion scattering need be considered. From (1.4.13) the scattering time is

$$\tau_{ii} = \frac{2\pi \varepsilon_0^2 m_i^2 U^3}{n e^4 \ln(\lambda_D / b_0)} \tag{5.3.7}$$

and so

$$\tau_{ii} = \frac{U^3}{U_c^3} \tau_s \tag{5.3.8}$$

The final term in equation (5.3.1) represents the source of injected fast ions. Since the neutral atoms are injected almost monoenergetically, their energy dependence takes the form of a δ-function, and the pitch-angular spread of the beam is represented by the function $K(\gamma)$. The source term for the fast ion distribution will have the same form if the collisions that ionise the injected neutrals do so without significantly altering their momentum. While this is true for electron impact ionisation, since in an individual collision between an electron and an atom the momentum of the nucleus is not significantly changed, it is not obvious that ion impact and charge exchange will conserve the form of the source function.

In deriving equation (5.3.1) it was assumed that trapped particle and energy diffusion effects can be neglected, in addition, in the calculations of collision times it was assumed that the test particles velocity was in the range

$$V_i << U << V_e \qquad (5.3.9)$$

The solution of equation (5.3.1) can be readily obtained, since the equation is separable in u and γ, and the differential operator which depends on γ is Legendre's equation. The distribution function, f, can be expressed as a sum of eigenfunctions of the form

$$a_n(\bar{u})\, P_n(\gamma) \qquad \bar{U} = \frac{U}{U_{inj}} \qquad (5.3.10)$$

where the eigenfunctions $P_n(\gamma)$ are Legendre polynomials with eigenvalues $\alpha_n = n(n+1)$. The functions $a_n(\bar{u})$ are determined from the separated equation in u, with the boundary condition $a_n(\bar{u})=0$ when $\bar{u}=1$. This boundary condition assumes that the effects of energy diffusion are negligible. The full solution of equation (5.3.1) can be written in the form

$$(U<1) \quad f = S_o \tau_s \sum_n \left(\frac{1+\bar{U}_c^3}{\bar{U}^3+\bar{U}_c^3} \right)^{\frac{\alpha_n}{6} - \frac{\tau_s}{3\tau_{cx}}} \frac{\bar{U}^{\frac{\alpha_n}{2}}}{\bar{U}^3+\bar{U}_c^3} K_n P_n(\gamma)$$

$$(U \geqslant 1) \quad f = 0 \tag{5.3.11}$$

where

$$K_n = \frac{2n+1}{2} \int_{-1}^{1} K(\gamma) P_n(\gamma) \, d\gamma \tag{5.3.12}$$

For the source function to be correctly normalised S must be of the form $S_o = \dot{n}_f / 2\pi v_{inj}^3$, where $\dot{n}_f$ is the fast-ion density input rate.

5.4. The Fast Ion Conductivity

Now that an equilibrium distribution has been obtained, the analysis of Chapters 3 and 4 can be repeated with the new driving terms obtained in section 5.2, the only further modifications being due to the use of spherical coordinates instead of cylindrical coordinates for the velocity space. In these coordinates

$$P = \frac{\partial f}{\partial v_\perp} = \frac{(1-\gamma^2)^{1/2}}{U_{inj}} \left(\frac{\partial f}{\partial \bar{U}} - \frac{\gamma}{\bar{U}} \frac{\partial f}{\partial \gamma} \right)$$

$$R = \frac{\partial f}{\partial v_z} = \frac{1}{U_{inj}}\left(\gamma \frac{\partial f}{\partial \bar{U}} + \frac{(1-\gamma^2)}{\bar{U}}\frac{\partial f}{\partial \gamma}\right)$$

$$Q = V_\perp R - V_z P = (1-\gamma^2)^{1/2}\frac{\partial f}{\partial \gamma} \qquad (5.4.1)$$

while

$$A_n = \frac{\bar{U}(1-\gamma^2)^{1/2}}{2ik_z(J_n - \bar{U}\gamma)} \qquad J_n = \frac{w + n w_c}{k_z U_{inj}} \qquad (5.4.2)$$

The velocity integrals are now complicated by the fact that they are no longer separable and so in principle the effects of the cyclotron resonances appear in both the γ and $\bar{U}$ integrals. A possible solution to this problem is to rewrite the velocity integrals

$$\int_0^1 \bar{U}^2 d\bar{U} \int_{-1}^1 d\gamma \; \frac{F_1(\bar{U}) F_2(\gamma) P_m(\gamma)}{J_n - \bar{U}\gamma} \qquad (5.4.3)$$

as

$$\sum_L F_L \int_0^1 \bar{U}\, d\bar{U}\, F_1(\bar{U}) \int_{-1}^1 \frac{P_L(\gamma) P_m(\gamma)\, d\gamma}{\frac{J_n}{\bar{U}} - \gamma} \qquad (5.4.4)$$

where

$$F_L = \frac{2L+1}{2}\int_{-1}^1 F_2(\gamma)\, P_L(\gamma)\, d\gamma \qquad (5.4.5)$$

and then use the following identity for Legendre polynomials.

$$\int_{-1}^1 \frac{P_n(\gamma) P_m(\gamma)\, d\gamma}{x - \gamma} = P_n(x) Q_m(x) \qquad n \le m \qquad (5.4.6)$$

This however raises the problem for resonances, defined by

$$|\zeta_n| < 1 \qquad (5.4.7)$$

that part of the contour of the $\bar{u}$ integration runs along the cut line $-1 < \frac{\zeta_n}{\bar{U}} < 1$ for which $Q_m(\frac{\zeta_n}{\bar{U}})$ is not properly defined. Customarily, the value of Q_m on the cut line is defined as the average of the value above and below the cut, but this would be analogous to taking the Cauchy principal part of the resonant integral. To include the contribution of the pole, the contour must be deflected below the cut line. The $\bar{u}$ integration then gives a cyclotron damping term of $i\pi$ times the residue at $\bar{u} = \zeta_n$. Displacing the contour in this manner is consistent with the displacement of contours used for Maxwellian distribution functions to produce the well known plasma dispersion function,

$$\int_{-\infty}^{\infty} \frac{e^{-\bar{v}_z^2}}{\zeta_n - \bar{v}_z}\, d\bar{v}_z = 2\sqrt{\pi}\, e^{-\zeta_n^2} \int_0^{\zeta_n} e^{t^2} dt + i\pi e^{-\zeta_n^2} \qquad (5.4.8)$$

Although the $\bar{u}$ integral may have to be evaluated numerically, the coefficients of the o.d.e's can in principle be calculated and so the effect of this distribution function on wave propagation can be modelled.

5.5. Parameter Problems

In obtaining equation (5.3.1) it was assumed that the fast ions formed only a small part of the equilibrium plasma. This assumption was made so that beam-beam collisions could be ignored, effectively linearising (5.3.1). Although some tokamak experiments have a large proportion of the plasma injected in this fashion, this does not necessarily imply that a large proportion of the plasma has a fast ion distribution. Only if the injection rate is high enough for the particles injected in a few collision times to form a significant fraction of the ion density will equation (5.3.1) become invalid.

The fact that the fast ion distribution only forms a small part of the plasma implies that, with the possible exceptions of fast ion resonances, the wave propagation will be similar to that of a plasma without fast ions. This has unfortunate consequences for the form of the o.d.e's, for parameters relevant to present day and future experiments.

The problem is that, while for thermal particles the small Larmor orbit approximation is valid, the much larger orbits of the fast ions encompass too great a variation in the perturbing field. For example, for the parameters used in Chapter 3, which were based on those of JET,

$$\underset{\text{fast wave}}{0.15} < \frac{k_x V_T}{2\omega_c} < \underset{\text{Bernstein}}{0.47} \qquad (5.5.1)$$

for thermal ions, however the neutral beam injection energy used for JET is 160keV, 32 times the energy of the particles included in the example. Therefore assuming that the perpendicular 'wavelength' is not dramatically altered by the presence of fast ions

$$\underset{\text{fast wave}}{0.85} < \frac{k_x V_{inj}}{2\omega_c} < \underset{\text{Bernstein}}{2.7} \qquad (5.5.2)$$

The slowly varying approximation is not valid for the fast ion distribution and so to model the effects of the fast ions accurately the extension to the theory made in Chapter 4 must be employed. This unfortunately implies that the order of the o.d.e's obtained will be considerably greater and so will cause major difficulties for numerical solution techniques. Again, as in Chapter 4, the conclusion is that to solve such problems without making impractical demands for computing resources, the direct differential equation approach must be modified.

Chapter 6

Spatially Inhomogenous Distribution

6.1. Introduction

The decision to deal first with only spatially homogeneous f_o was the result of two main considerations: the first was that the effects of explicit magnetic field gradient terms would be more easily identified if they were the only addition to the traditional equations; the second consideration was one of the inconsistency of a spatially inhomogenous equilibrium distribution function that was simultaneously isotropic in velocity space. The latter can be more clearly understood by examining the unperturbed Vlasov equation for an equilibrium distribution (2.3.1).

$$\underset{\sim}{v} \cdot \frac{\partial f_o}{\partial \underline{x}} + \frac{q}{m}\left(\underline{v} \times \underline{B}_o\right) \cdot \frac{\partial f_o}{\partial \underline{v}} = 0 \tag{2.3.1}$$

If the equilibrium is isotropic in velocity space

$$\underline{v} \times B_o \cdot \frac{\partial f_o}{\partial \underline{v}} = 0 \tag{6.1.1}$$

(which will be recognised as the same argument that allowed the perturbing magnetic field to be ignored in Chapter 2). The Vlasov equation therefore requires such an equilibrium to be spatially homogeneous.

Looking more closely at the $\underline{v} \times \underline{B}_0$ term reveals that not only isotropic distributions but in fact any velocity distribution with cylindrical symmetry about the equilibrium magnetic field is required by the Vlasov equation to be spatially homogeneous. Since, using the same change of velocity variables that was used in the perturbed Vlasov equation, equation (2.3.1) can be rewritten as

$$\underline{v} \cdot \frac{\partial f_0}{\partial \underline{x}} = \omega_c \frac{\partial f_0}{\partial \phi} \qquad (6.1.2)$$

it is clear that only a ϕ dependent f_0 can have a spatial dependence.

The problems raised by allowing f_0 to vary with position are therefore twofold: the fact that f_0 no longer commutes with the differential operator and the requirement (for consistency) that f_0 no longer be cylindrically symmetric.

With regard to the former problem it should be noted that at no stage in the manipulations between equations (3,2,1) and (3,2,4) was it required that f was homogeneous and therefore equation (3,2,4) is still consistent for f_0 which are functions of x. In the case of a two species plasma this problem can therefore be circumvented in a particularly simple fashion. The requirement of charge neutrality for the plasma equilibrium implies that the distribution functions of both species must have the same spatial dependence; therefore, simply by factoring the x dependence of the f_0 into a new $\underline{E}$ variable, the only alteration to the o.d.e's describing the evolution of the perturbing electric field would be in the vacuum field terms. The new o.d.e's could then be solved in the same way as before, with the real E simply obtained from the solution of the o.d.e's by dividing out the x dependence of the f_0.

$$\underline{E} \cdot \frac{\partial}{\partial \underline{v}} \left(X(x) \, V(\underline{v}) \right) \qquad (6.1.3)$$

would be replaced by

$$\underline{\tilde{E}} \cdot \frac{\partial V(\underline{v})}{\partial \underline{v}} \qquad (6.1.4)$$

the wave equation now being

$$\frac{c^2}{\omega^2} \nabla \times \nabla \times \left(\frac{\underline{\tilde{E}}}{X(x)} \right) - \frac{\underline{\tilde{E}}}{X(x)} - \frac{\underline{\underline{\sigma}} \cdot \underline{\tilde{E}}}{i \epsilon_0 \omega} = 0 \qquad (6.1.5)$$

6.2. General Equilibrium

The second difficulty, that of the loss of cylindrical symmetry, causes greater problems. The first step in solving this problem is to express f_0 as a Fourier series in ϕ, in the same way that f_1 was in Chapter 3, and with the same justification.

$$f_0(\phi) = \sum_{n=-\infty}^{\infty} f_{on}\, e^{in\phi} \tag{6.2.1}$$

Then, considering the coefficient of $e^{in\phi}$ in (6.1.2) gives

$$v_\perp \frac{\partial}{\partial x}\left(\frac{f_{o\,n+1} + f_{o\,n-1}}{2}\right) = i\omega_c n f_{on} \tag{6.2.2}$$

These equations, resulting as they do from the Vlasov equation, are necessary for an equilibrium distribution but not sufficient. For an equilibrium distribution to be valid on longer timescales, collisional effects would have to be considered. However, it is of some interest to examine what classes of f_o satisfy the Vlasov equation and how these new f_0 would modify the conductivity tensor.

The system of equations (6.2.2) clearly has an infinite number of solutions; however bearing in mind that each f_o introduces another tree of terms to the calculation of the conductivity tensor, it is reasonable to look first for finite Fourier series solutions of (6.2.2). The shortest possible series solution is the trivial one

$$f_{on} = \delta_{o,n}\, F(v_\perp^2, v_z) \tag{6.2.3}$$

The next simplest solution, and the simplest solution that is not spatially homogeneous, is

$$f_{on} = 0 \qquad |n| > 1$$

$$f_{01} = -f_{0-1} = \frac{v_\perp}{2i} F(v_\perp^2, v_z)$$

$$f_{00} = F(v_\perp^2, v_z) \int^x \omega_c \, d\tilde{x} \qquad (6.2.4)$$

Substituting (6.2.4) in (6.2.1) gives

$$f_0 = F(v_\perp^2, v_z)\left[v_y + \int^x \omega_c \, d\tilde{x} \right] \qquad (6.2.5)$$

The new factor will be recognised as one of the two 'additional constants of the motion'

$$v_y + \int^x \omega_c \, d\tilde{x}$$

$$v_x - \int^y \omega_c \, d\tilde{y} \qquad (6.2.6)$$

obtained from considerations of guiding centre motion (Krall and Trivelpiece, 1973). It might be hoped that new constants of the motion would be obtained by examining the next simplest series.

$$f_{0n} = 0 \qquad |n| > 2$$

$$f_{02} = f_{0-2} = \frac{v_\perp^2}{2} F(v_\perp^2, v_z)$$

$$f_{01} = -f_{0-1} = 2iv_\perp F(v_\perp^2, v_z) \int^x \omega_c \, d\tilde{x}$$

$$f_{00} = -4F(v_\perp^2, v_z) \int^x \omega_c \int^{\tilde{x}} \omega_c \, d\tilde{\tilde{x}} \, d\tilde{x}$$

(6.2.7)

However, the solution given by substituting these in (6.2.1)

$$f_o = F(v_\perp^2, v_z)\left[v_x^2 - v_y^2 - 4v_y\int^x \omega_c\, d\tilde{x} - 4\int^x \omega_c \int^{\tilde{x}} \omega_c\, d\tilde{\tilde{x}}\, d\tilde{x}\right] \tag{6.2.8}$$

can be recognised as a function of the previously obtained constants

$$F(v_\perp^2, v_z)\left[v_\perp^2 - 2\left(v_y + \int^x \omega_c\, d\tilde{x}\right)^2\right]$$

$$= F(v_\perp^2, v_z)\left(v_x^2 - v_y^2 - 4v_y\int^x \omega_c\, d\tilde{x} - 2\left[\int^x \omega_c\, d\tilde{x}\right]^2\right) \tag{6.2.9}$$

which is equivalent to (6.2.8) since

$$\frac{d}{dx}\left[2\left(\int^x \omega_c\, d\tilde{x}\right)^2\right] = 4\omega_c\int^x \omega_c\, d\tilde{x}$$

$$\Rightarrow 2\left[\int^x \omega_c\, d\tilde{x}\right]^2 = 4\int^x \omega_c \int^{\tilde{x}} \omega_c\, d\tilde{\tilde{x}}\, d\tilde{x} + \text{const} \tag{6.2.10}$$

and so (6.2.8) is merely a particular example of the fact that any function of the solutions of the Vlasov equation is also a solution.

Including constants of integration in (6.2.4) and (6.2.7) would be equivalent to adding multiples of the shorter series (6.2.3) and (6.2.4) respectively.

6.3. Current-Carrying Plasmas

Both (6.2.4) and (6.2.7) carry electric currents in the y direction, since both have $f_{01} = -f_{0-1} \neq 0$. This property allows the analysis of the case where the spatial dependence of B_0 is due to currents flowing in the plasma.

$$\frac{dB_0}{dx} = -\mu_0 J_y \qquad (6.3.1)$$

Considering first (6.2.4), it is clear that this class of equilibrium implies a constant J_y and therefore a linear B_0. Such a distribution would have been quite consistent with the magnetic field profile used in 3.5 . However (6.2.7) gives

$$\frac{d^2B_0}{dx^2} \propto B_0 \qquad (6.3.2)$$

and so is consistent with an exponentially varying or oscilliatory B_0.

Given the spatial dependence of B_0, (6.3.1) gives f_{0-1} and f_{01}, and these in turn give information about the number of terms in the Fourier series (6.2.1) and those terms' spatial dependence. Relations between B_0 and higher moments of f_0 can also be found. For example, from (6.2.2) it can be seen that any solution of the unperturbed Vlasov equation obeys

$$\frac{d}{dx}\left[\frac{mv_\perp^2}{2}\left(f_{00}+\frac{(f_{02}+f_{0-2})}{2}\right)\right]=\frac{q\,v_\perp B_0\,(f_{0-1}-f_{01})}{2i} \qquad (6.3.3)$$

Therefore the rate of change of pressure exerted by the species in the x direction

$$P_{x_\alpha}=\frac{m_\alpha}{2}\int_0^{2\pi}d\phi\int_0^\infty v_\perp dv_\perp\int_{-\infty}^{\infty}dv_z\,(v_\perp\cos\phi)^2 f_{0_\alpha} \qquad (6.3.4)$$

is simply the current carried by the species times the total magnetic field.

$$\frac{dP_{x_\alpha}}{dx}=-B_0 J_{y_\alpha} \qquad (6.3.5)$$

If the spatial dependence of B_0 is solely due to currents carried in the plasma, then summing over all the species in the plasma gives

$$P_x+\frac{B_0^2}{2\mu_0}=\mathrm{const} \qquad (6.3.6)$$

Since the ratio of plasma pressure to magnetic pressure is very small for a tokamak, if the plasma obeyed (6.3.6) then even the slight increase in magnetic field modelled in Chapter 3 would expel all of the plasma. Fortunately, in a tokamak the main magnetic field component is a toroidal field

$$B_0(R)=\frac{B R_0}{R} \qquad (6.3.7)$$

which, being curl free, does not require a current in the plasma. Therefore, for the tokamak inspired parameters of Chapter 3 the trivial equilibrium (6.2.3) is more suitable than (6.2.4) or (6.2.7).

6.4. The Wood Diagram

The effect that losing the cylindrical symmetry of f_o has on the perturbed Vlasov equation is the following modification to the 'driving term'

$$\frac{\partial f_o}{\partial \underline{v}} = \sum e^{in\phi} \begin{pmatrix} \cos\phi \dfrac{\partial f_{on}}{\partial v_\perp} - in\sin\phi \dfrac{f_{on}}{v_\perp} \\ \sin\phi \dfrac{\partial f_{on}}{\partial v_\perp} + in \dfrac{\cos\phi\, f_{on}}{v_\perp} \\ \dfrac{\partial f_{on}}{\partial v_z} \end{pmatrix} \tag{6.4.1}$$

Since the individual tree diagrams for each Fourier component of f_o overlap because of the $\cos\phi$ and $\sin\phi$ factors, it is more convenient to consider their sum as a wood diagram (figure 6.1) where

$$\begin{aligned} D_n = & \left[P_{n+1} + P_{n-1} + \frac{k_z}{\omega}\left(Q_{n+1} + Q_{n-1}\right) - \frac{1}{v_\perp}\left(1 - \frac{k_z v_z}{\omega}\right) W_n \right] \frac{qE_x}{mv_\perp} \\ & + i\left[P_{n+1} - P_{n-1} + \frac{k_z}{\omega}\left(Q_{n+1} - Q_{n-1}\right) + \frac{1}{v_\perp}\left(1 - \frac{k_z v_z}{\omega}\right) W_n - \frac{n f_{on}}{\omega}\frac{d}{dx} \right] \frac{qE_y}{mv_\perp} \\ & + \left[2R_n + \frac{i}{\omega}\left[Q_{n+1} + Q_{n-1} + \frac{v_z}{v_\perp} W_n \right] \frac{d}{dx} \right] \frac{qE_z}{mv_\perp} \end{aligned} \tag{6.4.2}$$

It should be emphasised that P_n and Q_n, are in no way related to the Legendre polynomials of the first and second kind used in Chapter 5, but are in fact the velocity derivatives of f_{on}.

$$P_n = \frac{\partial f_{on}}{\partial v_\perp} \qquad R_n = \frac{\partial f_{on}}{\partial v}$$

$$Q_n = v_\perp R_n - v_z P_n$$

$$W_n = (n-1) f_{on-1} - (n+1) f_{on+1} \qquad (6.4.3)$$

In order to calculate $\underline{J}_1$, and hence obtain the conductivity tensor, the 1, 0 and -1 components of f_1 are required. These components can be obtained from the general formula

$$f_{1n} = \sum_{l=0}^{\infty} \sum_{s=0}^{l} G_n{}^{n+s}{}_{2s+n-l} \, D_{2s+n-l} \qquad (6.4.4)$$

where l is the order of the coupling, i.e. the maximum order of the derivative of D_{2s+n-l} arising from the action of $G_n{}^{n+s}{}_{2s+n-l}$ (defined in Chapter 4).

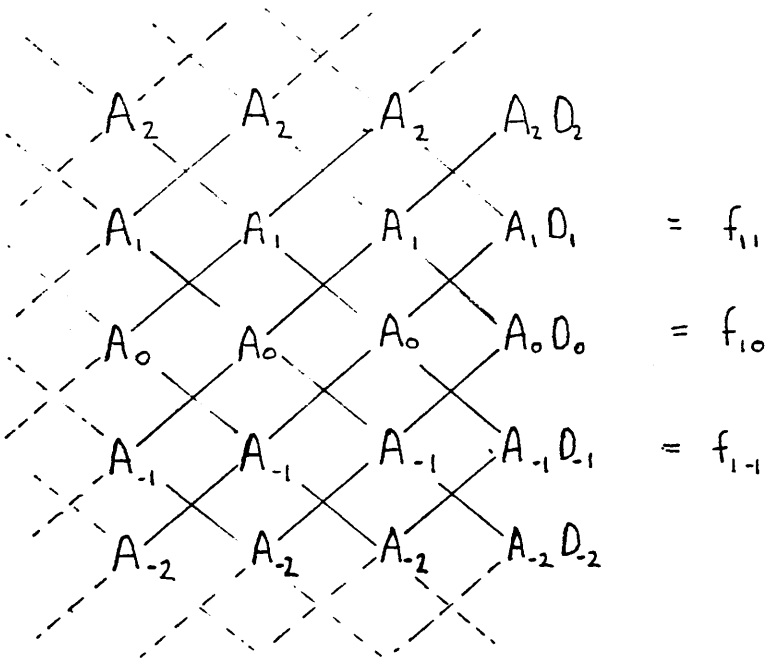


Figure 6.1 - Wood diagram showing the effects of the additional driving terms on the Fourier components of f_1.

Note that since D_{2s+n-l} has a contribution from curl $\underline{E}$, the maximum order of the derivatives of E_y and E_z from $G_{n\ 2s+n-l}^{n+s}$ is l+1.

6.5. The Conductivity Tensor

Using (6.4.4) $\underline{J}_1$ can be obtained in terms of the D_n and their derivatives. Then, by splitting the D_n into E_x, E_y and E_z terms the conductivity tensor can be obtained.

As an example, consider σ_{11}, which gives the dependency of $\underline{J}_{1x}$ on $\underline{E}_x$. J_{1x} is

$$\pi q \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{\infty} dv_z \left(f_{11} + f_{1-1} \right) \tag{6.5.1}$$

which, to first order in $\frac{dB_0}{dx}$, is

$$\pi q \sum_{l=0}^{\infty} \sum_{s=0}^{l} \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{} dv_z \left\{ \left[T_{1\ 2s+1-l}^{s+1} + S_{1\ 2s+1-l}^{s+1} \frac{d}{dx} \right] \frac{d^{l-1}}{dx^{l-1}} D_{2s+1-l} \right.$$
$$\left. + \left[T_{-1\ 2s-1-l}^{s-1} + S_{-1\ 2s-1-l}^{s-1} \frac{d}{dx} \right] \frac{d^{l-1}}{dx^{l-1}} D_{2s-1-l} \right\} \tag{6.5.2}$$

Therefore $J_x(E_x)$ can be obtained using (6.4.2).

$$J_x(E_x) = \frac{\pi q^2}{m} \sum_{l=0}^{\infty} \sum_{s=0}^{l} \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \left[T_{1\ 2s+1-l}^{s+1} \frac{d^{l-1}}{dx^{l-1}} \left(\alpha_{2s+1-l} E_x \right) \right.$$
$$+ S_{1\ 2s+1-l}^{s+1} \frac{d^l}{dx^l} \left(\alpha_{2s+1-l} E_x \right) + T_{-1\ 2s-1-l}^{s-1} \frac{d^{l-1}}{dx^{l-1}} \left(\alpha_{2s-1-l} E_x \right)$$
$$\left. + S_{-1\ 2s-1-l}^{s-1} \frac{d^l}{dx^l} \left(\alpha_{2s-1-l} E_x \right) \right] \tag{6.5.3}$$

where

$$\alpha_n = \left[P_{n+1} + P_{n-1} + \frac{k_z}{\omega}\left(Q_{n+1} + Q_{n-1}\right) - \frac{1}{V_\perp}\left(1 - \frac{k_z V_z}{\omega}\right) W_n \right] \tag{6.5.4}$$

and use has been made of the identity

$$T^{\,n}_{nn} = 0 \tag{6.5.5}$$

Using Leibnitz' formula and rearranging the summations to obtain the coefficients of the derivatives of E_x gives

$$\begin{aligned}\sigma_{11} = \frac{\pi q^2}{m} \sum_{r=0}^{\infty} \sum_{l=r}^{\infty} \binom{l}{r} \sum_{s=0}^{l+r} \int_0^\infty V_\perp \, dV_\perp \int_{-\infty}^{\infty} dV_z \Big[& S^{\,s+1}_{1\;2s+1-l} \frac{d^{\,l-r}}{dx^{l-r}} (\alpha_{2s+1-l}) \\ & + T^{\,s+1}_{1\;2s+1-l} \left(\frac{l-r}{r}\right) \frac{d^{\,l-r-1}}{dx^{l-r-1}} (\alpha_{2s+1-l}) + S^{\,s-1}_{-1\;2s-1-l} \frac{d^{\,l-r}}{dx^{l-r}} (\alpha_{2s-1-l}) \\ & + T^{\,s-1}_{-1\;2s-1-l} \left(\frac{l-r}{r}\right) \frac{d^{\,l-r-1}}{dx^{l-r-1}} (\alpha_{2s-1-l}) \Big] \quad \frac{d^r}{dx^r}\end{aligned} \tag{6.5.6}$$

The rest of the elements of $\underline{\underline{\sigma}}$ can be obtained in similar fashion.

The physical consequences on the conductivity of the plasma of allowing such general equilibria are the earlier appearance of cyclotron harmonic resonances in the coefficients of the o.d.e's. For example the $\omega = 2\omega_c$ resonance did not contribute to the conductivity tensor in section 3.5. until the second application of the differential operator, but given a significant f_{0-1}, the $\omega = 2\omega_c$ resonance is directly driven and so contributes to $\underline{\underline{\sigma}}$ after the first application of the differential operator. In general if $f_{0\text{-}n}$ is significant, the $\omega = (n+1)\omega_c$ resonance is directly driven and so appears in $\underline{\sigma}$ after only n rather than 2n differential operators.

Although a fully general equilibrium causes a great increase in the complexity of the algebra, there are two cases where the algebra is still feasible.

If the f_{on} and $\underline{E}$ are all slowly varying, then the wood diagram can be truncated at low l, just as the tree diagram was truncated. Since the vertical extent of the wood diagram is also controlled by l, only a few of the D_n would need to be considered. In such a case l should be chosen large enough to include any resonances (large A_n) or particularly large D_n.

A second case where the algebra would be tractible is when the equilibrium distribution function can be adequately modelled by a short Fourier series. If in addition, the f_{on} or $\underline{E}$ are slowly varying, the algebra will only be slightly more tedious than it is for trivial f_o (Chapter 4).

6.6. Summary

In this chapter, and to a lesser extent in Chapter 5, the analysis necessary to extend the theory of Chapter 4 to handle any equilibrium velocity distribution has been carried out. With no symmetries required, equilibria with particle, momentum and heat drifts etc can be handled.

Even equilibria that vary rapidly on the scale of the species Larmor orbit can be analysed consistently. This is because the f_{on} and their velocity derivatives appear only at the 'driving edge' of the tree diagram. Therefore, just as the factorials in Chapter 4 caused eventual convergence of the higher derivative terms for the case of rapidly varying $\underline{E}$, they will also cause convergence of the higher derivative terms for the case of rapidly varying f_o.

Unlike rapidly varying $\underline{E}$, rapidly varying f_o do not increase the order of the differential equation; however, they do increase the complexity of the coefficients of the o.d.e's quite dramatically.

Chapter 7

Finite k_y.

7.1. Introduction

In the introduction to Chapter 3, the possibility of including the effects of a finite k_y on the terms of the conductivity tensor was mentioned as one of the advantages of the formalism that was introduced in that chapter.

Historically k_y was ignored due to the use of homogeneous plasma dispersion relation techniques to obtain the terms in the conductivity tensor. Since in a homogeneous plasma there is only one preferred direction, that lying along the equilibrium magnetic field, the homogeneous system can be solved by choosing the coordinate system with the z axis along $\underline{B}_o$, and the direction of wave propagation lying in the x-z plane. In effect the y axis was redundant and so only k_x and k_z needed to be considered. This allowed a corresponding simplification of the algebra involved.

However, with the introduction of x dependency for B_o this cylindrical symmetry is lost. In the inhomogeneous system the dependency of B_o forms a second special direction. While the theory derived so far can describe a wave propagating in the plane formed by $\underline{B}_o$ and ∇B_o, it cannot, as it stands, describe waves outside this plane.

7.2. Vlasov Equation

Including a y variation in f_1 and then Fourier transforming the perturbed Vlasov equation in y,z and t gives

$$\omega_c \frac{\partial f_1}{\partial \phi} + i(\omega - k_z v_z) f_1 = i k_y v_\perp \sin\phi \, f_1 + v_\perp \cos\phi \frac{\partial f_1}{\partial x} + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B}_1) . \frac{\partial f_o}{\partial v} \quad (7.2.1)$$

Following the procedure of Chapter 3 the k_y is observed to cause a second coupling term between f_{1n}, f_{1n+1} and f_{1n-1}.

$$f_{1n} = A_n \frac{d}{dx} \frac{(f_{1n+1} + f_{1n-1})}{2} + A_n k_y \frac{(f_{1n-1} - f_{1n+1})}{2} + A_n D_n \quad (7.2.2)$$

The great similarity of the two coupling terms can now be exploited. Again, the tree diagram is useful, although now an asymmetry must be incorporated. Instead of all links representing

$$\frac{d}{dx} \quad (7.2.3)$$

those with positive slope represent

$$\frac{d}{dx} - k_y \quad (7.2.4)$$

while those with negative slope represent

$$\frac{\partial}{\partial x} + k_y \qquad (7.2.5)$$

If in addition to the restrictions listed in Chapter 3, on the rate of variation with x of all quantities involved, attention is restricted to the case of small k_y,

$$\left|\frac{k_y V_T}{2\omega_c}\right| << 1 \qquad (7.2.6)$$

then a simple perturbation technique as used in Chapter 3 is valid.

7.3. Homogeneous Case

It is significant, that while the gradient independent terms are simply calculated, even for perturbations that vary rapidly in the x and y directions, the result is not the trivial replacement of k_x^2 with $k_x^2 + k_y^2$ or equivalently replacing $\frac{d^2}{dx^2}$ with $\frac{d^2}{dx^2} - k_y^2$. This can be shown by consideration of the tree diagram elements. While the diamond elements

$$S_{nn}^{m} \left(\frac{\partial^2}{\partial x^2} - k_y^2\right)^{m-n} \qquad (7.3.1)$$

are clearly functions of $k_\perp^2$ or of $k_x^2 - \frac{d^2}{dx^2}$, the offset elements

$$S^m_{nl}\left(\frac{d}{dx}-k_y\right)^{m-n}\left(\frac{d}{dx}+k_y\right)^{m-l} \qquad (7.3.2)$$

are not. While the diamond elements have as many connections with positive slope as with negative slope (since there is no net 'height' gained or lost between the ends on the diamond) the offset elements clearly do not have this symmetry (since by definition there is a net 'height' gain or loss).

Consider $\underline{\underline{\sigma}}(m)$, the gradient free kernel of the conductivity tensor (4.2.10). It can be seen that, while the 3,3 component of $\underline{\underline{\sigma}}(m)$, depending as it does only on the S^m_{00} terms, will be a function of $k_\perp^2 = k_x^2 + k_y^2$ only, the other components of $\underline{\underline{\sigma}}(m)$ which also depend on offset elements will not have such a simple form.

It is very significant that the offset terms G^m_{-11} and G^m_{1-1} are not functions merely of $k_\perp^2$, since they form the difference between the 1,1 and 2,2 components of $\underline{\underline{\sigma}}(m)$. The physical need for a different dependence for these terms can be seen if we consider a wave in the y-z plane. If the elements of the conductivity tensor were functions of $k_\perp^2$ only then they would be no different from those for a wave in the x-z plane. This would lead to a different dispersion relation for k_y^2 from the one obtained for k_x^2 due to the difference between the σ_{11} term and the σ_{22} term, with the obviously unphysical result that the propagation of a wave would depend on the orientation of the axes chosen. It should be clear that the justification for $\sigma_{11} \neq \sigma_{22}$ is that the x direction was picked before the conductivity tensor was calculated. The conductivity in the x direction differs from that in the y direction because of the wave.

The effect of the k_y dependency on the 1,1 and 2,2 elements of the conductivity tensor can be expressed in a more convenient form, if the following manipulation is performed. This is particularly useful when standard textbook expressions are available for the $k_y = 0$ case. By expressing $G^m_{-11}+G^m_{1-1}$ and $G^{m+1}_{11}+G^{m-1}_{-1-1}$ in terms of sums and differences of σ_{11} and σ_{22} we obtain

$$\sum_m G^m_{-11} + G^m_{1-1} = \frac{1}{2}\left(c_{11}(k_x,k_y) - c_{22}(k_x,k_y)\right) \tag{7.3.3a}$$

$$\sum_m G^{m+1}_{11} + G^{m-1}_{-1-1} = \frac{1}{2}\left(c_{11}(k_x,k_y) + c_{22}(k_x,k_y)\right) \tag{7.3.3b}$$

Also, using (7.3.1) and (7.3.2) (noting the fact that $S^m_{-11} = S^m_{1-1}$)

$$G^m_{-11} + G^m_{1-1} = \left(S^m_{-11} + S^m_{1-1}\right) k_\perp^{2(m-1)} \left(k_x^2 - k_y^2\right)$$

$$G^{m+1}_{11} + G^{m-1}_{-1-1} = \left(S^{m-1}_{-1-1} + S^{m+1}_{11}\right) k_\perp^{2m} \tag{7.3.4}$$

Therefore (7.3.3) can be rewritten

$$\sum_m \left(S^m_{-11} + S^m_{1-1}\right) k_\perp^{2(m-1)} \left(k_x^2 - k_y^2\right) = \frac{1}{2}\left(c_{11}(k_x,k_y) - c_{22}(k_x,k_y)\right) \tag{7.3.5a}$$

$$\sum_m \left(S_{11}^{m+1} + S_{-1-1}^{m-1}\right) k_\perp^{2m} = \tfrac{1}{2}\left(c_{11}(k_x,k_y) + c_{22}(k_x,k_y)\right)$$

(7.3.5b)

Then by considering the case of $k_y=0$,

$$\sum_m \left(S_{-11}^{m} + S_{1-1}^{m}\right) k_\perp^{2m} = \tfrac{1}{2}\left(c_{11}(k_\perp,0) - c_{22}(k_\perp,0)\right) \qquad (7.3.6a)$$

$$\sum_m \left(S_{-1-1}^{m-1} + S_{11}^{m+1}\right) k_\perp^{2m} = \tfrac{1}{2}\left(c_{11}(k_\perp,0) + c_{22}(k_\perp,0)\right) \qquad (7.3.6b)$$

Comparing the left-hand sides of (7.3.6) and (7.3.7)

$$(7.3.5a) = \frac{k_x^2 - k_y^2}{k_\perp^2}\,(7.3.6a)$$

$$(7.3.5b) = (7.3.6b)$$

(7.3.7)

Therefore

$$c_{11}(k_x,k_y) = (7.3.5a) + (7.3.5b)$$

$$= \frac{k_x^2}{k_\perp^2}\, c_{11}(k_\perp,0) + \frac{k_y^2}{k_\perp^2}\, c_{22}(k_\perp,0) \qquad (7.3.8)$$

while

$$c_{22}(k_x,k_y) = (7.3.5b) - (7.3.5a)$$

$$= \frac{k_y^2}{k_\perp^2}\, c_{11}(k_\perp,0) + \frac{k_x^2}{k_\perp^2}\, c_{22}(k_\perp,0) \qquad (7.3.9)$$

The form of these expressions could have been obtained more simply by rotating the conductivity tensor about the symmetry axis ($\underline{B}_o$) of the physical system.

7.4. The Inhomogeneous Case

To understand the difficulties in calculating the high order derivatives in the inhomogeneous case now that there is this loss of symmetry between rising and falling operators, consider, once again, the 3,3 element of the conductivity tensor. The cold plasma term is of course unaltered, but now the modified element G'_{00} gives two new terms in addition to the parameter gradient term (4.3.10). The first modification,

$$\frac{d}{dx}\left[\left(\frac{i v_\perp}{2\omega_c}\right)^2 (A_1 - A_{-1})\right] k_y D_0 \qquad (7.4.1)$$

is in principle not very different from the b_m terms obtained in Chapter 4. Such an antisymmetric term would have been expected, proportional to the antisymmetric k_y coupling. However, in addition to producing symmetric coefficients of A_1 and A_{-1} the old symmetry also allowed the cancellation of certain second order poles. This was an important property, referred to in section 6 of Chapter 3. The different nature of this second new term is particularly obvious in the example being considered, since the second order pole produced is the A_0 or Landau damping term which cannot arise directly from differentiation of A_0 with respect to x because A_0 is not a function of x.

The new term produced is best expressed as

$$2\left(\frac{iV_L}{2\omega_c}\right)^2\left(\frac{\partial A_o}{\partial \zeta_o}\right)\left(\frac{\partial \zeta_1}{\partial x}\right) D_o\, k_y \qquad (7.4.2)$$

where

$$\frac{\partial \zeta_1}{\partial x} = \frac{\omega_c'}{k_z V_T} \qquad (7.4.3)$$

While there is no difficulty in calculating the contribution of this term to the conductivity tensor, e.g. for a Maxwellian f_o the velocity integral gives

$$i\varepsilon_o \frac{\omega_p^2}{k_z V_T}\frac{V_T^2}{2\omega_c^2}\left(\frac{\partial Y(\zeta_o)}{\partial \zeta_o}\right)\left(\frac{\partial \zeta_1}{\partial x}\right) k_y \qquad (7.4.4)$$

with

$$\frac{\partial Y(\zeta_o)}{\partial \zeta_o} = 2\left[2\zeta_o Z(\zeta_o)(1-\zeta_o^2) - 1 + 2\zeta_o^2\right] \qquad (7.4.5),$$

the existence of such terms which do not fit the existing patterns established in Chapter 4 causes considerable difficulties.

Fortunately, the new terms arising from the introduction of k_y can also be fitted into patterns of a similar nature to those obtained in Chapter 4. Once a general form for the magnetic field gradient terms had been found for the new terms, it could be used in a proof by induction, (appendix C) just as the simpler case (Chapter 4) was proved (appendix B). The full expression for the first order equilibrium magnetic field gradient term is

$$\sum_{j=n+l-m}^{m} \Bigg\{ \Bigg[\frac{2m-l-n}{2} P_{ln}^{m}(j) + \frac{n-l}{2} H_{ln}^{m}(j) \Bigg] A_j' \left(\frac{iV_L}{2\omega_c}\right)^{2m-l-n} \frac{d}{dx} +$$

$$\Bigg[\frac{n-l}{2} P_{ln}^{m}(j) + \frac{2m-l-n}{2} H_{ln}^{m}(j) \Bigg] A_j' \left(\frac{iV_L}{2\omega_c}\right)^{2m-l-n} k_y +$$

$$2(m-l)(m-n)\, P_{ln}^{m-1}(j)\, A_j^2 \left(\frac{iV_L}{2\omega_c}\right)^{2(m-1)-l-n} \left(\frac{iV_L}{2\omega_c}\right)' k_y +$$

$$\Bigg[\Big(n(n-l) + 2(m-l)(m-n)\Big) P_{ln}^{m}(j) + \frac{(n-l)(2m-l-n-1)}{2} H_{ln}^{m}(j) \Bigg] A_j \left(\frac{iV_L}{2\omega_c}\right)^{2m-l-n-1} \left(\frac{iV_L}{2\omega_c}\right)' \frac{d}{dx}$$

$$+ \Bigg\{ n(2m-l-n) P_{ln}^{m}(j) + \Bigg[(m-l)(m-n) + \frac{(2m-l-n)(2m-l-n-1)}{2} \Bigg] H_{ln}^{m}(j) \Bigg] A_j \left(\frac{iV_L}{2\omega_c}\right)^{2m-l-n-1} \left(\frac{iV_L}{2\omega_c}\right)' k_y \Bigg\}$$

$$\frac{\left(\frac{d}{dx} - k_y\right)^{m-l-1} \left(\frac{d}{dx} + k_y\right)^{m-n-1}}{(m-l)!\,(m-n)!} \qquad (7.4.6)$$

Some of the terms in (7.4.6) could have been predicted without resorting to appendix C. Of the (m-n) operators $(\frac{d}{dx} + k_y)$ and the (m-l) operators $(\frac{d}{dx} - k_y)$ only one is applied to an A. Therefore, there is a common factor (row 6 of (7.4.6)) in all terms. Having extracted this factor, all remaining terms must contain either a $\frac{d}{dx}$ or a k_y. Since the terms containing $\frac{d}{dx}$ are independent of k_y, then they must be those obtained in Chapter 4 in order to satisfy (7.4.6) (4.2.5) as $k_y \to 0$. Thus we have the form of rows 1 and 4 in (7.4.6).

Finally, since the $A^2 k_y$ terms arise from products of A that fail to cancel, they must disappear if m=n or m=l. In both cases the parallelogram G_{ln}^{m} contracts into a line in which no A_j

is repeated. Therefore the (m-n)(m-1) factor in row 3 could also be predicted.

The existence of the general form for the element G^{m}_{ln} (figure 4.1) allows the results of this chapter to be combined with those of Chapters 5 and 6 - including the effects of equilibria that do not have cylindrical symmetry, and which require Fourier components outside the 'normal' range 1,0,-1 to describe the 'driving' or zeroeth order terms of the tree diagram.

The only modification to the results of Chapters 5 and 6 necessary, is to include k_y in the expression for curl $\underline{B}_0$. This gives the new Lorentz force term

$$\underline{v} \times \underline{B}_1 = \frac{1}{\omega} \begin{pmatrix} -iv_y E_y' - v_y k_y E_x - v_z k_z E_x - iv_z E_z' \\ v_z k_y E_z - v_z k_z E_y + iv_x E_y' + v_x k_y E_x \\ v_x k_z E_x + iv_x E_z' - v_y k_y E_z + v_y k_z E_y \end{pmatrix} \tag{7.4.7}$$

The resultant driving terms for a cylindrically symmetric velocity distribution are

$$D_1 = \frac{2q}{mv_\perp}\left[\left(P + \frac{k_z}{\omega}Q\right)E_- + \frac{iQ}{2\omega}\left(\frac{d}{dx} + k_y\right)E_z\right]$$

$$D_0 = \frac{2q}{mv_\perp} R E_z$$

$$D_{-1} = \frac{2q}{mv_\perp}\left[\left(P + \frac{k_z}{\omega}Q\right)E_+ + \frac{iQ}{2\omega}\left(\frac{d}{dx} - k_y\right)E_z\right] \tag{7.4.8}$$

which reduces to (5.2.5) for $k_y=0$, while the new expression for the general driving term for a dependent velocity distribution is

$$
\begin{aligned}
D_n = \frac{q}{mV_\perp}\Bigg(& \Big[P_{n+1} + P_{n-1} + \frac{k_z}{\omega}\left(Q_{n+1} + Q_{n-1}\right) + \frac{1}{V_\perp}\left(\frac{k_z V_z}{\omega} - 1\right) W_n + \\
& \frac{2ik_y}{\omega} n f_{on} \Big] E_x + i\Big[P_{n+1} \quad P_{n-1} + \frac{k_z}{\omega}\left(Q_{n+1} - Q_{n-1}\right) - \\
& \frac{1}{V_\perp}\left(\frac{k_z V_z}{\omega} - 1\right) W_n - \frac{2n}{\omega} f_{on} \frac{d}{dx} \Big] E_y + \\
& \Big[2R_n + \frac{ik_y}{\omega}\left(Q_{n-1} - Q_{n+1} + \frac{V_z}{V_\perp} W_n\right) + \\
& \frac{i}{\omega}\left(Q_{n-1} + Q_{n+1} + \frac{V_z}{V_\perp} W_n\right) \frac{d}{dx} \Big] E_z \Bigg)
\end{aligned}
\tag{7.4.9}
$$

which reduces to (6.4.2) for $k_y=0$.

The effect of introducing k_y that would probably have greatest physical significance is the appearance of the harmonic resonances even in the lowest order coefficients of the differential equation. Such an effect would also arise if the equilibrium was not slowly varying.

Chapter 8

Future Work

8.1. Further Generalisation

In this thesis a formalism for obtaining a set of coupled o.d.e's describing a wave in an inhomogeneous plasma has been developed. First for very restricted types of wave in the simplest possible equilibrium plasma in Chapter 3; then in Chapter 4 most of the restrictions on the wave were removed and in Chapters 5 and 6 the allowed equilibrium distributions were fully generalised; finally in Chapter 7 wave propagation outside the $\underline{B}_0 - \nabla B_0$ plane was modelled. By this stage the equations describing most perturbations of almost any plasma equilibrium can be written down - although solving these equations can become difficult (section 8.5). There still remain perturbations for which the methods so far developed in this thesis are not yet sufficiently general to describe.

The most immediately desirable generalisation of the theory established in this thesis would be to remove the restriction to $|k_z L| \gg |n|$. This change would be particularly useful for two main reasons.

8.2. Small Kz

First, this restriction is 'new', arising clearly from the methods of Chapter 3, but missed by other less systematic approaches. As a result, some of the cases previously tackled less rigorously cannot be corrected until a way around this restriction can be found. Although such cases form only a small part of the spectrum of k_z used in experiments (since, for example, in a tokamak the restriction is equivalent to requiring that the toroidal mode number of the wave be greater than the number of the harmonic being excited) it is still hardly satisfactory to knock other theories down for these cases without putting forward a better theory in their place. These cases would also be of considerable interest because of the increasing importance of mode conversion as k_z is reduced. Since it is the amount of mode conversion obtained that is the biggest difference between the solutions of dispersion relation based equations and those of consistently derived o.d.e's, the cases where most mode conversion occurs are probably those where consistency is most important, although one possible exception to this 'rule' is considered below.

8.3. Wave Propagation Perpendicular to B

The second main reason why it would be useful to remove the restriction to $|k_z L| \gg |n|$ is that if all restrictions on k_z could be removed, it would allow accurate modelling of perpendicular wave propagation. Since this case is analysed using other techniques very frequently in the literature, a direct comparison of equations and results for $k_z = 0$ would be very interesting. Not least because mode conversion is very important in this case, where there is no cyclotron damping, and the differences in the results produced by including parameter gradient terms is, in the examples examined so far, predominantly in the amount of mode conversion obtained.

For the examples considered in this thesis the transmission of the fast wave has been almost completely unaltered by the inclusion of the parameter gradient terms, yet the amount of mode conversion is quite dramatically altered. However in the case of perpendicular wave propagation where there is no cyclotron damping, the mode conversion is simply calculated from the power lost from the fast wave. This poses the following question. Do the gradient terms alter the transmission of the fast wave significantly when they are sufficiently large or do they have no effect on the mode conversion factor when the resonance is sufficiently thin? It is by no means impossible that the case of purely perpendicular wave propagation is a limit in which the dispersion relation techniques might well be sufficient.

8.4. The Difficulties

The potential for progress in 8.2 is far from bleak: in effect the $|n| << |k_z L|$ restriction is similar in type to

$$\left| \frac{V_I}{2\omega_c} \frac{\partial E}{\partial x} \right| << |E| \tag{8.4.1}$$

and so by dint of a considerable amount of algebra $|n| \gtrsim |k_z L|$ could quite possibly also be treated, with a similar convergence led by a factorial in the denominator. The algebra involved would become considerably more complex if it was necessary to consider higher spatial derivatives of the equilibrium magnetic field B_o, since in such cases $A_n'' A_n$ is of a different form from $A_n' A_n'$ as now the former includes terms from higher derivatives of B_o than the latter.

$$A_n'' A_n = 2 \left[\frac{2\omega_c'}{iv_\perp} n A_n^2 \right]^2 + \frac{2\omega_c''}{iv_\perp} n A_n^3$$

$$A_n' A_n' = \left[\frac{2\omega_c'}{i v_\perp} n A_n^2 \right]^2 \tag{8.4.2}$$

On the bright side, this development would not, by itself, increase the order of the o.d.e's to be solved, and should therefore not cause major problems for conventional numerical techniques, although the increase in complexity of the coefficients of the o.d.e's would impose a time penalty.

The potential for progress in the limiting case of 8.3 is, however, far less promising, for in this case the resonant A_{-n} would have a pole, not in velocity space, but in physical space.

$$A_{-n} = \frac{v_\perp}{2i(\omega - n\omega_c)} \tag{8.4.3}$$

Far from forming a series that will eventually converge, the successive differentiation of A_{-n} would lead to the coefficients of the o.d.e's having essential singularities.

$$\frac{d^m}{dx^m} A_{-n} = \left(\frac{n\omega_c'}{\omega - n\omega_c}\right)^m m!\, A_{-n} + O\left\{\left(\frac{1}{\omega - n\omega_c}\right)^m\right\} \ldots \tag{8.4.4}$$

This limit would appear to be the breaking point of the techniques developed in this thesis. While there is good reason, on purely physical grounds, to assume that the electric and magnetic field perturbations will be well behaved, there is no corresponding reason for the coefficients of an o.d.e. to behave.

It would seem therefore that the case of $k_z=0$ must be dealt with using a very different approach.

8.5. A Twisted Magnetic Field

In most magnetic confinement devices, the magnetic field not only changes in intensity, but also in direction. For example, in the case of tokamaks and r.f.p's the different spatial dependence of the poloidal and toroidal components leads to the spatial dependence of the direction of the total magnetic field. Since the theoretical modelling of r.f. heating in tokamaks is a major objective of this research, it is clearly important to be able to include the effects of a twisted $\underline{B}_o$.

Part of the motivation for Chapter 7 was that a prerequisite for modelling the effects of a twisted magnetic field is the ability to deal with finite k_y. Clearly if the magnetic field rotates in the y-z plane through an angle θ, then the local values of $\tilde{k}_y$ and $\tilde{k}_z$ are

$$\tilde{k}_y = k_y \cos\theta + k_z \sin\theta \qquad \tilde{k}_z = k_z \cos\theta - k_y \sin\theta \tag{8.5.1}$$

However, in addition to the gradient terms already obtained in Chapter 7, the spatial dependence of $\tilde{k}_y$ and of $\tilde{k}_z$ should also be accounted for. A further complication arises when the new Vlasov equation is considered.

The form of the perturbed Vlasov equation was greatly simplified in earlier chapters by the choice of a coordinate system with one axis parallel to the magnetic field. This allowed the three velocity derivatives of f_1 to be replaced, first by two

$$\frac{q}{m}(\underline{v}\times\underline{B}_o)\cdot\frac{\partial f_1}{\partial \underline{v}} = \omega_c\left(v_y\frac{\partial f_1}{\partial v_x} - v_x\frac{\partial f_1}{\partial v_y}\right) \qquad (8.5.1)$$

and then, through the use of cylindrical coordinates for velocity, one

$$\omega_c\left(v_y\frac{\partial f_1}{\partial v_x} - v_x\frac{\partial f_1}{\partial v_y}\right) = -\omega_c\frac{\partial f_1}{\partial \phi} \qquad (8.5.2)$$

Such a choice of coordinates can now only be local, with the result that the global coordinate system now has a twist to follow the equilibrium magnetic field.

It can be seen that both of these effects are directly caused by the rate of twist of $\underline{B}_o$.

$$\underline{B}_o = (0, B_y(x), B_z(x))$$

$$\tan\theta = B_y(x)/B_z(x)$$

$$\frac{d\theta}{dx} = \frac{\underline{B}_o\cdot(\nabla\times\underline{B}_o)}{B_o^2} \qquad (8.5.3)$$

The second problem, the twist of the coordinate system, has already been tackled in MHD literature (Appert, Vaclavik and Villard, 1984) although the motivation was slightly different. In MHD the advantage in using a local coordinate system or 'magnetic coordinates' ($\underset{\sim}{e}_{\parallel}$ parallel to $\underset{\sim}{B}_o$, $\underset{\sim}{e}_n$ normal to the magnetic surface and $\underset{\sim}{e}_\wedge = \underset{\sim}{e}_{\parallel} \times \underset{\sim}{e}_n$) is that as in the slab model there is a high conductivity in the direction parallel to the magnetic field, and so the local electric field component parallel to $\underset{\sim}{B}_o$ is suppressed. Once again, this allows the set of equations to be reduced to the local 3,3 minor set, i.e. only the perpendicular electric fields need be considered. This reduces the differential equations for the fluid model to only second order, just as the kinetic model equations were reduced to a fourth order system in Chapter 3.

The expression in their paper is

$$\mathrm{rot}\,\mathrm{rot}\,\underset{\sim}{E} - \frac{\omega^2}{c^2}\,\underset{\approx}{\varepsilon}\cdot\underset{\sim}{E} \tag{8.5.4}$$

where rot is a local form of curl, and the operator

$$\underset{\approx}{\varepsilon} = \begin{pmatrix} \varepsilon_{\wedge\wedge} & \varepsilon_{n\wedge} \\ -\varepsilon_{n\wedge} & \varepsilon_{nn} \end{pmatrix} + \frac{c^2}{\omega^2}\,\frac{\underset{\sim}{B}_o\cdot\mathrm{rot}\,\underset{\sim}{B}_o}{B_o^2} \begin{pmatrix} \mathrm{rot}_n e_n & \mathrm{rot}_n e_\wedge \\ \mathrm{rot}_\wedge e_n & \mathrm{rot}_\wedge e_\wedge \end{pmatrix} \tag{8.5.5}$$

The first term in (8.5.5) is the cold plasma dielectric tensor in the local coordinates, while the second term clearly has its origins in the twist of the magnetic coordinates and is proportional to the twist of $\underset{\sim}{B}_o$.

8.6. Partial Differential Equations

As was mentioned in Chapter 5, the magnetic field in confinement devices such as the tokamak is not only twisted, but also no longer depends on only one spatial variable. The perturbing fields are therefore described by partial differential equations (p.d.e's) rather than the o.d.e's obtained in this thesis.

One method for retrieving a system of o.d.e's from such a problem uses the periodicity of angular coordinates to justify expanding the field quantities as Fourier series in those coordinates. For example, in the case of a tokamak, if toroidal symmetry is assumed, then the perturbing fields obey a p.d.e. in θ (the poloidal angle) and r (the minor radius); this equation can then be tackled (Smithe. Colestock, Kashuba and Kammash, 1987) by expanding the fields as Fourier series in θ, just as f_1 was expanded in ϕ in Chapter 3 to obtain a system of o.d.e's from a p.d.e. (the Vlasov equation).

Of course, this technique gives rise to a large number of o.d.e's which have to be solved simultaneously.

8.7. Solving the Differential Equations

While most of this thesis has been devoted to obtaining the correct differential equations, a subject of equal importance is the problem of solving these equations. That the solutions of these systems will, in general, require the use of numerical techniques at some stage is fairly obvious; however the simple minded approach of loading the system into a standard differential equation solver, like that used in Chapter 3, will not always be adequate.

8.8. Reduced Order Differential Equations

In Chapter 4 and Chapter 5 the difficulties involved in solving the very large systems of equations that arise when

$$\left| \frac{v_T}{2\omega_c} \frac{dE}{dx} \right| << |E| \tag{8.8.1}$$

is no longer true, were noted. To avoid these problems some method of reducing the order of the differential equations must be found. One way of achieving this objective was mentioned in Chapter 2 (Cairns and Lashmore-Davies,1983). Despite doubts about the methods used to obtain the original equation, the basic strategy, that of only considering the modes that are coupled in the particular region of space being modelled, has much to recommend it. Although simply calculating the eigenvalues and their derivatives for a large system of o.d.e's at all points on a spatial mesh would not be trivial, once these were known the coupling terms could be calculated. Even if it was not possible to split the plasma into regions of binary coupling, i.e. regions where only the coupling terms between two of the modes are significant, it should still be possible to greatly reduce the number of equations being solved in any one region, with a corresponding saving in computer space.

Certainly it is true that numerically solving a very large system of o.d.e's directly would be an inefficient method, particularly if only one or two of the solutions of the equations are of interest.

Further research into these methods might also clarify the robust nature of fast wave transmission coefficients. As was pointed out in Chapter 3, while the quantity of mode conversion produced by the consistently derived equations is dramatically different from that produced by equations from the homogeneous dispersion relation, the transmission factor for the fast wave for each case is practically identical. While this robustness was already evidenced by the variety of equations that have been used to give accurate values for fast wave transmission, it has perhaps not yet been satisfactorily explained, although a link between the cumulative change in the perpendicular wavenumber of a mode and the relative importance of parameter gradient terms in modelling the mode was suggested in the conclusions of Chapter 3.

Appendix A.

$$S_{ln}^{m} = \sum_{j=n+l-m}^{m} \frac{1}{(m-n)!(m-l)!} \binom{2m-l-n}{m-j} (-1)^{m-j} A_j \left(\frac{iv_\perp}{2\omega_c}\right)^{2m-l-n}$$

Clearly true for 2m-1-n=0

Assume true for 2m-1-n=r, then for 2m-1-n=r+1

$$S_{Ln}^{m} = A_L \left(S_{L+1\,n}^{m} + S_{L-1\,n}^{m-1}\right)$$

$$= \sum_{j=n+L-m}^{m} \frac{-(-1)^{m-j} A_L \,(2m-L-n-)!}{(m-L)!(m-n)!(m-j)!(m-L-n+j)!} \Big[(m-n)(m-j) -$$

$$(m-L)(m-L-n+j)\Big] A_j \left(\frac{iv_L}{2\omega_c}\right)^{2m-L-n-1}$$

$$= \frac{A_L}{(m-L)!(m-n)!} \left(\frac{iv_L}{2\omega_c}\right)^{2m-L-n-1} \sum_{j=n+L-m}^{m} \frac{(-1)^{m-j}(2m-L-n-1)!}{(m-j)!\,(m-L-n+j)!}$$

$$\left[2m-L-n\right](j-L)\, A_j$$

Noting that the coefficient of A_L inside the summation is 0

$$S_{Ln}^{m} = \sum_{\substack{j=n+L-m \\ j\neq L}}^{m} \frac{(-1)^{m-j}}{(m-L)!(m-n)!} \binom{2m-L-n}{m-j} \left(\frac{iv_\perp}{2\omega_c}\right)^{2m-L-n} (A_j - A_L)$$

then using

$$\sum_{j=n+L-m}^{m} (-1)^{m-j} \binom{2m-L-n}{m-j} = 0$$

$$S_{Ln}^{m} = \sum_{j=n+L-m}^{m} \frac{(-1)^{m-j}}{(m-n)!(m-L)!} \binom{2m-L-n}{m-j} \left(\frac{iv_L}{2\omega_c}\right)^{2m-L-n} A_j$$

Appendix B.

$$T^m_{Ln} = \frac{1}{(m-L)!(m-n)!} \sum_{j=L+n-m}^{m} \left\{ \left[\frac{2m-L-n}{2} P^m_{Ln}(j) + \frac{n-L}{2} H^m_{Ln}(j)\right] A_j'\left(\frac{iv_\perp}{2\omega_c}\right) + \right.$$

$$\left. \left(\frac{iv_\perp}{2\omega_c}\right)^{2m-L-n-1} \left[\left[n(n-L) + 2(m-L)(m-n)\right] P^m_{Ln}(j) + \frac{(n-L)(2m-L-n-1)}{2} H^m_{Ln}(j)\right] A_j\left(\frac{iv_\perp}{2\omega_c}\right)' \right\}$$

Clearly true for 2m-1-n=0

Assume true for 2m-1-n=r, then for 2m-1-n=r+1

$$T^m_{Ln} = A_L\left(T^m_{L+1\,n} + T^m_{L-1\,n} + \frac{d}{dx}\left[S^m_{L+1\,n} + S^m_{L-1\,n}\right]\right]$$

$$= A_L \sum_{j=L+n-m}^{m} c_1 \left(c_2 A_j'\left(\frac{iv_\perp}{2\omega_c}\right) + c_3 A_j\left(\frac{iv_\perp}{2\omega_c}\right)' \right]$$

$$c_1 = \frac{(2m-L-n-2)!\;(-1)^{m-j}}{(m-L)!(m-n)!(m-j)!(m-L-n+j)!} \left(\frac{iv_\perp}{2\omega_c}\right)^{2m-L-n-2}$$

$$c_2 = \left[\frac{(2m-L-n-1)^2}{2} + \frac{(n-L-1)(2j-L-n-1)}{2}\right](m-L)(m-L-n+j) -$$

$$\left[\frac{(2m-L-n-1)^2}{2} + \frac{(n-L+1)(2j-L-n+1)}{2}\right](m-n)(m-j) +$$

$$(2m-L-n-1)\left[(m-L)(m-L-n+j) - (m-n)(m-j)\right]$$

$$c_3 = \left[(2m-L-n-1)\left[n(n-L-1) + 2(m-L-1)(m-n)\right] + (n-L-1)(2m-L-n-2) \right.$$

$$\left. \frac{(2j-L-n-1)}{2}\right](m-L)(m-L-n+j) - (m-j)(m-n)\left[(2m-L-n-1)\left[n(n-L+1) + \right.\right.$$

$$\left.\left. 2(m-L)(m-n\;\;1)\right] + \frac{(2m-L-n-2)(n-L+1)(2j-L-n+1)}{2}\right] +$$

$$(2m-L-n-1)^2\left[(m-L)(m-L-n+j) - (m-n)(m-j)\right]$$

$$C_2 = \left[\frac{(2m-L-n-1)^2}{2} + \frac{(n-L)(2j-L-n)}{2} + \frac{1}{2} + (2m-L-n-1) \right]$$

$$\Big[(m-L)(m-L-n+j) - (m-n)(m-j)\Big] - (j-L)\Big[(m-L)(m-L-n+j) + (m-n)(m-j)\Big]$$

$$= \left(\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2} \right] (j-L)(2m-L-n) -$$

$$(j-L) \left(\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2} \right]$$

$$\underline{C_2 = (j-L)(2m-L-n-1)\left[\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2} \right]}$$

$j = L \qquad C_2 = 0$

$$j \neq L \qquad A_L A_j' = \left(\frac{iV_\perp}{2\omega_c}\right) \frac{A_j'}{j-L} + \frac{j(A_j - A_L)}{(j-L)^2} \left(\frac{iV_\perp}{2\omega_c}\right)'$$

Therefore $\displaystyle\sum_{j=L+n-m}^{m} C_1 A_L C_2 A_j' \left(\frac{iV_\perp}{2\omega_c}\right)$

gives the required expression for the coefficients of all the A terms, except for the A term.

In addition it gives a contribution to the terms,

$$\underline{\sum_{\substack{j=L+n-m \\ \neq L}}^{m} C_1 j \left(\frac{iV_\perp}{2\omega_c}\right)' \frac{A_j - A_L}{j-L} (2m-L-n-1)\left[\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2}\right]}$$

$$C_3 = \Big\{ (2m-L-n-1)\big[n(n-L) + 2(m-L)(m-n) + n - 2m\big] +$$

$$(2m-L-n-2)\left(\frac{(n-L)(2j-L-n)}{2} - (j-L) + \frac{1}{2} \right]\Big\} (m-L)(m-L-n+j)$$

$$-\Big\{ (2m-L-n-1)\big[n(n-L) + 2(m-L)(m-n) + n - 2m + 2L\big] +$$

$$(2m-L-n-2)\left(\frac{(n-L)(2j-L-n)}{2} + (j-L) + \frac{1}{2} \right]\Big\} (m-n)(m-j) +$$

$$(2m-L-n-1)^2(2m-L-n)(j-L)$$

$$C_3 = (j-L)(2m-L-n)(2m-L-n-1)\left[n(n-L)+2(m-L)(m-n)\right] +$$

$$\frac{(j-L)(2m-L-n-1)^2(n-L)(2j-L-n)}{2} - \frac{(j-L)(n-L)(2j-L-n)}{2}$$

$$+(j-L)(2m-L-n)(2m-L-n-1)(n-2m) + (j-L)(2m-L-n-1)^2(2m-L-n)$$

$$-(j-L)(2m-L-n-2)\left[\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2}\right] +$$

$$L(2m-L-n-1)\left[(2m-L-n)(j-L) - \frac{(2m-L-n)^2}{2} - \frac{(n-L)(2j-L-n)}{2}\right]$$

$$+\frac{(j-L)(2m-L-n)(2m-L-n-2)}{2}$$

$$= (j-L)(2m-L-n)(2m-L-n-1)\left(n(n-L)+2(m-L)(m-n)\right)$$

$$+\frac{(j-L)(2m-L-n-1)^2(n-L)(2j-L-n)}{2} + \frac{(j-L)(2m-L-n)^2}{2}$$

$$-(L+1)(2m-L-n-1)(2m-L-n)(j-L)$$

$$-j(2m-L-n-1)\left[\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2}\right]$$

$$+L(2m-L-n-1)(2m-L-n)(j-L)$$

$$+\frac{(j-L)(2m-L-n)(2m-L-n-2)}{2}$$

$$C_3 = (j-L)(2m-L-n)(2m-L-n-1)\left[n(n-L)+2(m-L)(m-n)\right]$$

$$+\frac{(j-L)(2m-L-n-1)^2(n-L)(2j-L-n)}{2}$$

$$-j(2m-L-n-1)\left[\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2}\right]$$

$$j = L \qquad C_3 = -L\,(2m-L-n-1)\left[\frac{(2m-L-n)^2}{2} - \frac{(n-L)^2}{2}\right]$$

Using
$$-L\,A_L^2\left(\frac{iV_\perp}{2\omega_c}\right)' = A_L'\left(\frac{iV_\perp}{2\omega_c}\right)^2$$

$$C_1 C_3\, A_L^2\left(\frac{iV_\perp}{2\omega_c}\right)'$$

gives the exact A_L' term required.

$$j \neq L \quad \sum_{j=L+n-m}^{m} C_1 A_L C_3 A_j \left(\frac{iV_\perp}{2\omega_c}\right)' \qquad \text{gives}$$

$$\frac{1}{(m-L)!\,(m-n)!} \sum_{\substack{j=L+n-m \\ \neq L}}^{m} \left(\frac{iV_\perp}{2\omega_c}\right)^{2m-L-n-1} \Big\{\big[n(n-L) + 2(m-L)(m-n)\big] P_{Ln}^{m}(j) +$$

$$\frac{(n-L)(2m-L-n-1)}{2} H_{Ln}^{m}(j)\Big\}\,(A_j - A_L)\left(\frac{iV_\perp}{2\omega_c}\right)'$$

$$-\sum_{\substack{j=n+L-m \\ j\neq L}}^{m} C_1\, j \left(\frac{iV_\perp}{2\omega_c}\right)' \frac{A_j - A_L}{j-L}\,(2m-L-n-1)\left[\frac{(2m-L-n)^2}{2} + \frac{(n-L)(2j-L-n)}{2}\right]$$

Therefore by cancelling the bottom term with the contribution from the $A_L A_j'$ terms, and using

$$\sum_{\substack{j=n+L-m \\ \neq L}}^{m} -P_{Ln}^{m}(j) = P_{Ln}^{m}(L) \qquad \sum_{\substack{j=n+L-m \\ \neq L}} -H_{Ln}^{m}(j) = H_{Ln}^{m}(L)$$

The expression for T_{Ln}^{m} is recovered exactly.

$$G_{Ln}^{m} = S_{Ln}^{m}\left(\frac{d}{dx}-k_y\right)^{m-L}\left(\frac{d}{dx}+k_y\right)^{m-n} + T_{Ln}^{m}\left(\frac{d}{dx}-k_y\right)^{m-L-1}\left(\frac{d}{dx}+k_y\right)^{m-n-1}\frac{d}{dx}$$
$$+ U_{Ln}^{m}\left(\frac{d}{dx}-k_y\right)^{m-L-1}\left(\frac{d}{dx}+k_y\right)^{m-n-1} k_y$$

$$U_{Ln}^{m} = \frac{1}{(m-L)!(m-n)!}\sum_{j=n+L-m}^{m}\left(\frac{n-L}{2}P_{Ln}^{m}(j) + \frac{2m-L-n}{2}H_{Ln}^{m}(j)\right)A_j'\left(\frac{iV_L}{2w_c}\right)^{2m-L-n} +$$

$$2(m-L)(m-n)\,P_{Ln}^{m-1}(j)\,A_j^2\left(\frac{iV_L}{2w_c}\right)^{2(m-1)-L-n}\left(\frac{iV_L}{2w_c}\right)' + \Big[n(2m-L-n)\,P_{Ln}^{m}(j) +$$

$$\left((m-L)(m-n) + \frac{(2m-L-n)(2m-L-n-1)}{2}\right)H_{Ln}^{m}(j)\Big]\,A_j\left(\frac{iV_L}{2w_c}\right)^{2m-L-n-1}\left(\frac{iV_L}{2w_c}\right)'$$

($P_{Ln}^{m}(j)$, $H_{Ln}^{m}(j)$, S_{Ln}^{m}, T_{Ln}^{m} defined in Chapter 4)

Expression for G_{Ln}^{m} is true for 2m-1-n=0

If true for 2m-1-n=r, then for 2m-1-n=r+1

$$S_{Ln}^{m} = A_L\left(S_{L+1\,n}^{m} + S_{L-1\,n}^{m-1}\right)$$

already proven in Appendix A

$$T_{Ln}^{m} = A_L\left(T_{L+1\,n}^{m} + T_{L-1\,n}^{m-1} + \frac{d}{dx}\left(S_{L+1\,n}^{m} + S_{L-1\,n}^{m-1}\right)\right)$$

already proven in Appendix B.

Therefore need only show that

$$U_{Ln}^{m} = A_L\left[U_{L+1\,n}^{m} + U_{L-1\,n}^{m-1} + \frac{d}{dx}\left(S_{L+1\,n}^{m} - S_{L-1\,n}^{m}\right)\right]$$

$$U_{Ln}^{m} = A_L \sum_{j=n+L-m}^{m} c_1 \left[c_2 A_j' \left(\frac{iv_\perp}{2\omega_c}\right)^2 + c_3 A_j^2 \left(\frac{iv_\perp}{2\omega_c}\right)' + c_4 A_j \left(\frac{iv_\perp}{2\omega_c}\right)\left(\frac{iv_\perp}{2\omega_c}\right)' \right.$$

$$c_1 = \frac{(2m-L-n-3)! \qquad (-1)^{m-j}}{(m-L)!\,(m-n)!\,(m-j)!\,(m-L-n+j)!} \left(\frac{iv_\perp}{2\omega_c}\right)^{2m-L-n-3}$$

$$c_2 = (2m-L-n-1)(2m-L-n-2)\left\{ \left(\frac{n-L-1}{2} + \frac{2j-L-n-1}{2} + 1\right)(m-L)(m-L-n+j) - \right.$$

$$\left(\frac{n-L+1}{2} + \frac{2j-L-n+1}{2} - 1\right)(m-n)(m-j)$$

$$= \underline{(j-L)(2m-L-n)(2m-L-n-1)(2m-L-n-2)\left(\frac{n-L}{2} + \frac{2j-L-n}{2}\right)}$$

Therefore

$$A_L \sum_{j=n+L-m}^{m} c_1 c_2 A_j' \left(\frac{iv_\perp}{2\omega_c}\right)^2$$

gives the required expression for all the A_j' terms. (N.B. there is no A_L' term). In addition it gives a contribution to the $\left(\frac{iv_\perp}{2\omega_c}\right)'$ term

$$\underline{\sum_{\substack{j=n+L-m \\ \neq L}}^{m} c_1 j \left(\frac{iv_\perp}{2\omega_c}\right)\left(\frac{iv_\perp}{2\omega_c}\right)' (A_j - A_L)\,(2m-L-n)(2m-L-n-1)(2m-L-n-2)}$$

$$c_3 = 2(m-L)(m-n)(m-j)(m-L-n+j)\left[(m-L-1)(m-L-n+j) - (m-n-1)(m-j-1)\right]$$

$$= \underline{(j-L)(2m-L-n-2)(m-j)(m-L-n+j)\,2(m-L)(m-n)}$$

Therefore

$$A_L \sum_{j=n+L-m}^{m} c_1 c_3 A_j^2 \left(\frac{iv_\perp}{2\omega_c}\right)'$$

gives the required expression for all A_j^2 terms except A_L^2. In addition it gives a contribution to the $\left(\frac{iv_\perp}{2\omega_c}\right)'$ term

$$\sum_{\substack{j=n+L-m \\ \neq L}}^{m} c_1 \left(\frac{iv_\perp}{2\omega_c}\right)\left(\frac{iv_\perp}{2\omega_c}\right)' \left(\frac{A_j - A_L}{j-L}\right)(2m-L-n-2)(m-j)(m-L-n+j)\,2(m-L)(m-n)$$

$$
\begin{aligned}
C_4 = {} & (2m-l-n-2)\Big\{ \Big[n(2m-l-n-1)^2 + (m-l-1)(m-n)(2j-l-n-1) + \\
& \frac{(2m-l-n-1)(2m-l-n-2)(2j-l-n-1)}{2} + (2m-l-n-1)^2 \Big](m-l)(m-l-n+j) - \\
& \Big[n(2m-l-n-1)^2 + (m-l)(m-n-1)(2j-l-n+1) - (2m-l-n-1)^2 + \\
& \frac{(2m-l-n-1)(2m-l-n-2)(2j-l-n+1)}{2} \Big](m-n)(m-j) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= {} & (2m-l-n-2)\Big\{ (j-l)\Big[n(2m-l-n)^2(2m-l-n-1) - n(2m-l-n)(2m-l-n-1) \\
& + (m-l)(m-n)(2m-l-n-1)(2j-l-n) + (m-l)(m-n)(2j-l-n) + \\
& \frac{(2m-l-n)(2m-l-n-1)^2(2j-l-n)}{2} - \frac{(2m-l-n-1)(2m-l-n)(2j-l-n)}{2} \Big] + \\
& \Big[(2m-l-n-1)^2 - (m-l)(m-n) - \frac{(2m-l-n-1)(2m-l-n-2)}{2} \Big]\Big[(m-l)(m-l-n+j) + (m-n)(m-j) \Big] \\
& + (m-l)(m-n)\Big[(m-j)(2j-l-n+1) - (m-l-n+j)(2j-l-n-1) \Big] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= {} & (2m-l-n-2)\Big\{ (j-l)\Big[n(2m-l-n)^2(2m-l-n-1) + (m-l)(m-n)(2m-l-n-1)(2j-l-n) + \\
& \frac{(2m-l-n)(2m-l-n-1)^2(2j-l-n)}{2} + (m-l)(m-n)\,2(j-n) + (n-l)(m-l)(m-n) \\
& - j(2m-l-n)(2m-l-n-1) + \frac{l-n}{2}(2m-l-n-1)(2m-l-n) \Big] + \\
& \Big[(2m-l-n-1)\frac{(2m-l-n)}{2} - (m-l)(m-n) \Big]\Big[2(m-l)(m-n) + (j-l)(n-l) \Big] + \\
& (m-l)(m-n)\Big[2m-l-n - (2j-l-n)^2 \Big] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= {} & (2m-l-n-2)\Big\{ (j-l)\Big[n(2m-l-n)^2(2m-l-n-1) + (m-l)(m-n)(2m-l-n-1)(2j-l-n) \\
& + \frac{(2m-l-n)(2m-l-n-1)^2(2j-l-n)}{2} - j(2m-l-n-1)(2m-l-n) \Big] + \\
& (m-l)(m-n)\Big[2(j-n)(j-l) + (2m-l-n)^2 - 2(m-l)(m-n) - (2j-l-n)^2 \Big] \Big\}
\end{aligned}
$$

$$C_4 = (2m-L-n-2)\Big\{ (j-L)(2m-L-n-1)\Big[n(2m-L-n)^2 + \Big[(m-L)(m-n) + \frac{(2m-L-n)(2m-L-n-1)}{2}\Big](2j-L-n) - j(2m-L-n)\Big] + 2(m-L)(m-n)(m-j)(m-L-n+j)\Big\}$$

For j=1

$$C_4 = 2(m-L)(m-n)(2m-L-n-2)(m-j)(m-L-n+j)$$

Therefore

$$A_L C_1 C_4 A_L \left(\frac{iv_\perp}{2\omega_c}\right)\left(\frac{iv_\perp}{2\omega_c}\right)'$$

gives the missing A_L^2 term.

For $j \neq 1$

$$A_L \sum_{j=n+L-m}^{m} C_1 C_4 A_j \left(\frac{iv_\perp}{2\omega_c}\right)\left(\frac{iv_\perp}{2\omega_c}\right)'$$

$$= \frac{\dfrac{1}{(m-L)!(m-n)!}\displaystyle\sum_{j=n+L-m \neq L}^{m}\Big(n(2m-L-n)P_{Ln}^{m}(j) + \big[(m-L)(m-n) + (2m-L-n)(2m-L-n-1)\big]H_{Ln}^{m}(j)\Big)}{\underline{(A_j - A_L)\left(\frac{iv_\perp}{2\omega_c}\right)'\left(\frac{iv_\perp}{2\omega_c}\right)^{2m-L-n-1}}},$$ minus a term

$$\sum_{\substack{j=n+L-m \\ \neq L}}^{m} C_1 j \left(\frac{iv_\perp}{2\omega_c}\right)\left(\frac{iv_\perp}{2\omega_c}\right)'(A_j - A_L)(2m-L-n)(2m-L-n-1)(2m-L-n-2)$$

which cancels with the contribution from C_2, plus a term

$$\sum_{\substack{j=n+L-m \\ \neq L}}^{m} C_1 \left(\frac{iv_\perp}{2\omega_c}\right)\left(\frac{iv_\perp}{2\omega_c}\right)'\left(\frac{A_j - A_L}{j-L}\right)(2m-L-n-2)(m-j)(m-L-n+j)\,2(m-L)(m-n)$$

which cancels with the contribution from C_3.

Finally, the identities

$$\sum_{\substack{j=n+L-m \\ \neq L}}^{m} P_{Ln}^{m}(j)(A_j - A_L) = \sum_{j=n+L-m}^{m} P_{Ln}^{m}(j)A_j \quad \text{and} \quad \sum_{\substack{j=n+L-m \\ \neq L}}^{m} H_{Ln}^{m}(j)(A_j - A_L) = \sum_{j=n+L-m}^{m} H_{Ln}^{m}(j)A_j$$

when applied to the underlined expression, give all the $\left(\frac{iv_\perp}{2\omega_c}\right)'$ terms, completing the proof.

Appendix D.

$$(1+x)^{m-n}(1+x)^{m+n} = (1+x)^{2m}$$

Applying the Binomial Theorem

$$\sum_{s=0}^{m-n}\binom{m-n}{s}x^s\sum_{t=0}^{m+n}\binom{m+n}{t}x^t = \sum_{r=0}^{2m}\binom{2m}{r}x^r$$

comparing coefficients of x^r

$$\sum_{s=0}^{m-n}\binom{m-n}{s}\binom{m+n}{r-s} = \binom{2m}{r}$$

$$\sum_{s=0}^{m-n}\frac{(-1)^{m-n}}{s!(m-n-s)!(r-s)!(m+n-r+s)!} = \frac{(-1)^{m-n}}{(m-n)!(m+n)!}\binom{2m}{r}$$

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